SYMBOLIC MODELLING AND DYNAMIC ANALYSIS OF FLEXIBLE MANIPULATORS

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Abstract : This paper presents a systematic method to symbolically derive the full nonlinear dynamic equations of motion of Multi-link flexible manipulators . Lagrange's-Assumed Modes method is used for the dynamic modelling and implemented via a commercially available symbolic manipulation program. Adaptation of the method suitable for symbolic manipulation and advantages are discussed. Simulation results for a two-link planar flexible arm presented.

Keywords : Robotic Manipulators, Modelling, Symbolic , Dynamics, Simulation .

I. Introduction :

Dynamics of a typical Industrial Manipulator, with six degrees of freedom , is governed by coupled highly nonlinear ordinary differential equations . These equations present as a very complicated problem from control system design point of view, mainly because the knowledge in Nonlinear Control System theory is very limited. Traditional independent servo controllers are designed based on the assumption that nonlinear coupling terms are negligible. However, this assumption is reasonable and the control system performance may be satisfactory only if the speed of manipulator is "relatively slow " . Increasing demand for higher industrial productivity requires manipulators that moves faster and more accurate. As a result the speed of manipulators increases and the independent linear servo controllers , designed based on the slow motion dynamics , performs unsatisfactorily. In recent years there has been considerable progress in the Adaptive Control of Robotic Manipulators. Compute Torque based methods are aimed at better performance by designing controllers based on more accurate models. However the performance and the capabilities of a system, i.e. maximum speeds etc. , are limited by the initial design of the overall system. A control system at best can utilize these capabilities in an optimum manner, but not better than what the system is capable of. In other words no control law can make the system move at speed which can not be afforded by the existing actuators. Apparently one way of designing manipulators that can move faster is to increase the actuator sizes. However, since actuators themselves are carried by the other actuators, increasing size also increases the effective inertia resulting in a very bulky structure. So this approach can be quickly self defeating and is not the ultimate answer. The next option is to design light weight systems. Light-weight systems will have the following advantages ; higher speed of operation ,less overall cost , less energy consumption , smaller actuator sizes, higher productivity. The drawback of such systems is the structural flexibility which deteriorates the accuracy and repeatability. Rigid body dynamic analysis will no longer be accurate and controllers based on this will not perform satisfactorily. Flexibilities has to be included in the analysis. The control problem of Flexible manipulators may be solved by combination of the following approaches ;

1. Design materials and shapes such that highest stiffness/mass ratio is achieved.
2. Active feedback control of flexible vibrations
3. Passive damping treatment of flexible elements to help to damp out the vibrations
4. Develop trajectory generation algorithms that designs tasks such that the excitation of flexible modes are minimized.

Background :

Modelling and Control of a single link flexible arm has been investigated by many authors. The system is essentially modeled as Bernoulli-Euler Beam and vibration coordinates are approximated by a finite number of assumed mode shapes. This allows the application of the whole finite dimensional linear control theory to the problem . The effect of controller (observer) based on the truncated model on the unmodelled modes ("spillover") is first presented by Balas [17]. The state space domain based controller performances are theoretically and experimentally investigated by Hastings and Book [14]. The effect of augmenting passive damping to the flexible link is found to be very effective at stabilizing the high frequency oscilations [15]. Cannon and Schmitz used frequency domain and LQR based controllers on a more flexible single link arm.[16]

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Multi-link flexible manipulator modelling and control has not been researched as much as single - link case. There are two main method used in the modelling : 1. Lagrange's-Finite Element based methods 2. Lagrange's- Assumed mode based methods. The end result of these methods are essentially the same. Many of the finite element based works on the analysis of closed chain mechanisms can be applied to the dynamic modelling of multi-link flexible arms.[1,2] . One of the well knows works of this class can be summarized as follows [4,5]

1. Derive the equation of motion of every individual element of the system with no joint constraints (Using one of the commercially available finite element packages, i.e. NASTRAN)
2. Reduce the order of every element by and order reduction method (i.e . Component Mode Synthesis [6,7,8])
3. Assemble the element equations with the " Compatibility Matrix " , imposing system constraints.

In [5] the nominal joint variable time histories are assumed to be known and the small vibrational dynamic model of the manipulators and mechanisms about nominal motions are developed. In [3] this assumption is removed and full dynamic model is derived.Pro's of this method are : a) very systematic b) Can be applied to complex shaped systems, applicable to a very wide class of problems c) many information about the system dynamics can be obtained . Con's are : a) requires a substantial amount of software organization, b) results in constrained model.
Static deflection modes are included in the modes to improve the accuracy of models with limited number of mode shapes [2] . Usuro et.al. investigated the performance of LQR with prescribed degree of stability on a two-link planar arm by simulations [13].

Lagrangian - Assumed modes method is used in the modelling of a two-link robotic manipulator in [19]. Distributed frequency domain analysis of non-planar manipulators using Transfer-Matrices has been developed at [12]. The work [10] by Hughes describes the dynamics of open loop chain flexible structures , but ignores the interactions between flexible deformations and angular rates . A recursive method using Homogeneous transformation matrices to generate FULL coupled nonlinear dynamics of multi-link flexible manipulators is presented at [11] . It was experienced by the authors that the application of this technique to multi-link manipulators works well , but with an important drawback : Algebraic complexity of intermediate steps. When carried out by hand the length of expressions becomes very large and very time consuming . In addition to that the possibility of making algebraic errors was quite high. On the other hand modelling method is easy to understand, recursive , does not require any dedicated special software and derives the full nonlinear dynamic model .

The symbolic manipulation programs are the answer to eliminate the major drawback of the method. Symbolic modelling allows one to model systems with large orders in a very short time, check the elements of the dynamic equations in explicit forms and manipulate them very conveniently .

The remaining part of the paper is organized as follows ;
Section II summarizes the modelling technique used.
Section III presents the symbolic manipulation adaptation of the method .
Section IV presents an example case application and simulation results. Conclusion forms Section V.

II. Lagrangian - Assumed Modes Method :

Kinematics : The first step in dynamic modelling of any mechanical system is to establish the kinematical relationships and be able to define fundamental vector quantities Position, Velocity and acceleration. Consider the kinematic structure shown in Fig.1 representing a manipulator with serial links and joints. Let the coordinate systems used for kinematics of the system be ;
O_0XYZ - Fixed to base (Global Coordinate Frame)
$O_i xyz$ - Fixed to the base of the link i
$O'_i xyz$ - Fixed to the end of link i

If arms are rigid then $O_i' xyz$ coordinates are not needed . The position vector of any point on link i can be expressed with respect to $O_i xyz$ as ;

$$^{i}h(x_i) = [x_i,0,0,1]^T + {}^{i}[w_x(x_i,t),w_y(x_i,t),w_z(x_i,t),0]^T$$

where , $w_x(x_i,t)$, $w_y(x_i,t)$,$w_z(x_i,t)$ are displacements of the flexible arm due to flexibility in respective directions. The dependence of w's on the spatial coordinates makes the system infinite dimensional, leading to coupled Ordinary and Partial differential equations of motion. However in general these are approximated by finite number of modal coordinates by truncation. Once decided on the number of modal coordinates used ;

$$w_\beta(\beta_i,t)= \sum_{j=1}^{m_j} \phi_j(\beta_i)\ \delta_j(t)$$

$^{i}h(x)$ is uniquely defined. Next we need to be able to transfer this position vector with respect to global coordinate frame to obtain absolute position vector. Let $^{0}W_i$ be the homogeneous matrix transformation from moving coordinate frame $O_i xyz$ to fixed inertial frame O_0XYZ. Then The absolute Position vector ,[Fig.2]

$$^{0}h(x_i) = {}^{0}W_i\ .\ {}^{i}h(x_i)$$

It is clear that the transformation W consists of two parts , due to joint variables and flexible deflations . More clearly,

$$^{0}W_i = {}^{0}W_{i-1}\ .\ E_{i-1} .\ A_i$$

where
A_i - the transformation between $O_{i+1}xyz$ and $O'_i xyz$ - joint transformation
E_{i-1} - the transformation from the end of the link coordinates to link base coordinates.
$^{0}W_{i-1}$ - the total transformation to the base coord.

The form of these transformation matrices are ;

$$ {}^{j}W_i = \begin{bmatrix} x_j \text{ component of } O_i & {}^{j}R_i \\ y_j \text{ component of } O_i & \\ z_j \text{ component of } O_i & \\ 1 & 0^T \end{bmatrix} $$

${}^{j}R_i$ is (3x3) matrix of direction cosines, 0^T (1x3) zero

$$ E_i = \begin{bmatrix} 1 & 0 & 0 & l_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \Sigma\, \delta_{ij} \begin{bmatrix} 0 & \Theta_{zij} & \Theta_{yij} & x_{ij} \\ \Theta_{zij} & 0 & \Theta_{xij} & y_{ij} \\ -\Theta_{yij} & \Theta_{xij} & 0 & z_{ij} \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

where $\Theta_{\beta ij}$'s are rotation components of link i due to mode j, assuming small rotations due to flexible deflections. l_i is the length of the link i.

Notice the homogeneous matrices are very convenient tool for kinematic description. Once the kinematic description of the system is set up , the process of obtaining the equations of motion is as follows ,

1. Pick generalized coordinates (natural choices joint variables and modal coordinates for every flexible elements)
2. Form the kinetic , and potential energy ,and virtual work for the system
3. Take the necessary derivatives of the Lagrangian Equations and assemble the equations.

If system has N_j number of joints and N_l number of flexible elements with m_i modal coordinate for each element , the dynamic model of the system will be governed by $N_j + \Sigma\, m_i$, i=1, ..,N_l set of coupled second order ordinary differential equations.

III. Symbolic Implementation of Lagrangian- Assumed Modes Method:

Here the modelling method is adopted in a way suitable for symbolic manipulation by a digital computer. Let us first specify some desired features of a modelling algorithm . First, the mode shapes and the mode shape dependent parameters should be easily varied by the analyst. The selection of " appropriate " or " best " mode shapes for a given flexible system is not a clearly answered problem [12]. One should be able to simulate the effect of different mode shapes on the system behavior easily. For the case of a simple beam under bending vibrations the mode shapes effectively determine the natural frequencies of the system. Effective mass and spring matrix elements are functions of mode shapes as ;(with simple boundary conditions)

$$ m_{ij} = \int \rho\, A(x)\, \phi_i(x)\, \phi_j(x)\, dx $$

$$ k_{ij} = \int E\, I(x)\, \phi_i''(x)\, \phi_j''(x)\, dx $$

Then if mode shapes are orhonormalized such that m_{ij} = 1 for i=j and 0 , for i≠j, then $k_{ij} = w^2_i$ for i=j. The most accurate way is to update the mode shapes [12] as the boundary conditions of the links varies as function of controller action.

Second, recursive algorithm would be very desirable. For instance , when the number of modal coordinates increased or additional links included , the dynamic modelling process should not be repeated all over.

The equations governing the dynamics of the system is given by;

$$ \frac{d}{dt}\frac{(\partial\, \Sigma\, KE)}{\partial \dot q_i} - \frac{(\partial\, \Sigma\, KE)}{\partial q_i} + \frac{(\partial\, \Sigma\, PE)}{\partial q_i} = Q_i $$

where

$$ \Sigma\, KE = \sum_{i=1}^{n} (KE)_i $$; n: total number of discrete element in the system (joints, links, payload).

$$ \Sigma\, PE = \sum_{i=1}^{n} (PE)_{i\ \text{gravitatinal}} + (PE)_{i\ \text{elastic}} $$

q_i is the generalized coordinates of every element, joint variables for joints, flexible modal coordinates for links.

Kinetic Energies for rotary joints , if considered as mass with rotarty inertia about axis of rotation :

$$ (KE)_{\text{joint } i} = 1/2\, m_j\, V_{gj}^2 + 1/2\, \vec{H}_{gj} \cdot \vec{w}_j $$

H_{gj} , angular momentum of joint with respect to center of mass, w_j is the total angular velocity of the joint.

Kinetic Energy of the flexible links;

$$ (KE)_i = 1/2 \int \rho_i(x)\, (\, \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\,)\, dx $$

Here we would like to indentify the all possible elements that are function of spatial variable and mode shapes so that the integration does not have to be taken explicitly at this level.

$$ (KE)_i = (\ ) \int \rho(x)\, dx + (\ ) \int \rho(x)\, x\, dx + (\ ) \int \rho(x)\, x^2\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta \cdot w_\alpha\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta \cdot w_\beta\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta \cdot \dot{x}\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta \cdot x\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta\, dx + (\ ) \int \rho(x)\, \Sigma\, w_\beta\, dx $$

Also for the Elastic Potential energy (gravitational potential energy is omitted here to save space)

$$ (PE)_{i\ els} = 1/2 \int \left(E \left(I_y \left(\frac{\partial^2 w_{yi}}{\partial x^2}\right)^2 + I_z \left(\frac{\partial^2 w_{zi}}{\partial x^2}\right)^2 \right) + E\, A(x) \left(\frac{\partial^2 w_{xi}}{\partial x^2}\right)^2 \right) dx $$

where $w_{\beta i} = \Sigma\, \phi_{ij}(x)\, \delta_{ij}(t)$ for β; x,y,z .

If all the modal coordinates and associated mode shapes were given , then the integration under spatial variable could be evaluated. However since the mode shapes and dependent parameters are desired to be inputed by the user for analysis purposes , we identify all possible elements that are function of spatial variable and assign them parametric names.

These values are evaluated in the simulation level for desired cases, as discussed above. Notice that if mode shapes for each type of vibration are chosen as orthogonal with respect to distributed mass and stiffness, many terms drops out in the above equations.

$\int \rho(x)\, dx = m_i$, $\int \rho(x)\ x\, dx = m_i\, l_i/2$, $\int \rho(x)\, x^2\, dx = Jo_i$

$$\int \rho(x)\, \Sigma\, w_\beta\, dx = \int \rho(x)\, \Sigma\, \Sigma\, \phi_{ij}x)\, \delta_{ij}(t)\ dx$$

$$= \Sigma\, \Sigma (\int \rho(x)\, \phi_{ij}(x)\, dx\,)\, \delta_{ij}(t)$$

$$= \Sigma\, \Sigma\, nq\beta_{ij}\, .\, \delta_{ij}(t)$$

for β : x,y,z ; $nq\beta_{ij}$: nqx_{ij} , nqy_{ij} , nqz_{ij}

Similarly the rest of the spatial variable dependent terms can be expanded in their modal coordinates.

$$\int \rho(x)\, \Sigma\, w_{\beta i}\, dx = \Sigma\, \Sigma\, nq\beta_{ij}\, .\, \delta_{ij}(t)$$

$$\int \rho(x)\, \Sigma\, w_{\beta i}\, .\, x\, .dx = \Sigma\, \Sigma\, nw\beta_{ij}\, .\, \delta_{ij}(t)$$

$$\int \rho(x)\, \Sigma\, w_{\beta i}\, .\, w_{\alpha i}\, dx = \Sigma\, \Sigma\, nm\beta_{ij}\, .\, \delta_{ij}{}^2(t)$$

and so on...

Now the next important topic is " can we design an algorithm that can work in a recursive manner and also suitable for digital computer in terms of memory requirements ". As the system dimension gets larger, carrying out the derivations using total Energy expressions can easily run into memory problems.

$$\frac{d}{dt}\frac{\partial}{\partial q_i}(\,\Sigma\, KE_j\,) - \frac{\partial}{\partial q_i}(\,\Sigma\, KE_j\,) + \frac{\partial}{\partial q_i}(\Sigma PE_j\,) = Q_j$$

$$\Sigma\,\left(\frac{d}{dt}\frac{\partial}{\partial q_i}(KE_j) - \frac{\partial}{\partial q_i}(KE_j) + \frac{\partial}{\partial q_i}(PE_j)\right) = Q_i$$

Due to serial nature of manipulator arm ;

$$\frac{\partial}{\partial q_i}(KE_j) = \frac{\partial}{\partial q_i}(KE_j) = \frac{\partial}{\partial q_i}(PE_j) = 0. \quad \text{for } i > j$$

Then the equations of motion of the system

$$\sum_{j=i}^{N}\left(\frac{d}{dt}\frac{\partial}{\partial q_i}(KE_j) - \frac{\partial}{\partial q_i}(KE_j) + \frac{\partial}{\partial q_i}(PE_j)\right) = Q_i \qquad (1)$$

Algorithm :

```
For j = 1 to N
   For i=1, to j
          Find and store KE_j , PE_j

          ∂            d  ∂           ∂
          ---(KE_j) , ---(---(KE_j)) , ----(PE_j)
          ∂q_i        dt ∂q_i         ∂q_i
   Next i
Next j
```

Given all the non-zero derivatives substitute these to equation (1) and assemble the equations in a convenient form for simulations and analysis purposes. Let us assume that after modelling a manipulator , it is desired to add another link to the model. Based on the above algorithm

```
For i=1, to N+1
    d   ∂                ∂                    ∂
   --- ---(KE_N+1) ,    ---( KE_N+1 )  ,     ---( PE_N+1 )
   dt  ∂q_i              ∂q_i                 ∂q_i
Next i
```

Let us assume previous model was assembled in the form;

$$[M_N]\, \ddot{\underline{q}} + \underline{f}_N = \underline{Q}_N \qquad (2)$$

The result of additional link contribution is of form

$$\begin{vmatrix} m_{nn} & m_{nn+1} \\ m^T_{nn+1} & m_{n+1} \end{vmatrix} \begin{vmatrix} q \\ q_{n+1} \end{vmatrix} + \begin{vmatrix} f_{nn+1} \\ f_{n+1} \end{vmatrix} = \begin{vmatrix} Q_{nn+1} \\ Q_{n+1} \end{vmatrix} \qquad (3)$$

Total equation of motion is obtained by the addition of (3) to (2) . The implementation adopted here has the following advantages a) Memory problems are not likely to occur, b) all unnecessary derivatives avoided c) It is recursive ,and d) Mode shape and dependent parameters can be easily varied.

IV.Application and Discussion of Simulation Results:

Here the described method is applied to a two-link planar flexible arm , with rotary joints and payload. Two mode shapes for each link is considered to represent the structural flexibilities. As noted earlier, mode shapes can be inputed in to the simulation program and the effect of different mode shapes on the dynamic response and the accuracy of modes can be checked. Joints and payload are considered as mass with rotary inertia. These inertial parameters can be set to zero as well. [Fig.3]

System input parameters for simulation ;

mj_1, mj_2 , jj_1 , jj_2 , m_p , j_p

m_1, m_2, l_1 , l_2 , EI_1 , EI_2

$\phi_{11}(x)$, $\phi_{12}(x)$, $\phi_{21}(x)$, $\phi_{22}(x)$, Gravity vector

Normalization procedure

Time independent parameters calculated at the initialization of the program only once per session (if mode shapes are up dated as function of changing boundary conditions, than these parameters need to be reevaluated)

nm11,nm12,nm21,nm22,nw11,nw12,nw21,nw22,nq11,nq12,nq21,nq22,kw11,kw12,kw21,kw22, $\phi_{11}(l_1)$, $\phi_{12}(l_1)$, $\phi_{21}(l_2)$, $\phi_{22}(l_2)$

$\frac{\partial}{\partial x}(\phi_{11})\big|$, $\frac{\partial}{\partial x}(\phi_{12}\,)\big|_{x=l1}$, $\frac{\partial}{\partial x}(\phi_{21}\,)\big|$, $\frac{\partial}{\partial x}(\phi_{22}\,)\big|_{x=l2}$

Free open loop response of the system is simulated for different values of parameters and for different mode shapes . In all results presented here , for the first link Clamped - clamped , for second link Clamped- Free mode shapes used [14]. Clearly the end point boundary condition of neither link is neither clamped nor free (if payload is used). Fig 4 shows the variation of mode shapes from free end to clamped end boundary condition as the end point mass- with rotary inertia - is increased. To verify the model, results of this model is compared with the simulation results of the same system with rigid arms and a two pendulum example. Corresponding parameters between rigid and flexible model are set same. The flexible model is simulated for different values of link flexural rigidity . As flexible model gets more and more rigid , joint variable responses of the system converges to the Rigid model responses as seen from Fig.5 and 6 a - c . Also from simple beam analysis , the Natural frequencies of the system are related to the characteristic values as ;

$$w_i = \beta_i^2 \left(\frac{EI}{\rho A} \right)^{1/2} \qquad (4)$$

So the frequency of oscillations of modal coordiantes should increase as EI_i 's of the links increases, but in two link case the relation ship is not exactly as Eqn.(4). As seen at Fig 7.a,b ,only the the first mode responses of each link is presented due to space limitation , this expectation is confirmed. The behavior of flexible coordinates in the over all motion time scale indicates that there is a significant coupling between joint variables and flexible variables. Although the full nonlinear model is simulated, since the simulation conditions are rather slow motion type and within a small configuration variation, It is expected that linear analysis would be reasonably accurate. At figure 8.b, the slow frequency content essentially resulting from the coupling of link 1 motion .

Conclusion :

From modelling technique point of view, it is shown that Lagrangian - Assumed modes method can be effectively used for multi-link flexible arms. The availability of Symbolic manipulation programs overcomes the algebraic complexity of derivation steps, and allows the researchers to obtain more complicated models in very short time. A systematic method is presented suitable for symbolic manipulation by digital computers.

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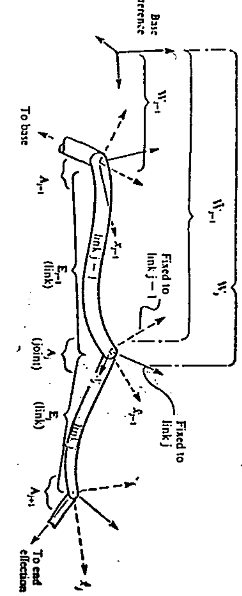


Fig. 1

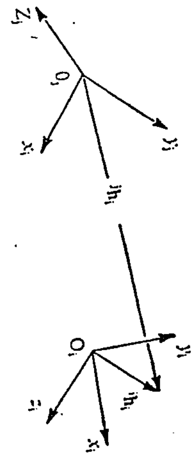


Fig. 2

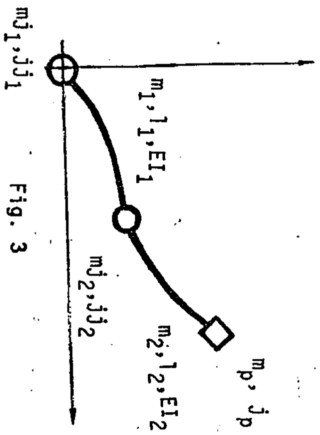


Fig. 3

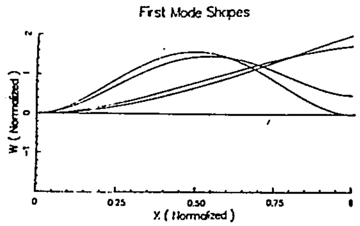


Fig. 4.a

$m_p/m = j_p/m = 0.1$

$= 1.0$

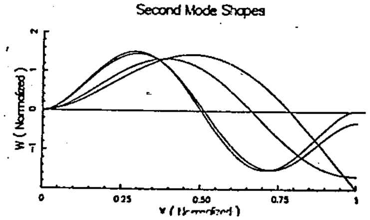


Fig. 4.b

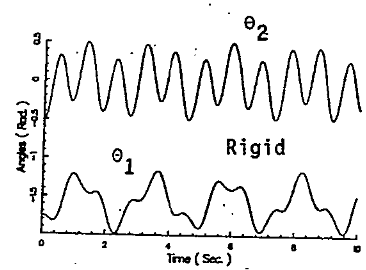


Fig.5

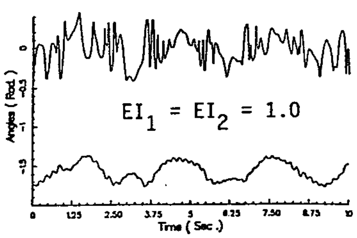


Fig. 6.a

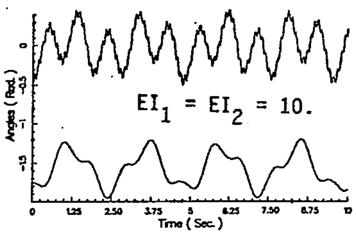


Fig. 6.b

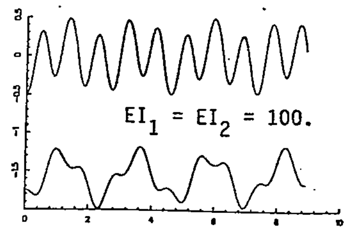
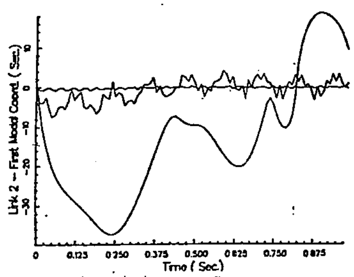


Fig. 6.c



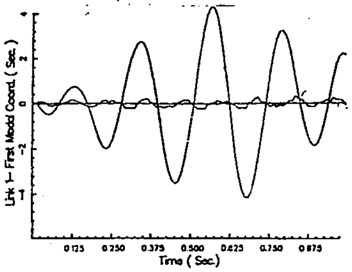


Fig. 7.a Fig. 7.b

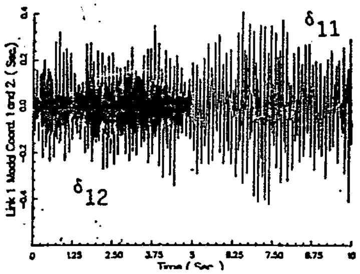


Fig. 8.a

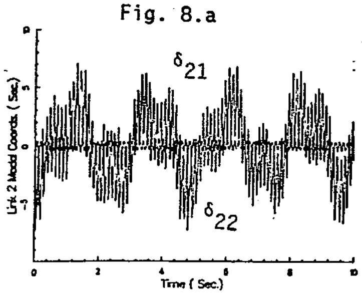


Fig. 8.b