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A NEW PROOF OF THE EXISTENCE AND UNIQUENESS OF THE HAAR INTEGRAL

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CHAPTER I

INTRODUCTION

In the process of solving D. Hilbert's fifth problem, mathematicians were confronted with the question of the existence of an invariant integral on a topological group. In 1933, A. Haar [1] established the existence of such an integral for compact groups which are separable. Haar's proof is both constructive and simple. One year later, J. von Neumann [2] succeeded in proving that the integral discovered by Haar is unique up to a positive factor. Although these results represented significant advances, the restrictions imposed on the group were too severe.

A major breakthrough occurred in 1938 when A. Weil [3] proved the existence and uniqueness of an invariant integral for an arbitrary locally compact Hausdorff group. Weil's existence proof is not constructive; the integral is obtained by an application of the Tychonoff product theorem. The integral, if it is to be unqiue up to a positive factor, should be established by an actual construction.

In 1940, H. Cartan [4] presented a constructive proof of the existence and uniqueness of an invariant integral for locally compact Hausdorff groups. Although Cartan succeeded in meeting the criticism of Weil's existence proof, his proof is unintuitive.

In 1963, E. Alfsen [5] succeeded in establishing the existence and uniqueness of an invariant integral for locally compact Hausdorff groups by an argument which is both constructive and intuitive.

It is the purpose of this work to discuss Alfsen's proof in detail.

In the brief outline of Alfsen's proof given below, the following notation is used: G denotes a fixed locally compact Hausdorff group, and L^+ denotes the class of all continuous, non-negative functions defined on G with compact supports.

In the second chapter, for each pair of elements of L^+, f and g, with $g \neq 0$, two approximations, denoted by $(\overline{f : g})$ and $(\underline{f : g})$, both of which give the relative "size" of f and g, are described.

In Chapter III, a certain separation property is derived. This property turns out to be the key to the uniqueness argument presented in Chapter IV.

In Chapter IV several ideas are developed. After formally defining an invariant integral J, a pre-ordering is associated with J as follows: if f and g are members of L^+, then $f \leq g \pmod J$ if, and only if, $J(f) \leq J(g)$. The following propositions are then proved.

(a) If J_1 and J_2 are invariant integrals on G such that $f \leq g \pmod{J_2}$ implies $f \leq g \pmod{J_1}$, then there is a positive constant α such that $J_1(f) = \alpha J_2(f)$ for every $f \in L^+$.

(b) If J is any invariant integral on G, then $J(f) \leq J(g)$ implies $(\underline{g : h}) \leq (\overline{f : h})$ for every $h \in L^+$, $h \neq 0$.

Thus it suffices to show the existence of an invariant integral I such that $f \leq g \pmod I$ if, and only if, $(\underline{f : h}) \leq (\overline{g : h})$ for every $h \in L^+$, $h \neq 0$. This is achieved by defining two generalized sequences in terms of the approximations defined in Chapter II, and then using the separation property to obtain the desired construction.

CHAPTER II

THE HAAR COVERING FUNCTIONS

Notation

The following notation shall be used in this study:

(a) G denotes a fixed locally compact Hausdorff topological group.

(b) L^+ denotes the class of all non-negative, real valued, continuous functions defined on G with compact supports; for each $f \in L^+$, the symbol $N(f)$ denotes the set $\{x \in G : f(x) > 0\}$, and the symbol supp (f) denotes the closure of $N(f)$.

(c) For each $A \subseteq G$, L_A^+ denotes the elements of L^+ whose supports are contained in A.

(d) Υ denotes the neighborhood filter of the identity e of G.

(e) For each $f \in L^+$ and $s \in G$, the symbols f^*, f_s, and f^s denote the functions defined by $f^*(x) = f(x^{-1})$, $f_s(x) = f(s^{-1}x)$, and $f^s(x) = f(xs)$, respectively.

(f) For each $f \in L^+$, $\|f\|$ denotes the supremum of the set $\{f(x) : x \in G\}$.

(g) For $f, g \in L^+$, $f \leq g$ shall mean $f(x) \leq g(x)$ for all $x \in G$.

(h) For each $f \in L^+$, and fixed real number $r \geq 0$, $(f - r)^+$ denotes the function defined by $(f - r)^+(x) = \sup \{f(x) - r, 0\}$.

The Lower Haar Covering Function

Proposition 2.1. Let f, $\varphi \in L^+$, $\varphi \neq 0$. Then there exist elements $s_1, \ldots, s_n$ of G and positive real numbers $\alpha_1, \ldots, \alpha_n$ such that

(1) $$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{si} .$$

Proof. Let u be an element of G such that $\varphi(u) > 0$. Choose ε so that $0 < \varepsilon < \varphi(u)$. For each $t \in G$, $\varphi_{tu^{-1}}(t) = \varphi(u) > 0$. The open set $U_t = \{x \in G : \varphi_{tu^{-1}}(x) > \varepsilon\}$ is a neighborhood of t. Let $a_t = \inf\{\varphi_{tu^{-1}}(x) : x \in U_t\}$. Then $a_t \geq \varepsilon$. Choose α_t such that $\alpha_t a_t \geq \|f\|$. Then

(2) $$\alpha_t \varphi_{tu^{-1}}(x) \geq \alpha_t a_t \geq f(x)$$

for every $x \in U_t$.

Now the collection of sets $\{U_t : t \in G\}$ is an open covering of G, and hence of supp (f). Since supp (f) is compact, a finite subcollection $U_{t_1}, \ldots, U_{t_n}$ covers supp (f). For each $t_i, i=1,\ldots,n$, there is an $\alpha_i > 0$ such that $\alpha_i \varphi_{t_i u^{-1}}(x) \geq f(x)$ for every $x \in U_{t_i}$. Since f vanishes off supp (f), it follows that

$$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{si} ,$$

where $s_i = t_i u^{-1}$, $i = 1, \ldots, n$. ∎

Definition 2.2. Let f, $\varphi \in L^+$, $\varphi \neq 0$. The lower Haar covering function of f relative to φ, denoted by $(\overline{f : \varphi})$, is defined as follows:

$$(3) \qquad (\overline{f:\varphi}) = \inf\left\{\sum_{i=1}^{n}\alpha_i : \alpha_i > 0,\ \ i = 1,\ldots,n, \text{ and there exist elements } s_1,\ldots,s_n \text{ of } G \text{ such that } f \le \sum_{i=1}^{n}\alpha_i\varphi_{s_i}\right\}.$$

Proposition 2.3. If f and φ are non-zero members of L^+, then $(\overline{f:\varphi}) > 0$.

Proof. Let x_1 and x_2 be elements of G such that $f(x_1) = \|f\|$ and $\varphi(x_2) = \|\varphi\|$. Then $f(x_1)$ and $\varphi(x_2)$ are both positive. Let $s_1,\ldots,s_n$ be elements of G and $\alpha_1,\ldots,\alpha_n$ be positive real numbers such that

$$f \le \sum_{i=1}^{n}\alpha_i\varphi_{s_i} .$$

Then

$$\|f\| \le \sum_{i=1}^{n}\alpha_i\varphi_{s_i}(x_1) \le \sum_{i=1}^{n}\alpha_i \quad \|\varphi\| .$$

Whence

$$\sum_{i=1}^{n}\alpha_i \ge \|f\|/\|\varphi\|$$

which implies that $(\overline{f:\varphi}) > 0$. ∎

Proposition 2.4. If f, g and φ are members of L^+, $\varphi \neq 0$, then $f \le g$ implies $(\overline{f:\varphi}) \le (\overline{g:\varphi})$.

Proof. Let $s_1, \ldots, s_n$ be elements of G and $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that $g \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i}$. Then $f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i}$ which implies that $(\overline{f : \varphi}) \leq \sum_{i=1}^{n} \alpha_i$. Hence $(\overline{f : \varphi}) \leq (\overline{g : \varphi})$. ∎

Proposition 2.5. If f, φ and ψ are members of L^+, where φ and ψ are non-zero, then

$$(4) \qquad (\overline{f : \varphi}) \leq (\overline{f : \psi})(\overline{\psi : \varphi}) .$$

Proof. Let $s_1, \ldots, s_n$ be elements of G and $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that

$$(5) \qquad f \leq \sum_{i=1}^{n} \alpha_i \psi_{s_i}$$

Similarly, let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be positive real numbers such that

$$(6) \qquad \psi \leq \sum_{i=1}^{m} \beta_j \varphi_{t_j} .$$

Combining (5) and (6),

$$f \leq \sum_{i=1}^{n} \alpha_i \left(\sum_{i=1}^{m} \beta_j \varphi_{s_i t_j} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \varphi_{t_j s_i} .$$

Consequently,

$$(\overline{f:\varphi}) \leq \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \beta_j \quad .$$

hence

$$(\overline{f:\varphi}) \leq (\overline{f:\psi})\ (\overline{\psi:\varphi}) \ . \ \blacksquare$$

<u>Proposition 2.6</u>. If $f, \varphi \in L^+$, $\varphi \neq 0$, and if $s \in G$, then

(7) $$(\overline{f_s:\varphi}) = (\overline{f:\varphi}) \ .$$

<u>Proof</u>. Let $s_1,\dots,s_n$ be elements of G and $\alpha_1,\dots,\alpha_n$ be positive real numbers such that

$$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \ .$$

Then

$$f_s \leq \sum_{i=1}^{n} \alpha_i \varphi_{ss_i} .$$

Hence $(\overline{f_s:\varphi}) \leq \sum\limits_{i=1}^{n} \alpha_i$ which implies

(8) $$(\overline{f_s:\varphi}) \leq (\overline{f:\varphi}) .$$

Now let $t_1,\dots,t_m$ be elements of G and $\beta_1,\dots,\beta_m$ be positive real numbers such that

$$f_s \leq \sum_{i=1}^{m} \beta_j \varphi_{t_j} \text{ .}$$

Then

$$f \leq \sum_{j=1}^{m} \beta_j \varphi_{s^{-1}t_j} \text{ .}$$

This last inequality implies $(\overline{f:\varphi}) \leq \sum_{j=1}^{m} \beta_j$. Hence

(9) $$(\overline{f:\varphi}) \leq (\overline{f_s:\varphi}) \text{ .}$$

The desired conclusion follows from (8) and (9). $\blacksquare$

Proposition 2.7. Let $f, \varphi \in L^+$, $\varphi \not\equiv 0$, and let $\alpha \geq 0$ be a constant. Then

(10) $$(\overline{\alpha f:\varphi}) = \alpha(\overline{f:\varphi}) \text{ .}$$

Proof. If $\alpha = 0$, then for any positive number β, $\alpha f \leq \beta\varphi$. Thus $(\overline{\alpha f:\varphi}) = 0 = \alpha(\overline{f:\varphi})$. Suppose that $\alpha > 0$. Let $s_1,\ldots,s_n$ be elements of G and $\alpha_1,\ldots,\alpha_n$ be positive real numbers such that

$$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \text{ .}$$

Then

$$\alpha f \leq \alpha \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \text{ .}$$

Thus

$$(\overline{\alpha f : \varphi}) \leq \alpha \sum_{i=1}^{n} \alpha_i$$

which implies

(11) $$(\overline{\alpha f : \varphi}) \leq \alpha(\overline{f : \varphi}) .$$

Now let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be positive real numbers such that

$$\alpha f \leq \sum_{j=1}^{m} \beta_j \varphi_{t_j} .$$

Since $\alpha > 0$,

$$f \leq \frac{1}{\alpha} \sum_{j=1}^{m} \beta_j \varphi_{tj} .$$

Hence $(\overline{f : \varphi}) \leq \frac{1}{\alpha} \sum_{j=1}^{m} \beta_j$ which implies

(12) $$\alpha(\overline{f : \varphi}) \leq (\overline{\alpha f : \varphi}) .$$

(11) and (12) together imply $(\overline{\alpha f : \varphi}) = \alpha(\overline{f : \alpha})$. ∎

<u>Proposition 2.8</u>. Let $f_i \in L^+$, $i = 1, \ldots, n$, and let $\varphi \in L^+$, $\varphi \neq 0$. Then

(13) $$\left(\overline{\sum_{i=1}^{n} f_i : \varphi}\right) \leq \sum_{i=1}^{m} (\overline{f_i : \varphi}) .$$

Proof. The proof is by induction on n. For $n = 1$, there is nothing to prove. Consider the case $n = 2$. Let $s_1,\ldots,s_n$ be elements of G and $\alpha_1,\ldots,\alpha_n$ be positive real numbers such that

$$f_1 \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} . \tag{14}$$

Similarly, let $t_1,\ldots,t_m$ be elements of G and $\beta_1,\ldots,\beta_m$ be positive real numbers such that

$$f_2 \leq \sum_{j=1}^{n} \beta_j \varphi_{t_j} . \tag{15}$$

Adding (14) and (15),

$$f_1 + f_2 \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} + \sum_{j=1}^{m} \beta_j \varphi_{t_j} .$$

Then

$$(\overline{f_1 + f_2 : \varphi}) \leq \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j .$$

Hence

$$(\overline{f_1 + f_2 : \varphi}) \leq (\overline{f_1 : \Phi}) + (\overline{f_2 : \varphi}). \tag{16}$$

Now suppose the assertion holds for k, where $2 \leq k < n$. Then by (16) and the inductive hypothesis,

$$\overline{\left(\sum_{i=1}^{k+1} f_i : \varphi\right)} = \overline{\left(\sum_{i=1}^{k} f_i + f_{k+1} : \varphi\right)}$$

$$\leq \overline{\left(\sum_{i=1}^{k} f_i : \varphi\right)} + (\overline{f_{k+1} : \varphi})$$

$$\leq \sum_{i=1}^{k} (\overline{f_i : \varphi}) + (\overline{f_{k+1} : \varphi})$$

$$= \sum_{i=1}^{k+1} (\overline{f_i : \varphi}) . \blacksquare$$

Proposition 2.9. Let $f_i \in L^+$, $i = 1,\ldots,n$, and write $f = \sum_{i=1}^{n} f_i$. Then for every $\lambda > 1$, there is a $V \in \mathcal{V}$ such that

(17) $$\sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq \lambda(\overline{f : \varphi})$$

for every $\varphi \in L_V^+$, $\varphi \neq 0$.

Proof. If f is identically zero, there is nothing to prove. Assume that $f \neq 0$. Let $f_o \in L^+$ be such that $f_o(x) = 1$ for every x in supp (f), and let λ_o be a real number such that $1 < \lambda_o < \lambda$. Choose $\delta > 0$ so that

(18) $$\delta(\overline{f_o : f}) \leq \lambda_o - 1.$$

Let $h = f + \delta f_o$. For each $i = 1,\ldots,n$, define h_i on G as follows:

$$h_i(x) = \begin{cases} \dfrac{f_i(x)}{h(x)} & \text{if } h(x) \neq 0, \\ 0 & \text{if } h(x) = 0. \end{cases}$$

Then for each $i = 1,\ldots,n$, h_i is continuous on $N(h)$, which contains supp (f). Also, h_i is continuous on the open set G- supp (f) since h_i is identically zero on G- supp (f). Since $G = (G\text{- supp}(f)) \bigcup \text{supp}(f) \subseteq (G\text{- supp}(f)) \bigcup N(h)$, it follows that $h_i \varepsilon L^+$, $i = 1,\ldots,n$.

Now for each $i = 1,\ldots,n$, $f_i(x) = 0$ implies $h_i(x) = 0$; if $f_i(x) \neq 0$, then $f_i(x) = h_i(x)\, h(x)$. Hence

(19) $$f_i(x) = h_i(x)\, h(x)$$

for all $x \varepsilon G$, $i = 1,\ldots,n$. Suppose $h(x) \neq 0$. Then

$$\sum_{i=1}^{n} h_i(x) = \sum_{i=1}^{n} \frac{f_i(x)}{h(x)}$$

$$= \frac{1}{f(x) + \delta f_o(x)} \sum_{i=1}^{n} f_i(x)$$

$$= \frac{f(x)}{f(x) + \delta f_o(x)}$$

$$\leq 1.$$

But if $h(x) = 0$, then $h_1(x) = \ldots = h_n(x) = 0$, so that

(20) $$\sum_{i=1}^{n} h_i(x) \leq 1$$

for all $x \varepsilon G$.

For each fixed $i = 1,\ldots,n$, corresponding to the positive number

$\varepsilon = \dfrac{\lambda - \lambda_o}{n\lambda_o}$, there is a $V_i \in \mathcal{V}$ such that

$$(21) \qquad |h_i(x) - h_i(y)| \leq \varepsilon$$

whenever $x^{-1}y \in V_i$. Let $V = V_1 \cap \ldots \cap V_n$. Then, if $x^{-1}y \in V$, (21) holds for every $i = 1,\ldots,n$. Let $\varphi \in L_V^+$, $\varphi \neq 0$. Let $s_1,\ldots,s_m$ be elements of G and $\alpha_1,\ldots,\alpha_m$ be positive real numbers such that

$$h \leq \sum_{j=1}^{m} \alpha_j \varphi_{s_j} .$$

Then for each $i = 1,\ldots,n$,

$$h(x)\ h_i(x) \leq \sum_{j=1}^{m} \alpha_j\ \varphi(s_j^{-1}\ x)\ h_i(x)$$

for every $x \in G$. For each $j = 1,\ldots,m$, if $s_j^{-1}\ x \notin V$, then $\varphi(s_j^{-1}\ x) = 0$; if $s_j^{-1}\ x \in V$, then from (21), $h_i(x) \leq \varepsilon + h_i(s_j)$, $i = 1,\ldots,n$. Whence

$$(22) \qquad h(x)\ h_i(x) \leq \sum_{j=1}^{m} \alpha_j \varphi_{s_j}(x)(h_i(s_j) + \varepsilon) ,$$

for all $x \in G$ and for each $i = 1,\ldots,n$. By (19),

$$(\overline{f_i : \varphi}) \leq \sum_{j=1}^{m} \alpha_j (h_i(s_j) + \varepsilon),\ i=1,\ldots,n.$$

Now summing this last inequality over i,

$$\sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_j (h_i(s_j) + \alpha)$$

$$= \sum_{j=1}^{m} \alpha_j \left(\sum_{i=1}^{n} (h_i(s_j) + \varepsilon) \right)$$

but from (20) $\sum_{i=1}^{n} h_i(s_j) \leq 1$, so that

$$\sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq (1 + n\varepsilon) \sum_{j=1}^{m} \alpha_j .$$

Therefore,

$$(23) \qquad \sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq (1 + n\varepsilon)(\overline{h : \varphi}) .$$

But by propositions 28, 27 and 25,

$$(1 + n\varepsilon)(\overline{h : \varphi}) = (1 + n\varepsilon)(\overline{f + \delta f_o : \varphi})$$

$$\leq (1 + n\varepsilon) \left\{ (\overline{f : \varphi}) + \delta (\overline{f_o : \varphi}) \right\}$$

$$\leq (1 + n\varepsilon) \left\{ (\overline{f : \varphi}) \; 1 + \delta (\overline{f_o : f}) \right\} .$$

Now applying (18) and substituting the value of ε,

$$(24) \qquad (1 + n\varepsilon)(\overline{h : \varphi}) \leq \lambda (\overline{f : \varphi}) .$$

The conclusion then follows from (23) and (24). ∎

<u>Proposition 2.10</u>. Let $\{f_i\}$ be a generalized sequence of functions in L^+ which converges uniformly to f, $f \neq 0$, and for which there is some fixed compact set K outside of which f_i vanishes for each index i.

Then

$$(25) \qquad \lim_i \frac{(\overline{f_i : \varphi})}{(\overline{f : \varphi})} = 1 .$$

when $\varphi \varepsilon L^+$, $\varphi \neq 0$; moreover, the convergence is uniform in φ.

Proof. Let $\varepsilon > 0$ be given. Choose $f_o \varepsilon L^+$ such that $f_o(x) = 1$ for every $x \varepsilon K$. From the hypothesis, there exists an index j such that

$$|f - f_i| < \varepsilon \ (\overline{f_o : f})^{-1}$$

for every $i \geq j$. Now noting that $f(x) = f_i(x) = 0$ for all $x \notin K$, and for each index i, it follows that

$$(26) \qquad |f - f_i| < \varepsilon \ (\overline{f_i : f})^{-1} \ f_o ,$$

for every $i \geq j$. Writing (26) as two inequalities one obtains

$$f < \varepsilon(\overline{f_o : f})^{-1} \ f_o + f_i ,$$

and

$$f_i < \varepsilon(\overline{f_o : f})^{-1} \ f_o + f$$

for every $i \geq j$. Applying propositions 2.4, 2.8 and 2.7 to these inequalities, one obtains

$$(\overline{f : \varphi}) \leq \varepsilon(\overline{f_o : f})^{-1} \ (\overline{f_o : \varphi}) + (\overline{f_i : \varphi})$$

and

$$(\overline{f_i : \varphi}) \leq \varepsilon(\overline{f_o : f})^{-1}(\overline{f_o : \varphi}) + (\overline{f_i : \varphi}),$$

for each $\varphi \in L^+$, $\varphi \neq 0$, and for every $i > j$. There two inequalities can be combined to obtain

(27) $$|(\overline{f_i : \varphi}) - (\overline{f : \varphi})| \leq \varepsilon \ (\overline{f_o : f})^{-1} \ (\overline{f_o : \varphi})$$

for each $\varphi \in L^+$, $\varphi \neq 0$, and for every $i \geq j$. Dividing (27) by $(\overline{f : \varphi})$,

$$\left| \frac{(\overline{f_i : \varphi})}{(\overline{f : \varphi})} - 1 \right| \leq \varepsilon \ (\overline{f_o : f})^{-1} \ (\overline{f : \varphi})^{-1} \ (\overline{f_o : \varphi}),$$

for each $\varphi \in L^+$, $\varphi \neq 0$, and for every $i \geq j$. But $(\overline{f_o : \varphi}) \leq (\overline{f_o : f})(\overline{f : \varphi})$, so

$$\left| \frac{(\overline{f_i : \varphi})}{(\overline{f : \varphi})} - 1 \right| \leq \varepsilon$$

for each $\varphi \in L^+$, $\varphi \neq 0$, and for every $i > j$. ∎

<u>Proposition 2.11</u>. If $\{f_i\}$ is a generalized sequence of functions in L^+ such that $f_i \uparrow f$, where $f \in L^+$, $f \neq 0$, then

(28) $$\frac{(\overline{f_i : \varphi})}{(\overline{f : \varphi})} \uparrow 1$$

uniformly in φ, where $\varphi \in L^+$, $\varphi \neq 0$.

<u>Proof</u>. From Dini's lemma, $\{f_i\}$ converges uniformly to f. Since $f_i \uparrow f$, it follows that supp $(f_i) \subseteq$ supp (f), for each index i. Letting K = supp (f), the conclusion follows from proposition 2.10. ∎

<u>The Upper Haar Covering Function</u>

Let f, $\varphi \in L^+$, $\varphi \neq 0$. There exist non-negative real numbers

$\beta_1,\ldots,\beta_m$ and elements $t_1,\ldots,t_m$ of G such that

(29) $$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f .$$

Let $\lambda > 1$ be given. By proposition 2.9, there is a $V \varepsilon \mathcal{V}$ such that

$$\sum_{j=1}^{m} (\overline{\beta_j \varphi_{t_j} : \psi}) \leq \lambda\left(\overline{\sum_{j=1}^{m} \beta_j \varphi_{t_j} : \psi}\right)$$

for every $\psi \varepsilon L_V^+$, $\psi \neq 0$. But $(\overline{\beta_j \varphi_{t_j} : \psi}) = \beta_j(\overline{\varphi_{t_j} : \psi}) = \beta_j(\overline{\varphi : \psi})$; for each $j = 1,\ldots,m$, and

$$\lambda\left(\overline{\sum_{j=1}^{m} \beta_j \varphi_{t_j} : \psi}\right) \leq \lambda(\overline{f : \psi}),$$

so that

$$(\overline{\varphi : \psi}) \sum_{j=1}^{m} \beta_j \leq \lambda(\overline{f : \psi}) .$$

But $(\overline{f : \psi}) \leq (\overline{f : \varphi})\ (\overline{\varphi : \psi})$, and $(\overline{\varphi : \psi}) > 0$ for every $\psi \varepsilon L_V^+$, $\psi \neq 0$. Thus

$$\sum_{j=1}^{m} \beta_j \leq \lambda(\overline{f : \varphi}) .$$

Since λ is arbitrary, one has

(30) $$\sum_{j=1}^{m} \beta_j \leq (\overline{f : \varphi}) .$$

Therefore, the collection of all sums of the form $\sum_{j=1}^{m} \beta_j$, $\beta_j \geq 0$, $j = 1,\ldots,m$, for which (29) holds for some $t_1,\ldots,t_m$ in G is bounded above. One has the following definition:

Definition 2.12. Let $f, \varphi \in L^+$, $\varphi \neq 0$. The upper Haar covering function of f relative to φ, denoted by the symbol $(\underline{f : \varphi})$, is defined to be the supremum of all sums of the form

$$\sum_{j=1}^{m} \beta_j, \qquad \beta_j \geq 0, \quad j = 1,\ldots,m$$

for which there exist elements $t_1,\ldots,t_m$ of G such that $\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f$.

From (30), the following proposition is true.

Proposition 2.13. If $f, \varphi \in L^+$, $\varphi \neq 0$, then $(\underline{f : \varphi}) \leq (\overline{f : \varphi})$.

Proposition 2.14. Let $f \in L^+$, $f \neq 0$. Then there exists a $V \in \mathcal{V}$ such that $(\underline{f : \varphi}) > 0$, for every $\varphi \in L_V^+$, $\varphi \neq 0$.

Proof. Let $u \in G$ be such that $f(u) > 0$. Choose ε so that $0 < \varepsilon < f(u)$. The set $U = \{x \in G : f(x) > \varepsilon\}$ is an open neighborhood of u. Let $V = u^{-1} U \in \mathcal{V}$. For each $\varphi \in L_V^+$, $\varphi \neq 0$, choose $\beta > 0$ such that $\beta \|\varphi\| \leq \varepsilon$. Then

$$\beta \varphi_u(x) \leq \beta \|\varphi_u\| = \beta\|\varphi\| \leq \varepsilon < f(x)$$

for every $x \in U$. But since φ_u vanishes off U,

$$\beta \varphi_u(x) \leq f(x)$$

for all $x \in G$. Consequently, $(\underline{f : \varphi}) > 0$.

Proposition 2.15. If f, $g \in L^+$, then $f \leq g$ implies $(\underline{f : \varphi}) \leq (\underline{g : \varphi})$ for every $\varphi \in L^+$, $\varphi \neq 0$.

Proof. Let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be non-negative real numbers such that

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f ,$$

where $\varphi \in L^+$, $\varphi \neq 0$. Then

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq g ,$$

which implies

$$\sum_{i=1}^{m} \beta_j \leq (\underline{g : \varphi}) .$$

Hence

$$(\underline{f : \varphi}) \leq (\underline{g : \varphi}). \blacksquare$$

Proposition 2.16. Let f, φ and ψ be members of L^+, with $\varphi \neq 0$ and $\psi \neq 0$, then

$$(\underline{f : \psi})\,(\underline{\psi : \varphi}) \leq (\underline{f : \varphi}) . \tag{31}$$

Proof. The assertion is obvious if $(\underline{f : \psi}) = 0$. Assume that $(\underline{f : \psi}) > 0$. Let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be positive real numbers such that

$$\sum_{j=1}^{m} \beta_j \psi_{t_j} \leq f \,. \tag{32}$$

Let $u_1,\ldots,u_n$ be elements of G and $\gamma_1,\ldots,\gamma_n$ be non-negative real numbers such that

$$\sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq \psi \,. \tag{33}$$

Combining (32) and (33)

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \gamma_i \varphi_{t_j u_i} \leq f \,,$$

which implies

$$\left(\sum_{j=1}^{m} \beta_j\right)\left(\sum_{i=1}^{n} \gamma_i\right) \leq (\underline{f : \varphi}) \,.$$

Thus

$$(\underline{f : \psi})\ (\underline{\psi : \varphi}) \leq (\underline{f : \varphi}) \,. \blacksquare$$

<u>Proposition 2.17</u>. If $f, \varphi \in L^+$, $\varphi \neq 0$, then

$$(\underline{f : \varphi}) = (\underline{f_s : \varphi}) \tag{34}$$

for every $s \in G$.

<u>Proof</u>. Let $t_1,\ldots,t_m$ be elements of G and $\beta_1,\ldots,\beta_m$ be non-negative

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f ,$$

Then

$$\sum_{j=1}^{m} \beta_j \varphi_{st_j} \leq f_s ,$$

which implies

$$\sum_{j=1}^{m} \beta_j \leq \underline{(f_s : \varphi)} .$$

Whence

(35) $$\underline{(f : \varphi)} \leq \underline{(f_s : \varphi)} .$$

Now let $u_1, \ldots, u_n$ be elements of G and $\gamma_1, \ldots, \gamma_n$ be non-negative real numbers such that

$$\sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq f_s .$$

Then

$$\sum_{i=1}^{n} \gamma_i \varphi_{s^{-1}u_i} \leq f ,$$

which implies

$$\sum_{i=1}^{n} \gamma_i \leq \overline{(f : \varphi)} .$$

Thus

(36) $$\underline{(f_s : \varphi)} \leq \underline{(f : \varphi)} \ .$$

(36) and (35) give the desired equality (34). ∎

<u>Proposition 2.18</u>. Let $f, \varphi \in L^+$, $\varphi \neq 0$, and let α be a non-negative constant. Then

(37) $$\underline{(\alpha f : \varphi)} = \alpha \underline{(f : \varphi)} \ .$$

<u>Proof</u>. If $\alpha = 0$, the assertion follows immediately. Suppose that $\alpha > 0$. Let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be non-negative real numbers such that

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f \ .$$

Then

$$\alpha \sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq \alpha f ,$$

which implies

$$\alpha \sum_{j=1}^{m} \beta_j \leq \underline{(\alpha f : \varphi)} \ .$$

Consequently,

(38) $$\alpha \ \underline{(f : \varphi)} \leq \underline{(\alpha f : \varphi)} \ .$$

Now let $u_1,\ldots,u_n$ be elements of G and $\gamma_1,\ldots,\gamma_n$ be non-negative real numbers such that

$$\sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq \alpha f .$$

Then

$$\frac{1}{\alpha} \sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq f .$$

Therefore,

$$\sum_{i=1}^{n} \gamma_i \leq \alpha(\underline{f : \varphi})$$

which implies

$$(\underline{\alpha f : \varphi}) \leq \alpha(\underline{f : \varphi}) . \tag{39}$$

The desired conclusion follows from (38) and (39). ∎

<u>Proposition 2.19</u>. If $f_i \in L^+$, $i = 1,\ldots,n$, and if $\varphi \in L^+$, $\varphi \neq 0$, then

$$\sum_{i=1}^{n} (\underline{f_i : \varphi}) \leq \left(\underline{\sum_{i=1}^{n} f_i : \varphi}\right) . \tag{40}$$

<u>Proof</u>. The proof is by induction on n. Equality holds for $n = 1$. Consider the case $n = 2$. Let $t_1,\ldots,t_m$ be elements of G and $\beta_1,\ldots,\beta_m$ be non-negative real numbers such that

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f_1 .$$

Let $u_1, \ldots, u_n$ be elements of G and $\gamma_1, \ldots, \gamma_n$ be non-negative real numbers such that

$$\sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq f_2 .$$

Then

$$\sum_{j=1}^{m} \beta_j \varphi_{t_j} + \sum_{i=1}^{n} \gamma_i \varphi_{u_i} \leq f_1 + f_2 .$$

which implies

$$\sum_{j=1}^{n} \beta_j + \sum_{i=1}^{n} \gamma_i \leq \underline{(f_1 + f_2 : \varphi)} .$$

Hence

$$\underline{(f_1 : \varphi)} + \underline{(f_2 : \varphi)} \leq \underline{(f_1 + f_2 : \varphi)} .$$

Now suppose the assertion were true for k, where $2 \leq k < n$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \underline{(f_i : \varphi)} &= \sum_{i=1}^{k} \underline{(f_i : \varphi)} + \underline{(f_{k+1} + \varphi)} \\
&\leq \underline{\left(\sum_{i=1}^{k} f_i : \varphi \right)} + \underline{(f_{k+1} : \varphi)} \\
&\leq \underline{\left(\sum_{i=1}^{k+1} f_i : \varphi \right)} .
\end{aligned}$$

CHAPTER III

THE SEPARATION PROPERTY

The purpose of this chapter is to prove the following separation property: If f and g are non-zero members of L^+ such that $f(x) < g(x)$ for all $x \in \text{supp}(f)$, then there exists a $V \in \mathcal{V}$ such that, for every $\varphi \in L_V^+$, $\varphi \neq 0$, there exist elements $s_1, \ldots, s_n$ of G and positive real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \leq g .$$

To this end, several preliminary results are required.

<u>Definition 3.9</u>. Let $f, g \in L^+$. For each $\varphi \in L^+$, $\varphi \neq 0$, the <u>convolution of f and g relative to φ</u> is defined by

(1) $$[f * g]_\varphi (x) = \overline{(f(g^*)_x : \varphi)} .$$

<u>Remark</u>. Note that $(f_{x^{-1}} g^*)_x (s) = f_{x^{-1}}(x^{-1}s)\, g^* (x^{-1}s) = f(s)(g^*)_x(s) = (f(g^*)_x)(s)$, hence, by proposition 2.6, one has

(2) $$[f * g]_\varphi (x) = \overline{(f_{x^{-1}} g^* : \varphi)} .$$

<u>Proposition 3.2</u>. Let f and g be non-zero members of L^+. For each $\varphi \in L^+$, $\varphi \neq 0$, define h_φ on G as follows:

(3) $$h_\varphi(x) = \frac{[f * g]_\varphi (x)}{\overline{(f : \varphi)}} .$$

Then the family of all functions defined by (3) is equicontinuous.

Proof. Let $\varepsilon > 0$ be given, and let x denote a fixed element of G. The proof is given in several steps.

(i) Let $\mathcal{F}$ denote the neighborhood filter of x, and let I denote the set $\{(U,y) : U \varepsilon \mathcal{F},$ and $y \varepsilon U\}$. For two elements (U,y), (V,z) of I define $(U,y) \leq (V,z)$ if, and only if, $V \subseteq U$. Then $(I, \leq)$ is a directed set. For each $(U,y) \varepsilon I$, define $\gamma(U,y)$ on G as follows:

$$\Upsilon_{(U,y)}(s) = g(s^{-1} y) .$$

Note that $\Upsilon_{(U,y)} \varepsilon L^{+}$, and that $\Upsilon_{(U,y)} = (g^{*})_{y}$ for each $(U,y) \varepsilon I$.

Let ε' be an arbitrary positive real number. There exists a $W \varepsilon \mathcal{V}$ such that

$$|g(u) - g(v)| < \varepsilon'$$

whenever $u^{-1} v \varepsilon W$. Let $U = x W \varepsilon \mathcal{F}$. Note that $(s^{-1}x)^{-1}(s^{-1}y) = x^{-1} y \varepsilon W$ for all $s \varepsilon G$ and $y \varepsilon U$; hence,

$$|g(s^{-1}x) - \Upsilon_{(V,z)}(s)| = |g(s^{-1}x) - g(s^{-1}z)| < \varepsilon'$$

for all $(V,z) \varepsilon I$ such that $(U, x) \leq (V, z)$. Consequently, the generalized sequence $\{\Upsilon_{(V,y)}\}$ converges uniformly to $(g^{*})_{x}$.

(ii) By (i), the generalized sequence $\{f\Upsilon_{(U,g)}\}$ converges uniformly to $f(g^{*})_{x}$. Note that $f\Upsilon_{(U,y)}$ vanishes off the compact set supp (f) for each $(U, y) \varepsilon I$.

Suppose that $[f * g]_{\varphi}(x) > 0$ for every $\varphi \varepsilon L^{+}$, $\varphi \neq 0$. Since

$[f * g]_\varphi (x) \leq [f * g]_f (x) \ (\overline{f : \varphi})$ for every $\varphi \in L^+$, $\varphi \neq 0$, it follows that $[f * g]_f(x) > 0$. By proposition 2.10, there is a $(U, y) \in I$ such that

$$\left| 1 - \frac{(\overline{f \gamma_{(U,z)} : \varphi})}{(\overline{f(g^*)_x : \varphi})} \right| < \frac{\varepsilon}{[f * g]_f(x)}$$

for every $z \in U$, $\varphi \in L^+$, $\varphi \neq 0$; whence

$$(4) \qquad \left| 1 - \frac{[f * g]_\varphi (z)}{[f * g]_\varphi (x)} \right| < \frac{\varepsilon}{[f * g]_f(\alpha)}$$

for every $z \in U$, $\varphi \in L^+$, $\varphi \neq 0$. Multiplying (4) by the inequality $[f * g]_\varphi(\alpha) \leq [f * g]_f(x) \ (\overline{f : \varphi})$, and then dividing by $(\overline{f : \varphi})$, one obtains

$$(5) \qquad \left| \frac{[f * g]_\varphi (z)}{(\overline{f : \varphi})} - \frac{[f * g]_\varphi (z)}{(\overline{f : \varphi})} \right| \leq \varepsilon$$

for every $z \in U$, $\varphi \in L^+$, $\varphi \neq 0$. This completes the proof in the case $[f * g]_\varphi(x) = 0$ for every $\varphi \in L^+$, $\varphi \neq 0$.

Now suppose that $[f * g]_\psi (x) = 0$ for some $\psi \in L^+$, $\psi = 0$. Then by proposition 2.3, $f(s) \ g(s^{-1}x) = 0$ for every $s \in G$. There exists a neighborhood W of x such that

$$(6) \qquad |g(s^{-1}x) - g(s^{-1}z)| \leq \varepsilon$$

for every $s \in G$ and $z \in W$. Since $f(s) \ g(s^{-1}x) = 0$, by (6), one has

$$f(s) \ g(s^{-1}z) \leq \varepsilon \ f(s)$$

for every $s \varepsilon G$ and $z \varepsilon W$, which implies

$$[f * g]_\varphi(z) \leq \varepsilon(\overline{f : \varphi})$$

for every $z \varepsilon W$, $\varphi \varepsilon L^+$, $\varphi \neq 0$. Consequently, (5) holds for every $z \varepsilon W$, $\varphi \varepsilon L^+$, $\varphi \neq 0$. ∎

Proposition 3.3. Let $f_1, \ldots, f_n$ and g be non-zero members of L^+, and write $f = \sum_{i=1}^{n} f_i$. For each $\varphi \varepsilon L^+$, $\varphi \neq 0$, define k_φ on G as follows:

$$k_\varphi(x) = \frac{\sum_{i=1}^{n} [f_i * g]_\varphi(x)}{(\overline{f : \varphi})}$$

Then the family $\{k_\varphi\}$ is equicontinuous.

Proof. Let $\varepsilon > 0$ be given, and let x be a fixed element of G. By proposition 3.2, for each $i = 1, \ldots, n$, there exists a neighborhood V_i of x such that

$$(7) \qquad \left| \frac{[f_i * g]_\varphi(x)}{(\overline{f_i : \varphi})} - \frac{[f_i * g]_\varphi(y)}{(\overline{f_i : \varphi})} \right| \leq \frac{\varepsilon}{n}$$

for all $y \varepsilon V_i$, $\varphi \varepsilon L^+$, $\varphi \neq 0$. Let $V = \bigcap_{i=1}^{n} V_i$. Then by the triangle inequality, proposition 2.4, and (7), one has

$$|k_\varphi(x) - k_\varphi(y)| \leq \sum_{i=1}^{n} \frac{1}{(\overline{f:\varphi})} |[f_i * g]_\varphi(x) - [f_i * g]_\varphi(y)|$$

$$\leq \sum_{i=1}^{n} \frac{1}{(\overline{f_i:\varphi})} |[f_i * g]_\varphi(x) - [f_i * g]_\varphi(y)|$$

$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{n}$$

$$= \varepsilon$$

for all $y \in V$, and $\varphi \in L^+$, $\varphi \neq 0$. ∎

<u>Proposition 3.4</u>. Let $f_1,\ldots,f_n$ and g be non-zero members of L^+, and write $f = \sum_{i=1}^{n} f_i$. Then given $\varepsilon > 0$, there exists a $V \in \mathcal{V}$ such that

$$\sum_{i=1}^{n} [f_i * g]_\varphi(x) - [f * g]_\varphi(x) < \varepsilon(\overline{f:\varphi})$$

for every $x \in G$, and $\varphi \in L_V^+$, $\varphi \neq 0$.

<u>Proof</u>. For each $\varphi \in L^+$, $\varphi \neq 0$, define r_φ on G as follows:

$$r_\varphi(x) = \frac{\sum_{i=1}^{n} [f_i * g]_\varphi(x)}{(\overline{f:\varphi})} - \frac{[f * g]_\varphi(x)}{(\overline{f:\varphi})}$$

Note that from proposition 2.8, $r_\varphi \geq 0$. By proposition 2.9, for each $x \in G$ corresponding to

(8) $$\lambda_x = 1 + \frac{\varepsilon}{2([f * g]_f(x) + 1)}$$

there exists a $V_x \varepsilon \mathcal{V}$ such that

$$\sum_{i=1}^{n} \overline{(f_i(g^*)_x : \varphi)} \leq \lambda_x \overline{(f(g^*)_x : \varphi)}$$

for every $\varphi \varepsilon L^+_{V_x}$, $\varphi \neq 0$. Therefore,

$$(9) \qquad r_\varphi(x) \leq (\lambda_x - 1) \frac{[f * g]_\varphi(x)}{\overline{(f : \varphi)}}$$

for every $\varphi \varepsilon L^+_{V_x}$, $\varphi \neq 0$. But $\overline{(f(g^*)_x : \varphi)} \leq \overline{(f(g^*)_x : \varphi)}\overline{(f : \varphi)}$; whence from (8) and (9), one has

$$(10) \qquad r_\varphi(x) \leq \frac{\varepsilon}{2}$$

for every $\varphi \varepsilon L^+_{V_x}$, $\varphi \neq 0$.

Now by the two previous propositions, the family $\{r_\varphi\}$ is equicontinuous; thus for each $x \varepsilon G$, there exists a neighborhood U_x of x such that

$$(11) \qquad |r_\varphi(x) - r_\varphi(y)| < \frac{\varepsilon}{2}$$

for every $y \varepsilon U_x$, and let $\varphi \varepsilon L^+$, $\varphi \neq 0$. Both (10) and (11) hold simultaneously if $y \varepsilon U_x$ and $\varphi \varepsilon L^+_{V_x}$, $\varphi \neq 0$; hence, from the triangle inequality, one obtains

$$(12) \qquad r_\varphi(y) < \varepsilon$$

if $y \varepsilon U_x$, and $\varphi \varepsilon L^+$, $\varphi \neq 0$.

Now suppose $r_\varphi(z) > 0$. Then there exists an $s \varepsilon G$ such that

$f(s)\ g(s^{-1}z) > 0$, which implies that $s \varepsilon \text{ supp } (f)$, and $s^{-1} z \varepsilon \text{ supp } (g)$. Hence $z \varepsilon \text{ supp } (f) \cdot \text{supp } (g)$. Consequently, r_φ vanishes of the set $K = \text{supp } (f) \cdot \text{supp } (g)$ for every $\varphi \varepsilon L^+$, $\varphi \neq 0$. Since K is compact, a finite subcollection $U_{x_1}, \ldots, U_{x_m}$ of $\{U_x : x \varepsilon G\}$ covers K. Let $V = \bigcap_{i=1}^{n} V_{x_i}$. Then from (12), $r_\varphi(x) < \varepsilon$ for every $x \varepsilon G$, and $\varphi \varepsilon L_V^+$, $\varphi \neq 0$. ■

<u>Proposition 3.5</u>. Let $g \varepsilon L^+$, and let $\varepsilon > 0$ be given. Then there exists a $u \varepsilon \Upsilon$ such that

$$\text{(i)} \quad \|[h * g]_\varphi - (\overline{h : \varphi})\ g_t\| \leq (\overline{h : \varphi})\ \varepsilon$$

whenever $t \varepsilon G$, $h \varepsilon L_{tU}^+$, and $\varphi \varepsilon L^+$, $\varphi \neq 0$. Similarly, there exists a $V \varepsilon \Upsilon$ such that

$$\text{(ii)} \quad \|[g * k]_\varphi - (\overline{k^* : \varphi})\ g^u\| \leq (\overline{k^* : \varphi})\ \varepsilon$$

whenever $u \varepsilon G$ $k \varepsilon L^+_{V_{u^{-1}}}$, and $\varphi \varepsilon L^+$, $\varphi \neq 0$.

<u>Proof</u>. (i) By right uniform continuity, there exists a $U \varepsilon \Upsilon$ such that

$$(13) \qquad |g(x) - g(y)| \leq \varepsilon$$

whenever $x\, y^{-1} \varepsilon U$. Let $t \varepsilon G$ and $h \varepsilon L_{tU}^+$. If $s \notin tU$, then $h(s) = 0$, or equivalently, if $(t^{-1}x)(s^{-1}x)^{-1} = t^{-1}s \notin U$, where x is any element of G, then $h(s) = 0$. From this argument and (14), it follows that

$$|h(s)\ g(s^{-1}x) - h(s)\ g(t^{-1}x)| \leq \varepsilon\, h(s)$$

for every $x \in G$. By an argument similar to the one used in the proof of proposition 2.10, this last inequality implies

$$(14) \qquad |[h * g]_\varphi(x) - (\overline{h : \varphi})g_t(x)| \leq \varepsilon(\overline{h : \varphi})$$

for every x and t in G, $h \in L^+_{tU}$, and $\varphi \in L^+$, $\varphi \neq 0$. Thus

$$\|[h * g]_\varphi - (\overline{h : \varphi})g_t\| \leq \varepsilon(\overline{h : \varphi})$$

whenever $t \in G$, $h \in L^+_{tU}$, and $\varphi \in L^+$, $\varphi \neq 0$.

(ii) By left uniform continuity, there exists a $V \in \mathcal{V}$ such that

$$(15) \qquad |g(x) - g(y)| \leq \varepsilon$$

whenever $x^{-1} y \in V$. Let $u \in G$, and let $k \in L^+_{V_{u^{-1}}}$. If $s^{-1} \notin V_{u^{-1}}$, then $k^*(s) = 0$, or equivalently, if $(xs)^{-1} (xu)^u = s^{-1} u \notin V$, where $x \in G$, then $k^*(s) = 0$. Hence, from (15) one has

$$|g(xs)k^*(s) - g(xu)k^*(s)| \leq \varepsilon\ k^*(s)$$

for every $x \in G$. Now by the remark following definition 3.1, and by an argument similar to the one used to obtain (14), it follows that

$$|[g * k]_\varphi(s) - (\overline{k^* : \varphi})g^u(x)| \leq \varepsilon(\overline{k^* : \varphi})$$

for every x and u of G, $k \in L^+_{V_{u^{-1}}}$ and $\varphi \in L^+$, $\varphi \neq 0$. Consequently,

$$\|[g * k]_\varphi - (\overline{k^* : \varphi})\ g^u\| \leq \varepsilon\ (\overline{k^* : \varphi})$$

for every $u \in G$, $k \in L^+_{V_{u^{-1}}}$, and $\varphi \in L^+$, $\varphi \neq 0$. ∎

Proposition 3.6. Let $f, g \in L^+$, $f \neq 0$, and let $\varepsilon > 0$ be given. Then there exists a $V \in \mathcal{V}$ such that, for every $\varphi \in L_V^+$, $\varphi \neq 0$, there exist elements $s_1, \ldots, s_n$ of the support of f and positive real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\| [f * g]_\varphi - \sum_{i=1}^{n} \alpha_i g_{s_i} \| \leq \varepsilon \, (\overline{f : \varphi}) .$$

Proof. From proposition 3.5, there exists an open set $U \in \mathcal{V}$ such that

$$(16) \qquad \| [h * g]_\varphi - (\overline{h : \varphi}) g_s \| \leq \frac{\varepsilon}{4} \, (\overline{h : \varphi})$$

whenever $s \in G$, $h \in L_{sU}^+$, and $\varphi \in L^+$, $\varphi \neq 0$. The collection of open sets $\{sU : s \in \text{supp}\,(f)\}$ is an open covering of supp (f). By the compactness of supp (f), a finite subcollection $s_1 U, \ldots, s_n U$ covers supp (f). By a partition of unity, f can be written as $f = \sum_{i=1}^{n} f_i$, where $f_i \in L_{s_i U}^+$, $f_i \neq 0$, $i = 1, \ldots, n$. By propositions 2.9 and 3.4, there exists a $V \in \mathcal{V}$ such that

$$(17) \qquad \sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq 2 (\overline{f : \varphi}),$$

and

$$(18) \qquad \| [f * g]_\varphi - \sum_{i=1}^{n} [f_i * g]_\varphi \| \leq \frac{\varepsilon}{2} \, (\overline{f : \varphi})$$

whenever $\varphi \in L_V^+$, $\varphi \neq 0$. Substituting f_i for h in (16), one has

$$(19) \qquad \|[f_i * g]_\varphi - (\overline{f_i : \varphi})\, g_{s_i}\| \leq \frac{\varepsilon}{4}(\overline{f_i : \varphi})$$

whenever $\varphi \varepsilon L^+$, $\varphi \neq 0$, and $i = 1,\ldots,n$. Therefore, combining (19) and (17),

$$\begin{aligned}\|\sum_{i=1}^{n}[f_i * g]_\varphi - \sum_{i=1}^{n}(\overline{f_i : \varphi})\, g_{s_i}\| &\leq \sum_{i=1}^{n}\left(\|[f_i * g]_\varphi - (\overline{f_i : \varphi})g_{s_i}\|\right)\\ &\leq \frac{\varepsilon}{4}\sum_{i=1}^{n}(\overline{f_i : \varphi})\\ &\leq \frac{\varepsilon}{2}(\overline{f : \varphi})\end{aligned}$$

for every $\varphi \varepsilon L_V^+$, $\varphi \neq 0$. Let φ denote a fixed member of L_V^+, and let $\alpha_i = (\overline{f_i : \varphi})$, $i = 1,\ldots,n$. This last inequality and (19) together with the triangle inequality, imply

$$\|[f * g]_\varphi - \sum_{i=1}^{n}\alpha_i\, g_{s_i}\| \leq \varepsilon\,(\overline{f : \varphi})$$

By proposition 2.3, $\varphi_i > 0$, $i = 1,\ldots,n$. ∎

<u>Proposition 3.7</u>. For every non-zero member f of L^+ and positive number ε, there exists a $V \varepsilon \mathcal{V}$ such that, for every $g \varepsilon L^+$, $g \neq 0$, there exist elements $s_1,\ldots,s_n$ in the support of f and positive real numbers $\alpha_1,\ldots,\alpha_n$ such that

$$\|f - \sum_{i=1}^{n}\alpha_i g_{s_i}\| \leq \varepsilon .$$

Proof. By the second part of proposition 3.5, there exists a $V \varepsilon$ such that

$$(20) \qquad \|[f * g]_\varphi - \overline{(g^* : \varphi)} f\| \leq \frac{1}{2} \overline{(g^* : \varphi)}$$

whenever $g \varepsilon L_V^+$, and $\varphi \varepsilon L^+$, $\varphi \neq 0$. Let g be a fixed non-zero member of L_V^+, and let W denote the member of $\mathcal{V}$ guaranteed to exist for f, g and $\frac{\varepsilon}{2} \overline{(f : g^*)}^{-1}$ in proposition 3.6. Then for a fixed $\varphi \varepsilon L_W^+$, $\varphi \neq 0$, there exists elements $s_1,\ldots,s_n$ of the support of f and positive real numbers $\gamma_1,\ldots,\gamma_n$ such that

$$(21) \qquad \|[f * g]_\varphi - \sum_{i=1}^{n} \gamma_i g_{s_i}\| \leq \frac{\varepsilon}{2} \overline{(f : \varphi)} \; \overline{(f : g^*)}^{-1} .$$

Using (20) and (21), and the triangle inequality one obtains

$$(22) \qquad \|\overline{(g^* : \varphi)} f - \sum_{i=1}^{n} \gamma_i g_{s_i}\| \leq \frac{\varepsilon}{2} \left(\overline{(g^* : \varphi)} + \overline{(f : \varphi)}\,\overline{(f : g^*)}^{-1}\right)$$

Let $\alpha_i = \overline{(g^* : \varphi)}^{-1} \gamma_i$, $i = 1,\ldots,n$. By proposition 2.5, $\overline{(f : \varphi)} \leq \overline{(f : g^*)} \; \overline{(g^* : \varphi)}$; whence from (22),

$$\|f - \sum_{i=1}^{n} \alpha_i g_{s_i}\| \leq \varepsilon \; . \blacksquare$$

Proposition 3.8. Let f and g be non-zero members of L^+ such that $f(x) < g(x)$ for every $x \varepsilon$ supp (f). Then there exists a $V \varepsilon \mathcal{V}$ such that, for every $\varphi \varepsilon L_V^+$, $\varphi \neq 0$, there exist elements $s_1,\ldots,s_n$ in the support of f and positive real numbers such that

$$f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \leq g .$$

Proof. (i) Let $\varepsilon = \frac{1}{4} \inf \{g(x) - f(x) : x \varepsilon \text{ supp } (f)\}$. Since supp (f) is compact, ε is a positive real number. Let $U = x \varepsilon G : g(x) > 2\varepsilon$. If $x \varepsilon$ supp (f), then $g(x) \geq 4\varepsilon + f(x) > 2\varepsilon$; hence, supp $(f) \subseteq U$. Since U is open, there exists a $V_1 \varepsilon \mathcal{V}$ such that supp $(f) \cdot V_1 \subseteq U$. Let A denote the set supp $(f) \cdot V_1$

Now suppose k is a non-zero number of L_A^+ such that $\| \frac{1}{2} (f + g) - k\| \leq \varepsilon$. Then

$$\frac{1}{2} (f(x) + g(x)) - \varepsilon \leq k(x) \leq \varepsilon + \frac{1}{2} (f(x) + g(x))$$

for every $x \varepsilon G$. It will be shown that

$$f(x) \leq k(x) \leq g(x) \tag{23}$$

for every $x \varepsilon G$. Two main possibilities occur: (i) $x \notin$ supp (f), and (ii) $x \varepsilon$ supp (f).

Suppose $x \notin (f)$. Certainly $f(x) \leq k(x)$. If $x \varepsilon U$ - supp (f), then $k(x) < \varepsilon + \frac{1}{2} (f(x) + g(x)) = \varepsilon + \frac{1}{2} g(x) < \frac{1}{2} g(x) + \frac{1}{2} g(x) = g(x)$; if $x \notin U$, then $k(x) = 0 \leq g(x)$. Thus (23) holds for all $x \notin$ supp (f).

If $x \varepsilon$ supp (f), then

$$f(x) < (3f(x) + g(x)) < \frac{1}{2} (f(x) + g(x) - \varepsilon \leq k(x),$$

and

$$k(x) \leq \varepsilon + \frac{1}{2}(f(x) + g(x)) \leq \frac{1}{4}(g(x) + f(x) < g(x) .$$

Consequently, (23) holds for all $x \varepsilon G$. Therefore, $k \varepsilon L_A^+$, $k \neq 0$, and $\| \frac{1}{2}(f + g) - k\| \leq \varepsilon$ imply that $f(x) \leq k(x) \leq g(x)$ for all $x \varepsilon G$.

(ii) By proposition 3.7, there is a $V_2 \varepsilon \mathcal{V}$ such that, for every $\psi \varepsilon L_{V_2}^+$, $\psi \neq 0$, there exist elements $t_1,\ldots,t_m$ in the support of f and positive real numbers $\beta_1,\ldots,\beta_m$ such that

$$\| \frac{1}{2}(f + g) - \sum_{j=1}^{m} \beta_j \psi_{t_j} \| \leq \varepsilon .$$

Let $V = V_1 \bigcap V_2$. If $\varphi \varepsilon L_V^+$, $\varphi \neq 0$, then there exist elements $s_1,\ldots,s_n$ in the support of f and positive real numbers $\alpha_1,\ldots,\alpha_n$ such that

$$\| \frac{1}{2}(f + g) - \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \| \leq \varepsilon .$$

Since supp $(\varphi_{s_i}) = s_i$ supp $(\varphi) \subseteq s_i V \subseteq$ supp $(f) \cdot V_1 = A$, it follows that $\varphi_{s_i} \varepsilon L_A^+$, $i = 1,\ldots,n$. The desired conclusion then follows from part one of the proof. ∎

CHAPTER IV

THE EXISTENCE AND UNIQUENESS OF THE HAAR INTEGRAL

Invariant Integrals

Definition 4.1. A non-negative functional J on L^+ is said to be an invariant integral provided J satisfies the following:

(i) $J \neq 0$;

(ii) if $f, g \in L^+$, then $f \leq g$ implies $J(f) \leq J(g)$;

(iii) if $\alpha \geq 0$, then $J(\alpha f) = \alpha J(f)$ for every $f \in L^+$;

(iv) if $f, g \in L^+$, then $J(f + g) = J(f) + J(g)$;

(v) if $f \in L^+$, then $J(f_s) = J(f)$ for every $s \in G$.

Proposition 4.2. Let J be an invariant integral on L^+. If f is a non-zero member of L^+, then $J(f) > 0$.

Proof. Let $\varphi \in L^+$, $\varphi \neq 0$, be such that $J(\varphi) > 0$. By proposition 2.1, there exist elements $s_1, \ldots, s_n$ of G and positive real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\varphi \leq \sum_{i=1}^{n} \alpha_i f_{s_i} .$$

Then

$$J(\varphi) \leq J\left(\sum_{i=1}^{n} \alpha_i f_{s_i}\right) .$$

But

$$J\left(\sum_{i=1}^{n} \alpha_i f_{s_i}\right) = \sum_{i=1}^{n} J(\alpha_i f_{s_i}) = \sum_{i=1}^{n} \alpha_i J(f_{s_i}) = \left(\sum_{i=1}^{n} \alpha_i\right) J(f) ,$$

whence

$$J(f) \geq \left(\sum_{i=1}^{n} \alpha_i\right)^{-1} J(\alpha) > 0 . \blacksquare$$

Definition 4.3. Let A be an arbitrary non-empty set. A subset R of $A \times A$ is said to be a pre-ordering of A if R is reflexive and transitive. If R and S are pre-orderings of A such that $(x, y) \varepsilon S$ implies $(x, y) \varepsilon R$, then R is said to be coarser than S.

Proposition 4.4. Let J be an invariant integral on L^+. Then the relation defined by

$$f \leq g (\text{mod } J) \quad \text{if, and only if, } J(f) \leq J(g),$$

where $f, g \varepsilon L^+$, is a pre-ordering of L^+.

Proof. Certainly $f \leq f$ (mod J) for every $f \varepsilon L^+$. Suppose that $f \leq f$ (mod J) and that $g \leq h$ (mod J). Then $J(f) \leq J(g)$ and $J(g) \leq J(h)$. Hence $J(f) \leq J(h)$ which implies that $f \leq h$ (mod J). $\blacksquare$

Definition 4.5. For every invariant integral J on L^+, the pre-ordering of L^+ associated with J is defined to be the one given in proposition 4.4.

Proposition 4.6. Let J_1 and J_2 be two invariant and integrals on L^+ such that the pre-ordering associated with J_2 is coarser than the one associated with J_1. Then there is a positive real number α such that $J_1(f) = \alpha J_2(f)$ for every $f \in L^+$.

Proof. Let g denote a fixed non-zero member of L^+. By proposition 4.2, there exists a positive real number α such that

(1) $$J_1(g) = \alpha\, J_2(g) .$$

For each $f \in L^+$, there is a $\beta \geq 0$ such that

(2) $$J_1(f) = \beta\, J_1(g) .$$

Whence

$$J_2(f) \leq \beta\, J_2(g) ,$$

and

$$\beta\, J_2(g) \leq J_2(f) .$$

Thus

(3) $$J_2(f) = \beta\, J_2(g) .$$

Combining (1), (2) and (3), one has

$$J_1(f) = \beta\, J_1(g) = \alpha\beta\, J_2(g) = \alpha J_2(f) . \blacksquare$$

The Haar Integral

Proposition 4.7. Let f and g be non-zero members of L^+. Then given $\varepsilon > 0$, there exists a $U \varepsilon \mathcal{V}$ such that, for each $\varphi \varepsilon L_U^+$, $\varphi \neq 0$, there exists a $V \varepsilon \mathcal{V}$ and a real number $c(\varphi) > 0$ such that

$$\left| \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} - c(\varphi) \frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \right| \leq \varepsilon$$

for every $\psi \varepsilon L_V^+$, $\psi \neq 0$.

<u>Proof</u>. Let U_1 be a fixed compact neighborhood of the identity C of G. Choose $f_1 \varepsilon L^+$ such that $\|f\| \leq 1$ and $f(x) = 1$ for all $x \varepsilon$ supp $(g) \circ U_1$. By proposition 3.7, corresponding to the positive real number

$$(4) \qquad \varepsilon' = \frac{\varepsilon}{1 + \overline{(f_1 : f)}}$$

there exists a $U \varepsilon \mathcal{V}$ such that for each $\varphi \varepsilon L_U^+$, $\varphi \neq 0$, there exist elements $s_1, \ldots, s_n$ of supp (g) and positive real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$(5) \qquad |g(x) - \sum_{i=1}^{n} \alpha_i \varphi_{s_i}(x)| \leq \varepsilon'$$

for all $x \varepsilon G$. It may be assumed that $U \subseteq U_1$. Since $\varphi_{s_i}, \ldots, \varphi_{s_n}$ and g each vanish off supp $(g) \circ U_1$, it follows that

$$(6) \qquad |g(x) - \sum_{i=1}^{n} \alpha_i \varphi_{s_i}(x)| \leq \varepsilon$$

for all $x \varepsilon G$. By an argument similar to the one used in the proof of proposition 2.10, (6) implies

(7)
$$\left| \frac{(\overline{g : \psi})}{(\overline{f : \psi})} - \frac{\left(\overline{\sum_{i=1}^{n} \alpha : \varphi_{s_i} : \psi}\right)}{(\overline{f : \psi})} \right| \leq \varepsilon' \frac{(\overline{f_1 : \psi})}{(\overline{f : \psi})}$$

for every $\psi \, \varepsilon \, L^+$, $\psi \neq 0$. By proposition 2.5, $(\overline{f_1 : \psi}) \leq (\overline{f_1 : f})(\overline{f : \psi})$, so that (7) may be replaced by

$$\frac{(\overline{g : \psi})}{(\overline{f : \psi})} - \frac{\left(\overline{\sum_{i=1}^{n} \alpha : \varphi_{s_i} : \psi}\right)}{(\overline{f : \psi})} \leq \varepsilon' \, (\overline{f_1 : f})$$

for every $\psi \, \varepsilon \, L^+$, $\psi \neq 0$.

Now by proposition 2.9, corresponding to the positive real number

$$\delta = \varepsilon' \left(\overline{\sum_{i=1}^{n} \alpha_i \, \varphi_{s_i} : f}\right)^{-1},$$

there exists a $V \, \varepsilon \, \mathcal{V}$ such that

$$\sum_{i=1}^{n} (\overline{\alpha_i \varphi_{s_i} : \psi}) \leq (1 + \delta) \left(\overline{\sum_{i=1}^{n} \alpha_i \varphi_{s_i} : \psi}\right)$$

for every $\psi \, \varepsilon \, L_V^+$, $\psi \neq 0$. Then by proposition 2.8,

$$\left| \frac{\sum_{i=1}^{n} (\alpha_i \varphi_{s_i} : \psi)}{(\overline{f : \psi})} - \frac{\left(\overline{\sum_{i=1}^{n} \alpha_i \varphi_{s_i} : \psi}\right)}{(\overline{f : \psi})} \right| \leq \delta \, \frac{\left(\overline{\sum_{i=1}^{n} \alpha_i \varphi_{s_i} : \psi}\right)}{(\overline{f : \psi})}$$

for every $\psi \in L_V^+$, $\psi \neq 0$. From proposition 2.5 and the choice of δ, it follows that

$$\left| \frac{\sum_{i=1}^{n} (\overline{\alpha_i \varphi_{s_i} : \psi})}{(\overline{f : \psi})} - \frac{\left(\overline{\sum_{i=1}^{n} \alpha_i \varphi_{s_i} : \psi}\right)}{(\overline{f : \psi})} \right| \leq \varepsilon'$$

for every $\psi \in L_V^+$, $\psi \neq 0$.

Now by propositions 2.7 and 2.8, $\sum_{i=1}^{n} (\overline{\alpha_i \varphi_{s_i} : \psi}) = \sum_{i=1}^{n} \alpha_i \ (\overline{\varphi : \psi})$; let $c(\varphi) = \sum_{i=1}^{n} \alpha_i$. Then combining (8) and (9), and using the triangle inequality, one has

$$\left| \frac{(\overline{g : \psi})}{(\overline{f : \psi})} - c(\varphi) \frac{(\overline{\varphi : \psi})}{(\overline{f : \psi})} \right| \leq \varepsilon' \ (1 + (\overline{f_1 : f}))$$

for every $\psi \in L_V^+$, $\psi \neq 0$. The desired conclusion follows from (4).∎

<u>Proposition 4.8</u>. Let f and g be non-zero members of L^+. Then given $\varepsilon > 0$, there exists a $V \in \mathcal{V}$ such that

$$\left| \frac{(\overline{g : \psi_1})}{(\overline{f : \psi_1})} - \frac{(\overline{g : \psi_2})}{(\overline{f : \psi_2})} \right| \leq \varepsilon$$

whenever ψ_1 and ψ_2 are non-zero members of L_V^+.

<u>Proof</u>. Let δ be a real number such that:

(10) $$0 < \delta < 1, \quad \text{and} \quad \frac{\delta}{1 - \delta} < \frac{\varepsilon}{2((\overline{g : f}) + 1)}.$$

In view of proposition 3.7, there exists a $U_1 \varepsilon \mathcal{V}$ such that for every $\varphi \varepsilon L_U^+$, $\varphi \neq 0$, there exists a $V_1(\varphi) \varepsilon \mathcal{V}$ and $c(\varphi) > 0$ such that

$$\left| \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} - c(\varphi) \frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \right| \leq \delta$$

for every $\psi \varepsilon L^+_{V(\varphi)}$, $\psi \neq 0$. Similarly, there exists a $U_2 \varepsilon \mathcal{V}$ such that for every $\varphi \varepsilon L^+_{U_2}$, $\varphi \neq 0$, there exists a $V_2(\varphi) \varepsilon \mathcal{V}$ and $d(\varphi) > 0$ such that

$$\left| 1 - d(\varphi) \frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \right| \leq \delta$$

for every $\psi \varepsilon L^+_{V_2(\varphi)}$, $\psi \neq 0$.

Now let $U = U_1 \cap U_2$. For each $\varphi \varepsilon L_U^+$, $\varphi \neq 0$, define $V(\varphi) = V_1(\varphi) \cap V_2(\varphi)$, and $r(\varphi) = c(\varphi)/d(\varphi)$. It follows from the previous paragraph that for every $\varphi \varepsilon L_U^+$, $\varphi \neq 0$, the inequalities

$$\left| \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} - c(\varphi) \frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \right| \leq \delta \tag{11}$$

and

$$\left| 1 - d(\varphi) \frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \right| \leq \delta \tag{12}$$

both hold for all $\psi \varepsilon L^+_{V(\varphi)}$, $\psi \neq 0$; and therefore, combining (11) and (12) and using the triangle inequality, the relation

$$\left| \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} - r(\varphi) \right| \leq \delta(1 + r(\varphi)) \tag{13}$$

holds for every $\psi \in L^+_{V(\varphi)}$, $\psi \neq 0$.

Let φ denote a fixed non-zero member of L^+_U. If $\psi \in L^+_{V(\varphi)}$, $\psi \neq 0$, then inequalities (11) and (12) yield

$$c(\varphi)\,\frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \leq \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} + \delta \,, \tag{14}$$

and

$$d(\varphi)\,\frac{\overline{(\varphi : \psi)}}{\overline{(f : \psi)}} \geq 1 - \delta \,, \tag{15}$$

so that dividing (14) by (15), one has

$$r(\varphi) \leq \frac{\overline{(g : \psi)}\,\overline{(f : \psi)}^{-1} + \delta}{1 - \delta}$$

but $\overline{(g : \psi)} \leq \overline{(g : f)}\,\overline{(f : \psi)}$, so that

$$r(\varphi) \leq \frac{\overline{(g : f)} + 1}{1 - \delta} \,. \tag{16}$$

Then by applying (16) to (13), one obtains

$$\left| \frac{\overline{(g : \psi)}}{\overline{(f : \psi)}} - r(\varphi) \right| \leq \delta + \frac{\delta(\overline{(g : f)} + 1)}{1 - \delta} = \left(\frac{2 - \delta + \overline{(g : f)}}{1 - \delta} \right) \delta < \left(\frac{2 + \overline{(g : f)}}{1 - \delta} \right) \delta < \frac{\varepsilon}{2} \tag{17}$$

for every $\psi \in L^+_{V(\varphi)}$, $\psi \neq 0$. If ψ_1 and ψ_2 are non-zero members of $V(\varphi)$, then from (17), one has

$$\left| \frac{(\overline{g : \psi_1})}{(\overline{f : \psi_1})} - \frac{(\overline{g : \psi_2})}{(\overline{f : \psi_2})} \right| \leq \varepsilon .$$

Taking V to be $V(\varphi)$, the proposition is proved. ∎

Let f_o denote a fixed non-zero member of L^+. By proposition 2.14, there exists a $V(f_o) \varepsilon \mathcal{V}$ such that $\underline{(f_o : \varphi)} > 0$ for every $\varphi \varepsilon L^+_{V(f_o)}$, $\varphi \neq 0$. Let

$$S = \left\{ i : i = (U, \varphi), \text{ where } U \varepsilon \mathcal{V},\ U \subseteq V(f), \text{ and } \varphi \varepsilon L^+_U, \varphi \neq 0 \right\}.$$

Define a relation on S as follows: given $i, j \varepsilon S$, where $i = (U, \varphi)$, $j = (V, \psi)$, $i \leq j$ if, and only if, $V \subseteq U$. Clearly, this relation is transitive. Let $i, j \varepsilon S$, where $i = (U, \varphi)$, $j = (V, \psi)$; define $k = (W, \Upsilon)$ as follows: $W = U \cap V$ and Υ is any non-zero member of L^+_W. Then $i \leq k$ and $j \leq k$. Therefore, $(S, \leq)$ is a directed set.

Now let $g \varepsilon L^+$, $g \neq 0$. For each $i = (U, \varphi) \varepsilon S$, define $Bg(i)$ as follows:

$$Bg(i) = \frac{(\overline{g : \varphi})}{(\overline{f_o : \varphi})} .$$

Let $\varepsilon > 0$ be given. By the previous proposition, there exists a $V \varepsilon \mathcal{V}$ such that $V \subseteq V(f_o)$ and

$$\left| \frac{(\overline{g : \psi_1})}{(\overline{f_o : \psi_1})} - \frac{(\overline{g : \psi_2})}{(f_o : \psi_2)} \right| \leq \varepsilon$$

whenever ψ_1 and ψ_2 are non-zero members of L^+_V. Let φ be a fixed non-zero member of L_V^+, and let $i = (V, \varphi)$. Then

$$\left| Bg(j) - Bg(k) \right| \leq \varepsilon$$

for every j, $k \varepsilon S$ such that $i \leq j$ and $i \leq k$. Therefore $\{Bg(i)\}_{i \varepsilon S}$ is a generalized Cauchy sequence; hence, the limit $\lim_{i \varepsilon S} Bg(i)$ exists for each $g \varepsilon \mathcal{L}^+$, $g \neq 0$.

<u>Proposition 4.9</u>. The functional I defined on L^+ by

$$I(g) = \begin{cases} \lim_{i \varepsilon S} Bg(i) & \text{if } g \neq 0\,, \\ 0 & \text{if } g = 0 \end{cases}$$

is an invariant integral.

<u>Proof</u>. (i) Certainly $I \neq 0$, since $I(f_o) = 1$. (ii) Suppose g, $h \varepsilon L^+$, and $g \leq h$. If $g = 0$, then $I(g) \leq I(h)$. If $g \neq 0$, then $h \neq 0$; by proposition 2.4, $I(g) \leq I(h)$. (iii) Let $g \varepsilon L^+$, and let $\alpha \geq 0$. If $\alpha = 0$, or if $g = 0$, then $I(\alpha g) = 0 = \alpha I(g)$. Suppose that $\alpha > 0$, and that $g \neq 0$. By proposition 2.4, $I(\alpha g) = \alpha I(g)$. (iv) Let g, $h \varepsilon L^+$. If $g + h = 0$, then $g = 0$, and $h = 0$. Hence $I(g + h) = 0 = I(g) + I(h)$. If $g = 0$, then $I(g + h) = I(h) = I(g) + I(h)$; similarly if $h = 0$. Suppose that $g \neq 0$, and that $h \neq 0$. By proposition 2.8, $I(g + h) \leq I(g) + I(h)$. Let $\lambda > 1$ be given. By proposition 2.9, there is a $V \varepsilon \mathcal{V}$ such that $V \_ V(f_o)$ and $(\overline{g : \varphi}) + (\overline{h : \varphi}) \leq \lambda(\overline{g + h : \varphi})$ for every $\varphi \varepsilon L_V^+$, $\varphi \neq 0$. Whence, $I(g) + I(h) \leq \lambda I(g + h)$. Since $\lambda > 1$ is arbitrary, one has that $I(g) + I(h) \leq I(g + h)$. Consequently, $I(g + h) = I(g)+I(h)$.

(v) Let $g \in L^+$, and let $s \in G$. Note that $g \neq 0$ if, and only if, $g_s \neq 0$. Then $I(g_s) = I(g) = 0$ if $g = 0$. If $g \neq 0$, then $I(g_s) = I(g)$ by proposition 2.6. ∎

Definition 4.10. The Haar integral is defined to be the integral I as defined in proposition 4.9

The Uniqueness of the Haar Integral

Proposition 4.11. Let g be a non-zero member of L^+, and define a generalized sequence $\{Cg(i)\}_{i \in S}$, as follows:

$$Cg(i) = \frac{\overline{(\overline{g : \varphi})}}{\underline{(f_o : \varphi)}} ,$$

where $i = (U, \varphi) \in S$. Then

$$\lim_{i \in S} Cg(i) = \lim_{i \in S} Bg(i) .$$

Proof. Let $\lambda > 1$ be given. For each $n = 1,2,\ldots,$ define $f_n = (f_o - \frac{1}{n})^+$. For each $n \geq 1$, $f_n \in L^+$, and $f_n \uparrow f_o$. By proposition 2.11, there is a positive integer m such that

$$-(1 - \lambda^{-1}) \leq \frac{(\overline{f_m : \varphi})}{(\overline{f_o : \varphi})} - 1$$

for every $\varphi \in L^+$, $\varphi \neq 0$. Whence

$$\lambda^{-1} \leq \frac{(\overline{f_m : \varphi})}{(\overline{f_o : \varphi})} \tag{18}$$

for every $\varphi \in L^+$, $\varphi \neq 0$.

Now applying proposition 3.8 to f_m and f and in place of f and g, there exists a $V \in \mathcal{V}$, $V \subseteq V(f_o)$, such that for every $\varphi \in L^+$, $\varphi \neq 0$, there exist elements $s_1, \ldots, s_k$ in the support of f_m and positive real numbers $\alpha_1, \ldots, \alpha_k$ such that

$$f_m \leq \sum_{i=1}^{k} \alpha_i \varphi_{s_i} \leq f .$$

From this last inequality, one obtains

$$(19) \qquad \overline{(f_m : \varphi)} \leq \underline{(f_o : \varphi)}$$

for every $\varphi \in L_V^+$, $\varphi \neq 0$. Now combining (18) and (19) with proposition 2.13, one obtains the relation

$$\lambda^{-1} \leq \frac{\overline{(f_m : \varphi)}}{\overline{(f_o : \varphi)}} \leq \frac{\underline{(f_o : \varphi)}}{\overline{(f_o : \varphi)}} \leq 1$$

for every $\varphi \in L_V^+$, $\varphi \neq 0$. Whence

$$(20) \qquad 1 \leq \frac{\overline{(f_o : \varphi)}}{\underline{(f_o : \varphi)}} \leq \lambda$$

for every $\varphi \in L_V^+$, $\varphi \neq 0$.

Now define a generalized sequence $\{Df(i)\}_{i \in S}$ as follows:

$$Df(i) = \frac{\overline{(f_o : \varphi)}}{\underline{(f_o : \varphi)}} .$$

where $i = (U, \varphi) \in S$. From (20), one has the following limit:

(21) $$\lim_{i \in S} Df(i) = 1 .$$

Now observe that,

$$Dg(i) = Bg(i) \cdot Df(i)$$

for each $i \in S$. The desired conclusion follows from the existence of the limit $\lim\limits_{i \in S} Bg(i)$ and (21). ∎

Proposition 4.12. Let g be a non-zero member of L^+. Then

$$\inf_{i \in S} \left\{ Cg(i) \right\} = \lim_{i \in S} Cg(i) ,$$

where $\left\{ Cg(i) \right\}_{i \in S}$ is as defined in proposition 4.11.

Proof. Let $\lambda > 1$ be given. By (21) of proposition 4.11, for each $\psi \in V(f_o)$, $\psi \neq 0$, there exists a $V \in \mathcal{V}$ and $V \subseteq V(f_o)$ such that

(22) $$\frac{\overline{(\psi : \varphi)}}{\underline{(\psi : \varphi)}} \leq \lambda$$

for every L_V^+, $\varphi \neq 0$. From propositions 2.5 and 2.13, one has

$$\frac{\overline{(g : \varphi)}}{\underline{(f_o : \varphi)}} \leq \frac{\overline{(g : \psi)}\,\overline{(\psi : \varphi)}}{\underline{(f_o : \psi)}\,\underline{(\psi : \varphi)}} ,$$

and hence by (22), it follows that

(23) $$\frac{\overline{(g : \varphi)}}{\underline{(f_o : \varphi)}} \leq \lambda \frac{\overline{(g : \psi)}}{\underline{(f_o : \psi)}}$$

for every $\varphi \in L_V^+$, $\varphi \neq 0$. Now keeping ψ fixed in (23), one has

$$\lim_{i \varepsilon S} \ Cg(i) \leq \lambda \frac{\overline{(g : \psi)}}{\underline{(f_o : \psi)}} .$$

and whence

$$\lim_{i \varepsilon S} \ Cg(i) \leq \lambda \inf_{i \varepsilon S} \left\{ Cg(i) \right\} .$$

But since $\lambda > 1$ is arbitrary, it follows that

$$(24) \qquad \lim_{i \varepsilon S} \ Cg(i) \leq \inf_{i \varepsilon S} \left\{ Cg(i) \right\} .$$

Since $\inf\limits_{i \varepsilon S} \left\{ Cg(i) \right\} \leq \lim\limits_{i \varepsilon S} \ Cg(i)$, the conclusion follows from (29). ∎

<u>Proposition 4.13</u>. Let g, $h \varepsilon L^+$. Then if $\underline{(g : \varphi)} \leq \overline{(h : \varphi)}$ for every $\varphi \varepsilon L^+$, $\varphi \neq 0$, then $I(g) \leq I(h)$.

<u>Proof</u>. If $g = 0$, there is nothing to prove. Assume that $g \neq 0$. By proposition 2.14 there is a $V(g) \varepsilon \mathcal{V}$ such that $V(g) \subseteq V(f_o)$ and $\underline{(g : \varphi)} > 0$ for every $\varphi \varepsilon L^+_{V(g)}$, $\varphi \neq 0$. Let

$$T = \left\{ i : i = (U, \varphi), \text{ where } U \varepsilon \mathcal{V}, U \subseteq V(g), \text{ and } \varphi \varepsilon L^+_U, \ \varphi \neq 0 \right\} .$$

For each $i \varepsilon T$, define a generalized sequences $\left\{ Eg(i) \right\}_{i \varepsilon T}$ and $\left\{ Fg(i) \right\}_{i \varepsilon T}$ as follows:

$$Eg(i) = \frac{\overline{(h : \varphi)}}{\overline{(g : \varphi)}} ,$$

and

$$Fg(i) = \frac{\overline{(h : \varphi)}}{\underline{(g : \varphi)}}$$

where $i = (U, \varphi)$. By propositions 4.11 and 4.12, one has

$$\lim_{i \in T} Eg(i) = \inf_{i \in T} \{Fg(i)\} \geq 1 . \tag{25}$$

Now by the definition of $I(h)$,

$$\begin{aligned} I(h) &= \lim_{i \in S} Bh(i) \\ &= \lim_{i \in T} Bh(i) \\ &= \lim_{i \in T} \{Bg(i) \cdot Eg(i)\} \\ &= I(g) \lim_{i \in T} Eg(i) . \end{aligned}$$

Hence by (25), one has

$$I(g) \leq I(h) . \blacksquare$$

<u>Proposition 4.14</u>. Let J be any invariant integral on L^{+}. Then $J(g) \leq J(h)$ implies $(\underline{g : \varphi}) \leq (\overline{h : \varphi})$ for every $\varphi \in L^{+}$, $\varphi \neq 0$.

<u>Proof</u>. Let g, $h \in L^{+}$ be such that $J(g) \leq J(h)$. Let $\varphi \in L^{+}$, $\varphi \neq 0$. Let $s_1, \ldots, s_n$ be elements of G and $\alpha_1, \ldots, \alpha_n$ be non-negative real numbers such that

$$\sum_{i=1}^{n} \alpha_i \varphi_{s_i} \leq g . \tag{31}$$

Let $t_1, \ldots, t_m$ be elements of G and $\beta_1, \ldots, \beta_m$ be positive real numbers such that

$$h \leq \sum_{j=1}^{m} \beta_j \varphi_{t_j} \, . \tag{32}$$

Combining (31) and (32), and using the properties of J, one has

$$\left(\sum_{i=1}^{m} \alpha_i\right) J(\varphi) \leq \left(\sum_{j=1}^{m} \beta_j\right) J(\varphi)$$

By proposition 4.2, $J(\varphi) > 0$, hence

$$\sum_{i=1}^{m} \alpha_i \leq \sum_{j=1}^{m} \beta_j \, ,$$

which implies $(\underline{g : \varphi}) \leq (\overline{h : \varphi})$. ∎

<u>Proposition 4.15</u>. Let J be any invariant integral on L^+. Then there is an $\alpha > 0$ such that $J(g) = \alpha I(g)$ for every $g \in L^+$.

<u>Proof</u>. By propositions 4.14 and 4.13, the pre-ordering associated with I is coarser than the one associated with J. From proposition 4.6, there is an $\alpha > 0$ such that $J(g) = \alpha I(g)$ for every $g \in L^+$. ∎

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