Emergent Pfaffian Relations in Quasi-Planar Models

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joint works with:
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Talk outline

1. Introduction and preliminary remarks
   a. A general feature of planar Ising spin systems (at zero mag. field)
   b. Pfaffians in Physics

2. The Ising model
   a. The spin perspective
   b. The model's Random Current representation
   c. The stochastic geometry of correlations

3. ~~Applications for results on the critical behavior beyond the solvable 2D case~~.

4. a. A key stochastic geometric relation
   b. An interesting contrast (between d>4 and d=2)

5. Emergent planarity - statement of the main result

6. 'Order-disorder' operators:
   a. their definition
   b. interpretation in Random-Current terms
   c. expression of correlations in terms of Kac-Ward amplitudes

7. Related observations, and questions

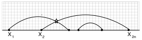
I. A general feature of planar Ising spin systems (at zero mag. field)

A planar Ising model: $\mathcal{G} = (V(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ a finite planar graph,

$$\boxed{Z_{\beta,h=0} \;=\; \sum_{\sigma \in \{-1,+1\}^{\mathcal{G}}} e^{\beta \sum J_{x,y}\sigma_x\sigma_y}}$$

and $\langle -- \rangle$ the corresponding equilibrium state average.

Theorem $^{(*)}$ *For any such system of Ising spins on a planar graph with a connected boundary segment* Γ, *and any collection of boundary sites* $\{x_1, ..., x_{2n}\} \subset \Gamma$



$$\boxed{\langle \prod_{j=1}^{2n} \sigma_{x_j} \rangle \;=\; \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \;\equiv\; Pf\left(S_2(x_j, x_k)\right)}$$

where $\varepsilon(\pi) = \pm 1$ *is the pairing's parity, relative to the boundary's cyclic order.*

* *Realized in increasing generality –* *for graphs with a regular transfer matrix: Schultz-Mattis-Lieb '64,* *in the above form: Groeneveld-Boel-Kasteleyn '78.*

II. The above relation is limited to planar models. Our main goal here is to explain the emergence of such relations in the scaling limits of $2D$ models with non-planar interactions, at their critical points (an example of "universality" in critical behavior).

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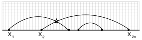

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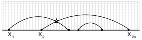

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Pfaffians showed earlier in statistical mechanics in the partition functions in certain exactly solvable models, on planar graphs (Kasteleyn, Fisher, Temperley '61-'63) :

- Dimer cover:

$$\boxed{\#\text{Dimer Covers } (\mathcal{G}) \;=\; \mathrm{Pf}\,(A) \;=\; \det\,(A)^{1/2}}$$

A – the Kasteleyn matrix

- Planar Ising model:

$$\boxed{Z_{\beta,h=0} \;=\; \sum_{\sigma\in\{-1,+1\}^{\mathcal{G}}} e^{\beta\sum J_{x,y}\sigma_x\sigma_y} \;=\; \det\,(\mathbb{1}-KW)^{1/2}_{\mathcal{E}_0\times\mathcal{E}_0}}$$

K, W – the Kac-Ward matrices

The dimensions of the matrices (or triangular arrays) appearing in the partition functions is of the order of the graph. In contrast, in the Pfaffian relations

$$\boxed{S_{2n}(x_1,\dots,x_{2n}) \;:=\; \langle\prod_{j=1}^{2n}\sigma_{x_j}\rangle \;=\; \mathrm{Pf}\,\big(S_2(x_j,x_k)\big)_{2n\times 2n}}$$

the matrix dimensions correspond to the number of particles involved in the given correlation function.

The Pfaffian structure of correlations is a characteristic of non-interacting fermions (for which it holds in any dimension). As such, it is indicative of the model's integrability.

- The relation

$$\boxed{\langle \prod_{j=1}^{2n} \sigma_{x_j} \rangle = \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \equiv \mathrm{Pf}\left(S_2(x_j, x_k)\right)}$$

is valid for boundary spin correlation functions on amorphous planar graphs, and arbitrary sets of pair couplings (not limited to ferromagnetic).

However, for rather simple reasons this relation does not hold for the spin correlation functions in the bulk, nor for non-planar models (Boel-Kast. '78)

- The relation does however extend to correlation functions of the order-disorder operators (which will be presented below).

- Both Pfaffian relations (of boundary spin correlations, and more general of order-disorder variables) have counterparts in monomer correlation functions of planar dimer cover models. (Priezzhev- Ruelle 08, Giuliani-Jauslin-Lieb '15, A-LainzValcazar-Warzel '16)

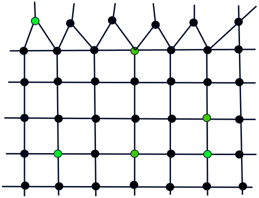
- Our proof & explanation of the relation (ADTW'16) utilizes the random current representation.

2 a. The Ising model – the spin perspective

Ising spins on a general graph $\mathcal{G}$:

$$\sigma : \mathcal{G} \to \{-1, +1\}$$

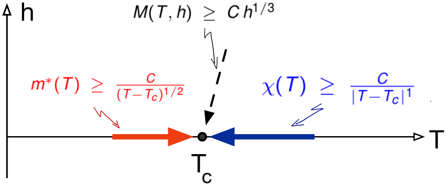
$$H(\sigma) = - \sum_{(x,y)\in\mathcal{E}} J_{x,y}\sigma_x\sigma_y - h\sum_{x\in\mathcal{G}}\sigma_x$$



Gibb's equil. measure

$$\boxed{\Pr_\Lambda(\sigma) = e^{-\beta H_\Lambda(\sigma)}/Z_\Lambda}$$

$$Z_\Lambda \;=\; \sum_{\sigma\in\{-1,1\}} e^{-\beta H_\Lambda(\sigma)}$$



The spontaneous magnetization:

$$m^*(T) \equiv M(T, h = 0+) := \langle\sigma_x\rangle_{T,h=0+}$$

is $\begin{cases} 0 & T > T_c \\ > 0 & T < T_c \end{cases}$

Phase diagram for $\boxed{\langle\cdot\rangle \;=\; \lim_{\Lambda\nearrow\mathcal{G}} \mathbb{E}_\Lambda[\cdot]}$

[On transitive graphs the corresponding critical exponents are bounded by their mean field values (ABF'87):
$\gamma \geq 1, \quad \beta \leq 1/2, \quad \delta \geq 3$.]

The (ferr.) Ising spin system on a graph $\mathcal{G}$ of edge set $\mathcal{E}$ (finite subsets $\Lambda \subset \mathcal{G}$) is:

$$\sigma : \mathcal{G} \mapsto \{-1, 1\}, \quad \mathrm{Pr}_\Lambda(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{Z_\Lambda}$$

with $H(\sigma) = -\sum_{(x,y)\in\mathcal{E}} J_{x,y}\,\sigma_x\sigma_y \;-\; h\sum_{x\in\mathcal{G}}\sigma_x$; $\quad J_{x,y} \geq 0$ (ferromag. interaction)

The Random Current representation (starting from the *high temp. exp., as GHS did*)

$\mathbf{n} : \mathcal{E} \mapsto \{0, 1, 2, \ldots\}$ $\quad \boxed{\partial\mathbf{n} := \{x \in \mathcal{G} \;:\; (-1)^{\sum_y n_{x,y}} = -1\}}$ - *the set of sources*

weights: $\boxed{w(n) := \prod_{b\in\mathcal{E}} (\beta J_b)^{n_b} / n_b!}$ *with "b" an alternative symbol for* $(x, y) \in \mathcal{E}$

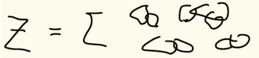
Basics relations (for $h = 0$):

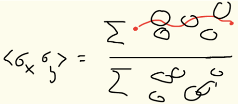
$$\boxed{Z \;:=\; \sum_\sigma e^{-\beta H(\sigma)} \;=\; \sum_{\mathbf{n}:\,\partial\mathbf{n}=\emptyset} w(\mathbf{n})}$$

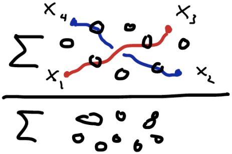
and for any $A \subset \mathcal{G}$:

$$\boxed{\langle \prod_{x\in A} \sigma_x \rangle \;=\; \sum_{\mathbf{n}:\,\partial\mathbf{n}=A} w(\mathbf{n})/Z}$$

pictorially:





$\langle \sigma_{x_1} \ldots \sigma_{x_4} \rangle =$ 

This yields a suggestive explanation of the phenomenon of upper critical dimension: in high dimensions (as it turns out $d > 4$), at large separations:

$$\langle \sigma_{x_1} \ldots \sigma_{x_4} \rangle \approx \left[\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle \right] [1 + o(1)]$$

Theorem (A 81) *For the n.n. Ising models on $\mathbb{Z}^d$ in $d > 4$, if for some $\kappa(\delta) \to \infty$ the scaled correlation functions converge (pointwise for $x_1, \ldots, x_{2n} \in \mathbb{R}^d$)*

$$S_{2n}(x_1, \ldots, x_{2n}) = \lim_{\delta \to 0} \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$$

then the limiting functions satisfy $\boxed{S_{2n}(x_1, \ldots, x_{2n}) = \sum_{\text{pairings}\,\pi} \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$

Under the above conditions also (A-Barsky-Fernandez '87) :

$$\boxed{\gamma = 1, \qquad \beta = 1/2, \qquad \delta = 3 .}$$

4. A key stochastic geometric relation

Defining $u_4(x_1, ..., x_4)$ so that:

$$\langle \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}\rangle = \left[\langle \sigma_{x_1}\sigma_{x_2}\rangle \langle \sigma_{x_3}\sigma_{x_4}\rangle + \langle \sigma_{x_1}\sigma_{x_3}\rangle \langle \sigma_{x_2}\sigma_{x_4}\rangle + \langle \sigma_{x_1}\sigma_{x_4}\rangle \langle \sigma_{x_2}\sigma_{x_3}\rangle\right] + u_4(x_1, ..., x_4)$$

we have:

Lemma: *For any Ising model on a finite graph:*

$$\boxed{u_4(x_1, ..., x_4) \ = \ -2\,\langle \sigma_{x_1}\sigma_{x_2}\rangle \langle \sigma_{x_3}\sigma_{x_4}\rangle \ Prob\left(C_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \ni x_2, x_3, x_4 \mid {}^{\partial \mathbf{n}_1=\{x_1,x_2\}}_{\partial \mathbf{n}_2=\{x_3,x_4\}}\right)}$$

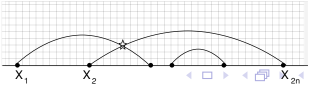
Note:

i) In situations where $|u_4(x_1, ..., x_4)|/\langle \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}\rangle \to 0$ one gets Gaussian limits

ii) If for intertwined pairs: Prob(...) $\to 1$, then one gets a fermonic expression.

ii) The argument has a simple extension to to all even-n boundary correlation functions (ADTW).

iv) Important here are not just the statistics, but the apparent “free-ness” (or integrability) of the model.



An interesting contrast

For $d > 4$ the critical correlations $S_{2n}(x_1, \ldots, x_{2n}) \; = \; \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$

satisfy $\boxed{S_{2n}(x_1, \ldots, x_{2n}) \; \approx \; \sum_{\pi} \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$ Gauss-Wick rule (Aiz 81)

with equality in the scaling limit ($\delta \to 0$, $\kappa(\delta) \to \infty$ adjusted so the limit exists),

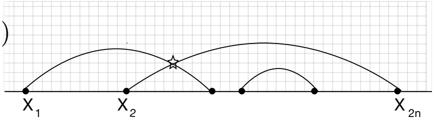
In $d = 2$ dimensions for any ferromag. Ising model on a planar graph, with a connected boundary segment, the boundary fields satisfy (SML 65, McCoy-Wu'73, GBK 78):

$$\boxed{S_{2n}(x_1, \ldots, x_{2n}) \; = \; \sum_{\pi} \varepsilon(\pi) \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$$

Fermi-Wick rule

$x_j \in [0, \infty) \times \mathbb{R}^{d-1}$

$$= \; \mathrm{Pf}\left(S_2(x_j, x_k)\right)$$



Curiously, both relation have a relatively simple explanation through the "random current representation". Using it, the Pfaffian structure of correlations (on which more can be read in Chelkak-Cimasoni-Kassel '15) appears as a consequence of elementary topological arguments (ADTW).

"Almost planar" – finite-range models on planar graphs.

For a class of such models we have the following statement of emergent planarity.

Theorem (ADTW '16) *In any finite range ferromagnetic Ising model in $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}_+$ whose couplings J are: i) translation invariant, ii) acyclic, iii) invariant under reflections: for any cyclicly ordered $(x_1, \ldots, x_{2n}) \in \partial\mathcal{G} := \mathbb{Z} \times \{0\}$*

$$\langle \sigma_{x_1} \cdots \sigma_{x_{2n}} \rangle_{\beta_c} = Pf_n \left(\left[\langle \sigma_{x_i} \sigma_{x_j} \rangle_{\beta_c} \right]_{1 \le i,j \le 2n} \right) (1 + o(1)) \tag{1}$$

where $o(1)$ is a quantity tending to zero with the smallest distance in $\mathcal{G}$ between any two x_i.

In the stochastic geometric argument the effective planarity emerges due to the critical random currents' fractal nature (at β_c).

Related universality results -stability of the law under weak perturbations- were previously derived using rigorous (perturbative) renormalization arguments by Pinson-Spencer and Giuliani-Greenblatt-Mastropietro '12.

6. Order-disorder operators

A natural question: Does the fermionic structure extend to variables in the bulk?

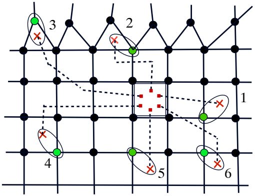
Our answer: "Yes / but": a natural extension is found in the order-disorder operators.

ℓ_j: dual lines linking sites of $\{x_j'\}$
with $x_0^* \in \mathcal{G}^*$ (the grand central).

Coupling-reversing transform's:

$$(R_\ell J)_{x,y} = -J_{x,y}$$

for edges $\{x, y\}$ crossed by ℓ.



The "order - disorder" variables $\boxed{\tau_{\hat{x}} = \sigma_x \mu_{x'}}$ are defined by:

$$\boxed{\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle := \sum_{\sigma} \left(\prod_{j=1}^{2n} \sigma_{x_j} \right) e^{-\beta R_{\ell_1} \ldots R_{\ell_j} \ldots H(\sigma)} / Z}$$

Of particular interest:
τ_j for neighboring pairs
$\hat{x}_j = (x_j, x'_j) \in \mathcal{G} \times \mathcal{G}^*$.

Theorem 4 (ADTW) In planar Ising models, of pair interaction $\mathcal{J}$ with $Z_{\mathcal{G}}(\mathcal{J}) \neq 0$, for any collection of "order - disorder" variables labeled cyclicly in terms of the disorder lines

$$\boxed{\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle = \sum_{\text{pairings}\,\pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \tau_{\hat{x}_{\pi(2j-1)}} \tau_{\hat{x}_{\pi(2j)}} \rangle \equiv \mathrm{Pf}\left(\langle \tau_{\hat{x}_j} \tau_{\hat{x}_k} \rangle \right)} \quad . \qquad (2)$$

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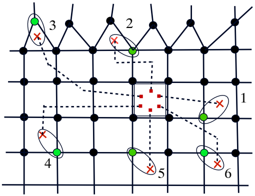

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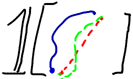
6 b. ‘Order-disorder’ operators’ stochastic geometric interpretation

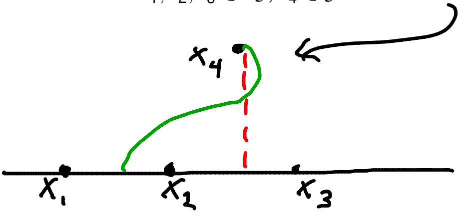
i) In terms of random currents:

$$\langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle = \sum_{\substack{\partial n_1 = \{x_1, x_2\} \\ n_2 = \emptyset}} \frac{w(n_1)}{Z} \frac{w(n_2)}{Z} (-1)^{(n_1, \gamma_{1,2})}$$



$\cancel{\sum} \; n_1 = n_2 = 0$

which in the case of $x_1, x_2, x_3 \in \partial\mathcal{G}$, $x_4 \in \mathcal{G}$ reduces to:



ii) In terms of the Kac-Ward (“parafermionic”) amplitudes

$$\begin{aligned} \langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle &= e^{i\angle(\tilde{x}_1, \tilde{x}_2)} \langle \bar{e}_2 | \, (\mathbb{1} - KW)^{-1} \, | e_1 \rangle \\ &= e^{i\angle(\tilde{x}_1, \tilde{x}_2)} \sum_{\gamma : e_1 \to \bar{e}_2} \chi_{KW}(\gamma) \; e^{i \int_\gamma dArg(e)/2} \end{aligned}$$

7. Related observations, and questions

1) The order-disorder operators form the Kaufman spinors, and are key elements in the Kadanoff - Ceva list.

Their product also yields the energy density operator. Through this relation, the above fermonic rule yields yet another intuitive explanation, a-la Kadanoff, of some of the (already well known) critical exponents, e.g.:

- the energy- energy correlations decay in 2D as $1/r^2$ (and hence the energy density has, in 2D, a logarithmic cusp at T_c).
- boundary spin correlators decay as $1/r$, etc.

Can this structure be understood more robustly (universality, etc)?

2) Emergent planarity: There is still room for a more complete mathematical grasp of the stochastic geometry of the critical models. This may adds some robust insight on the emergent planarity at criticality in two dimensional models with non-planar interactions / weights, supplementing the (perturbative) renormalization group analysis.

3) Among emergent structures/features of physical systems:

- fermionic excitations in classical Ising systems
- particles as collective excitations (including Majorana fermions, etc.)
- topological states of matter
- A timid question: could one day also the laws of QM be viewed as an emergent feature?

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- boundary spin correlators decay as $1/r$, etc.

Can this structure be understood more robustly (universality, etc)?

2) Emergent planarity: There is still room for a more complete mathematical grasp of the stochastic geometry of the critical models. This may adds some robust insight on the emergent planarity at criticality in two dimensional models with non-planar interactions / weights, supplementing the (perturbative) renormalization group analysis.

3) Among emergent structures/features of physical systems:

- fermionic excitations in classical Ising systems
- particles as collective excitations (including Majorana fermions, etc.)
- topological states of matter
- A timid question: could one day also the laws of QM be viewed as an emergent feature?

Thank you for your attention.