Acceleration Constraints in Modeling and Control of Nonholonomic Systems

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Acceleration Constraints in Modeling and Control of Nonholonomic Systems

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To my parents,

Dr. Hasan Bajodah and Dr. Fatimah Ramadan,

my wife and children,

Hibah Aboulfarage, Renad and Hasan Bajodah,

my sisters,

Ghada, Jehan, Nada, Dhoha.

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SUMMARY

Acceleration constraints are used to enhance modeling techniques for dynamical systems. In particular, Kane's equations of motion subjected to bilateral constraints, unilateral constraints, and servo-constraints are modified by utilizing acceleration constraints for the purpose of simplifying the equations and increasing their applicability.

The tangential properties of Kane's method provide relationships between the holonomic and the nonholonomic partial velocities, and hence allow one to describe nonholonomic generalized active and inertia forces in terms of their holonomic counterparts, i.e., those which correspond to the system without constraints. Therefore, based on the modeling process objectives, the holonomic and the nonholonomic vector entities in Kane's approach are used interchangeably to model holonomic and nonholonomic systems. When the holonomic partial velocities are used to model nonholonomic systems, the resulting models are full-order (also called nonminimal or unreduced) and separated in accelerations. As a consequence, they are readily integrable and can be used for generic system analysis. Other related topics are constraint forces, numerical stability of the nonminimal equations of motion, and numerical constraint stabilization.

Two types of unilateral constraints considered are impulsive and friction constraints. Impulsive constraints are modeled by means of a continuous-in-velocities and impulse-momentum approaches.

In controlled motion, the acceleration form of constraints is utilized with the Moore-Penrose generalized inverse of the corresponding constraint matrix to solve for the inverse dynamics of servo-constraints, and for the redundancy resolution of overactuated manipulators. If control variables are involved in the algebraic constraint equations, then these tools are used to modify the controlled equations of motion in order to facilitate control system design. An illustrative example of spacecraft stabilization is presented.

CHAPTER I

INTRODUCTION

1.1 Historical Perspective

The treatments of constraints in the field of dynamics have distinguished two traditional approaches to this science. These are the *Newtonian* approach, and the *analytical* approach.

The Newtonian approach (also called the vectorial approach) utilizes the laws of motion, discovered in the seventeenth and the eighteenth centuries [100, 39]. It is based on freeing the dynamical system from the influence of its constraints, and accounting for this influence by means of forces and moments called *constraint forces and moments*. The constraint forces are added vectorially to the forces applied to the system, and the sum is equated to the time rate of change of the linear momentum of the system. Similarly, the constraint moments and the moments of the constraint forces about the center of mass of the system are added vectorially to the applied moments and the moments of the applied forces about the center of mass of the system, and the sum is equated to the time rate of change of the angular momentum of the system about its center of mass. This explicit way of treating constraints renders this approach effective only to formulate equations of motion for dynamical systems with few or no constraints. Examples are six degrees of freedom aerospace vehicles.

The analytical approach (also called the Hamiltonian approach) is based on d'Alembert's extension of the principle of virtual displacements to dynamical problems [32]. This principle successfully formulates equations of motion for constrained dynamical systems, regardless of the nature of the constraint forces and moments. If the constraint forces and moments are ideal, i.e., workless on the system, then the net work done on the system in an arbitrarily chosen set of virtual displacements that comply with the constraints is solely the contribution of the remaining applied forces, in addition to the inertia forces. As a result, no consideration of constraint forces and moments is needed.

Utilizing the *virtual mechanical work* in d'Alembert's principle made it possible to use *kinetic energy* and *potential energy* of the dynamical system, in addition to some concepts in *variational calculus* to derive different types of equations of motion. These are Hamilton's equations [51], Lagrange's equations [85, 106], Maggi's equations [94, 102], and the Boltzmann-Hamel equations [50, 99].

Because of the disappearance of constraint forces from the formulations of the analytical approach, modeling dynamical systems that are more complicated in terms of the number of degrees of freedom and the number of constraints has become easier.

Despite the fact that the principle of virtual displacements has no limitations on the nature of constraints, the variational approach that is based on this principle has its limitations. Among these limitations are *nonholonomic constraints*, which continued to be a hurdle in the way of obtaining equations of motion for dynamical systems.

The reason for the difficulty in modeling systems with nonholonomic constraints is that among the trajectories that satisfy these constraints, the physical trajectories of the systems do not render the augmented Hamiltonian functions stationary [116]. This limitation is sometimes lacking awareness in the engineering communities [49, 109], and false formulations are frequently observed due to augmentations of nonholonomic constraints or nonintegrable kinematical relations with the Hamiltonians, e.g., [95, 112, 23, 48, 110].

Several attempts have been made to come up with generalizations of Hamilton's equations to include treatments of nonholonomic constraints. Some of these attempts yielded false extensions to some classes of nonholonomic constraints, e.g., [110], and some enhanced the knowledge of the problem by giving justifications for the difficulty, e.g., [116, 105]. However, the problem of general variational extension of Hamilton's equations to nonholonomic constraints appears to have been settled in the negative.

There are two ways in which nonholonomic constraints are treated in the analytical approach. The first way is using *Lagrange's multipliers* to adjoin the unconstrained differential equations with the algebraic constraint equations. The other way is *embedding* the constraint equations in the equations of motion of the unconstrained system, which is equivalent to eliminating the dependent virtual displacements in d'Alembert's principle, and reducing the number of equations of motion to the number of degrees of freedom. Both ways are disadvantageous, as discussed later in the chapter.

In the second half of the nineteenth century, Gibbs [43] and Appell [7] independently utilized Gauss' principle of least constraints [42] to derive their equations of motion, by minimizing the *acceleration energy* of the constrained dynamical system relative to the acceleration energy of its unconstrained counterpart, as stated by the principle. The advantage of using Gauss' principle to derive equations of motion is that the principle can be cast in a differential form, which avoids the difficulty of nonholonomicity that is related to the Hamiltonian variational approach. Other features of these equations, named later the Gibbs-Appell equations of motion, are using a differential form of the constraint equations, at the acceleration level, and using *quasi-coordinates*, which are linear combinations of the time rates of change of the generalized coordinates. Careful choice of the quasi-coordinates can remarkably reduce the complexity of the equations of motion.

Interestingly enough, both Gauss' principle and the Gibbs-Appell equations of motion have received very little attention, and except for rare mentioning in some books, they were almost forgotten. This remained the situation until very recently, when Udwadia and Kalaba derived their *generalized inverse equations of motion* from Gauss' principle [125, 126], and showed it to be a possible starting point for deriving the major equations that describe constrained motion [124]. The acceleration form of the constraint equations was used in deriving the equations, together with the Moore-Penrose generalized inverse of the constraint matrix. The Udwadia-Kalaba equations of motion added another feature beyond the Gibbs-Appell equations, which is the *irreduced motion space*. That is, the number of the acceleration variables in the set of equations is for the first time equal to the number of generalized coordinates, not to the number of degrees of freedom; and this is accomplished without employing Lagrange multipliers.

The Gibbs-Appell and the Udwadia-Kalaba equations of motion have set a different trend in analytical dynamics, not only because they descend from a different source than d'Alembert's principle, but also because of the way nonholonomic constraints are dealt with. The acceleration form of constraints creates a unifying framework for modeling holonomic and nonholonomic systems, and the appearance of the acceleration variables in the constraint equations in addition to the unconstrained equations of motion creates a mathematical conformity between the two sets of equations.

During the twentieth century, analytical dynamics had a significant influence on almost all disciplines of engineering and physics. Among these disciplines is control systems, which takes from analytical dynamics many of its concepts and methodologies, including variational and energy schemes, as well as the treatments of constraints, which appear as restrictions on system dynamics and/or available control authorities, and as requirements that have to be satisfied.

In control systems, nonholonomic constraints cause the dynamical system to go outside the capability of many of the methods used in the subject, like smooth stabilization [21] and feedback linearization [60], even if the dynamical system is completely controllable.

1.2 Kane's Equations of Motion

The need to formulate mathematical models for the purpose of dynamical system analysis, design, and control has brought to existence many modeling methodologies during the past three centuries. With the continuous increase in complexity of dynamical systems in industry, such as mechatronic systems, flexible multibodied structures, and aerospace vehicles, there is a growing need to enhance the capabilities of modeling techniques for such systems.

In classical dynamics, the modeling process starts by applying one of the physical principles, namely Newton's laws, d'Alembert's principle, or one of the derivatives of d'Alembert's principle. Then the process relies on available mathematical tools to cast the equations into the simplest useful form. This renders the modeling process to be dependent on both the application as well as the analyst's ability.

The advancement in digital computers has motivated the algorithmic approach of the modeling process. One of the key developments in this arena is the approach popularly referred to as "Kane's method" and the associated "Kane's equations." The major framework was first published in 1965 [72].

Kane's method combines the advantages of the two main approaches to the subject of dynamics. It has the simplicity of the Newtonian approach, because Kane's equations are actually force-moment balance equations. On the other hand, the *generalized active forces* and *generalized inertia forces* are obtained by scalar (dot) multiplications of the active and inertia forces, respectively, with vector entities that can be obtained for particles and bodies in the system, equal in number to the number of degrees of freedom, called *partial angular velocities* and *partial velocities*. This process delicately eliminates the contribution of constraint forces, provided that they are ideal, i.e., orthogonal to the constraint manifold. Kane's method shares this useful property with the analytical approach. For the purpose of bringing these forces into evidence, fictitious degrees of freedom that violate the constraints, may be introduced [68].

The equations resulting from Kane's approach are simple and effective in describing the motion of nonconservative and nonholonomic systems within the same framework, requiring neither energy methods nor Lagrange multipliers. Furthermore, the partial velocities adopted in Kane's method inspire useful geometric features of the constrained motion [91].

Kane's method implements the concept of *generalized speeds*, quasi-velocity coordinates as a way to represent motion, similar to what the concept of *generalized coordinates* does for the configuration. Generalized speeds are nonlinear combinations of the rates of changes of generalized coordinates, and the relation between the two sets of variables is chosen to be reversible for all permissible configurations. The implementation of generalized speeds allows one to focus on the motion aspects of dynamical systems rather than only on the configuration [68]. Therefore, it provides a suitable framework for treating nonholonomic constraints. Generalized speeds provide the formulation process with a desirable flexibility because they can be chosen to satisfy the needs and interests of the designer. The choice of the generalized speeds is crucial, for they significantly affect the simplicity of the resulting equations of motion [96]. Historically, it should be recognized that the use of generalized speeds goes back at least to the Gibbs-Appell equations [43, 7].

The similarities and differences between Kane's method and other well-established modeling methodologies have been the subject of interest and debate. One of the criticisms that the originality of Kane's method is subjected to is its similarity with the standard application of d'Alembert's principle for deriving dynamical equations of motion. This similarity is not in doubt. Actually, every modeling technique for dynamical systems subjected to ideal constraints has to use or to comply with d'Alembert's principle.

However, Kane's method bypasses several steps that are needed when directly using d'Alembert's principle. Expressions for virtual displacements can be lengthy for complex systems, especially if the displacements are represented in terms of moving reference frames. In this case, the variations of the unit vectors of the moving frame are needed also [13]. This difficulty can be eased by constructing virtual displacements using velocities expressions instead, because the time is held constant when forming virtual displacements. This is called the *kinematical Jourdanian approach*, versus the *analytical d'Alembertian approach* for forming virtual displacements [44]. In Kane's method, partial velocities are found trivially by inspecting the expressions of velocities for the coefficients of the generalized speeds.

The need to express dependent virtual displacements in terms of independent virtual displacements in d'Alembert's principle is waived in Kane's method, because the number of partial velocities of a particle in the dynamical system is equal to the number of degrees of freedom of the system. Also, it is difficult, if possible at all, to use quasi-coordinates with d'Alembert's principle. Using generalized speeds (general form of quasi-coordinates) is one of the main features of Kane's method, and a source of its power when using computers to form equations of motion and obtaining system response.

Another source of criticism is the similarity of Kane's equations with the Gibbs-Appell equations [33, 64, 34, 92, 78, 113, 10, 56, 35, 36, 120, 55]. The final forms of Kane's equations and the Gibbs-Appell equations are identical for the same choices of quasi-coordinates. However, the derivations of the two sets of equations are entirely distinct.

Deriving the Gibbs-Appell equations requires forming the Gibbsian, a function that is quadratic in the quasi-accelerations. These quasi-accelerations are the time derivatives of a subset of the quasi-velocities that is equal in number to the number of degrees of freedom of the system, and are formed by eliminating the remaining quasi-velocities with the aid of the constraint equations. The generalized inertia forces are obtained by taking the partial derivatives of the Gibbsian with respect to the quasi-accelerations. This partial differentiation suffers from the curse of dimensionality, and hence the procedure loses efficiency for large systems [13]. If the applied forces on the system are not from potential sources, then the principle of virtual displacements is needed to obtain the generalized active forces [93].

On the other hand, Kane's approach does not require differentiations of a scalar function or using the principle of virtual displacements. It is desirable for large multibodied systems because the scalar multiplications performed to obtain the generalized inertia and active forces are computationally efficient, and the procedure for deriving partial velocities can be mechanized and made suitable for computer implementation [13].

Kane's equations have been applied to the formulation of explicit equations of motion for complex flexible structures [65], as well as to formulate computationally efficient equations of motion in the area of robotics [66, 67]. Among the recent applications of Kane's method are the formulations of highly specialized computer-based methodologies for modeling and simulation of multi-rigid and flexible-body constrained systems [115, 114, 2, 3, 4, 5], and the structural dynamic analyses of these systems [89].

1.3 Acceleration Constraints

Acceleration constraints can be defined as *acceleration-involved quantities that are being conserved as the dynamical system evolves in time.* In a sense, kinetic equations of motion are acceleration constraints. Nevertheless, this definition will be taken to exclude quantities that contain control forces.

Acceleration constraints can be classified into the following categories:

1. *Holonomic Constraints*: geometrical and integrable kinematical constraints expressed at the acceleration level by taking the time derivatives of the constraint equations. Examples are configuration and integrable velocity relations between the particles and bodies comprising the dynamical system;

2. *Nonholonomic Constraints*: nonintegrable kinematical and dynamical constraints on the motion of the dynamical system. There are two types of nonholonomic acceleration constraints:

   (a) *first-order nonholonomic constraints*: kinematical nonholonomic constraints expressed at the acceleration level as the time derivatives of nonintegrable velocity relations between the system components, examples of which are energy and momentum integrals;

   (b) *second-order nonholonomic constraints*: nonholonomic dynamical constraints on the accelerations of the system components due to the difference between the number of degrees of freedom of the dynamical system and the number of independent control forces. Examples are the acceleration-involved relations that constrain the dynamics of underactuated systems. That is, if the dynamical equations of motion of a system that has p degrees of freedom are

$$\dot{x} = f(x) + g(x)\tau, \tag{1}$$

where $x \in \mathbb{R}^p$ is a column matrix containing the velocity variables, $\tau \in \mathbb{R}^l$ is a column matrix containing the control variables, $f \in \mathbb{R}^p$, $g \in \mathbb{R}^{p\times l}$, then a matrix $g^c(x) \in \mathbb{R}^{p\times(p-l)}$ that is an orthogonal complement of $g(x)$ can be defined such that $g^{cT}g = 0^{(p-l)\times l}$. Pre-multiplying the above equations by $g^c(x)$ yields the acceleration constraint equations

$$g^{cT}(x)[\dot{x} - f(x)] = 0, \tag{2}$$

which contain the acceleration variables $\dot{x}$, and are generally nonintegrable.

The number of degrees of freedom of the dynamical system may not be affected by energy integrals, momentum integrals, and second-order nonholonomic constraints, in contrast to all other types of constraints.

Constraints that change the number of degrees of freedom of the system can be classified according to the nature of the constraint forces as *passive* constraints or *servo-constraints*. Passive constraints are caused by the interaction of the dynamical system with its environment, while servo-constraints are enforced by control forces. The main difference between the two types is that the constraint forces of the first type are mostly ideal, while those of the second type are not ideal in general. In this work, passive constraints are considered until the end of the sixth chapter. The seventh chapter is concerned with servo-constraints, and the eighth chapter is concerned with constraints of both types, that involve control variables.

In this work, *nonholonomic control systems* are referred to controlled dynamical systems that involve nonholonomicity in the sense of servo-constraints, i.e., tracking functions that the dynamics of the (possibly holonomic) system is required to follow or to avoid. Nonholonomicity that results from underactuation (i.e., when the number of degrees of freedom to be controlled exceeds the number of independent actuators) is not considered.

Historically there has been little mention of systematic use of the acceleration form in modeling dynamical systems. One reason is perhaps its lack of a "physical" interpretation.

The general understanding was that most, if not all, physical constraints are either in the zeroth-order (i.e., finite) form or first-order form.

Among the exceptions, Ref. [106] adopted the acceleration form (which was called the third form of the fundamental equation) and explored a number of important applications. By suggesting the possibility of a large acceleration change, one is able to analyze problems in which the acceleration is discontinuous (such as a ball rolling off a table). In Ref. [106], the third form was also used to prove Gauss' principle [42].

More recently, the acceleration form of constraints was utilized in the methods of coordinate partitioning [134, 119] and undetermined multipliers [132, 57] for constrained dynamical systems. Both methods lead to elimination of the Lagrange multipliers.

Eliminating Lagrange's multipliers is advantageous. Beside the increase of dimensionality, when using Lagrange multipliers one runs into the difficulties of controlling differential algebraic equations and the costly process of solving for and/or controlling the multipliers. Nevertheless, many authors used Lagrange's multipliers to include the effect of constraints. For example, these variables were used in the control of nonholonomic systems [123], the solution of inverse dynamics problems [45, 46], the dynamic analysis of flexible structures [1, 137], and in conjunction with the finite element method for modeling flexible joints in multibody systems [16].

The equations obtained by the methods of coordinate partitioning and undetermined multipliers, together with the acceleration form of constraints, constitute two sets of differential equations. These equations can be integrated simultaneously. They can also be reduced to one set of differential equations in the independent accelerations by using the constraint equations to eliminate the dependent velocities and accelerations. However, the two sets of differential equations do not constitute, with the kinematical differential equations, a separated-in-accelerations, state-space model description of the dynamical system of the form $\dot{x} = f(x, t)$, because more than one acceleration term appears in the same equation. Therefore, this form cannot make use of certain techniques for studying the generic features of dynamical systems, like stability, chaos, bifurcation, etc. The reduced set is not useful when information about the dependent velocities and accelerations are needed.

Further work in adopting the acceleration form of constraint equations can be found in, e.g., Refs. [29, 127, 124, 61]. By taking advantage of the mathematical conformity of the acceleration form of constraints with the dynamical equations, explicit expressions for constraint forces were derived without any need to appeal to the free-body approach. This is particularly important in Lagrange's mechanics formulations when the active forces are dependent on the constraint forces, as it is in the case of friction forces [30].

1.4 Motivation

The purpose of this work is to make use of acceleration constraints and the mathematical conformity that they exhibit with dynamical equations, as tools of modeling, analysis, and synthesis of nonholonomic control systems. Kane's method is the framework chosen for that reason.

There are several reasons for choosing Kane's method. One reason is that it yields the same set of equations obtained by the approach of Gibbs and Appell. This set of equations is considered by many to be the most effective in treating nonholonomic constraints, and "the pinnacle of our understanding of the time evolution of constrained mechanical systems" [126]. In 1965, it was described by Pars as "probably the simplest and most comprehensive equations of motion so far discovered" [106]. With the additional advantages of Kane's approach in the intermediate derivations, it is not an exaggeration or advocation to describe Kane's method as the simplest and most practical in the history of man trying to model constrained motion.

Although not derived from an energy principle, the relation between Kane's equations and the Gibbs-Appell equations motivates a search for a possible adaptability with acceleration energy and acceleration constraints to modify the set of equations for the purpose of extending its applications.

Another reason for employing Kane's approach is its ease of implementation on digital computers, which made it the material of many computational procedures, and the main contribution to the field of dynamics and its applications in the second half of the twentieth century.

Kane's equations are equal in number to the number of degrees of freedom of the system. They can be put in full- or reduced-order form in terms of the dimension of the space of generalized speeds.

These equations, together with the kinematical differential equations and the constraint equations, can be utilized in dynamical system analysis and control system design in two ways. One way is with the aid of differential-algebraic equations analysis and control techniques. This may cause difficulties, since most of the available time-domain techniques related to these subjects are based on state-space models that are separated in the derivatives of the states (position and velocity variables). In this work, the acceleration form of constraint equations is utilized to resolve this difficulty.

The other way is to leave the equations in the reduced form and use the constraint equations to eliminate the dependent velocities from the kinematical differential equations. This is actually desirable because the resulting equations are free from the constraint drift problem that results from numerical integration. Nevertheless, there are applications under which the appearance of all acceleration terms is important. For example, stability deduced from the reduced form cannot guarantee a stable behavior of the dependent velocity variables.

For those reasons, it is highly desirable to obtain a mathematical model that is full-order, separated in the accelerations, and involves no Lagrange multipliers. This implies obtaining a non-deficient "constrained" inertia matrix that yields (upon inversion) an explicit description of each of the accelerations in terms of only the configuration and kinematic variables, and possibly time. The first step in this direction was the Udwadia-Kalaba equations [125, 126].

The derivation of Udwadia-Kalaba equations utilizes the acceleration form of the constraint equations, together with the generalized Moore-Penrose inverse of a scaled constraint matrix, and carries several desirable features. It presents a unified treatment of holonomic and nonholonomic constraints by using the acceleration form of constraints, and produces the concept of *nonminimal nonholonomic form.*

Similar to the Gibbs-Appell equations, the Udwadia-Kalaba equations were shown to be derivable from Gauss' principle [124]. This raises a question on a hidden relationship between Kane's equations and acceleration constraints.

The disadvantage of the Udwadia-Kalaba approach is that the Moore-Penrose generalized inverse is generally a discontinuous function in the matrix elements. This is a concern in regard to the existence and uniqueness of solutions to the resulting equations of motion [79], and was highlighted in the otherwise objectionable criticizing article by Bucy [23]. A special attention is given to this issue in the derivation of the nonminimal nonholonomic form presented in this work.

The present work takes the advantage of the acceleration form of the constraint equations together with the tangential properties of Kane's method to derive a version of Kane's equations for linear and nonlinear nonholonomic systems that is both full-order and separated in the derivatives of the generalized speeds, and extend the treatment to other types of constraints, namely impulsive constraints, friction constraints, and servo-constraints.

1.5 Overview

In the next two chapters, versions of Kane's equations for linear and nonlinear nonholonomic systems that are both full-order and separated in the derivatives of the generalized speeds are derived. A square matrix inversion that is needed in the derivations is shown always possible for all configurations and velocities that satisfy the constraints and all choices of generalized speeds, except for certain configurations of systems that involve toggle positions.

It is shown in the third chapter that manipulating linear nonholonomic constraint equations to yield nonlinear nonholonomic constraint equations substantially alters the constraint violation dynamics, and can reduce the deterioration in accuracy of the numerical simulations.

Another numerical issue is the constraint violation dynamics caused by the integration errors due to enforcing a differentiated form of the constraint equations. Although the numerical solution of the resulting equations of motion is not difficult, the propagated errors resulting from integrating the accelerations causes the displacements and velocities to violate the constraint equations, which makes it necessary to come up with a way to suppress this violation. The nonminimal equations are modified in the fourth chapter to suppress the resulting constraint drift, by augmenting Baumgarte stabilization terms with the constraint equations.

Two approaches for modeling impulsive constraints are employed in this work. The first approach is called the *continuous in velocities* approach, and is aimed at predicting the kinematics of the colliding bodies during the impact time (which is not assumed ignorable) and estimating the resulting impulsive constraint forces in terms of their relative displacements and velocities. For that purpose, certain coefficients that represent the material compliance and damping of the colliding bodies are needed. Some works assume that these coefficients are known constants of the materials. Examples are Refs. [37, 38, 131, 90]. Others use kinetic-elastic energy relations to calculate these coefficients. Examples are [54, 82, 86].

The continuous in velocities approach is used in the fifth chapter to model friction constraints also. Friction and impulsive constraints are the main types of *unilateral constraints*.

The other approach for modeling impulsive constraints is called the *impulse-momentum* approach. This approach was adopted in Kane's approach in Refs. [73, 26] for both holonomic and nonholonomic constraints, but was not until recently applied to multibody impulsive motion using Kane's method [27]. It was also applied to different modeling methodologies. Examples are the method of coordinate partitioning [135, 52] and Hamilton's equations of motion [87].

The basic assumption in the impulse-momentum approach is that the duration of the impact is very short compared to the time interval of the motion, such that the impact can be considered a discrete event, and the change in the configuration of the system during the impact is ignorable, although the changes in the velocities of the system components can be significant. This allows for converting the differential equations that govern the dynamics of the system to algebraic equations, through integrating the equations in general terms over the infinitesimal period of impact. The relationship between the velocities prior to and after the impact are given by an experimentally evaluated constant that is dependent on the material and the geometry of the collided surfaces, called the coefficient of restitution [47]. The sixth chapter uses the impulse-momentum approach for modeling impulsive constraints.

The first approach for modeling impulsive constraints is focusing on the short, but important period of time in which the constraint impulsive forces act. The second approach is suitable for studying the general behavior of dynamical systems encountering impulsive motion, without dealing with the details of impact. Both ways for analyzing impulsive motion are employed in the framework of the nonminimal nonholonomic form, whether the constraints apply only over an instant or over a finite time interval.

The seventh and the eighth chapters in the thesis are concerned with controlled motion. The nonminimal form of the equations of motion is utilized in the seventh chapter to solve for the control forces that are required to enforce servo-constraints, and to obtain the ideal form of these forces. This is used to solve the redundancy problem in overactuated systems (i.e., systems with number of control actuators exceeding number of controlled degrees of freedom), when the optimization criterion is the acceleration energy of the system, as stated by Gauss' principle. The acceleration form of constraints is used in the eighth chapter together with the Moore-Penrose generalized inverse to obtain simplified mathematical models that facilitate control design for dynamical systems subjected to passive constraints and servo-constraints, when the constraint equations involve control variables. For the later type, an illustrative example of spacecraft stabilization is presented, where the servo-constraint equation is a Lyapunov equation governing the desired decay in kinetic energy of the spacecraft.

The thesis is concluded with the ninth chapter, which discusses the main results, and suggests further ideas to proceed with.

CHAPTER II

NONMINIMAL KANE'S EQUATIONS OF MOTION FOR SIMPLE NONHOLONOMIC SYSTEMS

2.1 Introduction

During the past three decades, Kane's equations of motion were successfully applied to multibody systems numerical analysis and simulation. The original treatment of using the minimal (reduced) set of equations [67] becomes less useful when the multibody system is composed of a large number of bodies that are heavily constrained, as the resulting equations of motion increase in complexity, and hence become more difficult to analyze and less efficient for time simulations. To alleviate this problem, dynamical equations with orders exceeding the numbers of degrees of freedom were derived. Examples are Refs. [132, 2, 114, 11, 6].

The purpose of this chapter is to use the tangential properties of Kane's method together with the acceleration form of constraint equations to derive a version of Kane's equations for simple nonholonomic systems that is both full-order and separated in the derivatives of the generalized speeds. This implies obtaining a non-deficient "constrained" inertia matrix that yields (upon inversion) an explicit description of each of the accelerations in terms of only the configuration and kinematic variables, and possibly time.

2.2 Kane's Equations of Motion

Consider a p degrees of freedom nonholonomic dynamical system $\mathcal{S}$ consisting of a set of ν particles and μ rigid bodies, and let $\mathcal{R}$ be an inertial frame of reference in which the configuration of the system is described by a set of n *generalized coordinates* $q_1, \ldots, q_n$.

The velocity of a generic particle P of this nonholonomic system relative to $\mathcal{R}$ can be written as

$$^{\mathcal{R}}\mathbf{v}^P = \sum_{r=1}^{n} {^{\mathcal{R}}\mathbf{v}_r^P}(q,t)u_r + {^{\mathcal{R}}\mathbf{v}_t^P}(q,t), \tag{3}$$

where the *generalized speeds* $u_1 \ldots u_n$ are scalar variables satisfying some nonholonomic constraint relations, as discussed later in the chapter. They also satisfy the *kinematical differential equations*

$$\dot{q} = C(q,t)u + D(q,t). \tag{4}$$

In the above equations, q denotes a column matrix containing the n generalized coordinates, u denotes a column matrix containing the generalized speeds, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^n$, C^{-1} exists for all $q \in \mathbb{R}^n$, and all $t \in \mathbb{R}$, and $(\dot{\ }) = d(\,)/dt$. The *holonomic partial velocities* ${^{\mathcal{R}}\mathbf{v}_1^P} \ldots {^{\mathcal{R}}\mathbf{v}_n^P}$ in Eq. (3) are vector entities that can be obtained by inspecting the velocity expression of the particle for the coefficients of the generalized speeds. In a similar manner, the angular velocity of a generic body B of the system relative to $\mathcal{R}$ may be written as

$$^{\mathcal{R}}\boldsymbol{\omega}^B = \sum_{r=1}^{n} {^{\mathcal{R}}\boldsymbol{\omega}_r^B}(q,t)u_r + {^{\mathcal{R}}\boldsymbol{\omega}_t^B}(q,t), \tag{5}$$

where the vector entities ${^{\mathcal{R}}\boldsymbol{\omega}_1^B} \ldots {^{\mathcal{R}}\boldsymbol{\omega}_n^B}$ are named the *holonomic partial angular velocities* of the body, and can be obtained by inspecting the expression of the angular velocity of the body for the coefficients of the generalized speeds.

Remark. *A simple choice of generalized speeds is $u = \dot{q}$, obtained by setting C(q,t) to be the identity matrix $I_{n\times n}$, and D(q,t) a column matrix with n zero elements. For this choice of generalized speeds, the final form of Kane's equations is reduced to Lagrange's equations if equations are formed in full symbolic form. This choice does not usually yield the simplest form of the equations of motion.*

Let $\mathbf{R}_i$ be the resultant active force on the i^{th} particle, P_i. The resultant active forces on the i^{th} rigid body B_i are equivalent to a force $\mathbf{Z}_i$ on the center of mass of B_i, denoted by b_i, together with a torque $\mathbf{T}_i$ evaluated about point b_i. The *holonomic generalized active force*

F_r is [68]

$$\begin{aligned} F_r(q,u,t) &= \sum_{i=1}^{\nu}(F_r)_{P_i} + \sum_{i=1}^{\mu}(F_r)_{B_i} \\ &= \sum_{i=1}^{\nu} {}^{\mathcal{R}}\mathbf{v}_r^{P_i} \cdot \mathbf{R}_i + \sum_{i=1}^{\mu} {}^{\mathcal{R}}\mathbf{v}_r^{b_i} \cdot \mathbf{Z}_i + \sum_{i=1}^{\mu} {}^{\mathcal{R}}\boldsymbol{\omega}_r^{B_i} \cdot \mathbf{T}_i, \qquad r = 1,\ldots,n. \quad (6) \end{aligned}$$

Also, let m_{P_i} and m_{B_i} denote the masses of P_i and B_i, respectively, and let ${}^{\mathcal{R}}\mathbf{a}^{P_i}$, ${}^{\mathcal{R}}\mathbf{a}^{b_i}$, and ${}^{\mathcal{R}}\boldsymbol{\alpha}^{B_i}$ denote the acceleration of P_i, the acceleration of the center of mass of B_i, and the angular acceleration of B_i, relative to $\mathcal{R}$, respectively. The *central angular momentum of B_i in $\mathcal{R}$*, ${}^{\mathcal{R}}\mathbf{H}^{B_i}$, is

$$^{\mathcal{R}}\mathbf{H}^{B_i} = \underline{\mathbf{I}}^{B_i} \cdot {}^{\mathcal{R}}\boldsymbol{\omega}^{B_i}, \qquad (7)$$

where $\underline{\mathbf{I}}^{B_i}$ is the *central inertia dyadic of B_i relative to b_i* [68]. The inertia torque of B_i relative to $\mathcal{R}$ is [68]

$$\begin{aligned} {}^{\mathcal{R}}\mathbf{T}^{\star}_{B_i} &= -\frac{{}^{\mathcal{R}}d\,{}^{\mathcal{R}}\mathbf{H}^{B_i}}{dt} = -\underline{\mathbf{I}}^{B_i} \cdot \frac{{}^{\mathcal{R}}d\,{}^{\mathcal{R}}\boldsymbol{\omega}^{B_i}}{dt} - {}^{\mathcal{R}}\boldsymbol{\omega}^{B_i} \times {}^{\mathcal{R}}\mathbf{H}^{B_i}, \qquad (8) \\ &= -{}^{\mathcal{R}}\boldsymbol{\alpha}^{B_i} \cdot \underline{\mathbf{I}}^{B_i} - {}^{\mathcal{R}}\boldsymbol{\omega}^{B_i} \times \underline{\mathbf{I}}^{B_i} \cdot {}^{\mathcal{R}}\boldsymbol{\omega}^{B_i}. \qquad (9) \end{aligned}$$

The *holonomic generalized inertia force* $F_r^{\star}$ is [68]

$$\begin{aligned} F_r^{\star}(q,u,\dot{u},t) &= \sum_{i=1}^{\nu}(F_r^{\star})_{P_i} + \sum_{i=1}^{\mu}(F_r^{\star})_{B_i} \\ &= -\sum_{i=1}^{\nu} {}^{\mathcal{R}}\mathbf{v}_r^{P_i} \cdot m_{P_i}{}^{\mathcal{R}}\mathbf{a}^{P_i} - \sum_{i=1}^{\mu} {}^{\mathcal{R}}\mathbf{v}_r^{B_i} \cdot m_{B_i}{}^{\mathcal{R}}\mathbf{a}^{b_i} - \sum_{i=1}^{\mu} {}^{\mathcal{R}}\boldsymbol{\omega}_r^{B_i} \cdot {}^{\mathcal{R}}\mathbf{T}^{\star}_{B_i}, \\ & \qquad r = 1,\ldots,n. \quad (10) \end{aligned}$$

Remark. *Expanding the velocities and the angular velocities of the nonholonomic system $\mathcal{S}$ components in terms of the n generalized speeds allows to define quantities that are related to the corresponding holonomic system, i.e. the system obtained by removing the nonholonomic constraints. This is crucial for the present development, as it permits to construct the equations of motion for the nonholonomic system from the equations of motion of its holonomic counterpart.*

Other ways to write the velocities and angular velocities are in terms of the minimal set of generalized speeds, $u_1 \dots u_p$. The velocity of P relative to $\mathcal{R}$ can be written as

$$ {}^{\mathcal{R}}\mathbf{v}^P = \sum_{r=1}^{p} {}^{\mathcal{R}}\tilde{\mathbf{v}}_r^P(q,t)u_r + {}^{\mathcal{R}}\tilde{\mathbf{v}}_t^P(q,t) \, , \quad r = 1, \dots, p, \tag{11} $$

and the angular velocity of B relative to $\mathcal{R}$ can be written as

$$ {}^{\mathcal{R}}\omega^B = \sum_{r=1}^{p} {}^{\mathcal{R}}\tilde{\omega}_r^B(q,t)u_r + {}^{\mathcal{R}}\tilde{\omega}_t^B(q,t) \, , \quad r = 1, \dots, p. \tag{12} $$

The coefficients of the generalized speeds in Eqs. (11) and (12) are called the *nonholonomic partial velocities* of P relative to $\mathcal{R}$ and the *nonholonomic partial angular velocities* of B relative to $\mathcal{R}$, respectively. Their use in Eqs. (6) and (10) instead of the holonomic partial velocities and the holonomic partial angular velocities yields the definitions of the *nonholonomic generalized active forces* and the *nonholonomic generalized inertia forces* as

$$ \widetilde{F}_r(q,u,t) \;=\; \sum_{i=1}^{\nu} {}^{\mathcal{R}}\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i + \sum_{i=1}^{\mu} {}^{\mathcal{R}}\tilde{\mathbf{v}}_r^{b_i} \cdot \mathbf{Z}_i + \sum_{i=1}^{\mu} {}^{\mathcal{R}}\tilde{\omega}_r^{B_i} \cdot \mathbf{T}_i, \quad r = 1, \dots, p, \tag{13} $$

and

$$ \widetilde{F}_r^\star(q,u,\dot{u},t) = -\sum_{i=1}^{\nu} m_{P_i} {}^{\mathcal{R}}\tilde{\mathbf{v}}_r^{P_i} \cdot {}^{\mathcal{R}}\mathbf{a}^{P_i} - \sum_{i=1}^{\mu} m_{B_i} {}^{\mathcal{R}}\tilde{\mathbf{v}}_r^{B_i} \cdot {}^{\mathcal{R}}\mathbf{a}^{b_i} - \sum_{i=1}^{\mu} {}^{\mathcal{R}}\tilde{\omega}_r^{B_i} \cdot {}^{\mathcal{R}}\mathbf{T}_{B_i}^\star, $$
$$ r = 1, \dots, p. \tag{14} $$

Kane's dynamical equations of motion for nonholonomic systems are [68]

$$ \widetilde{F}_r(q,u,t) + \widetilde{F}_r^\star(q,u,\dot{u},t) = 0 \, , \quad r = 1, \dots, p. \tag{15} $$

2.3 Simple Nonholonomic Constraints

A simple nonholonomic constraint is defined by the requirement that a nonintegrable configuration and time dependent linear combination of generalized speeds is equal to a specified function of generalized coordinates and time. Therefore, if the system has n generalized coordinates and p degrees of freedom, the $(n-p)$ nonholonomic constraints can be written as

$$u_{p+r} = \sum_{s=1}^{p} A_{rs} u_s + B_r, \qquad r = 1, \ldots, n-p \tag{16}$$

where the scalars A_{rs} and B_r are functions of the generalized coordinates $q_1, \ldots, q_n$, and t. Let

$$u = \lfloor u_1 \ldots u_n \rfloor^T = \lfloor u_I{}^T \quad u_D{}^T \rfloor^T, \tag{17}$$

where $u_I = \lfloor u_1 \ldots u_p \rfloor^T$ and $u_D = \lfloor u_{p+1} \ldots u_n \rfloor^T$. Hence, the matrix representation of (16) is

$$u_D = A(q,t) u_I + B(q,t), \tag{18}$$

where $A \in \mathbb{R}^{(n-p)\times p}$, $B \in \mathbb{R}^{n-p}$. Eq. (18) can be written as

$$A_1(q,t) u = B(q,t), \tag{19}$$

where

$$A_1 = \begin{bmatrix} -A & I \end{bmatrix}. \tag{20}$$

Differentiating Eq. (18) with respect to t, and dropping the arguments of the matrices for simplicity, yields:

$$\dot{u}_D = \dot{A}\, u_I + A \dot{u}_I + \dot{B}, \tag{21}$$

which can be written as

$$A_1 \dot{u} = \dot{A}\, u_I + \dot{B}. \tag{22}$$

2.4 Holonomic vs. Nonholonomic Partial Velocities and Partial Angular Velocities

It is convenient to write Eq. (3) in the matrix form

$$^{\mathcal{R}}\mathbf{v}^P = {}^{\mathcal{R}}\mathbf{v_I}^P(q,t) u_I + {}^{\mathcal{R}}\mathbf{v_D}^P(q,t) u_D + {}^{\mathcal{R}}\mathbf{v}_t{}^P(q,t), \tag{23}$$

where

$$^{\mathcal{R}}\mathbf{v_I}^P = \lfloor {}^{\mathcal{R}}\mathbf{v}_1{}^P \ldots {}^{\mathcal{R}}\mathbf{v}_p{}^P \rfloor \tag{24}$$

and

$$^{\mathcal{R}}\mathbf{v_D}^P = \lfloor {}^{\mathcal{R}}\mathbf{v}_{p+1}{}^P \ldots {}^{\mathcal{R}}\mathbf{v}_n{}^P \rfloor. \tag{25}$$

Substituting the expression (18) for u_D in Eq. (23) yields

$$\begin{aligned} ^{\mathcal{R}}\mathbf{v}^P &= {}^{\mathcal{R}}\mathbf{v_I}^P(q,t)u_I + {}^{\mathcal{R}}\mathbf{v_D}^P(q,t)[A(q,t)u_I + B(q,t)] + {}^{\mathcal{R}}\mathbf{v}_t{}^P(q,t) &(26)\\ &= [{}^{\mathcal{R}}\mathbf{v_I}^P(q,t) + {}^{\mathcal{R}}\mathbf{v_D}^P(q,t)A(q,t)]u_I + {}^{\mathcal{R}}\mathbf{v_D}^P(q,t)B(q,t) + {}^{\mathcal{R}}\mathbf{v}_t{}^P(q,t). &(27)\end{aligned}$$

Also, it is convenient to write Eq. (11) in the matrix form

$$^{\mathcal{R}}\mathbf{v}^P = {}^{\mathcal{R}}\tilde{\mathbf{v}}^P(q,t)u_I + {}^{\mathcal{R}}\tilde{\mathbf{v}}_t^P(q,t), \tag{28}$$

where $^{\mathcal{R}}\tilde{\mathbf{v}}^P$ is the row matrix containing the nonholonomic partial velocities

$$^{\mathcal{R}}\tilde{\mathbf{v}}^P = \lfloor {}^{\mathcal{R}}\tilde{\mathbf{v}}_1^P \ldots {}^{\mathcal{R}}\tilde{\mathbf{v}}_p^P \rfloor. \tag{29}$$

Comparing the coefficients of u_I in Eqs. (27) and (28) gives the relations between the holonomic and the nonholonomic partial velocities of a particle in the system as

$$^{\mathcal{R}}\tilde{\mathbf{v}}_r^P = {}^{\mathcal{R}}\mathbf{v}_r^P + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\mathbf{v}_{p+s}^P A_{sr}(q,t) \quad r = 1, \ldots, p. \tag{30}$$

In a similar manner, the two expressions (5) and (12) for the angular velocity $^{\mathcal{R}}\boldsymbol{\omega}^B$ can be used together with Eq. (18) to obtain the relations between the holonomic and the nonholonomic partial angular velocities of a body in the system. These relations take the form

$$^{\mathcal{R}}\tilde{\boldsymbol{\omega}}_r^B = {}^{\mathcal{R}}\boldsymbol{\omega}_r^B + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\boldsymbol{\omega}_{p+s}^B A_{sr}(q,t) \quad r = 1, \ldots, p. \tag{31}$$

2.5 Holonomic vs. Nonholonomic Generalized Active and Inertia Forces

Using Eqs. (30) and (31) in the expressions (13) and (14) for the nonholonomic generalized active and inertia forces yields

$$\begin{aligned} \widetilde{F}_r(q,u,t) &= \sum_{i=1}^{\nu}[{}^{\mathcal{R}}\mathbf{v}_r^{P_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\mathbf{v}_{p+s}^{P_i}] \cdot \mathbf{R}_i + \sum_{i=1}^{\mu}[{}^{\mathcal{R}}\mathbf{v}_r^{b_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\mathbf{v}_{p+s}^{b_i}] \\ &\quad + \sum_{i=1}^{\mu}[{}^{\mathcal{R}}\boldsymbol{\omega}_r^{B_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\boldsymbol{\omega}_{p+s}^{B_i} A_{sr}(q,t)] \cdot \mathbf{T}_i, \quad r = 1, \ldots, p, \qquad (32)\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{F}_r^\star(q,u,\dot{u},t) &= -\sum_{i=1}^{\nu} m_{P_i}[{}^{\mathcal{R}}\mathbf{v}_r^{P_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\mathbf{v}_{p+s}^{P_i} A_{sr}(q,t)] \cdot {}^{\mathcal{R}}\mathbf{a}^{P_i} \\
&\quad - \sum_{i=1}^{\mu} m_{B_i}[{}^{\mathcal{R}}\mathbf{v}_r^{b_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\mathbf{v}_{p+s}^{b_i} A_{sr}(q,t)] \cdot {}^{\mathcal{R}}\mathbf{a}^{b_i} \\
&\quad - \sum_{i=1}^{\mu} [{}^{\mathcal{R}}\boldsymbol{\omega}_r^{B_i} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\boldsymbol{\omega}_{p+s}^{B_i} A_{sr}(q,t)] \cdot {}^{\mathcal{R}}\mathbf{T}_{B_i}^\star, \qquad r = 1,\ldots,p. \qquad (33)
\end{aligned}
$$

The above two relations are representations of the nonholonomic generalized active and inertia forces in terms of the holonomic generalized active and inertia forces (6) and (10), and can be written by omitting the arguments for simplicity as

$$
\widetilde{F}_r = F_r + \sum_{s=1}^{n-p} F_{p+s} A_{sr} \qquad (34)
$$

$$
\widetilde{F}_r^\star = F_r^\star + \sum_{s=1}^{n-p} F_{p+s}^\star A_{sr}, \qquad r = 1,\ldots,p. \qquad (35)
$$

Therefore, Eq. (15) can be written as

$$
F_r + F_r^\star + \sum_{s=1}^{n-p} \left(F_{p+s} + F_{p+s}^\star\right) A_{sr} = 0, \qquad r = 1,\ldots,p. \qquad (36)
$$

or in matrix form as

$$
A_2 F^\star = -A_2 F, \qquad (37)
$$

where

$$
A_2 := \begin{bmatrix} I & A^T \end{bmatrix}. \qquad (38)
$$

The accelerations and angular accelerations are linear in $\dot{u}$; it follows that the generalized inertia forces are as well. Consequently, $F^\star$ can be written in the form

$$
F^\star = -Q(q,t)\dot{u} - L(q,u,t), \qquad (39)
$$

where Q is a symmetric positive definite matrix. Then, Eqs. (37) become

$$
A_2(q,t)Q(q,t)\dot{u} = A_2 P(q,u,t), \qquad (40)
$$

where

$$P(q,u,t) = -L(q,u,t) + F(q,u,t), \tag{41}$$

and Q is the generalized inertia matrix of the system.

2.6 *Nonminimal System of Equations*

Eqs. (22) and (40) can be used to form the matrix system

$$T\dot{u} = V, \tag{42}$$

where $T := \begin{bmatrix} {A_1}^T & [A_2Q]^T \end{bmatrix}^T$, and $V := \begin{bmatrix} [\dot{A}u_I + \dot{B}]^T & [A_2P]^T \end{bmatrix}^T$. The matrix T is a *constrained generalized inertia matrix* for the nonholonomic system $\mathcal{S}$. It is invertible for all choices of generalized coordinates and generalized speeds that render the elements of the constraint matrix A finite. To show this, it is noticed that the row spaces of A_1 and A_2 are orthogonal complements. That is, both matrices are full row ranks, and $A_1{A_2}^T = 0$. The row space of A_2 is unaltered if the rows of A_2 are scaled by nonzero scalars. Therefore, T is full rank if the holonomic system (39) is diagonal, i.e. the inertia matrix Q is diagonal. This diagonalization is possible by a proper choice of generalized speeds, and can be performed starting from an arbitrary choice of generalized speeds, by a Graham-Schmidt orthogonalization of the corresponding partial velocities [91]. The invertibility of T for this special choice of generalized speeds, denoted say by w, implies the invertibility of T for any other choice of generalized speeds. This can be seen by equating the right sides of the equations

$$\dot{q} = C_1(q,t)w + D_1(q,t) \tag{43}$$

$$\dot{q} = C_2(q,t)u + D_2(q,t) \tag{44}$$

which gives

$$w = \Phi_1(q,t)u + \Phi_2(q,t) \tag{45}$$

where

$$\Phi_1 = C_1^{-1} C_2 \tag{46}$$

$$\Phi_2 = C_1^{-1}(D_2 - D_1). \tag{47}$$

Eq. (45) is a unique invertible transformation between the two sets of generalized speeds, which implies the equivalency of the existence of solution for one set and the existence of solution for the other. Therefore,

$$\dot{u} = T^{-1} V. \tag{48}$$

The required inversion of the matrix T in getting from form (42) to (48) can be done numerically online for the purpose of time simulations. However, to obtain the analytical form of the equations, the symbolic inversion of the matrix T is needed, which cannot be done for excessively large fully-populated matrices.

Remark. *The holonomic and nonholonomic partial velocities/angular velocities are related to each others by the scalars A_{sr}, as seen from Eqs. (30) and (31). This causes A to appear in the equations of motion, Eqs. (40), beside its appearance in the constraint equations, Eqs. (22). This is a key point in the derivation, since it provides a mathematical relationship between the constraints and the constrained motion.*

Remark. *The possible configurations of the dynamical system might involve singular configurations that cause numerical explosions of the constraint matrix A for a specific choice of the independent generalized speeds u_I, e.g. a toggle position of a four-bar linkage. In this case, different choices of u_I must be employed in the neighborhoods of these configurations such that finite values of the elements of the matrices A_1 and A_2 are obtained, and T remains invertible. The dependency of the constraint equations does not affect the invertibility of T, because A_1 and A_2 always have linearly independent rows, regardless what (finite) values the elements of A might have. Thus, one need not assume that the constraint equations are linearly independent, and any set of constraint equations can be included without the need to extract the largest independent set.*

The procedure of using the acceleration form of constraints in obtaining full order and decoupled equations of motion is summarized as follows:

Step 1. A set of generalized speeds is chosen. The dependency among the set is described by Eq. (18). The matrix A is used to construct the matrices A_1 and A_2. If configuration constraints are involved, the corresponding equations are differentiated in time to appear in the same kinematical form.

Step 2. Eq. (18) is differentiated in time, resulting in Eq. (22).

Step 3. Expressions are obtained for holonomic partial velocities/angular velocities by inspecting the corresponding expressions for linear/angular velocities, as the coefficients of the generalized speeds.

Step 4. Holonomic generalized active and inertia forces are found from the scalar (dot) product of the impressed and gravitational forces with the holonomic partial velocities/angular velocities, and used together with A_2 to form Eq. (40).

Step 5. Eqs. (22) and (40) are used to form the matrix equation, Eq. (42), and T is inverted to yield the resulting equations of motion, Eq. (48). The following two examples illustrate the procedure.

2.7 *Example 2.1: Motion of a particle subjected to holonomic constraints*

Consider the three-dimensional motion of a particle P of mass m as shown in Fig. 1. Let $\mathcal{R}$ be an inertial frame in which the coordinate system (X, Y, Z) is fixed, and let r be the radial distance from the origin of (X, Y, Z), θ be the polar angle from the X-axis, and ϕ be the cone angle from the Z-axis. The particle's motion is restricted by the ideal holonomic constraint $r\phi = c$, where c is a nonzero constant. This constraint is enforced by means of a surface (not shown in Fig. 1) defined by the holonomic relation. Because of the nature of the constraint, it is more convenient to use the spherical coordinate system (r,θ,ϕ). Let the particle be at the origin of the coordinate system (X', Y', Z') where X' is along r, attached to a unit vector $\mathbf{i}$, and pointing outward relative to O; Y' is parallel to the XY plane, attached to a unit vector $\mathbf{j}$ pointing to the positive rotation direction relative to the Z-axis; and Z' is perpendicular to both x' and y', attached to a unit vector $\mathbf{k}$ such that $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. The resultant active force on the particle is given by

$$\mathbf{F} = F_r\mathbf{i} + F_\theta\mathbf{j} + F_\phi\mathbf{k}. \tag{49}$$

The particle's velocity in $\mathcal{R}$ is

$${}^{\mathcal{R}}\mathbf{v}^P = \dot{r}\mathbf{i} + r\sin\phi\dot{\theta}\mathbf{j} - r\dot{\phi}\mathbf{k}. \tag{50}$$

Let the generalized speeds be $u_1 = \dot{r}$, $u_2 = r\dot{\theta}\sin\phi$, and $u_3 = -r\dot{\phi}$. Then one can write

$${}^{\mathcal{R}}\mathbf{v}^P = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \tag{51}$$

so that the partial velocities are $\mathbf{v_1} = \mathbf{i}$, $\mathbf{v_2} = \mathbf{j}$, and $\mathbf{v_3} = \mathbf{k}$. The acceleration of the particle in $\mathcal{R}$ is

$${}^{\mathcal{R}}\mathbf{a}^P = \frac{{}^{\mathcal{R}}d\,{}^{\mathcal{R}}\mathbf{v}^P}{dt} = \left(\dot{u}_1 - \frac{{u_2}^2 + {u_3}^2}{r}\right)\mathbf{i} + \left(\dot{u}_2 + \frac{u_1u_2}{r} - \frac{u_2u_3}{r\tan\phi}\right)\mathbf{j} + \left(\dot{u}_3 + \frac{u_1u_3}{r} + \frac{{u_2}^2}{r\tan\phi}\right)\mathbf{k}. \tag{52}$$

The generalized inertia forces are

$$\begin{aligned} F_1^\star &= -m\,{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v_1} = -m\left(\dot{u}_1 - \frac{{u_2}^2 + {u_3}^2}{r}\right) && (53) \\ F_2^\star &= -m\,{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v_2} \\ &= -m\left(\dot{u}_2 + \frac{u_1u_2}{r} - \frac{u_2u_3}{r\tan\phi}\right) && (54) \\ F_3^\star &= -m\,{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v_3} \\ &= -m\left(\dot{u}_3 + \frac{u_1u_3}{r} + \frac{{u_2}^2}{r\tan\phi}\right). && (55) \end{aligned}$$

In matrix form,

$$F^\star = \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{Bmatrix} m\left(\frac{{u_2}^2+{u_3}^2}{r}\right) \\ m\left(\frac{u_2u_3}{r\tan\phi} - \frac{u_1u_2}{r}\right) \\ m\left(-\frac{u_1u_3}{r} - \frac{{u_2}^2}{r\tan\phi}\right) \end{Bmatrix}. \tag{56}$$

The generalized active forces are

$$F_1 = \mathbf{F} \cdot \mathbf{v}_1 = F_r \tag{57}$$

$$F_2 = \mathbf{F} \cdot \mathbf{v}_2 = F_\theta \tag{58}$$

$$F_3 = \mathbf{F} \cdot \mathbf{v}_3 = F_\phi. \tag{59}$$

Differentiating the holonomic constraint $r\phi = c$ with respect to t yields

$$\dot{r}\phi + r\dot{\phi} = 0, \tag{60}$$

or

$$u_3 = \phi u_1. \tag{61}$$

Therefore,

$$A = \lfloor \phi \quad 0 \rfloor, \tag{62}$$

and

$$B = 0. \tag{63}$$

Differentiating the constraint matrix A with respect to t yields

$$\dot{A} = \left\lfloor -\frac{u_3}{r} \quad 0 \right\rfloor \tag{64}$$

Also,

$$A_1 = \left[-A \quad I \right] = \lfloor -\phi \quad 0 \quad 1 \rfloor. \tag{65}$$

Thus, Eq. (22) for this system is

$$\lfloor -\phi \quad 0 \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \left\lfloor -\frac{u_3}{r} \quad 0 \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}. \tag{66}$$

On the other hand,

$$\begin{aligned} A_2 &= \left[I \quad A^T \right] && (67) \\ &= \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix}, && (68) \end{aligned}$$

so that Eq. (40) for this system is

$$\begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{Bmatrix} m(\frac{u_2{}^2+u_3{}^2}{r}) \\ m(\frac{u_2 u_3}{r\tan\phi} - \frac{u_1 u_2}{r}) \\ m(-\frac{u_1 u_3}{r} - \frac{u_2{}^2}{r\tan\phi}) \end{Bmatrix} \right\} = - \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_r \\ F_\theta \\ F_\phi \end{Bmatrix}, \tag{69}$$

or

$$\begin{bmatrix} -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} -m(\frac{u_2{}^2+u_3{}^2}{r}) + m\phi(\frac{u_1 u_3}{r} + \frac{u_2{}^2}{r\tan\phi}) - F_r - \phi F_\phi \\ -m(\frac{u_2 u_3}{r\tan\phi} - \frac{u_1 u_2}{r}) - F_\theta \end{Bmatrix}. \tag{70}$$

Therefore, Eqs. (66) and (70) can be used to form the matrix system

$$\begin{bmatrix} -\phi & 0 & 1 \\ -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{u_1 u_3}{r} \\ -\frac{m}{r}(u_2{}^2 + u_3{}^2 - \phi u_1 u_3 - \phi \frac{u_2{}^2}{\tan\phi}) - F_r - \phi F_\phi \\ \frac{m}{r}(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}) - F_\theta \end{Bmatrix}. \tag{71}$$

Solving for $\dot{u}$,

$$\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{bmatrix} -\phi & 0 & 1 \\ -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix}^{-1} \begin{Bmatrix} -\frac{u_1 u_3}{r} \\ -\frac{m}{r}(u_2{}^2 + u_3{}^2 - \phi u_1 u_3 - \phi \frac{u_2{}^2}{\tan\phi}) - F_r - \phi F_\phi \\ \frac{m}{r}(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}) - F_\theta \end{Bmatrix}$$
$$= \frac{1}{a} \begin{bmatrix} -m^2\phi & -m & 0 \\ 0 & 0 & -m(\phi^2+1) \\ m^2 & -m\phi & 0 \end{bmatrix} \begin{Bmatrix} -\frac{u_1 u_3}{r} \\ -\frac{m}{r}(u_2{}^2 + u_3{}^2 - \phi u_1 u_3 - \phi \frac{u_2{}^2}{\tan\phi}) - F_r - \phi F_\phi \\ \frac{m}{r}(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}) - F_\theta \end{Bmatrix}, \tag{72}$$

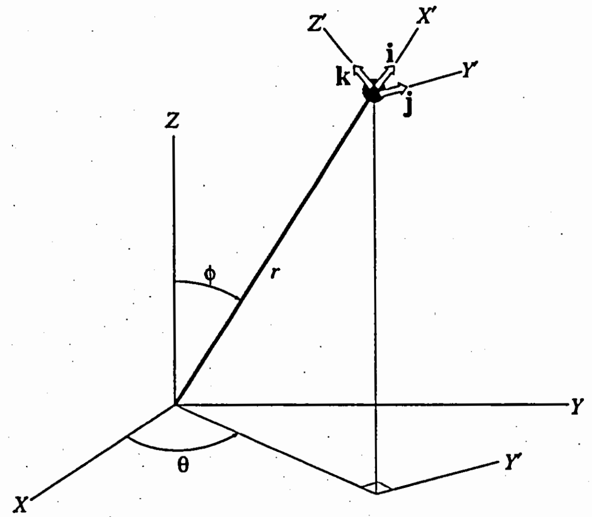

Figure 1: Schematic for Example 2.1

where $a := m^2(1+\phi^2)$. Therefore, the final form of the equations is

$$\begin{aligned} \dot{u}_1 &= \frac{m}{a}(F_r + \phi F_\phi) \\ &\quad + \frac{m^2}{ar}\left(u_2{}^2 + u_3{}^2 - \phi\frac{u_2{}^2}{\tan\phi}\right) \end{aligned} \tag{73}$$

$$\dot{u}_2 = \frac{F_\theta}{m} - \frac{1}{r}\left(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}\right) \tag{74}$$

$$\begin{aligned} \dot{u}_3 &= \frac{m\phi}{a}(F_r + \phi F_\phi) \\ &\quad + \frac{m^2\phi}{ar}\left(u_2{}^2 + u_3{}^2 - \phi\frac{u_2{}^2}{\tan\phi}\right) - \frac{u_1 u_3}{r}. \end{aligned} \tag{75}$$

Remark. *The particle is subjected to one holonomic constraint. Thus two generalized coordinates and two generalized speeds are sufficient to form the equations. However, since the holonomic constraint equation is differentiated with respect to time, the constraint is treated as nonholonomic, and one pseudo-generalized coordinate is added, together with one dependent generalized speed that satisfies Eq. (18).*

Remark. *The system of equations (73)-(75) satisfy the passive constraint equation $r\phi = c$, regardless what nature the generalized active forces F_r, F_ϕ, F_θ may have. For instance, these generalized active forces might be constants, time variants, or dependent on the generalized coordinates and/or generalized speeds. They may also be control forces that are* *intended to satisfy some objective(s), (but not to enforce the passive constraint). The equation set does not show the constraint forces that enforce the passive constraint. A procedure will be provided in the fourth chapter to bring these forces to evidence.*

2.8 Example 2.2: Motion of a particle subjected to nonholonomic constraints

Consider the three-dimensional motion of a particle P of mass m subjected to the nonholonomic constraint

$$\dot{y} = z\dot{x} + \alpha(t), \tag{76}$$

where $\alpha(t)$ is a prescribed smooth function of time. The active forces in the Cartesian coordinate system (x, y, z) are F_x, F_y, F_z, respectively. Assume that (x, y, z) is fixed to an inertial frame $\mathcal{R}$, and define the generalized speeds as $u_1 = \dot{x}$, $u_2 = \dot{z}$, $u_3 = \dot{y}$. Then, the above nonholonomic constraint equation can be written as

$$u_3 = \lfloor z \quad 0 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \alpha(t). \tag{77}$$

Therefore,

$$A = \lfloor z \quad 0 \rfloor, \qquad B = \alpha(t). \tag{78}$$

Differentiating the constraint matrix A with respect to time yields

$$\dot{A} = \lfloor u_2 \quad 0 \rfloor, \tag{79}$$

The matrix A_1 is therefore

$$A_1 = \begin{bmatrix} -A & I \end{bmatrix} = \lfloor -z \quad 0 \quad 1 \rfloor. \tag{80}$$

Thus, Eq. (22) for this system is

$$\lfloor -z \quad 0 \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \lfloor u_2 \quad 0 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \dot{\alpha} \tag{81}$$

The matrix A_2 is

$$A_2 = \begin{bmatrix} I & A^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix}. \tag{82}$$

The velocity of the particle in $\mathcal{R}$ is

$$\begin{aligned} {}^{\mathcal{R}}\mathbf{v}^P &= u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + u_3\mathbf{v}_3 & (83) \\ &= \dot{x}\mathbf{i} + \dot{z}\mathbf{k} + \dot{y}\mathbf{j} & (84) \end{aligned}$$

so that $\mathbf{v}_1 = \mathbf{i}$, $\mathbf{v}_2 = \mathbf{k}$, $\mathbf{v}_3 = \mathbf{j}$, where $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors parallel to the x, y, and z axes, respectively.

The generalized active forces are $F_1 = F_x$, $F_2 = F_z$, and $F_3 = F_y$. The generalized inertia forces are

$$\begin{aligned} F_1^\star &= -m\dot{u}_1 & (85) \\ F_2^\star &= -m\dot{u}_2 & (86) \\ F_3^\star &= -m\dot{u}_3. & (87) \end{aligned}$$

In matrix form

$$F^\star = \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}. \tag{88}$$

Hence, Eq. (40) for this system is

$$\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = - \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \\ F_y \end{Bmatrix} \tag{89}$$

or

$$\begin{bmatrix} -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = - \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \\ F_y \end{Bmatrix}. \tag{90}$$

Eqs. (81) and (90) can be put in the following matrix form

$$\begin{bmatrix} -z & 0 & 1 \\ -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} u_1 u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix}. \tag{91}$$

Solving for $\dot{u}$

$$\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{bmatrix} -z & 0 & 1 \\ -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix} \tag{92}$$

$$= \begin{bmatrix} -\frac{mz}{a} & \frac{-1}{a} & 0 \\ 0 & 0 & \frac{-1}{m} \\ \frac{m}{a} & \frac{-z}{a} & 0 \end{bmatrix} \begin{Bmatrix} u_1 u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix}, \tag{93}$$

where $a := m(1+z^2)$. Therefore, the final form of the equations of motion is

$$\dot{u}_1 = \frac{-mz}{a}(u_1 u_2 + \dot{\alpha}) + \frac{1}{a}(F_x + F_y z) \tag{94}$$

$$\dot{u}_2 = \frac{F_z}{m} \tag{95}$$

$$\dot{u}_3 = \frac{m}{a}(u_1 u_2 + \dot{\alpha}) + \frac{z}{a}(F_x + F_y z). \tag{96}$$

2.9 *The Generalized Inertia Matrix*

The nature of matrix Q is now considered by using the kinetic energy of the system. For a nonholonomic system of ν particles and μ rigid bodies, the rth nonholonomic generalized inertia force can be written as[68]

$$\begin{aligned} \tilde{F}_r^\star &= F_r^\star + \sum_{s=1}^{n-p} F_{p+s}^\star A_{sr} \\ &= -\sum_{s=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_s}\right) - \frac{\partial K}{\partial q_s} \right] \left(W_{sr} + \sum_{k=1}^{n-p} W_{s,p+k} A_{kr} \right), \qquad r = 1, \dots, p \end{aligned} \tag{97}$$

where $W = C^{-1}(q,t)$, and K is the system's kinetic energy relative to an inertial frame, and is given by

$$K = \sum_{i=1}^{\nu} k^{P_i} + \sum_{i=1}^{\mu} k^{B_i} \tag{98}$$

$$= \frac{1}{2}\sum_{i=1}^{\nu} m_{P_i}\mathbf{v}^{P_i}\cdot\mathbf{v}^{P_i} + \frac{1}{2}\sum_{i=1}^{\mu} m_{B_i}\mathbf{v}^{B_i^*}\cdot\mathbf{v}^{B_i^*} + \frac{1}{2}\sum_{i=1}^{\mu} \boldsymbol{\omega}^{B_i}\cdot\boldsymbol{I}_{B_i}\cdot\boldsymbol{\omega}^{B_i}. \tag{99}$$

The kinetic energy K can also be represented as

$$K = \frac{1}{2}\, u^T M(q,t)u + N(q,t)u + R(q,t), \tag{100}$$

where $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{1\times n}$, $R \in \mathbb{R}$. Hence,

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial K}{\partial u}\,\frac{\partial u}{\partial \dot{q}} \tag{101}$$

$$= \left[u^T M + N\right] W \tag{102}$$

Therefore, the matrix representation of Eq. (97) is

$$A_2 F^\star = -A_2 W^T\left(\frac{d}{dt}\left[W^T M u + W^T N^T\right] - {K_q}^T\right) \tag{103}$$

$$= -A_2 W^T\left(W^T M\dot{u} + \frac{d}{dt}\left[W^T M\right] u + \frac{d}{dt}\left[W^T N^T\right] - {K_q}^T\right) \tag{104}$$

where

$$K_q = \lfloor K_{q_1} \ldots K_{q_n} \rfloor \tag{105}$$

$$= \frac{\partial K}{\partial q}. \tag{106}$$

If Eq. (39) is multiplied by A_2 and compared with Eq. (104), we obtain

$$Q := W^T W^T M, \tag{107}$$

$$L := W^T\left(\frac{d}{dt}\left[W^T M\right] u + \frac{d}{dt}\left[W^T N^T\right] - {K_q}^T\right). \tag{108}$$

Although Q is not necessarily symmetric, its spectrum set satisfies

$$\mathrm{Spec}[Q] = \mathrm{Spec}\left[W^T M W\right] = \mathrm{Spec}[M]. \tag{109}$$

Therefore, provided that M is positive definite, Q is of full rank, and has positive real eigenvalues. Hence, the row subspaces of A_2 and A_2Q have the same dimension, and the invertibility of T implies that they are the same subspaces.

Remark. *It is not desirable to use kinetic energy to derive the matrices Q and L. It is easier to obtain $F_\tau^\star$ directly by constructing the appropriate acceleration and inertia torque vectors.*

2.10 Summary

By taking advantage of the conformity of the two sets, the acceleration form of the constraint equations is used with Kane's equations of motion, resulting in equations of motion that are both *full-order* and *separated* in the generalized accelerations. This means that the time derivatives of all generalized speeds appear in the equations, but only one in each equation. Thus, one obtains a single set of consistent equations without reducing the dimensionality of the space of generalized speeds from the number of generalized coordinates to the number of degrees of freedom. Furthermore, this full dimensionality is maintained without employing Lagrange multipliers.

The resulting nonminimal set of equations is effective in describing complex constrained motion. This is because of its full-order nature. It facilitates study of the effects of constraints on the behavior of the dynamical system, by comparing with the corresponding behavior of the same system without constraints. Further advantages include that there is no loss of information, as compared with the reduction of the dimension of the space of generalized speeds, because the whole set of generalized accelerations appear in the differential equations. The complexity is decreased relative to an analysis using Lagrange multipliers.

If the generalized speeds are chosen such that the constraint matrix is well-defined for all configurations, then the constrained inertia matrix inversion involved in the derivation is guaranteed for all values of configurations and velocities that satisfy the constraints.

Moreover, this is true without having to assume that the constraint equations are linearly independent, so that any set of constraint equations can be included without the need to extract the largest independent set. With the kinematical differential equations, the non-minimal form of Kane's equations compose a complete state-space representation of the system. The resulting formulation is simple and useful in performance analysis and control system design for constrained dynamical systems. Applications are presented in the seventh chapter.

The symbolic manipulator computer program AUTOLEV™ has been adapted to the present method and used to check the validity of the resulting equations of motion obtained in the examples introduced in this chapter. Simulation results confirm that the models are equivalent. However, it is noticed that enforcing the constraint equations at the acceleration level causes the numerical solutions of the resulting equations of motion to be sensitive to the finite precision and accuracy errors. It can thus cause continuous violations of the constraint equations, especially in the case of holonomic constraints, as the equations are twice integrated to obtain the generalized coordinates. A remedy to this problem is presented in the fourth chapter.

Finally, a study of the generalized inertia matrix is conducted with the aid of kinetic energy. The eigenvalues of this matrix are shown to be invariant under the choice of the generalized speeds. No conclusion is drawn on the sign-definiteness of the generalized inertia matrix. Its positivity remains a fact as there exists no known counterexample in nature.

CHAPTER III

NONMINIMAL KANE'S EQUATIONS OF MOTION FOR NONLINEARLY CONSTRAINED SYSTEMS

3.1 Introduction

Dynamicists have noticed the absence of nonlinear nonholonomic constraints from daily life observations, and some argued the nonexistence of known examples of such constraints in nature [111]. Among the few examples of dynamical systems with nonlinear nonholonomic constraints in the literature of analytical mechanics is the one due to Appell [8] and Hamel [50].

In the Appell-Hamel mechanism example, the constraints were modeled as nonlinear in the velocities by a limiting process on the nonholonomic constraint equations. However, the validity of the resulting equations of motion was criticized [99] because of the reduction in order associated with this limit condition, which results in a qualitative change in the system behavior and a huge difference in the corresponding results from the results associated with taking the limit after obtaining the equations of motion. The reinterpretation of nonholonomic constraints of the rolling type as nonlinear is originally due to Saletan and Cromer [118]. A further study of the Appell-Hamel problem for the purpose of analyzing nonlinear nonholonomic constraints in the context of Kane's method is found in Ref. [136].

An example of nonlinear nonholonomic constraints of the nonintegrable in accelerations type is due to Kitzka [84]. The system was dismissed in Ref. [41] as nonlinearly constrained, and an alternative derivation of the constraint equations as linear nonholonomic was presented.

Despite the controversy on the feasibility of nonlinear nonholonomic constraints of the passive type, active constraints (also called servo- or program constraints) certainly can be nonlinear nonholonomic. Furthermore, they need not be ideal [20] (of the Chetaev type) or limited to second order in the generalized coordinates [62]. The importance of understanding servo-constraints for control system analysis and synthesis can be considered the main reason for studying the various categories of constraints, including nonlinear nonholonomic constraints, the material of this chapter. A study of servo-constraints is presented in the seventh chapter.

Nonlinear nonholonomic constraints treatment in Kane's approach began with the extension of Passerello-Huston equation to include such constraints [58]. Later, Huston's method of undetermined multipliers [132, 133] was generalized [136] to include nonlinear nonholonomic constraints.

The use of the acceleration form of nonlinear constraint equations with Kane's equations was first depicted in Refs. [132, 136, 57], where the (nonunique) orthogonal complement of the constraint matrix is multiplied by the full-order, constrained form of Kane's equations to eliminate the contribution of the generalized constraint forces. It was shown in the previous chapter that a particular choice of the orthogonal complement matrix is embedded in the minimal Kane's equations and is obtained by expanding these equations in terms of the unconstrained generalized active and inertia forces. This particular choice implies the consistency among the governing equations because it guarantees the nondeficiency of the augmented matrix, which becomes a generalized "constrained" inertia matrix.

The purpose of this chapter is to extend the nonminimal formulation of Kane's equations of motion to nonlinearly constrained multibody systems, with the aid of the acceleration form of constraints. It is shown that the previous treatment is also capable of handling nonlinear constraints. This is exploited from the fact that the acceleration form of constraint equations is linear in the generalized accelerations, even if the nonholonomic constraint equations are nonlinear in the generalized speeds. On the other hand, the relations between the holonomic and the nonholonomic partial velocities and partial angular velocities of the system are preserved in the case of nonlinear nonholonomic constraints, and hence the special structure of the resulting constraint matrices A_1 and A_2 is also preserved.

Next, the system of ν particles and μ rigid bodies considered in the previous chapter is revisited, with a set of nonlinear nonholonomic constraints.

3.2. Constraints Involving Nonlinearity in Generalized Speeds

The m nonholonomic constraint equations take the form

$$\phi(q, u, t) = 0, \tag{110}$$

where ϕ is the column matrix

$$\phi = \lfloor \phi_1 \ldots \phi_m \rfloor^T, \tag{111}$$

in which $\phi(q, u, t)$ is in general nonlinear in its arguments. Differentiating the constraint equations, Eqs. (110), with respect to time t, one obtains

$$\dot{\phi}(q, u, \dot{u}, t) = \frac{\partial \phi}{\partial q}\dot{q} + \frac{\partial \phi}{\partial u}\dot{u} + \frac{\partial \phi}{\partial t} = 0. \tag{112}$$

Substitution of the kinematical differential equations, Eqs. (4), in the above equation results in the acceleration form of the constraint equations

$$\dot{\phi}(q, u, \dot{u}, t) = \frac{\partial \phi}{\partial u}\dot{u} + B_1(q, u, t)u + B_2(q, u, t) = 0, \tag{113}$$

where

$$B_1(q, u, t) = \frac{\partial \phi}{\partial q} C(q, t) \tag{114}$$

$$B_2(q, u, t) = \frac{\partial \phi}{\partial q} D(q, t) + \frac{\partial \phi}{\partial t}. \tag{115}$$

Let

$$u = \lfloor u_I{}^T \quad u_D{}^T \rfloor^T, \tag{116}$$

3.3 *Holonomic vs. Nonholonomic Partial Velocities and Partial Angular Velocities*

The velocity of the particle P of the system can be written in terms of the holonomic partial velocities as

$$^{\mathcal{R}}\mathbf{v}^{P} = {}^{\mathcal{R}}\mathbf{v_I}^{P}(q,t)u_I + {}^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)u_D + {}^{\mathcal{R}}\mathbf{v}_t^{P}(q,t), \tag{123}$$

where

$$^{\mathcal{R}}\mathbf{v_I}^{P} = \lfloor {}^{\mathcal{R}}\mathbf{v}_1^{P} \ldots {}^{\mathcal{R}}\mathbf{v}_p^{P} \rfloor \tag{124}$$

and

$$^{\mathcal{R}}\mathbf{v_D}^{P} = \lfloor {}^{\mathcal{R}}\mathbf{v}_{p+1}^{P} \ldots {}^{\mathcal{R}}\mathbf{v}_n^{P} \rfloor. \tag{125}$$

Hence, the acceleration of the particle relative to $\mathcal{R}$ is

$$\begin{aligned} ^{\mathcal{R}}\mathbf{a}^{P} = {}^{\mathcal{R}}\mathbf{v_I}^{P}(q,t)\dot{u}_I + {}^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)\dot{u}_D + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v_I}^{P}(q,t)]}{dt}u_I \\ + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)]}{dt}u_D + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v}_t^{P}(q,t)]}{dt}. \end{aligned} \tag{126}$$

Substituting Eq. (120) for $\dot{u}_D$ in Eq. (126) gives

$$\begin{aligned} ^{\mathcal{R}}\mathbf{a}^{P} &= [^{\mathcal{R}}\mathbf{v_I}^{P}(q,t) + {}^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)A(q,u,t)]\dot{u}_I \\ &\quad + {}^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)B(q,u,t) + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v_I}^{P}(q,t)]}{dt}u_I \\ &\quad + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v_D}^{P}(q,t)]}{dt}u_D + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\mathbf{v}_t^{P}(q,t)]}{dt}. \end{aligned} \tag{127}$$

Also, the velocity of the particle P can be written in terms of the nonholonomic partial velocities as

$$^{\mathcal{R}}\mathbf{v}^{P} = {}^{\mathcal{R}}\tilde{\mathbf{v}}^{P}(q,t)u_I + {}^{\mathcal{R}}\tilde{\mathbf{v}}_t^{P}(q,t), \tag{128}$$

where $^{\mathcal{R}}\tilde{\mathbf{v}}^{P}$ is the row matrix containing the nonholonomic partial velocities

$$^{\mathcal{R}}\tilde{\mathbf{v}}^{P} = \lfloor {}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P} \ldots {}^{\mathcal{R}}\tilde{\mathbf{v}}_p^{P} \rfloor. \tag{129}$$

Differentiating Eq. (128) with respect to time in $\mathcal{R}$ gives

$$^{\mathcal{R}}\mathbf{a}^{P} = {}^{\mathcal{R}}\tilde{\mathbf{v}}^{P}(q,t)\dot{u}_I + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\tilde{\mathbf{v}}^{P}(q,t)]}{dt}u_I + \frac{^{\mathcal{R}}d[^{\mathcal{R}}\tilde{\mathbf{v}}_t^{P}(q,t)]}{dt}. \tag{130}$$

where $u_I = \lfloor u_1 \ldots u_{n-m} \rfloor^T$ and $u_D = \lfloor u_{p+1} \ldots u_n \rfloor^T$. Define the $m \times p$ matrix

$$
\begin{aligned}
J_1(q,u,t) &:= \begin{bmatrix} \frac{\partial \phi_1}{\partial u_1} & \cdots & \frac{\partial \phi_1}{\partial u_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial u_1} & \cdots & \frac{\partial \phi_m}{\partial u_p} \end{bmatrix} \\
&= \frac{\partial \phi}{\partial u_I}
\end{aligned}
\tag{117}
$$

and the $m \times m$ matrix

$$
\begin{aligned}
J_2(q,u,t) &:= \begin{bmatrix} \frac{\partial \phi_1}{\partial u_{p+1}} & \cdots & \frac{\partial \phi_1}{\partial u_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial u_{p+1}} & \cdots & \frac{\partial \phi_m}{\partial u_n} \end{bmatrix} \\
&= \frac{\partial \phi}{\partial u_D}.
\end{aligned}
\tag{118}
$$

We assume that $u_{p+1}, \ldots, u_n$ can be chosen such that the matrix J_2 is nonsingular for all q, u, and t that satisfy the constraint equations, Eqs. (110). Thus, Eq. (113) can be written as

$$\dot{\phi}(q,u,\dot{u},t) = J_1 \dot{u}_I + J_2 \dot{u}_D + B_1(q,u,t)u + B_2(q,u,t) = 0. \tag{119}$$

Solving for $\dot{u}_D$ yields

$$\dot{u}_D = A(q,u,t)\dot{u}_I + B(q,u,t), \tag{120}$$

where

$$
\begin{aligned}
A(q,u,t) &:= -J_2^{-1} J_1 \\
B(q,u,t) &:= -J_2^{-1}[B_1(q,u,t)u + B_2(q,u,t)].
\end{aligned}
$$

Eq. (120) can be written in matrix form as

$$A_1(q,u,t)\dot{u} = B(q,u,t), \tag{121}$$

where

$$A_1 = \begin{bmatrix} -A & I \end{bmatrix}. \tag{122}$$

Comparing the coefficients of $\dot{u}_I$ in Eqs. (127) and (130) gives the relations between the holonomic and the nonholonomic partial velocities of a particle in the system as

$$^{\mathcal{R}}\tilde{\mathbf{v}}_r^P = ^{\mathcal{R}}\mathbf{v}_r^P + \sum_{s=1}^{n-p} {^{\mathcal{R}}\mathbf{v}_{p+s}^P} A_{sr}(q,u,t) \quad r = 1,\ldots,p. \tag{131}$$

Similarly, the relations between the holonomic and the nonholonomic partial angular velocities of a body in the system is found by matching the coefficients of $\dot{u}_I$ in two expressions of the angular acceleration $^{\mathcal{R}}\boldsymbol{\alpha}^B$. The first is obtained by taking the time derivative of $^{\mathcal{R}}\boldsymbol{\omega}^B$ given by the equation

$$^{\mathcal{R}}\boldsymbol{\omega}^B = \sum_{r=1}^{n} {^{\mathcal{R}}\boldsymbol{\omega}_r^B}(q,t)u_r + {^{\mathcal{R}}\boldsymbol{\omega}_t^B}(q,t)\ , \quad r = 1,\ldots,n, \tag{132}$$

resulting in

$$\begin{aligned} ^{\mathcal{R}}\boldsymbol{\alpha}^B &= \sum_{r=1}^{n} {^{\mathcal{R}}\boldsymbol{\omega}_r^B}(q,t)\dot{u}_r + \sum_{r=1}^{n} \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_r^B(q,t)]}{dt}u_r + \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_t^B(q,t)]}{dt}\ , \quad r = 1,\ldots,n \\ &= {^{\mathcal{R}}\boldsymbol{\omega}_I^B}(q,t)\dot{u}_I + {^{\mathcal{R}}\boldsymbol{\omega}_D^B}(q,t)\dot{u}_D + \sum_{r=1}^{n} \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_r^B(q,t)]}{dt}u_r + \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_t^B(q,t)]}{dt}, \end{aligned} \tag{133}$$

where

$$^{\mathcal{R}}\boldsymbol{\omega}_I^B = \lfloor {^{\mathcal{R}}\boldsymbol{\omega}_1^B} \ldots {^{\mathcal{R}}\boldsymbol{\omega}_p^B} \rfloor \tag{134}$$

and

$$^{\mathcal{R}}\boldsymbol{\omega}_D^B = \lfloor {^{\mathcal{R}}\boldsymbol{\omega}_{p+1}^B} \ldots {^{\mathcal{R}}\boldsymbol{\omega}_n^B} \rfloor. \tag{135}$$

Substituting the expression (120) for $\dot{u}_D$ in Eq. (133) yields

$$\begin{aligned} ^{\mathcal{R}}\boldsymbol{\alpha}^B &= [{^{\mathcal{R}}\boldsymbol{\omega}_I^B}(q,t) + {^{\mathcal{R}}\boldsymbol{\omega}_D^B}(q,t)A(q,u,t)]\dot{u}_I \\ &\quad + {^{\mathcal{R}}\boldsymbol{\omega}_D^B}(q,t)B(q,u,t) + \sum_{r=1}^{n} \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_r^B(q,t)]}{dt}u_r + \frac{d[^{\mathcal{R}}\boldsymbol{\omega}_t^B(q,t)]}{dt}. \end{aligned} \tag{136}$$

The second expression for $^{\mathcal{R}}\boldsymbol{\alpha}^B$ is obtained by taking the time derivative of $^{\mathcal{R}}\boldsymbol{\omega}^B$ represented in terms of the nonholonomic partial angular velocities,

$$^{\mathcal{R}}\boldsymbol{\omega}^B = \sum_{r=1}^{p} {^{\mathcal{R}}\tilde{\boldsymbol{\omega}}_r^B}(q,t)u_r + {^{\mathcal{R}}\tilde{\boldsymbol{\omega}}_t^B}(q,t)\ , \quad r = 1,\ldots,p, \tag{137}$$

resulting in

$$
\begin{aligned}
{}^{\mathcal{R}}\alpha^{B} &= \sum_{r=1}^{p} {}^{\mathcal{R}}\tilde{\omega}_r^B(q,t)\dot{u}_r + \sum_{r=1}^{p} \frac{d[{}^{\mathcal{R}}\tilde{\omega}_r^B(q,t)]}{dt} u_r + \frac{d[{}^{\mathcal{R}}\tilde{\omega}_t^B(q,t)]}{dt}, \quad r = 1,\ldots,p \\
&= {}^{\mathcal{R}}\tilde{\omega}_I^B(q,t)\dot{u}_I + \sum_{r=1}^{n} \frac{d[{}^{\mathcal{R}}\tilde{\omega}_r^B(q,t)]}{dt} u_r + \frac{d[{}^{\mathcal{R}}\tilde{\omega}_t^B(q,t)]}{dt},
\end{aligned}
\tag{138}
$$

where

$$
{}^{\mathcal{R}}\tilde{\omega}_I^B = \lfloor {}^{\mathcal{R}}\tilde{\omega}_1^B \ldots {}^{\mathcal{R}}\tilde{\omega}_p^B \rfloor. \tag{139}
$$

Comparing the coefficients of $\dot{u}_I$ in Eqs. (136) and (138) gives the relations

$$
{}^{\mathcal{R}}\tilde{\omega}_r^B = {}^{\mathcal{R}}\omega_r^B + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\omega_{p+s}^B A_{sr}(q,u,t) \quad r = 1,\ldots,p. \tag{140}
$$

Remark. *The relation between $\dot{u}_I$ and $\dot{u}_D$ given by Eq.(120) is similar to the relation between u_I and u_D in a simple nonholonomic system, except that the matrices A and B are functions of u. This results in relations between the holonomic and the nonholonomic partial velocities and partial angular velocities for nonlinearly constrained nonholonomic systems that are similar to their relations in a simple nonholonomic system [68], except that the matrix A is a function of u also, as given by Eqs. (131) and (140).*

3.4 Generalized Active and Inertia Forces

Eqs. (131) and (140) can be used to represent the nonholonomic generalized active and inertia forces in terms of the holonomic generalized active and inertia forces. Omitting the arguments for simplicity, these relations become

$$
\tilde{F}_r = F_r + \sum_{s=1}^{n-p} F_{p+s} A_{sr} \tag{141}
$$

$$
\tilde{F}_r^\star = F_r^\star + \sum_{s=1}^{n-p} F_{p+s}^\star A_{sr}, \qquad r = 1,\ldots,p. \tag{142}
$$

Therefore, Kane's equations of motion can be written as

$$
F_r + F_r^\star + \sum_{s=1}^{n-p} \left(F_{p+s} + F_{p+s}^\star\right) A_{sr} = 0, \qquad r = 1,\ldots,p. \tag{143}
$$

or in matrix form as

$$A_2 F^\star = -A_2 F, \tag{144}$$

where

$$A_2 := \begin{bmatrix} I & A^T \end{bmatrix}. \tag{145}$$

The accelerations and angular accelerations are linear in $\dot{u}$; it follows that the generalized inertia forces are as well. Consequently, $F^\star$ can be written in the form

$$F^\star = -Q(q,t)\dot{u} - L(q,u,t), \tag{146}$$

where Q is a symmetric positive definite matrix. Then, Eqs. (144) become

$$A_2(q,u,t)Q(q,t)\dot{u} = A_2 P(q,u,t), \tag{147}$$

where

$$P(q,u,t) = -L(q,u,t) + F(q,u,t), \tag{148}$$

and Q is the generalized inertia matrix of the system.

3.5 *Nonminimal System of Equations*

Eqs. (121) and (147) can be used to form the matrix system

$$T\dot{u} = V, \tag{149}$$

where $T := \begin{bmatrix} {A_1}^T & [A_2 Q]^T \end{bmatrix}^T$, and $V := \begin{bmatrix} B^T & [A_2 P]^T \end{bmatrix}^T$. The matrix T is a *constrained generalized inertia matrix* for the nonholonomic system $\mathcal{S}$. It is invertible for all choices of generalized coordinates and generalized speeds that render the elements of the constraint matrix A finite. Therefore,

$$\dot{u} = T^{-1}V. \tag{150}$$

Remark: *The appearance of the constraint matrix A in the dynamical equations (144) as well as in the constraint equations (121) exploits the feature of deriving the nonminimal* *form of equations for a nonholonomic system by simple manipulations of the equations of motion for the corresponding holonomic system.*

The two sets of ordinary differential equations (4) and (150) form a complete separated-in-accelerations state-space model for the constrained dynamical system, and involves no reduction in the dimension of the space of generalized speeds from the number of generalized coordinates to the number of degrees of freedom. Furthermore, this is obtained without employing Lagrange multipliers. Therefore, it enables the use of system analysis and control techniques that are related to state-space model representation, in a unified treatment of holonomic and nonholonomic constraints. This complements the previous differential algebraic equations (DAE) approach [107].

The procedure of using the acceleration form of constraints in obtaining a consistent set of separated in accelerations equations of motion for systems with nonlinear nonholonomic constraints is summarized as follows:

1. A set of generalized speeds satisfying Eq. (4) is chosen, and the nonlinear nonholonomic constraints, Eq. (110) are differentiated with respect to time. The set of generalized speeds is partitioned according to Eq. (116), and the dependency among the set is described at the acceleration level by Eq. (120). If holonomic constraints are involved, the corresponding equations are twice differentiated in time to appear in the same acceleration form.

2. The matrix A is used to construct the matrices A_1 and A_2, Eq. (122) and (145).

3. Expressions are obtained for holonomic partial velocities/angular velocities by inspecting the corresponding expressions for linear/angular velocities, as the coefficients of the generalized speeds.

4. Holonomic generalized active and inertia forces are found from the scalar (dot) product of the impressed and gravitational forces with the holonomic partial velocities/angular velocities, and used together with A_2 to form Eq. (147).

5. Eqs. (121) and (147) are used to form the matrix equation, Eq. (149), and T is inverted to yield the resulting equations of motion (150).

3.6 *Example 3.1: The Appell-Hamel Problem*

The mechanism shown in Fig. 2 consists of a frame with two legs that slide without friction on the x-y plane and supports two massless pulleys that are a distance ρ apart. A thread is passed around the pulleys, hanging a weight P that is idealized as a particle of mass m, and its movement is restricted to be along the vertical bar of the frame. The thread is wound around a drum of radius b, which is fixed to a wheel W of radius a, mass M, mass center W^*. The wheel rolls on the x-y plane, where ϕ is its angle of rotation in its own plane. For simplicity, it is specified that the wheel has equal axial and polar moments of inertia, I. The plane of W makes the angle θ with the x axis, and the frame keeps it vertical relative to the x-y plane. Let x, y, z be the coordinates of the center of mass of P in the xyz coordinate system, which is fixed to an inertial frame of reference. The configuration parameters can be chosen as x, y, z, θ, and ϕ. Finally, let $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ be unit vectors parallel to the positive x, y, and z directions, respectively, and let $\mathbf{i}_w$, $\mathbf{j}_w$, and $\mathbf{k}_w$ be wheel-fixed unit vectors parallel to $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, respectively when $\theta = 0$ and $\phi = 0$. The no slip condition of W on the plane xy gives rise to two relations that describe the velocity of the center of the wheel o,

$$\dot{x}_o = a\dot{\phi}\cos\theta \tag{151}$$

$$\dot{y}_o = a\dot{\phi}\sin\theta. \tag{152}$$

The velocity of o can also be described in terms of the velocity of P by the relations

$$\dot{x}_o = \dot{x} + \rho\dot{\theta}\sin\theta \tag{153}$$

$$\dot{y}_o = \dot{y} - \rho\dot{\theta}\cos\theta. \tag{154}$$

The relations (151) and (152) can be manipulated in order to create the nonlinear nonholonomic constraint equation [118, 9]

$$\dot{x}_o^2 + \dot{y}_o^2 = a^2\dot{\phi}^2, \tag{155}$$

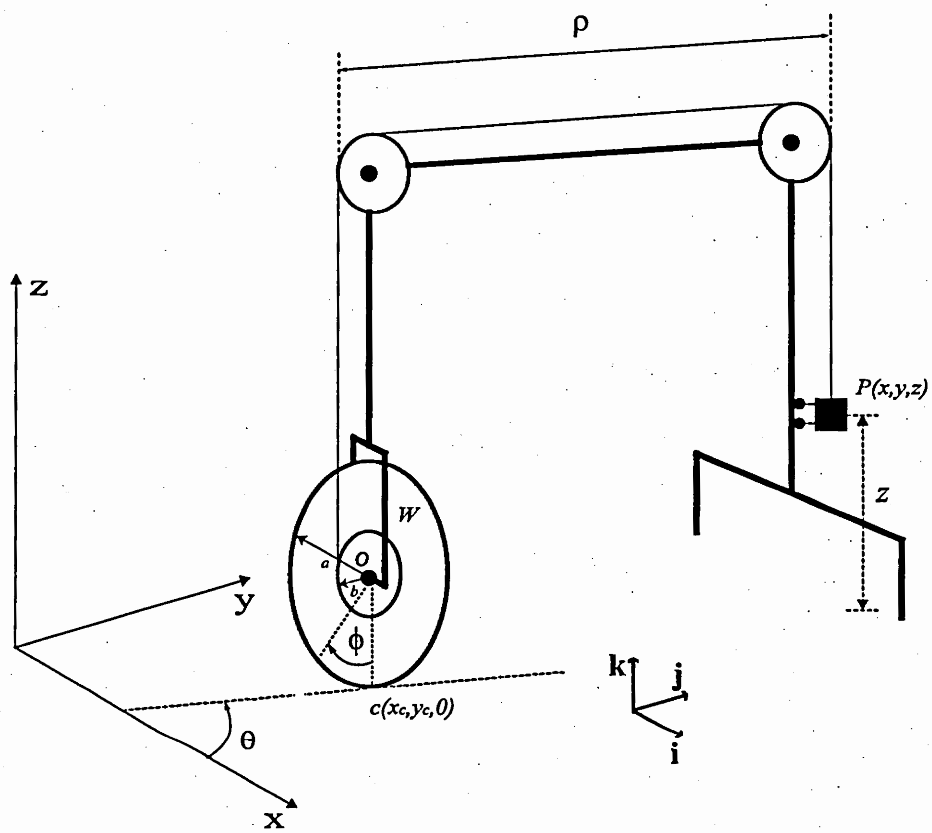

Figure 2: Schematic for the Appell-Hamel mechanism

and the linear nonholonomic constraint equation

$$\dot{x}_o \sin\theta - \dot{y}_o \cos\theta = 0. \tag{156}$$

Substituting Eqs. (153) and (154) into Eqs. (155) and (156) yields

$$(\dot{x} + \rho\dot{\theta}\sin\theta)^2 + (\dot{y} - \rho\dot{\theta}\cos\theta)^2 - a^2\dot{\phi}^2 = 0, \tag{157}$$

and

$$\dot{x}\sin\theta - \dot{y}\cos\theta + \rho\dot{\theta} = 0. \tag{158}$$

The inextensibility of the thread gives rise to the holonomic constraint equation

$$z = -b\phi + z_0, \tag{159}$$

where z_0 is a constant. Hence, the system has two degrees of freedom. Considering the generalized speeds $u_1 = \dot{\theta}$, $u_2 = \dot{\phi}$, $u_3 = \dot{x}$, $u_4 = \dot{y}$, $u_5 = \dot{z}$, and taking the time derivatives of the constraint equations (157)-(159), the acceleration form of the constraint equations is

$$\begin{aligned}(u_3 + \rho u_1 \sin\theta)(\dot{u}_3 + \rho\dot{u}_1 \sin\theta + \rho u_1^2 \cos\theta)& \\ + (u_4 - \rho u_1 \cos\theta)(\dot{u}_4 - \rho\dot{u}_1\cos\theta + \rho u_1^2 \sin\theta) &= a^2 u_2 \dot{u}_2\end{aligned} \tag{160}$$

$$\dot{u}_3 \sin\theta + u_1 u_3 \cos\theta - \dot{u}_4 \cos\theta + u_1 u_4 \sin\theta + \rho\dot{u}_1 = 0 \tag{161}$$

$$\dot{u}_5 = -b\dot{u}_2. \tag{162}$$

Let

$$u_I = \lfloor u_1 \;\; u_2 \rfloor \tag{163}$$

$$u_D = \lfloor u_3 \;\; u_4 \;\; u_5 \rfloor, \tag{164}$$

then the matrices J_1 and J_2 for the system are

$$J_1 = \begin{bmatrix} n_1 & n_2 \\ \rho & 0 \\ 0 & b \end{bmatrix} \tag{165}$$

$$J_2 = \begin{bmatrix} n_3 & n_4 & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{166}$$

where

$$n_1 = \rho^2 u_1 + \rho u_3 \sin\theta - \rho u_4 \cos\theta \tag{167}$$

$$n_2 = -a^2 u_2 \tag{168}$$

$$n_3 = u_3 + \rho u_1 \sin\theta \tag{169}$$

$$n_4 = u_4 - \rho u_1 \cos\theta. \tag{170}$$

The matrices A and B in Eq. (120) for the system are

$$A(q,u,t) = \frac{1}{n_5} \begin{bmatrix} -\rho n_4 - n_1 \cos\theta & -n_2 \cos\theta \\ \rho n_3 - n_1 \sin\theta & -n_2 \sin\theta \\ 0 & -n_5 b \end{bmatrix} \tag{171}$$

and

$$B(q,u,t) = \frac{n_6}{n_5} \begin{bmatrix} -u_1(n_4 + \rho u_1 \cos\theta) \\ u_1(n_3 - \rho u_1 \sin\theta) \\ 0 \end{bmatrix}, \tag{172}$$

where

$$n_5 = n_3 \cos\theta + n_4 \sin\theta \tag{173}$$

$$n_6 = u_3 \cos\theta + u_4 \sin\theta. \tag{174}$$

Therefore, the matrices A_1 and A_2 are

$$A_1 = \begin{bmatrix} \frac{\rho n_4 + n_1\cos\theta}{n_5} & \frac{n_2\cos\theta}{n_5} & 1 & 0 & 0 \\ \frac{-\rho n_3 + n_1\sin\theta}{n_5} & \frac{n_2\sin\theta}{n_5} & 0 & 1 & 0 \\ 0 & b & 0 & 0 & 1 \end{bmatrix} \tag{175}$$

and

$$A_2 = \begin{bmatrix} 1 & 0 & \frac{-\rho n_4 - n_1\cos\theta}{n_5} & \frac{\rho n_3 - n_1\sin\theta}{n_5} & 0 \\ 0 & 1 & \frac{-n_2\cos\theta}{n_5} & \frac{-n_2\sin\theta}{n_5} & -b \end{bmatrix}. \tag{176}$$

The inertial velocity of P is

$$\mathbf{v}^P = u_3\mathbf{i} + u_4\mathbf{j} + u_5\mathbf{k}, \tag{177}$$

and its inertial acceleration is

$$\mathbf{a}^P = \dot{u}_3\mathbf{i} + \dot{u}_4\mathbf{j} + \dot{u}_5\mathbf{k}. \tag{178}$$

The applied force on P is

$$\mathbf{F}_P = -mg\mathbf{k}, \tag{179}$$

where g is the gravitational constant. Hence, the generalized active forces on P are contained in the column matrix

$$F_P = \lfloor 0 \;\; 0 \;\; 0 \;\; 0 \;\; -mg \rfloor^T. \tag{180}$$

Similarly, the generalized inertia forces are contained in the column matrix

$$F_P^\star = \lfloor 0 \;\; 0 \;\; -m\dot{u}_3 \;\; -m\dot{u}_4 \;\; -m\dot{u}_5 \rfloor^T. \tag{181}$$

Since the applied forces acting on W are all in the vertical direction, they do not contribute to the generalized active forces. The inertial angular velocity of W is

$$\boldsymbol{\omega}^W = -u_2\sin\theta\mathbf{i} + u_2\cos\theta\mathbf{j} + u_1\mathbf{k}, \tag{182}$$

and its angular acceleration is

$$\boldsymbol{\alpha}^W = (-\dot{u}_2\sin\theta - u_1u_2\cos\theta)\,\mathbf{i} + (\dot{u}_2\cos\theta - u_1u_2\sin\theta)\,\mathbf{j} + \dot{u}_1\mathbf{k}. \tag{183}$$

The velocity of the center of mass of the wheel is

$$\mathbf{v}^o = (u_3 + \rho u_1 \sin\theta)\mathbf{i} + (u_4 - \rho u_1 \cos\theta)\mathbf{j}, \tag{184}$$

and its acceleration is

$$\mathbf{a}^o = (\dot{u}_3 + \rho \dot{u}_1 \sin\theta + \rho u_1^2 \cos\theta)\mathbf{i} + (\dot{u}_4 - \rho \dot{u}_1 \cos\theta + \rho u_1^2 \sin\theta)\mathbf{j}. \tag{185}$$

The generalized inertia forces of the wheel are given by

$$F^{\star}_{W_r} = \mathbf{F}^{\star}_o \cdot \mathbf{v}^o_r + \mathbf{T}^{\star}_W \cdot \boldsymbol{\omega}^W_r, \tag{186}$$

where the inertia force $\mathbf{F}^{\star}_o$ is

$$\mathbf{F}^{\star}_o = -M\mathbf{a}^o, \tag{187}$$

and the inertia torque $\mathbf{T}^{\star}_W$ is

$$\mathbf{T}^{\star}_W = -\boldsymbol{\alpha}^W \cdot \underline{\mathbf{I}}^W - \boldsymbol{\omega}^W \times \underline{\mathbf{I}}^W \cdot \boldsymbol{\omega}^W. \tag{188}$$

Here, $\underline{\mathbf{I}}^W$ denotes the central inertia dyadic of W [68]. The relation between the wheel-fixed and the inertial frame-fixed unit vectors is given by:

$$\begin{Bmatrix} \mathbf{i}_w \\ \mathbf{j}_w \\ \mathbf{k}_w \end{Bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}. \tag{189}$$

Hence,

$$\begin{aligned} \underline{\mathbf{I}}^W &= I(\mathbf{i}_w\mathbf{i}_w + \mathbf{j}_w\mathbf{j}_w + \mathbf{k}_w\mathbf{k}_w) & (190) \\ &= I(\mathbf{ii} + \mathbf{jj} + \mathbf{kk}), & (191) \end{aligned}$$

and the inertia torque is,

$$\mathbf{T}^{\star}_W = -I\dot{u}_1\mathbf{k} - I\dot{u}[\cos\theta\mathbf{j} - \sin\theta\mathbf{i}] - u_1u_2I[-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}]. \tag{192}$$

Therefore, the contribution of the wheel to the generalized inertia forces is given by

$$F^\star_{W1} = -M(\rho^2\dot{u}_1 + \rho\dot{u}_3\sin\theta - \rho\dot{u}_4\cos\theta) - I\dot{u}_1 \tag{193}$$

$$F^\star_{W2} = -I\dot{u}_2 \tag{194}$$

$$F^\star_{W3} = -M(\dot{u}_3 + \rho\dot{u}_1\sin\theta + \rho {u_1}^2\cos\theta) \tag{195}$$

$$F^\star_{W4} = -M(\dot{u}_4 - \rho\dot{u}_1\cos\theta + \rho {u_1}^2\sin\theta) \tag{196}$$

$$F^\star_{W5} = 0. \tag{197}$$

The generalized inertia forces for the system are given by

$$F^\star = F^\star_P + F^\star_W. \tag{198}$$

Therefore, with the above mentioned choice of generalized speeds,

$$Q = \begin{bmatrix} M\rho^2 + I & 0 & M\rho\sin\theta & -M\rho\cos\theta & 0 \\ 0 & I & 0 & 0 & 0 \\ M\rho\sin\theta & 0 & M+m & 0 & 0 \\ -M\rho\cos\theta & 0 & 0 & M+m & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \tag{199}$$

$$P = \begin{Bmatrix} 0 \\ 0 \\ -M\rho {u_1}^2\cos\theta \\ -M\rho {u_1}^2\sin\theta \\ -mg \end{Bmatrix}. \tag{200}$$

Forming Eqs. (121) and (147) for this system and augmenting the two equations yields Eq. (149). The equations of motion are

$$\frac{\rho n_4 + n_1\cos\theta}{n_5}\dot{u}_1 + \frac{n_2\cos\theta}{n_5}\dot{u}_2 + \dot{u}_3 = -\frac{n_6}{n_5}u_1(n_4 + \rho u_1\cos\theta) \tag{201}$$

$$\frac{-\rho n_3 + n_1\sin\theta}{n_5}\dot{u}_1 + \frac{n_2\sin\theta}{n_5}\dot{u}_2 + \dot{u}_4 = \frac{n_6}{n_5}u_1(n_3 - \rho u_1\sin\theta) \tag{202}$$

$$b\dot{u}_2 + \dot{u}_5 = 0 \tag{203}$$

$$\begin{aligned}
&\left[M\rho^2 + I + \frac{-\rho n_4 - n_1\cos\theta}{n_5}M\rho\sin\theta - \frac{\rho n_3 - n_1\sin\theta}{n_5}M\rho\cos\theta\right]\dot{u}_1 \\
&+M\rho\sin\theta + (M+m)\left[\frac{-\rho n_4 - n_1\cos\theta}{n_5}\right]\dot{u}_3 \\
&+\left[-M\rho\cos\theta + \frac{\rho n_3 - n_1\sin\theta}{n_5}(M+m)\right]\dot{u}_4 \\
&= -\left[\frac{-\rho n_4 - n_1\cos\theta}{n_5}\right]M\rho {u_1}^2\cos\theta - \left[\frac{\rho n_3 - n_1\sin\theta}{n_5}\right]M\rho {u_1}^2\sin\theta
\end{aligned} \tag{204}$$

$$\begin{aligned}
&\left[\frac{-n_2\cos\theta}{n_5}M\rho\sin\theta - \frac{-n_2\sin\theta}{n_5}M\rho\cos\theta\right]\dot{u}_1 + \dot{u}_2 \\
&+\left[\frac{-n_2\cos\theta}{n_5}(M+m)\right]\dot{u}_3 + \left[\frac{-n_2\sin\theta}{n_5}(M+m)\right]\dot{u}_4 - \frac{b}{n_5}m\dot{u}_5 \\
&= \frac{n_2\cos\theta}{n_5}M\rho {u_1}^2\cos\theta + \frac{n_2\sin\theta}{n_5}M\rho {u_1}^2\sin\theta + \frac{b}{n_5}mg.
\end{aligned} \tag{205}$$

$\dot{u}$ can be obtained by inverting the coefficient matrix $T := \left[{A_1}^T \quad [A_2Q]^T\right]^T$. This can be done for all values of generalized coordinates and generalized speeds that give nonzero values of n_5.

The inversion of the matrix T can be done either numerically or symbolically. The symbolic inversion results in lengthy expressions that are not needed for our purpose. The time simulations must run with initial conditions that satisfy the constraint equations. These are chosen to be $\dot{\theta} = \dot{\phi} = 1.0$ rad/sec, $\dot{x} = 1.0$ m/sec, $\dot{y} = 5.0$ m/sec, $\dot{z} = -0.5$ m/sec, $z = 30$ m, and zero initial conditions for the remaining generalized coordinates. Time simulations are performed with $a = 1.0$ m, $b = 0.5$ m, $\rho = 5.0$ m, $m = 1.0$ Kg, $M = 5.0$ Kg, $z_0 = 30$ m. Figs. 3 and 4 show the responses of θ and ϕ respectively. The responses of the time rates of change of these angles, $\dot{\theta}$ and $\dot{\phi}$ are shown in Figs. 5 and 6 respectively. The angle θ tends to reach a constant steady state value as the wheel continues to roll over

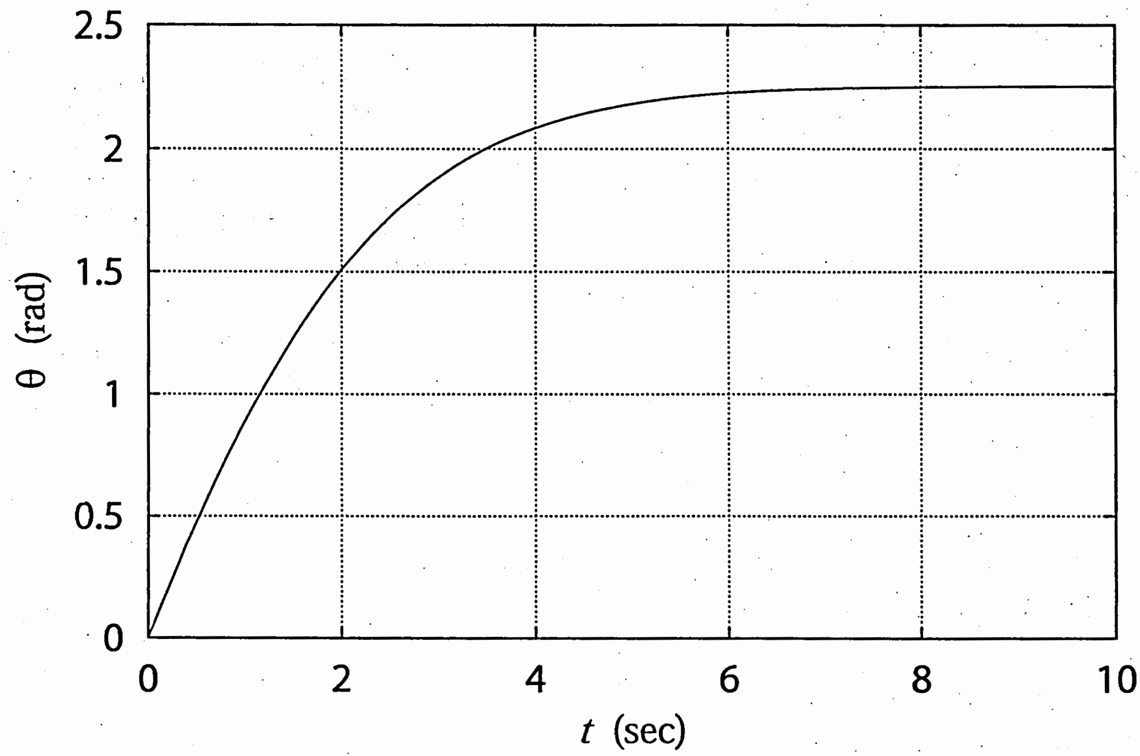

Figure 3: Example 3.1: θ versus t

the x-y plane, as shown in Fig. 7. Also, the load P intercept on the x-y plane is shown in Fig. 8, and the time history of its height z is shown in Fig. 9.

3.7 Nonlinear Nonholonomic Constraints and Numerical Stability of the Equations of Motion

The problem of numerical drift of constraints and integrals of motion is well known in the solutions of differential equations subjected to constraints. Several methods were introduced in remedy for this problem. Every method has its advantages and disadvantages, but all these methods involve modifications to the dynamical equations in order to suppress the numerical violation. Stabilizing the constraint equations and the dynamical equations are interfering, and one should be careful when implementing a constraint stabilizing scheme, as the modification can alter the dynamics of the whole system in addition to its effect on

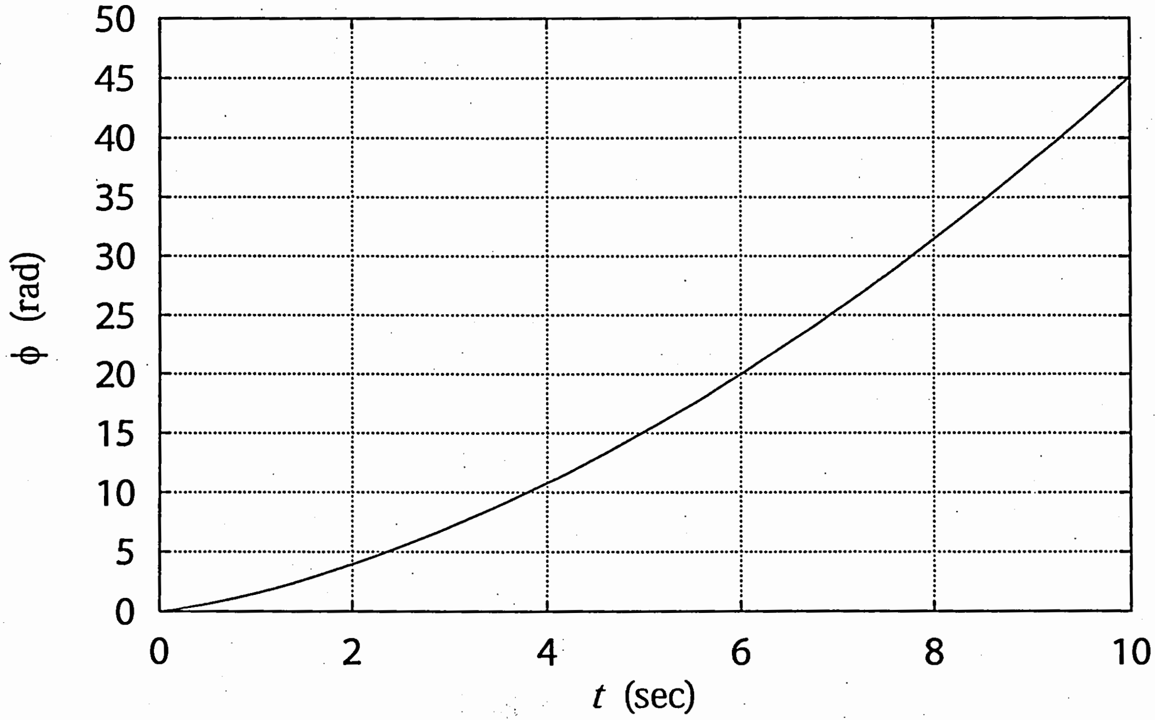

Figure 4: Example 3.1: ϕ versus t

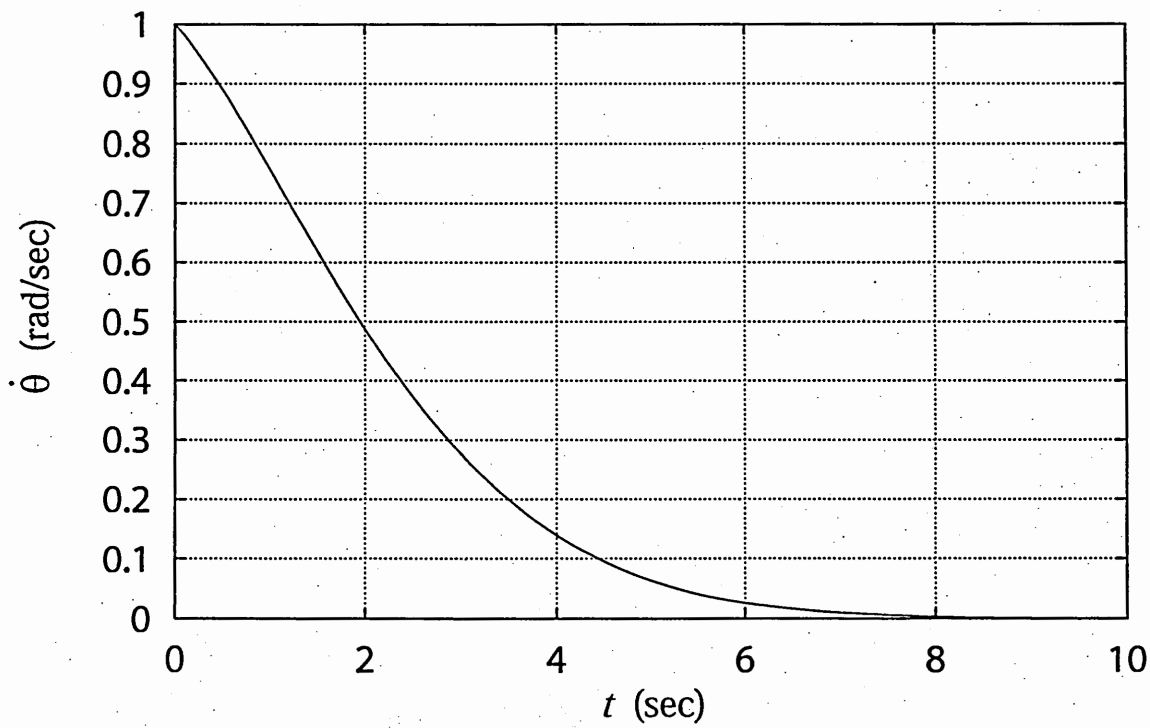

Figure 5: Example 3.1: $\dot{\theta}$ versus t

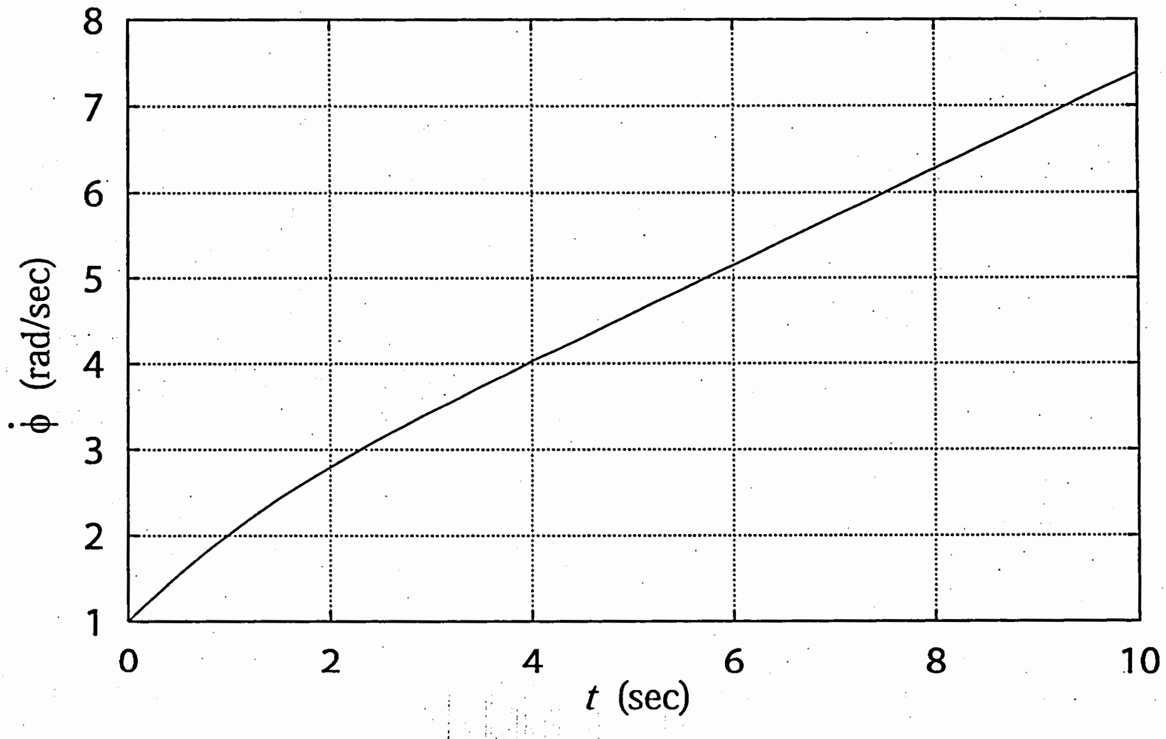

Figure 6: Example 3.1: $\dot{\phi}$ versus t

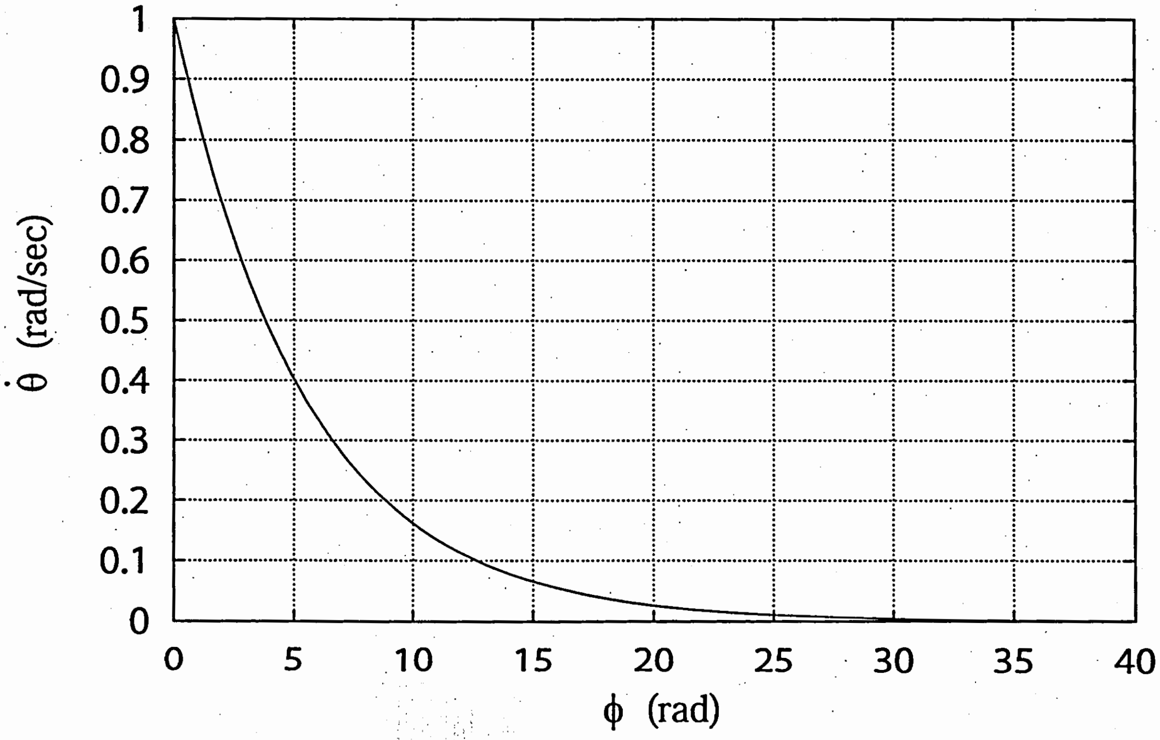

Figure 7: Example 3.1: $\dot{\theta}$ versus ϕ

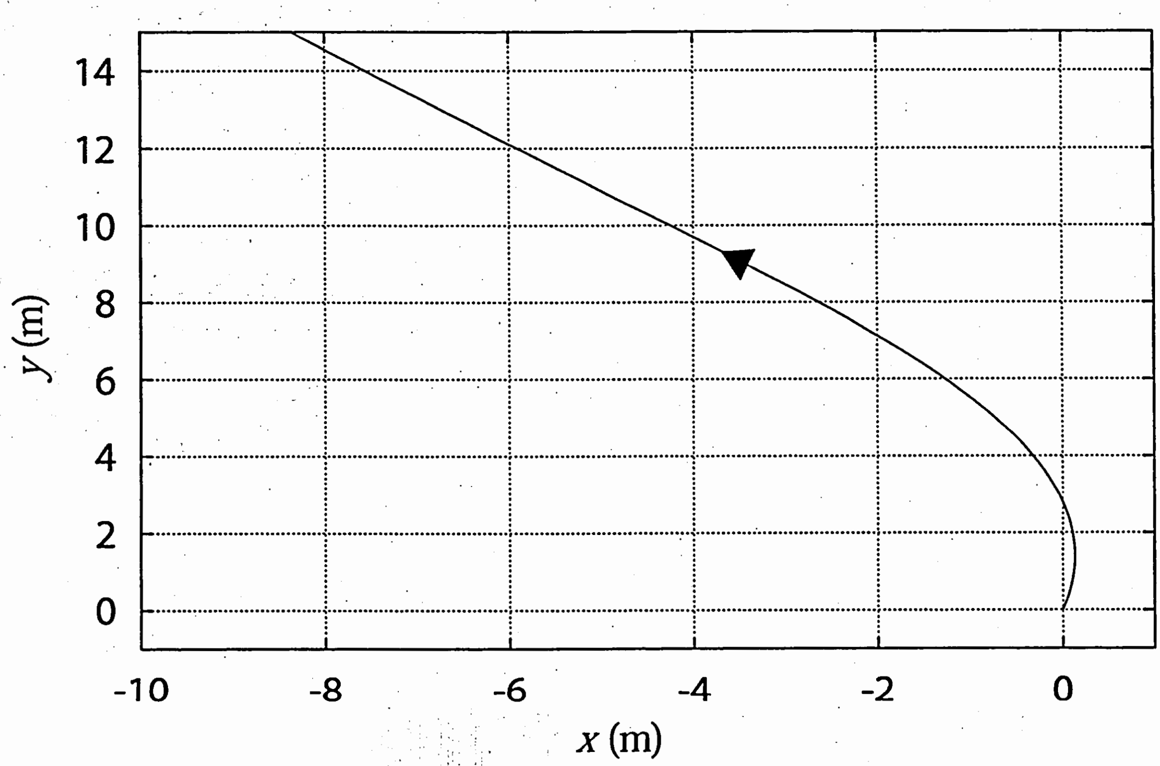

Figure 8: Example 3.1: y versus x

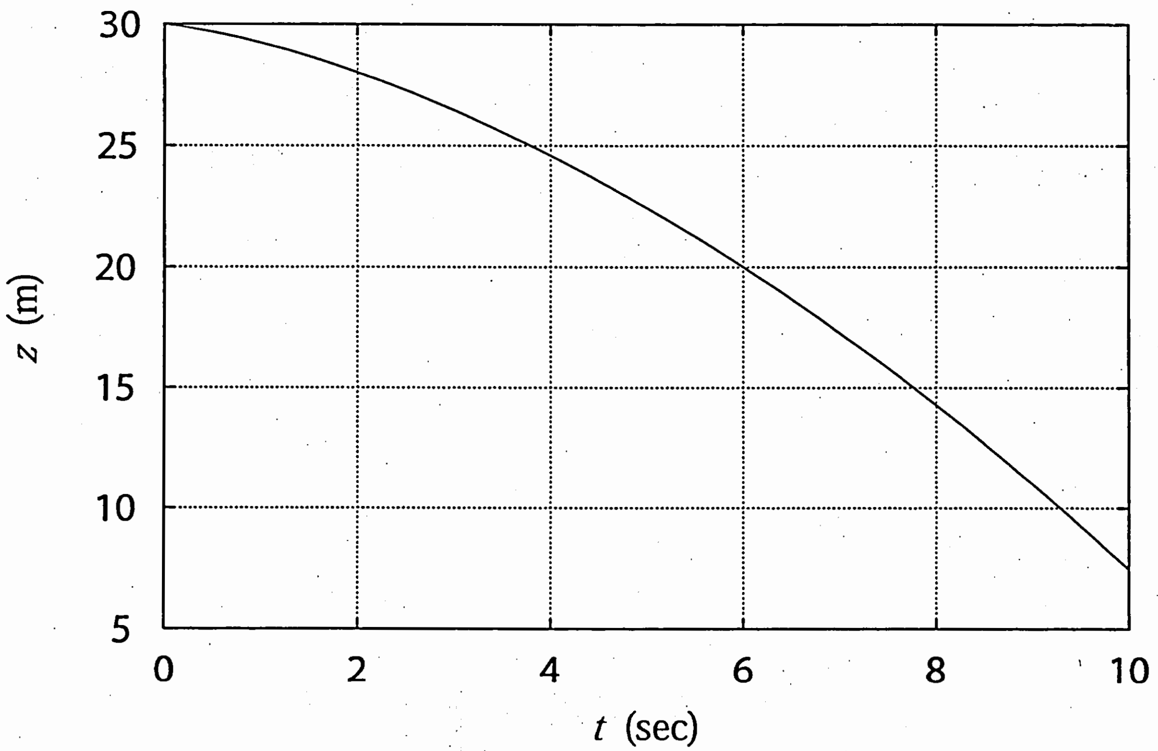

Figure 9: Example 3.1: z versus t

the constraint dynamics. Alternatively, it is possible to consider the issue of numerical stability during the modeling phase, to avoid the need to correct the motion by modifying the already formulated equations of motion.

For a given system, modeling the constraints to be nonlinear in the velocities results in equations of motion that are different in appearance from the equations of motion resulting from modeling the constraints as linear in the velocities. However, the solutions of the resulting equations of motion and the time simulations should not be different, irrespective of the way the constraint equations are manipulated in order to be augmented with the dynamical equations. Nevertheless, the numerical stability of the solution is certainly affected by the constraints modeling. In that regard, it can be beneficial to use the nonlinearity of the constraint equations as a passive tool to suppress the numerical errors. To illustrate that, we create the linear nonholonomic constraint equations by equating the equations (151) and (152) with the equations (153) and (154). The resulting constraint equations are

$$a\dot{\phi}\cos\theta - \dot{x} - \rho\dot{\theta}\sin\theta = 0 \tag{206}$$

$$a\dot{\phi}\sin\theta - \dot{y} + \rho\dot{\theta}\cos\theta = 0. \tag{207}$$

The holonomic constraint equation in the kinematical form is

$$\dot{z} = -b\dot{\phi}. \tag{208}$$

Taking the time derivatives of Eqs. (206)-(208), the same procedure can be used to obtain the equations of motion for the Appell-Hamel mechanism with the constraints modeled as linear nonholonomic. The constraint matrices are

$$A_1 = \begin{bmatrix} \rho\sin\theta & -a\cos\theta & 1 & 0 & 0 \\ -\rho\cos\theta & -a\sin\theta & 0 & 1 & 0 \\ 0 & b & 0 & 0 & 1 \end{bmatrix} \tag{209}$$

$$A_2 = \begin{bmatrix} 1 & 0 & -\rho\sin\theta & \rho\cos\theta & 0 \\ 0 & 1 & a\cos\theta & a\sin\theta & b \end{bmatrix} \tag{210}$$

$$B = \begin{bmatrix} -au_1u_2\sin\theta - \rho{u_1}^2\cos\theta \\ -\rho{u_1}^2\sin\theta + au_1u_2\cos\theta \\ 0 \end{bmatrix}, \tag{211}$$

and the equations of motion become

$$\rho\sin\theta\dot{u}_1 - a\cos\theta\dot{u}_2 + \dot{u}_3 = -au_1u_2\sin\theta - \rho{u_1}^2\cos\theta \tag{212}$$

$$-\rho\cos\theta\dot{u}_1 - a\sin\theta\dot{u}_2 + \dot{u}_5 = -\rho{u_1}^2\sin\theta + au_1u_2\cos\theta \tag{213}$$

$$b\dot{u}_2 + \dot{u}_5 = 0 \tag{214}$$

$$\dot{u}_1 - m\rho\sin\theta\dot{u}_3 + m\rho\cos\theta\dot{u}_4 = 0 \tag{215}$$

$$\begin{aligned} &\dot{u}_2 + a\cos\theta(M+m)\dot{u}_3 + a\sin\theta(M+m)\dot{u}_4 + bm\dot{u}_5 \\ &= -aM\rho{u_1}^2 - bmg. \end{aligned} \tag{216}$$

By running the time simulations for both systems of equations, a common numerical violation measure can be tested, that is the total energy E of the mechanism. Considering the xy plane as the datum for computing the potential energy, E is given as

$$\begin{aligned} E = \frac{I}{2}({u_1}^2 + {u_2}^2) + \frac{M}{2}[(u_3 + \rho u_1\sin\theta)^2 + (u_4 - \rho u_1\cos\theta)^2] \\ + \frac{M}{2}ga + \frac{m}{2}({u_3}^2 + {u_4}^2 + {u_5}^2) + mgz. \end{aligned} \tag{217}$$

Fig. 10 shows the plots of E by using the state variables obtained from integrating the equations of motion that correspond to the two types of constraint modeling, where ΔE is the difference between the computed value of the energy and its initial value. It is noticed that the nonlinearity in the constraint equations subdues the growing deviation in the total energy of the mechanism.

Nevertheless, the error dynamics for nonlinear systems depends on the initial errors in the state variables, and on the input forces on the system. These can vary substantially during the simulation process, and manipulating the constraint equations provides no guarantee of error convergence, which frequently necessitates an implementation of a corrective

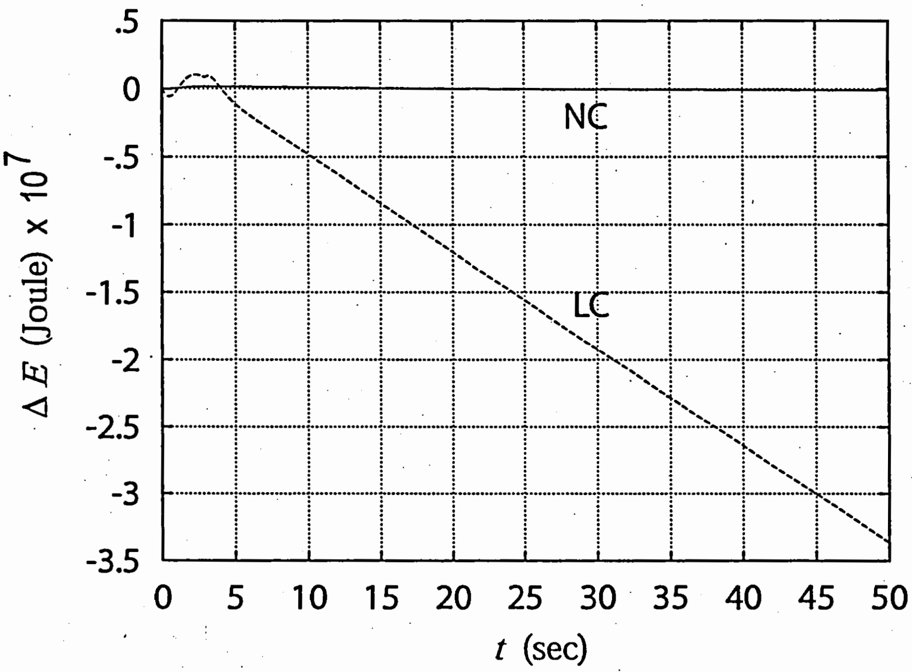

Figure 10: Example 3.1: Energy integral, LC: linear constraints, NC: nonlinear constraints

scheme. A constraint numerical stabilizing scheme for the derived nonminimal equations is introduced in the fourth chapter.

3.8 Summary

Nonholonomic constraint equations that are nonlinear in velocities are incorporated with Kane's dynamical equations by utilizing the acceleration form of constraints, resulting in Kane's nonminimal equations of motion, i.e., the equations that involve the full set of generalized accelerations. Together with the kinematical differential equations, these equations form a state-space model that is full-order, separated in the derivatives of the states, and involves no Lagrange multipliers. The method is illustrated by using it to obtain nonminimal equations of motion for the classical Appell-Hamel problem when the constraints are modeled as nonlinear in the velocities. It is shown that this fictitious nonlinearity has a predominant effect on the numerical stability of the dynamical equations, and hence it is possible to use it for improving simulations accuracy. Although the numerical accuracy may increase, the complexity in the dynamical equations resulting from this nonlinearity makes it difficult to use the equations for system analysis. For the purpose of system analysis, the simpler model that corresponds to the linear nonholonomic constraint equations is preferred.

CHAPTER IV

CONSTRAINT FORCES

4.1 Introduction

Similar to Lagrange's fundamental equations, Kane's equations do not identify the constraint forces directly. The way introduced in Ref. [68] to bring these forces into evidence is to define a set of generalized speeds that violate the constraints, without considering more generalized coordinates. This results in an increase in the number of partial velocities, and the number of the governing equations, from which the constraint forces and moments are determined. The choice of these generalized speeds is not unique.

In this chapter, the acceleration form of the constraint equations is used to simply and systematically obtain explicit analytical expressions for the constraint forces and moments, without the introduction of auxiliary generalized speeds or Lagrange multipliers. The process is outlined with an illustrative example.

In spite of the advantages of modeling multibody systems using the acceleration form of constraint equations, the accuracy of numerical simulations may degenerate due to constraint violations caused by enforcing the constraint equations at the acceleration level. This is especially true for the case of holonomic constraints, as the equations must be numerically integrated twice to obtain the generalized coordinates. The explicit appearance of the acceleration form of constraint equations facilitates the augmentation of Baumgarte type damping terms [18] prior to using the acceleration form of the constraint equations with the dynamical equations and inverting the generalized inertia matrix, in case it is necessary to modify the dynamical equations in order to suppress this violation. The procedure is illustrated by stabilizing the nonminimal equations for a holonomically constrained pendulum with a varying length.

4.2 *Analytical Expressions for Constraint Forces*

Since the constraint equations are differentiated with respect to time in our framework, it can be assumed, without loss of generality, that the dynamical system is subjected to nonholonomic constraints only; hence, in this context, "holonomic" and "unconstrained" have the same meaning. The equation

$$F(q,u,t) + F^{\star}(q,u,\dot{u},t) = 0 \tag{218}$$

is considered the equation for constraint-free systems in this sense.

Consider now a constrained system S consisting of a set of ν particles and μ rigid bodies. Let the resultant forces and moments exerted on S by the constraints be the force ${\mathbf{F}^{c}}_{P_i}$ on the ith particle, ${\mathbf{F}^{c}}_{Q_j}$ on the point Q_j of the jth body, and the moment ${\mathbf{M}^{c}}_{B_j}$ on the jth body, $i = 1 \ldots \nu$, $j = 1 \ldots \mu$. Also, let $\mathbf{v}_r^{P_i}$, $\mathbf{v}_r^{Q_j}$, $\omega_r^{B_j}$ be the rth partial velocity of the ith particle, the rth partial velocity of the point Q_j, and the rth partial angular velocity of the jth body, respectively. These holonomic (unconstrained) partial velocities and partial angular velocities satisfy the relations

$$\mathbf{v}^{P_i} = \sum_{r=1}^{n} u_r \mathbf{v}_r^{P_i} + \mathbf{v}_t^{P_i} \tag{219}$$

$$\mathbf{v}^{Q_j} = \sum_{r=1}^{n} u_r \mathbf{v}_r^{Q_i} + \mathbf{v}_t^{Q_j} \tag{220}$$

$$\omega^{B_j} = \sum_{r=1}^{n} u_r \omega_r^{B_j} + \omega_t^{B_j}, \tag{221}$$

where n is the number of generalized coordinates, the u_r's are the generalized speeds defined for the system, and $\mathbf{v}_t^{P_i}$, $\mathbf{v}_t^{Q_j}$, $\omega_t^{B_j}$ are possibly time-varying functions of the generalized coordinates but are independent of the generalized speeds. We define F^c as the *generalized constraint force* the rth component of which is given by

$$F_r^c = \sum_{i=1}^{\nu} {\mathbf{F}^{c}}_{P_i} \cdot \mathbf{v}_r^{P_i} + \sum_{j=1}^{\mu} {\mathbf{F}^{c}}_{Q_j} \cdot \mathbf{v}_r^{Q_i} + \sum_{j=1}^{\mu} {\mathbf{M}^{c}}_{B_j} \cdot \omega_r^{B_j}, \quad r = 1, \ldots, n. \tag{222}$$

Since the deviation in $\dot{u}$ from an unconstrained value can occur only because of the constraint force, an equation for constrained motion can be written by adding F^c to the left hand side of Eq. (218), so that

$$F(q,u,t) + F^{\star}(q,u,\dot{u},t) + F^{c}(q,u,t) = 0. \tag{223}$$

Substituting the representation (146) for generalized inertia forces

$$F^{\star} = -Q\dot{u} - L(q,u,t) \tag{224}$$

into Eq. (223) yields

$$F(q,u,t) - Q\dot{u} - L(q,u,t) + F^{c}(q,u,t) = 0 \tag{225}$$

Substituting the expression (48) for $\dot{u}$ into Eq. (225), and solving for F^c gives

$$F^{c} = -F(q,u,t) + QT^{-1}V + L(q,u,t). \tag{226}$$

Remark. *Generalized constraint forces derived here are inertially based, because the velocities and accelerations used to derive Eq. (226) are in an inertial frame. In general, active forces may contain acceleration terms [28]. Examples are control forces that depend on the accelerations of the controlled plants. In such cases, the generalized constraint forces would be dependent on $\dot{u}$. For simplicity, we consider the active forces to be dependent only on q, u, and t, and independent of $\dot{u}$.*

The procedure for finding constraint forces is summarized:

1. Generalized active and inertia forces satisfying Eq. (218) are derived for the corresponding unconstrained system.

2. Expressions for generalized accelerations are obtained by the procedures outlined in chapters 2 and 3.

3. Generalized accelerations and generalized inertia and active forces are substituted into Eq. (223) to solve for generalized constraint forces.

4.3 Example 4.1: Motion of a particle on an elliptical path

The two-dimensional motion of a particle P of mass m in the horizontal plane is shown in Fig. 11. The particle is forced to follow the frictionless elliptic track formed by a rigid wire fixed to an inertial frame $\mathcal{R}$. The coordinate system (x, y) is fixed to $\mathcal{R}$, with its origin at O. A spring with a stiffness constant K is attached from its two ends at the particle P and the point O. We are interested in finding the inplane constraint force exerted by the wire on the particle P. The elliptic motion of P defines the holonomic constraint

$$a\bar{x}^2 + by^2 = c, \tag{227}$$

where $\bar{x} = x - x_0$, such that x_0, a, b, and c are positive constants. Differentiating the above equation with respect to t yields:

$$a\bar{x}\dot{x} + by\dot{y} = 0. \tag{228}$$

Let the generalized speeds be $u_1 = \dot{x}$, and $u_2 = \dot{y}$. Eq. (228) can be written as

$$u_2 = -\epsilon u_1, \tag{229}$$

where $\epsilon = \frac{a\bar{x}}{by}$. Therefore,

$$A = -\epsilon, \quad B = 0. \tag{230}$$

The matrix A_1 is

$$A_1 = \begin{bmatrix} -A & I \end{bmatrix} = \lfloor \epsilon \quad 1 \rfloor. \tag{231}$$

Thus, Eq. (22) for this system is

$$\lfloor \epsilon \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \frac{abu_1(-yu_1 + \bar{x}u_2)}{(by)^2}. \tag{232}$$

The matrix A_2 is

$$A_2 = \begin{bmatrix} I & A^T \end{bmatrix} = \lfloor 1 \quad -\epsilon \rfloor. \tag{233}$$

The velocity of P in $\mathcal{R}$ is

$${}^{\mathcal{R}}\mathbf{v}^P = u_1\mathbf{i} + u_2\mathbf{j}, \tag{234}$$

where $\mathbf{i}$ and $\mathbf{j}$ are are unit vectors parallel to the positive x and y axes, respectively. Hence, the partial velocities are $\mathbf{v}_1 = \mathbf{i}$, and $\mathbf{v}_2 = \mathbf{j}$. P is subjected to the spring force

$$\mathbf{F}_s = -K(x\mathbf{i} + y\mathbf{j}). \tag{235}$$

Hence, the generalized active forces on P are

$$F_1 = -Kx, \quad F_2 = -Ky. \tag{236}$$

The acceleration of P in $\mathcal{R}$ is

$$^{\mathcal{R}}\mathbf{a}^P = \dot{u}_1\mathbf{i} + \dot{u}_2\mathbf{j}. \tag{237}$$

The generalized inertia forces on P are

$$F_1^\star = -m{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v}_1 = -m\dot{u}_1 \tag{238}$$

$$F_2^\star = -m{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v}_2 = -m\dot{u}_2. \tag{239}$$

Therefore, the generalized constraint force F^c can be written from Eq. (223) as

$$\begin{Bmatrix} F_1^c \\ F_2^c \end{Bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + K \begin{Bmatrix} x \\ y \end{Bmatrix}. \tag{240}$$

Eq. (37) for this system is

$$\lfloor -m \quad m\epsilon \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = -F_1 + \epsilon\, F_2 = \left(1 - \frac{a}{b}\right) Kx + \frac{a}{b}\, Kx_0. \tag{241}$$

Eqs. (232) and (241) can be used to form the matrix system

$$S\dot{u} = U, \tag{242}$$

where

$$S = \begin{bmatrix} \epsilon & 1 \\ -m & m\epsilon \end{bmatrix}, \tag{243}$$

and

$$U = \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} \tag{244}$$

where

$$e_1 := \frac{abu_1(-yu_1 + \bar{x}u_2)}{(by)^2} \tag{245}$$

$$e_2 := \left(1 - \frac{a}{b}\right) Kx + \frac{a}{b} Kx_0 = K\left[x - \frac{a}{b}\bar{x}\right]. \tag{246}$$

Solving for $\dot{u}$

$$\dot{u} = S^{-1}U. \tag{247}$$

This yields

$$\dot{u}_1 = \frac{1}{md}(m\epsilon e_1 - e_2) \tag{248}$$

$$\dot{u}_2 = \frac{1}{md}(me_1 + \epsilon e_2), \tag{249}$$

where

$$d := \epsilon^2 + 1. \tag{250}$$

Therefore, the constraint forces are found from Eq. (240) as

$$F_1^c = \frac{1}{d}(m\epsilon e_1 - e_2) + Kx \tag{251}$$

$$F_2^c = \frac{1}{d}(me_1 + \epsilon e_2) + Ky. \tag{252}$$

In vector form

$$\mathbf{F}^c = \left[\frac{1}{d}(m\epsilon e_1 - e_2) + Kx\right]\mathbf{i} + \left[\frac{1}{d}(me_1 + \epsilon e_2) + Ky\right]\mathbf{j}. \tag{253}$$

For the purpose of comparison, we use the method introduced in Ref. [68] to perform the same task. The system can be described in terms of one generalized coordinate. However, we will consider the same set of generalized coordinates and generalized speeds as above, using Eq. (228) to put the system in simple nonholonomic form. Therefore,

$$^{\mathcal{R}}\mathbf{v}^P = u_1\mathbf{i} + u_2\mathbf{j} \tag{254}$$

$$= u_1\left(\mathbf{i} - \frac{a\bar{x}}{by}\mathbf{j}\right) \tag{255}$$

$$= u_1(\mathbf{i} - \epsilon\mathbf{j}). \tag{256}$$

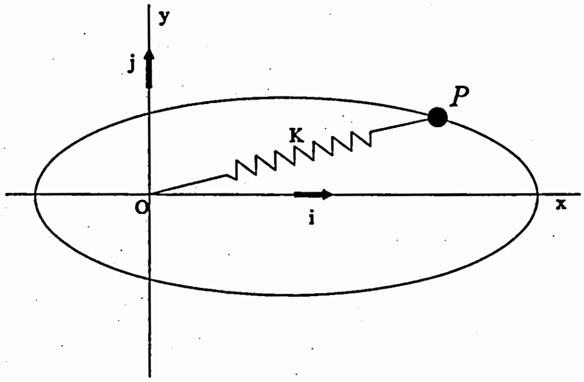

Figure 11: Schematic for Example 4.1

Hence, the partial velocity is

$$\mathbf{v_1} = \mathbf{i} - \epsilon\mathbf{j}. \tag{257}$$

The acceleration of P is

$$\begin{aligned} {}^{\mathcal{R}}\mathbf{a}^P &= \dot{u}_1(\mathbf{i} - \epsilon\mathbf{j}) + u_1\frac{ab(\bar{x}u_2 - yu_1)}{(by)^2}\,\mathbf{j} && (258) \\ &= \dot{u}_1\mathbf{i} + (-\dot{u}_1\epsilon + e_1)\mathbf{j}. && (259) \end{aligned}$$

Therefore, the generalized inertia force for the system is

$$\begin{aligned} F^\star &= -m{}^{\mathcal{R}}\mathbf{a}^P \cdot \mathbf{v_1} && (260) \\ &= -m[d\dot{u}_1 - e_1\epsilon]. && (261) \end{aligned}$$

The generalized active force for the system is

$$\begin{aligned} F = \mathbf{F_s} \cdot \mathbf{v_1} &= -K(x\mathbf{i} + y\mathbf{j}) \cdot (\mathbf{i} - \epsilon\,\mathbf{j}) && (262) \\ &= -e_2. && (263) \end{aligned}$$

Formulating Kane's equation and solving for $\dot{u}_1$ yields

$$\dot{u}_1 = \frac{1}{d}e_1\epsilon - \frac{e_2}{md}. \tag{264}$$

which is identical to Eq. (248). Now, we introduce an auxiliary generalized speed w such that

$${}^{\mathcal{R}}\mathbf{v}^P = u_1\mathbf{v_1} + w\boldsymbol{\tau} \tag{265}$$

where the partial velocity $\boldsymbol{\tau}$ is a unit vector orthogonal to the elliptic path of P, defined by the holonomic constraint,

$$\boldsymbol{\tau} = \frac{\epsilon\,\mathbf{i} + \mathbf{j}}{\sqrt{d}}. \tag{266}$$

The constraint force $\mathbf{F}^c = F^c\boldsymbol{\tau}$ is found from the equation

$$(-m^{\mathcal{R}}\mathbf{a}^P + \mathbf{F} + \mathbf{F}^c)\cdot\boldsymbol{\tau} = 0 \tag{267}$$

which gives

$$-\frac{1}{\sqrt{d}}[me_1 + K(\epsilon x + y)] + F^c = 0. \tag{268}$$

Therefore,

$$\begin{aligned}\mathbf{F}^c &= \frac{1}{\sqrt{d}}[me_1 + K(\epsilon x + y)]\boldsymbol{\tau} \\ &= \frac{\epsilon}{d}[me_1 + K(\epsilon x + y)]\mathbf{i} + \frac{1}{d}[me_1 + K(\epsilon x + y)]\mathbf{j}.\end{aligned} \tag{269}$$

Figs. 12 and 13 show the system response to initial conditions satisfying Eqs. (227) and (229). Also, time simulations for Eqs. (253) and (269) are shown in Fig. 14, and identical values are noticed for constraint forces throughout the trajectory of the system for the same initial conditions. However, the introduction of the auxiliary generalized speed is not needed in the present treatment. An algebraic verification of the equivalency of Eqs. (253) and (269) can be done also. Subtracting the two expressions of constraint forces yields

$$\begin{aligned}&\left[\frac{1}{d}\left(m\epsilon e_1 - e_2\right) + Kx - \frac{\epsilon}{d}[me_1 + K(\epsilon x + y)]\right]\mathbf{i} \\ &\qquad + \left[\frac{1}{d}\left(me_1 + \epsilon e_2\right) + Ky - \frac{1}{d}[me_1 + K(\epsilon x + y)]\right]\mathbf{j}\end{aligned} \tag{270}$$

$$= \left[\frac{-e_2}{d} + Kx - \frac{\epsilon}{d}[K(\epsilon x + y)]\right]\mathbf{i} + \left[\frac{\epsilon e_2}{d} + Ky - \frac{K}{d}(\epsilon x + y)\right]\mathbf{j} \tag{271}$$

$$\begin{aligned}&= \left[-\frac{K}{d}\left[x - \frac{a}{b}\bar{x}\right] + Kx - \frac{K}{d}\left[\epsilon^2 x + \frac{a}{b}\bar{x}\right]\right]\mathbf{i} \\ &\qquad + \left[\frac{\epsilon}{d}K[x - \frac{a}{b}\bar{x}] + Ky - \frac{K}{d}(\epsilon x + y)\right]\mathbf{j}\end{aligned} \tag{272}$$

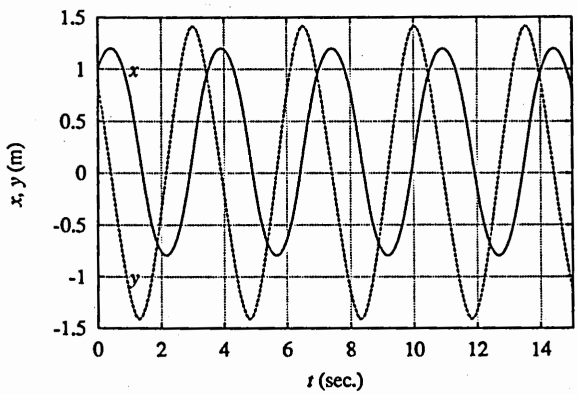

Figure 12: Example 4.1: Time history of generalized coordinates

$$= \left[-\frac{Kx}{d} + Kx - \epsilon^2 \frac{Kx}{d}\right] \mathbf{i} + \left[\frac{\epsilon}{d} K[-\frac{a}{b}\bar{x}] + Ky - \frac{K}{d} y\right] \mathbf{j} \tag{273}$$

$$= \left[-\frac{Kx}{\epsilon^2+1} + Kx - \epsilon^2 \frac{Kx}{\epsilon^2+1}\right] \mathbf{i} + \left[\frac{\epsilon}{\epsilon^2+1} K[-\frac{a}{b}\bar{x}] + K\frac{a\bar{x}}{b\epsilon} - \frac{K}{\epsilon^2+1}\frac{a\bar{x}}{b\epsilon}\right] \mathbf{j} \tag{274}$$

$$= \frac{-Kx + Kx(\epsilon^2+1) - \epsilon^2 Kx}{\epsilon^2+1} \mathbf{i} + \left[\frac{\epsilon}{\epsilon^2+1} K[-\frac{a}{b}\bar{x}] + [1 - \frac{1}{\epsilon^2+1}] K\frac{a}{b\epsilon}\bar{x}\right] \mathbf{j} \tag{275}$$

$$= 0\,\mathbf{i} + \left[\frac{\epsilon}{\epsilon^2+1}[-K\frac{a}{b}\bar{x}] + \frac{\epsilon}{\epsilon^2+1}[K\frac{a}{b}\bar{x}]\right] \mathbf{j} \tag{276}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j}. \tag{277}$$

The values of generalized coordinates and generalized speeds that result from integrating Eqs. (248) and (249) together with the kinematical relations should satisfy the holonomic constraint equation, Eq. (227), and its kinematical equivalent, Eq. (229), for initial conditions that abide by these equations. However, small violations of these equations are noticed due to the accumulative numerical errors. This problem is common when the equations of motion subjected to constraints are numerically integrated, and becomes tangible for long-time simulations. A strategy for overcoming this problem is presented later in the chapter.

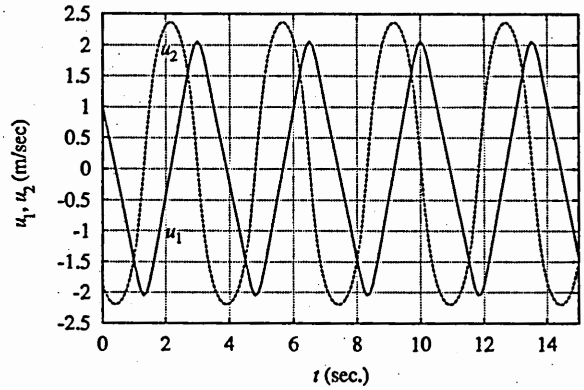

Figure 13: Example 4.1: Time history of generalized speeds

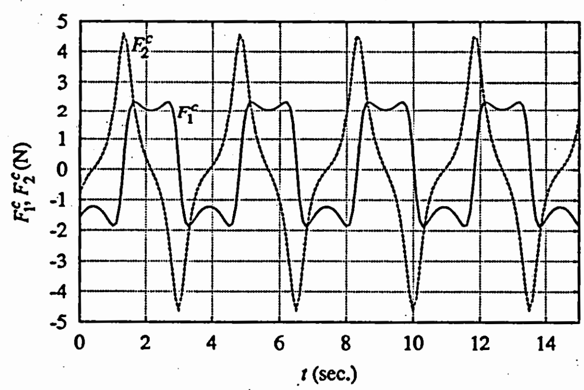

Figure 14: Example 4.1: Time history of constraint forces

4.4 Example 4.2

Now let us revisit Example 2.1 (section 2.7) to obtain the constraint forces that enforce the passive constraint $r\phi = c$. The generalized active forces on the particle are given by

$$F(q,u,t) = \begin{Bmatrix} F_r \\ F_\theta \\ F_\phi \end{Bmatrix}, \tag{278}$$

and the generalized inertia forces are given by

$$F^\star(q,u,\dot{u},t) = \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{Bmatrix} m\left(\frac{u_2{}^2+u_3{}^2}{r}\right) \\ m\left(\frac{u_2u_3}{r\tan\phi} - \frac{u_1u_2}{r}\right) \\ m\left(-\frac{u_1u_3}{r} - \frac{u_2{}^2}{r\tan\phi}\right) \end{Bmatrix}. \tag{279}$$

Therefore, the generalized constraint forces are obtained from Eqs. (223) as

$$F^c(q,u,t) = -F(q,u,t) - F^\star(q,u,\dot{u},t), \tag{280}$$

where $\dot{u}$ corresponds to the constrained system. Substituting the expressions (503)-(505) for the generalized accelerations in the above equation yields

$$\begin{aligned} F_1^c(q,u,t) &= \frac{m^2-a}{a}F_r + \frac{m^2\phi}{a}F_\phi - \frac{m^3\phi}{ar}\frac{u_2{}^2}{\tan\phi} \\ &\quad + \frac{(m^3-am)\left(u_2{}^2+u_3{}^2\right)}{ar} && (281) \\ F_2^c(q,u,t) &= 0 && (282) \\ F_3^c(q,u,t) &= \frac{m^2\phi}{a}F_r + \frac{m^2\phi^2-a}{a}F_\phi + \frac{m^3\phi}{ar}\left(u_2{}^2+u_3{}^2-\phi\frac{u_2{}^2}{\tan\phi}\right) \\ &\quad + \frac{mu_2{}^2}{r\tan\phi}. && (283) \end{aligned}$$

Remark: *The generalized constraint forces do not depend on the second component of the generalized active forces F_θ, and the second component of the generalized constraint forces $F_2^c(q,u,t)$ is zero. This implies that $\dot{u}_2$, the component of the generalized accelerations* *along the second partial velocity $\mathbf{v}_2 = \mathbf{j}$ is the same for both the constrained and the unconstrained systems, provided that the generalized active forces are the same for both systems.*

4.5 Elimination of Constraint Drift

The problem of constraint drift is generally unavoidable when a differential form of the constraint equations is used to formulate the equations of motion. Such a problem becomes more serious for the case in which the acceleration form of holonomic constraints is enforced, since two integrations are needed at each time step to obtain the generalized coordinates. The purpose of this section is to modify the derived equations of motion to suppress the errors resulting from this integration process. The explicit use of the acceleration form of constraint equations suggests the employment of the classical numerical stabilization method by Baumgarte [18]. Let ϕ be the column matrix

$$\phi = \lfloor \phi_1 \dots \phi_m \rfloor^T, \tag{284}$$

in which $\phi(q, u, t)$ is in general nonlinear in its arguments. Instead of using the acceleration form of the constraint equations

$$\phi(q, u, t) = 0, \tag{285}$$

the equations used are the Baumgarte type equations

$$\dot{\phi}(q, u, \dot{u}, t) - \Gamma\phi(q, u, t) = 0. \tag{286}$$

Here $\Gamma \in \mathbb{R}^{m \times m}$ is a matrix that has eigenvalues with strictly negative real parts, and

$$\dot{\phi}(q, u, \dot{u}, t) = \frac{\partial \phi}{\partial q}\dot{q} + \frac{\partial \phi}{\partial u}\dot{u} + \frac{\partial \phi}{\partial t}. \tag{287}$$

Substitution of the kinematical differential equations, Eqs. (4), in the above equation results in

$$\dot{\phi}(q, u, \dot{u}, t) = \frac{\partial \phi}{\partial u}\dot{u} + B_1(q, u, t)u + B_2(q, u, t), \tag{288}$$

where

$$
\begin{aligned}
B_1(q,u,t) &= \frac{\partial \phi}{\partial q} C(q,t) && (289)\\
B_2(q,u,t) &= \frac{\partial \phi}{\partial q} D(q,t) + \frac{\partial \phi}{\partial t}. && (290)
\end{aligned}
$$

Letting

$$
u = \begin{bmatrix} u_I{}^T & u_D{}^T \end{bmatrix}^T, \tag{291}
$$

where $u_I = \lfloor u_1 \ldots u_p \rfloor^T$ and $u_D = \lfloor u_{p+1} \ldots u_n \rfloor^T$, and defining the $m \times p$ matrix $J_1(q,u,t)$ as in Eq. (117) and the matrix $J_2(q,u,t)$ as in Eq. (118), and assuming that $u_{p+1}, \ldots, u_n$ can be chosen such that the matrix J_2 is nonsingular for all q, u, and t that satisfy the constraint equations, Eqs. (285), Eq. (288) can be written as

$$
\dot{\phi}(q,u,\dot{u},t) = J_1 \dot{u}_I + J_2 \dot{u}_D + B_1(q,u,t)u + B_2(q,u,t). \tag{292}
$$

Therefore, Eqs. (286) become

$$
J_1 \dot{u}_I + J_2 \dot{u}_D - \Gamma \phi(q,u,t) + B_1(q,t)u + B_2(q,t) = 0. \tag{293}
$$

Solving Eq. (293) for $\dot{u}_D$ yields

$$
\dot{u}_D = A(q,t)\dot{u}_I + \hat{B}(q,u,t), \tag{294}
$$

where

$$
\begin{aligned}
A(q,u,t) &= -J_2^{-1} J_1 && (295)\\
\hat{B}(q,u,t) &= B(q,u,t) + J_2^{-1} \Gamma \phi(q,u,t). && (296)
\end{aligned}
$$

Eq. (294) can be written as

$$
A_1(q,u,t)\dot{u} = \hat{B}(q,u,t), \tag{297}
$$

where

$$
A_1(q,u,t) = \begin{bmatrix} -A(q,u,t) & I \end{bmatrix}. \tag{298}
$$

Eqs. (297) together with Eqs. (147) form the matrix system

$$T\dot{u} = \hat{V}, \tag{299}$$

where $T := \begin{bmatrix} {A_1}^T & [A_2Q]^T \end{bmatrix}^T$ and $\hat{V} := \begin{bmatrix} \hat{B}^T & [A_2P]^T \end{bmatrix}^T$. Therefore,

$$\dot{u} = T^{-1}\hat{V}. \tag{300}$$

A similar treatment can be developed for holonomic constraint equations. Instead of the acceleration form of the constraint equations, the following constraint equations are used:

$$\ddot{\phi}(q,u,\dot{u},t) - \Gamma_1\dot{\phi}(q,u,t) - \Gamma_2\phi(q,t) = 0, \tag{301}$$

where Γ_1 and Γ_2 are chosen such that the dynamics of Eq. (301) is stable. In this case, $\hat{B}$ in Eq. (297) becomes

$$\hat{B}(q,u,t) = B(q,u,t) + J_2^{-1}[\Gamma_1\dot{\phi}(q,u,t) + \Gamma_2\phi(q,t)]. \tag{302}$$

Remark: *The matrices Γ, Γ_1, Γ_2 can be thought of as feedback gains of a control system that is aimed to regulate the constraint functions ϕ at the zero value. In order to obtain from the numerical integration scheme a true and accurate state of the dynamical system, these gains must also keep the entire system of Eqs. (300) stable. The choice of the gain matrices can affect the stability of Eqs. (300), beside its effect on the convergence rate of ϕ. For adaptive choices of gain matrices for Baumgarte type of constraint violation stabilization, the reader is referred to Ref. [25].*

The procedure for deriving nonminimal form of Kane's equations of motion that is free of constraint drift for dynamical systems subjected to nonholonomic constraints is summarized as follows:

1. Stable constraint dynamics equations are constructed by augmenting the constraint functions ϕ with their differentiated forms by means of the matrix Γ in case ϕ is nonholonomic, resulting in Eq. (286). In case ϕ is holonomic, both $\dot{\phi}$ and $\ddot{\phi}$ are augmented with ϕ by means of Γ_1 and Γ_2, resulting in Eq. (301). In both cases, these matrices are chosen to damp out any nonzero values of ϕ.

2. A generalized speeds partitioning according to Eqs. (291) is used to put the equations in the form (294), which results in the upper subsystem of the nonminimal form, Eqs. (297).

3. Eqs. (297) are used with Eqs. (147) to form the system of equations (299), which can be solved for $\dot{u}$ by inverting the matrix T to obtain Eqs. (300).

The following example illustrates the procedure for numerically stabilizing a holonomic constraint equation.

4.6 *Example 4.3: Pendulum of Varying Length:*

A pendulum that consists of a massless rod of variable length $l = \sqrt{x^2 + y^2}$ is shown in Fig. 15. The x-y coordinate system is fixed in an inertial reference frame $\mathcal{R}$, and a particle P of mass m is attached to the end of the rod. The pendulum adjusts its length according to the holonomic constraint equation

$$\phi(x, y) = y + \frac{x^2}{l_0} - l_0 = 0, \tag{303}$$

where l_0 is a positive constant equal to the rod length when either $x = 0$ or $y = 0$. Consider x and y as the configuration parameters, and let the generalized speeds be $u_1 = \dot{x}$ and $u_2 = \dot{y}$. Also, let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors in the x and y directions, respectively. The position vector of the particle P is

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j}. \tag{304}$$

The inertial velocity of the particle is

$${}^{\mathcal{R}}\mathbf{v}^{P} = u_1\mathbf{i} + u_2\mathbf{j}.$$

The velocity form of the constraint equation above is thus

$$\dot{\phi}(x, u) = u_2 + \frac{2}{l_0} x u_1 = 0,$$

and the acceleration form is

$$\ddot{\phi}(x, u, \dot{u}) = \dot{u}_2 + \frac{2}{l_0}(u_1^2 + x\dot{u}_1) = 0.$$

Choosing $u_I = u_1$,

$$\begin{aligned} A_1 &= \lfloor \tfrac{2}{l_0}x \quad 1 \rfloor, \\ A_2 &= \lfloor 1 \quad -\tfrac{2}{l_0}x \rfloor, \end{aligned} \tag{305}$$

and

$$B = -\frac{2}{l_0}u_1^2. \tag{306}$$

The unconstrained equations of motion are

$$\begin{aligned} \dot{u}_1 &= 0 \\ \dot{u}_2 &= g. \end{aligned} \tag{307}$$

Eq. (42) for this system is

$$\frac{2}{l_0}x\dot{u}_1 + \dot{u}_2 = -\frac{2}{l_0}u_1^2 \tag{308}$$

$$\dot{u}_1 - \frac{2}{l_0}x\dot{u}_2 = -\frac{2}{l_0}gx. \tag{309}$$

Given the parameter l_0 =1.0 m, solving for $\dot{u}_1$ and $\dot{u}_2$ yields,

$$\dot{u}_1 = \frac{-2x(g + 2u_1^2)}{1 + 4x^2} \tag{310}$$

$$\dot{u}_2 = \frac{4gx^2 - 2u_1^2}{1 + 4x^2}. \tag{311}$$

Setting the initial condition $x(0)$=1.0 m, and using the Kutta-Merson numerical integration scheme, the above system of equations is solved for x, y, u_1, and u_2, with the constraint violation ϕ evaluated at each time step. The time history of ϕ after 500 seconds of simulation time is plotted for two small values of the time integration step Δt, as shown in Figs. 16 and 17. It is clearly seen that ϕ grows with time. Both the pattern and the magnitude of ϕ are affected by the choice of Δt. Reducing Δt reduces the growth of ϕ, at the cost of increasing the required time to perform the simulation. The same is concluded for a constant

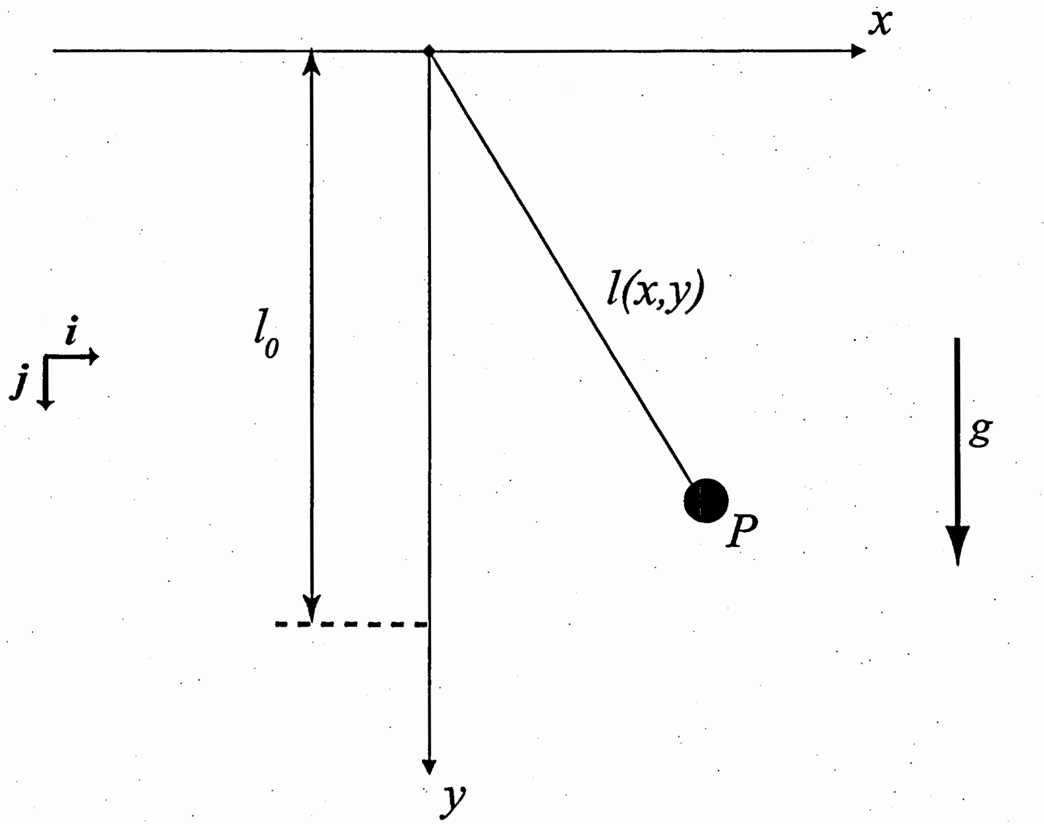

Figure 15: Schematic for Example 4.3

of motion of this conservative system, namely the energy integral E. This is simply the sum of kinetic and potential energies of the system,

$$E = K + V = \frac{1}{2}m(u_1^2 + u_2^2) + mg(l_0 - y), \tag{312}$$

where the datum for computing the potential energy is chosen as $y_d = l_0$. For m = 1.0 Kg and l_0 = 1, Figs. 18 and 19 show the deviations in E for the two choices of Δt after 500 seconds of simulation time.

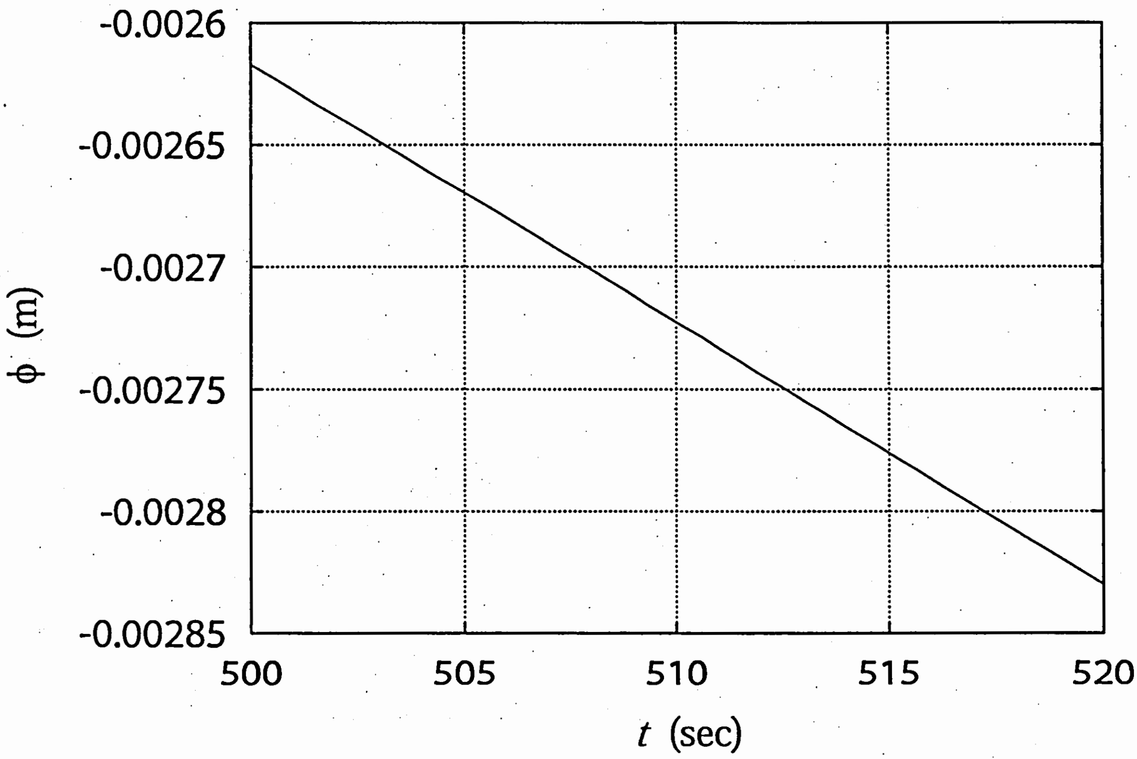

Figure 16: Example 4.3: Constraint violation, ϕ: $\Delta t = 0.01$ sec.

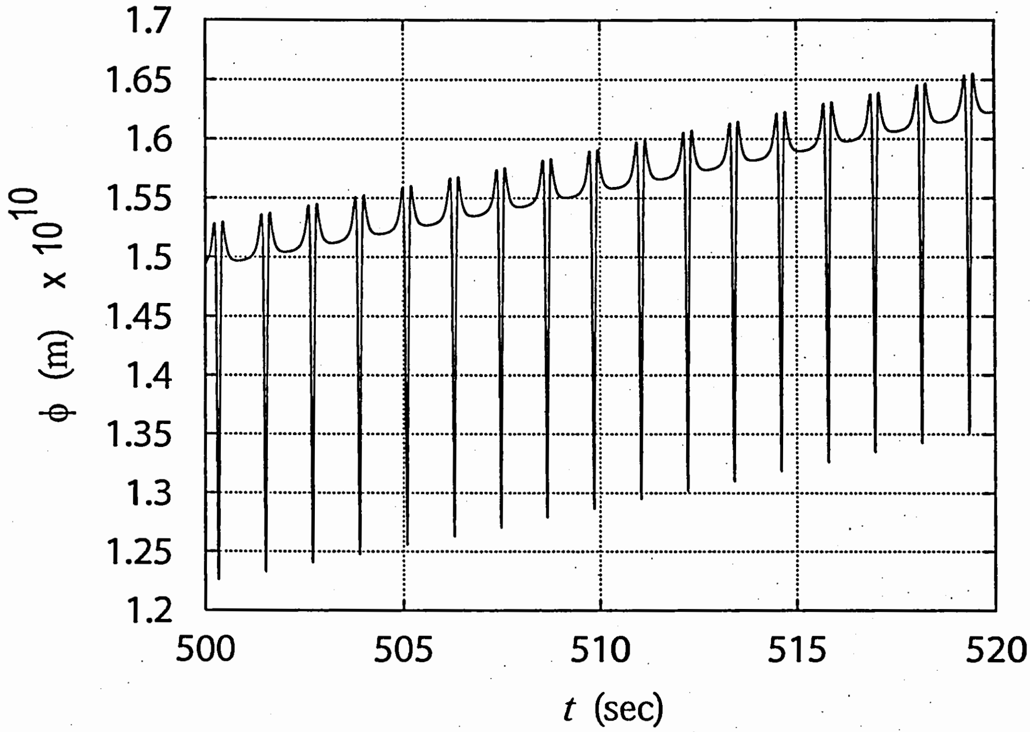

Figure 17: Example 4.3: Constraint violation, ϕ: Δt= 0.001 sec.

Next, Eq. (301) for this system is used. The resulting Eq. (299) with l_0 =1.0 m becomes

$$\begin{aligned} 2x\dot{u}_1 + \dot{u}_2 &= -2u_1^2 + \Gamma_1\dot{\phi}(x, u) + \Gamma_2\phi(x, y, u) \\ \dot{u}_1 - 2x\dot{u}_2 &= -2gx. \end{aligned} \tag{313}$$

Solving for $\dot{u}$ yields

$$\dot{u}_1 = 2x\frac{-\Gamma_2(1 - y - x^2) - g - 2u_1^2 + \Gamma_1(u_2 + 2xu_1)}{1 + 4x^2} \tag{314}$$

$$\dot{u}_2 = \frac{4gx^2 - \Gamma_2(1 - y - x^2) - 2u_1^2 + \Gamma_1(u_2 + 2xu_1)}{1 + 4x^2}. \tag{315}$$

The time history of ϕ after 500 seconds is shown in Fig. 20. It is noticed that the constraint violation becomes bounded during the time simulation period. Arbitrarily small bounds can be obtained by increasing the values of Γ_1 and Γ_2. However, this also increases the relative magnitudes of the damping terms, which results in an increase in the stiffness of

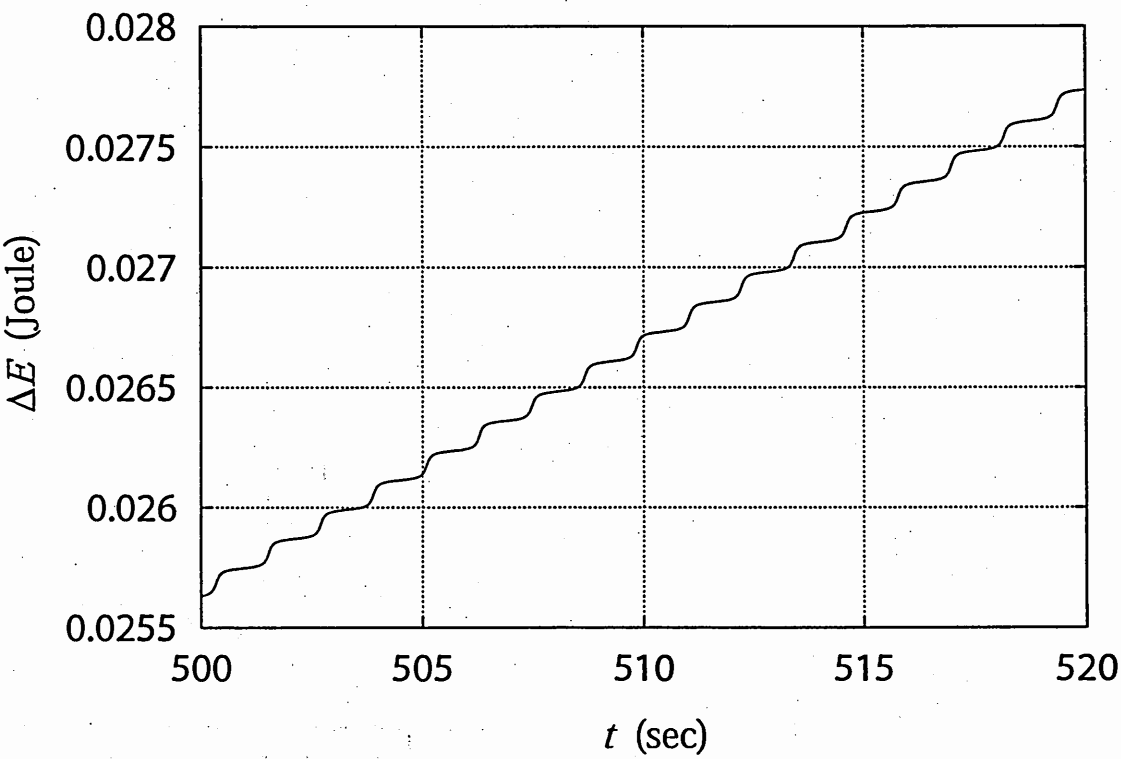

Figure 18: Example 4.3: Energy integral: $\Delta t = 0.01$ sec.

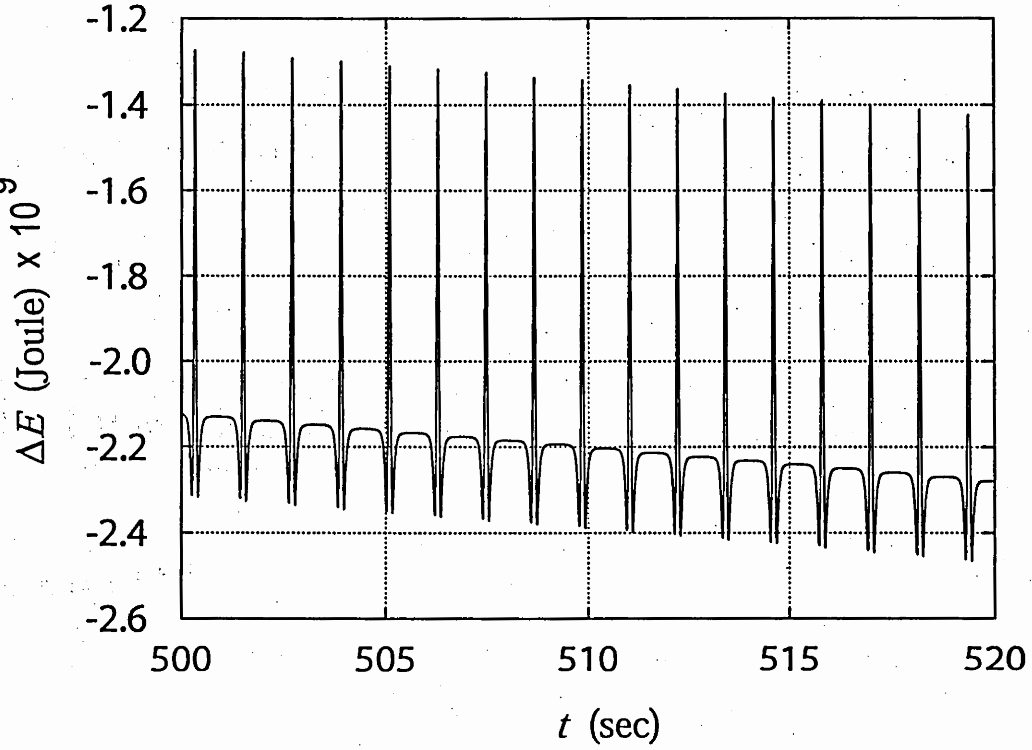

Figure 19: Example 4.3: Energy integral: $\Delta t = 0.001$ sec.

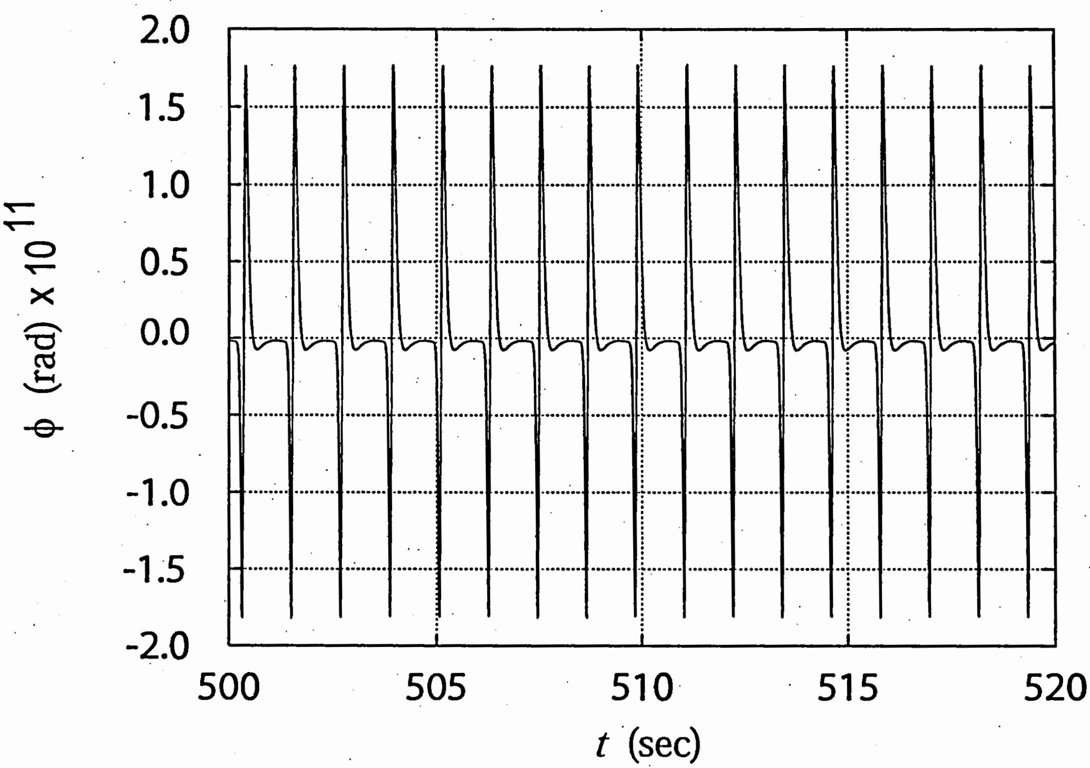

Figure 20: Example 4.3: Constraint violation, ϕ: $\Gamma_1 = -20, \Gamma_2 = -100, \Delta t = 0.001$ sec.

the differential equations, and requires smaller time steps. For Γ_1 = -20 sec^{-1} and Γ_2=-100 sec^{-2}, a bound of $|\phi| < 2.0 \times 10^{-11}$ m is obtained for Δt = 0.001 seconds. The choice of Γ_1 and Γ_2 affects the numerical stability of the whole nonminimal system of equations, as discussed below. For the purpose of comparison, Kane's minimal equation may be derived. The solution of this equation is free from the constraint drift, because the constraint equation is used in its algebraic form. Substituting y from Eq. (303) into Eq. (304) yields

$$\mathbf{p} = x\mathbf{i} + [1 - x^2]\mathbf{j}. \tag{316}$$

The first and second time derivatives of the above equation relative to $\mathcal{R}$ are the inertia velocity and acceleration of P, represented in terms of u_1 and $\dot{u}_1$,

$$\begin{aligned} {}^{\mathcal{R}}\mathbf{v}^P &= u_1\mathbf{i} + [1 - 2xu_1]\mathbf{j} && (317) \\ &= u_1[\mathbf{i} - 2x\mathbf{j}], && (318) \end{aligned}$$

and

$$^{\mathcal{R}}\mathbf{a}^P = \dot{u}_1\mathbf{i} - [2{u_1}^2 + 2x\dot{u}_1]\mathbf{j}. \tag{319}$$

The coefficient of u_1 in Eq. (318) is the holonomic partial velocity of P,

$$^{\mathcal{R}}\mathbf{v}_1^P = \mathbf{i} - 2x\mathbf{j}. \tag{320}$$

The holonomic generalized active force F on the pendulum is the contribution of gravity, given by

$$\begin{aligned} F &= mg\mathbf{j} \cdot {}^{\mathcal{R}}\mathbf{v}_1^P && (321)\\ &= -2mgx, && (322) \end{aligned}$$

and the generalized inertia force is

$$\begin{aligned} F^\star &= -m{}^{\mathcal{R}}\mathbf{a}^P \cdot {}^{\mathcal{R}}\mathbf{v}_1^P && (323)\\ &= -m[\dot{u}_1 + 2x(2{u_1}^2 + 2x\dot{u}_1)]. && (324) \end{aligned}$$

Kane's dynamical equation of motion is

$$F + F^\star = 0, \tag{325}$$

yielding to

$$2gx + \dot{u}_1 + 2x(2{u_1}^2 + 2x\dot{u}_1) = 0. \tag{326}$$

To illustrate the improvement in the numerical solution of the nonminimal equations resulting from the augmentation of the damping terms, the solutions for y after 1 hour of simulation time obtained from the integrations of Eqs. (310)-(311) and Eqs. (314)-(315) are compared with the most accurate one obtained from the integration of Eq. (326), as shown in Fig. 21. To reduce the computer memory and time costs, a larger integration step Δt of 0.1 seconds is chosen. Clearly, the damping of constraint violations is crucial for accurate fast long-term simulations of such systems. It should be noticed, however, that despite the correction in the computed holonomic constraint violation, this does not imply

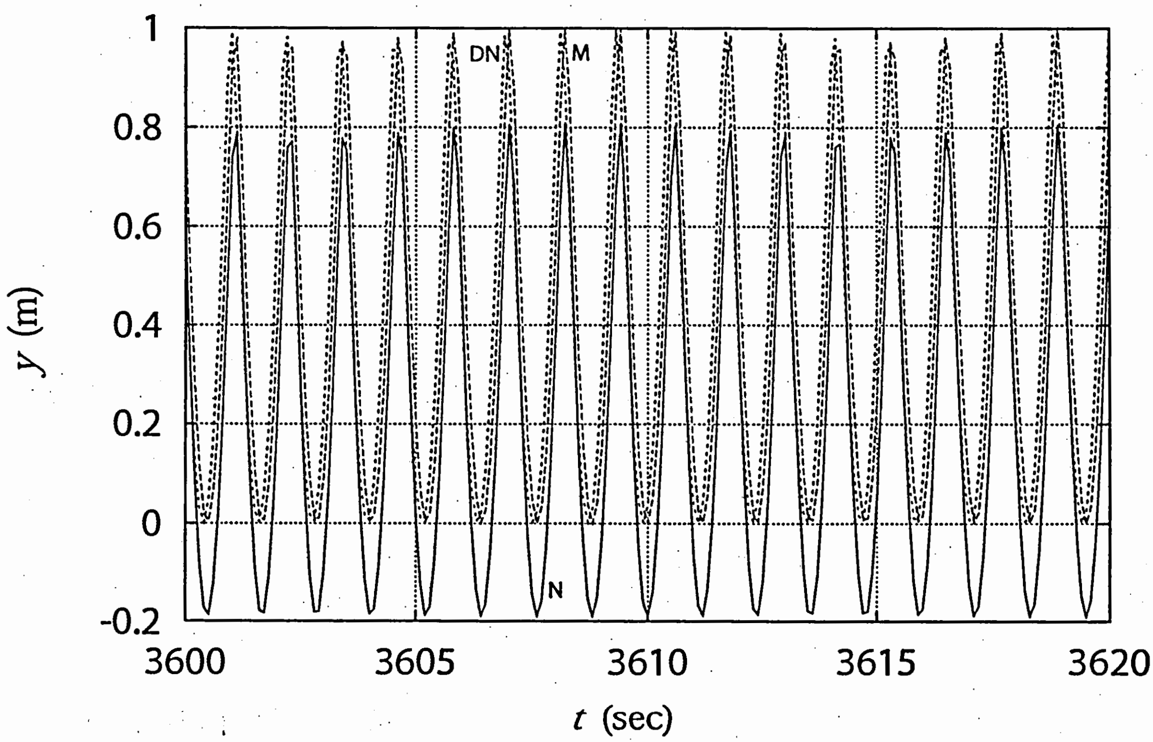

Figure 21: Example 4.3: y solution after 1 hr, $\Delta t = 0.1$ sec.: M minimal, N nonminimal, DN damped nonminimal

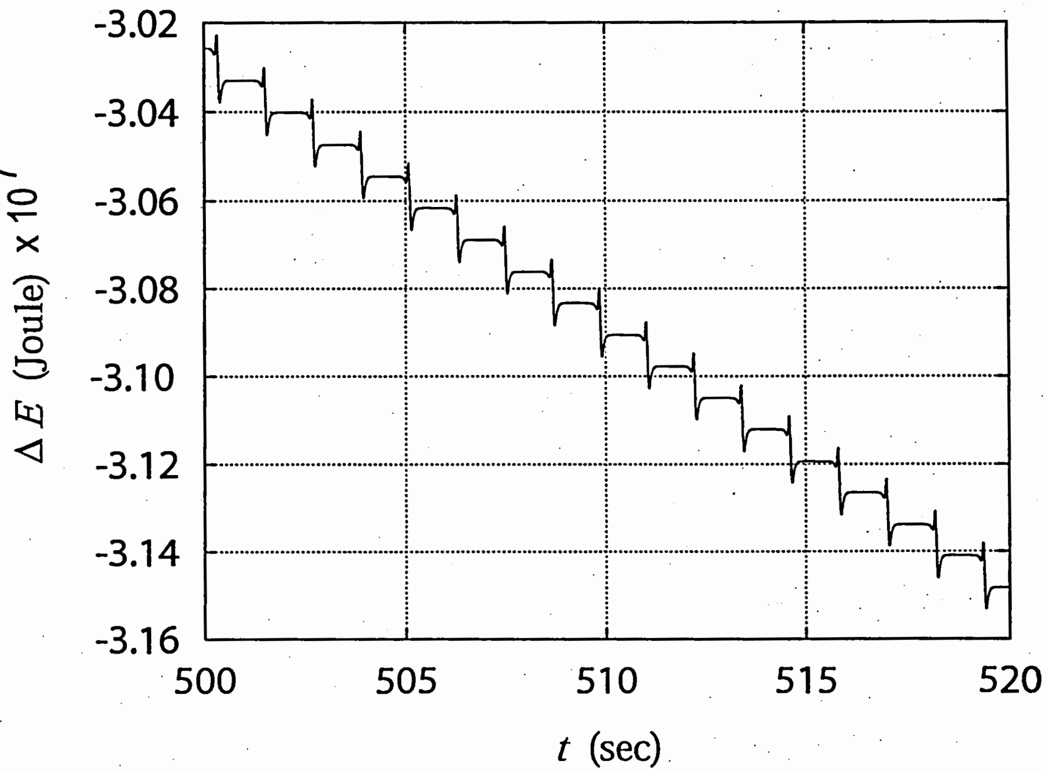

Figure 22: Example 4.3: Energy integral deviation ΔE: $\Gamma_1 = -20, \Gamma_2 = -100$, $\Delta t =$ 0.001 sec.

necessarily an improvement in the accuracy of the individual states. The energy integral provides a check on the stability of the whole system of nonminimal equations of motion, and is independent from the constraint violation measure ϕ. Fig. 22 shows the deviation in the energy for the constraint-stabilized system. This deviation represents a deterioration in accuracy compared to the constraint-unstabilized system for the same Δt, as noticed by comparing with Fig. 19. Careful choice of Γ_1 and Γ_2 is therefore important to preserve the stability of all the states. Nevertheless, the only tangible effect of the constraint stabilization in this example is favorable on y, as illustrated in Fig. 21.

4.7 Summary

Explicit expressions for constraint forces are derived in the first part of the chapter, without the introduction of auxiliary generalized speeds. This may complement existing approaches based on introduction of auxiliary generalized speeds, (i.e., fictitious degrees of freedom) or using Lagrange multipliers.

In the second part of the chapter, a systematic procedure is provided to modify the equations of motion for the purpose of suppressing the constraint violation due to numerical integration errors, by augmenting the constraint equations with damping terms of the Baumgarte type. The associated coefficients can be chosen to obtain any desired constraint dynamics without affecting the invertibility of the generalized constrained inertia matrix, as the coefficients of the acceleration terms in both the dynamical and the constraint equations remain unaltered. An illustrative example shows a significant reduction in a holonomic constraint violation, although the effect on the whole system of equations is slightly destabilizing. Therefore, the coefficients of the stabilizing terms must be chosen such that the improvement in the numerical stability of the constraint equations does not deteriorate the numerical stability of the resulting nonminimal system of equations.

CHAPTER V

UNILATERAL CONSTRAINTS

5.1 *Introduction*

Unilateral constraints (also named one-sided constraints) are constraints that can be modeled by inequalities involving the state and/or control of the dynamical system. Therefore, a unilateral constraint is best described by a constrained model that is active in some phases of motion, and inactive in others. Either part of the inequality " $\leq$ ", i.e. the strict equality and inequality parts, can be the one representing the constrained phase, depending on the application.

A constrained phase of motion corresponds to a fewer degrees of freedom and different equations of motion. Hence, it is necessary to form several models to describe the dynamics of the system at different phases. During the time instants when the strict part of a unilateral constraint is active, the dynamics of the system is governed by the corresponding constrained model, while the unconstrained model is the one that govern the dynamics of the system when the strict parts of all constraints are inactive. The criterion for switching between a model and another is the tendency of the system to obey or to violate the corresponding constraint.

Traditional techniques for modeling constrained dynamical systems yield differential equations that are equal in number to the number of degrees of freedom of the dynamical system during the corresponding phase, together with algebraic equations that describe the constraints. Due to the difference in nature and in number of equations of motion that correspond to each phase, both the formulation and the solution of these equations can be difficult and cumbersome for systems with numerous degrees of freedom and constraints.

The nonminimal nonholonomic form can be utilized in modeling a unilateral constraint by forming constrained models that alternate in action with the unconstrained one. All these models have the same number of differential equations, equal to the number of generalized coordinates of the dynamical system, and separated in the generalized accelerations. The activation of each set of equations is performed by simple and readily computer implementable manipulations of the matrices of the unconstrained model and the corresponding constraints.

However, a critical issue in modeling unilaterally constrained dynamical systems is the treatment of singularities at the switching points, i.e., when the sudden changes in the dynamical equations occur, which imply discontinuities in the accelerations for some configurations. These are accompanied with abrupt changes in velocities in very short periods of time, and hence large changes in momentum and impulsive forces that may affect the high-frequency characteristics of structures, and cause vibrations or even structural failures. Examples are impact forces in landing gears at the instants of ground touches, the contact forces generated when mechanical gears are meshed one with another, and the high-frequency phenomenon related to friction when transitions from rolling to slipping or from slipping to rolling occur.

Most classical models of unilaterally constrained motion idealize the changes in velocities at the discontinuities to be instantaneous, and the relations between the velocities before and after contacts to be governed either by geometrical and kinematical constraints or by means of the coefficient of restitution. However, the changes in velocities during the moments of contact are not really instantaneous, and ignoring the velocity behavior in these periods implies ignoring one of the most important phenomena related to impulsive motion, the impulsive forces that take place in the moment of contact.

The modified constrained phase dynamical equations derived in chapter 4 can be used to resolve the above mentioned concern. The constraint stabilization property of these equations can be utilized in approximating rapid changes in velocities, and hence augmenting the impact short period dynamics to the overall motion, and showing the different in time scales behaviors in an integrated manner, within the same numerical simulation scheme. This is performed by setting a tolerance for the unconstrained phase such that the modified constrained dynamical equations are activated whenever this tolerance is exceeded. The intermittent motion and the corresponding changes in velocities and impulsive forces take place in this tolerance, where the continuous velocity behavior at the time instant the strict part of the unilateral constraint becomes active is exhibited.

5.2 Unilaterally Constrained Dynamical Systems

Several types of constrained dynamical systems that appear to be different in the nature of constraints can be modeled using unilateral constraints. A unilateral constraint that affects the motion of a dynamical system falls under one of two classes. The first class is impulsive constraints. The constrained phase for this class of constraints is represented by the equality part. The second class is friction constraints. The inequality part is the one representing the constrained phase of motion for this class. The relation between these different types of motions was first investigated in Ref. [53].

In this chapter, the continuous in velocities approach in modeling motion constrained by unilateral constraints is used for the two classes of unilateral constraints, in the context of the nonminimal equations of motion. It is emphasized that the continuity meant here is in velocities and configurations, and does not imply using the same model to describe different phases, as the accelerations are generally discontinuous.

5.3 Impulsive Constraints

This class refers to constraints that are suddenly applied to, or removed from the course of motion of the dynamical system. These activations or deactivations might be isolated events in time, where the constraints are added or deleted within sufficiently long periods of time. On the other hand, the constraint addition and deletion can be consequent events that take place in very short periods of time. The corresponding motion is referred to as the *impact motion.*

The continuous velocity distribution approach is aimed at predicting the kinematics of the colliding bodies during the impact time (which is not assumed ignorable) and estimating the resulting impulsive constraint forces in terms of their relative displacements and velocities. For that purpose, certain coefficients that represent the material compliance and damping of the colliding bodies are needed. Some works assume that these coefficients are known constants of the materials. Examples are Refs. [37, 38, 131, 90]. Others use kinetic-elastic energy relations to calculate these coefficients. Examples are [54, 82, 86].

5.4 Constraint Activation and Deactivation

The motion of the unconstrained dynamical system from time $t = 0$ to $t = t_i$ is governed by the unconstrained model. If the constraint

$$\phi(q, t) = 0 \tag{327}$$

is suddenly activated at time t_i, then there exists a small positive number δt such that for all time $t \geq t_i + \delta t$, the motion of the dynamical system is governed by Eq. (48). In the time period $[t_i, t_i + \delta t]$, there exist rapid changes in the velocities of the dynamical system, such that the time derivative of the constraint, $\dot{\phi} = 0$, is satisfied for $t \geq t_i + \delta t$. This implies an abrupt change in momentum, which can occur only by the influence of an impulsive force. This is unavoidable unless the above mentioned restrictive derivative condition is satisfied before the time instant $t = t_i$.

For the numerical simulation to capture the impulsive force and velocity distributions immediately after the moment the constraint is activated, a tolerance $\delta\phi$ is now provided to the constraint function ϕ, such that the unconstrained model continues to govern the dynamics if $\phi - \delta\phi \leq 0$. Otherwise, the unconstrained model is augmented with constraint equations of the form

$$\ddot{\phi}(q, u, \dot{u}, t) - \Delta_1 \dot{\phi}(q, u, t) - \Delta_2 \phi(q, t) = 0, \tag{328}$$

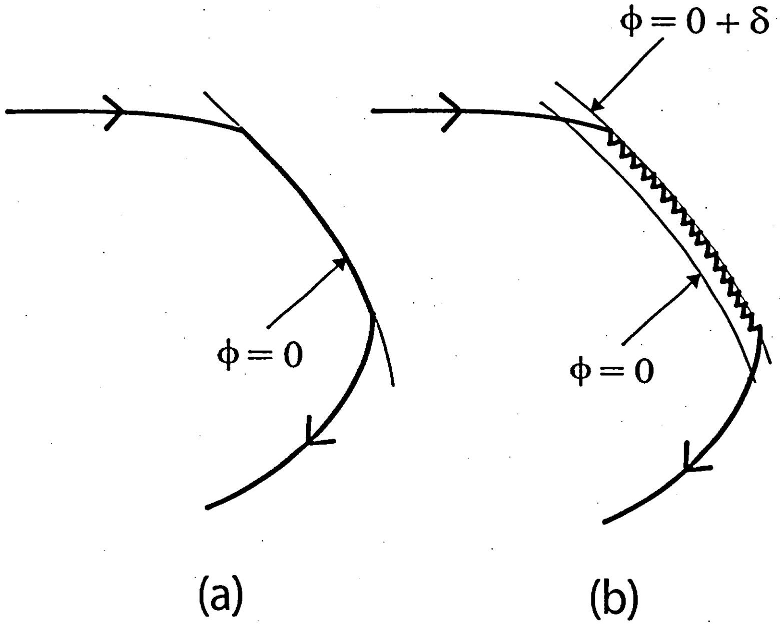

Figure 23: Unilateral constraints: (a) actual trajectory; (b) modified trajectory

and is activated to draw the trajectory towards the strictly constrained field, $\phi = 0$, as illustrated in Figure 23. The constants Δ_1 and Δ_2 are dependent on the materials of the bodies in contact, and their values approximate the dynamics during the instants of contact. For the purpose of this work, they are assumed to be known in advance. By using this acceleration form of constraint equations, the discontinuity is kept at the acceleration level, and a realistic continuous behavior of velocity is attained.

Remark *A similar analysis can be done if the constraint relation $\phi = 0$ is nonholonomic. In this case, the constrained model is obtained by augmenting the unconstrained model with the equations*

$$\dot{\phi}(q, u, \dot{u}, t) - \Delta\phi(q, u, t) = 0. \tag{329}$$

5.5 Example 5.1: Geometrically constrained double pendulum

The mechanism shown in Fig. 24 consists of a disc D of radius R and mass m_D, and a uniform bar B that is attached to the disc at the point c, and has mass m_B and length L. The disc is free to rotate in the vertical plane about its mass center o. The central moments of inertia of the disc D and the bar B are I_D and I_B, respectively. The disc is driven by a motor of controlled torque τ_1 applied at the point o, and the bar B is acted upon by the control torque τ_2 at the point c, by means of a motor that is mounted on the disc. The point o is fixed to an inertial reference frame $\mathcal{N}$. The mechanism configuration is constrained according to the relation

$$R\cos\theta_1 + L\cos\theta_2 \leq R + \frac{3L}{4}, \tag{330}$$

which models an inertially fixed and rigid obstacle that restrains the movement of the double pendulum. Let the generalized coordinates be θ_1 and θ_2. The angular velocities of D and B relative to $\mathcal{N}$ are

$$\begin{aligned} {}^{\mathcal{N}}\boldsymbol{\omega}^D &= \dot{\theta}_1\mathbf{k} && (331)\\ {}^{\mathcal{N}}\boldsymbol{\omega}^B &= \dot{\theta}_2\mathbf{k}. && (332) \end{aligned}$$

The velocities of the points c and a relative to $\mathcal{N}$ are

$$\begin{aligned} {}^{\mathcal{N}}\mathbf{v}^c &= \dot{\theta}_1 R(-\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) && (333)\\ {}^{\mathcal{N}}\mathbf{v}^a &= {}^{\mathcal{N}}\mathbf{v}^c + \dot{\theta}_2\frac{L}{2}(-\sin\theta_2\mathbf{i} + \cos\theta_2\mathbf{j}). && (334) \end{aligned}$$

If the generalized speeds are chosen as $u_1 = \dot{\theta}_1$ and $u_2 = \dot{\theta}_2$, then the angular velocities of the disc D and the bar B relative to $\mathcal{N}$ can be written as

$$\begin{aligned} {}^{\mathcal{N}}\boldsymbol{\omega}^D &= u_1\mathbf{k} && (335)\\ {}^{\mathcal{N}}\boldsymbol{\omega}^B &= u_2\mathbf{k}, && (336) \end{aligned}$$

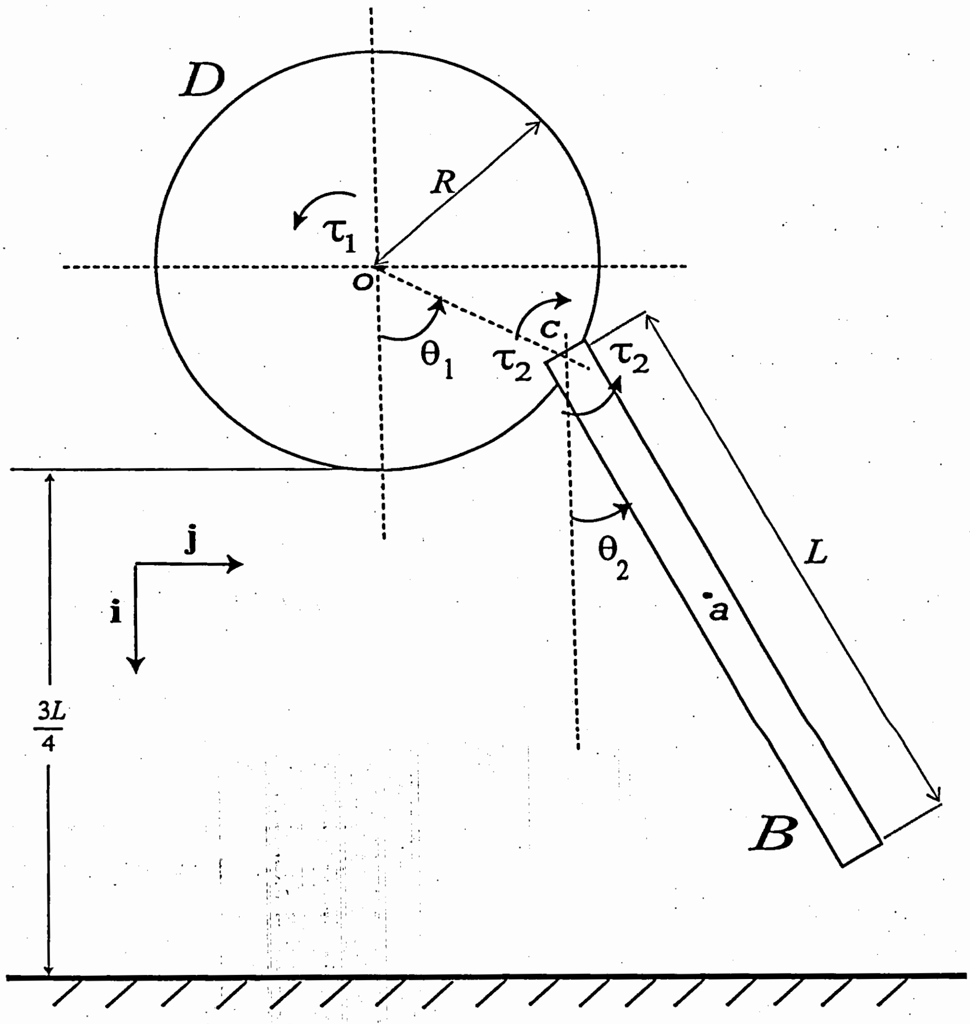

Figure 24: Schematic for example 5.1

from which the partial angular velocities of the disc and the bar are found to be

$$
\begin{aligned}
{}^{\mathcal{N}}\boldsymbol{\omega}_1^D &= {}^{\mathcal{N}}\boldsymbol{\omega}_2^B = \mathbf{k} && (337)\\
{}^{\mathcal{N}}\boldsymbol{\omega}_2^D &= {}^{\mathcal{N}}\boldsymbol{\omega}_1^B = 0. && (338)
\end{aligned}
$$

The velocities of the points c and a can be written as

$$
\begin{aligned}
{}^{\mathcal{N}}\mathbf{v}^c &= u_1 R(-\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) && (339)\\
{}^{\mathcal{N}}\mathbf{v}^a &= {}^{\mathcal{N}}\mathbf{v}^c + u_2\frac{L}{2}(-\sin\theta_2\mathbf{i} + \cos\theta_2\mathbf{j}) && (340)
\end{aligned}
$$

subject to $R(1-\cos\theta_1) > L(\cos\theta_2 - \frac{3}{4})$, from which the partial velocities of the points c and a are found to be

$$
\begin{aligned}
{}^{\mathcal{N}}\mathbf{v}_1^c &= {}^{\mathcal{N}}\mathbf{v}_1^a = R(-\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) && (341)\\
{}^{\mathcal{N}}\mathbf{v}_2^c &= 0 && (342)\\
{}^{\mathcal{N}}\mathbf{v}_2^a &= \frac{L}{2}(-\sin\theta_2\mathbf{i} + \cos\theta_2\mathbf{j}). && (343)
\end{aligned}
$$

The angular accelerations of the disc and bar are

$$
\begin{aligned}
{}^{\mathcal{N}}\boldsymbol{\alpha}^D &= \dot{u}_1\mathbf{k} && (344)\\
{}^{\mathcal{N}}\boldsymbol{\alpha}^B &= \dot{u}_2\mathbf{k}, && (345)
\end{aligned}
$$

and the inertial accelerations of the points c and a are

$$
\begin{aligned}
{}^{\mathcal{N}}\mathbf{a}^c &= \dot{u}_1 R[-\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}] - {u_1}^2 R[\cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j}] && (346)\\
{}^{\mathcal{N}}\mathbf{a}^a &= {}^{\mathcal{N}}\mathbf{a}^c + \frac{L}{2}\dot{u}_2[-\sin\theta_2\mathbf{i} + \cos\theta_2\mathbf{j}] - \frac{L}{2}{u_2}^2[\cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j}]. && (347)
\end{aligned}
$$

The motion of the double pendulum can be analyzed by two phases:

Unconstrained Phase: When the double pendulum is in the unconstrained region, the generalized active forces are

$$
\begin{aligned}
F_1 &= -m_B g R\sin\theta_1 + \tau_1 - \tau_2 && (348)\\
F_2 &= -m_B g\frac{L}{2}\sin\theta_2 + \tau_2, && (349)
\end{aligned}
$$

and the generalized inertia forces are

$$
\begin{aligned}
F_1^\star &= -m_B\{R^2\dot{u}_1 + \frac{L}{2}R\dot{u}_2\cos(\theta_1-\theta_2) + \frac{L}{2}R{u_2}^2\sin(\theta_1-\theta_2)\} - I_D\dot{u}_1 && (350)\\
F_2^\star &= -m_B\{\frac{L^2}{4}\dot{u}_2 + \frac{L}{2}R\dot{u}_1\cos(\theta_1-\theta_2) + \frac{L}{2}R{u_1}^2\sin(\theta_2-\theta_1)\} - I_B\dot{u}_2. && (351)
\end{aligned}
$$

Therefore, the unconstrained phase of dynamics can be represented by the model

$$
Q(q,t)\dot{u} = P(q,u,t) + G(q,u,t)\tau, \tag{352}
$$

where the matrices Q, P, and G are

$$
\begin{aligned}
Q &= \begin{bmatrix} m_BR^2 + I_D & m_B\frac{L}{2}R\cos(\theta_1-\theta_2) \\ m_B\frac{L}{2}R\cos(\theta_1-\theta_2) & m_B\frac{L^2}{4} + I_B \end{bmatrix} && (353)\\
P &= \begin{Bmatrix} -m_BgR\sin\theta_1 - m_B\frac{L}{2}R{u_2}^2\sin(\theta_1-\theta_2) \\ -m_Bg\frac{L}{2}\sin\theta_2 + m_B\frac{L}{2}R{u_1}^2\sin(\theta_1-\theta_2) \end{Bmatrix} && (354)\\
G &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. && (355)
\end{aligned}
$$

Constrained Phase: When the configuration is such that the double pendulum tends to violate the constraint, the equality part of the constraint becomes active. This corresponds to the case when the tip of the rod B is on the boundary of the admissible region. Therefore, the strict part, ϕ, of the unilaterally constraint is

$$
\phi = R\cos\theta_1 + L\cos\theta_2 - R - \frac{3L}{4}. \tag{356}
$$

The velocity and acceleration forms of the constraint equation are respectively

$$
\begin{aligned}
\dot{\phi} &= -Ru_1\sin\theta_1 - Lu_2\sin\theta_2, && (357)\\
\ddot{\phi} &= -R\sin\theta_1\dot{u}_1 - L\sin\theta_2\dot{u}_2 - R\cos\theta_1{u_1}^2 - L\cos\theta_2{u_2}^2. && (358)
\end{aligned}
$$

Therefore, letting $u_I = u_1$ and $u_D = u_2$, the constraint equation (328) can be written in the form

$$
\dot{u}_D = A\dot{u}_I + B, \tag{359}
$$

where A and B are

$$A = -\frac{R\sin\theta_1}{L\sin\theta_2} \tag{360}$$

$$B = -\frac{1}{L\sin\theta_2}\{R\cos\theta_1 {u_1}^2 + L\cos\theta_2 {u_2}^2 + \Delta_1[-Ru_1\sin\theta_1 - Lu_2\sin\theta_2] \tag{361}$$

$$+\Delta_2[R\cos\theta_1 + L\cos\theta_2 - R - \frac{3L}{4}]\}. \tag{362}$$

Hence, the constrained equations of motion are

$$\dot{u} = \begin{bmatrix} A_1 \\ A_2Q \end{bmatrix}^{-1} \begin{Bmatrix} B \\ A_2[P+G\tau] \end{Bmatrix}, \tag{363}$$

where

$$A_1 = \begin{bmatrix} \frac{R\sin\theta_1}{L\sin\theta_2} & 1 \end{bmatrix} \tag{364}$$

$$A_2 = \begin{bmatrix} 1 & -\frac{R\sin\theta_1}{L\sin\theta_2} \end{bmatrix}. \tag{365}$$

To simulate instantaneous changes of velocities when the bar touches the obstacle, large values of the constants Δ_1 and Δ_2 are assumed; $\Delta_1 = -3 \times 10^2$ and $\Delta_2 = -10^2$, respectively.

With initial conditions $\theta_1 = \theta_2 = 1$ rad., Figures 25 and 26 show the responses of θ_1, $u_1 = \dot{\theta}_1$, and θ_2, $u_2 = \dot{\theta}_2$ to constant applied torques $\tau_1 = 1.0$ N m. and $\tau_2 = 3.0$ N m., respectively. For the purpose of overall kinematical analysis, $\dot{\theta}_1$ and $\dot{\theta}_2$ can be viewed as discontinuous when contacts take place. However, adopting this continuous velocity modeling scheme has the benefit of increasing the accuracy of the model by capturing more physics of the motion, through exhibiting the generalized constraint forces behavior during these violent incidents. To achieve that, the constraints are relaxed by adding the generalized constraint forces F^c to the right hand side of Eqn. (352) and solving

$$F^c = Q(q,t)\dot{u} - P(q,u,t) - G(q,u,t)\tau, \tag{366}$$

where $\dot{u}$ is the column matrix that contains the generalized accelerations of the constrained system, obtained directly from Eqs. (363). Figures 27 and 28 are plots of the obtained generalized constraint forces.

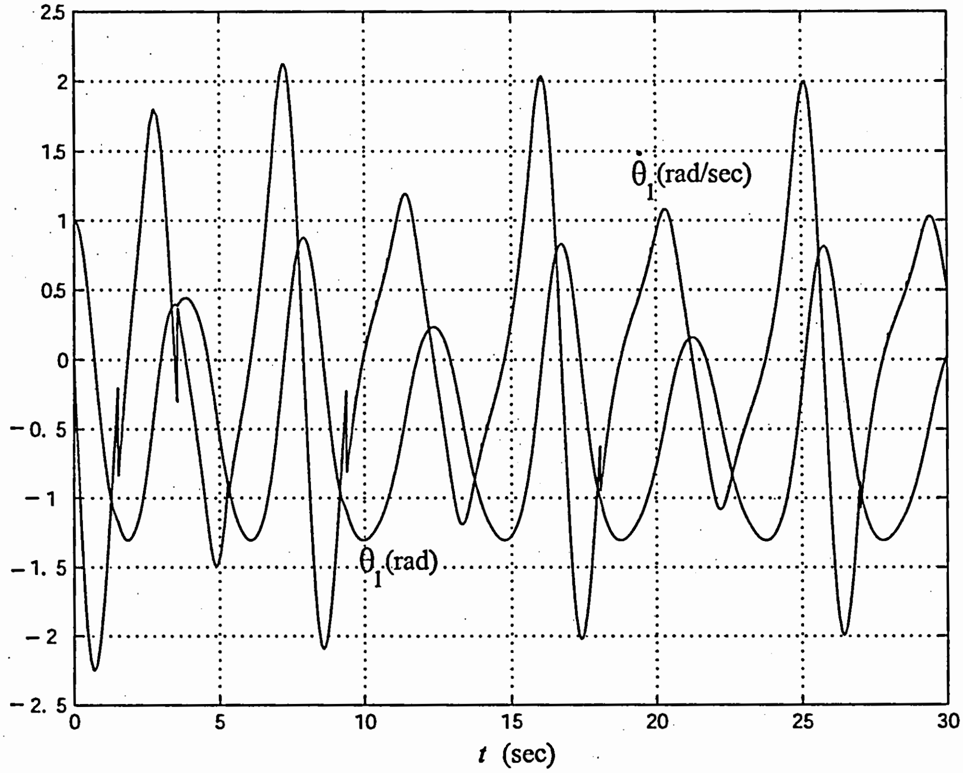

Figure 25: Example 5.1: $\theta_1, \dot{\theta}_1$ vs. t

An advantage of the present method is showing the indirect effect of the contact, particulary at its initial time when the impulsive action occurs, on the dynamics of components that are not involved in the process of contact, the disc in this example. Figures (29) and (30) show approximate "microscopic" views of the generalized contact forces at the initial moments of the first contact cycle. The spikes that take place at the moment of contact, $t = 1.49$ seconds, are clearly shown. The intensities of the spikes depend on the values of Δ_1 and Δ_2, which are dependent on the elastic properties and geometries of the materials of the bodies in contact. More realistic models of contact dynamics are considered next, in the context of impact.

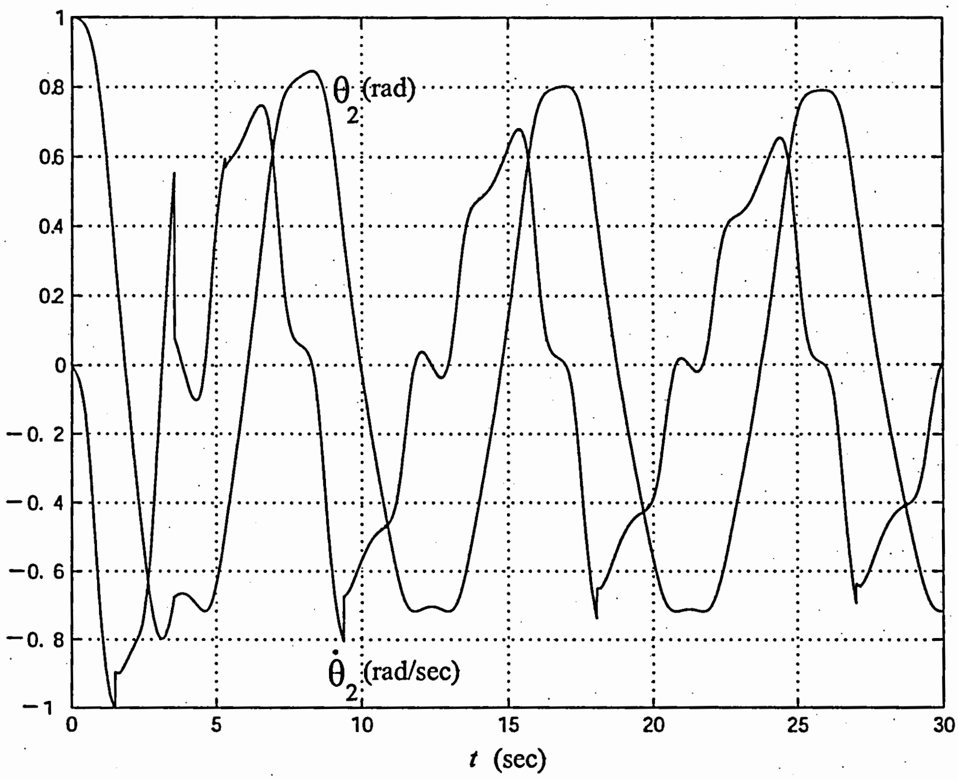

Figure 26: Example 5.1: θ_2, $\dot{\theta}_2$ vs. t

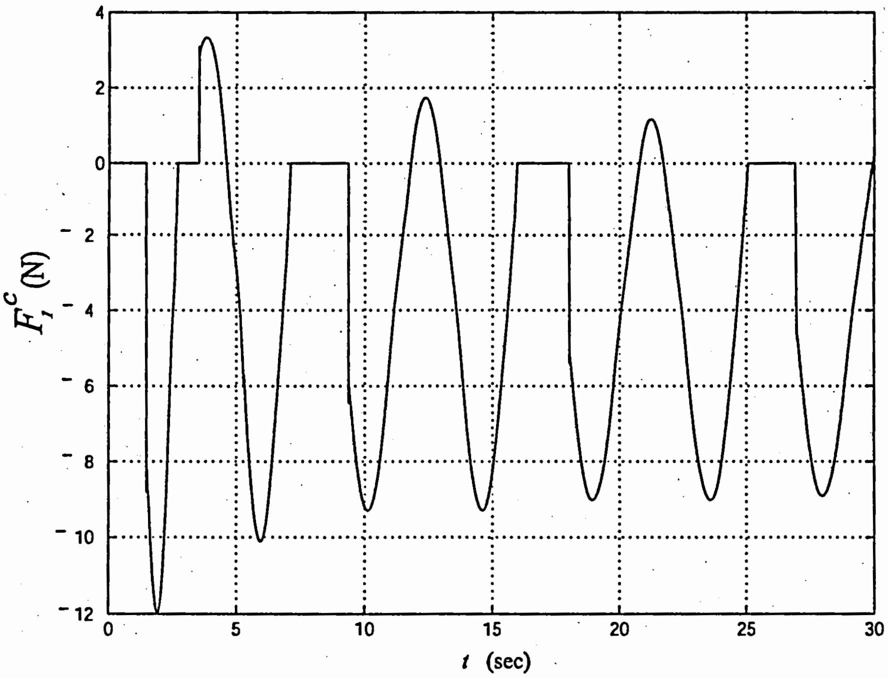

Figure 27: Example 5.1: F_1^c vs. t

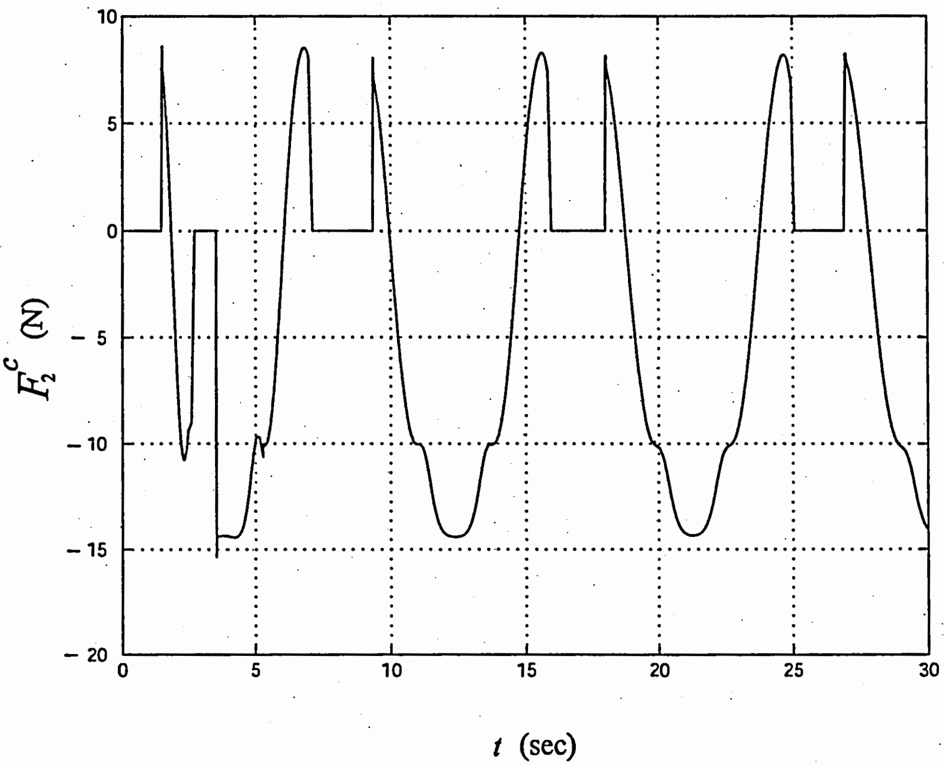

Figure 28: Example 5.1: F_2^c vs. t

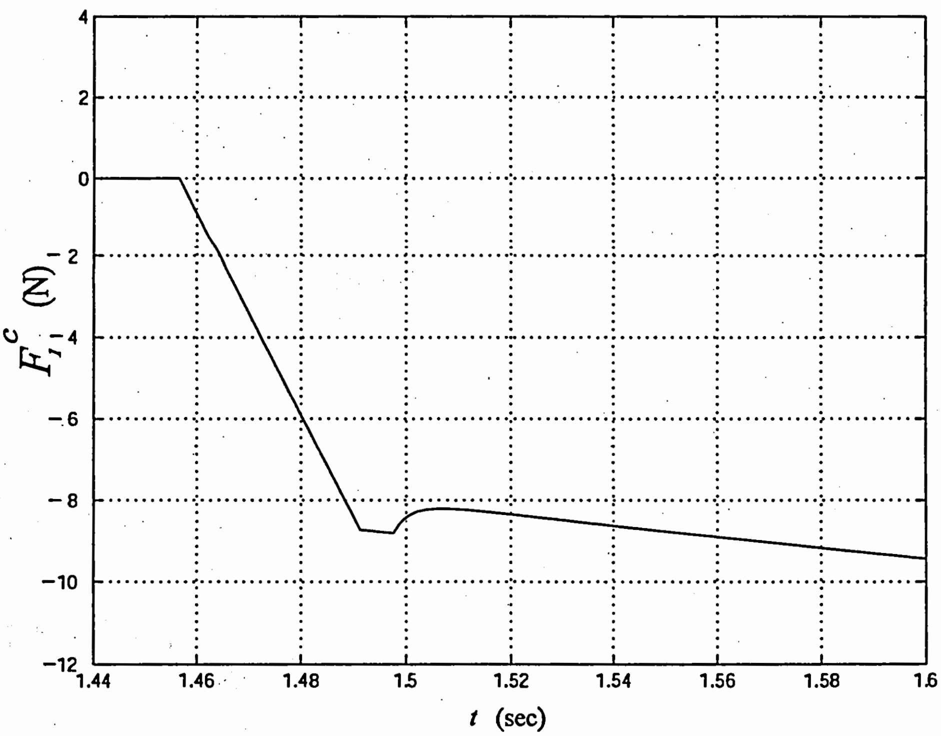

Figure 29: Example 5.1: F_1^c vs. t; Initial moment of 1^{st} contact cycle

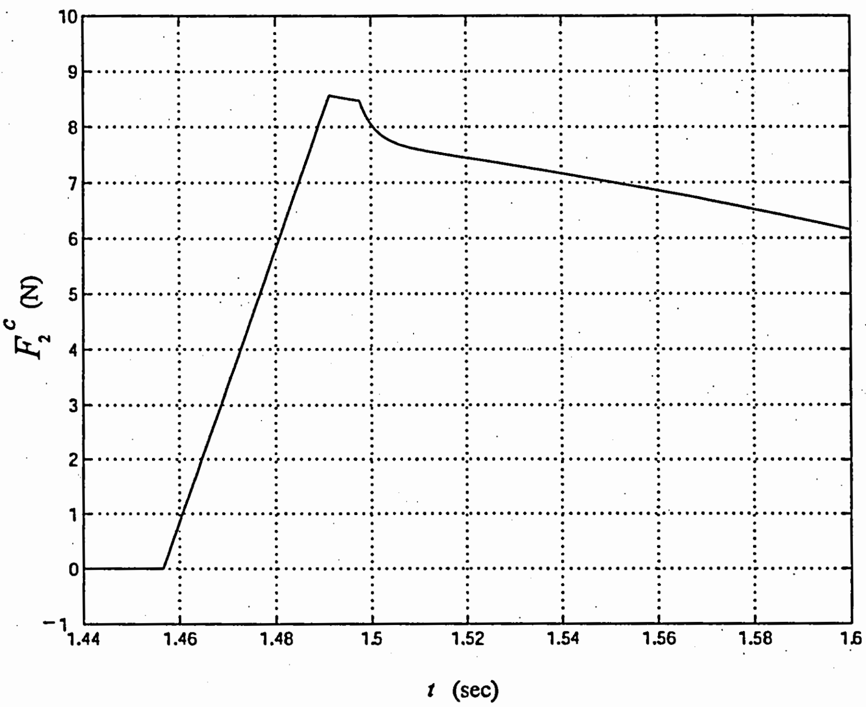

Figure 30: Example 5.1: F_2^c vs. t; Initial moment of 1^{st} contact cycle

5.6 Impact

The acceleration form of constraints can be used to approximate the collision dynamics according to the Kelvin-Voigt second-order model [40]

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{367}$$

where the damping coefficient c and the stiffness coefficient k are dependent on the materials and the geometries of the colliding objects, and x is the relative displacement due to penetration of one body in another. The advantage of the Kelvin-Voigt is its simplicity. However its validity was criticized in Ref. [54], as it contradicts some of the physics of impact by indicating a tensile force between the bodies before separation. Also, it violates the known Goldsmith impacting velocity-damping energy loss relation [47], by predicting it to be quadratic instead of cubic. Instead, the damping and stiffness were modified such that the collision dynamics takes the form

$$m\ddot{x} + (\lambda x^n)\dot{x} + kx^n = 0, \tag{368}$$

where n is a material constant, λ is an empirical damping constant that can be estimated with the aid of kino-elastic relations to be [54]

$$\lambda = n\alpha k, \tag{369}$$

and α is a material constant. By using any of the above two impact models, the nonminimal nonholonomic form can be used to solve the problem of interaction between the impacting bodies and other mechanical components that constitute the dynamical system, because the second-order relations (367) or (368) are augmented in the formulation with the dynamical equations of the bodies without contact, to obtain a realistic representation of the dynamics.

5.7 *Example 5.2: Continuous velocity impact model of unactuated double pendulum*

The double pendulum in example 5.1 is considered, where $\tau_1 = \tau_2 = 0$. It is assumed that the impact of the rod with the obstacle can be modeled by Eq. 368, where $n = \frac{3}{2}$, $\alpha = -5 \times 10^7$, and $k = 1$. Hence, $\lambda = -7.5 \times 10^7$, and the second-order nonlinear impact model (368) becomes

$$\ddot{x} - (7.5 \times 10^7 x^{\frac{3}{2}})\dot{x} + x^{\frac{3}{2}} = 0, \tag{370}$$

where x is the normal penetration displacement of the tip of the rod in the horizontal surface of the obstacle,

$$x = R\cos\theta_1 + L\cos\theta_2 - R - \frac{L}{3}. \tag{371}$$

When there is no contact between the rod and the surface, the governing equations are (352). During contact, a constrained model is obtained by augmenting equation (370) with the unconstrained model equations (352). The constraint equation (370) can be put in the form

$$\dot{u}_D = A\dot{u}_I + B, \tag{372}$$

where $u_I = u_1$ and $u_D = u_2$. The variables A and B are

$$A = -\frac{R\sin\theta_1}{L\sin\theta_2} \tag{373}$$

$$B = -\frac{1}{L\sin\theta_2}[R\cos\theta_1 {u_1}^2 + L\cos\theta_2 {u_2}^2$$

$$+ 7.5\times 10^7 x^{3/2}(-R\sin\theta_1 u_1 - L\sin\theta_2 u_2) - x^{3/2}]. \quad (374)$$

The constrained model takes the form (363), where the constraint matrices A_1 and A_2 are given by (364) and (365).

Time simulations are shown next. Figures 31 and 32 show the angular displacement θ_1 and angular rate $\dot{\theta}_1$ of the disc. The corresponding quantities for the rod, θ_2 and $\dot{\theta}_2$ are shown in Figures 33 and 34. The vertical displacement of the tip of the rod, x, is shown in Figure 35. The penetration of the rod in the obstacle is exhibited by using a small displacement axis scale for x, as shown in Figure 36. The value of x does not reach exactly zero because of numerical inaccuracy. Because of the fact that the friction between the rod and the obstacle is not modeled, it is noticed that the mechanism retains an undamped harmonic behavior as the penetration dynamics vanishes. Modeling of friction is the subject of the next section.

5.8 *Friction*

Friction is a contact related phenomenon that involves complex interaction mechanisms between the surfaces in contact [108, 101]. An accurate model of a dynamical system must involve a consideration of friction forces, as well as the resulting changes in system configuration, velocities, and accelerations. Friction is a highly nonlinear phenomenon. Therefore, a realistic model of friction is highly nonlinear, and simulations of models involving friction are difficult, because of sudden changes in the velocities and accelerations of the system, that vary from zero and infinitesimal to high values.

There exist several models to approximate the contact-with-friction process. These models can be classified as continuous and discontinuous models. Among the continuous models are the Dahl friction model [31], the friction circle model [17], and the viscous friction model [121]. The main discontinuous model is the classical Coulomb dry friction

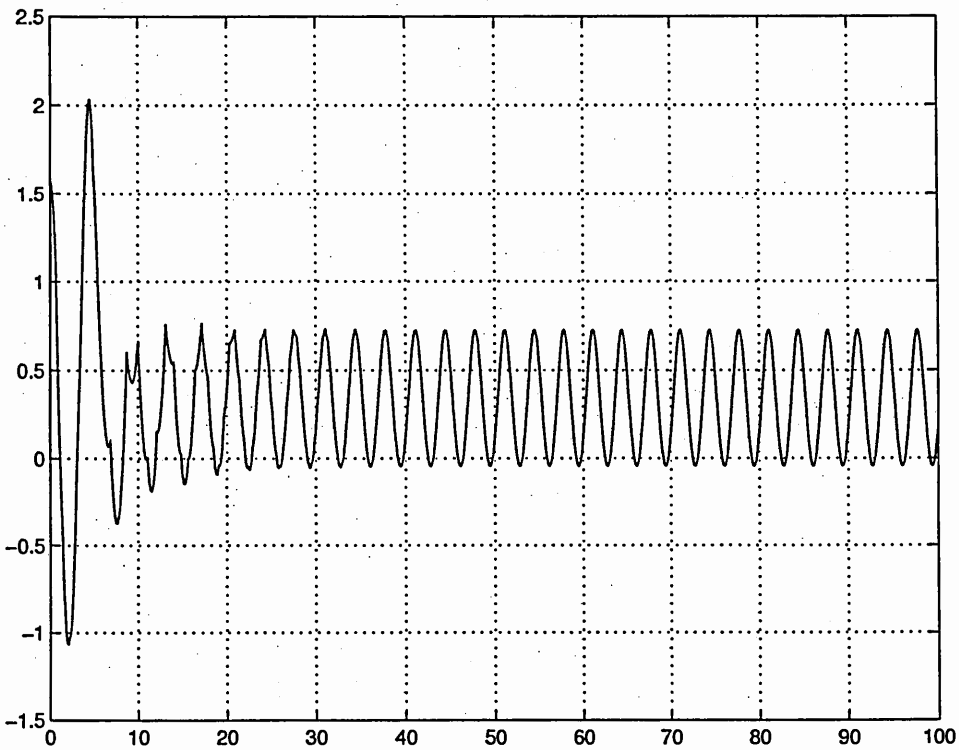

Figure 31: Example 5.2: θ_1 (rad) vs. t (sec)

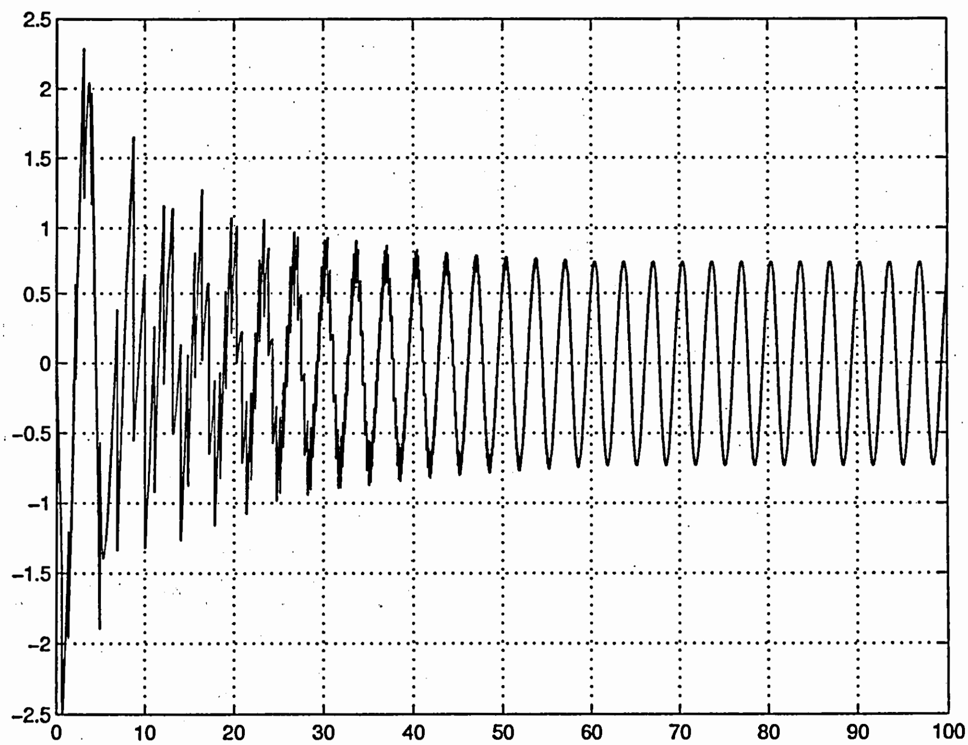

Figure 32: Example 5.2: $\dot{\theta}_1$ (rad/sec) vs. t (sec)

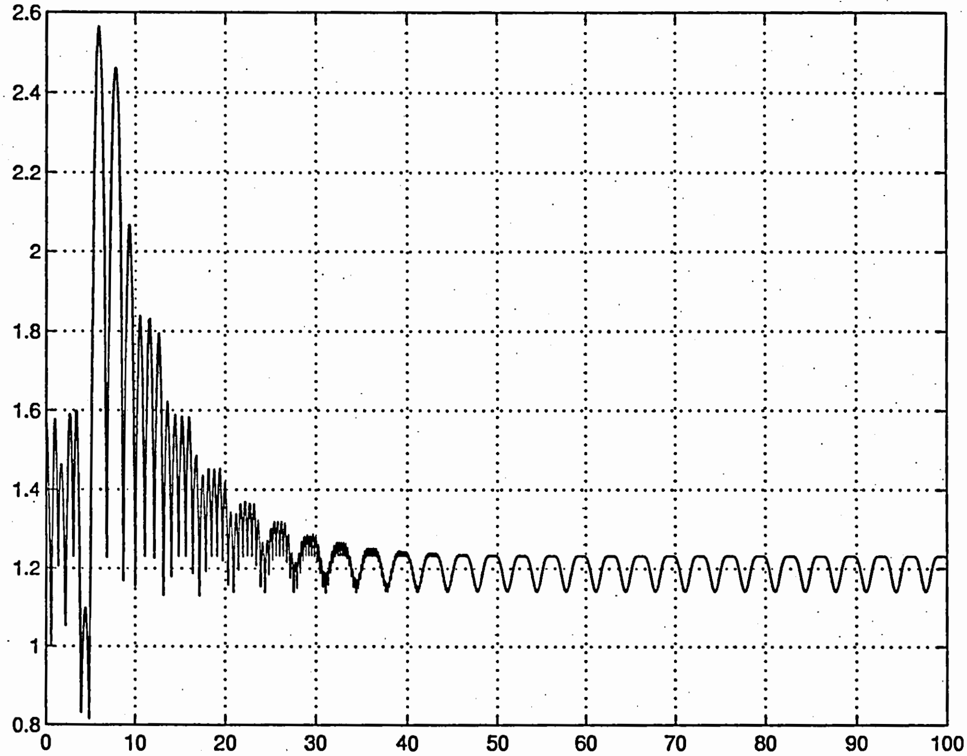

Figure 33: Example 5.2: θ_2 (rad) vs. t (sec)

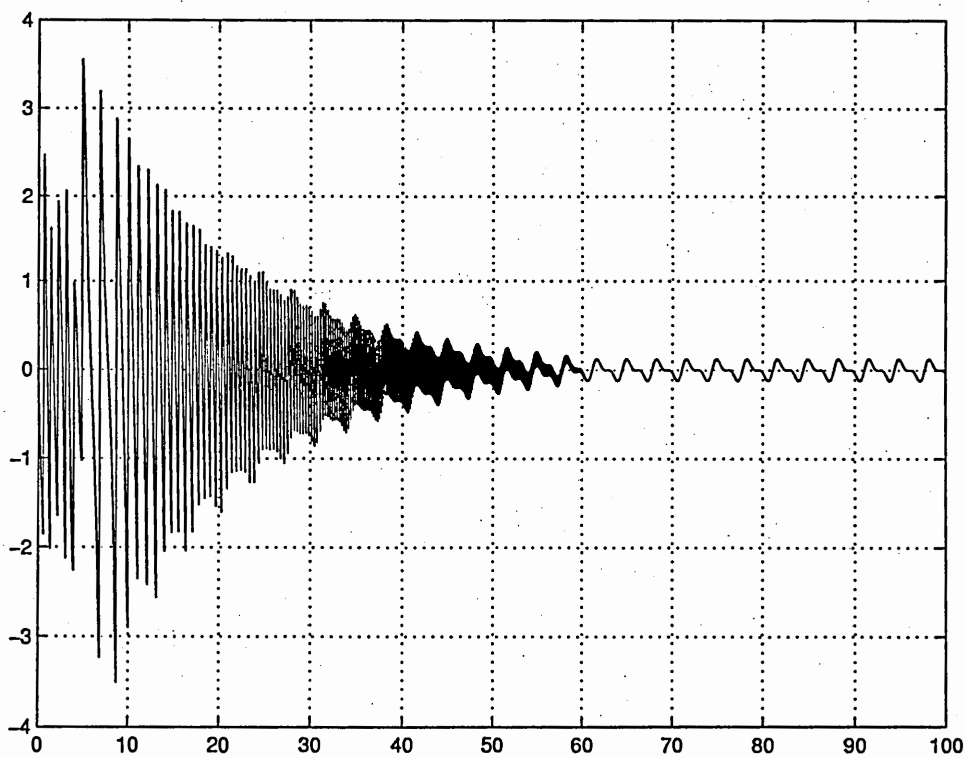

Figure 34: Example 5.2: $\dot{\theta}_2$ (rad/sec) vs. t (sec)

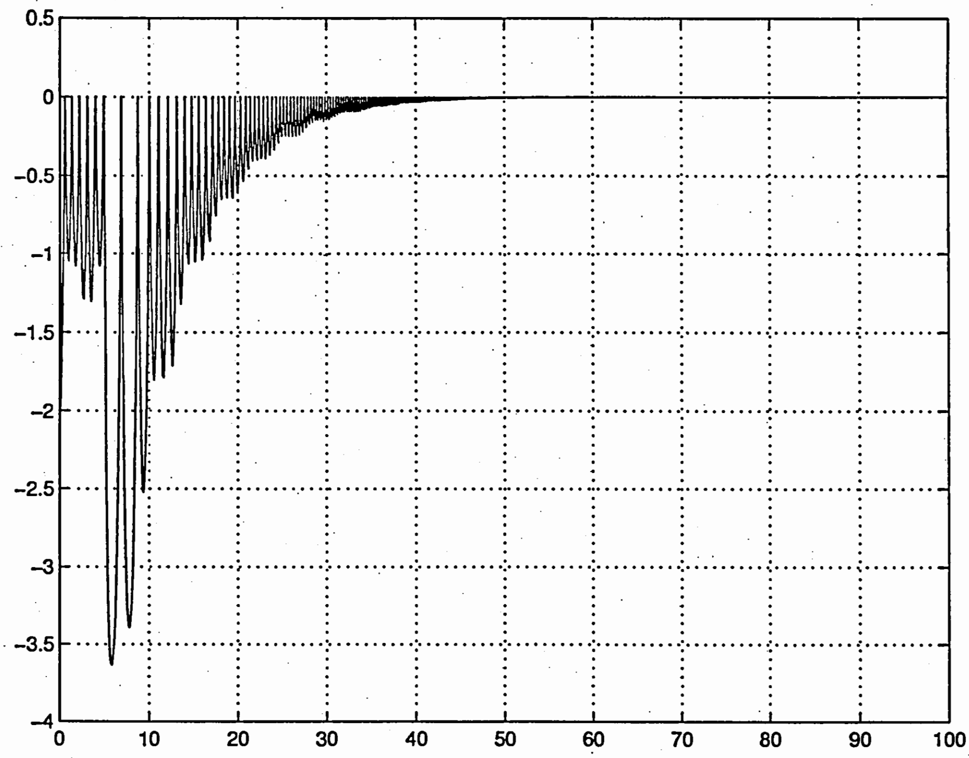

Figure 35: Example 5.2: x (m) vs. t (sec)

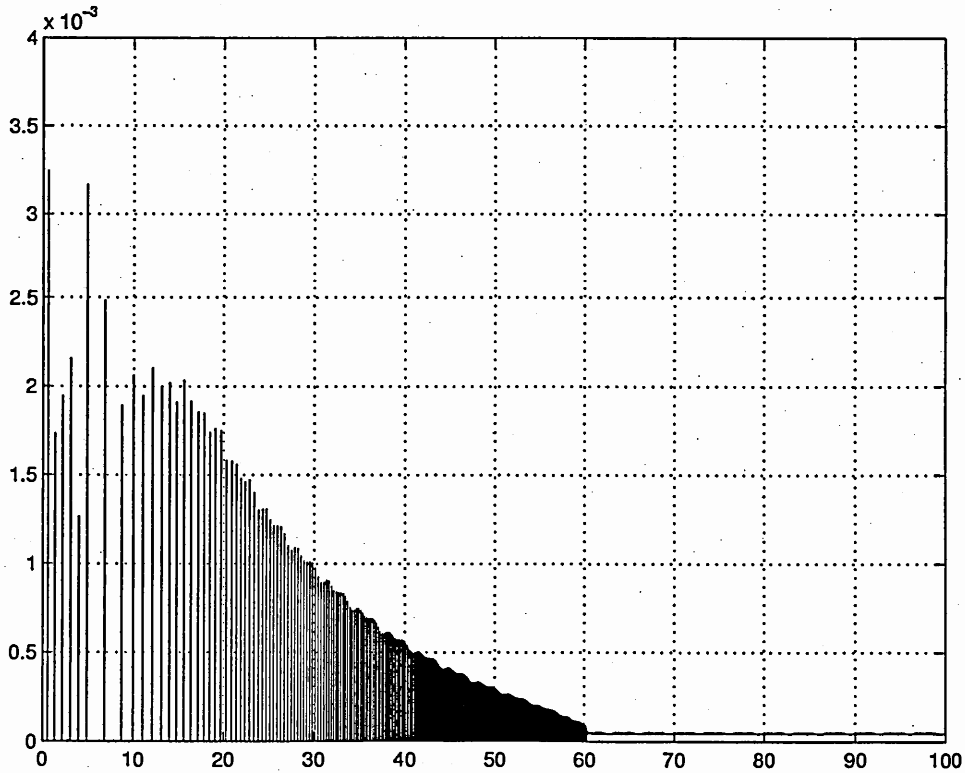

Figure 36: Example 5.2: x (m) vs. t (sec); small displacement scale

model [108]. Comparisons between these two classes are found in Refs. [97, 14].

The aim from modeling friction as a continuous process is to avoid the difficulty encountered in simulations as the relative velocity between the contact surfaces vanishes. Nevertheless, approximating the friction induced discontinuous relation between the relative contact velocity and the tangent-to-surface force as continuous is disadvantageous, for several reasons:

1. It does not provide detail information about the stick-slip nature of the motion, because using the approximating smoothing functions makes it impossible to satisfy the condition of vanishing relative velocity between the bodies in contact.

2. The smoothing of the stick-slip process alters the physical behavior of the dynamical system, and hence affects the accuracy of simulations.

3. It causes numerical difficulties for large deviations between static and kinetic coefficients of friction, because the continuous friction law does not distinguish between kinetic and static friction laws [15].

The above reasons leave the Coulomb friction model to be the highly desirable approach.

Several attempts were made to avoid the above mentioned numerical difficulty related to the Coulomb friction model. In [75] and [12], an intermediate phase of motion between sticking and slipping is introduced, named "the transition phase". The aim from introducing this phase is to reduce the numerical simulation impeding that is related to the vanishing contact velocity, by providing a small tolerance that replaces the zero value of the constraint function at small values of contact velocities. Despite the success of this model for relatively small systems, it causes a complexity as the number of bearings or surfaces of contact increases. To solve this difficulty, a continuous version of the transition-slip phases was introduced in Ref. [97]. However, this model runs into the same difficulties related to continuous friction models.

In this section, the nonminimal form of the equations of motion is used to generate a friction model that avoids some of the disadvantages related to the two classes of friction modeling mentioned above. To perform this task, the motion is divided into two phases. The first phase is the no slipping phase, taking place when there is no relative motion between the surfaces of contact of the bodies, because of the domination of the friction forces over the inertia and the applied forces. The governing equations of motion corresponding to this phase are constrained by the holonomic stiction condition or the nonholonomic rolling without slipping condition. The second phase is the sliding phase, taking place when the inertia and the applied forces dominate the friction forces. The unconstrained equations of motion are formed, and the friction forces take their dynamic values.

To avoid singularities at vanishing sliding velocities, a tolerance is provided for the activation of the no slipping condition, such that the sticking (or rolling) is activated at creeping values instead of exactly zero values. While solving the singularity problem, this waives the need to introduce the transition phase, keeping the number of equation sets equal to two. To damp out the creeping velocities in the no slipping phase, damping terms may be augmented with the corresponding equations of motion.

The activation criterion of the no slipping condition is the decrease of the relative velocity of the bodies in contact below the tolerance value. The activation criterion of the sliding or the rolling with slipping phase is the increase of the net inertia and applied forces over the friction force.

5.9 Example 5.3: Springs-masses system

The two connected blocks B_1 and B_2 shown in Figure 37 slide on a rough surface. The blocks are connected to each other by a spring s_2, and to the wall by a spring s_1. The masses of the blocks are m_1 and m_2, respectively. The static and kinetic coefficients of friction of the surfaces of contact of the blocks B_1 and B_2 with the surface are μ_{s_1}, μ_{k_1}, and μ_{s_2}, μ_{k_2}, respectively. The block B_2 is acted upon by the tangential sinusoidal force

$T = A \sin \omega t$, where A is a positive constant, and ω is the excitation frequency of T.

Letting the generalized coordinates be the displacements of the blocks from the springs unstretched positions, the kinematical differential equations are chosen as

$$\dot{q}_1 = u_1 \tag{375}$$

$$\dot{q}_2 = u_2. \tag{376}$$

The spring forces are

$$F_{s_1} = K_1 q_1 \tag{377}$$

$$F_{s_2} = K_2(q_2 - q_1), \tag{378}$$

For simplicity, the static and dynamic friction coefficients are assumed to be equal and identical for both blocks, and have the value μ. The friction forces f_1 and f_2 are

$$f_1 = \mu m_1 g \tag{379}$$

$$f_2 = \mu m_2 g. \tag{380}$$

The dynamical equation of motion for the block B_1 is dependent on its velocity relative to the surface, and on the magnitude the net spring force acting on it relative to the friction force. Introducing the small tolerance δ, if $|u_1| > \delta$, then

$$\dot{u}_1 = \frac{1}{m_1}[F_{s_2} - F_{s_1} - \mathrm{sgn}(u_1) f_1], \tag{381}$$

where sgn(.) is the sign function:

$$\mathrm{sgn}(x) = +1 \quad \text{if} \quad x > 0 \tag{382}$$

and

$$\mathrm{sgn}(x) = -1 \quad \text{if} \quad x <= 0. \tag{383}$$

The above expression for $\dot{u}_1$ is still valid when $|u_1| \leq \delta$ provided that $f_1 < |F_{s_2} - F_{s_1}|$. If both $u_1 \leq |\delta|$ and $f_1 \geq |F_{s_2} - F_{s_1}|$ then

$$\dot{u}_1 = -a u_1, \tag{384}$$

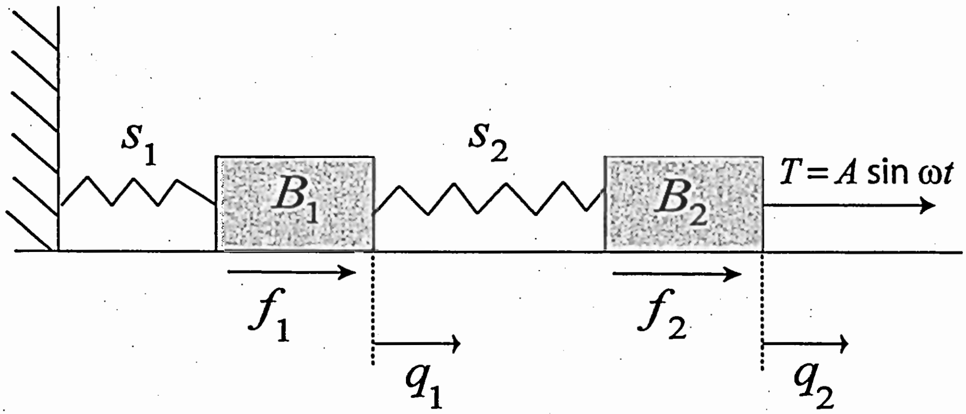

Figure 37: Schematic for Example 5.3

where a is a positive real constant such that the small creeping value of u_1 is caused to damp out. Similarly, when $|u_2| > \delta$, the dynamical equation for the block B_2 is

$$\dot{u}_2 = \frac{1}{m_2}[T - F_{s_2} - \text{sgn}(u_2) f_2], \tag{385}$$

which still holds true if both $|u_2| \leq \delta$ and $f_2 < |T - F_{s_2}|$. Otherwise,

$$\dot{u}_2 = -bu_2, \tag{386}$$

where b is a positive constant, introduced to damp out the creeping velocity of block B_2. The values of the constants chosen are A = 10 N, ω = 1.5 rad/sec., $m_1 = m_2$ = 0.1 kg, $K_1 = K_2$ = 5.0 N/m, μ = 1.0, g = 9.810 m/sec^2. Figures 38 and 39 show the forced response displacements and velocities of the blocks B_1 and B_2, for zero initial conditions on the displacements and velocities q_1, q_2, u_1, u_2. The damping constants are chosen to be $a = b = 2$.

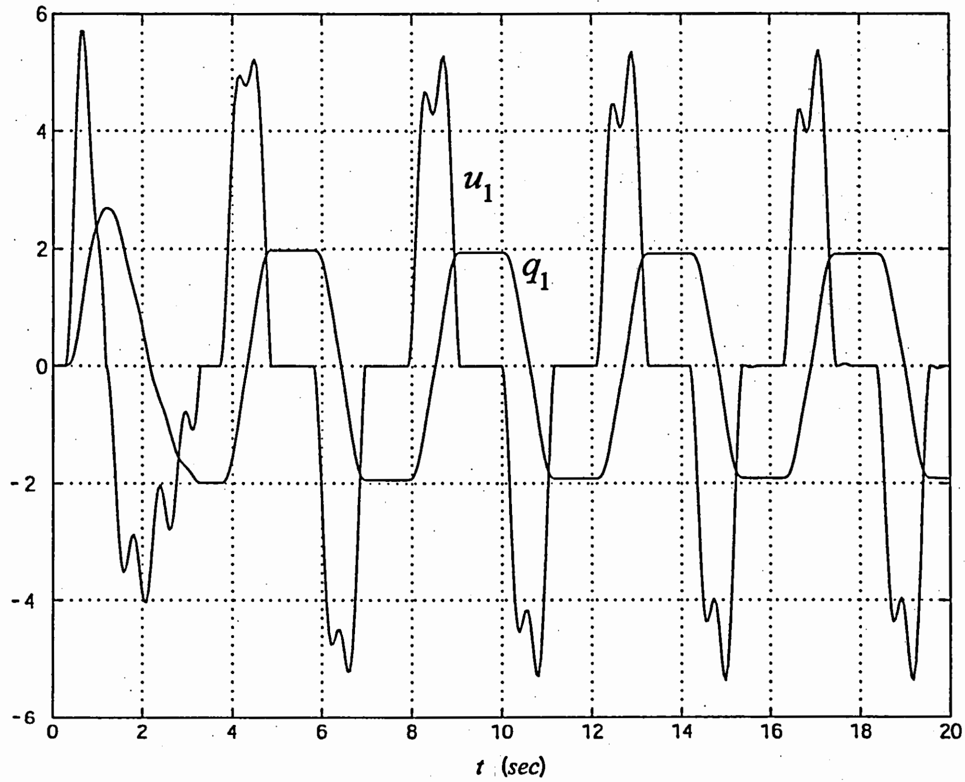

Figure 38: Example 5.3: Displacement and velocity of block B_1

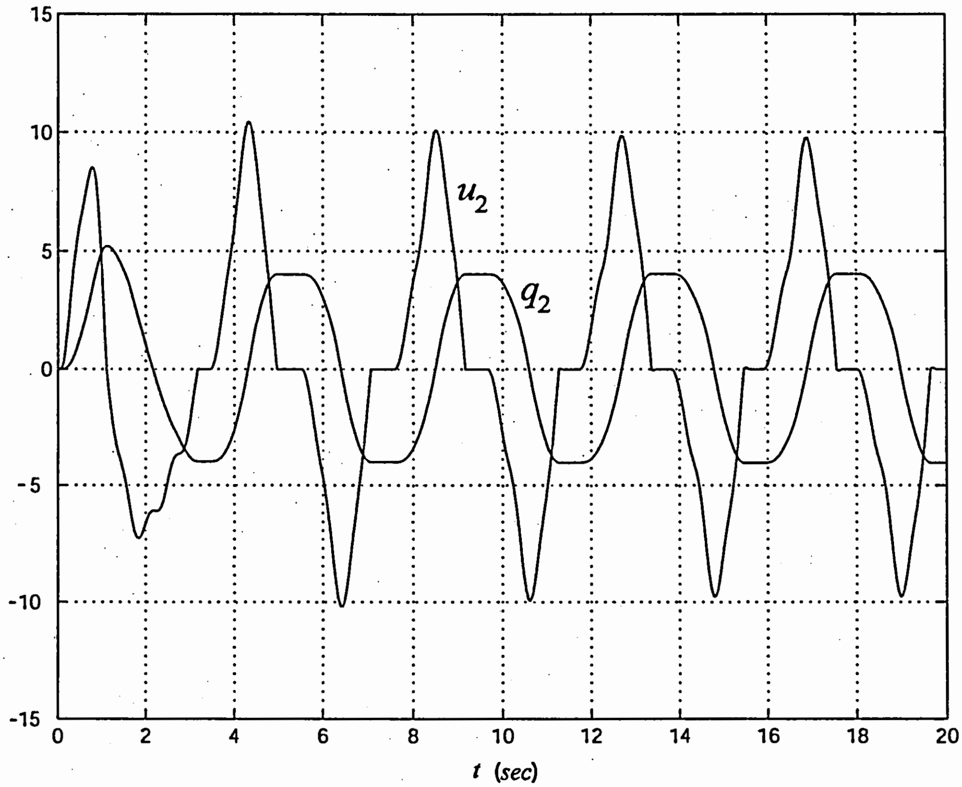

Figure 39: Example 5.3 :Displacement and velocity of block B_2

5.10 Summary

A unifying framework for modeling dynamical systems subjected to unilateral constraints is presented, by using the nonminimal dynamical equations of motion. The two types of unilateral constraints considered are impulsive constraints and friction constraints.

For impulsive constraints, the acceleration form of constraints is utilized to approximate the continuous force-velocity relation during the impulsive action. This is performed by differentiating the strict equality constraint equation and using it with the lower order forms, provided that the duration of contact is sufficiently long. The numerical impedance problem is solved by providing a small tolerance, such that the constrained model is activated only if this tolerance is exceeded.

In a problem involving impact, a model with continuous velocity distribution can be achieved by utilizing a force-deformation relationship for deformable bodies, like the Hertz law, together with a kino-elastic energy balance relationship, to obtain the acceleration level constraint equation (Kelvin-Voigt model for instance). This equation is used with the dynamical equations to obtain the nonminimal constrained model.

For friction constraints, it is recommended when modeling bearings, joints, and other mechanisms that are affected by friction to employ a discontinuous Coulomb friction model, and enforce the sticking constraint while avoiding the numerical singularity at the discontinuities. This is also achieved in the context of the nonminimal equations of motion, by replacing the stiction condition by a damped dynamics that is activated when the relative velocity falls below a predetermined small tolerance, and deactivated when the friction force is not capable of holding stiction.

An advantage of using the nonminimal equations to model the intermittent motion of a multibody dynamical system is to show the effect of the sudden activation of constraints on all the elements of the system, by solving a state-space model that is equal in order to the unconstrained one, and is obtained from the unconstrained model by simple matrix manipulations, and augmentations with the unilateral constraint matrices.

CHAPTER VI

EXTENSION TO THE IMPULSE-MOMENTUM APPROACH

6.1 *Introduction*

The impulse-momentum approach for modeling impact was adopted by Kane [73, 26] in a general form for both holonomic and nonholonomic constraints. The basic assumption in the approach is that the duration of the impact is very short compared to the time interval of the motion, such that the impact can be considered a discrete event, and the change in the configuration of the system during impact is ignorable, although the changes in velocities of the system components can be significant [68]. This allows for converting the differential equations that govern the dynamics of the system to algebraic equations, through integrating the equations in general terms over the infinitesimal period of impact.

The impulse-momentum approach does not assume that energy is conserved during impact [74]. The relationships between the velocities prior to and after impact are given by an experimentally evaluated constant that is dependent on the material and the geometry of the collided surfaces, called the coefficient of restitution [47].

The impulse-momentum approach for modeling impact was applied to different modeling methodologies. Examples are the method of coordinate partitioning [135, 52] and the Hamilton equations of motion [87].

The impulse-momentum approach was followed successfully in the area of multibody system dynamics to model the intermittent motion of both of rigid [52] and flexible [81] systems, but it is interesting to notice that using the approach in the context of Kane's equations of motion was not done until recently [27].

In this chapter, the same approach is extended to the nonminimal nonholonomic form, by explicitly including the effect of nonholonomic constraints on the rapid changes of the generalized speeds. This is achieved by integrating the acceleration form of constraints over the time period during which the impulsive forces act.

The advantages of using the nonminimal form of the equations of motion apply here as well. In particular, this form provides a convenient way to analyze the intermittent motion of both nonholonomic systems and complex holonomic systems with numerous configuration settings but relatively low numbers of degrees of freedom. The latter case pertains to analyses in which pseudo-generalized coordinates (i.e. additional configuration variables) are needed to facilitate the formulation, and hence more holonomic constraints are added.

In the next section, nonholonomic generalized impulses and momenta are defined, and are related to their holonomic counterparts by means of the constraint matrix.

6.2 Generalized Impulse and Momentum

An inertial reference frame $\mathcal{R}$ is considered, in which n generalized coordinates are used to describe the configuration of a set of ν particles and μ rigid bodies forming a nonholonomic system S possessing p degrees of freedom. Let $\mathbf{R}_i$ be the resultant active force on the ith particle, P_i. The resultant active forces on the ith rigid body B_i are equivalent to a force $\mathbf{Z}_i$ on a point Q_i on B_i, together with a torque $\mathbf{T}_i$. Also, let t_1 and t_2 be the initial and final instants of time that are close enough such that the generalized coordinates $q_1 \dots q_n$ can be considered as constants throughout the interval bounded by t_1 and t_2. The rth *nonholonomic generalized impulse* $\tilde{\mathcal{I}}_r$ is defined as [68]

$$\begin{aligned}\tilde{\mathcal{I}}_r(q,u,t_1) &= \sum_{i=1}^{\nu}\int_{t_1}^{t_2}\tilde{\mathbf{v}}_r^{P_i}\cdot\mathbf{R}_i dt+\sum_{i=1}^{\mu}\int_{t_1}^{t_2}\tilde{\mathbf{v}}_r^{Q_i}\cdot\mathbf{Z}_i dt+\sum_{i=1}^{\mu}\int_{t_1}^{t_2}\tilde{\omega}_r^{B_i}\cdot\mathbf{T}_i dt\\ &= \int_{t_1}^{t_2}\tilde{F}_r(q,t)dt, \quad r=1,\dots,p, \end{aligned} \tag{387}$$

where $\tilde{\mathbf{v}}_r^{P_i}$, $\tilde{\mathbf{v}}_r^{Q_i}$ are the rth nonholonomic *partial velocities* of P_i and Q_i, respectively; and $\tilde{\omega}_r^{B_i}$ is the rth nonholonomic *partial angular velocity* of B_i [68]. Alternatively, a full-order impulse variable can be introduced for the system S. The rth component of the *holonomic generalized impulse* is defined as

$$\begin{aligned}
\mathcal{I}_r(q,u,t_1) &= \sum_{i=1}^{\nu}\int_{t_1}^{t_2} \mathbf{v}_r^{P_i}\cdot\mathbf{R}_i dt + \sum_{i=1}^{\mu}\int_{t_1}^{t_2} \mathbf{v}_r^{Q_i}\cdot\mathbf{Z}_i dt + \sum_{i=1}^{\mu}\int_{t_1}^{t_2} \omega_r^{B_i}\cdot\mathbf{T}_i dt \\
&= \int_{t_1}^{t_2} F_r(q,t)dt, \quad r = 1,\ldots,n, \qquad (388)
\end{aligned}$$

where $\mathbf{v}_r^{P_i}$, $\mathbf{v}_r^{Q_i}$ are the rth holonomic *partial velocities* of P_i and Q_i, respectively; and $\omega_r^{B_i}$ is the rth holonomic *partial angular velocity* of B_i [68]. By using the relation between the *holonomic* and *nonholonomic* partial velocities and partial angular velocities given by Eqs. (131) and (140), Eq. (387) can be written as

$$\begin{aligned}
\tilde{\mathcal{I}}_r(q,u,t_1) &= \sum_{i=1}^{\nu}\int_{t_1}^{t_2}\left(\mathbf{v}_r^{P_i} + \sum_{s=1}^{n-p}\mathbf{v}_{p+s}^{P_i}A_{sr}\right)\cdot\mathbf{R}_i \\
&\quad +\sum_{i=1}^{\mu}\int_{t_1}^{t_2}\left(\mathbf{v}_r^{Q_i} + \sum_{s=1}^{n-p}\mathbf{v}_{p+s}^{Q_i}A_{sr}\right)\cdot\mathbf{Z}_i \\
&\quad +\sum_{i=1}^{\mu}\int_{t_1}^{t_2}\left(\omega_r^{B_i} + \sum_{s=1}^{n-p}\omega_{p+s}^{B_i}A_{sr}\right)\cdot\mathbf{T}_i \qquad (389) \\
&= \int_{t_1}^{t_2}\left[F_r(q,t) + \sum_{s=1}^{n-p}F_{p+s}(q,t)A_{sr}(q,t)\right]dt \qquad (390) \\
&= \mathcal{I}_r(q,u,t_1) + \sum_{s=1}^{n-p}\mathcal{I}_{p+s}(q,u,t_1)A_{sr}(q,t), r = 1,\ldots,p. \qquad (391)
\end{aligned}$$

Remark *If the dynamical system is holonomic or constrained by simple nonholonomic constraints, then the constraint matrix A is only dependent on the generalized coordinates $q_1,\ldots,q_n$ and t, and independent of the generalized speeds $u_1,\ldots,u_n$. This implies that A can be regarded constant in the interval $[t_1,t_2]$ during the evaluation of the generalized impulses, and can be taken to be its value at the interval entry, $t=t_1$.*

Let $\mathbf{L}^{P_i}$, $\mathbf{L}^{B_i}$, and $\mathbf{H}^{B_i}$ be respectively the linear momentum of the ith particle, the linear momentum of the ith body, and the angular momentum of the ith body of the system. If

m_{P_i} is the mass of the ith particle P_i, m_{B_i} is the mass of the ith body B_i, $B_i^\star$ is its center of mass, and $\underline{\mathbf{I}}^{B_i}$ is its central inertia dyadic, then

$$\begin{aligned} \mathbf{L}^{P_i} &= m_{P_i}\mathbf{v}^{P_i}(t) && (392) \\ \mathbf{L}^{B_i} &= m_{B_i}\mathbf{v}^{B_i^\star}(t) && (393) \\ \mathbf{H}^{B_i} &= \underline{\mathbf{I}}^{B_i}\cdot\boldsymbol{\omega}^{B_i}(t). && (394) \end{aligned}$$

The rth *nonholonomic generalized momentum* $\tilde{\mathcal{P}}_r$ is defined as

$$\tilde{\mathcal{P}}_r(q,u,t) = \tilde{\mathbf{v}}_r^{P_i}\cdot\mathbf{L}^{P_i} + \tilde{\mathbf{v}}_r^{B_i^\star}\cdot\mathbf{L}^{B_i} + \tilde{\boldsymbol{\omega}}_r^{B_i}\cdot\mathbf{H}^{B_i}, \quad r = 1,\ldots,p. \tag{395}$$

Therefore,

$$\tilde{\mathcal{P}}_r(q,u,t) = \sum_{i=1}^{\nu} m_{P_i}\tilde{\mathbf{v}}_r^{P_i}\cdot\mathbf{v}^{P_i}(t) + \sum_{i=1}^{\mu} m_{B_i}\tilde{\mathbf{v}}_r^{B_i^\star}\cdot\mathbf{v}^{B_i^\star}(t) + \sum_{i=1}^{\mu}\tilde{\boldsymbol{\omega}}_r^{B_i}\cdot\underline{\mathbf{I}}^{B_i}\cdot\boldsymbol{\omega}^{B_i}(t),$$
$$r = 1,\ldots,p. \tag{396}$$

The use of the full set of generalized speeds in the expressions of velocities and angular velocities of the particles and bodies comprising a nonholonomic system makes it feasible to introduce the rth *holonomic generalized momentum*,

$$\mathcal{P}_r(q,u,t) = \mathbf{v}_r^{P_i}\cdot\mathbf{L}^{P_i} + \mathbf{v}_r^{B_i^\star}\cdot\mathbf{L}^{B_i} + \boldsymbol{\omega}_r^{B_i}\cdot\mathbf{H}^{B_i}, \quad r = 1,\ldots,n. \tag{397}$$

Therefore,

$$\mathcal{P}_r(q,u,t) = \sum_{i=1}^{\nu} m_{P_i}\mathbf{v}_r^{P_i}\cdot\mathbf{v}^{P_i}(t) + \sum_{i=1}^{\mu} m_{B_i}\mathbf{v}_r^{B_i^\star}\cdot\mathbf{v}^{B_i^\star}(t) + \sum_{i=1}^{\mu}\boldsymbol{\omega}_r^{B_i}\cdot\underline{\mathbf{I}}^{B_i}\cdot\boldsymbol{\omega}^{B_i}(t),$$
$$r = 1,\ldots,n. \tag{398}$$

Similar to the generalized impulse, the relations (131) and (140) between the holonomic and nonholonomic partial angular velocities and partial velocities can be used to obtain the nonminimal representation of the nonholonomic generalized momentum

$$\tilde{\mathcal{P}}_r = \mathcal{P}_r + \sum_{s=1}^{n-p}\mathcal{P}_{p+s}A_{sr}, \quad r = 1,\ldots,p. \tag{399}$$

Considering the kinetic energy K of the system,

$$
\begin{aligned}
K &= \sum_{i=1}^{\nu} k^{P_i} + \sum_{i=1}^{\mu} k^{B_i} \\
&= \frac{1}{2}\sum_{i=1}^{\nu} m_{P_i}\mathbf{v}^{P_i}\cdot\mathbf{v}^{P_i} + \frac{1}{2}\sum_{i=1}^{\mu} m_{B_i}\mathbf{v}^{B_i^\star}\cdot\mathbf{v}^{B_i^\star} + \frac{1}{2}\sum_{i=1}^{\mu} \boldsymbol{\omega}^{B_i}\cdot\underline{\mathbf{I}}^{B_i}\cdot\boldsymbol{\omega}^{B_i}, \quad (400)
\end{aligned}
$$

the following identities relate both of the holonomic and nonholonomic generalized momenta to the kinetic energy, and can be verified by evaluating the partial derivatives of the kinetic energy expression (400) with respect to generalized speeds.

$$\tilde{\mathcal{P}}_r = \frac{\partial K}{\partial u_r}, \quad r = 1, \ldots, p \tag{401}$$

$$\mathcal{P}_r = \frac{\partial K}{\partial u_r}, \quad r = 1, \ldots, n, \tag{402}$$

where the generalized speeds $u_1, \ldots, u_p$ in Eq. (401) are the coefficients of the partial velocities in the nonholonomic representation of generic particles of the system,

$$\mathbf{v} = \sum_{r=1}^{p} \tilde{\mathbf{v}}_r(q,t)u_r + \tilde{\mathbf{v}}_t(q,t), \tag{403}$$

and $u_1, \ldots, u_n$ in Eq. (402) are the coefficients of the partial velocities in the holonomic representation

$$\mathbf{v} = \sum_{r=1}^{n} \mathbf{v}_r(q,t)u_r + \mathbf{v}_t(q,t). \tag{404}$$

The above usage of kinetic energy often facilitates the evaluation of generalized momenta [68].

6.3 *Nonminimal Impulse-Momentum Relations*

Both the holonomic and the nonholonomic generalized momenta can be related to the generalized inertia forces of system. The rth generalized inertia force of the particles and bodies comprising a nonholonomic system is given by

$$\tilde{F}_r^\star(q,u,\dot{u},t) = -\tilde{\mathbf{v}}_r^{P_i}\cdot\frac{d\mathbf{L}^{P_i}}{dt} - \tilde{\mathbf{v}}_r^{B_i^\star}\cdot\frac{d\mathbf{L}^{B_i}}{dt} - \tilde{\boldsymbol{\omega}}_r^{B_i}\cdot\frac{d\mathbf{H}^{B_i}}{dt}, \quad r = 1\ldots p. \tag{405}$$

Integrating the above expression of $\tilde{F}_r^\star$ from t_1 to t_2, one obtains

$$\int_{t_1}^{t_2} \tilde{F}_r^\star(q,u,\dot{u},t)dt = -\int_{t_1}^{t_2}\left[\tilde{\mathbf{v}}_r^{P_i}\cdot\frac{d\mathbf{L}^{P_i}}{dt} + \tilde{\mathbf{v}}_r^{B_i^\star}\cdot\frac{d\mathbf{L}^{B_i}}{dt} + \tilde{\boldsymbol{\omega}}_r^{B_i}\cdot\frac{d\mathbf{H}^{B_i}}{dt}\right]dt. \tag{406}$$

Because $t_1 \approx t_2$ such that the configuration of the system is invariant in the period $[t_1, t_2]$, the partial angular velocities and partial velocities of the particles and bodies comprising the system are considered to be constants, as these quantities are dependent on the generalized coordinates, and independent of the generalized speeds. Therefore, Eq. (395) can be used to write the above equations as

$$\int_{t_1}^{t_2} \tilde{F}_r^\star(q, u, \dot{u}, t)dt = \tilde{\mathcal{P}}_r(q, u(t_1)) - \tilde{\mathcal{P}}_r(q, u(t_2)). \tag{407}$$

Similar relations between the holonomic generalized active forces and the holonomic generalized momenta can be derived as

$$\int_{t_1}^{t_2} F_r^\star(q, u, \dot{u}, t)dt = \mathcal{P}_r(q, u(t_1)) - \mathcal{P}_r(q, u(t_2)). \tag{408}$$

The equations of motion for impulsive motion are

$$\int_{t_1}^{t_2} \left[\tilde{F}_r(q, u, t) + \tilde{F}_r^\star(q, u, \dot{u}, t)\right] dt = 0. \tag{409}$$

From Eqs. (387) and (407), the above equations becomes

$$\tilde{\mathcal{I}}_r(q, u, t_1) = \tilde{\mathcal{P}}_r(q, u(t_2)) - \tilde{\mathcal{P}}_r(q, u(t_1)), \quad r = 1 \ldots p. \tag{410}$$

If all the constraints are holonomic, then the above equations are written as

$$\mathcal{I}_r(q, u, t_1) = \mathcal{P}_r(q, u(t_2)) - \mathcal{P}_r(q, u(t_1)), \quad r = 1 \ldots n. \tag{411}$$

From Eqs. (391) and (399), Eq. (410) can be written as

$$\begin{aligned}\mathcal{P}_r(q, u(t_2)) + \sum_{s=1}^{n-p} \mathcal{P}_{p+s}(q, u(t_2))A_{sr}(q, t_1) &= \mathcal{P}_r(q, u(t_1)) + \mathcal{I}_r(q, u, t_1) \\ &+ \sum_{s=1}^{n-p} \left[\mathcal{P}_{p+s}(q, u(t_1)) + \mathcal{I}_{p+s}(q, u, t_1)\right] A_{sr}(q, t_1), \quad r = 1, \ldots, p. \end{aligned} \tag{412}$$

6.4 *Constraints Effect on Impulsive Motion*

If the constraint equations are simple nonholonomic, i.e., satisfy relations (16), then the acceleration form of constraints takes the form of Eqs. (22). To obtain the effect of constraints on the impulsive motion, this form is integrated from t_1 to t_2. If time t does not appear explicitly in matrix A, then the time dependency of A is implicit in $q(t)$. Therefore,

$$\int_{t_1}^{t_2} A_1 \dot{u} dt = \int_{t_1}^{t_2} \dot{A}\ u_I dt + \int_{t_1}^{t_2} \dot{B} dt. \tag{413}$$

The integral on the left hand side is

$$\int_{t_1}^{t_2} A_1 \dot{u} dt = A_1(q)u(t_2) - A_1(q)u(t_1). \tag{414}$$

The integrals on the right hand sides of Eq. (413) are

$$\begin{aligned} \int_{t_1}^{t_2} \dot{A}\ u_I dt &= A\ [u_I(t_2) - u_I(t_1)] - \int_{t_1}^{t_2} A\ \dot{u}_I dt &(415) \\ &= A\ [u_I(t_2) - u_I(t_1)] - A\ [u_I(t_2) - u_I(t_1)] &(416) \\ &= 0, &(417) \end{aligned}$$

$$\int_{t_1}^{t_2} \dot{B} dt = B(t_2) - B(t_1) = 0. \tag{418}$$

This implies that the effect of constraints during the impulsive motion of the dynamical system is such that

$$A_1(q)u(t) = c, \tag{419}$$

where c is a constant that can be determined from the values of the generalized coordinates and the generalized speeds at the beginning of the time interval of the impulsive action.

6.5 Impulsive Dynamical Equations of Motion

The nonminimal form of the impulsive equations of motion can be obtained by using Eqs. (412) with Eqs. (419). It is noticed that Eqs. (412) can be put in the form

$$A_2(q)\mathcal{P}_r(q, u(t_2)) = A_2(q)\left[\mathcal{P}_r(q, u(t_1)) + \mathcal{I}_r(q, u, t_1)\right] \quad r = 1, \ldots, n. \tag{420}$$

Furthermore, integrating the expression (39) of $F^\star$ from t_1 to t_2, and using (408), the following equality is obtained

$$\mathcal{P}_r(q, u(t_1)) - \mathcal{P}_r(q, u(t_2)) = \int_{t_1}^{t_2} F^\star dt = -\int_{t_1}^{t_2} \left[Q(q,t)\dot{u} + L(q,u,t)\right] dt. \tag{421}$$

The second term on the right hand side is an integral of a variable quantity over the time interval $[t_1, t_2]$. Nevertheless, it is a multivariable polynomial in the generalized speeds, and its integral over a finite interval is zero. Hence Eq. (421) becomes

$$\mathcal{P}_r(q, u(t_1)) - \mathcal{P}_r(q, u(t_2)) = Q(q, t)\left[u(t_1) - u(t_2)\right]. \tag{422}$$

Substituting Eq. (422) in Eq. (420) yields

$$A_2(q)Q(q, t)\left[u(t_2) - u(t_1)\right] = A_2(q)\mathcal{I}(q, u, t_1) \tag{423}$$

Eqs. (419) and (423) form the matrix system

$$\begin{bmatrix} A_1(q) \\ A_2(q)Q(q, t_1) \end{bmatrix}\left[u(t_2) - u(t_1)\right] = \begin{bmatrix} \mathbf{0} \\ A_2(q) \end{bmatrix}\mathcal{I}(q, u, t_1). \tag{424}$$

Remark *The coefficient matrix in Eq. (424) is the same as the matrix T in Eq. (42). The invertibility of T for all admissible configurations and velocities is guaranteed.* Therefore,

$$u(t_2) = T^{-1}(q, t_1)E(q, u, t_1), \tag{425}$$

where

$$T = \begin{bmatrix} A_1(q) \\ A_2(q)Q(q, t_1) \end{bmatrix} \tag{426}$$

$$E = \begin{bmatrix} A_1(q)u(t_1) \\ A_2(q)\left[Q(q, t_1)u(t_1) + \mathcal{I}(q, u, t_1)\right] \end{bmatrix}. \tag{427}$$

6.6 *Example 6.1: Four-Bar linkage:*

The four-bar linkage shown in Figure (40) moves in the vertical plane, and has the dimensions $L_1 = 1.0$ m, $L_2 = 3.0$ m, $L_3 = L_4 = 2.5$ m. The linkage OQ is fixed to the inertial frame $\mathcal{R}$. Two particles P_1 and P_2 have masses m_1 and m_2, respectively, are fixed to two vertices of the linkage, as shown in the figure. The mechanism is at rest at $q_1 = 0.0$ rad. when the particle P_2 is struck by an impulse force $\mathbf{M} = 10^3[\mathbf{i} + \mathbf{j}]$KN. It is required to

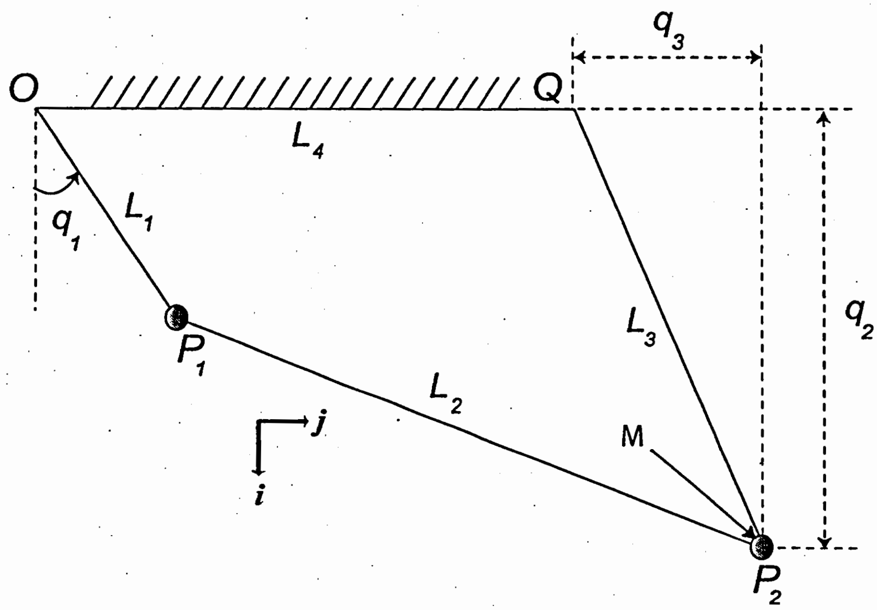

Figure 40: Schematic for Four-Bar Linkage

obtain the resulting changes in the values of the generalized speeds due to the impulsive action.

The mechanism has one degree of freedom. Nevertheless, three configuration variables, q_1, q_2, and q_3 are defined for convenience, as shown in the figure. The generalized speeds are defined as

$$u_1 = \dot{q}_1 L_1, \tag{428}$$

$$u_i = \dot{q}_i, \quad i = 2, 3. \tag{429}$$

The velocities of the particles relative to the inertial frame $\mathcal{R}$ are

$${}^{\mathcal{R}}\mathbf{v}^{P_1} = u_1(\cos q_1 \mathbf{j} - \sin q_1 \mathbf{i}), \tag{430}$$

$${}^{\mathcal{R}}\mathbf{v}^{P_2} = u_2 \mathbf{i} + u_3 \mathbf{j}, \tag{431}$$

such that the partial velocities of the particles are

$${}^{\mathcal{R}}\mathbf{v}_1^{P_1} = \cos q_1 \mathbf{j} - \sin q_1 \mathbf{i}, \quad {}^{\mathcal{R}}\mathbf{v}_2^{P_1} = {}^{\mathcal{R}}\mathbf{v}_3^{P_1} = 0, \tag{432}$$

$${}^{\mathcal{R}}\mathbf{v}_1^{P_2} = 0, \quad {}^{\mathcal{R}}\mathbf{v}_2^{P_2} = \mathbf{i}, \quad {}^{\mathcal{R}}\mathbf{v}_3^{P_2} = \mathbf{j}. \tag{433}$$

The corresponding accelerations relative to $\mathcal{R}$ are

$$
\begin{aligned}
{}^{\mathcal{R}}\mathbf{a}^{P_1} &= \left[-\dot{u}_1 \sin q_1 - \frac{{u_1}^2}{L_1} \cos q_1\right] \mathbf{i} + \left[\dot{u}_1 \cos q_1 - \frac{{u_1}^2}{L_1} \sin q_1\right] \mathbf{j} && (434) \\
\mathbf{a}^{P_2} &= \dot{u}_2 \mathbf{i} + \dot{u}_3 \mathbf{j}. && (435)
\end{aligned}
$$

The generalized inertia forces are

$$
\begin{aligned}
-m_1 \mathbf{a}^{P_1} \cdot \mathbf{v}_1^{P_1} - m_2 \mathbf{a}^{P_2} \cdot \mathbf{v}_1^{P_2} &= -m_1 \dot{u}_1 && (436) \\
-m_1 \mathbf{a}^{P_1} \cdot \mathbf{v}_2^{P_1} - m_2 \mathbf{a}^{P_2} \cdot \mathbf{v}_2^{P_2} &= -m_2 \dot{u}_2 && (437) \\
-m_1 \mathbf{a}^{P_1} \cdot \mathbf{v}_3^{P_1} - m_2 \mathbf{a}^{P_2} \cdot \mathbf{v}_3^{P_2} &= -m_2 \dot{u}_3. && (438)
\end{aligned}
$$

Therefore, the matrix Q in Eq. (424) is

$$
Q = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \tag{439}
$$

Two holonomic constraint equations relating the generalized coordinates are

$$
\begin{aligned}
&{q_2}^2 + {q_3}^2 - {L_3}^2 = 0 && (440) \\
&L_4 + q_3 - \sqrt{{L_2}^2 - (q_2 - L_1 \cos q_1)^2} - L_1 \sin q_1 = 0. && (441)
\end{aligned}
$$

Differentiation of the constraints leads to:

$$
q_2 u_2 + q_3 u_3 = 0 \tag{442}
$$

and

$$
X(u_1 \sin q_1 + u_2) + u_3 - u_1 \cos q_1 = 0, \tag{443}
$$

where

$$
X = \frac{q_2 - L_1 \cos q_1}{\sqrt{{L_2}^2 - (q_2 - L_1 \cos q_1)^2}}. \tag{444}
$$

Letting $u_I = u_1$ and $u_D = \lfloor u_2 \quad u_3 \rfloor^T$, Eqs. (442) and (443) can be used to form the matrix system

$$
\begin{bmatrix} q_2 & q_3 \\ X & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cos q_1 - X \sin q_1 \end{Bmatrix} u_1. \tag{445}
$$

Therefore, the constraint matrices A and B of Eq. (18) are

$$A = \begin{bmatrix} q_2 & q_3 \\ X & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \cos q_1 - X \sin q_1 \end{Bmatrix} \quad (446)$$

$$= \frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} \begin{Bmatrix} -q_3 \\ q_2 \end{Bmatrix} \quad (447)$$

$$B = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (448)$$

The impulse force $\mathbf{M}$ can be written as

$$\mathbf{M} = M_1\mathbf{i} + M_2\mathbf{j}. \quad (449)$$

Therefore, the holonomic generalized impulses are

$$\mathcal{I}_1 = \mathbf{M} \cdot \mathbf{v}_1^{P_2} = 0 \quad (450)$$

$$\mathcal{I}_2 = \mathbf{M} \cdot \mathbf{v}_2^{P_2} = M_1 = 10^3\text{KN} \quad (451)$$

$$\mathcal{I}_3 = \mathbf{M} \cdot \mathbf{v}_3^{P_2} = M_2 = 10^3\text{KN}. \quad (452)$$

The given geometric condition is $q_1 = 0.0$ rad. The remaining generalized coordinates are found from Eqs. (440) and (441) to be

$$q_2 = -1.8773409 \text{ m} \quad (453)$$

$$q_3 = -1.6509363 \text{ m}. \quad (454)$$

The corresponding value of X is -3.388800, and the constraint matrix A is

$$A = \begin{Bmatrix} -0.2209487 \\ 0.2512489 \end{Bmatrix}. \quad (455)$$

Specify $m_1 = m_2 = 1.0$ kg, and given that the entry conditions for the generalized speeds are $u_i(t_1) = 0$, $i = 1 \ldots 3$, the matrices T and E are

$$T = \begin{bmatrix} 0.2209487 & 1 & 0 \\ -0.2512489 & 0 & 1 \\ 1 & -0.2209487 & 0.2512489 \end{bmatrix} \tag{456}$$

$$E = \begin{bmatrix} 0 \\ 0 \\ -0.2209487M_1 + 0.2512489M_2 \end{bmatrix}. \tag{457}$$

With $M_1 = M_2 = 10^3$ KN, the generalized speeds just after the action of the impulsive force are

$$u(t_2) = T^{-1}E, \tag{458}$$

or

$$u_1(t_2) = 2.7250 \times 10^4 \text{ m/sec.} \tag{459}$$

$$u_2(t_2) = -0.6021 \times 10^4 \text{ m/sec.} \tag{460}$$

$$u_3(t_2) = 0.6846 \times 10^4 \text{ m/sec.} \tag{461}$$

It is possible to use different numbers of configuration variables to derive different forms of the impulse-momentum equations. However, the complexity of the resulting equations increases as the number of configuration variables decreases. For example, it becomes very difficult to use one generalized coordinate to derive a minimal impulse-momentum equation for this one degree of freedom mechanism.

Nevertheless, to preserve the minimality in the number of impulse-momentum equations, the mechanism may be treated as simple nonholonomic by considering the same generalized coordinates and eliminating two of the generalized speeds in favor of the third. Therefore, if the generalized speeds are chosen as Eqs. (428) and (429), then the velocities of P_1 and P_2 relative to $\mathcal{R}$ are given by Eqs. (430) and (431). Letting

$$u_I = u_1 \tag{462}$$

$$u_D = \lfloor u_2 \quad u_3 \rfloor^T, \tag{463}$$

and using the expressions (447) and (448) for the matrices A and B to eliminate u_2 and u_3, ${}^{\mathcal{R}}\mathbf{v}^{P_2}$ becomes

$$
{}^{\mathcal{R}}\mathbf{v}^{P_2} = u_1 \frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} \left[-q_3 \mathbf{i} + q_2 \mathbf{j} \right]. \tag{464}
$$

Hence, the nonholonomic partial velocities of the particles are

$$
\begin{aligned}
{}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P_1} &= \cos q_1 \mathbf{j} - \sin q_1 \mathbf{i} && (465)\\
{}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P_2} &= \frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} \left[-q_3 \mathbf{i} + q_2 \mathbf{j} \right]. && (466)
\end{aligned}
$$

The nonholonomic generalized momenta before and after the impact are

$$
\begin{aligned}
\tilde{\mathcal{P}}(t_1) &= 0 && (467)\\
\tilde{\mathcal{P}}(t_2) &= m_1^{\mathcal{R}}\mathbf{v}^{P_1} \cdot {}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P_1} + m_2^{\mathcal{R}}\mathbf{v}^{P_2} \cdot {}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P_2} \\
&= m_1 u_1(t_2) + m_2 u_1(t_2) \left[\frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} \right]^2 [q_2^2 + q_3^2]. && (468)
\end{aligned}
$$

The nonholonomic generalized impulse is

$$
\begin{aligned}
\tilde{I}_1 &= \mathbf{M} \cdot {}^{\mathcal{R}}\tilde{\mathbf{v}}_1^{P_2} && (469)\\
&= \frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} [M_1 \mathbf{i} + M_2 \mathbf{j}] \cdot [-q_3 \mathbf{i} + q_2 \mathbf{j}] && (470)\\
&= \frac{\cos q_1 - X \sin q_1}{q_2 - X q_3} [-M_1 q_3 + M_2 q_2]. && (471)
\end{aligned}
$$

Therefore, at $q_1 = 0.0$ deg., an impulsive action of $\mathbf{M} = 10^3\mathbf{i} + 10^3\mathbf{j}$ KN implies a nonholonomic generalized impulse of $\tilde{I}_1 = 2.7251 \times 10^4$ N, resulting in a change in the generalized momentum of $\tilde{P}_1(t_2) = 1.1119\ u_1(t_2)$ N.sec. Eq. (410) can be solved for $u_1(t_2)$, giving the same value of Eq. (459). Solving Eq. (18) results in the values of u_2 and u_3 given by Eqs. (460) and (461).

The two mentioned methods for writing impulse-momentum equations are comparable in the required effort and in complexity of the resulting equations. However, writing all exit generalized speeds (at $t = t_2$) as an explicit vector field that is dependent on generalized coordinates, generalized impulses, and all inlet generalized speeds (at $t = t_1$) is more beneficial whenever the Jacobian of this vector field is needed. An example is minimizing a function of exit generalized speeds over the admissible configuration settings for some known values of inlet generalized speeds and generalized momenta.

6.7 Summary

With the aid of the acceleration form of constraint equations, the discontinuous-in-velocities approximation approach to studying impulsive motion is followed in the context of the nonminimal equations. For that purpose, the concepts of holonomic generalized impulse and holonomic generalized momentum are defined and related by means of the constraint matrix to their nonholonomic counterparts. These quantities are used to expand the standard Kane's impulse-momentum equations, prior to their augmentation with the acceleration form of the constraint equations.

Based on the assumption of unchanging generalized coordinates during the action of impulsive forces, the resulting nonminimal Kane's impulse-momentum equations appear in an algebraic form. The equations can be used either to study the effects of impacts, or to study the effects of sudden activations and deactivations of holonomic and/or nonholonomic constraints. In the later case, the constraints can be instantaneously applied and removed from action, or applied and continued to act. In all cases, the equations are used in a discrete manner to describe the "no transient time" change in velocities.

The nonminimal impulse-momentum equations form an alternative way to study impulsive constraints, and gain the advantage of simplicity over the equations presented in Ref. [68] if the difference between the number of configuration and motion parameters is large. It also eliminates the need of using nonholonomic partial velocities to simplify the resulting equations. The explicit representation of all exit generalized speeds in terms of quantities that are dependent on all inlet generalized speeds is particularly beneficial in studying the effects of configuration settings, initial velocities, and impulsive loadings on the velocities of various parts of mechanisms and structures.

on two related matrices. The required controls are found with the aid of the generalized inverse of one of these matrices.

Another subject is obtaining the ideal form of servo-constraints for the purpose of emulating passive constraints and solving the redundancy resolution of redundant manipulators. In this chapter, the constrained full order state-space model derived in previous chapters is used for that purpose. An illustrative example is presented.

7.2 Servo-constraints Realization

Servo-constraints realization is the problem of moving the state of the dynamical system in a pre-specified constraint manifold with the aid of the available control forces. In this section, the acceleration form of constraints is used to solve for the forces required to realize servo-constraints, where the generalized inverse of the constraint matrix derived from the servo-constraint equations is the one utilized to express the redundancy. The considered dynamical equations of motion of a controlled dynamical system is of the form

$$\dot{q} = C(q,t)u + D(q,t) \tag{472}$$

$$Q(q,t)\dot{u} = P(q,u,t) + G(q,u,t)\tau, \tag{473}$$

where $q, u \in \mathbb{R}^n$ denote the column matrices containing the configuration parameters and the velocity parameters, $\dot{q}$ and $\dot{u}$ are the derivatives of q and u with respect to t, respectively. The square matrices involved in the two equations above are $C, Q \in \mathbb{R}^{n\times n}$, such that C^{-1}, Q^{-1} exist for all generalized coordinates and for all $t \in \mathbb{R}$. The control matrix $G \in \mathbb{R}^{n\times l}$ is such that $l \leq n$, and the column matrices $D, P \in \mathbb{R}^n$. The column matrix $\tau \in \mathbb{R}^l$ contains the control variables. Equations (472) and (473) form a complete state-space model.

The acceleration form of the constraint equations can be used to determine the control forces that are necessary to realize servo-constraints for the dynamical system above. Assume that the dynamical system is required to track the prescribed velocity-dependent trajectory described by the m nonholonomic constraint equations

$$\psi(q,u,t)=0, \qquad \psi\in\mathbb{R}^m, \tag{474}$$

where the servo-constraints ψ may be multi-objective, i.e. represent simultaneous requirements. The purpose is to find the control forces that are necessary to enforce the above equations, and to relate them to the available control authority. The acceleration form of the servo-constraint equations is

$$\dot{\psi}(q,u,\dot{u},t)=\frac{\partial\psi}{\partial u}\dot{u}+X(q,u,t)=0, \tag{475}$$

where X is found from Eq. (472) to be

$$X(q,u,t)=\frac{\partial\psi}{\partial q}C(q,t)u+\frac{\partial\psi}{\partial q}D(q,t)+\frac{\partial\psi}{\partial t}. \tag{476}$$

The following modified constraint equations at the acceleration level are considered

$$\dot{\psi}(q,u,\dot{u},t)-\Theta\psi(q,u,t)=0, \tag{477}$$

where $\Theta\in\mathbb{R}^{m\times m}$ is a prescribed matrix that has strictly negative-real eigenvalues. Substituting $\dot{u}$ from Eq.(473) into Eq.(477) yields

$$S(q,u,t)\tau=z(q,u,t), \tag{478}$$

where

$$S = \frac{\partial\psi}{\partial u}Q^{-1}G \tag{479}$$

$$z = -\frac{\partial\psi}{\partial u}Q^{-1}P-X+\Theta\psi. \tag{480}$$

If the above system of equations is consistent at some specific values of generalized coordinates and generalized speeds, i.e. z is in the range space of S, then it is possible to solve for τ,

$$\tau=S^{+}z+(I-S^{+}S)y, \tag{481}$$

where the superscript "+" refers to the Moore-Penrose generalized inverse, and $y\in\mathbb{R}^l$ is arbitrary. Therefore, depending on the nature of S and z, the servo-constraints realization problem can be categorized as one of the following:

1. *The problem has a unique solution*: z is in the range space of S, and $l \leq m$.

2. *The problem has no solution*: z is not in the range space of S.

3. *The problem has infinite number of solutions*: z is in the range space of S, and the null space of S^T is not trivial. In this case, the flexibility provided by y can be used to achieve further requirements beside realization of servo-constraints.

The procedure for enforcing servo-constraints, Eqs. (474), is summarized in the following steps:

1. The expression for $\dot{u}$ obtained from the dynamical equations of motion (473) is substituted in Eqs. (475).

2. The resulting expression for $\dot{\psi}$ is used to form Eqs. (477), where Θ is chosen such that the first-order servo-constraint dynamics is stable. Eqs. (477) is put in the form of Eqs. (478).

3. Using the generalized Moore-Penrose inverse of S, the expression for τ, Eqs. (481), is formed, where the column matrix y can be chosen arbitrarily.

A similar treatment for holonomic servo-constraints can be done. In this case, the servo-constraint equations take the form

$$\psi(q, t) = 0. \tag{482}$$

The above equations are twice differentiated, and the desired dynamics takes the form

$$\ddot{\psi}(q, u, \dot{u}, t) - \Theta_1 \dot{\psi}(q, u, t) - \Theta_2 \psi(q, t) = 0, \tag{483}$$

where Θ_1 and Θ_2 are chosen such that the servo-constraints dynamics is stable. The above equations can be put in the form of Eqs. (478), from which the procedure follows.

7.3 Redundancy Resolution

If the number of independent actuators is more than the necessary to enforce servo-constraints, then the set of required control forces is not unique. This redundancy has been studied extensively for over three decades [22] in the area of robotics, at both kinematic and dynamic levels. The Jacobian matrix of the manipulators and its generalized inverse are the main tools in these studies.

The redundancy resolution problem is concerned with finding the control forces that are necessary to enforce a predetermined dynamics of the system based on optimizing some criterion, like the required control effort [77], the kinetic energy of the mechanism [80], or the distance from a desired trajectory [117]. This dynamics may involve holonomic and/or nonholonomic constraints. Also, the desired motion can be a combination of several requirements, e.g., tracking some prescribed trajectory while preserving the total energy of the dynamical system. Removing the redundancy implies that the required control forces are unique, although the system trajectories are not unique.

The equations of motion derived in the previous section can be used to solve the inverse dynamics for the natural control forces, i.e. those equivalent to passive joint reactions. This is equivalent to minimizing the instantaneous acceleration energy of the dynamical system relative to its unconstrained status, at every configuration and velocity [63]. In doing that, the accelerations of the generated nonminimal constrained model and the controlled equations of motion are matched. Equating the expressions of $\dot{u}$ from equations (150) and (473) yields

$$G(q,u,t)\tau = \mathcal{R}(q,u,t), \tag{484}$$

where

$$\mathcal{R}(q,u,t) = QT^{-1}V - P. \tag{485}$$

For some specific values of generalized coordinates and generalized speeds, if the matrix

$\mathcal{R}$ is in the range space of G, then there exists a solution of τ that is given by

$$\tau = G^{+}\mathcal{R}, \tag{486}$$

where $G^{+}G = I$ holds true because $l \leq n$. If the dynamical system is fully-actuated, i.e. the number of independent control variables is equal to the number of degrees of freedom, we have that $l = n$, and the matrix G is of full-rank. In this case, τ is given by

$$\tau = G^{-1}\mathcal{R}. \tag{487}$$

Remark *If the solution to this inverse dynamics problem exists, then the solution is unique. The control forces $G\tau$ in Eq. (484) compensate for reaction forces that correspond to equivalent passive constraints on the dynamical systems. This implies that these control forces satisfy d'Alembert's principle, and the accelerations of the controlled system satisfy Gauss' principle of least constraints.*

The inverse dynamics for the ideal control forces can be viewed as a specialization of the servo-constraints realization problem, where the matrices G and $\mathcal{R}$ stand for S and z, respectively, and the second term in the right hand side of Eq. (481) vanishes because $G^{+}G = I$. Since the nonminimal nonholonomic form that is used to obtain $\mathcal{R}$ results in the accelerations of an equivalent passively constrained system, the interaction between the servo-constraints and the dynamics of the system is ideal, i.e., the reaction forces are normal to the constraint manifold, with the corresponding virtual displacements satisfying the principle of virtual displacements. The procedure for solving the inverse dynamics problem by using the nonminimal nonholonomic form is summarized in the following steps:

1. The expressions for $\dot{u}$ obtained from Eqs. (150) and (473) are equated, resulting in Eqs. (484).

2. Eqs. (484) are solved for τ, resulting in expressions (486) or (487), depending on the degree of actuation of the dynamical system.

7.4 Example 7.1: Double pendulum

Reconsidering the mechanism shown in Fig. 24, but without the constraint (formed by the rigid obstacle). It is required to determine the necessary controls to bring the total energy of the mechanism to a prescribed value E_f.

Let the generalized coordinates be θ_1 and θ_2, and the generalized speeds be ${}^{\mathcal{N}}\omega^D$ and ${}^{\mathcal{N}}\omega^B$, Eqs. (472) for the mechanism are

$$\dot{\theta}_1 = {}^{\mathcal{N}}\omega^D \tag{488}$$

$$\dot{\theta}_2 = {}^{\mathcal{N}}\omega^B, \tag{489}$$

and the matrices Q, P, and G in Eqs. (473) are the same as (353), (354), and (355), respectively. The total energy of the system is given by

$$\begin{aligned} E &= K+V \\ &= \frac{1}{2}I_D\,{}^{\mathcal{N}}\omega^D\cdot{}^{\mathcal{N}}\omega^D+\frac{1}{2}I_B\,{}^{\mathcal{N}}\omega^B\cdot{}^{\mathcal{N}}\omega^B+\frac{1}{2}m_B\,{}^{\mathcal{N}}\mathbf{v}^a\cdot{}^{\mathcal{N}}\mathbf{v}^a+ \\ &\quad +m_Bg\left[R(1-\cos\theta_1)+\frac{L}{2}(1-\cos\theta_2)\right] \\ &= \frac{1}{2}I_D{u_1}^2+\frac{1}{2}I_B{u_2}^2+\frac{1}{2}m_B\left[R^2{u_1}^2+\frac{L^2}{4}{u_2}^2-\frac{L}{2}Ru_1u_2\cos(\theta_1-\theta_2)\right] \\ &\quad +m_Bg\left[R(1-\cos\theta_1)+\frac{L}{2}(1-\cos\theta_2)\right], \end{aligned}$$

where the datum for calculating the potential energy is the vertical position of the center of mass of the bar when $\theta_1=\theta_2=0$. The servo-constraint equation is

$$\psi = E - E_f = 0. \tag{490}$$

Taking the time derivative of ψ, X in Eq. (475) is found to be

$$X = m_B\frac{L}{4}R({u_1}^2u_2-u_1{u_2}^2)\sin(\theta_1-\theta_2)+m_Bg\left[R\sin\theta_1u_1+\frac{L}{2}\sin\theta_2u_2\right],$$

and the desired servo-constraint dynamics, Eq.(477), is

$$\dot{E}-\Theta(E-E_f)=0,\quad \Theta<0. \tag{491}$$

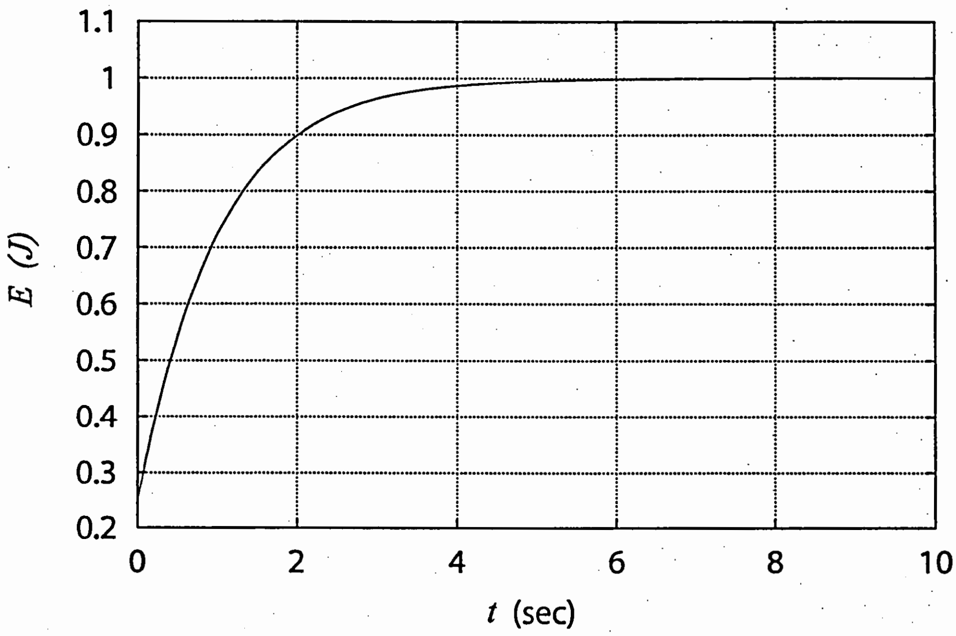

Figure 41: Example 7.1: Servo-constraint dynamics

The expressions (479) and (480) for S and z are now formed, and Eq. (481) is used to solve for τ, where S^+ for the row matrix S is given by [128]

$$S^+ = \frac{S^T}{||S||_2^2}, \tag{492}$$

where $||S||_2$ is the Euclidian norm of the row matrix S. The column matrix y can be chosen arbitrarily. For $\Theta = -1$, $E_f = 1$, the enforced servo-constraint dynamics is shown in Figure 41. The servo-constraints can be enforced by infinite number of ways, depending on the choice of y. Each choice results in different responses of the generalized coordinates and generalized speeds, but all choices yield to the same servo-constraint dynamics, as given by Eq. (491).

Nevertheless, an interesting choice of the control forces is the "ideal" one. The non-minimal nonholonomic form can be used to solve for this special type of control forces. Let $u_I = u_1$, and $u_D = u_2$. The servo-constraint dynamics can be put in the form

$$\dot{u}_D = A\dot{u}_I + B, \tag{493}$$

where the matrices A and B for this system are

$$A = \frac{-I_D u_1 - m_B R[R u_1 - \frac{L}{4} u_2 \cos(\theta_1 - \theta_2)]}{I_B u_2 + m_B \frac{L}{4}[L u_2 - R u_1 \cos(\theta_1 - \theta_2)]} \tag{494}$$

$$B = \frac{z}{I_B u_2 + m_B \frac{L}{4}[L u_2 - R u_1 \cos(\theta_1 - \theta_2)]}, \tag{495}$$

and

$$\begin{aligned} z = \quad & -\frac{L}{4} m_B R u_1 u_2 \sin(\theta_1 - \theta_2)(u_1 - u_2) - \frac{1}{2} I_D {u_1}^2 - \frac{1}{2} I_B {u_1}^2 - m_B g R \sin\theta_1 u_1 \\ & - m_B g \frac{L}{2} \sin\theta_2 u_2 - m_B g \left[R(1 - \cos\theta_1) + \frac{L}{2}(1 - \cos\theta_2) \right] \\ & - \frac{1}{2} m_B \left[R^2 {u_1}^2 + \frac{L^2}{4} {u_2}^2 - \frac{L}{2} R u_1 u_2 \cos(\theta_1 - \theta_2) \right]. \end{aligned} \tag{496}$$

The nonlinear nonminimal nonholonomic form, Eq. (149), can be constructed for the system, and the matrices T and V are used to form $\mathcal{R}$, where T is finite and invertible for all configurations and velocities, except at $u_1 = u_2 = 0$.

Next, the corresponding expression, Eq. (487), for the control torques matrix τ is formed. Figures 42 and 43 show the responses of the generalized coordinates θ_1 and θ_2 and the generalized speeds u_1 and u_2, respectively, and Figure 44 shows the corresponding control torques. It is noticed that, if the servo-constraint dynamics reaches its steady state, then the required control torques reach the zero values. This agrees with the fact that, if the sources of nonconservatism are removed, then the total energy of the system remains unchanged, and confirms that the nonminimal nonholonomic form (150) is the natural choice to enforce the servo-constraint dynamics.

Remark *Although the redundancy in the control system can be utilized in different manners, the servo-constraints must be kinematically and geometrically possible, for every possible configuration and velocity. For example, if the controls are required in addition to regulating the total energy of the double pendulum to cause it to track the prescribed trajectory*

$$R\cos\theta_1 + L\sin\theta_2 = 0, \tag{497}$$

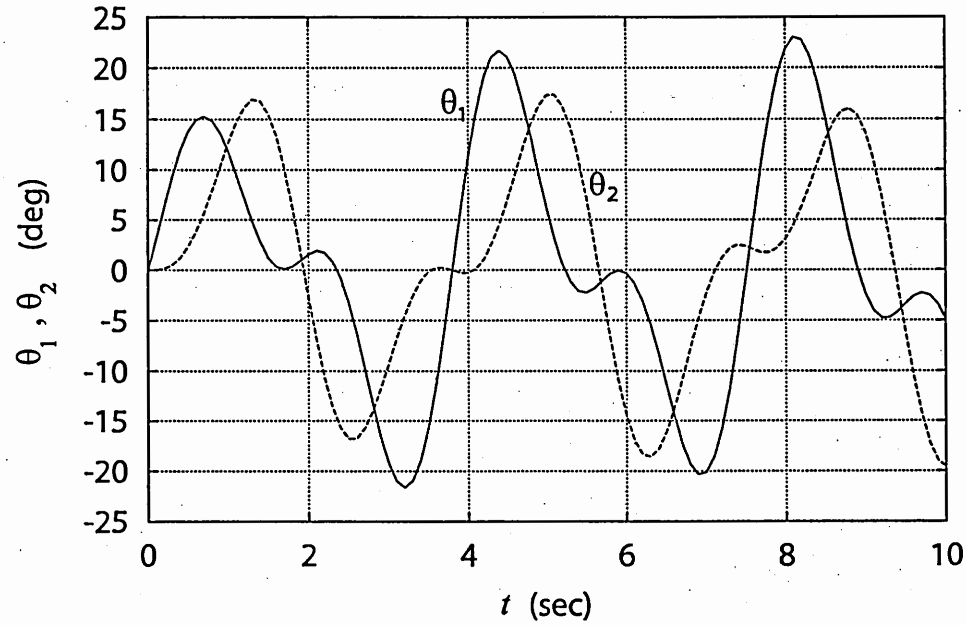

Figure 42: Example 7.1: Generalized coordinates; ideal controls case

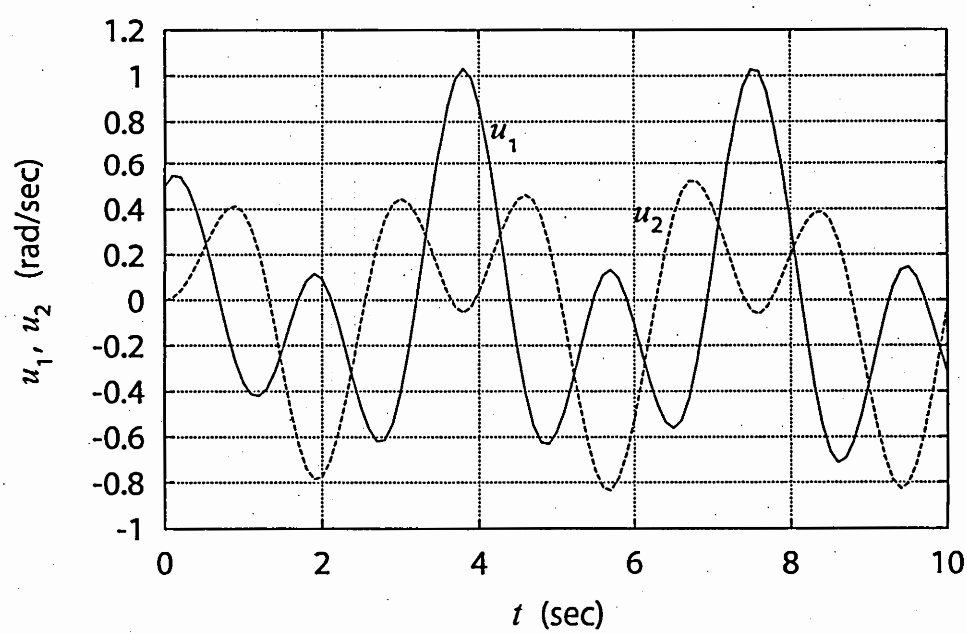

Figure 43: Example 7.1: Generalized speeds; ideal controls case

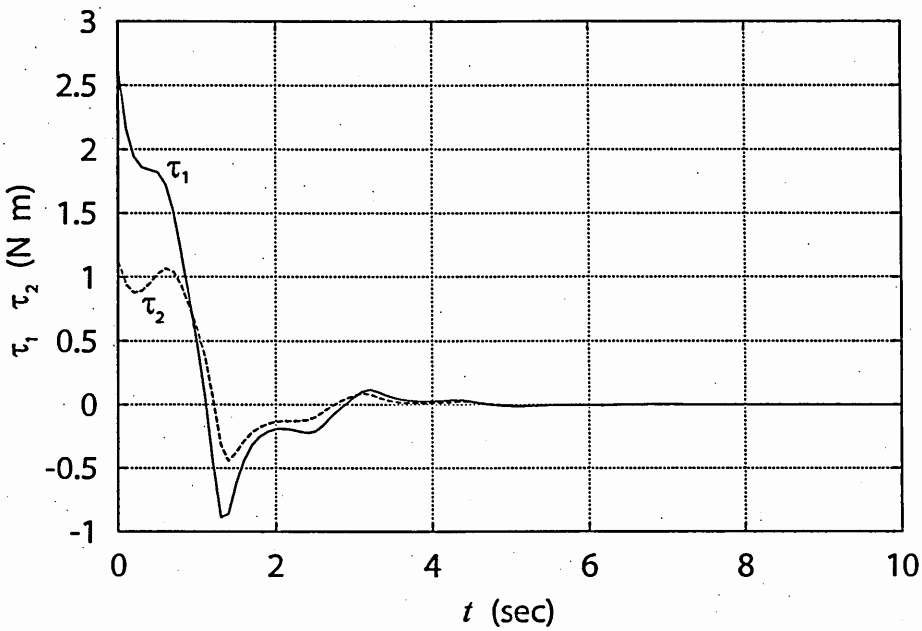

Figure 44: Example 7.1: Ideal control torques

then the requirements can be written as

$$\dot{\psi}_1(q,u,\dot{u}) - \Theta_1\psi_1(q,u) = 0 \tag{498}$$

$$\ddot{\psi}_2(q,u,\dot{u}) - \Theta_{21}\dot{\psi}_2(q,u) - \Theta_{22}\psi_2(q) = 0, \tag{499}$$

where

$$\psi_1 = E - E_f \tag{500}$$

$$\psi_2 = R\cos\theta_1 + L\sin\theta_2. \tag{501}$$

The resulting linear in accelerations equations form the matrix system

$$\begin{bmatrix} \frac{\partial\psi_1}{\partial u_1} & \frac{\partial\psi_1}{\partial u_2} \\ \frac{\partial\psi_2}{\partial u_1} & \frac{\partial\psi_2}{\partial u_2} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial\psi_1}{\partial q}u + \Theta_1\psi_1 \\ -u^T\frac{\partial^2\psi_2}{\partial q^2}u + \Theta_{21}\frac{\partial\psi_2}{\partial q}u + \Theta_{22}\psi_2 \end{Bmatrix}, \tag{502}$$

for some chosen values of Θ_1, Θ_{21}, *and* Θ_{22}. *It can be verified that the above matrix system has no solution. A sufficient condition for the satisfaction of servo-constraints to render the motion possible is that* $m < n$.

7.5 *Example 7.2: Trajectory Tracking*

Reconsidering the passively constrained dynamical system in Example 2.1 (section 2.7). The dynamics is governed by the equations

$$\begin{aligned}
\dot{u}_1 &= \frac{m}{a}(F_r + \phi F_\phi) + \\
&\quad + \frac{m^2}{ar}\left({u_2}^2 + {u_3}^2 - \phi\frac{{u_2}^2}{\tan\phi}\right) && (503)\\
\dot{u}_2 &= \frac{F_\theta}{m} - \frac{1}{r}\left(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}\right) && (504)\\
\dot{u}_3 &= \frac{m\phi}{a}(F_r + \phi F_\phi) \\
&\quad + \frac{m^2\phi}{ar}\left({u_2}^2 + {u_3}^2 - \phi\frac{{u_2}^2}{\tan\phi}\right) - \frac{u_1 u_3}{r}. && (505)
\end{aligned}$$

where $a := m(1+z^2)$. It is required to solve for the control forces F_r, F_θ, F_ϕ that drive the particle to the vertical plane defined by the servo-constraint equation

$$\theta - k = 0, \tag{506}$$

where k is a predetermined constant. Following the previous definition of generalized speeds

$$\begin{aligned}
u_1 &= \dot{r} && (507)\\
u_2 &= r\dot{\theta}\sin\phi && (508)\\
u_3 &= -r\dot{\phi}, && (509)
\end{aligned}$$

the time derivative of the above servo-constraint equation is

$$\frac{u_2}{r\sin\phi} = 0. \tag{510}$$

Differentiating one more time yields the acceleration form of constraints

$$\frac{r\sin\phi\dot{u}_2 - [u_1\sin\phi - u_3]u_2}{[r\sin\phi]^2} = 0. \tag{511}$$

The servo-constraint dynamics (483) is therefore,

$$\frac{r\sin\phi\dot{u}_2 - [u_1\sin\phi - u_3]u_2}{[r\sin\phi]^2} - \Theta_1\left(\frac{u_2}{r\sin\phi}\right) - \Theta_2(\theta - k) = 0, \tag{512}$$

where Θ_1 and Θ_2 are chosen such that the dynamics of

$$\psi := \theta - k \tag{513}$$

is stable. Substituting the expression (504) in Eq. (512) yields

$$\frac{r\sin\phi\left[\frac{F_\theta}{m} - \frac{1}{r}\left(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}\right)\right] - \left[u_1\sin\phi - u_3\right]u_2}{\left[r\sin\phi\right]^2} - \Theta_1\left(\frac{u_2}{r\sin\phi}\right) - \Theta_2\left(\theta - k\right) = 0. \tag{514}$$

Hence,

$$\tau = F_\theta \tag{515}$$

$$S = \frac{1}{mr\sin\theta} \tag{516}$$

$$z = \frac{r\sin\phi\left[-\frac{1}{r}\left(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}\right)\right] - \left[u_1\sin\phi - u_3\right]u_2}{\left[r\sin\phi\right]^2} \tag{517}$$

$$+\Theta_1\left(\frac{u_2}{r\sin\phi}\right) + \Theta_2\left(\theta - k\right). \tag{518}$$

Eq. (481) can now be constructed, where

$$S^+ = \frac{1}{S} = mr\sin\theta. \tag{519}$$

For $\Theta_1 = -2$, $\Theta_2 = -1$, and $k = 1$, the required control τ is

$$F_\theta = mr\sin\theta\left\{\frac{r\sin\phi\left[-\frac{1}{r}\left(u_1 u_2 - \frac{u_2 u_3}{\tan\phi}\right)\right] - \left[u_1\sin\phi - u_3\right]u_2}{\left[r\sin\phi\right]^2} - \frac{2u_2}{r\sin\phi} - (\theta - 1)\right\}. \tag{520}$$

Remark *It is noticed that the matrix S is a scalar. Therefore, S^+S is unity, and the last term on the right hand side of Eq. (481) vanishes, resulting in a unique control law. It is noticed also that the servo-constraint dynamics do not depend on $\dot{u}_1$ and $\dot{u}_3$, which causes this control law to be independent of F_r and F_ϕ.*

7.6 *Summary*

The acceleration form of constraint equations is utilized in this chapter to solve for the inverse dynamics of servo-constraints. A condition for the existence of controls that enforce servo-constraints is derived, together with a parametrization of the solution for these controls in terms of the generalized Moore-Penrose inverse.

In the case of redundant manipulators, the separation in accelerations of the nonminimal nonholonomic form provides a convenient way to obtain the ideal control forces, and minimizing acceleration energy of the dynamical system. The corresponding time marching of the generalized coordinates and the generalized speeds shows the way that the dynamical system will evolve in time if the constraints were passive ideal.

The present approach complements and generalizes the computed torque methods, in a unified treatment of holonomic and nonholonomic servo-constraints, where the constraints-free nature of the equations alleviates the need to measure, or to incorporate the effects of the constraint forces.

CHAPTER VIII

CONSTRAINTS INVOLVING CONTROLS

8.1 Introduction

The developments in the previous chapters deal with constraints that depend on the state of the dynamical system, i.e., the set of generalized coordinates and generalized speeds. The nonminimal equations of motion for the constrained system in these cases have the same order as the equations of motion of the dynamical system without constraints.

The constraint equations might involve control variables also. In this case, the acceleration form of constraint equations developed in the second and third chapters involves first derivatives of these control variables, and the nonminimal form contains both of the control variables and their derivatives. It is convenient in these cases to consider the control variables to be state variables in the nonminimal equations of motion, and introduce their derivatives as new control variables by adding new equations, equal in number to the number of control variables. This forms an augmented separated in accelerations state-space model, in which the states become the generalized coordinates, the generalized speeds, and the control variables.

If the dynamics of the system is fast, then an approximate model can be constructed that involves control variables in the kinematical differential equations. In this case, the second derivatives of these variables appear in the acceleration form of constraints, and the augmented state variables include the derivatives of the control variables also. The new control variables become the second derivatives of the true control variables.

The advantage of the above mentioned procedures becomes clear when it is preferable to use a single set of equations of motion to represent the dynamical system and its constraints. An example is the path-constrained optimal control problem, when the path constraints involve control variables. The single set of equations in this case waives the necessity to introduce another set of Lagrange's multipliers to augment the Lagrangean of the system with the path constraints.

For servo-constraints, the acceleration form of constraints can be utilized in constructing a separated in accelerations state-space model by using the Moore-Penrose generalized inverse of the matrix of coefficients of the control variables time derivatives, and parameterizing all control variables that enforce the constraints in terms of free parameters column matrix. In this chapter, this latter method for treating servo-constraints involving control variables is adopted.

In the next section, the methodology for deriving equations of motion for dynamical systems with controls-involved servo-constraint equations is presented.

8.2 Controls-Involved Constraints and Servo-constraints

The following equations of motion for a controlled dynamical system are considered

$$\dot{q} = C(q,t)u + D(q,t) \tag{521}$$

$$\dot{u} = f(q,u,\tau,t), \tag{522}$$

where $q, u \in \mathbb{R}^n$ denote the column matrices containing the configuration parameters and the velocity parameters, $\dot{q}$ and $\dot{u}$ are the derivatives of q and u with respect to t, respectively. The square matrix $C \in \mathbb{R}^{n\times n}$ is such that C^{-1} exists for all generalized coordinates and for all $t \in \mathbb{R}$, and $D \in \mathbb{R}^n$. The column matrix $\tau \in \mathbb{R}^l$ contains the control variables, and $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}^n$ is a vector field mapping its arguments to the generalized accelerations column matrix $\dot{u}$. The above sets of equations form a complete state-space model. It is assumed that the system is required to follow the servo-constraint equations

$$\phi(q,u,\tau,t) = 0, \tag{523}$$

where $\phi \in \mathbb{R}^m$. A solution for τ is obtained by differentiating equations (523) with respect to time to obtain

$$\frac{\partial \phi}{\partial q}\dot{q} + \frac{\partial \phi}{\partial u}\dot{u} + \frac{\partial \phi}{\partial \tau}\dot{\tau} + \frac{\partial \phi}{\partial t} = 0. \tag{524}$$

Substituting expression (521) for $\dot{q}$ into Eq. (524) gives

$$\mathcal{A}\dot{\tau} = \mathcal{B}, \tag{525}$$

where the matrices $\mathcal{A} \in \mathbb{R}^{m \times l}$ and $\mathcal{B} \in \mathbb{R}^m$ are

$$\mathcal{A} = \frac{\partial \phi}{\partial \tau} \tag{526}$$

$$\mathcal{B} = -\frac{\partial \phi}{\partial q}Cu - \frac{\partial \phi}{\partial q}D - \frac{\partial \phi}{\partial u}f(q, u, \tau, t) - \frac{\partial \phi}{\partial t}. \tag{527}$$

Solving for $\dot{\tau}$,

$$\dot{\tau} = \mathcal{A}^+\mathcal{B} + [I - \mathcal{A}^+\mathcal{A}]y, \tag{528}$$

where $\mathcal{A}^+ \in \mathbb{R}^{l \times m}$ is the Moore-Penrose generalized inverse of $\mathcal{A}$, and $y \in \mathbb{R}^l$ is arbitrary at a specific point, provided that the point is in the controllability subspace of the dynamical system, i.e., the matrix $\mathcal{B}$ is in the range space of the matrix $\mathcal{A}$ at that point. The example in the following section illustrates the method.

8.3 *Example 8.1: Spacecraft stabilization*

The following Euler equations form a mathematical model for a spacecraft.

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1 \tag{529}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2 \tag{530}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3, \tag{531}$$

where ω_1, ω_2, and ω_3 are the angular velocities about the principal axes of the spacecraft. The control variables for the system are the applied torques τ_1, τ_2, τ_3 about the corresponding axes. The principal moments of inertia of the spacecraft are I_1, I_2, and I_3. The servo-constraint equation used to stabilize the spacecraft is the Lyapanov equation

$$\dot{K} + aK = 0, \quad a > 0, \tag{532}$$

where K is the kinetic energy of the spacecraft

$$K = \frac{1}{2}\left[I_1{\omega_1}^2 + I_2{\omega_2}^2 + I_3{\omega_3}^2\right]. \tag{533}$$

Therefore, Eq. (532) is

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 + \frac{a}{2}\left[I_1{\omega_1}^2 + I_2{\omega_2}^2 + I_3{\omega_3}^2\right] = 0. \tag{534}$$

Substituting expressions (529)-(531) in the above equation, one obtains

$$\begin{aligned} &I_1\omega_1\tau_1 + I_2\omega_2\tau_2 + I_3\omega_3\tau_3 = \\ &\qquad -\frac{a}{2}\left[I_1{\omega_1}^2 + I_2{\omega_2}^2 + I_3{\omega_3}^2\right] - \omega_1\omega_2\omega_3\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \end{aligned} \tag{535}$$

Differentiating the above equation gives

$$\begin{aligned} &I_1\omega_1\dot{\tau}_1 + I_2\omega_2\dot{\tau}_2 + I_3\omega_3\dot{\tau}_3 = \\ &\qquad - I_1\dot{\omega}_1\tau_1 - I_2\dot{\omega}_2\tau_2 - I_3\dot{\omega}_3\tau_3 - aI_1\omega_1\dot{\omega}_1 - aI_2\omega_2\dot{\omega}_2 - aI_3\omega_3\dot{\omega}_3 \\ &\qquad - \left[\dot{\omega}_1\omega_2\omega_3 + \omega_1\dot{\omega}_2\omega_3 + \omega_1\omega_2\dot{\omega}_3\right]\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \end{aligned} \tag{536}$$

Substituting the expressions (529)-(531) for angular accelerations in the above equation gives

$$\begin{aligned} &I_1\omega_1\dot{\tau}_1 + I_2\omega_2\dot{\tau}_2 + I_3\omega_3\dot{\tau}_3 = \\ &- I_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\tau_1 - I_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\tau_2 - I_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\tau_3 \\ &- aI_1\omega_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right] - aI_2\omega_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right] - aI_3\omega_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right] \\ &- \left[\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\omega_2\omega_3 + \omega_1\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\omega_3 + \omega_1\omega_2\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \end{aligned} \tag{537}$$

Hence, the matrices $\mathcal{A}$ and $\mathcal{B}$ for the system are

$$\mathcal{A} = \lfloor I_1\omega_1 \quad I_2\omega_2 \quad I_3\omega_3 \rfloor \tag{538}$$

$$\begin{aligned}\mathcal{B} = &-I_1\left[\frac{I_2-I_3}{I_1}\omega_2\omega_3+\tau_1\right]\tau_1 - I_2\left[\frac{I_3-I_1}{I_2}\omega_3\omega_1+\tau_2\right]\tau_2 - I_3\left[\frac{I_1-I_2}{I_3}\omega_1\omega_2+\tau_3\right]\tau_3\\ &- aI_1\omega_1\left[\frac{I_2-I_3}{I_1}\omega_2\omega_3+\tau_1\right] - aI_2\omega_2\left[\frac{I_3-I_1}{I_2}\omega_3\omega_1+\tau_2\right] - aI_3\omega_3\left[\frac{I_1-I_2}{I_3}\omega_1\omega_2+\tau_3\right]\\ &- \left[\left[\frac{I_2-I_3}{I_1}\omega_2\omega_3+\tau_1\right]\omega_2\omega_3 + \omega_1\left[\frac{I_3-I_1}{I_2}\omega_3\omega_1+\tau_2\right]\omega_3 + \omega_1\omega_2\left[\frac{I_1-I_2}{I_3}\omega_1\omega_2+\tau_3\right]\right]\\ &\left[\frac{I_2-I_3}{I_1}+\frac{I_3-I_1}{I_2}+\frac{I_1-I_2}{I_3}\right].\end{aligned} \tag{539}$$

The Moore-Penrose generalized inverse of $\mathcal{A}$ is

$$\mathcal{A}^+ = \frac{1}{(I_1\omega_1)^2+(I_2\omega_2)^2+(I_3\omega_3)^2}\begin{Bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{Bmatrix}. \tag{540}$$

The expression (528) for $\dot{\tau}$ is now formed. For some specific choice of y, integrating these equations together with Euler's equations in time gives the trajectories of angular velocities of the dynamical system and the required control torques. All choices of y result in satisfying the servo-constraint equation (532).

The initial conditions of the control variables should satisfy the servo-constraint equation, and the constant a can be any positive number. Increasing the value of a increases the damping rate of K. The simulations are performed for I_1 = 10 Kg.m^2, I_2 = 6.3 Kg.m^2, I_3 = 8.5 Kg.m^2, and a = 1, and the initial conditions on angular velocities $\omega_1(0) = \omega_2(0) = \omega_3(0) = 0.1$ rad./sec. The initial conditions on the control variables that satisfy Eq. (532) are chosen to be $\tau_1(0) = \tau_2(0) = 0.1$ N.m, $\tau_3(0) = -0.3376$ N.m. The first order dynamic of K is shown Fig. (45).

Although the servo-constraint dynamics is satisfied irrespective of the choice of y, some choices may result in unsatisfactory performance of the controlled system. For example, choosing $y_1 = y_2 = y_3 = 0$ results in the angular velocities shown in Figs. (46)-(48),

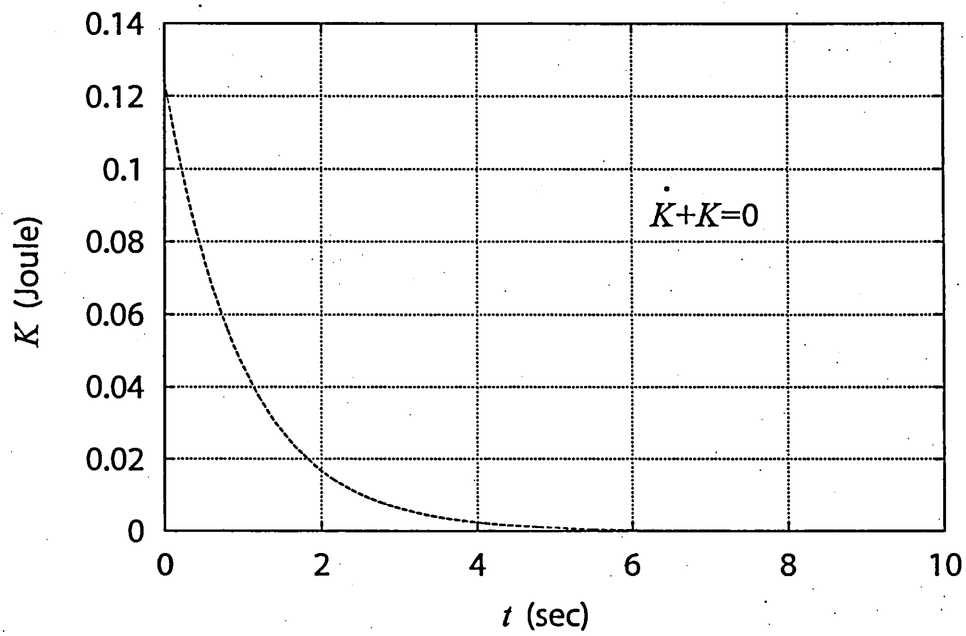

Figure 45: Servo-constraint (desired kinetic energy decay profile)

and the required control variables shown in Figs. (49)-(51). Clearly, the chattering of the control variables and the angular velocities are undesirable.

A better choice of y is $y_i = -\tau_i,\ i = 1 \ldots 3$ shown in Figs. (52)-(54). This choice is made based on the structure of the controls dynamics given by Eqs. (528). These equations can be written as

$$\dot{\tau} = y + \mathcal{A}^{+}\left[\mathcal{B} - \mathcal{A}y\right]. \tag{541}$$

Hence, the first term on the right hand side will have a stabilizing effect if $y_i = -\tau_i,\ i = 1 \ldots 3$ are chosen. This stabilizing effect dominates the dynamics of the system, as noticed from the corresponding smooth behaviors of the control variables and the angular velocities shown in Figs. (55)-(60).

8.4 Summary

An extension to the procedure of deriving nonminimal equations for constrained motion is introduced in this chapter for passive constraint equations containing control variables,

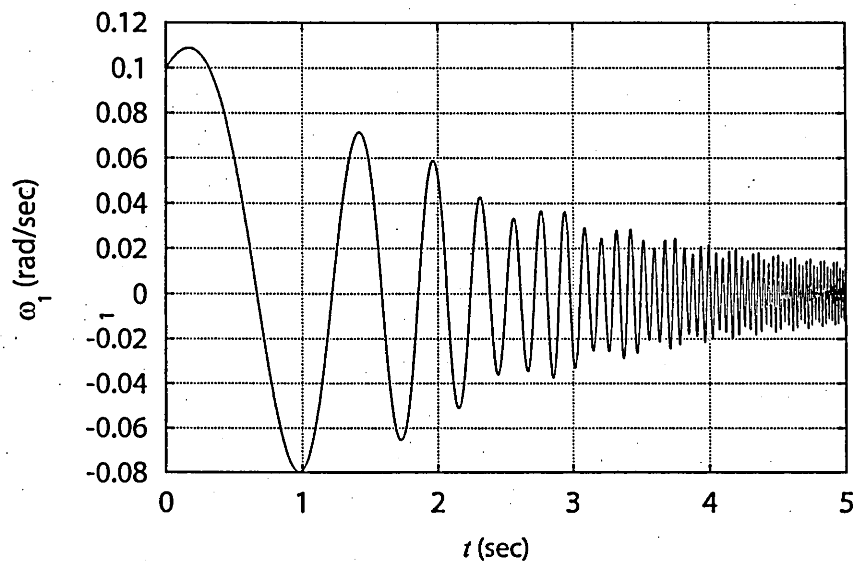

Figure 46: Angular velocity component about spacecraft body axis x; $y_1 = y_2 = y_3 = 0$

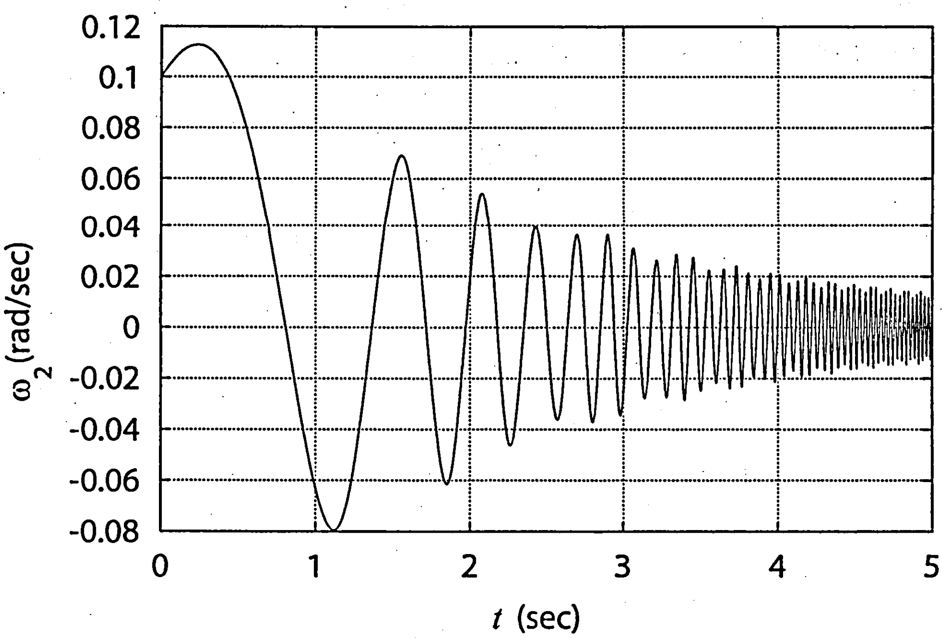

Figure 47: Angular velocity component about spacecraft body axis y; $y_1 = y_2 = y_3 = 0$

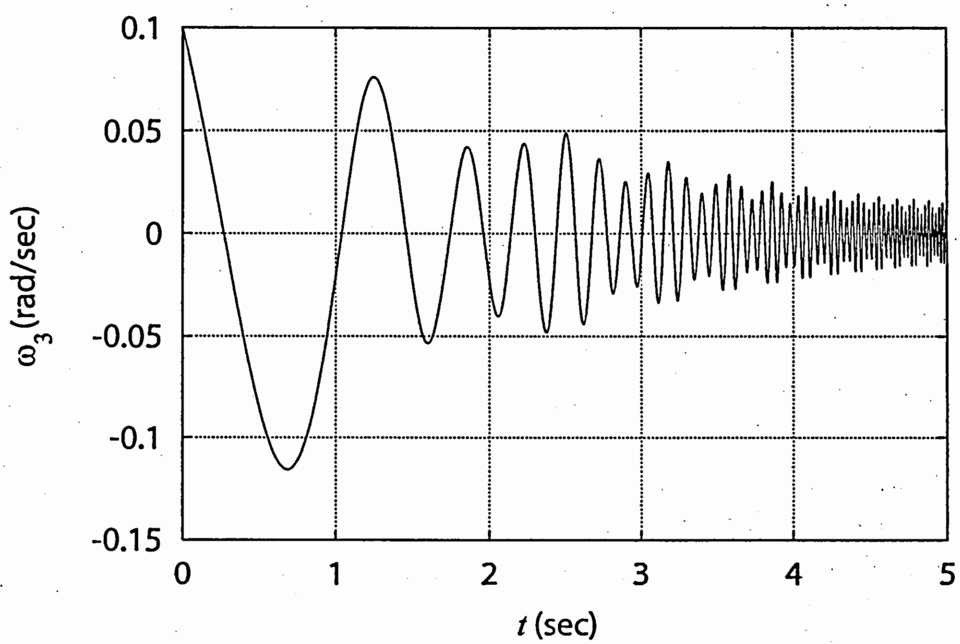

Figure 48: Angular velocity component about spacecraft body axis z; $y_1 = y_2 = y_3 = 0$

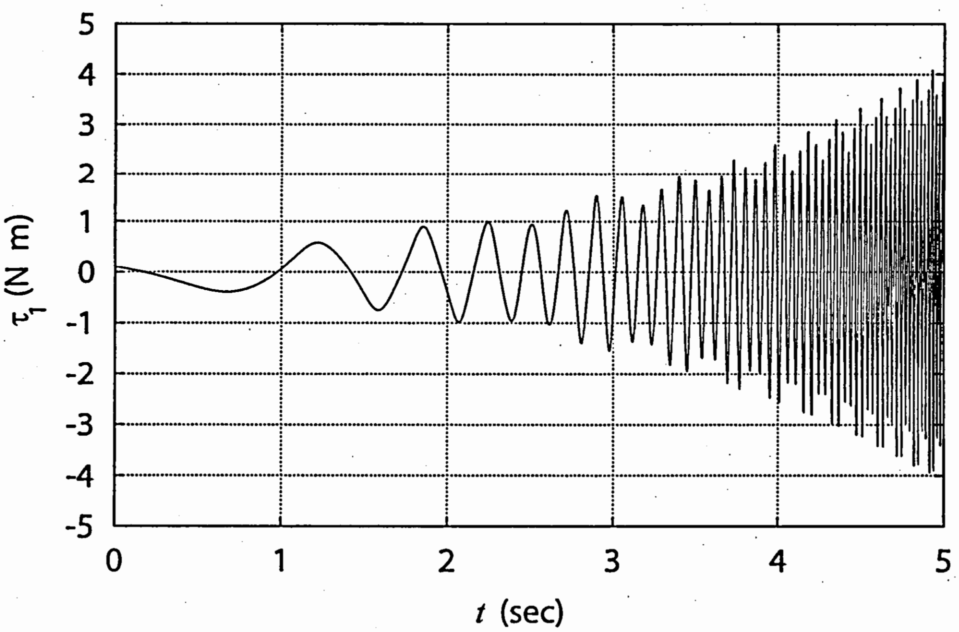

Figure 49: Torque about spacecraft body axis x; $y_1 = y_2 = y_3 = 0$

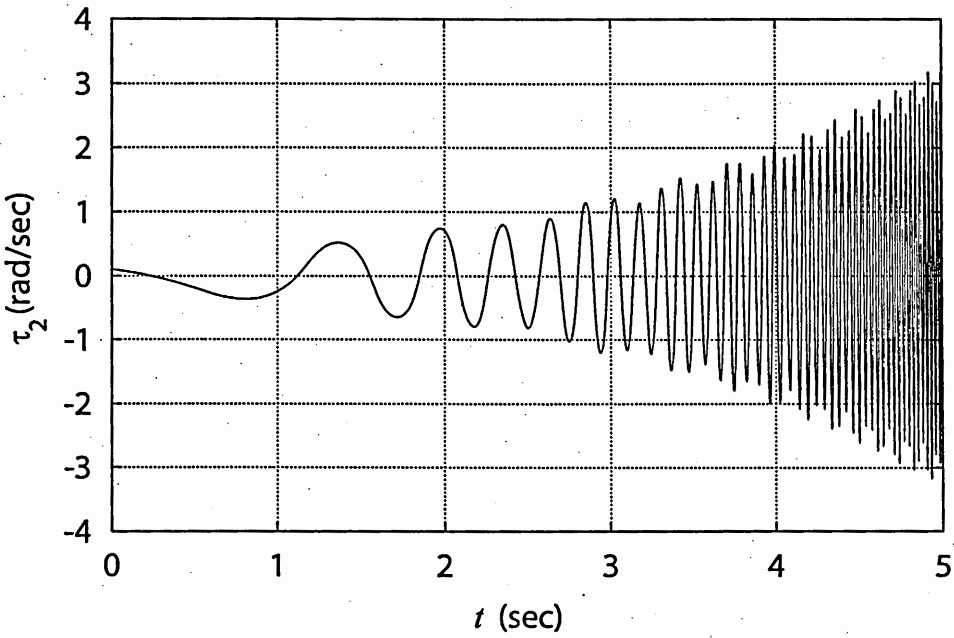

Figure 50: Torque about spacecraft body axis y; $y_1 = y_2 = y_3 = 0$

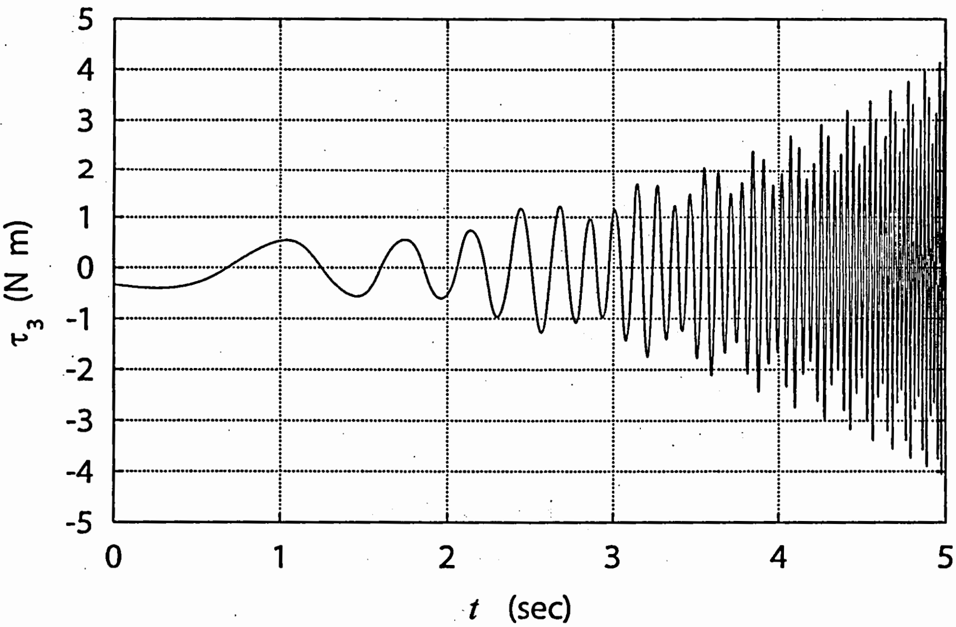

Figure 51: Torque about spacecraft body axis z; $y_1 = y_2 = y_3 = 0$

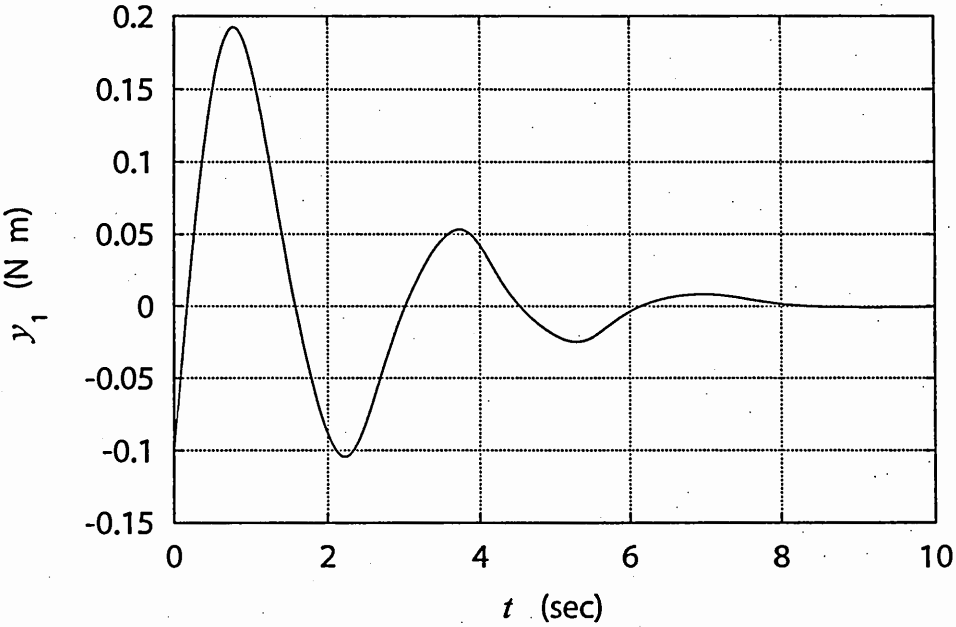

Figure 52: Parameter $y_1 = -\tau_1$

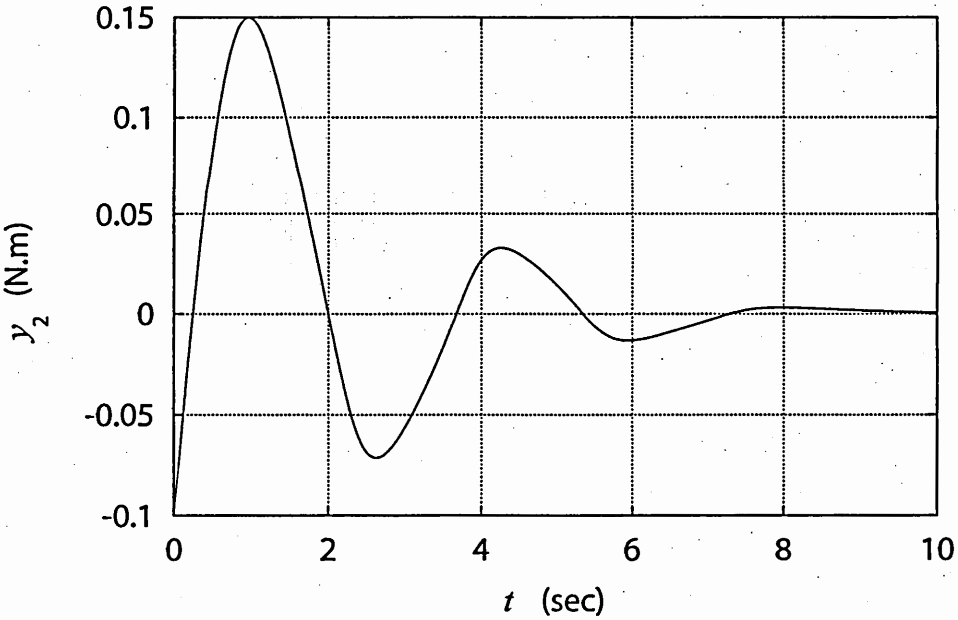

Figure 53: Parameter $y_2 = -\tau_2$

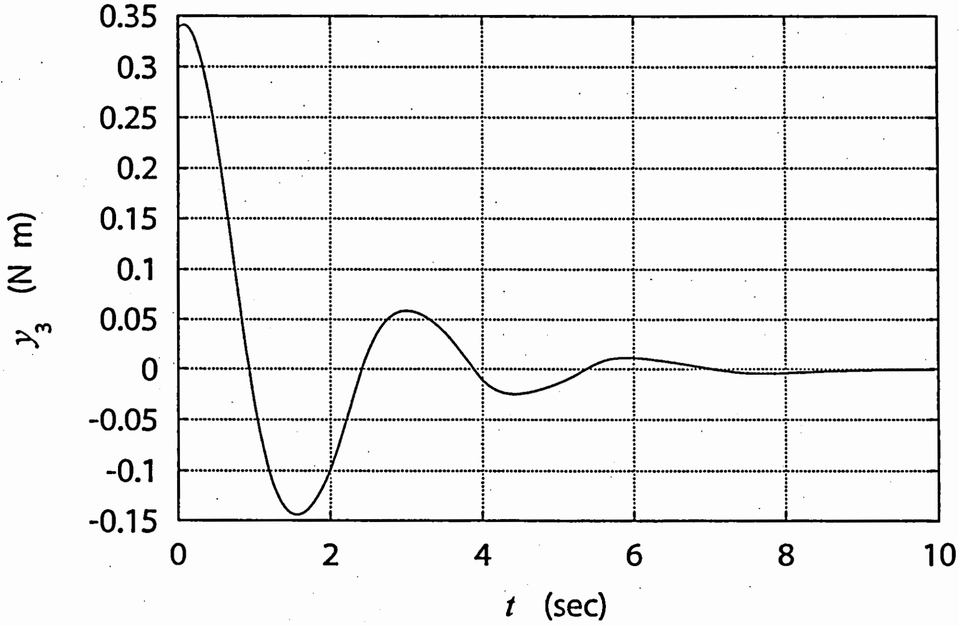

Figure 54: Parameter $y_3 = -\tau_3$

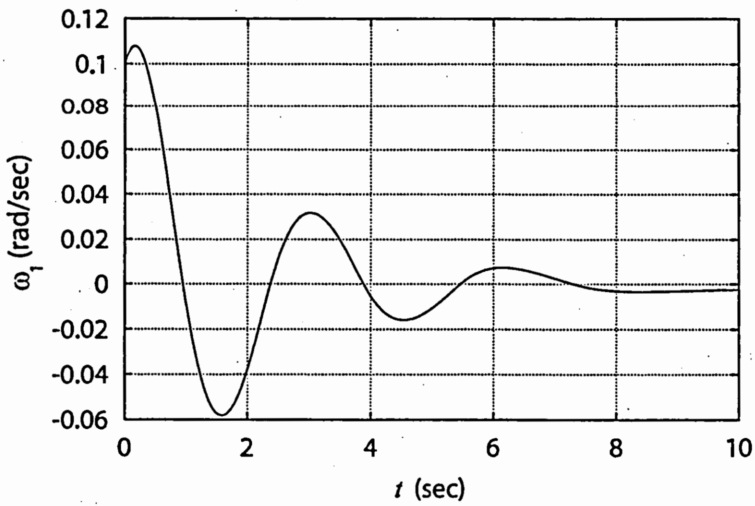

Figure 55: Angular velocity component about spacecraft body axis x; $y = -\tau$

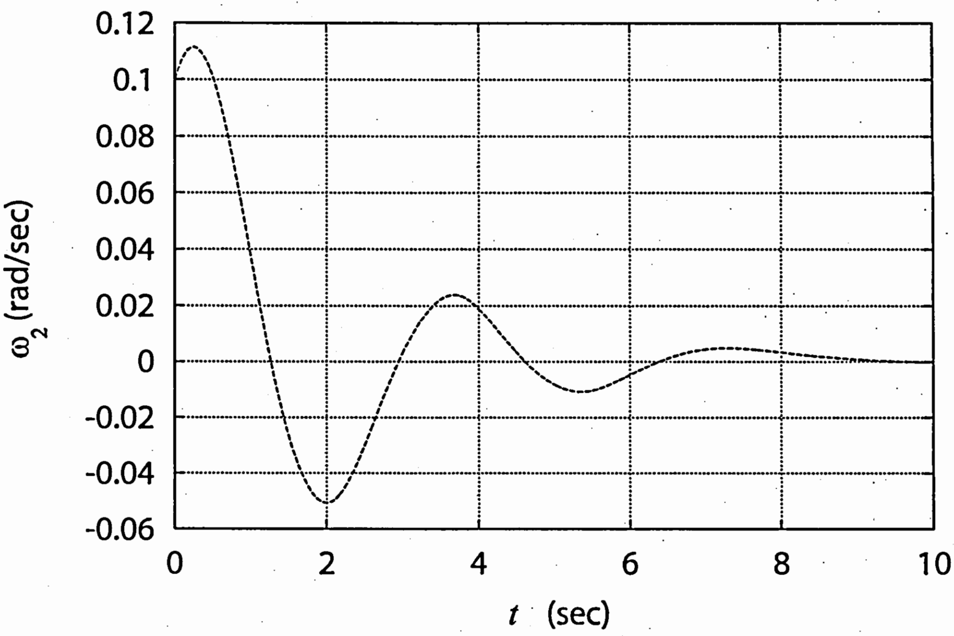

Figure 56: Angular velocity component about spacecraft body axis y; $y = -\tau$

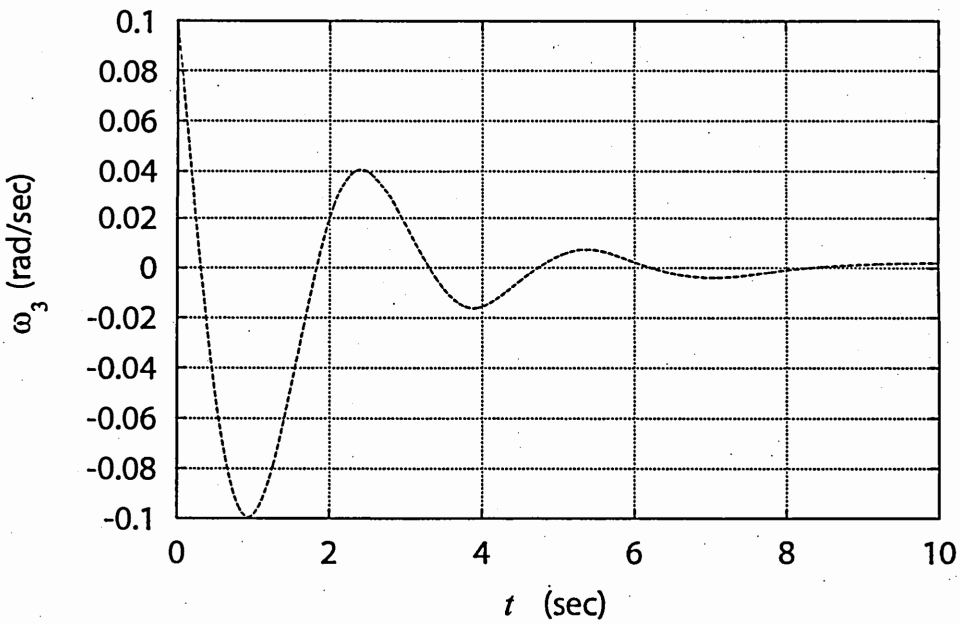

Figure 57: Angular velocity component about spacecraft body axis z; $y = -\tau$

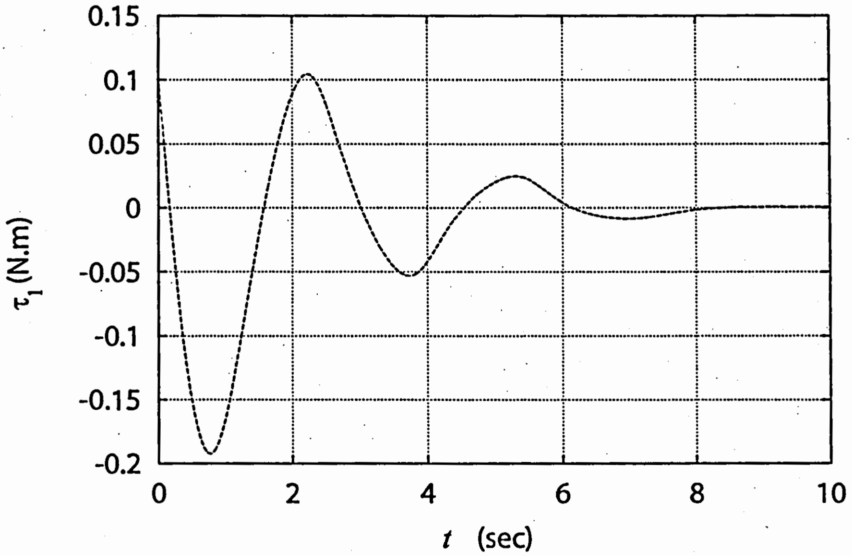

Figure 58: Torque about spacecraft body axis x; $y = -\tau$

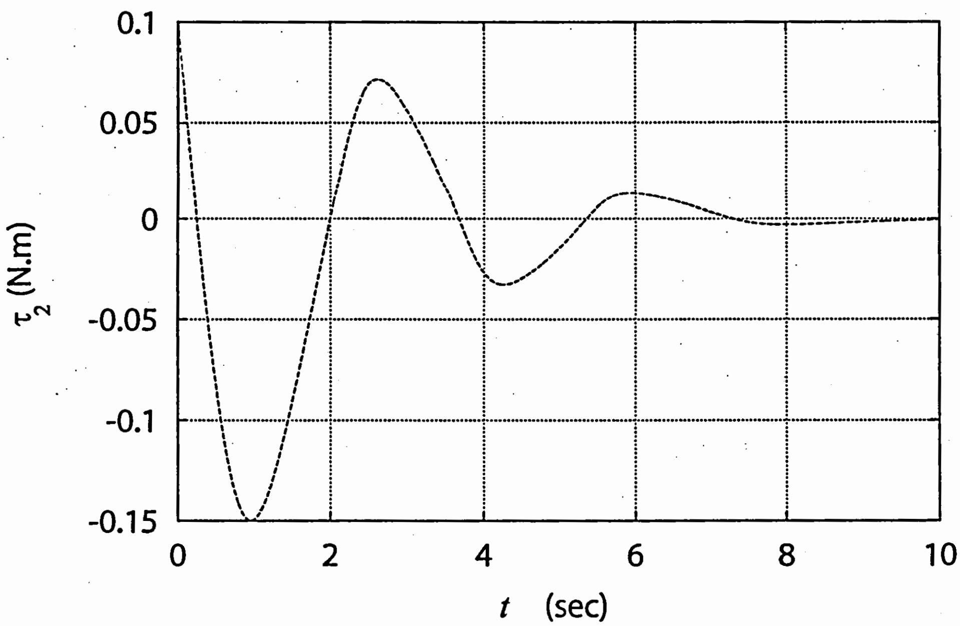

Figure 59: Torque about spacecraft body axis y; $y = -\tau$

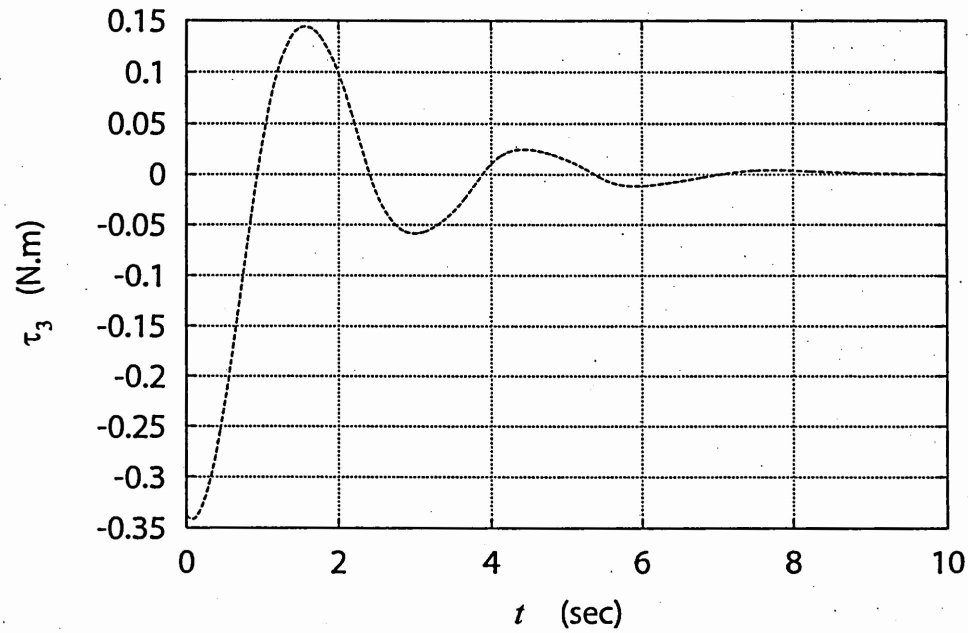

Figure 60: Torque about spacecraft body axis z; $y = -\tau$

resulting in elimination of algebraic path constraints.

Another procedure is introduced for converting algebraic servo-constraint equations involving control variables into dynamic constraint equations that complement the state-space model of the dynamical system, provided that the variables are in the controllable subspace of the dynamical system.

The introduction of the parameters y as fictitious control variables is beneficial in affining the control problem, i.e., making the state-space model linear in the control variables, and hence allowing using the wealth of related methodologies for control systems analysis and design.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

9.1 Conclusions

This work is concerned with formulating mathematical models for constrained dynamical systems, by focusing on increasing the simplicity and the applicability of these models. Kane's approach is adopted for that reason, making use of its flexibility and algorithmic nature.

Dividing the work into chapters is done mainly on the basis of the type of constraints considered, and the common feature among all parts is using a differentiated form of the constraint equations. The two main classes of constraints considered are passive constraints and servo-constraints. The five chapters after the introduction chapter are concerned with passive constraints, the seventh chapter is concerned with servo-constraints, and the eighth is concerned with constraints of both types, involving control variables.

Simple nonholonomic and nonlinear nonholonomic constraints are treated in the second and third chapters. A method for identifying the corresponding constraint forces is introduced in the fourth chapter. The resulting equations of motion are explicit in the generalized accelerations, and involve no Lagrange multipliers. The derivation is based on simple mathematical operations on the unconstrained equations of motion. This is particularly advantageous in the case where the equations are already derived and more constraints are to be added to the system for the purpose of improving its design or studying its performance, because the method does not require a totally new derivation.

Obtaining a constrained model that has the same order as the unconstrained one is advantageous. It facilitates, for instance, solving Euler-Lagrange equations or forming

Hamilton-Jacobi-Bellman equations for closed loop path-constrained optimal control systems, by using the standard procedures for single set unconstrained dynamical equations of motion, without the need to use more Lagrange's multipliers to augment the Lagrangean with the path constraints. This also waives the need to take higher derivatives of the constraint equations to enforce the appearance of control variables, because their appearance becomes unnecessary to begin with. The material of these chapters form the backbone of the work, and the basis for all later developments.

Unilateral constraints are considered in the fifth chapter. The two types of unilateral constraints considered are the impulsive and the friction constraints. A reconsideration of impulsive constraints in the context of the impulse-momentum approach is the subject of the sixth chapter.

Unilateral constraints can be described by multiple models that are activated in different phases of motion. Nevertheless, the two primary issues in modeling unilateral constraints are the treatment of singularities that impede the switching between a model and another, and the velocity behavior at the switching points. These issues are solved in the fifth chapter, and an approximation of the velocity as discontinuous as implied by the impulse-momentum approach is followed in the sixth chapter, both in the context of the nonminimal equations of motion.

The seventh chapter is concerned with the inverse dynamics of servo-constraints. That is, to obtain, if feasible, the control forces that enforce servo-constraints, and particularly the ideal control forces. This is achieved by matching the controlled system equations of motion with the constrained nonminimal equations of motion, and solving for the control variables.

In the eighth chapter, procedures are introduced for including constraint equations that involve control variables in the nonminimal equations of motion, with the aid of the acceleration form of constraints. For passive constraints, this results in a single set of dynamical equations with no accompanying algebraic constraint equations, which is advantageous in several applications. For servo-constraints, the resulting extended models facilitate synthesizing control laws to stabilize dynamical systems.

9.2 *Recommendations*

The nonminimal nonholonomic form is a single set of constrained equations of motion that is separated in the acceleration variables.

Any system analysis or control method related to this type of modeling will benefit from this form. For example, stability of the constrained motion can be determined directly by applying Lyapunov's second method to this form. The possibility of the existence of limit cycles can be tested by applying the Poincare-Bendixon theorems this form. The chaos behavior of dynamical systems can be studied as well by using this form. For time simulations, this form is recommended because it is readily integrable by using many available numerical schemes.

In optimal control, this form together with the cost index and the boundary conditions are what are needed to write Euler-Lagrange equations and Hamilton-Jacobi-Bellman equations. Using algebraic constraint equations is more expensive for deriving and solving Euler-Lagrange equations and might make it impossible to write Hamilton-Jacobi-Bellman equations.

Linearization of the constrained equations of motion does not preserve exact satisfaction of constraints, unless both of the unconstrained equations and the constraint equations are linear to begin with. This fact makes it impossible to employ linearized models to track nonlinear constraints. In model reference adaptive control, the reference model is usually linear. The nonminimal nonholonomic form provides a constrained nonlinear model that may be used as a model reference for nonlinear tracking.

Expanding the work of the sixth chapter to include nonlinear nonholonomic impulsive constraints may also be investigated.

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