Extreme Points of Unital Quantum Channels

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joint work with U. Haagerup and M. Musat

QMATH13 October, 2016

Overview

- Review: Stinespring, extreme points conds, etc.
- Family of factorizable extreme UCPT maps

  extreme mixed states with max mixed quant marginals
- Extreme points of CPT and UCP with Choi-rank d

  Kraus ops are partial isometries and generalization
- Example for $d = 2\nu + 1$ odd
- Universal example

  Reformulate linear independence as eigenvalue problem

  Associate eigenvectors (lin dep) with irreps of S_n

Complete positivity

Def: $\Phi : M_{d_A} \mapsto M_{d_B}$ is completely positive (CP) if $\Phi \otimes \mathcal{I}_{d_E}$ preserves positivity $\forall\, d_E$. Suffices to consider $d_E = \min\{d_A, d_B\}$

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Thm: (Choi) Φ is CP $\Leftrightarrow J_\Phi = \sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi\big(|e_j\rangle\langle e_k|\big) \geq 0$

Quantum Channel: Φ is CP and trace-preserving (CPT)

TP means $\operatorname{Tr} \Phi(A) = \operatorname{Tr} A \quad \forall\ A \in M_{d_A}$

Φ UCP if unital, i.e., $\Phi(I_{d_A}) = I_{d_B}$ and CP

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$\Phi : M_{d_A} \mapsto M_{d_B}$ is TP $\Leftrightarrow$ $\widehat{\Phi} : M_{d_B} \mapsto M_{d_A}$ is unital

$\widehat{\Phi}$ adjoint wrt Hilb-Schmidt inner prod. $\mathrm{Tr}\,[\widehat{\Phi}(A)]^* B = \mathrm{Tr}\,A^* \Phi(B)$

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Choi obtained F_k by "stacking" e-vec of J_Φ with non-zero evals

Thm: (Choi) Φ is extreme in set of CP maps with $\sum_k F_k^* F_k = X$

$\Leftrightarrow$ $\{F_j^* F_k\}$ is linearly independent.

$\Rightarrow \Phi = \sum_k F_k A F_k^*$ extreme CPT map $\Leftrightarrow$ $\{F_j^* F_k\}$ is lin indep.

$\Rightarrow \Phi = \sum_k F_k A F_k^*$ extreme UCP map $\Leftrightarrow$ $\{F_j F_k^*\}$ is lin indep.

Cor: extreme CPT $\Rightarrow$ $d_E \leq d_B$ extreme UCP $\Rightarrow$ $d_E \leq d_A$

Factorizable maps on matrix algebras

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Question: "small environment"

For $\Phi : M_d \mapsto M_d$ can one make environment $d_E \leq d$ if replace $|\phi\rangle\langle\phi|$ by DM γ s. t. $\Phi(\rho) = \mathrm{Tr}_E U\big(\rho \otimes \gamma\big) U^*$

More general: arbitrary γ rather than max mixed $\frac{1}{d} I_d$

More restrictive: $\rho \in M_d$ rather than higher dim environment

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Question: "small environment" two groups showed false $\approx$ 1999

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UCPT maps

Question: Are there UCPT maps $\Phi : M_d \mapsto M_d$ not extreme in either UCP or CPT maps, but are extreme in UCPT maps.

Thm: (Landau-Streater) $\Phi : M_d \mapsto M_d$ is extreme in set of UCPT maps $\Leftrightarrow$ $\{A_j^* A_k \oplus A_k A_j^*\}$ linearly independent $\Phi(\rho) = \sum_k A_k \rho A_k^*$

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By C-J isomorphism convex set of UCPT maps isomorphic to bipartite states with maximally mixed quantum marginals

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Def: Entanglement of Formation

$$\mathrm{EoF}(\rho_{AB}) = \inf \Big\{ \sum_k x_k E(\psi_k) : \sum_k x_k |\psi_k\rangle\langle\psi_k| = \rho_{AB} \Big\}$$

$$E(\psi_{AB}) = S(\rho_A), \quad \rho_A = \mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|, \quad S(\rho) = -\mathrm{Tr}\, \rho \log \rho$$

Known results about extreme points of CPT maps

- Qubit channels $\Phi : M_2 \mapsto M_2$
  - * Ruskai, Szarek Werner (2002) all extreme points
  - * UCPT much earlier, essent conj with I_2 or Pauli matrix

    correspond to max entangled Bells states – tetrahedron
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- General UCPT $\Phi : M_d \mapsto M_d$ unitary conj are extreme
- Few other results — very special
  - * $d = 3$ Werner-Holevo channel and symmetric variant ext.

    not true for Werner-Holevo when $d > 3$
  - * Arveson-Ohno examples – few high rank in low dims

    one low rank family using partial isometries

Family of high rank extreme points of UCPT maps

$$\Phi_{\alpha,\beta}(\rho) = \sum_{k=1}^{4} A_k^* \rho A_k$$

Def: For $|\alpha|^2 + |\beta|^2 = 1$ let

$$A_1 = \alpha|e_1\rangle\langle e_1| + |e_2\rangle\langle e_3| \qquad A_2 = \beta|e_1\rangle\langle e_3| + |e_3\rangle\langle e_2|$$
$$A_3 = |e_1\rangle\langle e_2| + \overline{\beta}|e_3\rangle\langle e_1| \qquad A_4 = |e_2\rangle\langle e_1| + \overline{\alpha}|e_3\rangle\langle e_3|$$

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Observe $U = \begin{pmatrix} A_1 & A_2 \\ -A_3 & A_4 \end{pmatrix}$ is unitary $\in M_3 \otimes M_2$

$$\Rightarrow \quad \Phi_{\alpha,\beta}(\rho) = \sum_{k=1}^{4} (\mathcal{I}_3 \otimes \mathrm{Tr}\,)\big(U^* \big(\rho \otimes \tfrac{1}{2} I_2\big) U\big)$$ factorizable

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Thm: $\Phi_{\alpha,\beta}$ is an extreme UCPT map for $\alpha, \beta \neq 0, \frac{1}{2}, 1$

corresponds to N and S poles and equator on Bloch sphere

Sketch proof:

$\alpha = \cos\theta, \ \beta = \sin\, \theta e^{i\phi}$ $\Phi_{\alpha,\beta}$ assoc with qubit pure state

$$\begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix} = \tfrac{1}{2}\big[I + \sin 2\theta\, \sin\phi\ \sigma_x - \sin 2\theta\, \cos\phi\ \sigma_y + \cos 2\theta\ \sigma_z\big]$$

$j = k$ can verify $\{A_k^* A_k \oplus A_k A_k^*\} \in \mathrm{span}\{|e_j\rangle\langle e_j|\}$ linearly indep

Direct calc shows $A_j^* A_k \oplus A_k A_j^*$ for $j \neq k$ splits into 4 disjoint sets

can verify lin indep $\Leftrightarrow \alpha, \beta, \neq 0, \frac{1}{2}, 1$ calc det of 3×3

$$\mathrm{EoF}(\rho_{AB}) = \tfrac{1+|\alpha|^2}{3} h\big(\tfrac{1}{1+|\alpha|^2}\big) + \tfrac{2-|\alpha|^2}{3} h\big(\tfrac{1}{2-|\alpha|^2}\big)$$

$$\tfrac{2}{3} \leq \mathrm{EoF}(\rho_{AB}) \leq h\Big(\frac{1}{3}\Big) = 0.918296 < 1 = \log 2 < \log 3$$

poles equator

Extreme points of UCP and CPT maps with rank d

$\{V_1, V_2, \ldots V_d\}$ unitary $\in M_{d-1}$, $\quad S = \sum_k |e_k\rangle\langle e_{k+1}|$ cyclic shift

$$A_m = \frac{1}{\sqrt{d-1}} S^m \begin{pmatrix} V_m & 0 \\ 0 & 0 \end{pmatrix} S^{d-m}$$

$$A_m^* A_m = A_m A_m^* = \tfrac{1}{d-1}(I_d - |e_m\rangle\langle e_m|)$$

$$\Rightarrow \; \textstyle\sum_m A_m^* A_m = A_m A_m^* = I_d$$

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Generalize $\qquad A_m = \frac{1}{\sqrt{d-1+t^2}} S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m}$

For $t \in (-1, 1)$, $\quad A_m^* A_m = A_m A_m^* = \frac{1}{d-1}\big[I_d - (1-t^2)\big]|e_m\rangle\langle e_m|)$

Choi rank d which suggests extreme

Almost always extreme

Thm: For $t \in (-1,1)$ fixed and $A_m = S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m}$ if $\exists$ one example with $\{A_m^* A_n\}$ linearly indep, then for almost every choice of unitary $V_1, V_2, \ldots V_d$ the set $\{A_m^* A_n\}$ is lin indep.

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Proof idea: $\{v_j\}$ lin indep iff gram matrix $g_{jk} = \langle v_j, v_k \rangle$ non-sing

For matrices $\{A_m A_n^*\}$ with Hilbert-Schmidt inner prod this is

$$g_{jk,mn} = \operatorname{Tr}(A_j A_k^*)(A_m A_n^*)^* = \operatorname{Tr} A_j A_k^* A_n A_m^*$$

det G is a poly in elements u_{jk}^m of matrices V_m.

If poly not ident. zero, roots an algebraic variety of measure zero

Aside

Notation: $|\mathbb{1}_d\rangle$ denotes the vector whose elements are all $d^{-1/2}$.

If $x_{jk} = \begin{cases} \alpha, & j = k \\ \beta & j \neq k \end{cases}$ then $X = d\beta|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + (\alpha - \beta)I_d$

$\Rightarrow$ e-vals of X are $\alpha - \beta$ with mult $d - 1$ and $\alpha + (d-1)\beta$

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$$A_m = S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m} \qquad A_m^* A_m = I_d - (1 - t^2)|e_m\rangle\langle e_m| \text{ diag.}$$

Matrix with rows given by these diags is $d|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + (t^2 - 1)I_d$

with e-vals $t^2 + (d-1)$ and $t^2 - 1 \neq 0$ for $t \in (-1, 1)$.

$\Rightarrow \{A_m^* A_m\}$ lin indep and $\Rightarrow \text{span}\{A_m^* A_m\} = \text{span}\{|e_j\rangle\langle e_j|\}$

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$\Rightarrow \{A_m^* A_m\}$ lin indep and $\Rightarrow$ $\text{span}\{A_m^* A_m\} = \text{span}\{|e_j\rangle\langle e_j|\}$

$\Rightarrow$ For purpose of determining lin indep of $\{A_m A_n^*\}$ can make arbitrary modifications to diagonal of $A_m A_n^*$

Main example

$S = \sum_k |e_k\rangle\langle e_{k+1}|$ cyclic shift

unitary $V_1 = V_2 = \dots V_d = V \equiv 2|\mathbb{1}_{d-1}\rangle\langle\mathbb{1}_{d-1}| - I_{d-1}$

$$A_1 = \begin{pmatrix} t & 0 \\ 0 & V \end{pmatrix} \qquad A_m = S^{-m} A_1 S^m = S^{-1} A_{m-1} S$$

Note that $A_m = A_m^* \;\Rightarrow\;$ suffices to consider lin indep of $\{A_m A_n\}$

Thm: For $d \geq 3$ and $t \in (-1,1)$ and $t \neq -\frac{1}{d-1}$
the set $\{A_m A_n\}$ is linearly independent

Cor: For $d \geq 3$, $t \in (-1,1)$, $t \neq -\frac{1}{d-1}$ map $\Phi(\rho) = \sum_m A_m \rho A_m^*$
is an extreme point of both the UCP and CPT maps.

More refined results

a) For $d \geq 3$ and $t = 1$, the sets $\{A_m^2\}_{m=1}^d$ and $\{A_m A_n - A_n A_m\}_{m<n}$ are each separately linearly dependent.

b) For $d \geq 3$, $t = -1$, the set $\{A_m^2\}_{m=1}^d$ is linearly dependent but the set $\{A_m A_n\}_{m \neq n}$ is linearly independent.

c) For $d \geq 4$, $t = \frac{-1}{d-1}$, the set $\{A_m^2\}_{m=1}^d$ is linearly independent, but $\sum_{m \neq n} A_m A_n$ is a multiple of I_d so that $\{A_m A_n\}$ is linearly dependent. Moreover, $\{A_m A_n - A_n A_m\}_{m<n}$ and $\{A_m A_n + A_n A_m\}_{m<n}$ are each linearly dependent.

d) For $d = 3, t = \frac{-1}{d-1}$, the set $\{A_m^2\}_{m=1}^d$ is linearly independent, but $\sum_{m \neq n} A_m A_n = 0 \Rightarrow \{A_m A_n + A_n A_m\}_{m<n}$ is linearly depend. Moreover, $\{A_m A_n - A_n A_m\}_{m<n}$ is also linearly dependent.

$d = 2\nu + 1$ odd

$$P_m = \sum_{j=1}^{d} |e_j\rangle\langle e_{2m-j}| \qquad A_m = P_m - (1-t)|e_m\rangle\langle e_m|$$

$P_m = P_m^*$ is perm matrix for ν swaps $(m+k, m-k) \;\Rightarrow\; P_m^2 = I_d$

$A_m A_{m+\ell} = S^{2\ell} - (1-t)\big(|e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|\big) + \delta_{\ell,0}(1-t)^2|e_m\rangle\langle e_m|$

linear independence of $\{A_m A_n\}$ reduces to lin indep of vectors

$|e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|$ with ℓ fixed – reduce to prob in $\mathbf{C}_d$

Find $\Phi(\rho) = \sum_m A_m \rho A_m^*$ is extreme in both CPT and UCP maps.

Fixed point $|e_m\rangle\langle e_m| \;\mapsto\; t|e_m\rangle\langle e_m|$ plays central role

Does not generalize to even $d = 2\nu$ in natural way

$d = 2\nu + 1$ odd (cont).

$$S_\nu = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ \vdots & & \ddots & \vdots & \iddots & & \vdots \\ 0 & 0 & \dots & t & \dots & 0 & 0 \\ \vdots & & \iddots & \vdots & \ddots & & \vdots \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix} \qquad S_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Existence of "fixed point" on skew diagonal seems key

Return to main example — form of A_m

$$A_1 = \frac{1}{d-1} \begin{pmatrix} t(d-1) & 0 & \dots & & \dots & 0 \\ 0 & d-3 & 2 & \dots & & 2 \\ 0 & 2 & d-3 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & 2 & \dots & & 2 & d-3 \end{pmatrix}$$

$$A_d = \frac{1}{d-1} \begin{pmatrix} d-3 & 2 & \dots & & 2 & 0 \\ 2 & d-3 & 2 & \dots & 2 & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ & & & & & \\ 2 & \dots & & 2 & d-3 & 0 \\ 0 & \dots & & \dots & 0 & t(d-1) \end{pmatrix}$$

Sketch proof for main example

$$V = 2|\mathbb{1}_{d-1}\rangle\langle\mathbb{1}_{d-1}| - I_{d-1}, \qquad A_m = S^{-m}\begin{pmatrix} t & 0 \\ 0 & V \end{pmatrix} S^m$$

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$$A_1 A_d = \frac{1}{(d-1)^2}\begin{pmatrix} \tau & b & \dots & b & \dots & b & 0 \\ a & & & & & & b \\ \vdots & & & & & & \vdots \\ a & & & \widetilde{V}^2_{d-2} & & & b \\ \vdots & & & & & & \vdots \\ a & & & & & & b \\ u & a & \dots & a & \dots & & \tau \end{pmatrix}$$

$$a = 2(d-3),\ \ b = 2t(d-1),\ \ \tau = -t(d-1)(d-3)$$

$$u = 4(d-2), \qquad \widetilde{V}^2_{d-2} = -4c|\mathbb{1}_{d-2}\rangle\langle\mathbb{1}_{d-2}| + bI_{d-2}$$

First reformulation

$$\sum_m \sum_n A_m A_n = p_d(t)|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + q_d(t) I_d$$

$q(t) \neq 0$ if $d > 3$ $\qquad$ $p(t) \neq 0$ if $t \neq \frac{-1}{d-1}$

$\Rightarrow \{A_m A_n\}$ lin dependent for $t = \frac{-1}{d-1}$

First reformulation

$$\sum_m \sum_n A_m A_n = p_d(t)|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + q_d(t) I_d$$

$q(t) \neq 0$ if $d > 3$ $\qquad$ $p(t) \neq 0$ if $t \neq \frac{-1}{d-1}$

$\Rightarrow \{A_m A_n\}$ lin dependent for $t = \frac{-1}{d-1}$

For purpose of linear independ for $d > 3, t \neq \frac{-1}{d-1}$ can replace

$$A_m A_n \mapsto X_{mn} \equiv (d-1)^2 A_m A_n + 4d|\mathbb{1}_d\rangle\langle\mathbb{1}_d|$$

$$= \widehat{u}|e_m\rangle\langle e_n| + \widehat{a} \sum_{j \neq m,n} \big(|e_j\rangle\langle e_m| + |e_n\rangle\langle e_j|\big) + \widehat{b} \sum_{j \neq m,n} \big(|e_m\rangle\langle e_j| + |e_j\rangle\langle e_n|\big)$$

where $\widehat{a} = 2(d-1)$, $\widehat{u} = 4(d-1)$, $\widehat{b} = 2t(d-1) + 4$.

Results hold for $d = 3$ but proofs need special handling.

$$X_{mn} = \begin{array}{cc} & \begin{array}{cccccccc} & & & m & & n & & \end{array} \\ \begin{array}{c} \\ \\ \\ m \\ \\ \\ \\ n \\ \\ \\ \end{array} & \begin{pmatrix} \cdot & \cdot & \cdot & \widehat{a} & \cdot & \widehat{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \widehat{a} & \cdot & \widehat{b} & \cdot & \cdot \\ \widehat{b} & \dots & \widehat{b} & 0 & \widehat{b} & 0 & \widehat{b} & \dots \\ \cdot & \cdot & \cdot & \widehat{a} & \cdot & \widehat{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \widehat{a} & \cdot & \widehat{b} & \cdot & \cdot \\ \widehat{a} & \dots & \widehat{a} & \widehat{u} & \widehat{a} & 0 & \widehat{a} & \dots \\ \cdot & \cdot & \cdot & \widehat{a} & \cdot & \widehat{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot \end{pmatrix} \end{array}$$

Permutational symmetry

Observe $\{A_m A_n\}$ linearly dep $\Leftrightarrow$ $\exists$ a matrix C such that

$$0 = \widehat{u} c_{jk} + \sum_m \Big[\widehat{a}\big(c_{mj} + c_{km}\big) + \widehat{b}\big(c_{jm} + c_{mk}\big) \Big]$$

Moreover, such C form a subspace $\mathcal{N}$ of M_d with properties

- $C \in \mathcal{N} \;\Rightarrow\; C^* \in \mathcal{N}$
- $C \in \mathcal{N} \;\Rightarrow\; P^* C\, P \in \mathcal{N} \quad \forall$ permutation matrices P

Decompose into subspaces $\mathcal{N}_\nu$ assoc with irreducible rep of S_d

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Now define $\qquad X^{\pm}_{mn} = X_{mn} \pm X^*_{mn}$

Consider linear indep. of $X_{mn} + X^*_{mn}$ and $X_{mn} - X^*_{mn}$ separately

$$
X_{mn}^{\pm}(x) = \begin{array}{c c}
 & \begin{array}{cccccccc} & & & m & & n & & \end{array} \\
\begin{array}{c} \\ \\ \\ m \\ \\ \\ \\ n \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \pm1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \pm1 & \cdot & 1 & \cdot & \cdot \\
1 & \ldots & 1 & 0 & 1 & {\color{blue}x} & 1 & \ldots \\
\cdot & \cdot & \cdot & \pm1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \pm1 & \cdot & 1 & \cdot & \cdot \\
\pm1 & \ldots & \pm1 & \pm{\color{blue}x} & \pm1 & 0 & \pm1 & \ldots \\
\cdot & \cdot & \cdot & \pm1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \vdots & \cdot & \vdots & \cdot & \cdot
\end{array} \right)
\end{array}
$$

Can ignore factor of $(\widehat{a} \pm \widehat{b})$ Main interest $x = w_d^{\pm}(t)$

Reformulate as eigenvalue problem

$$\begin{aligned} X^{\pm}_{mn}(w^{\pm}_d) &= \pm(\widehat{a} \pm \widehat{b})\Big[w^{\pm}_d \big(|e_m\rangle\langle e_n| \pm |e_n\rangle\langle e_m|\big) \\ &+ \sum_{j\neq m,n} \big(|e_j\rangle\langle e_m| + |e_n\rangle\langle e_j|\big) \pm \sum_{k\neq m,n} \big(|e_m\rangle\langle e_k| + |e_k\rangle\langle e_n|\big)\Big] \end{aligned}$$

$$w^{+}_d(t) = \tfrac{2d}{d+1+t(d-1)} \qquad w^{-}_d(t) = \tfrac{2(d-2)}{(d-3)-t(d-1)}$$

Let $\Omega^{\pm}_d(x)$ be $\frac{1}{2}d(d-1) \times \frac{1}{2}d(d-1)$ matrix with rows given by elements of $X^{\pm}_{mn}(x)$ above diagonal in lexigraphic order

Elements of $\Omega^{\pm}_d(x)$ are $\begin{cases} x & \text{on diagonal} \\ 0, \pm 1 & \text{otherwise} \end{cases}$

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Thm: $\{X_{mn}^{\pm}\}_{m<n}$ lin depend $\Leftrightarrow -w_d^{\pm}(t)$ an eigenvalue of $\Omega(0)$.

Find eigenvals: Mathematica for $d = 3, 4, 5, 6$ then educated guess

Proof: Exhibit lin indep eigenvecs of "at least" desired multiplicity

Eigenvalues

Thm: $\{X^{\pm}_{mn}\}_{m<n}$ lin indep $\Leftrightarrow -w^{\pm}_d(t)$ not eigenvalue of $\Omega(0)$.

Thm: The eigenvalues of $\Omega^{-}_{d}(0)$ are

- $d-2$ with multiplicity $d-1$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline k \\ \cline{1-1}\end{array}$ $\quad t=1$

- -2 with mullt $\frac{1}{2}(d-2)(d-1)$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array}$ $\quad t=\frac{-1}{d-1}$

Eigenvalues

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Thm: The eigenvalues of $\Omega^{+}_{d}(0)$ are

- $2(d-2)$ non-degenerate $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & & & & d \\ \hline\end{array}$

- $d-4$ with multiplicity $d-1$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline k \\ \cline{1-1}\end{array}$

- -2 with multiplicity $\binom{d-1}{2}-1=\frac{1}{2}d(d-3)$ $\quad t=\frac{-1}{d-1}$

Symmetric eigenvectors for $t = \frac{-1}{d-1}$

- $x = 2 = w_d^+\left(\frac{-1}{d-1}\right)$ eigenvecs $C_{jk,mn} = B_{jk,mn} + B^*_{jk,mn}$

$$B_{jk,mn} \equiv |e_m\rangle\langle e_j| - |e_m\rangle\langle e_k| - |e_n\rangle\langle e_j| + |e_n\rangle\langle e_k|$$

$\{B_{2k,1n} : 3 \le n < k \le d\} \cup \{B_{2k,13} : k = 4, 5 \ldots d\}$

lin indep $\Rightarrow$ $\frac{1}{2}d(d-3)$ lin indep eigenvecs $C_{jk,mn}$

Young tableaux

1	2		...	
n	k			

and

1	3		...	
2	k			

$$C_{34,12} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \qquad C_{24,13} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$$A_mA_j + A_jA_m - A_mA_k - A_kA_m - A_nA_j - A_jA_n + A_nA_k + A_kA_n = 0$$

lin dep A_mA_n not directly trans. to X^+_{mn} for $t = \frac{-1}{d-1}$ but still OK