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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO $x''(t) + p(t)x(t) = 0$

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CHAPTER I

INTRODUCTION

This thesis is a study of the behavior of solutions to the differential equation

$$x''(t) + p(t)x(t) = 0, \quad t \geq \alpha, \tag{1}$$

under various restrictions on the function p. However, p is always assumed to be continuous and real valued on some closed interval $[\alpha,\infty)$. This assumption on p is sufficient to guarantee the existence and uniqueness of solutions to initial value problems for (1) on the interval $[\alpha,\infty)$. By placing additional requirements on the function p, one can deduce properties of the solutions to (1).

1.1 Definition: A *solution to (1)* is a function with a continuous second derivative that satisfies the differential equation (1) at every point of $[\alpha,\infty)$.

There are discussed, below, criteria for answering the following questions: Under what hypotheses on p

(1) will the solutions to the differential equation (1) oscillate?

(2) will the solutions to the differential equation (1) be bounded?

(3) will the solutions to the differential equation (1) approach zero as $t \to +\infty$?

The significance of such a qualitative investigation into the behavior of solutions to (1) can be illustrated by the following example. Consider a particle which is at rest a unit distance from the origin. Suppose that it is subjected to a force directed toward the origin such that the particle's displacement x(t) is governed by the differential equation

$$x''(t) + (\exp t)\, x(t) = 0, \quad t \geq 0.$$

Solving the equation in a power series, one finds that

$$x(t) = 1 - \frac{1}{2} t^2 - \frac{1}{6} t^3 + \frac{1}{40} t^5 + \cdots .$$

Thus, it is difficult to determine from the series that the particle oscillates with decreasing amplitude about the origin and that the limiting position of the particle as time becomes infinite is the origin. By making a qualitative study, however, one is able to verify these assertions.

The differential equation (1) arises often in physical applications. If p is a positive constant, the differential equation (1) is the equation of motion of the simple harmonic oscillator, which consists of a particle of unit mass on a frictionless horizontal surface attached to a linear spring. When p is nonconstant, the differential equation (1) is the rectilinear equation of motion of a particle which is subject to a time-dependent central force.

Thus, as could be expected, equation (1) arises in the study of planetary orbits. In 1877, G. W. Hill investigated the lunar perigee by using equation (1) with p a periodic function. Because of his investigation the differential equation (1) with p periodic is commonly called Hill's equation. The case where p is periodic (except for the trivial periodic case where p is a constant) will not be treated here. The interested reader is referred to [13] for a discussion of this case. Except for special situations, the cases where p changes signs infinitely many times will also not be considered.

Certain elementary facts concerning properties of solutions to differential equations in general and equation (1) will be presented below. These results are needed for several proofs in the present work.

1.2 Proposition: If u is any nontrivial solution to equation (1), then in any interval where u does not vanish, all solutions to (1) can be expressed in the form

$$c_1u(t) + c_2u(t) \cdot \int [u(s)]^{-2}ds,$$

where $\int [u(s)]^{-2}ds$ is any antiderivative of $[u(t)]^{-2}$.

Proof: If u is a nontrivial solution and v is a second linearly independent solution, then by Abel's formula

$$\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = \begin{vmatrix} u(\alpha) & v(\alpha) \\ u'(\alpha) & v'(\alpha) \end{vmatrix} = c,$$

where c is a nonzero constant.

Hence $v'(t)u(t) - u'(t)v(t) = c$.

Since it was assumed that u does not vanish,

$$v'(t) - v(t) \cdot \frac{u'(t)}{u(t)} = \frac{c}{u(t)} .$$

It follows that

$$v(t) = cu(t) \int [u(s)]^{-2} ds,$$

where $\int [u(s)]^{-2} ds$ is any antiderivative of $[u(t)]^{-2}$. Since every solution of a linear second order differential equation can be expressed as a linear combination of any two linearly independent solutions, the desired conclusion is obtained. ▯

The next result, which is commonly known as the Gronwall inequality, is used to study the growth of solutions to (1).

1.3 Proposition (Gronwall Inequality): If u and v are positive valued continuous functions on $[\alpha,\infty)$, if c is a positive constant, and if

$$u(t) \le c + \int_\alpha^t u(s)v(s)ds \quad \text{for} \quad t \ge \alpha, \tag{2}$$

then

$$u(t) \le c \exp\left\{\int_\alpha^t v(s)ds\right\}.$$

<u>Proof</u>: Let $k(t) = c + \int_{\alpha}^{t} u(s)v(s)ds$.

Then $k'(t) = u(t)v(t) \leq v(t)k(t)$ by (2). Now $k(t) \geq c > 0$, thus $k'(t)/k(t) \leq v(t)$, and by integrating and taking exponentials, one sees that

$$k(t) \leq c \cdot \exp\left[\int_{\alpha}^{t} v(s)ds\right] .$$

But

$$u(t) \leq k(t) \leq c \cdot \exp\left[\int_{\alpha}^{t} v(s)ds\right] . \qquad \square$$

CHAPTER 2

OSCILLATORY BEHAVIOR

In this chapter answers to the following questions are sought:

(1) What properties must p satisfy in order that equation (1) be oscillatory?

(2) What properties must p satisfy in order that no (some) solutions to (1) will be oscillatory?

Since the definitions of oscillatory (nonoscillatory) solutions and oscillatory (nonoscillatory) equations vary in the references, we shall adopt, as a matter of convenience, definitions which are equivalent to those in Hille [9].

2.1 Definitions: A solution to (1) is *oscillatory* if it has an infinite number of zeros in $[\alpha,\infty)$. Equation (1) is *oscillatory* if every solution to (1) is oscillatory. A solution to (1) is *nonoscillatory* if it has at most a finite number of zeros in $[\alpha,\infty)$. Equation (1) is *nonoscillatory* if every nontrivial solution is nonoscillatory.

One usually thinks of functions defined on $[\alpha,\infty)$ as being oscillatory if their values not only equal zero an infinite number of times but also change signs at any zero. It will now be shown that the value of any nontrivial solution will change sign at any zero. Hence, if a nontrivial solution is oscillatory according to this definition, then it will also possess this additional property. It follows then, that if a nontrivial solution has value zero at a point, t_0, it will possess property (P) and not property (Q) as illustrated in Figure 1.

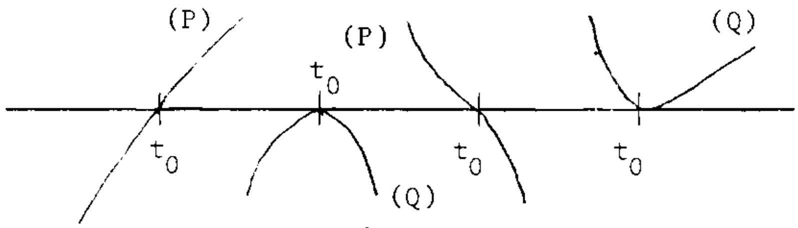

Figure 1

2.2 Lemma: If $x = u(t)$, $\alpha \le t < \infty$, is a nontrivial solution to (1) and if $u(k) = 0$, $\alpha \le k < \infty$, then either

(1) $u'(k)<0$, and there exists an interval (a,k) such that $u(t)>0$ on (a,k) and an interval (k,b) such that $u(t)<0$ on (k,b), or

(2) $u'(k)>0$, and there exists an interval (a,k) such that $u(t)<0$ on (a,k) and an interval (k,b) such that $u(t)>0$ on (k,b).

Proof: Since $x = u(t)$ is not the trivial solution, then $u'(k) \nequiv 0$ since solutions to initial value problems are unique. Thus there is an interval (a,b) contained in $[\alpha,\infty)$ with k in (a,b) such that $u'(t) \nequiv 0$ on (a,b). Suupose that $u'(t)>0$ on (a,b). For an arbitrary point c in (a,k),

$$u(c) - u(k) = u(c) = u'(s)(c-k) < 0$$

for some s in (c,k). Thus $u(t)<0$ on (a,k). By a similar argument one can show that $u(t)>0$ on (k,b), and hence property (2) holds. By assuming the alternative, $u'(k)<0$, one can show that property (1) must hold. □

A useful theorem for the study of oscillations of solutions is the Sturm comparison theorem.

2.3 Theorem (Sturm Comparison Theorem): Let u and v be nontrivial solutions of

$$x'' + p(t)x = 0 \tag{3}$$

and

$$x'' + q(t)x = 0, \tag{4}$$

respectively, where $p(t) \geq q(t)$. Then $u(t)$ equals zero at least once between any two zeros of v unless $p \equiv q$ in which case $u = cv$ for some nonzero constant c.

Proof: Let t_1 and t_2 be successive zeros of v so that $v(t_1) = v(t_2) = 0$ and suppose that u is nonzero on (t_1,t_2). Then by replacing u and (or) v by their negatives if necessary, one could find solutions u and v with positive values on (t_1,t_2). Then

$$W(t_1) = u(t_1)v'(t_1) \geq 0 \quad \text{and} \tag{5}$$

$$W(t_2) = u(t_2)v'(t_2) \leq 0, \tag{6}$$

where $W(t)$ denotes the Wronskian of u and v. Since u and v are positive on (t_1,t_2), however,

$$W'(t) = [p(t)-q(t)]u(t)v(t) \geq 0 \quad \text{on } (t_1,t_2).$$

Hence $W(t)$ is nondecreasing, thus a contradiction of (5) and (6) results unless $p(t) - q(t) \equiv 0$. In this event, $u(t) = cv(t)$ since the Wronskian of two linearly independent solutions is nonzero. □

2.4 Theorem: Let $p(t) \geq q(t)$ on $[\alpha,\infty)$. If Equation (3) is nonoscillatory, then Equation (4) is nonoscillatory. Similarly, if Equation (4) is oscillatory, then Equation (3) is oscillatory.

Proof: The result follows immediately from the Sturm comparison theorem. □

The Sturm comparison theorem is very useful in the case $p(t) \leq 0$ on $[\alpha,\infty)$.

2.5 Theorem: If $p(t) \leq 0$, then no nontrivial solution has more than one zero.

Proof: Suppose that u is a solution of (1) such that $u(t_1) = u(t_2) = 0$. By the Sturm comparison theorem, the solution $y(t) \equiv 1$ to $y''(t) = 0$ would have to vanish at least once in (t_1,t_2). Hence a contradiction results, and no solution can vanish more than once. □

A restatement of the above theorem in terms of oscillatory equations gives the following.

2.6 Corollary: If $p(t)\leq 0$ on $[\alpha,\infty)$, then Equation (1) is nonoscillatory.

The case where $p(t)\geq 0$ on $[\alpha,\infty)$ is examined next.

2.7 Theorem: If $p(t)\geq 0$ on $[\alpha,\infty)$ and if

$$\int_{\alpha}^{\infty} p(s)ds = +\infty,$$

then Equation (1) is oscillatory.

Proof (Bellman [2]): Suppose, for contradiction, that v is a solution to (1) which is positive valued for $t>k$, where $k>\alpha$ is a constant. Then $v''(t) = -p(t)v(t) \leq 0$ and $v'(t)$ is monotone nonincreasing for $t>k$. There are three cases to consider:

(1) $v'(t) \geq 0$ for all $t>k$.

(2) $v'(t) \geq 0$ for t in $(k,f]$, $v'(t)<0$ for $t>f$ for some constant $f>k$.

(3) $v'(t) < 0$ for $t>k$.

First, suppose that (1) holds. Then

$$v'(t) = -\int_{k}^{t} p(s)v(s)ds + v'(k),$$

and by the mean value theorem for integrals,

$$v'(t) = -v(c) \int_k^t p(s)ds + v'(k)$$

for some constant c in (k,t). But

$$\int_\alpha^\infty p(s)ds = +\infty,$$

hence v'(t) is negative for some t and a contradiction results. Thus (1) cannot hold. Now suppose (2) holds. Let $m>f$ so that $v'(m)<0$. Since $v''(t)\leq 0$, then $v'(t)\leq v'(m)$ for $t>m$. Thus

$$v(t) - v(m) = \int_m^t v'(s)ds \leq v'(m)(t-m).$$

Since $v'(m)(t-m) \to -\infty$ as $t\to+\infty$, a contradiction is obtained. Case (3) follows in a manner similar to Case (2) with $m>k$ an arbitrary constant. Thus v(t) cannot remain positive for all $t>k$. It follows from Lemma 2.2 that v(t) becomes negative. In a similar manner, one can show that v(t) cannot remain negative for $t>k$ for some constant $k>\alpha$. Therefore all solutions are oscillatory. ▯

2.8 Corollary: If $p(t)\geq a^2>0$, then Equation (1) is oscillatory.

Proof: The result follows immediately from the previous theorem.

2.9 Example (Cesari [4]): The restriction $p(t) \geq 0$ is not alone sufficient condition to guarantee oscillations of solutions. Solutions to the differential equation

$$x'' + (m/t^2)x = 0,$$

where m is a positive constant and $\alpha > 0$, are of the form

$$c_1 t^{1/2} \sin(k \ln t) + c_2 t^{1/2} \cos(k \ln t),$$

where k is a positive constant if $m > 1/4$. They are of the form

$$c_1 t^{k_1} + c_2 t^{k_2},$$

where k_1 and k_2 are constants if $m < 1/4$. If $m > 1/4$, Equation (1) is oscillatory; and if $m < 1/4$, Equation (1) is nonoscillatory. Thus, for the case where $p(t) \geq 0$ on $[\alpha,\infty)$ and

$$\int_\alpha^\infty p(s)ds < +\infty,$$

Equation (1) may or may not be oscillatory. The Sturm comparison theorem can be used to determine the behavior of Equation (1) in this case. If m is a positive constant, for example, it is easily seen that if $p(t) \geq (1+m)t^2/4$ on $[\alpha,\infty)$ then Equation (1) is oscillatory, and if $p(t) \leq (1-m)t^2/4$ on $[\alpha,\infty)$, then Equation (1) is nonoscillatory.

The last case considered is that where p is not necessarily of constant sign and remains "sufficiently close" to zero. If p is "sufficiently close" to zero, then one would expect solutions to (1) to be asymptotic to those of $x''(t) = 0$, and hence Equation (1) would be nonoscillatory.

2.10 Definition: The function f *is asymptotic to the function* g as $t \to +\infty$ if

$$\lim_{t\to+\infty} \frac{f(t)}{g(t)} = 1.$$

2.11 Theorem (Bellman [2]): If $\int_\alpha^\infty s|p(s)|ds < \infty$, then any solution to (1) is asymptotic to $a+mt$ as $t\to+\infty$, where at least one of the constants a and m is nonzero.

Proof: Let $A(t) = -p(t)$. Then (1) is transformed into

$$x''(t) = A(t)x(t). \tag{7}$$

We next choose a constant b satisfying the following:

(i) $b > \alpha$

(ii) $b > 1$

(iii) $\int_b^\infty s|A(s)|ds < 1/3$.

The existence of a constant b is guaranteed by hypothesis. Next let v denote the unique solution to (7) satisfying $v(b) = 0$, $v'(b) = 1$.

Integrating (7) from b to t, one obtains

$$v'(t) = 1 + \int_b^t A(s)v(s)ds. \tag{8}$$

A second integration yields the relation

$$v(t) = (t-b) + \int_b^t \int_b^u A(s)v(s)dsdu. \tag{9}$$

Changing the order of integration of the integral, one finds that

$$\int_b^t \int_b^u A(s)v(s)dsdu = \int_b^t \int_s^t A(s)v(s)duds = \int_b^t (t-s)A(s)v(s)ds.$$

But $t>b>1$, hence (9) becomes

$$|v(t)| \leq t + t\int_b^t |A(s)||v(s)|ds.$$

Dividing by t and then applying the Gronwall inequality one finds that

$$\frac{|v(t)|}{t} \leq \exp\left[\int_b^t |A(s)|sds\right] \leq \exp\left[\int_b^\infty |A(s)|sds\right]. \tag{10}$$

Denote $\exp\left[\int_b^\infty |A(s)|sds\right]$ by K.

But $$\left|\int_b^t A(s)v(s)ds\right| \leq \int_b^t |A(s)||v(s)|ds$$

and using from (10) the result that $\frac{|v(t)|}{t} \leq K$, one obtains

$$\left|\int_b^t A(s)v(s)ds\right| \leq K\int_b^t s|A(s)|ds .$$

Since

$$\lim_{t\to+\infty} \left|K\int_b^t s|A(s)|ds\right| \leq \frac{1}{3}\left(\exp\frac{1}{3}\right) < 1,$$

$$\lim_{t\to+\infty} \left|\int_b^t A(s)v(s)ds\right| < 1.$$

By (8), $v'(t)$ has a finite positive limit as $t\to+\infty$. Denote the limit by q. Then there is a T such that for $t>T$, $v(t)>0$. Hence, for $t>T$,

$$w(t) = v(t) \cdot \int_t^\infty \frac{ds}{v^2(s)}$$

is a second solution of (1) which is linearly independent of $v(t)$. By L'Hôpital's rule,

$$\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} \frac{\int_t^\infty \frac{ds}{v^2(s)}}{\frac{1}{v(t)}} = \lim_{t\to+\infty} - \frac{1}{v'(t)} = - \frac{1}{q} .$$

Hence $w(t)$ approaches a limit as $t\to+\infty$. Since a general solution to (7) can be expressed as a linear combination of $v(t)$ and $w(t)$, all non-trivial solutions are asymptotic to $a+mt$ where at least one of the constants a and m is not zero. Thus no solution to (1) can have infinitely many zeros, and Equation (1) is nonoscillatory. This result is recorded as Theorem 2.12.

Theorem 2.12: If $\int_\alpha^\infty s|p(s)|ds < \infty$, then Equation (1) is nonoscillatory.

CHAPTER III

THE CASE $p(t) < 0$

Theorem 2.11 showed that if $p(t) < 0$ on $[\alpha,\infty)$ and $\int_\alpha^\infty s\,p(s)ds$ converges, then every solution to (1) is asymptotic to $a+bt$ where at least one of the constants a and b is not zero. Thus, some solutions are bounded, but no nontrivial solution approaches zero as $t\to+\infty$.

The next case is that where $p(t) < 0$ and $\int_\alpha^\infty |p(s)|ds = +\infty$. First, a lemma is established which is required in the proof of the next theorem.

3.1 Lemma: Let c and d be positive [negative] constants. If $p(t) < 0$ on $[\alpha,\infty)$ and $\int_\alpha^\infty |p(s)|ds = +\infty$, then for any solution v to (1) with $v(\alpha) = c$ and $v'(\alpha) = d$,

$$\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} v'(t) = +\infty\ [-\infty].$$

Proof (Sansoné [17]): Let $a(t) = -p(t)$. For any nontrivial solution to (1),

$$\frac{d}{dt}[v(t)v'(t)] = a(t)[v(t)]^2 + [v'(t)]^2 > 0.$$

Thus for any nontrivial solution $v(t)v'(t) \geq v'(\alpha)v(\alpha) > 0$ for $t>\alpha$. Note that this implies that $v(t) \not\equiv 0$, $v'(t) \not\equiv 0$ for $t>\alpha$. Writing (1) in the form $v''(t) = a(t)v(t)$, multiplying by $v'(t)$ and integrating, one obtains

$$[v'(t)]^2 = [v'(\alpha)]^2 + 2\int_\alpha^t a(s)[v(s)v'(s)]ds.$$

But $v'(t)v(t) \geq v'(\alpha)v(\alpha) > 0$ for $t>\alpha$. Denote $v'(\alpha)v(\alpha)$ by f. Hence

$$[v'(t)]^2 = [v'(\alpha)]^2 + 2f\int_\alpha^t a(s)ds.$$

By hypothesis,

$$\lim_{t\to+\infty} \int_\alpha^t a(s)ds = +\infty,$$

hence

$$\lim_{t\to+\infty} v'(t) = +\infty.$$

It follows necessarily that $\lim_{t\to+\infty} v(t) = +\infty$. The case where c and d are negative constants is similar. ▯

3.2 <u>Example</u>: The requirement

$$\int_\alpha^\infty |p(s)|ds = +\infty$$

is necessary to guarantee that $\lim_{t\to+\infty} x'(t) = +\infty[-\infty]$. $x(t) = [t+t(\arctan t) - \frac{1}{2}\ln(t^2+1) + 1]$ is a solution to $x''(t) =$

$x(t)[(1+t^2)(t+t(\arctan t) - \frac{1}{2}\ln(t^2+1) + 1)]^{-1}$ such that $x(0) = 1$, $x'(0) = 1$, $a(t) > 0$ for $t>0$, $x(t) \to +\infty$ as $t \to +\infty$; nevertheless $x'(t)$ is bounded on $[0,\infty)$. This example corrects an error in [4] on p. 83. In [4] the conclusion of Lemma 3.1 was stated without the hypothesis $\int_\alpha^\infty |p(s)|ds = +\infty$.

3.3 <u>Theorem</u>: Let k be any nonzero constant. Then under the same hypotheses on p as in the last lemma, there exists a unique solution to (1) with $w(\alpha) = k$ such that

$$\lim_{t\to+\infty} w(t) = 0,$$

while for any other solution x satisfying $x(\alpha) = k$, either

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = +\infty \quad \text{or} \quad \lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = -\infty.$$

<u>Proof (Sansoné [17])</u>: Without loss of generality, suppose $k>0$. Let $v(t)$ be the unique solution to (1) satisfying $v(\alpha) = k$, $v'(\alpha) = 1$. Then, as shown in the lemma,

$$\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} v'(t) = +\infty,$$

and $v(t)$ does not vanish on $[\alpha,\infty)$. By Proposition 2 any solution x to (1) is of the form

$$x(t) = c_1v(t) + c_2v(t)\int_\alpha^t [v(s)]^{-2}ds \quad \text{for} \quad t\geq\alpha.$$

Since we require that $x(\alpha) = k$, x is of the form

$$x(t) = v(t)\left[1 + c\int_{\alpha}^{t} \frac{ds}{v^2(s)}\right]$$

for some constant c.

Since $\lim\limits_{t\to+\infty} v'(t) = +\infty$, $\int_{\alpha}^{\infty} \frac{ds}{v^2(s)}$ converges. Denote $\int_{\alpha}^{\infty} \frac{ds}{v^2(s)}$ by n. All possible cases for the values of c are considered now.

<u>Case 1</u>. $c<n<0$. Suppose $c = \left[-\int_{\alpha}^{b} \frac{ds}{v^2(s)}\right]^{-1}$ for some constant $b>\alpha$. At $t=b$, $x(b)=0$, and since for $t<b$, $x(t)>0$, then $x'(b)<0$. Hence for some interval (b,f), $f>b$, $x'(t)<0$. Thus there is a point g in (b,f) such that $x(g)<0$ and $x'(g)<0$. By the lemma,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = -\infty.$$

<u>Case 2</u>. Suppose $c=n$. Now

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}\left[\frac{1 + n\int_{\alpha}^{t} \frac{ds}{[v(s)]^2}}{\frac{1}{v(t)}}\right].$$

Using L'Hôpital's rule, one finds that

$$\lim_{t\to+\infty}\left[\frac{\frac{-m}{v^2(t)}}{\frac{v'(t)}{v^2(t)}}\right] = \lim_{t\to+\infty} \frac{n}{v'(t)} = 0$$

since v(t) was chosen such that $\lim_{t\to+\infty} v'(t) = +\infty$. Therefore $\lim_{t\to+\infty} x(t) = 0$.

Case 3. Suppose $c \geq 0$. Then clearly $\lim_{t\to+\infty} x(t) = +\infty$. Differentiating, one obtains

$$x'(t) = v'(t) + c\, v'(t) \int_{\alpha}^{t} \frac{ds}{v^2(s)} + \frac{c}{v'(t)} .$$

Since $v'(t) \to +\infty$ as $t\to+\infty$ and since the other terms are nonnegative, then $\lim_{t\to+\infty} x'(t) = +\infty$.

Case 4. Suppose $n<c<0$. Then

$$\lim_{t\to+\infty} x(t) = \left[\lim_{t\to+\infty} v(t)\right] \cdot \left[\lim_{t\to+\infty} 1 + c \int_{\alpha}^{t} \frac{ds}{[v(s)]^2}\right] = +\infty$$

since $\lim_{t\to+\infty} v(t) = +\infty$ and since the other limit is positive.
Differentiating,

$$x'(t) = v'(t)\left[1 + c \int_{\alpha}^{t} \frac{ds}{v^2(s)}\right] + \frac{c}{v(t)} .$$

Since $\lim_{t\to+\infty} c \cdot \left(\int_{\alpha}^{t} \frac{ds}{[v(s)]^2}\right) > -1$, then $\lim_{t\to+\infty} x'(t) = +\infty$.

When $c=n$, the unique solution to (1) with $x(\alpha) = k$ such that $\lim_{t\to+\infty} x(t) = 0$ is obtained. For any choice of $c \neq n$, either $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = +\infty$ or $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = -\infty$. If $k<0$, then a similar argument holds where v(t) is chosen to be the solution to (1) satisfying $v(\alpha) = k$ and $v'(\alpha) = -1$. $\square$

3.4 <u>Example</u>: The requirement $\int_\alpha^\infty |p(s)|ds = +\infty$ is necessary to guarantee that all solutions to (1) either approach zero or become unbounded as $t\to+\infty$. Observe that $x(t) = \frac{k}{2}[1 + e^{-t}]$ is a solution to

$$x'' = [1 + \exp t]^{-1}x$$

with $x(0) = k$ $(k \neq 0)$ such that $\lim_{t\to+\infty} x(t) = \frac{k}{2}$. This corrects the misprint of Sansone's result (Theorem 3.3 in this work) on p. 84 in [4] where $\int_\alpha^\infty |p(s)|ds = +\infty$ was printed as $\int_\alpha^\infty |p(s)|ds < +\infty$.

3.5 <u>Comment</u>: Consider again Equation (1) with $p(t) < 0$ and $\int_\alpha^\infty |p(s)|ds = +\infty$. Let $k \neq 0$ be arbitrary, and let w denote the unique solution to (1) satisfying $w(\alpha) = k$ such that $\lim_{t\to+\infty} w(t) = 0$. Denote $w'(\alpha)$ by m. If v is any solution to (1) satisfying $v(\alpha) = k$ and $v'(\alpha) \neq m$, then $\lim_{t\to+\infty} v(t) = \infty$. The Wronskian of u and v at $t=\alpha$ is non-zero, hence u and v are two linearly independent solutions to (1). Thus every solution to (1) will be a linear combination of u and v. Therefore, if a solution is bounded as $t\to+\infty$, it has limit zero as $t\to+\infty$.

CHAPTER IV

THE CASE $p(t) > 0$

The case where p is a positive function on $[\alpha,\infty)$ requires a more detailed analysis. First, results are obtained to insure that all solutions are bounded on $[\alpha,\infty)$.

Boundedness of Solutions

The first theorem, originally due to Dini and Hukuhara, shows that if for a particular function p, all solutions to (1) and their derivatives are bounded, then for any function "sufficiently close to p" all solutions to (1) and their derivatives for the new function would also be bounded. Vector-matrix notation will be convenient for proving this theorem. The necessary details are included in this work; however, for a more complete exposition see [7].

4.1 Definitions: Let $y = (y_1, y_2, \ldots, y_n)^T$ be a vector in R^n. Then $\|y\| = \sum_{k=1}^{n} |y_k|$. Let $A = (a_{ij})$ be an $n \times n$ square matrix. Then $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$. The symbol I will denote the identity matrix of appropriate dimension.

4.2 Lemma: Let $Y(t)$ denote the matrix solution to $Y'(t) = A(t)Y(t)$, $Y(\alpha) = I$, and let $X(t)$ be a solution of $X'(t) = A(t)X(t)$, $X(\alpha) = Y_0$. Then the solution to $Z'(t) = [A(t) + B(t)]Z(t)$, $Z(\alpha) = Y_0$, satisfies the integral equation

$$Z(t) = X(t) + \int_{\alpha}^{t} Y(t)Y^{-1}(s)B(s)Z(s)ds. \tag{11}$$

Proof: The result is easily verified by direct substitution.

4.3 Theorem: Consider the differential equations

$$x'' + f(t)x = 0 \tag{12}$$

and

$$y'' + [f(t) + g(t)]y = 0 \tag{13}$$

where

$$\int_{\alpha}^{\infty} |g(s)|ds < +\infty.$$

Then all solutions to (12) and their derivatives are bounded if and only if all solutions to (13) and their derivatives are bounded.

Proof: Assume that all solutions to (12) and their derivatives are bounded. Converting (13) to a system by the change of variables $z_1 = y$, $z_2 = y'$, one obtains

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -f & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \qquad ' = \frac{d}{dt}$$

Let $$A = \begin{bmatrix} 0 & 1 \\ -f & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix}.$$

By assumption, all solutions to $Y' = A(t)Y$ are bounded. Hence $\|Y(t)\|$ is bounded, and since $\mathrm{Tr}A(t) = 0$, $\|Y^{-1}(t)\|$ is bounded. Overestimating the right side of Equation (11), one obtains

$$\|z(t)\| \le \|Y(t)\| + \int_{\alpha}^{t} \|Y(t)\| \, \|Y^{-1}(s)\| \, \|B(s)\| \, \|z(s)\| ds. \tag{14}$$

Using the estimates derived above, one observes that

$$\|z(t)\| \le c_1 + c_2 \int_{\alpha}^{t} \|B(s)\| \, \|z(s)\| ds.$$

It follows from application of the Gronwall inequality that

$$\|z(t)\| \le c_1 \exp\left|\int_{\alpha}^{t} c_2 \|B(s)\| ds\right| = c_1 \exp\left[\int_{\alpha}^{t} c_2 |g(s)| ds\right].$$

By assumption $\int_{\alpha}^{\infty} |g(s)| ds$ is convergent. Therefore $\|z(t)\|$ is bounded above. Hence

$$\|z(t)\| = |z_1(t)| + |z_2(t)| = |y(t)| + |y'(t)| \le M$$

where y is a solution to (13). Thus, all solutions to (13) and their derivatives are also bounded.

The proof of the converse statement of the theorem follows immediately. $\square$

4.4 Corollary: If $p = a^2 + g$, a a positive constant,

$$\int_{\alpha}^{\infty} |g(t)|dt < +\infty,$$

then all solutions to $x'' + p(t)x = 0$ and their derivatives are bounded.

Proof: Solutions to the differential equation $x'' + a^2x = 0$ are of the form $x(t) = c_1 \sin at + c_2 \cos at$ for arbitrary constants c_1 and c_2. It follows immediately from Theorem 4.3 that all solutions to $x'' + (a^2+g(t))x = 0$ and their derivatives are also bounded. []

The next two examples show that the previous theorem and corollary are about the best results possible.

4.5 Example: If the constant a in the previous corollary is zero, then the corollary is not necessarily true. Consider

$$x'' + \frac{1}{8t^2}x = 0, \quad t \geq 1,$$

which has for solutions

$$x(t) = c_1 t^{\frac{2+\sqrt{2}}{4}} + c_2 t^{\frac{2-\sqrt{2}}{4}}$$

for some constants c_1 and c_2. Although

$$\int_1^{\infty} (8t)^{-2}dt < \infty, \quad \text{if } c_1^2 + c_2^2 \neq 0,$$

$x(t)$ becomes unbounded as $t \to +\infty$. Hence the requirement $a^2 > 0$ is necessary. Note that no nontrivial solution is bounded. However, p does not satisfy the hypotheses of Theorem 3.1 which guarantee that every solution of (1) is asymptotic to $c+bt$ with $c^2 + b^2 \neq 0$.

4.6 Example (Bellman [2]): The hypothesis

$$\int_{\alpha}^{\infty} |g(s)| ds < \infty$$

in Theorem 4.3 and Corollary 4.4 cannot be weakened to the requirement that $|g(t)| \to 0$ as $t \to +\infty$. The differential equation

$$x'' + \left(1 + \frac{4 \cos t \sin t}{t} + \frac{\cos^2 t \sin^2 t}{t^2}\right) x = 0, \quad t \geq 8$$

has an unbounded solution although p is positive on $[8, \infty)$, $\lim_{t \to +\infty} p(t) = 1$ and $\lim_{t \to +\infty} p'(t) = 0$. An unbounded solution is

$$x(t) = \exp\left[\int_8^t \frac{\cos^2(s)}{s} ds\right] \cdot \cos t.$$

Since

$$\int_8^{\infty} \left| \frac{4 \cos t \sin t}{t} + \frac{\cos^2 t \sin^2 t}{t^2} \right| dt$$

does not exist, neither the hypotheses of the theorem nor those of the corollary are satisfied.

In case $g(t)$ approaches a^2 monotonically and g' exists, however, the requirement $\int_{\alpha}^{\infty} |g(t)|dt < +\infty$ can be eliminated. The next lemma, which is a consequence of Theorem 3.3, will be used in proving these results.

4.7 Lemma: If $\int_{0}^{\infty} |q(b)|db < \infty$, then all solutions to $y'' + q(s)y' + y = 0$ and their derivatives are bounded on $[0,\infty)$.

Proof: Converting to a system by means of the change of variables $z_1 = y$, $z_2 = y'$, one obtains

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -q(s) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

All solutions to

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

are bounded, and since trace $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0$, then Equation (14) holds in this case. The rest of the proof is similar to that of Theorem 4.3. ▯

4.8 Theorem: If $p(t) \geq a > 0$ and $p'(t) \leq 0$ on $[\alpha,\infty)$, then all solutions to (1) and their derivatives are bounded on $[\alpha,\infty)$.

Proof: The substitution $y(s) = x(t)$,

$$s = \int_{\alpha}^{t} \sqrt{p(b)}\, db, \quad t \text{ in } [\alpha,\infty),$$

transforms (1) into

$$y'' + q(s)y' + y = 0 \tag{15}$$

where

$$q(s) = \frac{p'(t)}{2[p(t)]^{3/2}} \quad \text{and} \quad s = \int_{\alpha}^{t} \sqrt{p(b)}\, db.$$

Note that since $p(t) \geq a > 0$, the transformation is one-to-one and $\lim_{t\to+\infty} s(t) = +\infty$. Now,

$$\int_{0}^{\infty} |q(s)|ds = \int_{\alpha}^{\infty} \frac{-p'(k)}{2[p(k)]^{3/2}} \cdot \sqrt{p(k)}\, dk =$$

$$\lim_{k\to\infty} -\frac{1}{2}\left\{\ln p(k) - \ln\bigl(p(\alpha)\bigr)\right| = \text{Constant},$$

since $p(k)$ is bounded away from zero. Therefore all solutions to (15) and their derivatives are bounded on $[0,\infty)$. Hence

$$|y(s)| + |y'(s)| \leq M$$

for some constant M. But $y(s) = x(t)$ and $y'(s) = \dfrac{x'(t)}{\sqrt{p(t)}}$. Thus

$$|x(t)| \leq M \quad \text{and} \quad |x'(t)| \leq M\sqrt{p(t)} \leq M\sqrt{p(\alpha)} ,$$

hence all solutions to (1) and their derivatives are bounded on $[\alpha,\infty)$. □

4.9 Theorem: If p is positive and bounded above by some constant R on $[\alpha,\infty)$, and if p' is non-negative on $[\alpha,\infty)$, then all solutions to (1) and their derivatives are bounded on $[\alpha,\infty)$.

Proof: One makes the same change of variables as in Theorem 4.8 to obtain Equation (15). Since p' is nonnegative then $|q(s)| = \dfrac{p'(t)}{2p^{3/2}(t)}$. Hence,

$$\int_0^\infty |q(s)|ds = \lim_{k\to\infty}\left[\frac{1}{2}\ln p(k) - \frac{1}{2}\ln p(\alpha)\right] = \text{Constant},$$

since p is bounded above on $[\alpha,\infty)$. The rest of the proof follows exactly as that of Theorem 4.8. □

4.10 Example: Theorem 4.8 and Theorem 4.9 show that all solutions to $x'' + \left(4 + \frac{1}{t}\right)x$, $t\geq 1$, and $x'' + \left(4 - \frac{1}{t}\right)x = 0$, $t\geq 1$, and their derivatives are bounded on $[1,\infty)$. Note that this result is not obtainable from Theorem 4.3 since $\int_1^\infty \frac{ds}{s}$ diverges.

Theorems 4.8 and 4.9 can be generalized to obtain the following result.

4.11 Theorem: If $a \leq p(t) \leq M$ on $[\alpha,\infty)$ where a and M are positive constants, if p' exists on $[\alpha,\infty)$, and it

$$\int_\alpha^\infty \left|\frac{p'(s)}{p(s)}\right|ds$$

converges, then all solutions to (1) and their derivatives are bounded on $[\alpha,\infty)$.

Proof: Note that $\int_0^\infty |q(s)|ds = \int_\alpha^\infty \left|\frac{p'(k)}{p(k)}\right| dk$.

If $\int_0^\infty |q(s)|ds$ converges, then all solutions to (1) and their derivatives are bounded. ▯

The case where p(t) approaches infinity monotonically is now considered.

4.12 Theorem: If p is positive and approaches infinity monotonically, then all solutions to (1) are bounded.

Proof (Bellman [2]): Multiplying (1) by x' and integrating by parts, one obtains

$$\frac{[x'(t)]^2}{2} + p(t)\,\frac{[x(t)]^2}{2} = \frac{[x'(\alpha)]^2}{2} + \frac{p(\alpha)[x(\alpha)]^2}{2} + \int_\alpha^t \frac{[x(s)]^2}{2}\,d\,p(s).$$

Dropping nonnegative terms on the left results in the inequality

$$p(t)[x(t)]^2 \le C + \int_\alpha^t \frac{x^2(s)p(s)\;d\,p(s)}{[p(s)]}\,.$$

Application of the Gronwall inequality yields

$$p(t)[x(t)]^2 \le C \exp \int_\alpha^t \frac{d\,p(s)}{[p(s)]}$$

$$= C\,[p(t)-p(\alpha)] \le C\;p(t)$$

since $p(\alpha) > 0$. Hence $[x(t)]^2 \leq C$ and the result is obtained. $\square$

The Existence of Solutions Having Limit Zero as $t \to +\infty$

For the case where p becomes infinite as $t \to +\infty$, the problem of determining whether any or all nontrivial solutions to (1) have limit zero has been of current interest. For example, see [11], [12], [14], [15] and [16]. First, sufficient conditions for at least one nontrivial solution to tend toward zero as $t \to +\infty$ are shown.

4.13 Lemma (Lazer): If $p(t)>0$ and p' exists on $[\alpha,\infty)$, $p(t)$ approaches infinity as $t \to +\infty$, and if

$$\lim_{t \to +\infty} \int_{\alpha}^{t} \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds$$

exists, then there exists a nontrivial solution u to (1) such that $\lim_{t \to +\infty} u(t) = 0$.

Proof: Let

$$K(x(t)) = \left[\frac{x'(t)}{\sqrt{p(t)}}\right]^2 + [x(t)]^2. \tag{16}$$

Then

$$K'(x(t)) = -p'(t)\left[\frac{x(t)}{p(t)}\right]^2,$$

and integrating one obtains

$$K[x(t)] = K[x(\alpha)] - \int_{\alpha}^{t} \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds. \tag{17}$$

By hypothesis,

$$\lim_{t\to+\infty} \int_{\alpha}^{t} \left[\frac{x'(s)}{p(s)}\right]^2 \cdot p'(s)ds$$

exists, therefore $\lim_{t\to+\infty} K[x(t)]$ exists. Since $0 \le [x(t)]^2 \le K[x(t)]$, then we conclude immediately that all solutions to (1) are bounded. Let $V_1(t)$ and $V_2(t)$ be two linearly independent solution satisfying $V_1(\alpha)V_2'(\alpha) - V_2(\alpha)V_1'(\alpha) = 1$. Then by Abel's formula,

$$V_1(t)V_2'(t) - V_2(t)V_2'(t) \equiv 1 \quad \text{for } t\ge\alpha. \tag{18}$$

Suppose that $V_1(t)$ does not have limit zero as $t\to+\infty$. Since $p(t)$ approaches $+\infty$ as $t\to+\infty$, then all solutions are oscillatory so let $t_1 < t_2 < t_3 \cdots$ be the successive values at which $V_1(t)$ obtains its maximum value. Hence for each t_n, $V_1'(t_n) = 0$, $K[V_1(t_n)] = [V_1(t_n)]^2$, and $\lim_{n\to+\infty} t_n = +\infty$. Since $\lim_{t\to+\infty} K[V_1(t)]$ exists, and each $V_1(t_n) > 0$, then $\lim_{n\to+\infty} V_1(t_n)$ exists and, by assumption, is some positive number c. Let N be so large that for all positive integers $n\ge N$, then $V_1(t_n) > c/2$. But from (16) $V_1(t_n)V_2'(t_n) = 1$, hence $|V_2'(t_n)| \le c/2$ for all $n\ge N$. Now $V_2(t)$ is also bounded on $[\alpha,\infty)$ and by the Bolzano-Weierstrass Theorem there exists a subsequence $\{t_{nj}\}$ of the sequence $\{t_n\}$ such that $\{V_2(t_{nj})\}$ converges to a number b. It shall now be shown that the non-trivial solution $Y(t) = V_2(t) - (b/c)\, V_1(t)$ has limit zero. Then

$$\lim_{n_j\to\infty} Y(t_{nj}) = 0$$

and

$$|Y'(t_{nj})| = |V_2'(t_{nj})| \le 2/c.$$

Hence

$$0 \le K[Y(t_{nj})] \le \frac{4}{c^2 p(t_{nj})} + [Y(t_{nj})]^2 \quad \text{for } n_j \ge N.$$

Since $\lim_{t\to+\infty} p(t) = +\infty$, it follows that $\lim_{n_j\to\infty} K[Y(t_{nj})] = 0$. The existence of $\lim_{t\to\infty} K[Y(t)]$ was proved earlier in the lemma. Hence $\lim_{t\to\infty} K[Y(t)] = \lim_{n_j\to\infty} K[Y(t_{nj})] = 0$. Since $0 \le [Y(t)]^2 \le K[Y(t)]$, it follows that $\lim_{t\to+\infty} Y(t) = 0$. ▯

Of course, it is sufficient to check that $\lim_{t\to\infty} \int_\alpha^\infty \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds$ exists for two linearly independent solutions V_1 and V_2 since any other solution of (1) is a linear combination of V_1 and V_2. The lemma will not help if one cannot determine V' for two linearly independent solutions. Therefore a criterion is needed which will determine the answer from the nature of p alone.

4.14 <u>Theorem</u>: If p is positive and p' exists on $[\alpha,\infty)$, if $\lim_{t\to+\infty} p(t) = +\infty$, and if

$$\lim_{t\to+\infty} \int_\alpha^t \frac{|p'(s)| - p'(s)}{p(s)}\, ds$$

exists, then there exists at least one nontrivial solution to (1) with limit zero as $t\to+\infty$.

Proof (Lazer): From the previous lemma, it is sufficient to prove that

$$\lim_{t\to+\infty} \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds$$

exists for every solution x of (1).

Let

$$[p'(t)]^+ = \frac{|p'(t)| + p'(t)}{2}$$

and

$$[p'(t)]^- = \frac{|p'(t)| - p'(t)}{2}\ .$$

Now

$$\int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds = \tag{19}$$

$$\int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^+ds - \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^-ds.$$

If the limits of the two integrals on the right exist at $t\to+\infty$, then

$$\lim_{t\to+\infty} \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds$$

exists at $t\to+\infty$. If x is any solution to (1), then

$$0 \le \left[\frac{x'(t)}{\sqrt{p(t)}}\right]^2 + [x(t)]^2 \equiv K[x(t)].$$

Thus

$$0 \leq \left[\frac{x'(t)}{\sqrt{p(t)}}\right]^2 \leq K[x(t)] = K[x(\alpha)] - \tag{20}$$

$$- \int_\alpha^t \left(\frac{x'(s)}{p(s)}\right)^2 [p'(s)]^+ ds + \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^- ds.$$

The last equality above follows from Equation (17) in the previous lemma. Dropping the negative term on the right of (20), one obtains

$$\left[\frac{x'(t)}{\sqrt{p(t)}}\right]^2 \leq K[x(\alpha)] + \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^- ds.$$

Application of the Gronwall inequality yields

$$\frac{[x'(t)]^2}{p(t)} \leq K[x(\alpha)] \exp\left[\int_\alpha^t \frac{[p'(s)]^-}{p(s)} ds\right]$$

$$\leq K[x(\alpha)] \exp\left[\int_\alpha^\infty \frac{[p'(s)]^-}{p(s)} ds\right] .$$

By hypothesis the integral in the previous equation exists, hence $\frac{[x'(t)]^2}{p(t)}$ is bounded, say by M, on $[\alpha,\infty)$. Therefore

$$\int_\alpha^\infty \left[\frac{x'(t)}{p(t)}\right]^2 [p'(s)]^- ds \leq M \int_\alpha^\infty \frac{[p'(s)]^-}{p(s)}$$

which exists. But, from (20),

$$0 \leq \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^+ ds \leq K[x(\alpha)] + \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^- ds.$$

Since the limit on the right exists and the integrand $\left[\frac{x'(s)}{p(s)}\right]^2 \cdot [p'(s)]^+$ is positive,

$$\int_\alpha^\infty \left[\frac{x'(s)}{p(s)}\right]^2 [p'(s)]^+ ds$$

exists. Since the integrals on the right side of (19) have limits as $t \to +\infty$,

$$\lim_{t\to+\infty} \int_\alpha^t \left[\frac{x'(s)}{p(s)}\right]^2 p'(s)ds$$

exists, and the desired conclusion is reached. $\square$

An important consequence of the previous theorem is the following: If p is positive and p' exists on $[\alpha,\infty)$, and if $\lim_{t\to+\infty} p(t) = +\infty$, then $|p'(t)| - p'(t) \equiv 0$ and hence

$$\int_\alpha^\infty \frac{|p'(s)| - p'(s)}{p(s)} ds = 0.$$

Using Theorem 4.14, one can conclude that at least one nontrivial solution to (1) has limit zero.

4.15 <u>Corollary</u>: If p is positive on $[\alpha,\infty)$ and becomes infinite as $t \to +\infty$, and if p' exists and is nonnegative on $[\alpha,\infty)$, then at least one nontrivial solution to (1) has limit zero as $t \to +\infty$.

4.16 <u>Example</u>: There exists at least one nontrivial solution of $x'' + (\exp t)x = 0$ and of $x'' + t^n x = 0$, $n>0$, which has limit zero as

$t \to +\infty$. The boundedness of all solutions to the above two equations is a consequence of Theorem 4.12.

One might ask, "If p satisfied some condition in addition to those in Corollary 4.15, then would all solutions to (1) have limit zero as $t \to +\infty$?" The answer is "Yes," but the additional hypotheses on p are fairly stringent. However, the following theorem, due to Willett, indicates the reason for the hypotheses complexity.

4.17 Theorem: Let $b(t)$ be a given positive nondecreasing continuous function on $[\alpha, \infty)$. Then there exists a positive function $p(t)$ with a continuous derivative such that $p'(t) \geq b(t)$ and Equation (1) has at least one solution $x = u(t)$ such that

$$\limsup_{t \to +\infty} |u(t)| > 0.$$

Proof: See Willett [20].

The previous theorem shows that no restriction on the growth of $p(t)$ will alone be sufficient to guarantee that all solutions to (1) approach zero as $t \to +\infty$.

In 1936 Sansoné developed sufficient criterion for establishing that all solutions to (1) approach zero as $t \to +\infty$.

4.18 Theorem: Let $p(t)$ is a positive, nondecreasing function with continuous derivative and $\lim_{t \to +\infty} p(t) = +\infty$. If for every sequence $\{t_n\}$ satisfying the conditions

$$t_n < t_{n+1}, \; t_{n+1} - t_n \leq t_n - t_{n-1} \qquad n=1,2,\cdots,$$

$$\lim_{n\to+\infty} t_n = +\infty, \qquad \lim_{n\to+\infty} (t_{n+1}-t_n) = 0,$$

$$\limsup_{n\to+\infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = 1,$$

it happens that

$$\sum_{n=1}^{\infty} (t_{n+1}-t_n)\left[\min_{t_n\leq t\leq t_{n+1}} \frac{p'(t)}{p(t)}\right] = +\infty,$$

then all solutions to (1) have limit zero as $t\to+\infty$.

Proof: See Sansoné [18].

4.19 Example: Let $p(t) = \exp t$. Then for any sequence $\{t_n\}$ satisfying the above restrictions, $\left[\min_{t_n\leq t\leq t_{n+1}} \frac{p'(t)}{p(t)}\right] = 1$ since $p'(t) = p(t)$, and

$$\sum_{n=1}^{\infty} (t_{n+1}-t_n)\left[\min_{t_n\leq t\leq t_{n+1}} \frac{p'(t)}{p(t)}\right] = \sum_{n=1}^{\infty} (t_{n+1}-t_n) = +\infty \; .$$

Since $p(t) = \exp t$ satisfies the other hypotheses of the theorem, all solutions to $x'' + \exp t \; x = 0$ have limit zero as $t\to+\infty$.

It is usually difficult to verify that an arbitrary function satisfies the hypothesis of Theorem 4.19. A problem of much recent interest has been the development of hypotheses that are easier to verify. The most general result in this area at present is the following theorem and lemma due to Willett, Wong, and Meir.

4.20 Theorem: Let p be a positive, nondecreasing function with continuous derivative on $[\alpha,\infty)$ such that $\lim_{t\to+\infty} p(t) = +\infty$. If any one of the following criterion is satisfied, then all solutions to (1) have limit zero as $t\to+\infty$.

(1) There exists a positive, continuous a(t) on $[\alpha,\infty)$ such that

$$\int_{\alpha}^{\infty} a^{-1}(s)ds = +\infty,$$

$$\liminf_{t\to+\infty} \frac{a'(t)}{a(t)p^{1/2}(t)} \geq 0, \quad \text{and} \quad \liminf_{t\to+\infty} \frac{a'(t)p(t)}{a(t)} > 0$$

(2) $\liminf_{t\to+\infty} p'(t) > 0$ and $\int_{\alpha}^{\infty} p^{-1}(s)ds = +\infty$.

(3) $\liminf_{t\to+\infty} \frac{t\, p'(t) \ln t}{p(t)} > 0.$

(4) p has a continuous second derivative, $p' > 0$ on $[\alpha,\infty)$ and

$$\limsup_{t\to+\infty} p''(t) \leq 0.$$

4.21 Lemma: Let p be a positive nondecreasing function with continuous derivative on $[\alpha,\infty)$ such that $\lim_{t\to+\infty} p(t) = +\infty$. If there exist a positive, nondecreasing function q with a continuous second derivative on $[\alpha,\infty)$ such that $q(t) \to +\infty$ as $t\to+\infty$,

$$\lambda = \limsup_{t\to+\infty} \frac{1}{q(t)} \int_\alpha^t \frac{q''(s)ds}{p^{1/2}(s)} < \frac{1}{3} \tag{21}$$

and

$$\lim_{t\to+\infty} \frac{1}{q(t)} \int_\alpha^t \left[\max\left(2m\frac{q'(s)}{q(s)} - \frac{p'(s)}{p(s)}, 0\right)\right] \cdot q(s)ds = 0 \tag{22}$$

for some positive constant m, then all solutions to (1) have limit zero as $t\to+\infty$.

<u>Proof of the Lemma</u>: For any nontrivial solution x of (1), let

$$R(t) = x^2(t) + [x'(t) \cdot p^{-1/2}(t)]^2.$$

Note that $R'(t) = -\dfrac{[x'(t)]^2 p'(t)}{p^2(t)} \leq 0$. If R(t) has limit zero as $t\to+\infty$, then x(t) has limit zero as $t\to+\infty$. Suppose $R(t) \to S > 0$ as $t \to +\infty$. For every $e>0$, there exists a $t_e > \alpha$ such that for all $t > t_e$,

$$S + e \geq R(t) \geq S.$$

Suppose that (22) is true for some constant m, and let K be any constant in (0,m). Note that

$$0 \leq \max[2Kq'/q - p'/p, 0] \leq \max[2mq'/q - p'/p, 0]$$

and since q is positive,

$$0 \le \frac{1}{q(t)} \int_{\alpha}^{t} \max[2Kq'(s)/q(s) - p'(s)/p(s),0]q(s)ds \le$$

$$\le \frac{1}{q(t)} \int_{\alpha}^{t} \max[2mq'(s)/q(s) - p'(s)/p(s),0]q(s)ds.$$

Therefore

$$\lim_{t\to+\infty} \frac{1}{q(t)} \int_{\alpha}^{t} \max[2Kq'(s)/q(s) - p'(s)/p(s),0]q(s)ds = 0.$$

Thus (22) is true for all constants in (0,m). Hence, one can always assume that the constant m is in (0,1). A little algebra shows that

$$(qR)' = (1-m)q'R - mq'(xx')'p^{-1} + p^{-1}q(x')^2(2mq'/q-p'/p). \qquad (23)$$

Equation (23) is now integrated term by term with estimates made on each of the terms.

$$qR': \quad \int_{t_e}^{t} [q(s)R'(s)]'ds = q(t)R(t) - q(t_e) \cdot R(t_e) \ge$$

$$\ge q(t)S - q(t_e)R(t_e) \qquad ' = \frac{d}{ds}$$

$$(1-m)q'R: \quad \int_{t_e}^{t} (1-m)q'(s)R(s)ds \le (S+\varepsilon) \int_{t_e}^{t} (1-m)q'(s)ds =$$

$$= (S+\varepsilon)(1-m)[q(t)-q(t_e)].$$

$p^{-1}q(x')^2(2mq'/q-p'/p)$:

$$\int_{t_e}^{t} [x'(s)]^2 p^{-1}(s)\{2mq'(s)/q(s) - p'(s)/p(s)\}q(s)ds \le$$

$$\le (S+\varepsilon) \int_{t_e}^{t} \max[2mq'(s)/q(s) - p'(s)/p(s),0] \cdot q(s)ds$$

since $[x'(t)]^2 p^{-1}(t) \le R(t) \le (S+\varepsilon)$ for $t \ge t_e$.

$mq'[xx']'p^{-1}$: Integration by parts yields

$$\int_{t_e}^{t} q'(s)[x(s)x'(s)]'p^{-1}(s)ds = q'(s)p^{-1}(s)x(s)x'(s)\Big|_{t_e}^{t} \tag{24}$$

$$- \int_{t_e}^{t} q''(s)p^{-1}(s)x(s)x'(s)ds + \int_{t_e}^{t} q'(s)p'(s)p^{-2}(s)x(s)x'(s)ds.$$

Estimates are now made on the right side of the previous equation. Since $(x \pm x'p^{-1})^2 \ge 0$, it follows that

$$2|p^{-1}xx'| \le x^2 + p^{-1}(x')^2 \le S+\varepsilon \quad \text{for } t \ge t_e. \tag{25}$$

Because $p^{1/2}(t) \ge p^{1/2}(s)$ for all s in (t_e,t) and

$$\int_{t_e}^{t} |q''(s)|ds \ge q'(t) - q(t_e),$$

then

$$q'(t)p^{-1/2}(t) \le \int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds + O(1). \tag{26}$$

Thus

$$q'p^{-1}xx'\Big|_{t_e}^{t} \le \frac{(S+\varepsilon)}{2}\int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds + \underline{O}(1).$$

Also

$$\int_{t_e}^{t} q''(s)p^{-1}(s)x(s)x'(s)ds = \int_{t_e}^{t} p^{-1/2}(s)x(s)x'(s)q''(s)p^{-1/2}(s)\,ds \le$$

$$\le \left(\frac{S+\varepsilon}{2}\right)\int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds \quad \text{by (25).}$$

Now

$$\int_{t_e}^{t} q'(s)p'(s)p^{-2}(s)x(s)x'(s)ds \le \left(\frac{S+\varepsilon}{2}\right)\int_{t_e}^{t} q'(s)p'(s)p^{-3/2}(s)ds.$$

Integration by parts yields

$$\left(\frac{S+\varepsilon}{2}\right)\int_{t_e}^{t} q'(s)p'(s)p^{-3/2}(s)ds =$$

$$(S+\varepsilon)\left[q'(s)p^{-1/2}(s)\Big|_{t_e}^{t} - \int_{t_e}^{t} q''(s)p^{-1/2}(s)ds\right]$$

But by (26), it follows that

$$\left|\int_{t_e}^{t} q''(s)p'(s)p^{-1}(s)x'(s)x(s)ds\right| \le 2(S+\varepsilon)\int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds + O(1).$$

Combining the previous estimates on (24), one obtains

$$\left|\int_{t_e}^{t} mp^{-1}(s)[x(s)x'(s)]' ds \leq 3m(S+\varepsilon) \int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds + O(1).\right.$$

The estimates on the integrals of each of the terms of Equation (23) are now completed. One obtains

$$\begin{aligned} q(t)S - q(t)R(t_0) &\leq (S+\varepsilon)(1-m)[q(t)-q(t_0)] + \\ &+ (S+\varepsilon) \int_{t_e}^{t} \max[2mq'(s)/q(s)-p'(s)/p(s),0]q(s)ds + \\ &+ 3m(S+\varepsilon) \int_{t_e}^{t} |q''(s)|p^{-1/2}(s)ds + O(1). \end{aligned} \tag{27}$$

Dividing through by q and taking the limit as $t \to +\infty$, one finds with the use of (21) and (22) that

$$S \leq (S+\varepsilon)(1-m) + 0 + 3\lambda m(S+\varepsilon).$$

Solving for e, one obtains

$$e \geq \frac{(1-3\lambda)Sm}{1 - (1-3\lambda)Sm} > 0.$$

But e is arbitrary. Hence a contradiction of the fact that S is positive in obtained. □

Proof of the Theorem:

(1) Let $q(t) = 1 + \int_{\alpha}^{t} a^{-1}(s)ds$. It will be shown that q

satisfies the hypotheses of the lemma. From the hypotheses on a it follows easily that q is positive and becomes infinite as $t \to +\infty$. Further q' is nonnegative. If

$$\lim_{t \to +\infty} \int_{\alpha}^{t} \frac{|q''(s)|}{p^{1/2}(s)} ds$$

is finite, then (21) clearly holds. Suppose the integral does not exist. Then

$$\int_{\alpha}^{t} |q''(s)| p^{-1/2}(s) ds = \int_{\alpha}^{t} |a'(s)| p^{-1/2}(s) a^{-2}(s) ds =$$

$$\int_{\alpha}^{t} a'(s) a^{-2}(s) p^{-1/2}(s) ds + 2 \int_{\alpha}^{t} \max[-a'(s), 0] \cdot a^{-2}(s) p^{-1/2}(s) ds.$$

Integration by parts shows that

$$\int_{\alpha}^{t} |q''(s)| p^{-1/2}(s) ds \le$$

$$p^{-1/2}(\alpha) a^{-1}(\alpha) + 2 \int_{\alpha}^{t} \max[-a'(s), 0] a^{-2}(s) p^{-1/2}(s) ds.$$

Dividing by $q(t)$, one obtains

$$0 \le \frac{1}{q(t)} \int_{\alpha}^{t} |q''(s)| p^{-1/2}(s) ds \le$$

$$\le \frac{1}{q(t)} \left[p^{-1/2}(\alpha) a^{-1}(\alpha) + 2 \int_{\alpha}^{t} \max[-a'(s), 0] a^{-2}(s) p^{-1/2} ds \right].$$

Since $\int_\alpha^t |q''(s)|p^{-1/2}(s)ds$ is not finite, L'Hôpital's rule is valid for the last term of the previous expression. Therefore

$$0 \leq \lim_{t\to+\infty} \frac{2\int_\alpha^t \max[-a'(s),0] \cdot a^{-2}(s)p^{-1/2}(s)ds}{q(t)} \leq$$

$$\leq \limsup_{t\to+\infty} \frac{2\max[-a'(t),0]}{a(t)\ p(t)}$$

By hypothesis $\liminf_{t\to+\infty} \frac{a'(t)}{a(t)p(t)} \geq 0$, hence

$$\max\left[\lim_{t\to+\infty} \frac{-a'(t)}{a(t)\ p(t)}\ ,\ 0\right] = 0.$$

Therefore

$$\lim_{t\to+\infty} \frac{\int_\alpha^t |q''(s)|p^{-1/2}(s)ds}{q(t)} = 0$$

and (21) is verified with $\lambda = 0$. For any $m > 0$,

$$2mq'(t)/q(t) - p'(t)/p(t) =$$

$$q'(t)/q(t)\left[2m - p'(t)a(t)\left(1 + \int_\alpha^t a^{-1}(s)ds\right)/p(t)\right].$$

But since $\liminf_{t\to+\infty} p'(t)a(t)p^{-1}(t) > 0$ and since the integral becomes positively unbounded as $t\to+\infty$, then there is a $T>\alpha$, so that for all $t\geq T$,

$$[2mq'(t)/q(t) - p'(t)/p(t)] \leq 0,$$

Therefore (22) holds. Hence $q(t) = 1 + \int_{\alpha}^{t} a^{-1}(s)ds$ satisfies the hypothesis of the Lemma and condition (1) in the statement of the Theorem is valid.

(2) Let $a(t) = p(t)$. Then $a(t)$ satisfies the hypotheses of condition (1) and hence (2) is valid.

(3) Let $a(t) = (t+2) \ln (t+2)$. Then $a(t)$ satisfies the hypotheses of condition (1).

(4) Let $a(t) = \frac{p(t)}{p'(t)}$

Then

$$\liminf_{t \to +\infty} \frac{p'(t)}{p(t)a^{1/2}(t)} = \frac{p'(t)[1 - p(t)p''(t)]}{p^{3/2}(t)} \geq 0.$$

The other criteria of condition (1) are easily verified and hence condition (4) is valid. $\square$

4.22 Example: Let $p(t) = t^n$, $n>0$. Then all solutions to (1) have limit zero as $t \to +\infty$. Since $tp'(t)p^{-1}(t)\log t = n \log t$ and $\liminf_{t \to +\infty} n \log t > 0$, then $p(t)$ satisfies the hypotheses of the theorem and condition (3). Therefore all solutions have limit zero as $t \to +\infty$.

The previous theorem has given sufficient criteria to insure that all solutions to (1) have limit zero as $t \to +\infty$. These criteria are relatively easy to check in practice. Necessary conditions to insure that all solutions have limit zero do not seem to be known.

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