DETERMINATION AND CLASSIFICATION OF AIRPLANE TRANSFER FUNCTIONS

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DETERMINATION AND CLASSIFICATION

OF AIRPLANE TRANSFER FUNCTIONS

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SYMBOLS AND DEFINITIONS

R Amplitude of output produced by unit amplitude oscillation of input

R_a Amplitude of aircraft oscillation produced by unit amplitude oscillation of control surface

R_p Amplitude of aircraft oscillation required to produce unit control surface deflection by autopilot

θ *Angle between airplane longitudinal axis and horizontal, radians*

δ Control deflection, radians, subscripts refer to: a-aileron, e-elevator, r-rudder

E Difference between actual value and reference value of quantity being controlled, radians

ϵ Phase angle in degrees between θ and δ , positive when θ is ahead of δ ; subscripts are: a-aircraft, p-autopilot

ϕ Angle of roll, radians

ψ Angle of yaw, radians

β Angle of side slip, $\beta = \tan^{-1}\left(\frac{\text{side-slip velocity}}{V}\right)$ radians

γ Angle between flight path of center of gravity and horizontal, radians

r Yawing angular velocity $\frac{d\psi}{dt}$, radians per second

p Rolling angular velocity $\frac{d\phi}{dt}$, radians per second

q Pitching angular velocity, $\frac{d\theta}{dt}$

t time, seconds

s Non-dimensional time unit

ω Angular frequency, radians per span or chord-length traveled

ω_c Angular frequency at which $\epsilon_a = \epsilon_p$

Z-axis In plane of symmetry and perpendicular to relative wind

X-axis In plane of symmetry and perpendicular to Z-axis

Y-axis Perpendicular to plane of symmetry

k_X Radius of gyration about X-axis, feet

k_Y Radius of gyration about Y-axis, feet

k_Z Radius of gyration about Z-axis, feet

b Span of aircraft, feet

c Mean aerodynamic chord of aircraft, feet

ρ Air density, slugs per cubic foot

S Reference wing area of aircraft, square feet

V Velocity along flight path, feet per second

q Dynamic pressure, $\frac{\rho V^2}{2}$, pounds per square foot

D Differential operator $\frac{d}{dt}$ or $\frac{d}{ds}$

i Imaginary unit, $i = \sqrt{-1}$

C_n Yawing moment coefficient, $\frac{\text{yawing moment}}{qSb}$

C_l Rolling moment coefficient, $\frac{\text{rolling moment}}{qSb}$

C_Y Side force coefficient, $\frac{\text{side force}}{qS}$

C_L Lift coefficient, $\frac{\text{lift}}{qS}$

C_m Pitching moment coefficient, $\frac{\text{pitching moment}}{qSc}$

C_{n_β}, C_{l_β}, C_{Y_β} indicate $\frac{\partial C_n}{\partial \beta}$ $\frac{\partial C_l}{\partial \beta}$ $\frac{\partial C_Y}{\partial \beta}$

C_{n_r} , C_{l_r} indicate $\dfrac{\partial C_n}{\partial \frac{rb}{2v}}$ $\dfrac{\partial C_l}{\partial \frac{rb}{2v}}$

C_{n_p} , C_{l_p} indicate $\dfrac{\partial C_n}{\partial \frac{pb}{2v}}$ $\dfrac{\partial C_l}{\partial \frac{pb}{2v}}$

C_{n_δ} , C_{l_δ} indicate $\dfrac{\partial C_n}{\partial \delta}$ $\dfrac{\partial C_l}{\partial \delta}$

C_{L_α} Lift curve slope $\dfrac{\partial C_L}{\partial \alpha}$

C_{m_α} Pitching moment curve slope $\dfrac{\partial C_m}{\partial \alpha}$

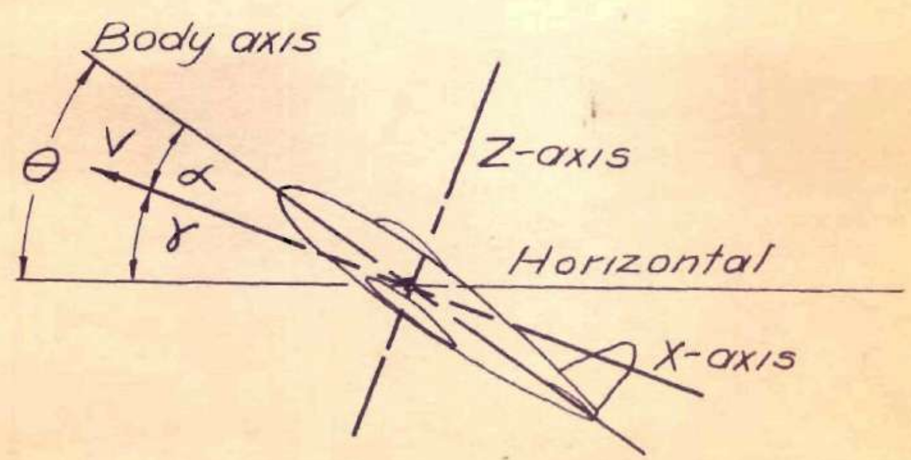
C_{m_q} Damping in pitch $\dfrac{\partial C_m}{\partial \frac{qc}{2v}}$

K G $(i\omega)$ Indicates open loop transfer function of system

α Angle of attack

μ Relative density factor $\dfrac{\text{mass of airplane}}{\rho Sb}$

or $\dfrac{\text{mass of airplane}}{\rho Sc}$



Relation Between θ , α , and γ

DETERMINATION AND CLASSIFICATION OF AIRPLANE TRANSFER FUNCTIONS

SUMMARY

A theoretical investigation has been made of airplane transfer functions. The airplane configuration chosen for analysis was typical of modern high speed aircraft. The stability characteristics of the airplane were expressed in terms of non-dimensional parameters, thus making the results applicable to aircraft of any size.

The stability parameters were substituted in differential equations of motion and the response of the airplane as a function of the driving frequency was obtained. The response as a function of frequency was obtained by use of the operator $D = i\omega$ and the derivation of this method is presented in detail. Calculations were made to determine the response in the following quantities; pitch angle, angle of attack, flight path angle, normal acceleration, roll angle, and yaw angle.

The results are presented in the form of frequency response curves and transfer function curves for the each various quantities. The responses in pitch angle, angle of attack, and flight path angle were calculated for a range of airplane inertia values.

INTRODUCTION

During and since World War II the performance of military aircraft, and potentially of commercial aircraft, has been greatly improved. Most of the improvement, however, has been evident in higher top speeds and longer ranges, rather than in those features which would tend to reduce pilot fatigue and error. The higher top speeds and their accompanying evil, higher rates of fuel consumption, have increased the effects of small errors in navigation and at the same time reduced the time available for navigational work. The longer ranges have also meant longer periods of active flying by the pilot. These features have caused considerable effort to be devoted to the development of devices intended to relieve the pilot of routine duties and allow him to conserve his energies for the more urgent periods of combat or other emergencies.

Practically all of the devices designed to aid the pilot involve some form of automatic control. Unfortunately, the theory of servo-mechanisms is not one with which most aeronautical engineers are well acquainted. This fact has caused the burden of designing such devices to be thrown upon the electronic engineers who have for some years been using automatic control theory in the design of feed-back amplifiers. As might be expected, considerable confusion can arise because of the dissimilar terminology in the two fields and the of aerodynamic experience on the part of the electronic engineers. There seems to exist a need for the dynamic characteristics of airplanes to be expressed in terms of quantities which can be readily handled by the existing servo-analysis methods.

It is the purpose of this paper to present the results of calculations made to determine those airplane characteristics which would be necessary in the rational design of automatic pilots, and to classify those characteristics in such a manner as to make them intelligible to a servo engineer who is unfamiliar with airplane dynamics.

The required airplane characteristics have been calculated from differential equations of motion by the use of operational mathematics. The basic values of the airplane parameters used in the equations are representative of current airplanes; in some cases certain parameters have been varied over wide ranges in order to indicate the effect of such changes and to increase the applicability of the results.

The types of characteristics calculated are as follows: the response in pitch angle, angle of attack, flight path angle and normal acceleration to elevator deflection and the response in roll and yaw to both aileron and rudder deflections. The longitudinal responses were calculated for a range of inertia and static stability values. A total of seventeen airplane responses are presented.

The results are presented in the form of frequency response curves and transfer function curves for each of the above mentioned cases. The curves are classified according to the form of their mathematical expression and also, as well as is possible, according to the type of autopilot response necessary to insure a stable airplane-autopilot combination.

GENERAL DESCRIPTION OF FREQUENCY RESPONSE METHOD

Frequency Response Curves

The frequency response method of autopilot-aircraft analysis involves the use of four (4) quantities as shown by the curves on Figure 1. These four consist of two pairs. One pair of curves describes what a forcing oscillation of the control surface (input) does to the motion of the aircraft (output). The other pair of curves describes how the autopilot moves the control surface (output) in response to the motion of the airplane (input). The two quantities plotted to describe each response are the amplitude ratio between the airplane motion and the control surface motion, R, and the phase angle, ϵ , between the airplane motion and the control surface motion. The relationship of the body motion both in amplitude and phase (time relationship) to the forcing control deflection depends solely upon airplane parameters. The relationship of the enforced control notion to the body motion both in amplitude and phase depends solely upon autopilot parameters. These response curves are usually determined for a range of frequencies from zero to about three times the natural frequency of the airplane. Both sets may be determined either experimentally or by use of equations of motion. Generally the response of the airplane is calculated and the response of the autopilot is measured.

The response of the aircraft is obtained by the calculation of a steady state solution of the equations of motion. The procedure involves the replacement of the differential operator D by a sinusoidal frequency variable '$i\omega$'. This permits the response of the airplane per unit disturbance to be plotted as a function of the periodic disturbance.

The response data for the autopilot are obtained by similar calculations or by oscillating the autopilot at various frequencies and amplitudes and measuring the control or servo motion produced.

The two sets of curves on figure 1 were calculated for a typical airframe and autopilot. They are used to determine stability as follows. The intersection of the ϵ curves is determined. This intersection, or critical frequency, determines the frequency at which a neutrally stable oscillation may exist. The values of the R curves at the critical frequency, ω_c determine if the oscillation can exist. Thus, if, at the critical frequency, the R values are equal, i. e., the R curves intersect at ω_c, then there will be a neutrally stable oscillation. If, however, R_p is greater than R_a as in this example the oscillation will be damped, and conversely if R_a is greater than R_p instability will be indicated. The existence of a stable or unstable oscillation is the only definite conclusion that can be drawn directly from the two sets of curves. By definition, see below, the response curves are valid only for the condition of zero damping of the oscillation and, therefore, if the R curves do not intersect at the same frequency as the ϵ curves, the intersection of the ϵ curves no longer determines the frequency of oscillation. This can be stated in another way - if the airplane-autopilot combination is either stable or unstable, the basic response curves will indicate the fact, but will not define the frequency or the damping of the oscillation.

The frequency response data mentioned above may be considered in another manner which increases the applicability of the data, and is more usual in servo-mechanism analysis. The total response may be considered as a vector quantity of amplitude R and phase angle ϵ . A plot of the locus of the head of this vector as a function of the forcing frequency, ω , is known as a transfer function curve. There is a significant difference in the form of the response curves and in the transfer function curve. The response curves for both airplane and autopilot must be expressed in terms of the same ratio. That is, they must be (using the pitch response as an example) curves of θ/δ and E/δ vs. ω or they must be curves of δ/θ and δ/E vs. ω . (The latter form requires a slightly different interpretation from that used on the θ/δ and E/δ curves) The transfer function curve, however, is always the ratio of output to input and thus the transfer function for the airplane will be in terms of θ/δ while that for the autopilot will be given in terms of δ/E.

The method of predicting the stability of a system through use of transfer function curves is outlined below.

1. The transfer function of the entire system is obtained (for single loop systems) by taking the product of the transfer function of the component parts.

Thus $\quad (\theta/E)_{\text{complete system}} = (\theta/\delta)_{\text{airplane}} \times (\delta/E)_{\text{autopilot}}$

2. This system transfer function is plotted and the Nyquist criterion is applied. In its simplest form this criterion requires that the system transfer function curve lie between the point 1.0 and

the origin where it crosses the 0° ray. This is reversed from the criterion as usually used by electrical engineers, the reversal being due to the fact that aeronautical convention requires that a plus sign be assigned that control deflection used to correct a plus deviation.

METHOD OF ANALYSIS

Calculation of the Airplane Transfer Function Curves

If an airplane is considered as a rigid body, it has six possible degrees of freedom and a set of six differential equations of motion is required to represent the motion. Because of the symmetry of the airplane in the X-Z plane there will be a negligible amount of coupling between the lateral (rolling, yawing, and sideslipping) motions and the longitudinal (pitching, vertical, and forward) motions, provided the oscillations are small. The assumption of zero coupling between the above mentioned motions allows the six equations to be divided into two sets (one lateral, one longitudinal) of three equations each. The validity of this assumption is discussed at length by Jones[1]. In some special cases where further simplifying assumptions may be made it is possible to eliminate one or two of the equations and thus reduce the set of three equations to two or one.

The equations of motion, as presented in this report are completely non-simensional. Not only Are they written in terms of dimensionless coefficients but the derivatives themselves are taken with respect to a

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[1] Jones, B. Melville - "Dynamics of the Airplane" - Aerodynamic Theory W.F. Durand, Editor. Julius Springer, Berlin, Div. N, Vol.V pp123

non-dimensional time unit. The definition of this time unit and its derivatives are given below.

Dimensional time, t, is equated to a non-dimensional unit, s, multiplied by the ratio of an airplane dimension to the forward velocity: that is,

$$t = \frac{sb}{v} \quad \text{for lateral equations}$$

or

$$t = \frac{sc}{v} \quad \text{for longitudinal equations.}$$

From these definitions the following relationships for the derivatives are obtained

$$D_t = \frac{v}{b} D_s \quad \text{or} \quad D_t^2 = \frac{v^2}{b^2} D_s^2 \text{ for lateral equations} \qquad (1)$$

$$D_t = \frac{v}{c} D_s \qquad D_t^2 = \frac{v^2}{c^2} D_s^2 \text{ for longitudinal equations} \qquad (2)$$

The use of this non-dimensional time unit causes the solutions of the equations to be in terms of span or chord lengths traveled. Thus the frequency of the forcing function is always in units of radians per span or chord length traveled.

Longitudinal equations

The three longitudinal equations mentioned above describe a motion that is usually the sum of two well defined oscillatory motions. One of these, the phugoid, is a lightly damped, long period motion in which the angle of attack remains nearly constant, and the forward speed and the horizontal inclination vary. The other motion is a short period, heavily damped motion in which the forward speed remains constant and θ and α vary.

The phugoid is easily controlled, or rather eliminated, by proper manipulation of the controls. Its period is so long (30 seconds - 100 seconds) that a human pilot 'flies it out' without conscious effort and therefore it is not involved in the flying qualities requirements as given by Gilruth[2].

On the other hand, the period of the shorter oscillation is of the order of one second and its damping is quite important in determining the flying qualities of the airplane. It thus seems reasonable to assume that any autopilot which does not cause an unstable short period oscillation will have a sufficiently rapid response to completely mask any phugoid characteristics which might be inherent in the uncontrolled airplane. On this basis it is only the response characteristics of the airplane in θ and α that need be investigated to obtain information for use in autopilot design. The elimination of the forward speed variable, , reduces the number of longitudinal equations to two and greatly simplifies the computations necessary in obtaining response curves. The longitudinal equations of motion, as used in this paper, are given below.

$$\left(2\mu\left(\frac{k_Y}{c}\right)^2 D^2 - \frac{1}{2} C_{m_q} D\right)\theta \quad - C_{m_\alpha}\alpha = C_{m_\delta}\delta \tag{3}$$

$$2\mu D\theta \quad -\left(2\mu D + C_{L_\alpha}\right)\alpha \quad = C_{L_\delta}\delta \tag{4}$$

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[2] Gilruth, Robest R. "Requirements for Satisfactory Flying Qualities of Airplanes" U.S. National Advisory Committee for Aeronautics Wartime Report L-276

Lateral Equations

The lateral equations of motion used to calculate the response presented herein are taken from a paper by Sternfield[3]. The equations as presented have been somewhat simplified by the assumption that the principal axes of inertia are co-incident with the stability axes. This condition is not often met exactly on actual airplanes, but it does represent a rather median condition. The lateral equations are given below.

$$\left(2\mu\left(\frac{k_x}{b}\right)^2 D^2 - \frac{1}{2} C_{l_p} D\right)\phi - \frac{1}{2} C_{l_r} D\psi - C_{l_\beta} \beta = C_{l_\delta} \delta \tag{5}$$

$$-\frac{1}{2} C_{n_p} D\phi + \left(2\mu\left(\frac{k_z}{b}\right)^2 D^2 - \frac{1}{2} C_{n_r} D\right)\psi - C_{n_\beta} \beta = C_{n_\delta} \delta \tag{6}$$

$$-C_L \phi + (2\mu D - C_L \tan \gamma_0)\psi + (2\mu D - C_{Y_\beta} \beta = C_{Y_\delta} \delta \tag{7}$$

Solution of equations for sinusoidal forcing function

The term transfer function, as used herein, is taken to mean that function of frequency which describes the above mentioned airplane response curves. The transfer function curves could be obtained by computing several discrete solutions to the equations of motion using sinusoidal forcing control motions of several different frequencies. It is much more convenient, however, to use a method that gives the response directly as a function of the forcing frequency. This may be done by setting up a solution for the desired response and substituting the relation $D = i\omega$. For example

[3] Sternfield, Leonard "Effect of Product of Inertia on Lateral Stability" U.S. National Advisory Committee for Aeronautics Technical Note 1193, Washington D.C.

if $$f(D)\theta = K\delta$$

then $$\frac{\theta}{\delta} = \frac{K}{f(D)} \tag{8}$$

and substituting $D = i\omega$ gives

$$\frac{\theta}{\delta} = \frac{K}{f(i\omega)} \tag{9}$$

The presence of both odd and even powers in $f(D)$ will cause the $f(i\omega)$ to be a complex number and the expression may be written

$$\frac{\theta}{\delta} = \frac{K}{a+ib} = \frac{K}{Re^{i\epsilon}} \quad \frac{Ke^{-i\epsilon}}{R}$$

where $$R = \sqrt{a^2+b^2} \quad \text{and} \quad \epsilon = \tan^{-1}\frac{b}{a} \tag{10}$$

The above expression describes θ/δ as a complex quantity of amplitude $\frac{K}{R}$ and phase ($-\epsilon$), both of which are functions of frequency.

The setting of D equal to $i\omega$ is a device often used, but seldom explained. The following justification for this substitution has been amplified somewhat from that given by Brown and Campbell[4].

Consider again a system having the equation of motion

$$f(D)\,\theta = K\delta$$

The ratio of output to input may be written

$$\frac{\theta}{\delta} = \frac{K}{f(D)} \tag{8}$$

and when $$\delta = \delta_0 \sin \omega t$$

$$\frac{\theta}{\delta} = \frac{K \sin\omega t}{f(D)} \quad . \tag{11}$$

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[4] Brown, Gordon S. and Campbell, Donald P.,"Principles of Servo-mechanisms", John Wiley and Sons New York Chapter IV

If the Laplace transform, as defined by Gardner and Barnes[5], is applied to the equation it becomes

$$\left(\frac{\theta}{\delta}\right)_{(s)} = \frac{K}{f(s)\,(s^2+\omega^2)}, \tag{12}$$

and the characteristic equation, $f(s)\ (s^2+\omega^2) = 0$, may be factored into $(s-s_1)\ (s-s_2) \ldots\ldots\ldots (s-i\omega)\ (s+i\omega) = 0$. When the inverse Laplace transform is applied, the general form of the solution will be

$$\left(\frac{\theta}{\delta}\right)_{(t)} = K_1 e^{s_1 t} + K_2 e^{s_2 t} \ldots\ldots K_{i\omega} e^{i\omega t} + K_{-i\omega} e^{-i\omega t} \tag{13}$$

For a consideration of the steady state behavior those terms having real exponents may be discarded leaving

$$\left(\frac{\theta}{\delta}\right)_{ss(t)} = K_{i\omega} e^{i\omega t} + K_{-i\omega} e^{-i\omega t} \tag{14}$$

If the expression for $\left(\frac{\theta}{\delta}\right)_{(s)}$ is broken up into partial fractions the value of the coefficients will become

$$K_{i\omega} = \frac{K}{f(i\omega)2i} \qquad \text{and} \qquad K_{-i\omega} = -\frac{K}{f(-i\omega)2i} \tag{15}$$

The steady state solution now becomes

$$\left(\frac{\theta}{\delta}\right)_{ss(t)} = \frac{K e^{i\omega t}}{f(i\omega)2i} - \frac{K e^{-i\omega t}}{f(-i\omega)2i}$$

which reduces to

$$\left(\frac{\theta}{\delta}\right)_{ss(t)} = \frac{\dfrac{K e^{i\omega t}}{f(i\omega)} - \dfrac{K e^{-i\omega t}}{f(-i\omega)}}{2i}\,. \tag{16}$$

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[5] Gardner, Murray F. and Barnes, John L., "Transients in Linear Systems" John Wiley and Sons New York 1942. Chapter V

The terms may be written as

$$\frac{1}{f(i\omega)} = \frac{1}{a+ib} = \frac{1}{\sqrt{a^2+b^2}}\, e^{-i\epsilon} \tag{17}$$

and

$$\frac{1}{f(-i\omega)} = \frac{1}{a-ib} = \frac{1}{\sqrt{a^2+b^2}}\, e^{i\epsilon} \tag{17a}$$

where $\epsilon = \tan^{-1}\frac{b}{a}$ and a and b are functions of ω.

This gives for $\left(\frac{\theta}{\delta}\right)_{ss(t)}$

$$\left(\frac{\theta}{\delta}\right)_{ss(t)} = \frac{K}{\sqrt{a^2+b^2}} \times \frac{e^{i(\omega t-\epsilon)} - e^{-i(\omega t-\epsilon)}}{2i} \tag{18}$$

which by use of Euler's formula may be changed to

$$\left(\frac{\theta}{\delta}\right)_{ss(t)} = \frac{K}{\sqrt{a^2+b^2}} \times \sin(\omega t-\epsilon) \tag{19}$$

The maxium value of the above expression will occur when $\sin(\omega t-\epsilon) = 1$. This will, of course, occur at a phase angle ϵ away from the time at which $\sin\omega t = 1$.

Thus the expression for $\left(\frac{\theta}{\delta}\right)_{max}$ in the steady state, and as a function of frequency, may be written as

$$\left(\frac{\theta}{\delta}\right) = \frac{K}{\sqrt{a^2+b^2}} \quad \text{at a phase angle } (-\epsilon)$$

or

$$\left(\frac{\theta}{\delta}\right) = \frac{K}{R} \angle{-\epsilon} \quad \text{, where both R and } \epsilon \text{ are functions of} \tag{20}$$

frequency.

It will be noticed that this is the same result as was obtained by substituting $D = i\omega$ in the response equation (8).

Estimation of stability derivatives

The airplane for which the aerodynamic derivatives have been estimated is described by the following dimensions.

Wing aspect ratio	6	μ	105 - - - longitudinal
Taper ratio	0.5	μ	17.5 - - lateral
Horizontal tail area	0.20S	$\frac{k_x}{b}$	0.17
Vertical tail area	0.10S	$\frac{k_Y}{c}$	1.0, 1.414, 0.50
Tail aspect ratio	3	$\frac{k_Z}{b}$	.20
Tail length	2c or $\frac{b}{3}$		

The values of the derivatives were estimated by use of methods now current in airplane stability analysis. These methods are outlined and extensive bibliographies on the subject are given by Donlan,[6] and Campbell[7]. The values of the derivatives used in this paper are given in table I.

Estimation of autopilot characteristics.

The autopilot curves shown on figure 1 describe the response characteristics typical of current electric-servo autopilots. The autopilot is assumed to contain both rate and displacement gyroscopes and to be equipped with synchro pick-offs and feedback links which feed into a phase sensitive amplifier. The output of the amplifier

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[6] Donlan, Charles J. "Factors affecting Longitudinal Stability and Control" NACA-University Conference Collection U.S, National Advisory Committee for Aeronautics. 1949 pp 187-202

[7] Campbell, John P. "Factors Affecting Lateral Stability" NACA University Conference Collection U.S. National Advisory Committee for Aeronautics, 1947 pp 203-229

A system of this sort may be represented by the following equation.

$$\delta/_{E} = \frac{K_4 \cdot K_5 \cdot (K_1 + K_2 D)}{D^2 + K_3 D + K_6}$$

A block diagram of the system is shown in figure 2.

Methods of Classification of Transfer Functions

Low frequency characteristics

It quite often happens that an airplane which has been carefully trimmed (i. e., put into equilibrium as regards forces and moments) will subsequently develop out of trim moments. Such moments may be caused by the using of fuel, the dropping of bombs, the accumulation of ice or the shifting of passenger load. These moments, unless cancelled out, will cause the airplane to deviate from the desired course.

The tendency of any airplane-autopilot system to maintain zero error (that is stay exactly on course) under a steady disturbing load may be determined qualitatively from the shape of the system transfer function curve. This is done by considering the physical menaing of the behavior of the curve at low frequencies. At or near zero frequency the R value of the transfer function may be likened to a sensitivity constant. Thus if at $\omega = 0$ the value of R goes to infinity the slightest deviation from the input reference will cause an infinite response from the airplane in a direction such as to reduce the deviation.

Mathematically to have zero error required that θ out $= \theta$ in, or that $\frac{\theta \text{ out}}{\theta \text{ in}} = 1$. It is shown by Brown and Campbell[8] that $\frac{\theta \text{ out}}{\theta \text{ in}} =$

$$= \frac{K \cdot G(i\omega)}{1 + K \cdot G(i\omega)}$$

[8] Brown and Campbell Op. cit. Chapter 6, section 8.

and thus that for a steady input angle

$$\theta_o/\theta_i = 1 \quad \text{if} \lim_{\omega \to 0} K \cdot G(i\omega) = \infty$$

and for a steady input velocity

$$\theta_o/\theta_i = 1 \quad \text{if} \lim_{\omega \to 0} K \cdot G(i\omega) i\omega = \infty$$

Obviously the case of the airplane with a steady out-of-trim load is not covered by the criterion for a steady input angle. It can be seen, however, that the criterion for a steady input velocity will cover the case of a system subjected to a steady extended load. If the system, be it airplane or not, is subjected to a steady input velocity there will be created on the system a steady load due to the output velocity and caused by the viscous damping always present in actual systems. If the system can overcome this load and maintain zero position error it can also evercome any external steady load and still maintain zero error.

This sort of an analysis then leads to the possibility of classifying airplane transfer functions by their behavior and that of their first derivatives as ω goes to zero. It will usually not be necessary to consider the system (airplane and autopilot) transfer function because a proportional autopilot (one in which there is a definite ratio between the error signal and the central deflection under static conditions) will never have an infinite response at zero frequency and thus all infinities in the system transfer function must come from the airplane.

Maximum allowable autopilot gain

The gain in the autopilot determines the amount of corrective control deflection per unit error that will be called for by the autopilot. The higher the gain the stiffer the system will be, that is with a high gain a small deviation from course will cause a large corrective control moment to be applied and this may be likened to a stiff spring which creates a large restoring force when subjected to a small deflection.

In systems which are inherently zero error systems at zero frequency such high stiffness is unnecessary but in other systems a high gain is sometimes quite desirable in order to prevent out-of-trim moments from causing objectionably large static errors.

Unfortunately the use of high gain in the autopilot is quite likely to cause the system to become unstable. This is shown by the curves on figure 1. If the gain in the autopilot were increased δ/E would increase and E/δ would decrease, thus decreasing R_p and lessening the stability of the system. The limiting value of R_p at ω_c is that value which will make the fraction $\frac{R_p}{R_\alpha} < 1$. It can be noticed that this limiting value could be changed considerably if ω_c were changed. It is evident that there is an infinity of solutions to the problem of designing a stable airplane-autopilot system and it is impossible to make any definite classification of airplane transfer functions unless the analysis is made using data for a specific autopilot and airplane.

RESULTS AND DISCUSSION

General Characteristics of Airplane Transfer Function

The most evident feature of all the longitudinal response curves, shown in figures 3-7 is the fact that they all bear a resemblance to the response curves for a one degree of freedom system consisting of a spring, a mass and a dashpot. The second resonant peak characteristic of two degrees of freedom appears in a degenerate form in the θ and γ responses as an infinite point at zero frequency, but does not show up at all in the α and $D\gamma$ (or normal acceleration) responses. This indicates the possibility that a good approximation to the α and $D\gamma$ curves might be obtained by using a single equation of motion.

Another general observation which may be made is that the θ and γ responses exhibit rather low resonant peaks, thus indicating that in these two variables the conventional airplane is nearly critically damped. On the other hand the responses in α and $D\gamma$ indicate rather low damping. It may be concluded from this that autopilots sensitive to θ or γ would not be as seriously affected by shifts away from the design phase characteristics as those autopilots sensitive to α or $D\gamma$ for the following reasons. A change in the phase responas of the autopilot would change ω_c; if the resonant peak is sharp a small change in ω_c can cause a large change in the value of R_a at ω_c and thus completely alter the stability picture.

Those responses computed with variations in pitching inertia and static stability (on figures 3-6) show about what would be expected from a knowledge of simple forced vebrations; namely, an increase in inertia lowers the frequency at which resonance occurs and an increase in static stability increases the frequency of resonance.

The lateral curves (see figures 8-11) in general do not exhibit as sharp peaks in the amplitude responses as do the longitudinal. In fact the roll amplitude response curves might be approximated except, at zero frequency, by a one degree of freedom system having only a mass and viscous damping, even though they actually represent motion in three degrees of freedom. The closeness of this approximation is illustrated by the dotted curve on figure 8. The amplitude response curves in yaw are also somewhat similar in shape to the approximate roll curves but the yaw phase angle curves are quite differnnt from any others. Experience gained in computing the curves indicates that by only slight changes in the airplane parameters the phase angel curve may be made to reverse its downward trend at about 180° and shoot up sharply to finally approach 90°. It may be noticed that the fact that the phase angles drop sharply down to lagging or negative values will tend to make ω_c occur at a relatively low frequency. As zero frequency is approached the R_a curve rises quite rapidly and thus tends to reduce the allowable value of autopilot gain.

Low frequency characteristics

As mentioned above the chief point of interest in the low frequency region is the behavior of either the amplitude curve or of the complete transfer function as zero frequency is approached. The mathematical form of each transfer function and the value of the derivative as ω goes to zero are presented in table II. These values of the derivative limits indicate that, for any of the types of motion studied here, any airplane-autopilot combination will have some steady state error under steady load unless the autopilot is equipped with an integrator.

Permissable autopilot gain

The fact that an airplane-autopilot system will inherently be subject to steady state errors makes the use of high autopilot gain more desirable. It has previously been shown that, for the airplane response on figurel, the value of R_p will have to be 1.0 or more if ω_c is at 0.08. This when put in terms of $\delta/_E$, means that the control deflection produced per degree of airplane deviation must be less than one degree. If the same autopilot is applied to the roll response data on figure 8 it can be seen that $R_p$would have to be greater than 3.0. This would mean that to insure stability the autopilot could produce only 0.33° aileron deflection per degree of airplane deviation.

The above amounts of control deflection are relatively small compared to those which would be required to trim an airplane subjected to battle damage or out-of-trim moments caused by asymmetrical use of fuel. Therefore if an automatically stabilized airplane is to have satisfactory characteristics under steady out-of-trim conditions it will probably be necessary to use an autopilot which incorporates powerful phase-compensating devices. Such devices would make the critical frequency higher and thus allow the use of higher autopilot gain.

CONCLUDING REMARKS

The problem of attaining all desirable characteristics in an airplane-autopilot system is not one which can be solved by a consideration of the airplane characteristics alone. However this analysis indicates that by a study of typical airplane transfer functions it is possible to obtain a general view of the problem and to propose plausible solutions. The final answer, though, must in all cases depend upon a detailed study of the particular airplane and autopilot involved.

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APPENDIX I

Tables

TABLE I

STABILITY DERIVATIVES

Longitudinal Derivatives

Response	C_{m_α}	C_{L_α}	C_{m_q}	μ	$\frac{k_Y}{c}$	C_{m_δ}	C_{L_δ}
θ/δ	-.464	4.64	-6	105	1.0	-1.032	-.516
θ/δ	-.464	4.64	-6	105	1.414	-1.037	-.516
α/δ	-.464	4.64	-6	105	1.0	-1.032	-.516
α/δ	-.464	4.64	-6	105	1.414	-1.032	-.516
α/δ	-.464	4.64	-6	105	.5	-1.032	-.516
θ/δ	-.464	4.64	-6	105	.5	-1.032	-.516
θ/δ	-.058	4.64	-6	105	1.0	-1.032	-.516
γ/δ	-.464	4.64	-6	105	1.0	-1.032	-.516
γ/δ	-.464	4.64	-6	105	1.414	-1.032	-.516
γ/δ	-.464	4.64	-6	105	.5	-1.032	-.516
$D\gamma/\delta$	-.464	4.64	-6	105	1.0	-1.032	-.516
θ/δ	-.232	4.64	-6	105	1.0	-1.032	-.516

TABLE I - Cont.

Lateral Derivatives

	C_{n_β}	C_{l_β}	C_{Y_β}	C_{l_p}	C_{n_p}	C_{l_r}	C_{n_r}	μ	$\frac{k_Z}{b}$	$\frac{k_Y}{b}$	$C_{l_{\delta a}}$	$C_{n_{\delta r}}$	$C_{Y_{\delta r}}$	C_L
ϕ/δ	.08	-.025	-.4745	-.45	-.0267	.116	-.071	17.5	.20	.17	-.12	0	0	.5
ψ/δ	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0	0	↓
ϕ/δ	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0	0	↓
ϕ/δ	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	.046	-.137	↓
ψ/δ	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	.046	-.137	↓

TABLE II

FORMS OF AIRPLANE TRANSFER FUNCTIONS

Response	Transfer Function	Limit of $i\omega KG(i\omega)$ as $\omega \to 0$
θ/δ	$\dfrac{a + ib\omega}{c\omega^2 + i(d\omega^3 - e\omega)}$	$-\dfrac{a}{e}$
α/δ	$\dfrac{a\omega^2 + ib\omega}{c\omega^2 + i(d\omega^2 - e\omega)}$	0
γ/δ	$\dfrac{(a\omega^2 + b) + ic\omega}{d\omega^2 + i(e\omega^3 - f\omega)}$	$-\dfrac{b}{f}$
$D\gamma/\delta$	$\dfrac{(a\omega^2 + b) + ic\omega}{(d\omega^2 - e) - if\omega}$	$-\dfrac{b}{f}$
ϕ/δ_a one-degree-of freedom	$\dfrac{-a}{-b\omega^2 + ic\omega}$	$-\dfrac{a}{c}$
ϕ/δ_a	$\dfrac{a\omega^2 + i(b\omega^3 - c\omega)}{(d\omega^4 - e\omega^2) + i(f\omega^5 - g\omega^3 + h\omega)}$	0
ψ/δ_a	$\dfrac{(a\omega^2 - b) + i(c\omega^3 + d\omega)}{(e\omega^4 + f\omega^2) + i(g\omega^5 - h\omega^3 - k\omega)}$	$\dfrac{b}{k}$
ϕ/δ_{a+r}	$\dfrac{a\omega^2 + i(b\omega^3 - c\omega)}{(d\omega^4 - e\omega^2) + (f\omega^5 - g\omega^3 + h\omega)}$	0
ψ/δ_{a+r}	$\dfrac{-(a\omega^2 + b)\ \ i(-c\omega^3 + d\omega)}{(e\omega^4 - f\omega^2) + i(g\omega^5 - h\omega^3 - k\omega)}$	$\dfrac{b}{k}$

APPENDIX II

Figures

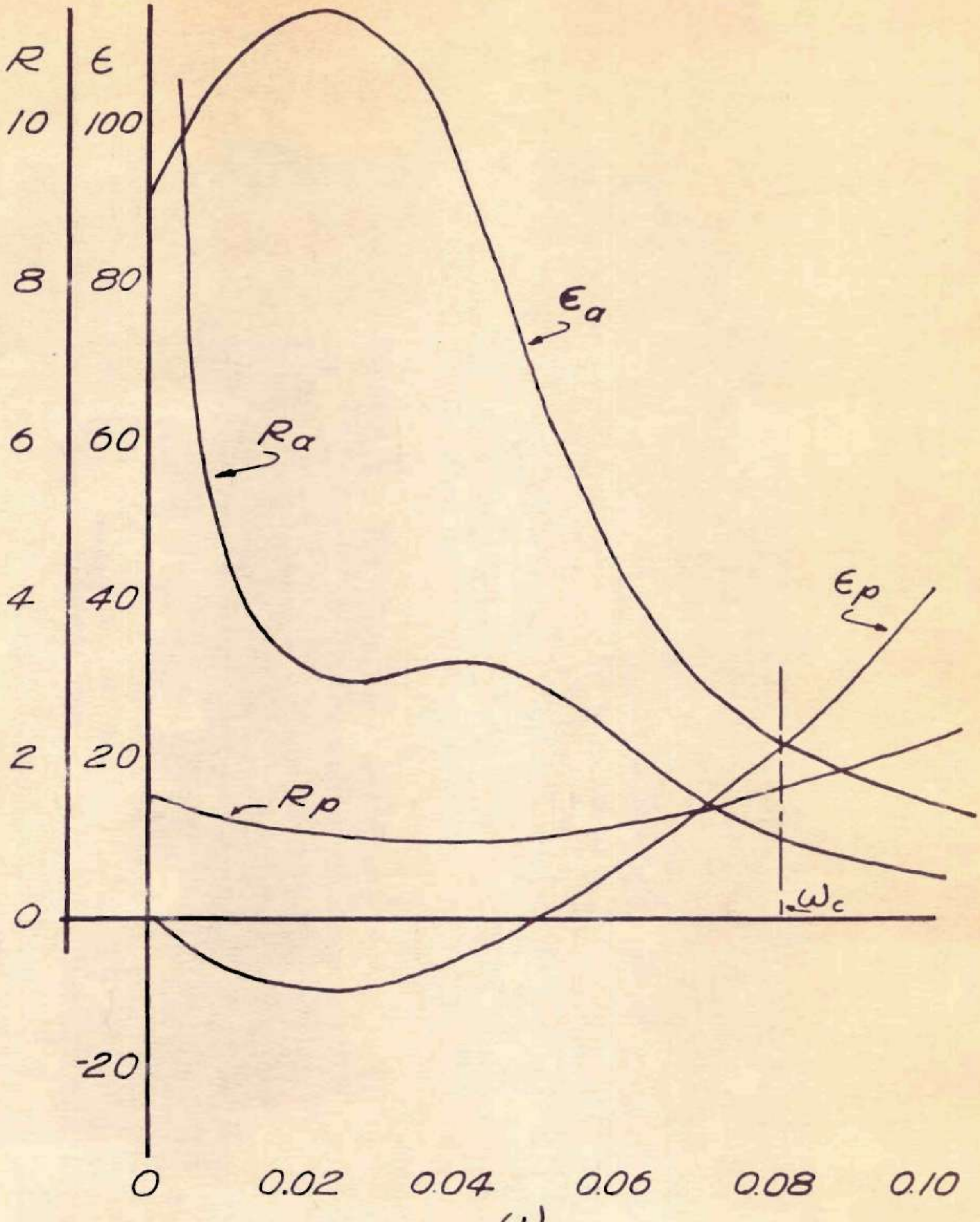

Figure 1. Response curves for autopilot and airplane in pitch. Curves indicate stable system.

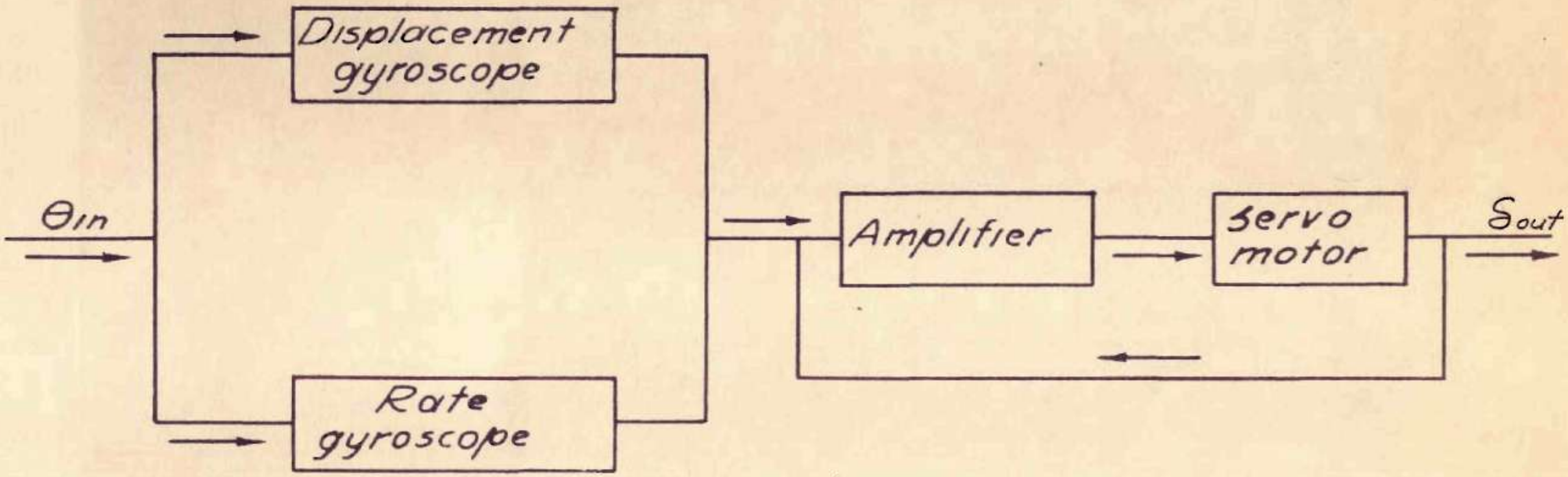

Figure 2. Block diagram of sample autopilot

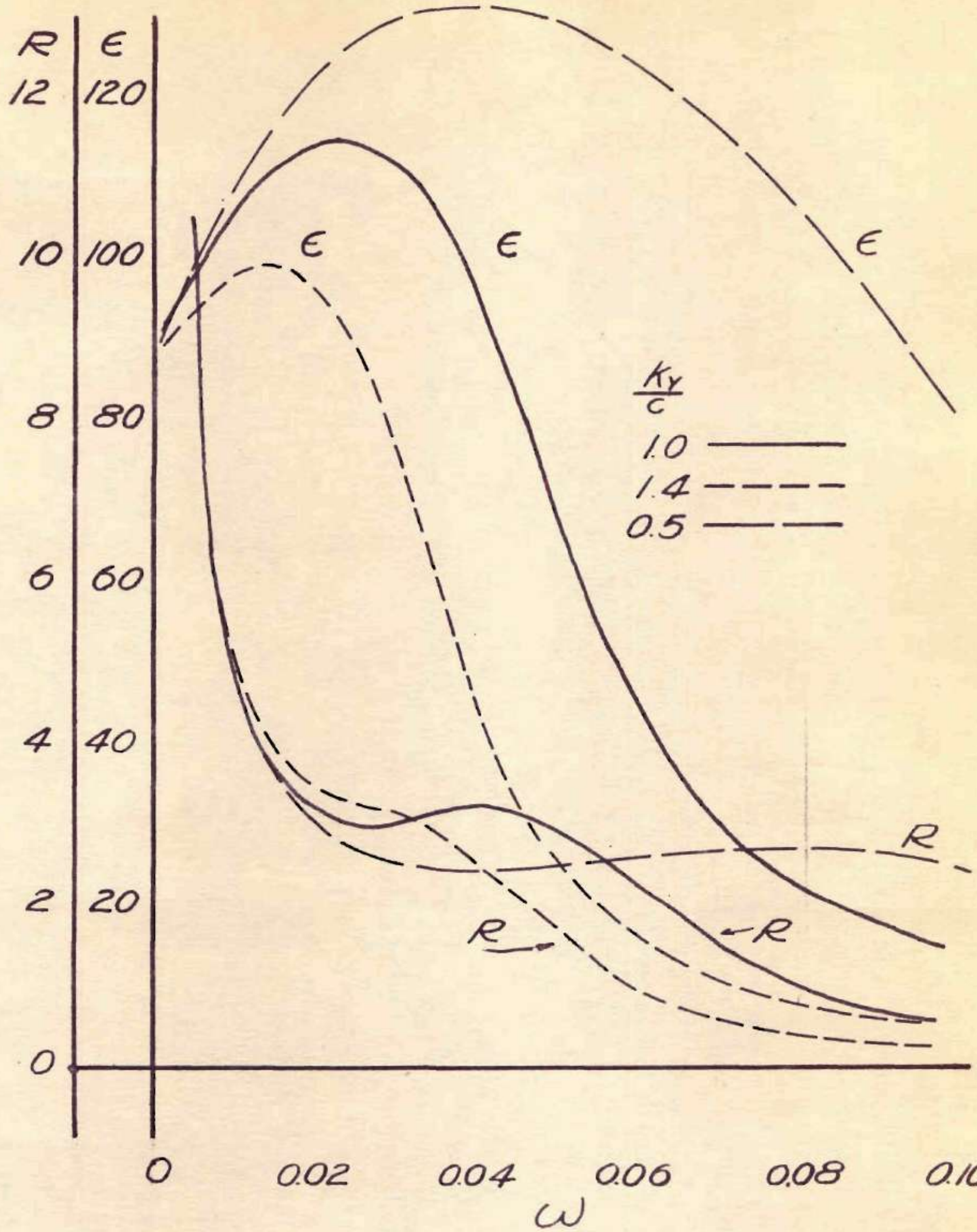

Figure 3. Effect of inertia on response in pitch.

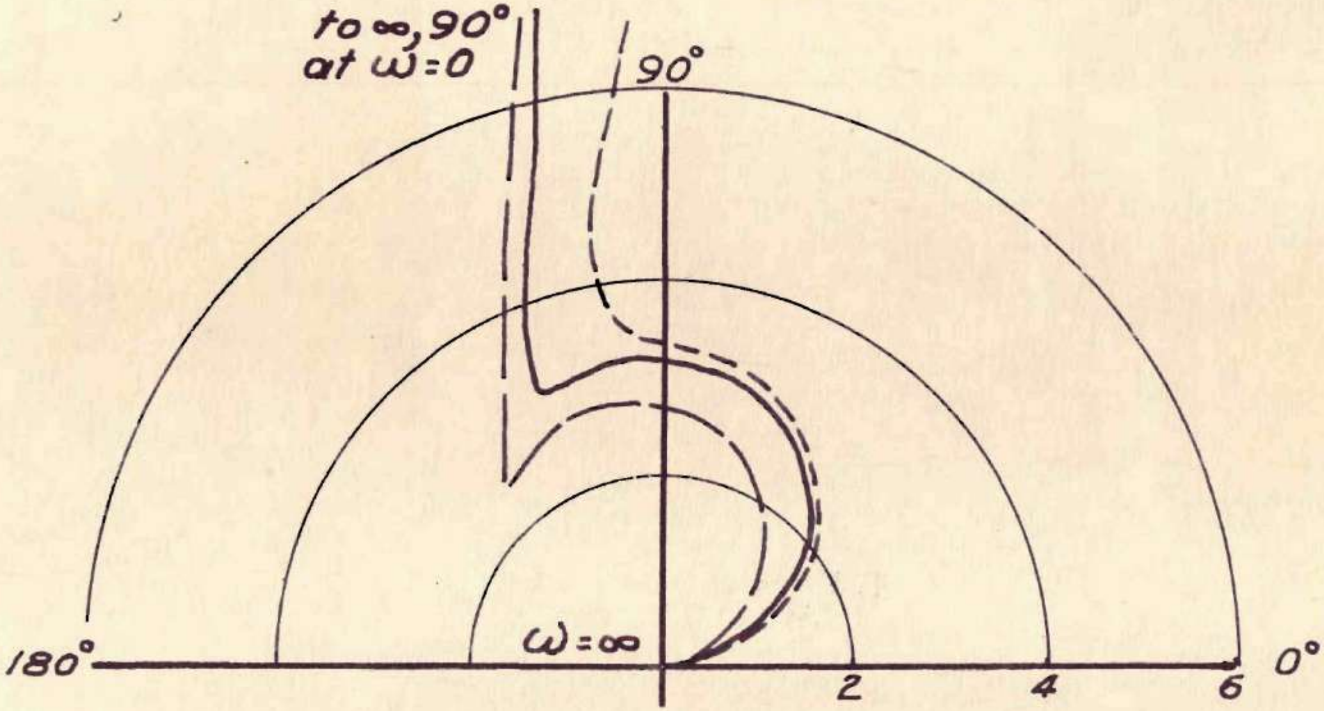

Figure 3-cont.

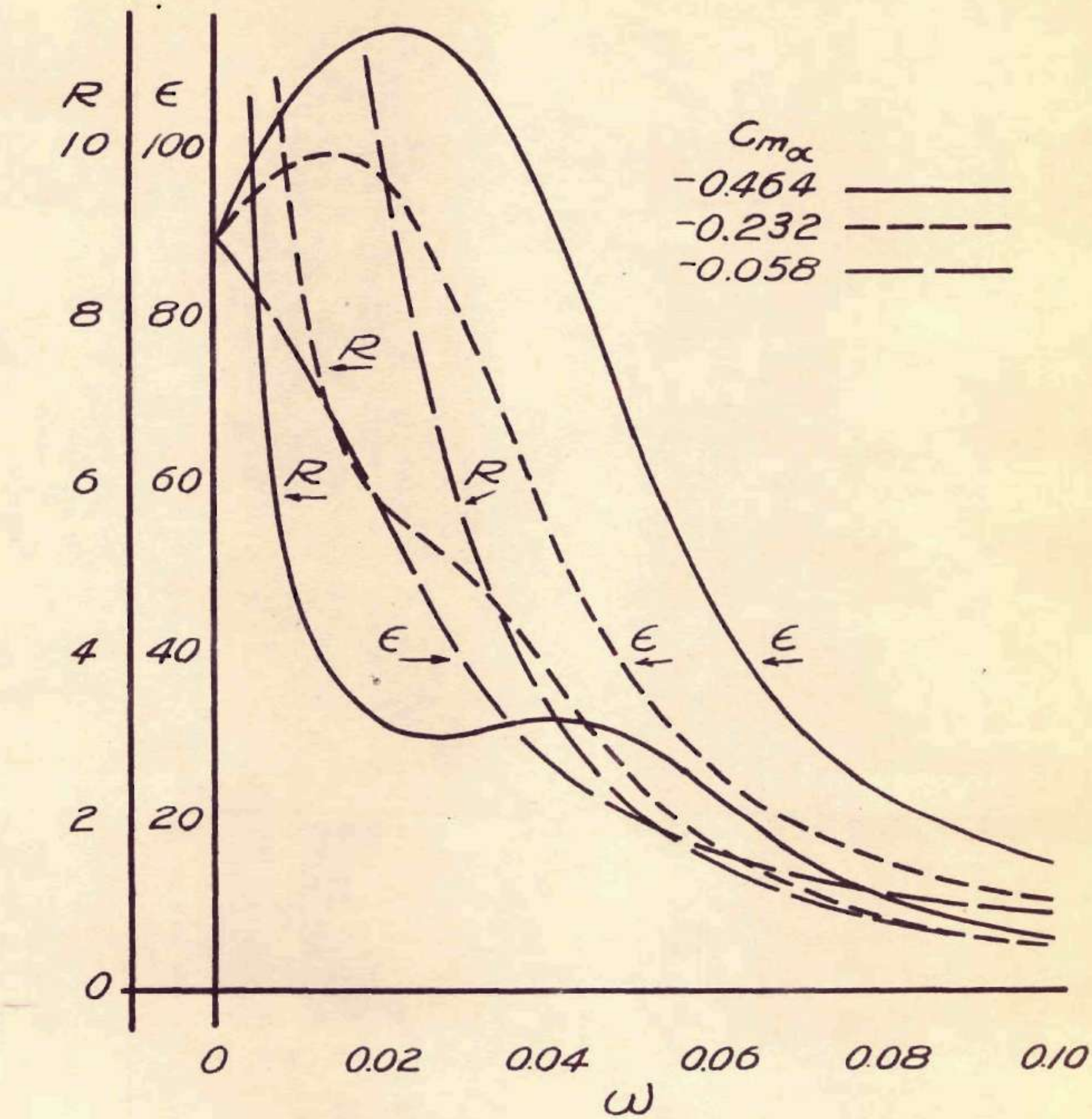

Figure 4 Effect of static stability on response in pitch

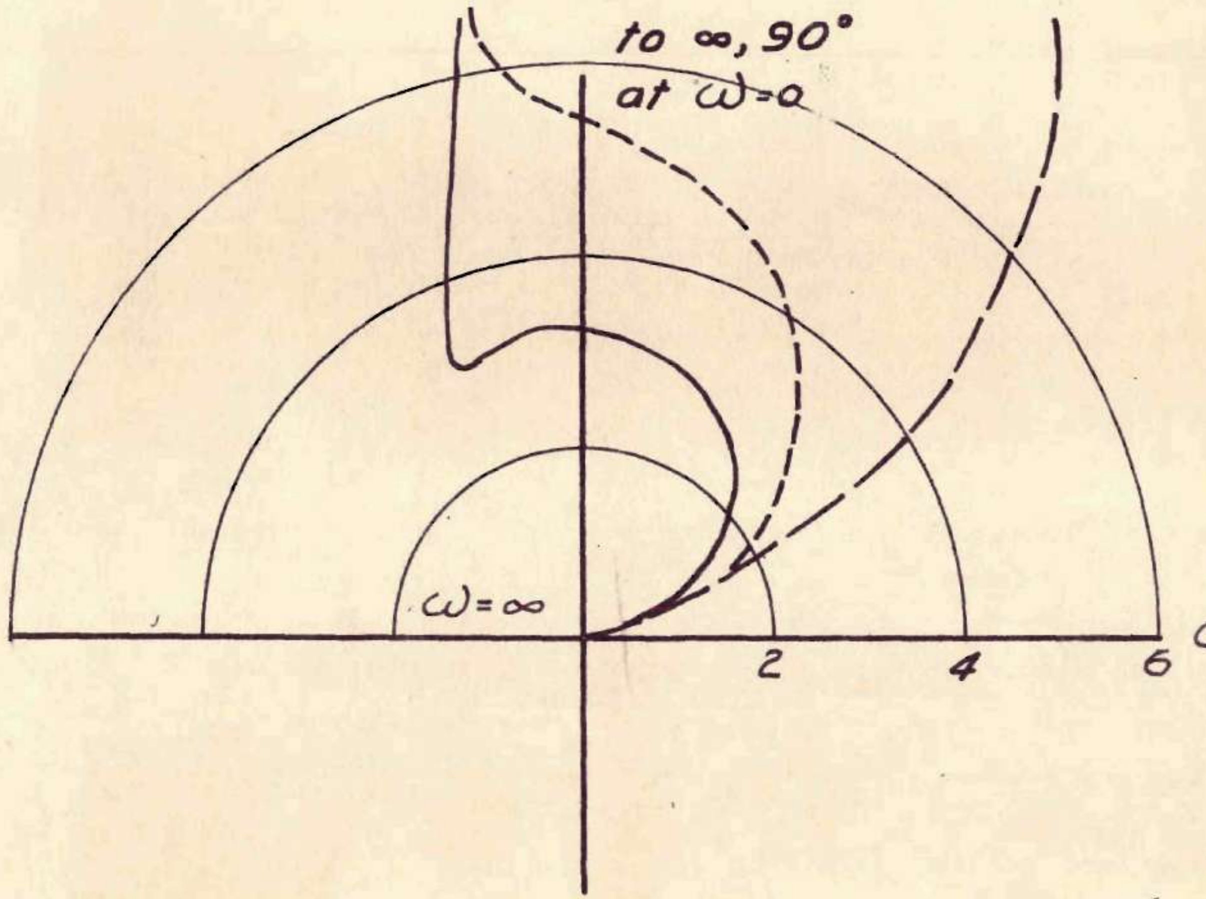

Figure 4 cont.

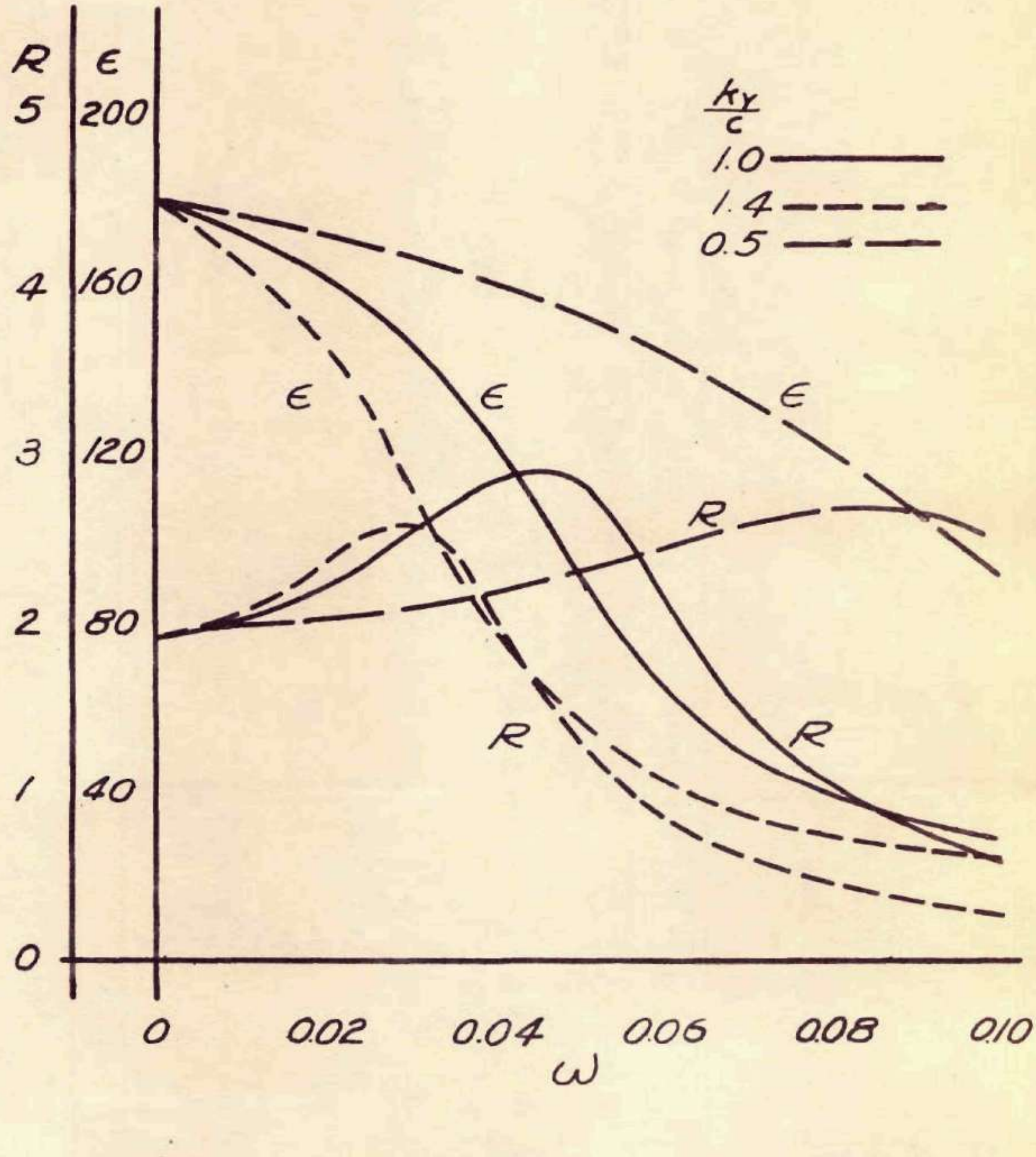

Figure 5. Effect of inertia on response in angle of attack

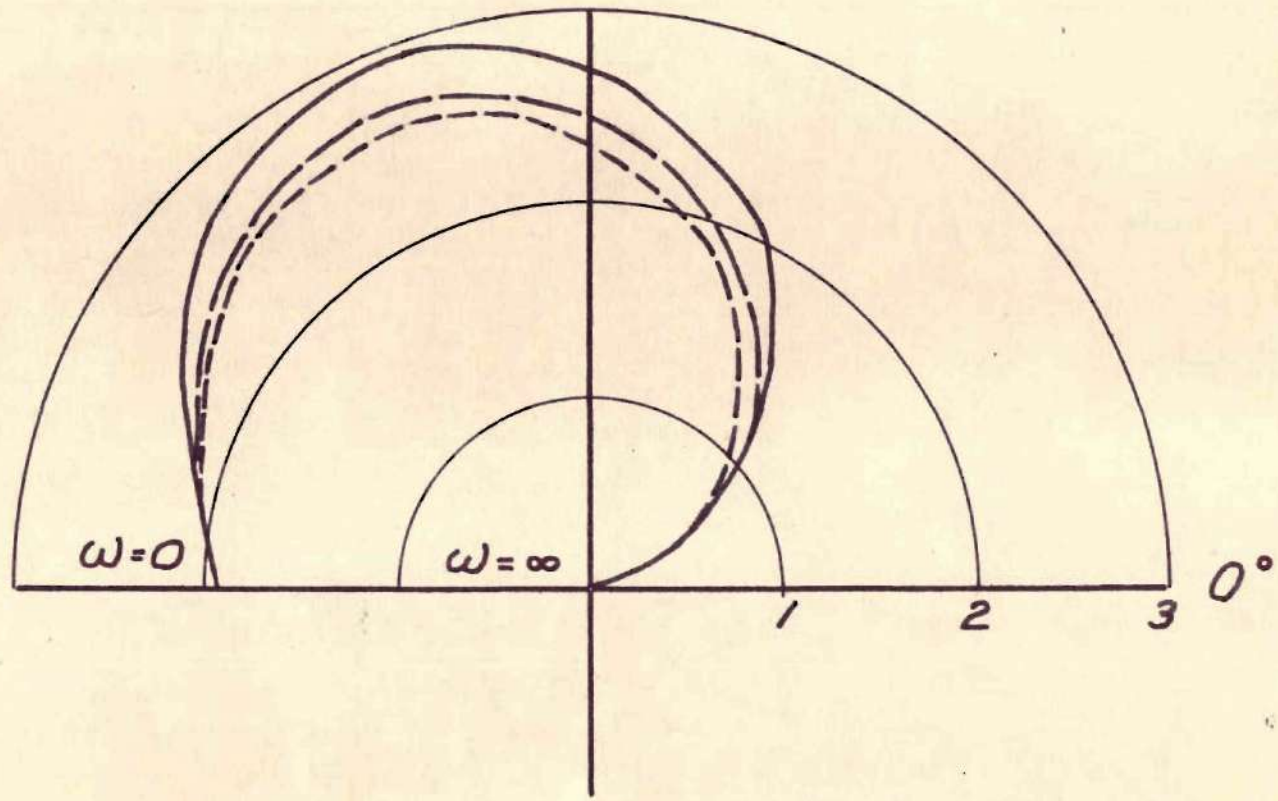

Figure 5 cont.

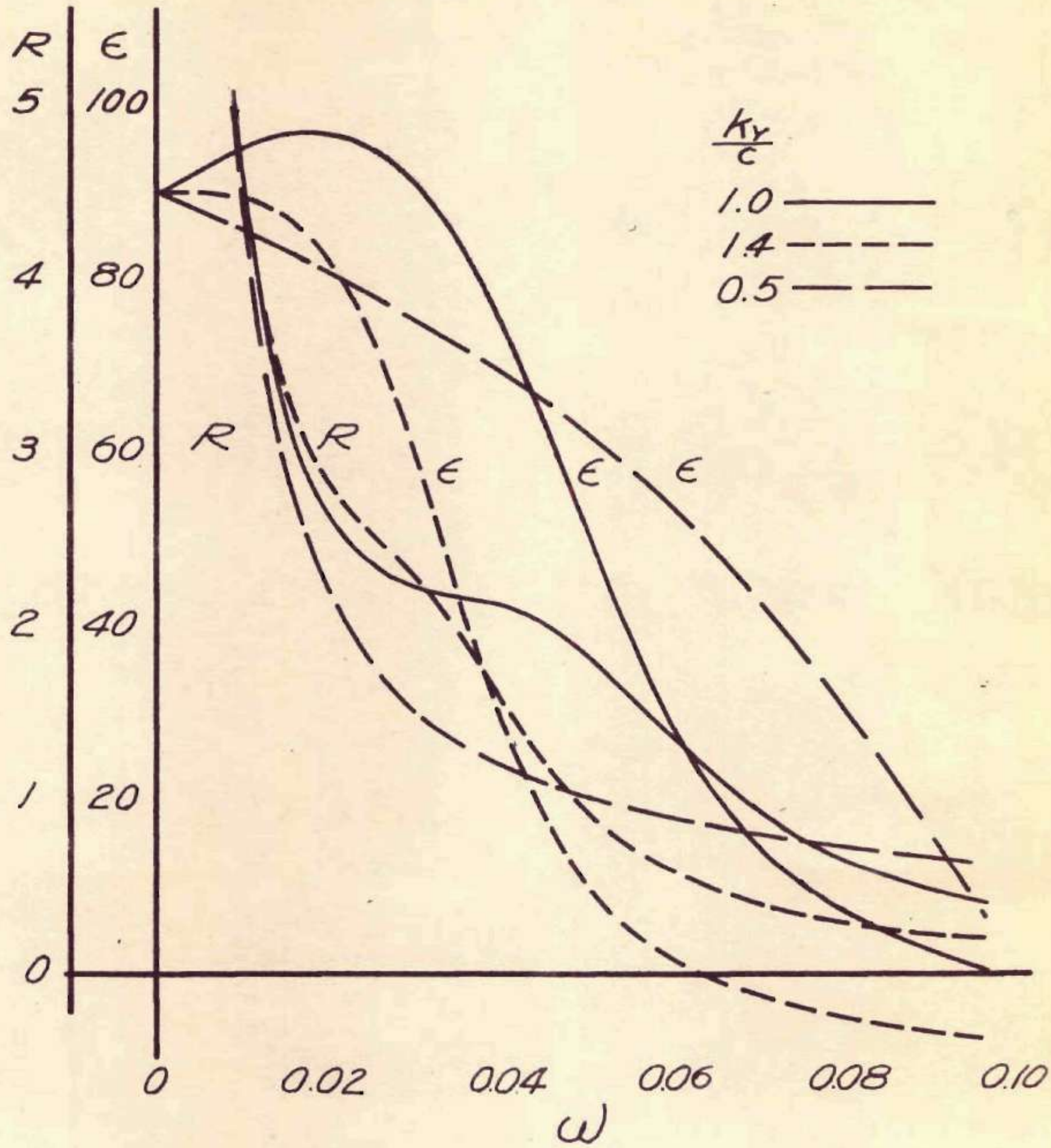

Figure 6. Effect of inertia on response in flight path angle, γ.

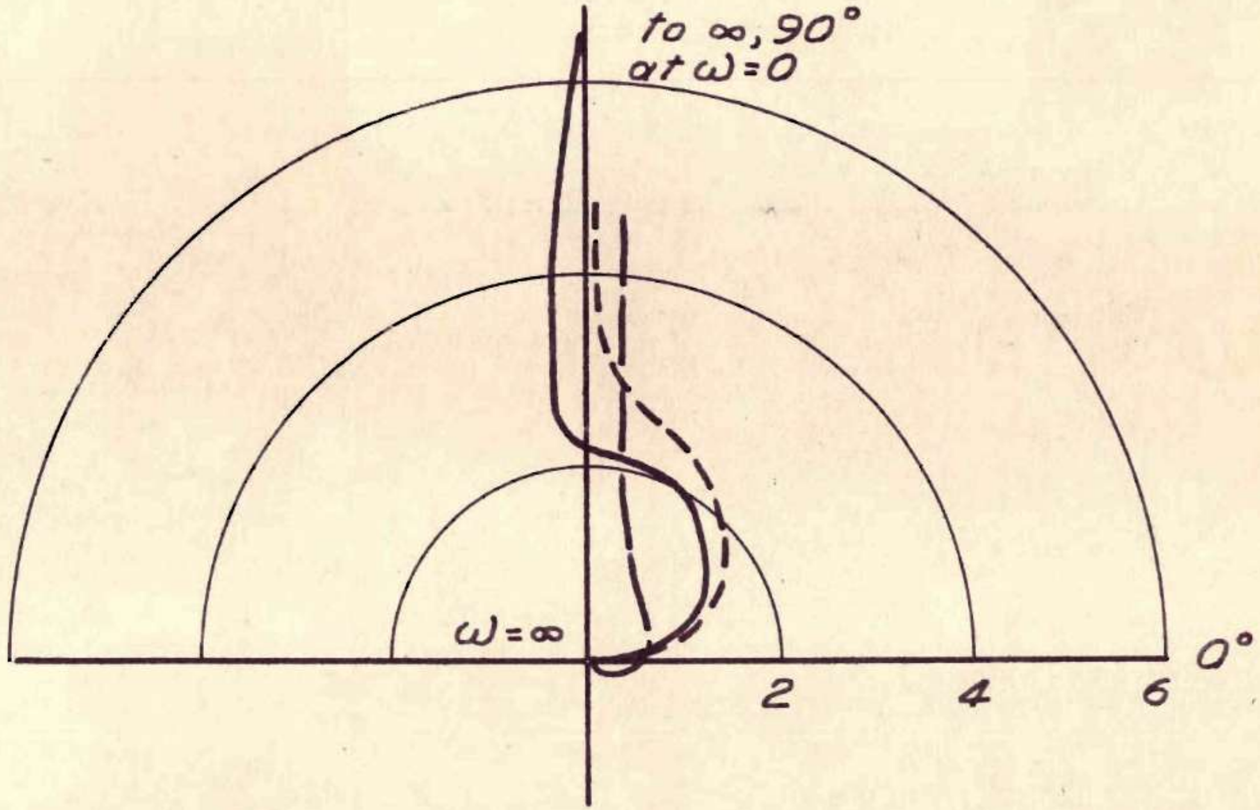

Figure 6 cont.

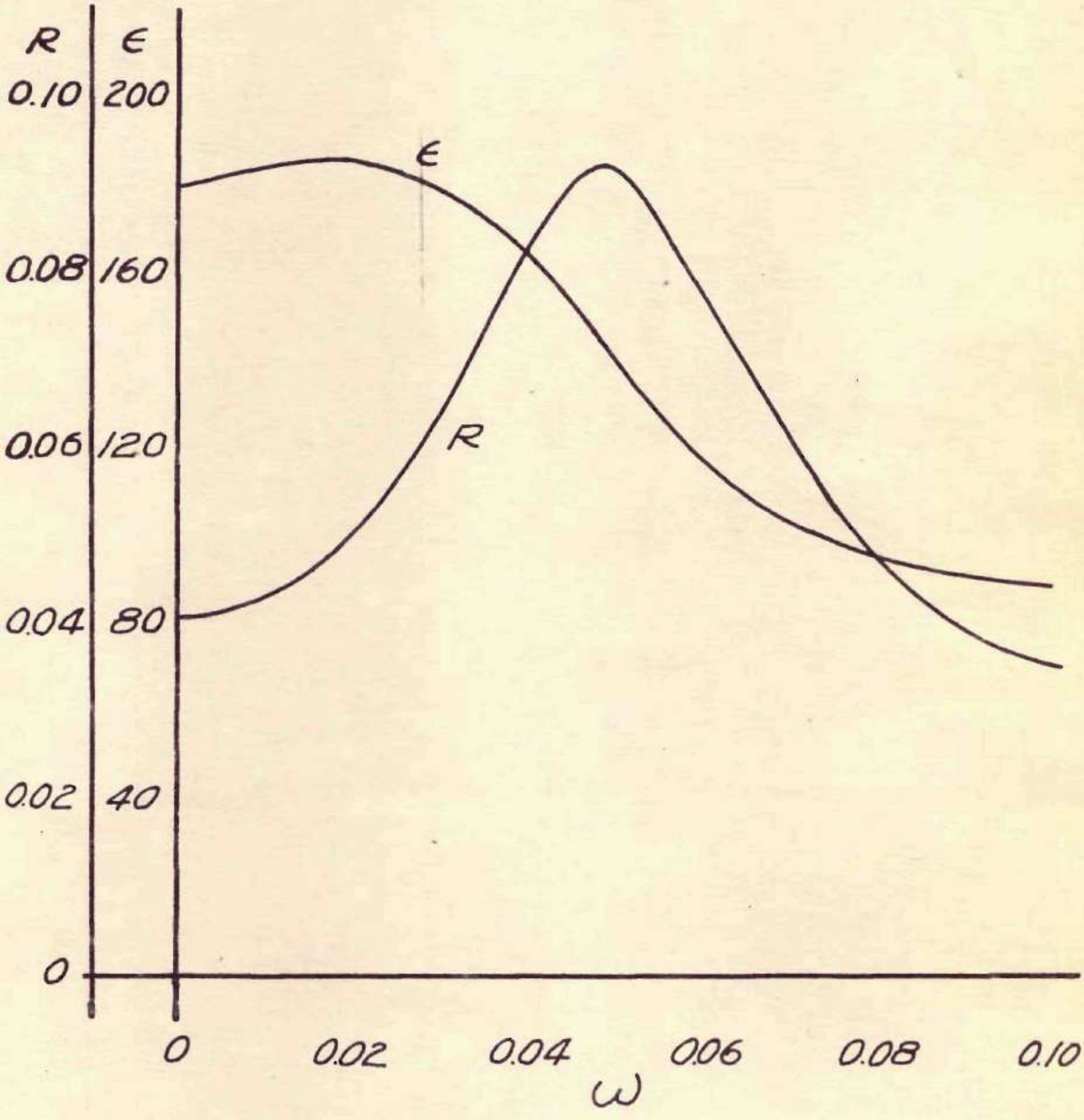

Figure 7 Response in derivative of flight path angle, $D\gamma$.

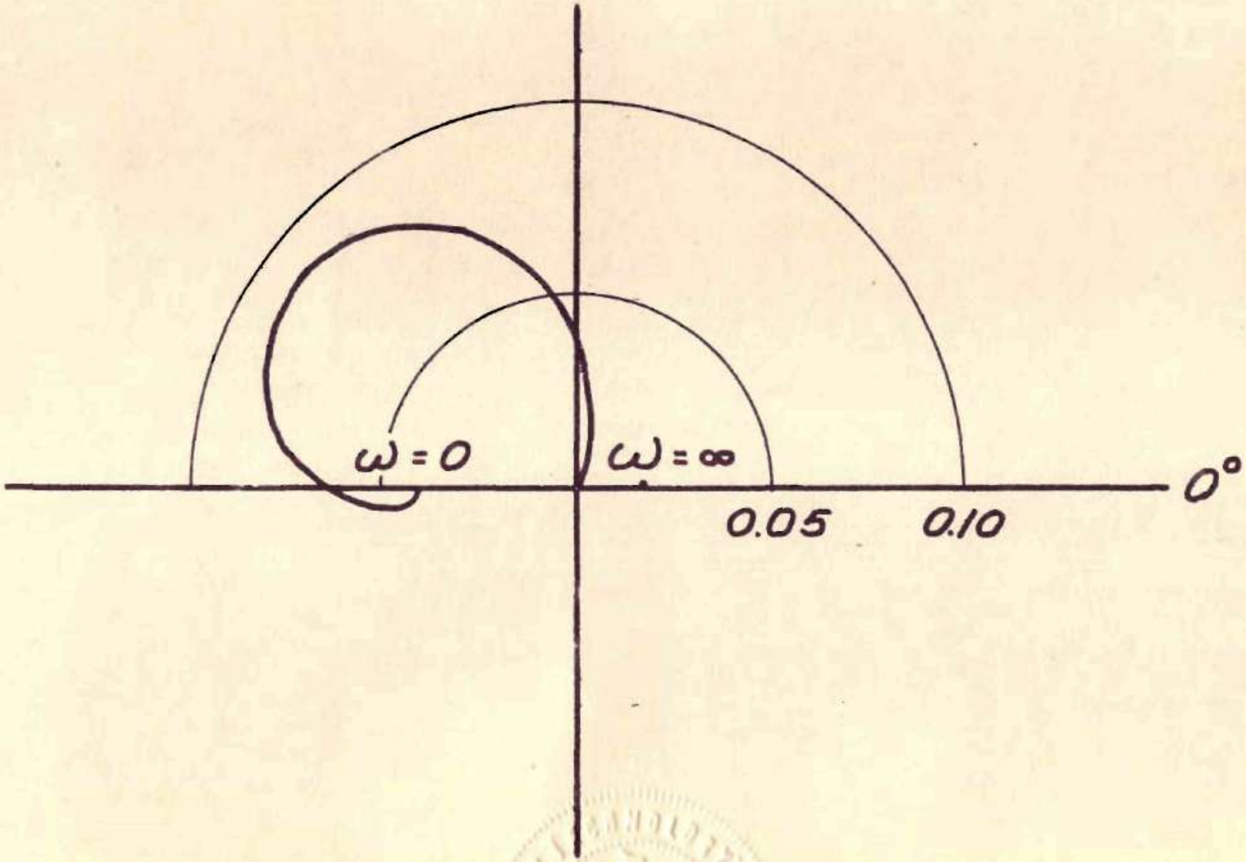

Figure 7 cont.

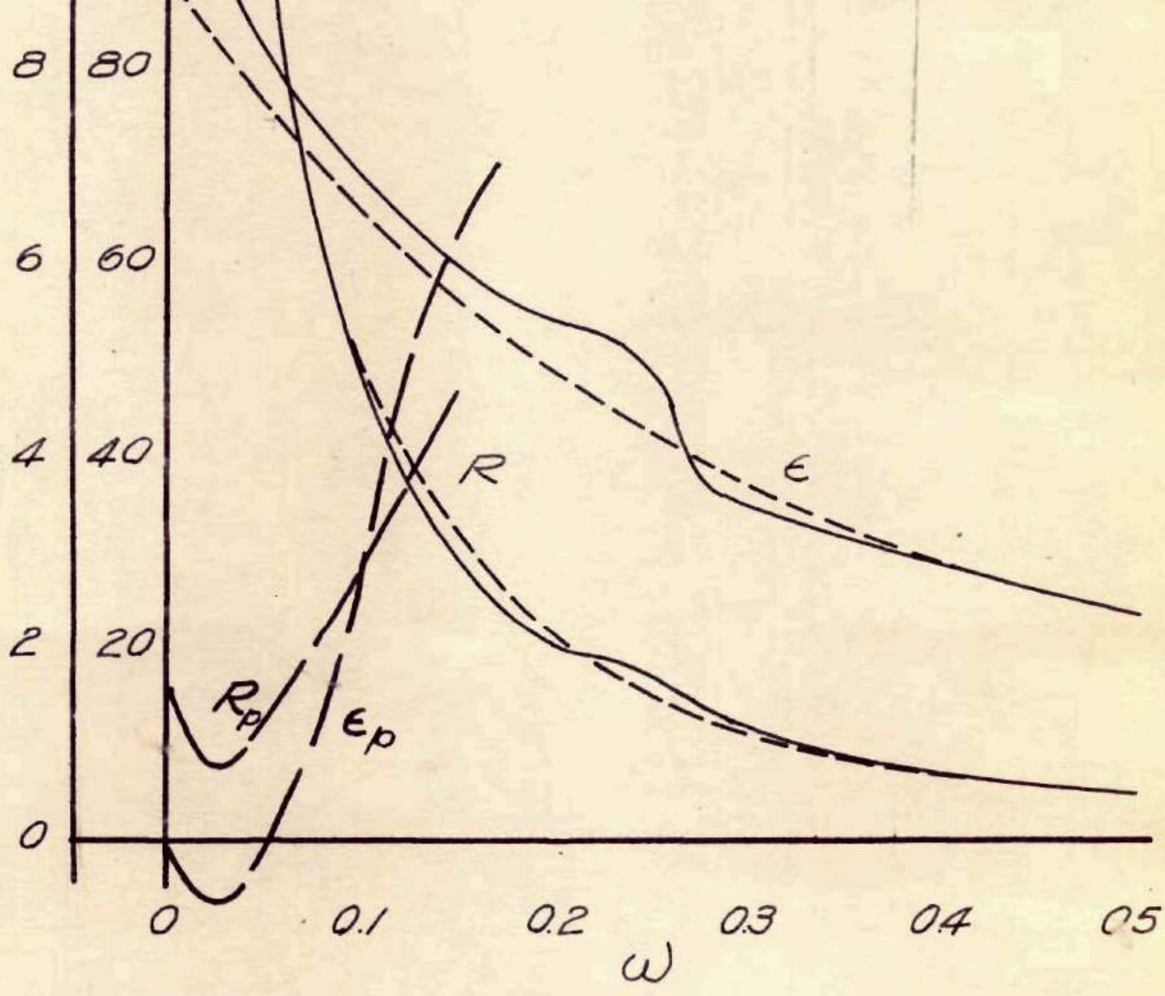

Figure 8. Response in roll, ϕ, by two methods
Aileron control only.

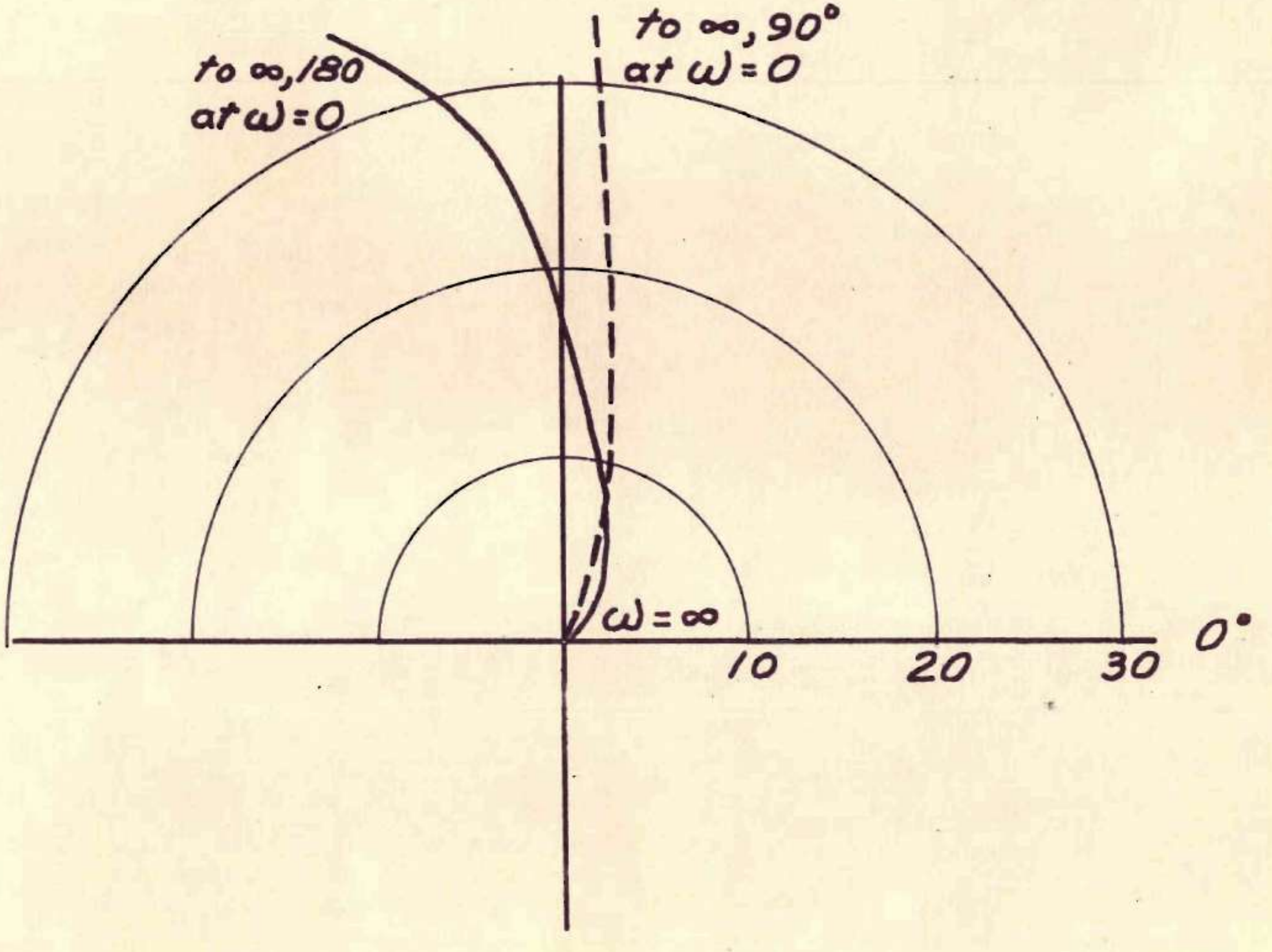

Figure 8 cont.

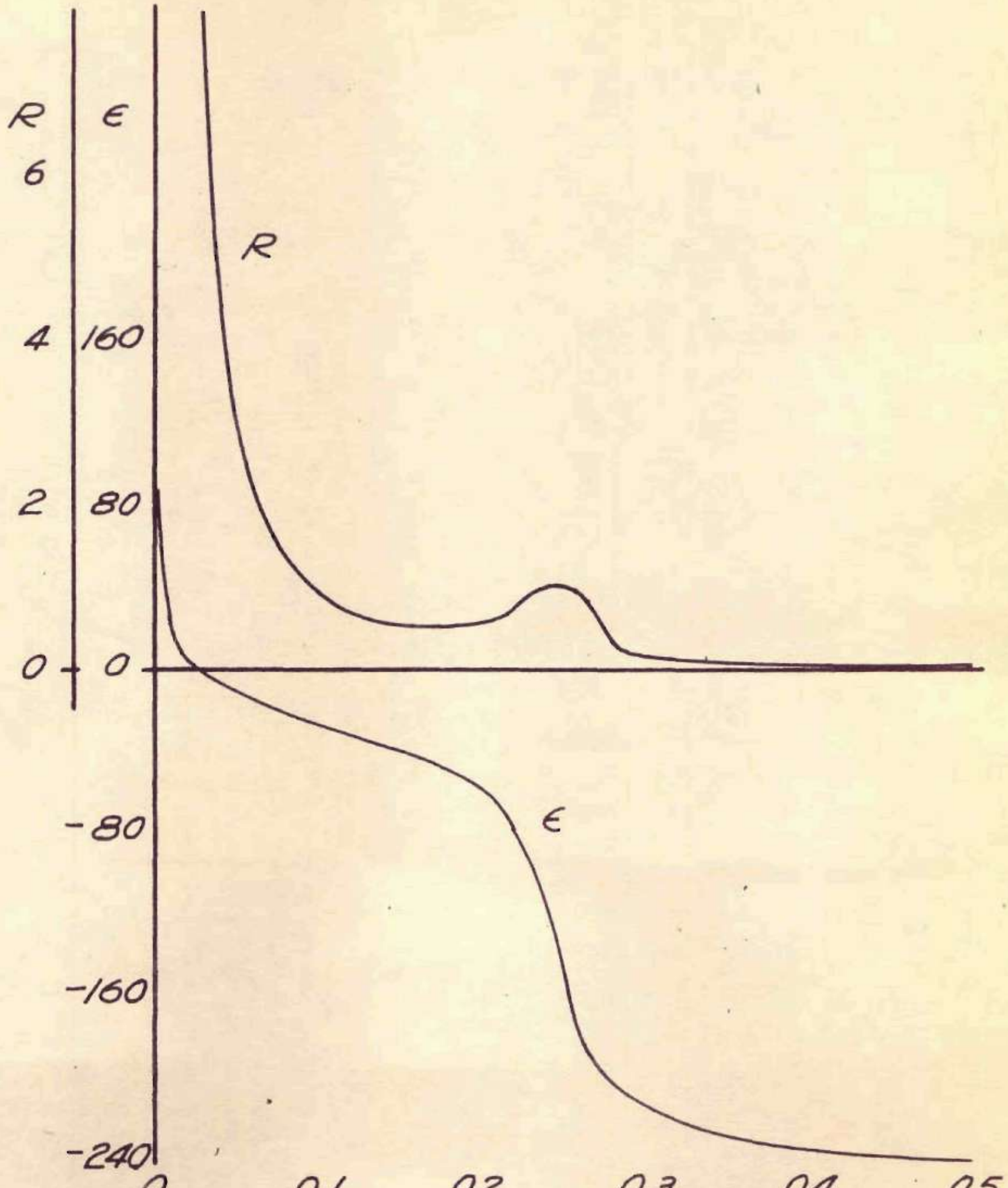

Figure 9 Response in yaw. Aileron control only.

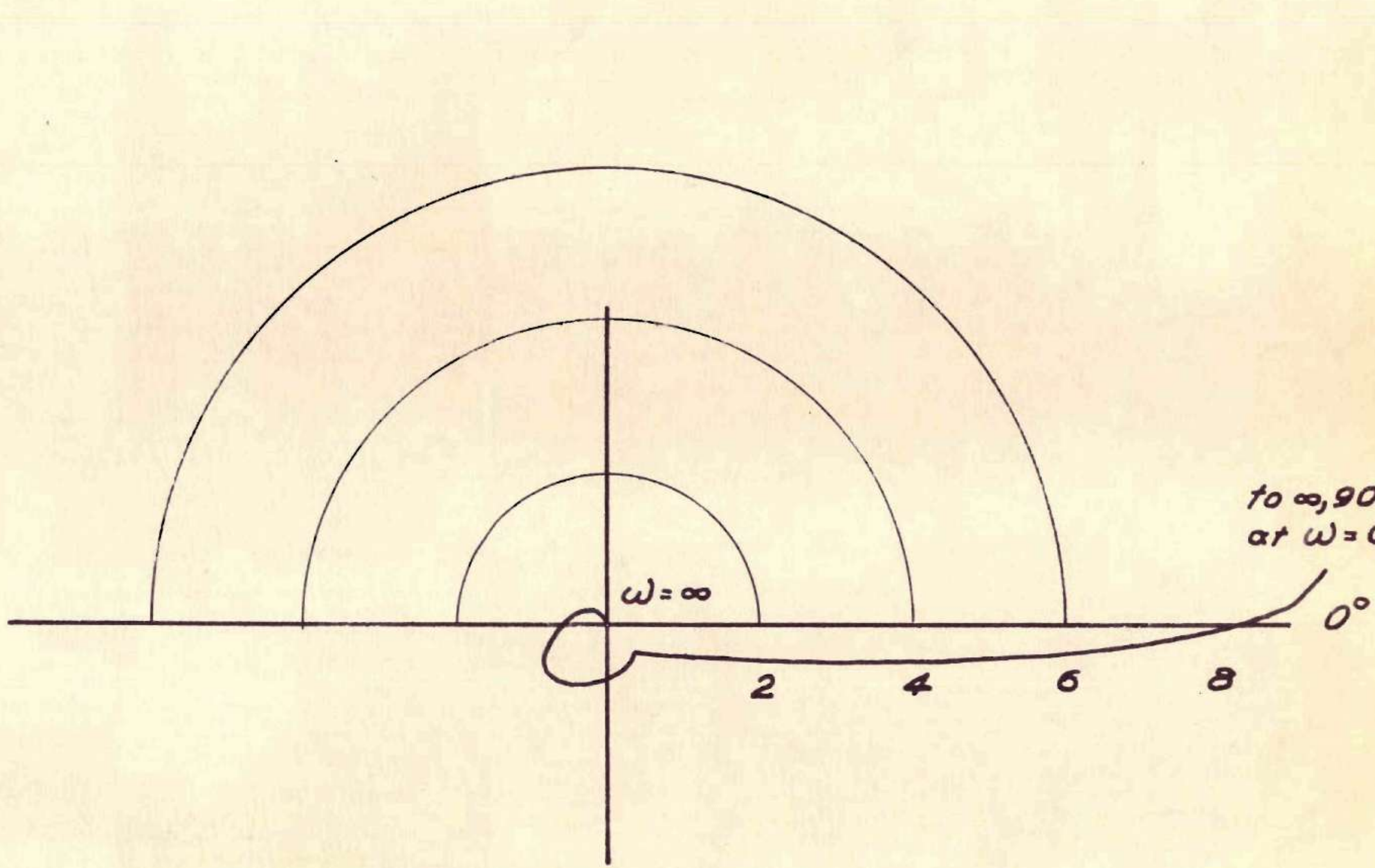

Figure 9 cont.

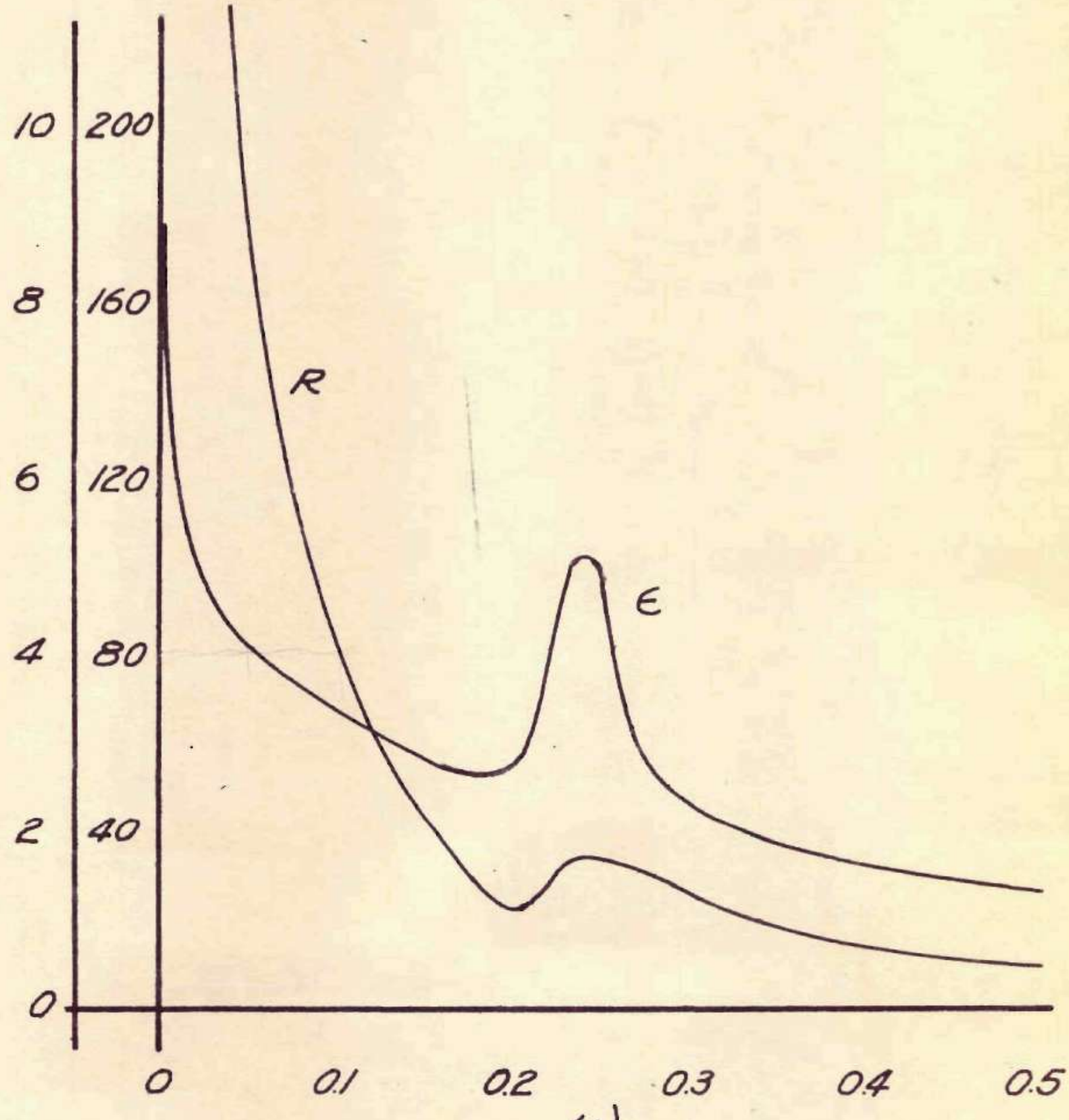

Figure 10 Response in roll, ϕ. Aileron and rudder control.

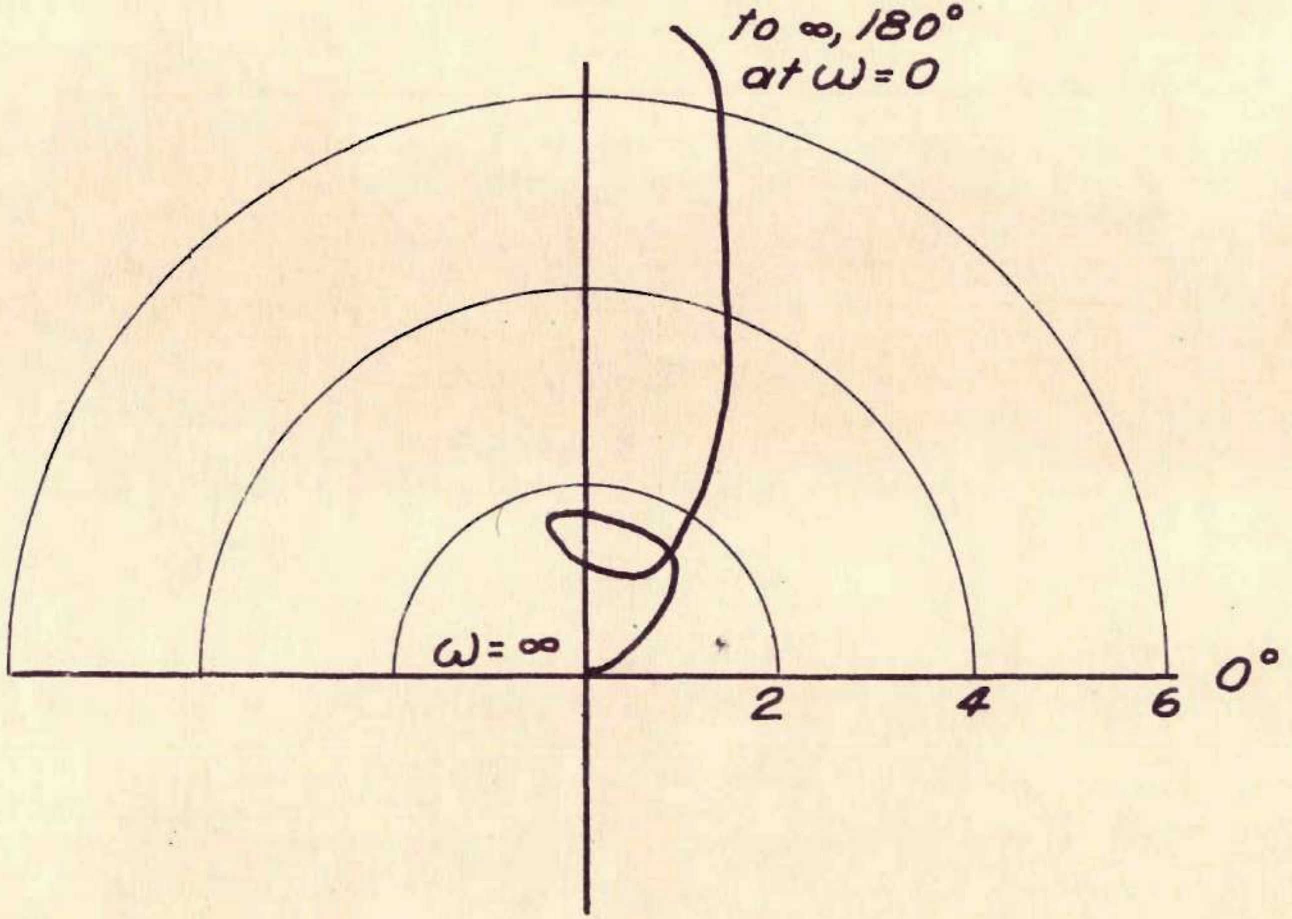

Figure 10 cont.

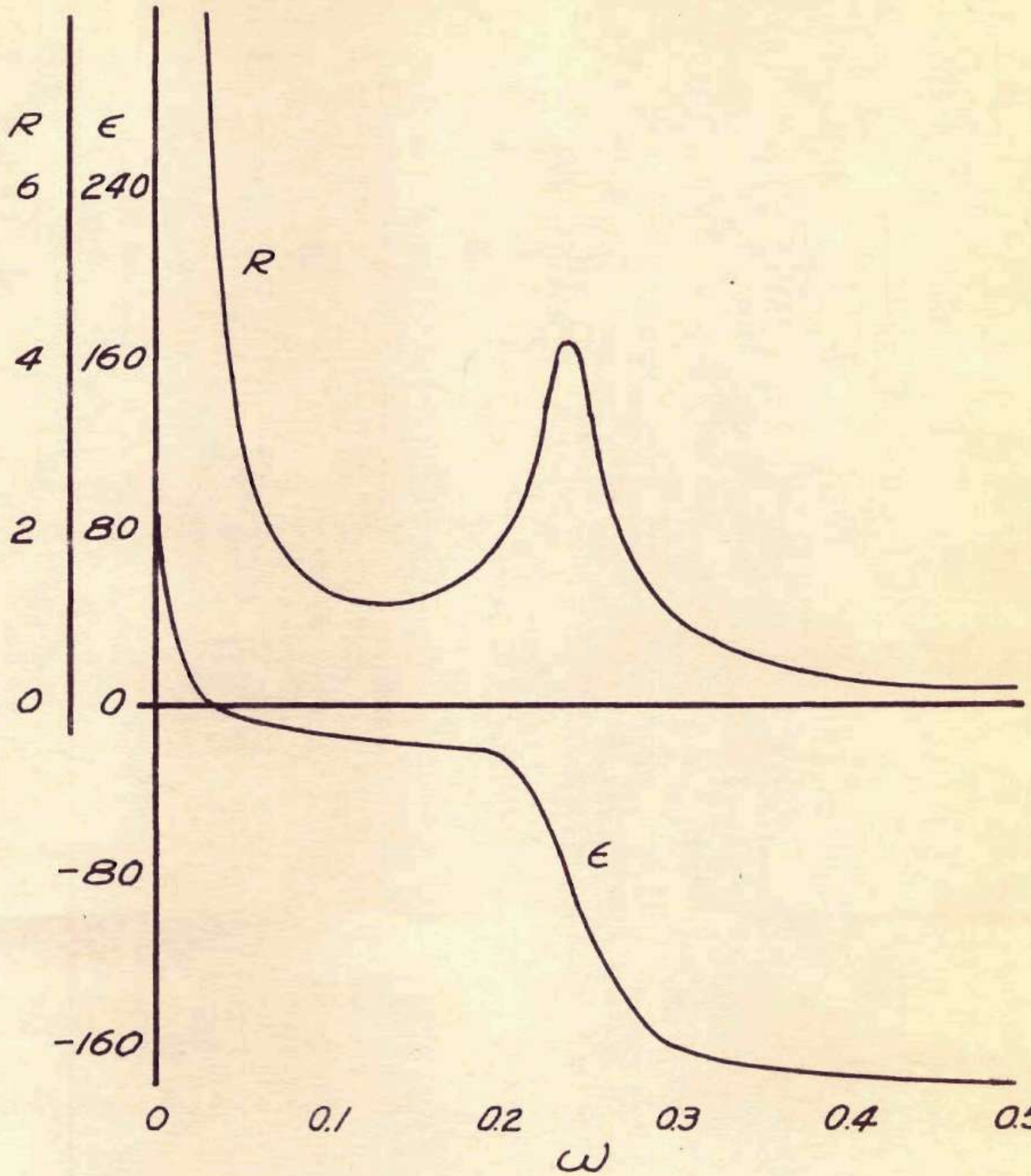

Figure 11 Response in yaw, ψ. Aileron and rudder control.

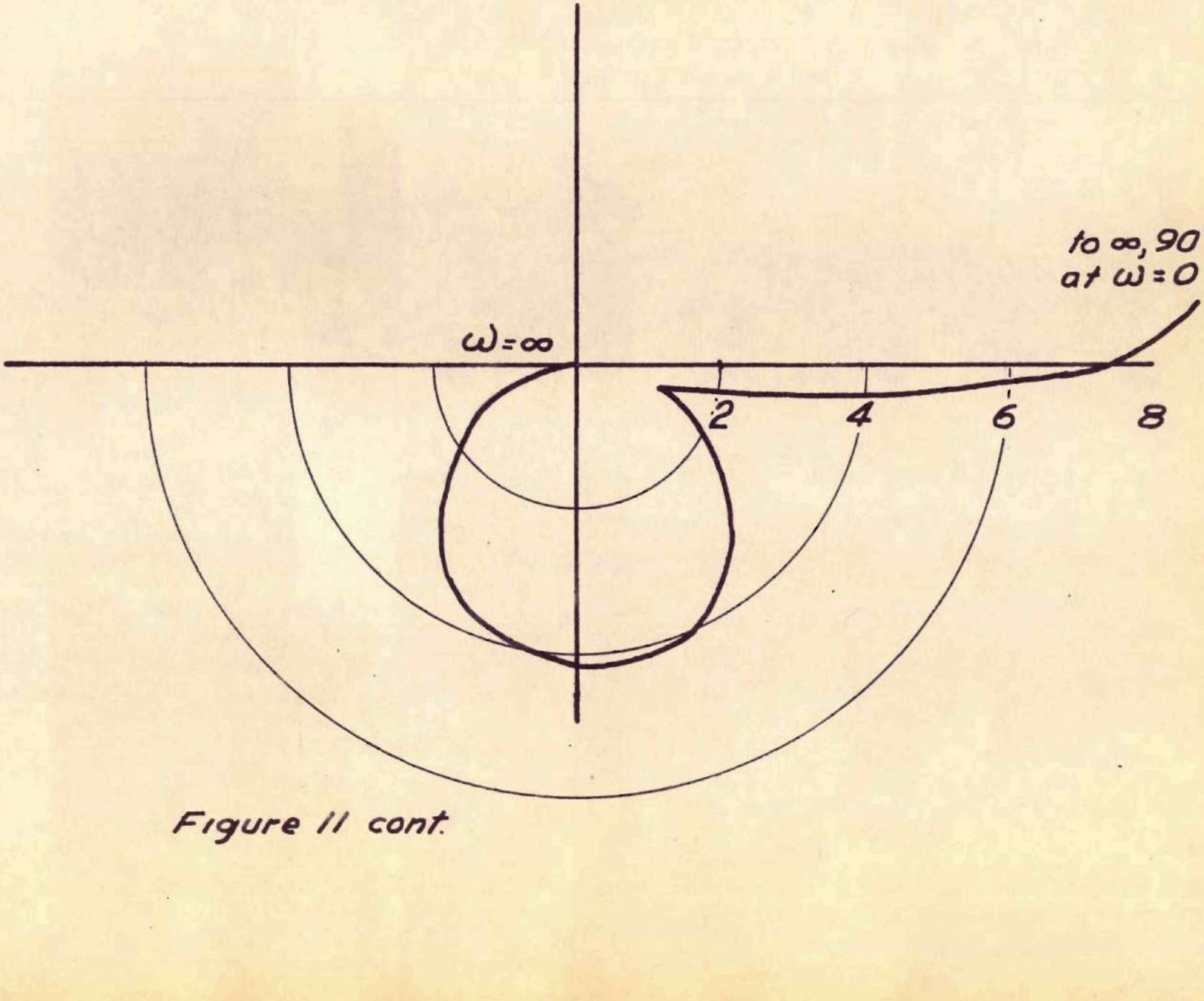

Figure 11 cont.