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A VARIATIONAL PRINCIPLE AND ITS APPLICATIONS

TO PROBLEMS IN ELECTROMAGNETIC THEORY

A THESIS

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SUMMARY

The theory of electricity and magnetism is basically the theory of Maxwell's equations. In specialized forms these equations provide the basis for the analysis of lumped-element circuits, antennas, waveguides, transmission lines, etc. Any problem in each of these specialized areas has Maxwell's equations as a starting point, and is usually solved by applying suitable boundary conditions. However, the task of applying boundary conditions at a surface of discontinuity in space quite often proves to be mathematically difficult, if not impossible, unless the discontinuity corresponds to one of the coordinates of the chosen frame of reference. For this reason, applications of the theory of Maxwell's equations are usually restricted to those problems which involve mathematically convenient boundaries.

The purpose of this investigation is to develop a new method of formulating problems in electromagnetic field theory. This method is based upon a variational principle which asserts that

$$\delta\iiiint \left(\frac{1}{2}\,\epsilon|\overline{E}|^2 - \frac{1}{2}\,\mu|\overline{H}|^2 + \overline{J}\cdot\overline{A} - \phi\rho\right)dx_o\,dy_o\,dz_o\,dt = 0 \qquad (1)$$

where x_o, y_o, z_o are space variables, t is time, and, in traditional notation, $\overline{E}$ and $\overline{H}$ are the intensities of the field, ϕ and $\overline{A}$ the potentials, and $\overline{J}$ and ρ the current and charge densities, respectively. The first term in the integrand of equation (1) is recognized to be the energy density of the electric field; the second term, the energy density

of the magnetic field; the third term, the work required to set up the current density $\overline{J}$ in the field $\overline{A}$; and finally, the fourth term, the work done in bringing ρ from infinity to a point in the field of ϕ.

It is shown mathematically that the variational principle expressed by equation (1) is a direct consequence of Maxwell's equations. In free space $J = \rho = 0$ and, therefore,

$$\delta \iiiint \left(\frac{1}{2}\,\epsilon\,|\overline{E}|^2 - \frac{1}{2}\,\mu\,|\overline{H}|^2\right) dx_o dy_o dz_o dt = 0 \tag{2}$$

In other words, the distributions of charge and current in a system of conductors are such that, if it extends over sourcefree regions, the integral of the difference between the energy stored in the electric field and the energy stored in the magnetic field has a stationary value as compared with the value of the same integral when the integration extends over the same volume in free space and over the same time interval, but for all other nearby varied current and charge distributions satisfying the boundary conditions for the prescribed conducting system. An illustrative example of the principle is provided by uniform plane waves which are characterized by the condition

$$\frac{1}{2}\,\epsilon\,|\overline{E}|^2 - \frac{1}{2}\,\mu\,|\overline{H}|^2 = 0$$

which states that the energy content of the electric field is numerically equal to the energy content of the magnetic field.

Now, according to Maxwell's equations, there can exist but one distribution of sources in a given system of conductors with prescribed excitation. This unique distribution is guaranteed by Maxwell's equations to be such that both the conditions demanded at the boundaries and the condition expressed by equation (2) are met. Therefore, instead of solving Maxwell's equations subject to the conditions demanded at the boundaries, a completely equivalent procedural substitute is to integrate the point source solution of Maxwell's equations and, subsequently, constrain the distribution of the sources to be such that the energy content of the electric field is as nearly equal to the energy content of the magnetic field as the conditions at the boundaries allow. Thus, the fundamental assertion of this work is that boundary value problems may be replaced by equivalent variational problems and, in so doing, the mathematical difficulties associated with the mechanics of applying boundary conditions are avoided.

The manner of formulating electromagnetic field theory problems using the principle expressed by equation (2) may be illuminated by considering the general case. Assume, for example, that currents and charges are distributed throughout a closed region R, and let (x, y, z) be the coordinates of a point in R and

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2} \tag{3}$$

its distance to a fixed point of observation (x_o, y_o, z_o) outside R. In terms of the source distributions, the scalar and vector potentials of the field are obtained using, respectively, the formulas

$$\phi(x_o, y_o, z_o, t) = \frac{1}{4\pi\epsilon} \iiint_R \frac{1}{r}\, \rho(x, y, z, t - \frac{r}{v})\, dx\, dy\, dz \tag{4}$$

$$\overline{A}\ (x_o,\ y_o,\ z_o,\ t) = \frac{\mu}{4\pi} \iiint_R \frac{1}{r}\, \overline{J}(x,\ y,\ z,\ t - \frac{r}{v})\ dx\ dy\ dz \qquad (5)$$

in which the ratio r/v denotes the time required for electromagnetic waves to travel the distance r. Now the field intensities are computed using the expressions

$$\overline{E} = -\nabla\phi - \frac{\partial \overline{A}}{\partial t} \qquad (6)$$

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A} \qquad (7)$$

where the differentiations denoted by the operator ∇ are with respect to x_o, y_o, z_o. Inspection of expressions (4), (5), (6), and (7) shows clearly that $\overline{E}$ and $\overline{H}$ are functions of the source distributions $\overline{J}$ and ρ. But, since the latter cannot be specified independently and, in fact, since they must always be such as to satisfy the equation of continuity

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (8)$$

it is clear that the field intensities $\overline{E}$ and $\overline{H}$ and, consequently, the integrand in equation (2) are functions of the components of the current density $\overline{J}$. Hence, the resulting variational problem expressed by (2) implies as many integral equations as $\overline{J}$ has components. Solving these equations simultaneously yields, in theory at least, the unknown source distribution from which the field can be calculated in the usual manner.

The foregoing procedure is used in this work to treat three specific examples. The first of these deals with a hollow cylindrical antenna sustaining an axial current sheet the magnitude of which is prescribed at the antenna's driving point. The application of the variational principle at points along the axis of the antenna results in an integral equation which is very similar in form to the equation derived when the cylindrical antenna problem is formulated by the so-called method of Hallén. Hallén's method is a procedure in which the electric field intensity is calculated on the basis of an assumed current distribution. Once the source distribution is assumed, the simplifying assumption of a perfectly conducting, thin-walled cylinder is made in order to make the axial component of the resulting electric field intensity identically zero on the metal surface of the antenna. This condition leads to an integral equation the solution of which has been the main topic for much of the literature of the past two decades on the subject of cylindrical antenna theory. But, in spite of intensive efforts, this equation has never been solved exactly. Usually, in order to put the equation in a manageable form, its kernel is so approximated that the resulting integral equation is now found to be identical with the equation obtained directly by the new variational method. Therefore, existing approximate solutions of an equation that has all along appeared to be approximate are, in fact, solutions of an exact equation.

The second example treated in this work has to do with a rectangular aperture antenna. Classical formulation cannot be employed in this case because no actual physical boundary exists. This problem is formulated here for the first time on a rigorous mathematical basis. The resulting integral equation is found to have precisely the same form as the equation associated with the cylindrical antenna problem. A method for solving the new equation is presented.

The treatment of the third and final problem shows that the natural behavior of center-driven linear arrays, consisting of an odd number of equally-spaced short antennas, is characterized by the fact that the field at distant points on the axis of the array vanishes identically. This characterization is accompanied by analytical expressions for the magnitude and phase angles of the currents induced in the parasitic elements.

Since circuit relations are just special cases of Maxwell's equations, the investigation concludes with a brief introduction to the Lagrangian treatment of the network problem including analyses of two illustrative examples by the dynamical method. Finally, a general proof of the theorem of constant flux linkages by variational methods is found to eliminate certain ambiguities hitherto associated with this tool of circuit analysis.

CHAPTER I

INTRODUCTION

The classical approach to the electromagnetic field problem is based upon the solution of Maxwell's equations subject to the conditions imposed by appropriately chosen boundary conditions. This study will show that it is possible to treat the same problem by a second general method which to this date has not been explored.

The simplest problem of electromagnetic theory arises in electrostatics. It is common knowledge that the electrostatic problem essentially requires solutions to Laplace's or Poisson's equations which guarantee that each conductor surface is maintained at equal potential. With the exception of a small number of relatively simple configurations the general problem in electrostatics, unfortunately, presents quite a formidable task. The fundamental reason for this is that usually the surfaces of the conductors do not correspond to a particular value of one coordinate of the chosen coordinate system. As a result, analyses of many configurations lacking mathematically attractive boundaries have not been carried out.

The transition from electrostatics to electrodynamics compounds even more the complexity of the corresponding problem because time is introduced as an additional variable. The most familiar problems in electrodynamics are those which fall under the categories of a-c circuits, guided waves, and antennas. Since the regions in which solutions to the antenna problem are sought are infinite in extent and since, furthermore, retardation effects are not neglected, this is without doubt one of the most difficult and complex problems in the field of electromagnetic theory. The problem of determining the electromagnetic field associated with even the simplest type of radiators is not trivial. The reason, of course, is that the field can be calculated only when the antenna current distribution is known; but since the latter can be established only if a sufficiently complete description of the electromagnetic field is given, a serious dilemma arises in that the solution to the problem needs to be known before the problem can be solved. Nevertheless, if the current distribution in question can somehow be established, the associated radiation field can be obtained by integration of the point-source solution of the radiation problem over the physical confines of the antenna. Clearly, the difficulty is not in finding the radiation field corresponding to a given source distribution, but rather in determining the current and charge distributions. In the absence of such information, it is necessary at times to assume a distribution in order to calculate the field. But since the accuracy of such an assumption can be checked only through experiment, this approach leaves much to be desired.

The task of bridging the wide gap separating, even today, antenna theory from engineering practice has taxed the ability and imagination of several investigators. An excellent account of their accomplishments both from the historical and the qualitative point of view is presented by King (1). The essential features of the three most general methods of attack which have been developed are outlined in one of the classical books on antennas and electromagnetic theory by Jordan (2). The first of these methods is a procedure in which the antenna is conceived as an open-ended waveguide, and the problem is solved in terms of waves transmitted along the antenna. The second method is a procedure in which the free, or natural, modes of the antenna are established first and, once this is done, the solution is obtained as an infinite series of these natural modes with the coefficients determined so as to satisfy the driving conditions. The third method proceeds with the solution by obtaining general expressions for the field which correspond to an assumed current distribution and, upon application of the boundary conditions on the metal surfaces of the antenna, yields an integral equation whose solution provides the required current distribution.

Over the last two decades considerable effort has been concentrated on the solution of the cylindrical antenna problem by the third method which, actually, was originally conceived and developed in conjunction with that particular antenna shape (3). The succeeding treatments of the same problem have contributed only to the mechanics of solving the same integral equation (4). Naturally, in the process some refinement of the final solution has been achieved.

Applications of the three methods of attack to actual problems have been restricted to relatively few antenna shapes: the biconical antenna, the prolate spheroid, and the cylindrical antenna. The reason for this poor dividend, poor in proportion to the effort expended, is that the resulting equations are so complex that their solution amounts to a challenge even to the most capable applied mathematician.

The work reported here will by no means provide the ultimate answer to the extremely complex problem of electrodynamics, best exemplified by the antenna. Instead, it will hopefully shed some light upon a previously unexplored path leading to the solution of this same general problem. Thus it will be shown that, in place of solving Maxwell's equations subject to the conditions imposed at the boundaries, a completely equivalent approach is to integrate the point source solution of Maxwell's equations (that is, the field generated by a Hertzian dipole) over the entire conducting region, and then demand that the difference in the energy content (density) of the electric field and the energy content of the magnetic field have the minimum value allowed by the boundary conditions. The last statement expresses a principle which, as it will be shown, governs the behavior of electromagnetic fields in sourcefree regions, and one which is obviously analogous to Hamilton's principle in mechanics.

Based upon this principle, the formulation of the electrodynamical problem leads, as it will be shown, to integral equations which, when solved, yield the unknown source distributions. In this work specific antenna problems will be formulated to illustrate the application of the method. Included among these is an aperture type problem which has not been formulated rigorously by means of any other existing method.

The use of this principle in establishing unknown source distributions parallels that of Hamilton's principle which is used in arriving at the equations of motion of a mechanical system acted upon by some prescribed system of forces. To illustrate the general procedure let (x, y, z) be the rectangular coordinates of a point in a closed region R, and

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2} \qquad (1)$$

its distance to some point (x_o, y_o, z_o) of observation outside R. Let t represent the time of observation of an electromagnetic disturbance at (x_o, y_o, z_o) caused by currents and charges distributed throughout the conducting region R. If v denotes the velocity of wave propagation, the quantity $(t - \frac{r}{v})$ expresses the time of generation of the disturbance at the point (x, y, z), and, as a result, the charge and current densities may be expressed thus:

$$\rho(x, y, z, t - \frac{r}{v}) \qquad (2)$$

$$\bar{J}(x, y, z, t - \frac{r}{v}) = \sum_{n=1}^{3} \bar{i}_n J_n(x, y, z, t - \frac{r}{v}) \qquad (3)$$

The scalar and vector potentials at (x_o, y_o, z_o) are obtained using the formulas

$$\phi(x_o, y_o, z_o, t) = \frac{1}{4\pi\epsilon} \iiint_R \frac{1}{r} \rho(x, y, z, t - \frac{r}{v}) d\tau \qquad (4)$$

$$\bar{A}(x_o, y_o, z_o, t) = \frac{\mu}{4\pi} \iiint_R \frac{1}{r} \bar{J}(x, y, z, t - \frac{r}{v}) d\tau \qquad (5)$$

the integrations extending throughout the volume of the region R. In terms of these potentials the field vectors $\bar{E}$ and $\bar{H}$ are given by

$$\bar{E} = - \nabla \phi - \frac{\partial \bar{A}}{\partial t} \qquad (6)$$

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A} \tag{7}$$

where the differentiations denoted by the operator ∇ are with respect to x_o, y_o, z_o at the point of observation.

In an isotropic medium, characterized by the scalar permittivity ϵ and the scalar permeability μ, the density of the energy stored in the electric field is given by

$$W_E = \frac{1}{2} \overline{D} \cdot \overline{E} = \frac{1}{2} \epsilon \left|\overline{E}\right|^2 \tag{8}$$

and that stored in the magnetic field by

$$W_H = \frac{1}{2} \overline{H} \cdot \overline{B} = \frac{1}{2} \mu \left|\overline{H}\right|^2 \tag{9}$$

By virtue of expressions (4) through (7) the energy density expressions (8) and (9) are, to be sure, functions of $\overline{J}$ and ρ. But inasmuch as $\overline{J}$ and ρ are related to one another by the equation of continuity

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0 \tag{10}$$

it follows that the difference

$$W_E - W_M \tag{11}$$

may be regarded as a function of the current density $\overline{J}$ alone or, when $\overline{J}$ consists of a single component, of the charge density ρ alone. The form of the equation of continuity is such that, when $\overline{J}$ is made up of two or three components, it becomes mathematically impossible to express each of them independently in terms of the charge density ρ. Therefore, the difference (11) may always be regarded as a function of $\overline{J}$ alone, but not necessarily of ρ alone.

Now the principle under consideration demands that, at every point in a sourcefree region where the quantities W_E and W_M are defined, the variation

$$\delta \iiiint (W_E - W_M) dx_o \, dy_o \, dz_o \, dt = 0 \tag{12}$$

Equation (12) expresses a constraint on $\overline{J}$ which implies as many integral equations as there are components comprising the current density function $\overline{J}$. These components can be found by solving simultaneously the set of equations implied by the condition (12). Since the problem of a driven conducting system involves not only the requirement expressed by equation (12) but also the driving conditions, the components of $\overline{J}$ must be such that they satisfy the boundary conditions at the location of the source.

Thus formulated, the problem is expected to have a unique solution on grounds that the electromagnetic field is also unique.

Evidently, the complexity of each individual problem depends, to a large extent, upon whether $\overline{J}$ consists of one, two, or three components requiring the solution of as many simultaneous equations. The complexity of the problem also depends upon the conductor configuration, though to a lesser extent. Nevertheless, since the requirement expressed by equation (12), as shown later, is equivalent to the requirements imposed upon the field by the boundary conditions, the variational principle renders any electrodynamical problem conducive to formulation regardless of the degree of compatibility existing between the physical configuration of the conductors and the chosen frame of reference.

Since circuit relations are special cases of more general field equations, it is reascnable to suspect that a variational principle equivalent to the one expressed by equation (12) governs the behavior of lumped-element networks. Hence this study included among its objectives an investigation into the network energy relations from the variational point of view.

CHAPTER II

A NOTE CONCERNING THE ORIGIN OF THE MAGNETIC FIELD

The variational principle to be discussed in the next chapter may be developed logically by a thought process which is based upon the concept that mass and energy are manifestations of the same physical phenomenon. Such a development is preceded by a simple exercise which shows that the magnetic effect is a direct consequence of relative motion between an observer and a system of charges which are static in their own frame of reference but are otherwise arbitrary.

In proving this assertion the point of departure is equation (8), repeated here in modified form:

$$W_E = \frac{1}{2}\,\overline{D}\cdot\overline{E} \tag{13}$$

Thus, every element of volume dV of the space in which the field is defined contains a quantity of energy equal to

$$\frac{1}{2}\,\overline{D}\cdot\overline{E}\,dV$$

Since mass and energy are manifestations of the same natural phenomenon, it may alternately be imagined that space is completely filled with an hypothetical mass of density dm_o such that

$$dm_o = \frac{W_E}{c^2} \tag{14}$$

where c is the velocity of light.

Let an observer move relative to the electrostatic field with a uniform velocity v. In the observer's frame of reference

$$dm = \frac{dm_o}{\sqrt{1 - k^2}} \tag{15}$$

where

$$k = \frac{v}{c} \tag{16}$$

However, since length suffers the Lorentz contraction in the direction of motion, the actual mass density observed has a magnitude given by

$$dm = \frac{dm_o}{1 - k^2} \tag{17}$$

Thus, owing to relative motion, the energy density appears to have increased by an amount

$$\Delta W_E = dmc^2 - dm_o c^2 = \frac{dm_o}{1 - k^2} c^2 - dm_o c^2 = W_E \frac{k^2}{1 - k^2} \tag{18}$$

Let ϵ be the permittivity of the medium and introduce a parameter μ such that

$$\mu\epsilon = \frac{1}{c^2} \tag{19}$$

Also let θ be the angle between the velocity vector $\overline{v}$ and the electric field intensity $\overline{E}$. Evidently, θ is a scalar point function. Now since

$$\sin^2 \theta + \cos^2 \theta = 1$$

it follows that

$$\Delta W_E = \Delta W_E \cos^2\theta + \Delta W_E \sin^2\theta = \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \cos^2\theta + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \sin^2\theta \tag{20}$$

and, therefore,

$$W_E + \Delta W_E = \frac{1}{2} \overline{D}\cdot\overline{E} + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \cos^2\theta + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \sin^2\theta \tag{21}$$

Combining the first and third terms in the right-hand member of (21) gives

$$\begin{aligned}
\frac{1}{2} \overline{D}\cdot\overline{E} + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \sin^2\theta &= \frac{1}{2} \overline{D}\cdot\overline{E} \left[1 + k^2 \frac{\sin^2\theta}{1 - k^2} \right] \\
&= \frac{1}{2} \overline{D}\cdot\overline{E} \left[\frac{1 - k^2\cos^2\theta}{1 - k^2} \right] \\
&= \frac{1}{2} \overline{D}\cdot\overline{E} \left[\frac{\sin^2\theta + \cos^2\theta - k^2\cos^2\theta}{1 - k^2} \right] \\
&= \frac{1}{2} \overline{D}\cdot\overline{E} \cos^2\theta + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{\sin^2\theta}{1 - k^2}
\end{aligned}$$

and, therefore, equation (21) may be put in the form

$$W_E + \Delta W_E = \frac{1}{2} \overline{D}\cdot\overline{E} \frac{k^2}{1 - k^2} \cos^2\theta + \frac{1}{2} \overline{D}\cdot\overline{E} \cos^2\theta + \frac{1}{2} \overline{D}\cdot\overline{E} \frac{\sin^2\theta}{1 - k^2} \tag{21a}$$

Usually $v \ll c$ so that $k \ll 1$ and

$$W_E + \Delta W_E \approx \frac{1}{2}\,\overline{D}\cdot\overline{E}\,\cos^2\theta + \frac{1}{2}\,\overline{D}\cdot\overline{E}\,\frac{\sin^2\theta}{1 - k^2} \tag{22}$$

Examination of equation (22) suggests the existence of a quasi-electrostatic field intensity E' consisting of a component parallel to $\overline{v}$ which is equal in magnitude to that of the static field,

$$E'_{||} = E_{||} \tag{23}$$

plus a normal component

$$E'_{\perp} = \frac{E_{\perp}}{\sqrt{1 - k^2}} \tag{24}$$

It is surprising to find that equations (23) and (24) are in complete agreement with established relativistic field transformations, as given by Stratton (5).

Now since

$$|\overline{v} \times \overline{D}| = vD\,\sin\theta$$

it follows that

$$\Delta W_E\,\sin^2\theta = \frac{1}{2}\,\overline{D}\cdot\overline{E}\,\frac{k^2}{1 - k^2}\,\sin^2\theta = \frac{1}{2}\left(\frac{\overline{v} \times \overline{D}}{\sqrt{1 - k^2}}\right)\cdot\left(\mu\,\frac{\overline{v} \times \overline{D}}{\sqrt{1 - k^2}}\right) \tag{25}$$

Define

$$H = \frac{\overline{v} \times \overline{D}}{\sqrt{1 - k^2}} \tag{26}$$

and

$$\overline{B} = \mu\overline{H} = \mu \frac{\overline{v} \times \overline{D}}{\sqrt{1 - k^2}} = \frac{\overline{v} \times \overline{E}}{c^2 \sqrt{1 - k^2}} \tag{27}$$

If in direct analogy to the field of gravity it is conjectured that the change in "potential" energy (25) is balanced by an equal but opposite change in "kinetic" energy, then the scalar product $\frac{1}{2}\ \overline{H}\cdot\overline{B}$ must be the energy associated with a new field identified by the vectors $\overline{H}$ and $\overline{B}$. Such a conjecture may be supported by the fact that (26) is the generic formula of the so-called Biot-Savant law. Thus, from the theory of static electricity it is known that a point charge dq generates a radial field

$$\overline{D} = \frac{dq}{4\pi r^2}\ \overline{u}_r$$

Hence

$$|\overline{v} \times \overline{D}| = \frac{ds}{dt}\frac{dq}{4\pi r^2} \sin\theta = \frac{I ds \sin\theta}{4\pi r^2}$$

The vector $(\overline{v} \times \overline{D})$ is normal to the plane defined by $\overline{v}$ and $\overline{u}_r$ and it points in the direction of advance of a right-hand threaded screw when $\overline{v}$ is rotated into $\overline{u}_r$. Therefore, if $v \ll c$, equation (26) gives

$$d\bar{H} = \frac{I\, d\bar{s} \times \bar{u}_r}{4\pi r^2} \qquad (28)$$

This is the well-known mathematical statement of the Biot-Savant law.

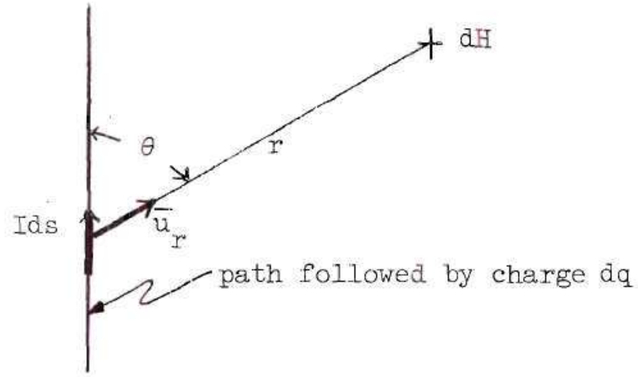

Figure 1. Motion of a Charged Particle Giving Rise to a Magnetic Field in Accordance with the Biot-Savant Law

Next, consider a rectangular coordinate system S' moving along the z-axis of another system S with velocity v, as shown in Figure 2. Chosen as a field source is a single point charge located at the origin

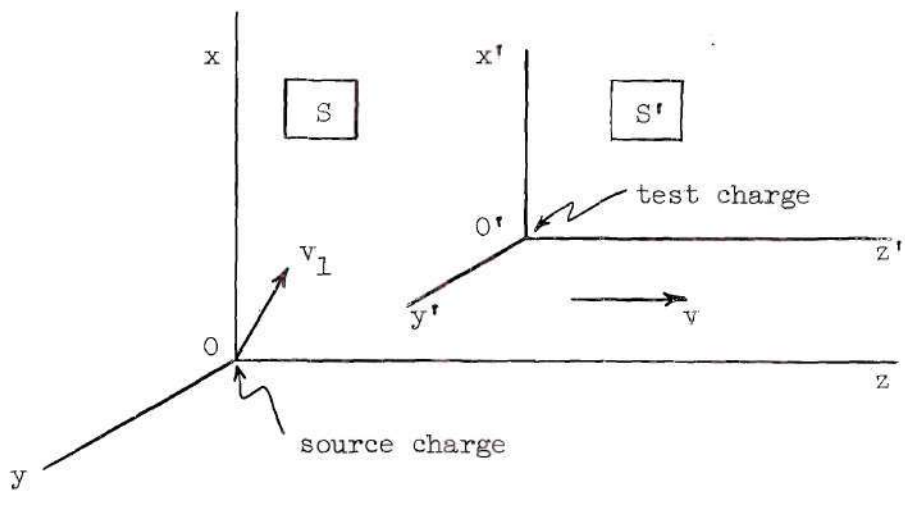

Figure 2. Charges in Relative Motion in an Inertial Frame of Reference

of S and moving with velocity

$$\bar{v}_1 = v_x \, \bar{i} + v_y \, \bar{j} + v_z \, \bar{k} \tag{29}$$

This choice of the field source does not destroy the generality of the situation since it is possible to construct the most general field by superposition of simpler fields generated by point charges. To an observer O', stationary in S', the source charge appears to be moving at a velocity $\bar{v}_1'$ with its components related to those of $\bar{v}_1$ according to the formulas

$$v_x' = (1 - k^2) \frac{v_x}{1 - \frac{v_z v}{c^2}} \tag{30}$$

$$v_y' = (1 - k^2) \frac{v_y}{1 - \frac{v_z v}{c^2}} \tag{31}$$

$$v_z' = \frac{v_z - v}{1 - \frac{v_z v}{c^2}} \tag{32}$$

These relations may be found, among other places, in reference (6). As before

$$k = \frac{v}{c} \tag{16}$$

Let

$$k_1 = \frac{v_1}{c} \tag{33}$$

$$k_1' = \frac{v_1'}{c} \tag{34}$$

$$k_z = \frac{v_z}{c} \tag{35}$$

Then

$$1 - (k_1')^2 = 1 - \frac{1}{c^2}(v_1')^2 \tag{36}$$

$$= 1 - (1 - k^2)\frac{v_x{}^2 + v_y{}^2}{c^2(1 - \frac{v_z v}{2})^2} - \frac{(v_z - v)^2}{c^2(1 - \frac{v_z v}{2})^2}$$

$$= \frac{(1 - k^2)(1 - k_1{}^2)}{(1 - k_z k)^2}$$

Let E, D, dm_o be defined as before. The mass density measured by observer O' is

$$dm' = \frac{dm_o}{1 - (k_1')^2} \tag{37}$$

When referred to the system S, this mass density becomes

$$dm = (1 - k^2)\frac{dm_o}{1 - (k_1')^2} \tag{38}$$

Combining (36) and (38) gives

$$dm = dm_o \frac{(1 - k_z k)^2}{1 - k_1^2} \tag{39}$$

Since (39) reduces to (17) whenever $v_z = 0$, it is permissible, without loss in generality, to set

$$v_x = v_y = 0 \tag{40}$$

and, thereupon, express (39) in the modified form

$$dm = dm_o \frac{(1 - k_1 k)^2}{1 - k_1^2} \tag{41}$$

In the zx plane of S the electric field intensity vector $\overline{E}$ consists of two components only, E_x and E_z. By comparing it with (17), equation (41) may now be interpreted in terms of a modified electrostatic field such that

$$E_x{}^* = E_x(1 - k_1 k) \tag{42}$$

$$E_z{}^* = E_z(1 - k_1 k) \tag{43}$$

The term $(- k_1 k \; E_x)$ in (42) is approximately equal to $(\overline{v} \times \overline{B})$ because

$$\overline{B} \approx \frac{\overline{v}_1 \times \overline{E}}{c^2} = \frac{1}{c^2} \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & v_1 \\ E_x & 0 & E_z \end{vmatrix} = \overline{j} \frac{v}{c^2} E_x \tag{44}$$

and

$$\overline{v} \times \overline{B} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & v \\ 0 & \frac{v}{c^2} E_x & 0 \end{vmatrix} = - \overline{i} \frac{v_1 v}{c^2} E_x = - \overline{i} \; k_1 k \; E_x \tag{45}$$

Therefore $(\overline{v} \times \overline{B})$ represents a force exerted upon the test charge as a consequence of its motion in the field of the moving point charge.

Thus, without recourse to the invariance of charge with respect to coordinate transformations, it has been shown that the so-called magnetic field is a second-order effect which may be derived using Coulomb's law, the Lorentz and mass transformations, plus the postulate that mass and energy are different manifestations of the same physical phenomenon. Such a postulate, incidentally, has another interesting implication in that, if the domain of an electric field is envisioned as a continuum permeated by an hypothetical mass distributed according to the density formula (14), then such a continuum must of necessity behave much like an ordinary mechanical system.

One of the laws which governs the behavior of conservative mechanical systems is known as Hamilton's principle; this principle asserts that the natural motion of a mechanical system is characterized by the fact that the time integral of the difference between the total kinetic and total potential energy taken between two configurations of the system is minimum. Reflection upon Hamilton's principle gives birth to a strong intuitive belief that charges and currents which generate electromagnetic fields must be so distributed on conductors that the energy associated with the field must obey some physical law similar to the one expressed by Hamilton's principle. Indeed, this is the case, and this topic is treated in the next chapter.

CHAPTER III

THE VARIATIONAL PRINCIPLE

Statement of the Principle.--In media which are linear, homogeneous, isotropic, and sourcefree the behavior of electromagnetic fields, generated by currents and charges distributed throughout a prescribed system of conductors, is characterized by the fact that the time integral of the difference between the energy stored in the electric field and the energy stored in the magnetic field has a stationary value as compared with the value of the same integral when the integration extends over the same volume in space and over the same time interval, but for all other nearby varied current and charge distributions satisfying the required boundary conditions for the prescribed conducting system.

In discussion of the principle, Maxwell's equations provide the obviously inescapable point of departure. In familiar notation they are:

$$\nabla \times \overline{E} + \frac{\partial \overline{B}}{\partial t} = 0 \tag{46}$$

$$\nabla \times \overline{H} - \frac{\partial \overline{D}}{\partial t} = \overline{J} \tag{47}$$

$$\nabla \cdot \overline{B} = 0 \tag{48}$$

$$\nabla \cdot \overline{D} = \rho \tag{49}$$

For the purposes of the work which follows, their solution, as given by Stratton (7), is restricted to sourcefree regions, that is, regions characterized by the condition $J = \rho = 0$. Further, it is assumed that the region of interest is homogeneous and isotropic.

Hence,

$$\overline{D} = \epsilon \, \overline{E} \tag{50}$$

and

$$\overline{B} = \mu \, \overline{H} \tag{51}$$

In terms of the potentials the general solution of Maxwell's equation is given by the set

$$\overline{B} = \nabla \times \overline{A} \tag{52}$$

$$E = - \nabla \phi - \frac{\partial \overline{A}}{\partial t} \tag{53}$$

The potentials ϕ and $\overline{A}$ satisfy the respective wave equations

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = - \frac{\rho}{\epsilon} \tag{54}$$

$$\nabla^2 \overline{A} - \mu\epsilon \frac{\partial^2 \overline{A}}{\partial t^2} = - \mu \, \overline{J} \tag{55}$$

which, when $J = \rho = 0$, become

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{56}$$

$$\nabla^2\overline{A} - \mu\epsilon \frac{\partial^2\overline{A}}{\partial t^2} = 0 \tag{57}$$

The potentials also satisfy the so-called Lorentz condition

$$\nabla\cdot\overline{A} + \mu\epsilon \frac{\partial\phi}{\partial t} = 0 \tag{58}$$

Now under the stated assumptions it is to be shown that $\overline{E}$ and $\overline{H}$, subject to prescribed boundary conditions, are solutions of the variational problem

$$\delta \iiiint \left(\frac{1}{2}\,\epsilon|\overline{E}|^2 - \frac{1}{2}\,\mu|\overline{H}|^2\right)\,dx\,dy\,dz\,dt = 0 \tag{59}$$

in which x, y, z are the variables of some arbitrarily chosen rectangular coordinate system, and t denotes time.

First, substituting expressions (52) and (53) in (59) gives

$$\iiiint \frac{1}{2}\left[\epsilon\,|-(\overline{i}\phi_x + \overline{j}\phi_y + \overline{k}\phi_z)-(\overline{i}\dot{A}_1 + \overline{j}\dot{A}_2 + \overline{k}\dot{A}_3)|^2 - \frac{1}{\mu}|\nabla\times A|^2\right] dx\,dy\,dz\,dt$$

where the subscripts x, y, z denote partial differentiations with respect to the respective space variables, the dot over the letters denotes partial differentiation with respect to time, and in terms of its components

$$\overline{A} = \overline{i}\,A_1 + \overline{j}\,A_2 + \overline{k}\,A_3$$

Now

$$\nabla\times\overline{A} = \overline{i}(A_{3y} - A_{2z}) + \overline{j}(A_{1z} - A_{3x}) + \overline{k}(A_{2x} - A_{1y})$$

so that the integral to be extremized becomes

$$I = \iiiint \frac{1}{2} \{\epsilon(\phi_x + \dot{A}_1)^2 + \epsilon(\phi_y + \dot{A}_2)^2 + \epsilon(\phi_z + \dot{A}_3)^2 - \frac{1}{\mu}(A_{3y} - A_{2z})^2 \qquad (60)$$

$$- \frac{1}{\mu}(A_{1z} - A_{3x})^2 - \frac{1}{\mu}(A_{2x} - A_{1y})^2\} \, dx \, dy \, dz \, dt.$$

Inspection of the integrand in (60) shows that it involves four independent variables, x, y, z, t, and four dependent variables ϕ, A_1, A_2, A_3. Let F denote this integrand. A necessary condition that the integral be stationary is that the variation

$$\delta I = \iiiint \Big\{ \left[\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{d\phi_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{d\phi_y}\right) - \frac{d}{dz}\left(\frac{\partial F}{d\phi_z}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\phi}}\right)\right] \delta\phi \qquad (61)$$

$$+ \left[\frac{\partial F}{\partial A_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial A_{1x}}\right) - \frac{d}{d_y}\left(\frac{\partial F}{\partial A_{1y}}\right) - \frac{d}{dz}\left(\frac{\partial F}{\partial A_{1z}}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}_1}\right)\right] \delta A_1$$

$$+ \left[\frac{\partial F}{\partial A_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial A_{2x}}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial A_{2y}}\right) - \frac{d}{dz}\left(\frac{\partial F}{\partial A_{2z}}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}_2}\right)\right] \delta A_2$$

$$+ \left[\frac{\partial F}{\partial A_3} - \frac{d}{dx}\left(\frac{\partial F}{\partial A_{3x}}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial A_{3y}}\right) - \frac{d}{dz}\left(\frac{\partial F}{\partial A_{3z}}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}_3}\right)\right] \delta A_3 \Big\} \, dx \, dy \, dz \, dt$$

vanish (8). Now the coefficient of $\delta\phi$ in (61) may be expressed as follows:

$$\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right) - \frac{d}{dz}\left(\frac{\partial F}{\partial \phi_z}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\phi}}\right) \qquad (62)$$

$$= \epsilon[- \frac{d}{dx}(\phi_x + \dot{A}_1) - \frac{d}{dy}(\phi_y + \dot{A}_2) - \frac{d}{dz}(\phi_z + \dot{A}_3)]$$

$$= - \epsilon[\phi_{xx} + \phi_{yy} + \phi_{zz} + \dot{A}_{1x} + \dot{A}_{2y} + \dot{A}_{3z}]$$

$$= - \epsilon[\nabla^2\phi + \frac{\partial}{\partial t} (\nabla\cdot\overline{A})]$$

But equation (58) implies that

$$\frac{\partial}{\partial t} (\nabla\cdot\overline{A}) = - \mu\epsilon \frac{\partial^2\phi}{\partial t^2}$$

and, therefore, by virtue of the wave equation (56), the right-hand member of (62) becomes

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\overline{A}) = \nabla^2\phi - \mu\epsilon \frac{\partial^2\phi}{\partial t^2} = 0 \qquad (63)$$

Thus, it is seen that the coefficient of $\delta\phi$ in (61) vanishes identically. The general procedure of evaluating the coefficients of δA_1, δA_2, δA_3 in (61) may be illustrated by considering the treatment of the coefficient of δA_1.

$$\frac{\partial F}{\partial A_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial A_{1x}} \right) - \frac{d}{dy}\left(\frac{\partial F}{\partial A_{1y}} \right) - \frac{d}{dz}\left(\frac{\partial F}{\partial A_{1z}} \right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}_1} \right) \qquad (64)$$

$$= \left[- \frac{1}{\mu} \frac{d}{dy} (A_{2x} - A_{1y}) + \frac{1}{\mu} \frac{d}{dz}(A_{1z} - A_{3x}) - \epsilon \frac{d}{dt}(\phi_x + \dot{A}_1)\right]$$

$$= \frac{1}{\mu} \left[(A_{1yy} + A_{1zz} - \mu\epsilon \frac{\partial^2 A_1}{\partial t^2}) - (A_{2xy} + A_{3xz} + \mu\epsilon\, \dot{\phi}_x)\right]$$

But equation (58) implies that

$$A_{1xx} + A_{2yx} + A_{3zx} + \mu\epsilon\, \dot{\phi}_x = 0$$

so that, by virtue of equation (57), the right-hand member of (64) becomes

$$\frac{1}{\mu}\left[\nabla^2 A_1 - \mu\epsilon\,\frac{\partial^2 A_1}{\partial t^2}\right] = 0 \tag{65}$$

In view of these results, the integrand in (61) vanishes identically and, therefore, the condition

$$\delta I = 0 \tag{66}$$

now guarantees that the integral in equation (59) meets the first necessary requirement for the existence of a relative extremum. To show that the extremum exists or that, in fact, is a minimum is a very difficult task. However, for practical problems of the type considered in this research it will not be absolutely necessary to know whether or not the value of the integral in equation (59) is actually minimum. Instead, it will suffice to know that its variation vanishes. Nonetheless, asserting without proof that the extremum implied by equation (59) is indeed a minimum would not be entirely out of the ordinary since many problems in engineering and physics are usually treated on the basis of assumptions which either from past experience or through physical intuition appear reasonable enough as to expect with some confidence that the results obtained in the forms of theoretical solutions will be correct. Hildebrand (9) lends further support to the argument by stating that in physically motivated problems of the type treated in this investigation considerations relating to sufficiency conditions may frequently be avoided. After all, water flows downhill.[1]

The General Variational Principle.--When the condition $J = \rho = 0$ does not hold, the integrand in equation (59) must be modified to include two additional terms, as suggested by Morse and Feshbach (11). Thus, in the general case Maxwell's equations guarantee that

$$\delta \iiiint \left(\frac{1}{2} \epsilon |\overline{E}|^2 - \frac{1}{2} \mu |\overline{H}|^2 + J \cdot A - \phi\rho\right) dx\, dy\, dz\, dt = 0 \tag{67}$$

The proof of this statement parallels the one already presented with the exception that equations (54) and (55) must now be used in place of equations (56) and (57), respectively. Specifically, the expression which corresponds to (63) is

$$\epsilon \left[\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} \right] - \rho \tag{68}$$

which vanishes by virtue of equation (54). Furthermore, since

$$\overline{J} \cdot \overline{A} = J_1 A_1 + J_2 A_2 + J_3 A_3$$

inclusion of the term $(\overline{J} \cdot \overline{A})$ in equation (67) implies that the left-hand member of equation (65) becomes

$$\frac{1}{\mu} \left[\nabla^2 A_1 - \mu\epsilon \frac{\partial^2 A_1}{\partial t^2} \right] + J_1 \tag{69}$$

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[1]The principle was conceived and proved in the foregoing manner with no prior knowledge of other treatments of the topic, such as the one by Morse and Feshbach (10).

which, in accordance with equation (55), is identically zero. Therefore, the proof of equation (67) is now complete.

Corollaries.--When the field sources consist of charges at rest the variational principle states that in free space the value of the integral

$$\iiiint \frac{1}{2} \epsilon |E|^2 \, dx \, dy \, dz \, dt \tag{70}$$

is minimum, or, in other words, the distribution of the charges on the surfaces of the conductors is such as to minimize the energy of the resultant electrostatic field. In electrostatics this result is commonly referred to as Thomson's theorem (12). The fact that Thomson's theorem is simply a degenerate case of a more general variational principle is interesting but not surprising because the field of gravity has a very similar and, indeed, completely analogous characteristic. Thus, Hamilton's principle tacitly implies that a body at rest in the field of gravity has the least possible potential energy. Naturally its kinetic energy is zero, but at the same time its potential energy is as close to zero as physical barriers permit. The correspondence in the two cases is apparent.

A second principle, which does not appear in any of the better-known texts on electromagnetic theory, follows directly from the variational problem expressed by equation (59). If a current is caused to flow in some conducting region by the motion of charge which is time-invariant and which, furthermore, does not result in the accumulation of free charge anywhere in the conducting region, then the current lines close upon themselves, and since $\rho = 0$ it follows that at points outside the conducting region $\overline{E} = 0$. Therefore, the so-called magnetostatic field is characterized by the fact that the distribution of current in the conducting system is such that in free space the value of the integral

$$\iiiint \frac{1}{2} \mu |\overline{H}|^2 \, dx\, dy\, dz\, dt \qquad (71)$$

is minimum.

CHAPTER IV

DISCUSSION OF THE METHOD

Careful reflection upon the variational principle under consideration gives rise to a series of questions regarding its interpretation and manner of application. The purpose of this chapter is to state and discuss these questions.

Interpretation of the Variation Principle.--In Chapter III it was shown that Maxwell's equations for free space correspond to the requirement that

$$\delta \iiiint \left(\frac{1}{2}\, \epsilon |\overline{E}|^2 - \frac{1}{2}\, \mu |\overline{H}|^2 \right) dx\, dy\, dz\, dt = 0 \qquad (59)$$

A careful examination of the derivation of equation (59) makes it quite clear that it is satisfied by every solution of Maxwell's equations, and that it is valid for any volume in free space regardless of its size.

Now in order to solve any problem in electromagnetic theory dealing with driven conducting systems, time variations must be specified a priori. Accordingly, when this information is made available relative to equation (59), the integration with respect to time may be performed initially. Then, since the integrand becomes a continuous function of the space coordinates there must exist small regions in which the integrand does not change sign, so that if the integral is to have a minimum value it is sufficient that the integrand be as small as possible. Thus, equation

(59) implies that, if currents and charges exist in a conducting region R, the resulting electromagnetic field is such that the magnitude of the difference

$$W_E - W_M = \frac{1}{2}\,\epsilon|\overline{E}|^2 - \frac{1}{2}\,\mu|\overline{H}|^2 \tag{72}$$

is minimum at every point of free space outside R.

At this point a question that might be asked is, can the distribution of charges and currents in the region R be determined as the solution of the variational problem (59)? At first thought, the answer to this question is negative because the distribution of the sources has no direct bearing upon the derivation of equation (59), so that as long as the formulas

$$\phi = \frac{1}{4\pi\epsilon}\iiint_R \frac{\rho}{r}\,d\tau$$

$$\overline{A} = \frac{\mu}{4\pi}\iiint_R \frac{\overline{J}}{r}\,d\tau$$

are used to evaluate the potentials, Maxwell's equations and, hence, condition (59) will be satisfied automatically throughout free space regardless of the manner in which the changes and currents distribute themselves in the conducting region R.

On second thought, however, it may be reasoned that since there is but one distribution of sources for which the wave equations

$$\nabla^2\,\phi - \mu\epsilon\,\frac{\partial^2\phi}{\partial t^2} = 0 \tag{56}$$

$$\nabla^2 \bar{A} - \mu\epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = 0 \tag{57}$$

are satisfied outside the prescribed conducting region R, and since these equations form an essential link in the development of equation (59), it follows that the unknowns $\bar{J}$ and ρ are solutions of the variational problem defined by equation (59). In other words, since Maxwell's equations require the existence of a unique current distribution satisfying the boundary conditions on the surface of the prescribed conducting system, and since Maxwell's equations also guarantee the validity of equation (59), the latter may be used in place of the former to determine the unknown source distributions. Furthermore, since the field is a unique function of the source distribution, the variation in equation (59) may be calculated in terms of the current density instead of directly in terms of the field.

The same question may be raised from a different point of view as follows: The variational principle is to be applied in order to find charge and current distributions by formulating the problem thus: Determine the distribution of current and charge which among all other distributions that can possibly exist in a conducting region R, subject only to the requirements imposed by the driving conditions and, of course, Maxwell's equations, satisfies equation (59). However, because, as demanded by Maxwell's equations, there is but one distribution for which the boundary conditions on the conductor surfaces are satisfied, and because, as already stated, Maxwell's equations automatically imply condition (59), the formulation of the problem on the basis of equation (59) seems to lack a firm foundation. In other words, the formulation requires that a choice be made when, in reality, there is no alternative.

At first thought, this last consideration may be discounted on grounds that the proposed approach would parallel the development of equation (59). However, the restrictions imposed in the case of equation (59) are on the potentials and not on the sources. Specifically, a glance at the derivation of equation (59) shows that ϕ and $\overline{A}$ are required to be such as to satisfy equations (56) and (57), respectively, and, of course, they are assumed to be continuously differentiable, but are otherwise arbitrary. Hence, in the case of equation (59) a choice can definitely be made.

In view of the foregoing considerations the situation appears hopeless. Yet it is reasonable to ask, if there is but one source distribution and thus one field, the integral in equation (59) is extreme as compared to what other possible values? One thought is that, perhaps, the value of the integral does not change with variations in source distributions. Were this to be the case, then either

$$W_E - W_M = \text{constant} \neq 0 \tag{73}$$

or

$$W_E - W_M = 0 \tag{74}$$

for all possible combinations of $\overline{J}$ and ρ. Equation (73) can never hold for it does not hold when $\overline{J} = \rho = 0$. On the other hand, were equation

(74) to hold in all cases, it would demand the simultaneous existence of electric and magnetic fields under all possible conditions, a virtual impossibility, and, therefore, W_E is not always equal to W_M. As a result, it must be concluded that the value of the integral in equation (59) does indeed change with variations in $\overline{J}$ and ρ, and that, consequently, the true $\overline{J}$ and the true ρ render the integral minimum as compared to all nearby varied current and charge distributions.

The notion of nearby variations may be clarified by considering a similar problem in elementary calculus. The roots of the equation

$$\frac{df(x)}{dx} = 0$$

specify those points on the x axis at which some differentiable function f attains its relative extremes as compared to the values it takes on when x deviates somewhat either above or below the root around which the comparison is made. This is illustrated in Figure 3. At $x = x_1$ the function f has a relative minimum, and at $x = x_2$ it has a relative maximum

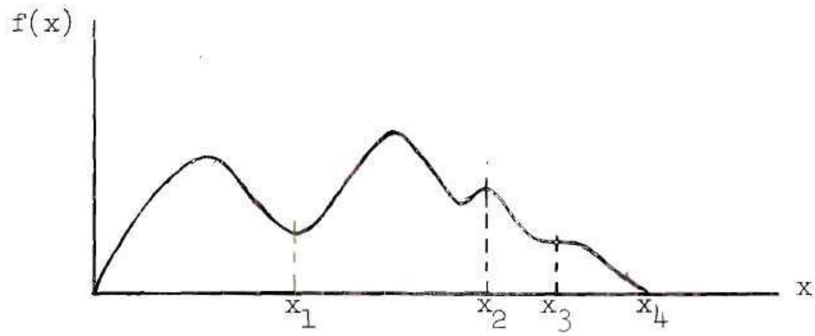

Figure 3. Relative Maximum and Minimum Values of f(x)

neither of which is, respectively, the minimum or the maximum value which the function f attains in the interval $[0, x_4]$. Note, in passing, that at $x = x_3$ the derivative of f vanishes even though the value of f is not extreme.

Still a third expression of the same question that was posed earlier is based upon the following line of reasoning. According to the calculus of variations the wave equations (56) and (57) are in essence the so-called Euler equations of the variational problem (59). As such, their solution constitutes the next step in the treatment of the problem. But, since the potentials do indeed comprise the complete solution of equations (56) and (57), the situation again appears hopeless. Fortunately, it is no more hopeless than the situation encountered in solving any electromagnetic field problem by the classical method, because such a solution is normally effected through the application of boundary conditions which express nothing more than Maxwell's equations at the boundaries. In other words, both the classical method of solution and the proposed method of solution have one and the same starting point, namely Maxwell's equations.

In view of the foregoing discussion it becomes clear that problems in electromagnetic theory may be solved by first finding the potentials of the field in terms of the unknown current density $\overline{J}$ and, subsequently, demanding that the source distribution be such as to satisfy both the prescribed driving conditions and condition (59). This approach constitutes a valid procedural substitute for the classical method of solution.

Special Cases of the Difference in Energy Contents.--It has been shown that the difference in electric and magnetic energy contents cannot always be zero. It is interesting to observe, nevertheless, that the requirement

$$\frac{1}{2}\,\epsilon|\overline{E}|^2 - \frac{1}{2}\,\mu|\overline{H}|^2 = 0$$

implies that the ratio of the field intensities

$$\left|\frac{\overline{E}}{\overline{H}}\right| = \sqrt{\frac{\mu}{\epsilon}}$$

is equal to the so-called intrinsic impedance Z_c of the lossless medium. The ratio of the field intensities is equal to $\sqrt{\mu/\epsilon}$ in traveling waves, common examples of which are uniform plane waves propagated in sourcefree and unbounded media, and TEM waves sustained along uniform transmission lines. In both cases the energy content of the electric field is exactly equal to the energy content of the magnetic field, as it may be verified by consulting references (13) and (14). If the difference vanishes the medium may be conceived to be in a relaxed state. In most practical cases the difference is not identically zero and the medium may be envisioned in a state of tension. A transmission line with reflections furnishes a simple example of an electromagnetic system under tension.

The Point Source Solution and the Far Field.--For purposes of further discussion of the principle, and by way of introduction to the treatment of specific examples, consider the point source solution of Maxwell's equations, that is, the electromagnetic field of an oscillating electric dipole, as given by Jordan (15). For a current element $Id\ell\, e^{j\omega t}$, located at the origin of a right-hand spherical coordinate system with variables r, θ, ϕ in that sequence, the expressions for the field intensities are

$$E_\theta = \frac{Z_o \, I d\ell \sin\theta \, e^{-j\beta r}}{4\pi r} \left(j\beta + \frac{1}{r} + \frac{1}{j\beta r^2} \right)$$

$$E_r = \frac{Z_o \, I d\ell \cos\theta \, e^{-j\beta r}}{4\pi r} \left(\frac{2}{r} + \frac{2}{j\beta r^2} \right) \tag{75}$$

$$H_\phi = \frac{I d\ell \sin\theta \, e^{-j\beta r}}{4\pi r} \left(j\beta + \frac{1}{r} \right)$$

where $\beta = \omega/v$, $Z_o = \sqrt{\mu/\epsilon}$ and the time factor $e^{j\omega t}$ is understood. The distant, or radiation field is by definition the one obtained by deleting those terms in the set (75) that involve inverse powers of r larger than one. Hence, the radiation field is given by

$$E_\theta = j\beta \, \frac{Z_o \, I d\ell \sin\theta \, e^{-j\beta r}}{4\pi r}$$

$$E_r = 0$$

$$H_\phi = j\beta \, \frac{I d\ell \sin\theta \, e^{-j\beta r}}{4\pi r}$$

Inspection of these simplified expressions shows that

$$E_\theta = Z_o \, H_\phi$$

which means that the radiation field of a current element acts locally like a uniform plane wave and, therefore,

$$W_E - W_M = 0 \tag{74}$$

It is interesting to inquire as to the validity of equation (74) in complicated situations involving actual antennas. Since the total field at any point is obtained by a linear process, namely by integration of the point source solution, the ratio of the total electric to the total magnetic field is maintained equal to Z_o. Therefore, any radiation field satisfies equation (74) automatically regardless of the manner in which the sources distribute themselves in the conductors and, as a result, it is impossible to formulate the strictly distant field on the basis of the variational principle. When using the principle to formulate physically realizable problems, the entire point source solution must always be used in the evaluation of the fields.

CHAPTER V

THE CYLINDRICAL ANTENNA PROBLEM

An illuminating example of the manner in which the variational principle may be applied is furnished by the cylindrical antenna problem. The fundamental reason for the selection of this type of antenna is that, an mentioned earlier, its investigation has been carried out as completely as the presently available tools of mathematical analysis permit. A favorable comparison with existing results should provide the basis for some degree of confidence in the new approach.

Formulation of the Problem.--Consider a hollow cylindrical antenna, radius a, embedded in a dielectric which is characterized by the scalar permittivity ϵ and the scalar permeability μ. The pertinent geometry is shown in Figure 4. Let a total axial current $I(\xi, t - \frac{r}{v})$ be uniformly

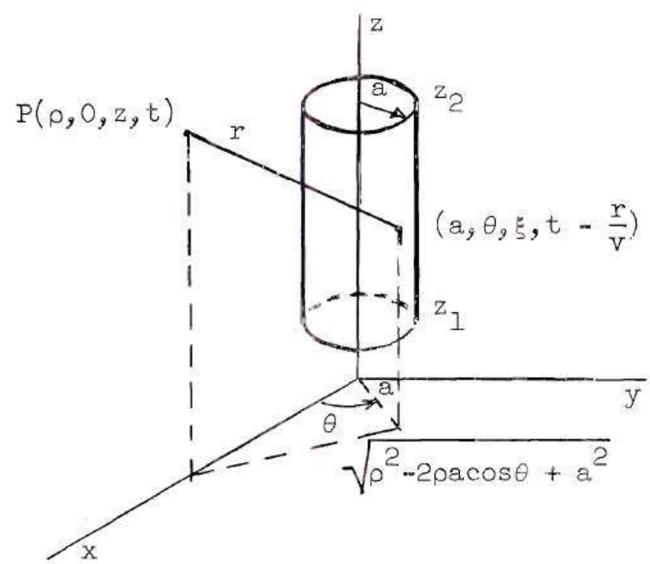

Figure 4. The Geometry of a Cylindrical Antenna

distributed around the antenna. Because of cylindrical symmetry the point of observation P may, without loss in generality, be assumed to be located in the $\theta = 0$ plane. Inspection of Figure 4 shows that the distance from P to the point (a, θ, ξ) on the surface of the cylinder is given by

$$r = \sqrt{(\xi - z)^2 + \rho^2 - 2\rho a \cos\theta + a^2} \tag{76}$$

The vector magnetic potential at P is given by

$$\overline{A} = \frac{\mu}{4\pi}\iiint \frac{\overline{J}}{r}\, d\tau = \overline{u}_z \frac{\mu}{4\pi}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{I(\xi, t - \frac{r}{v})}{r}\frac{d\theta}{2\pi}\, d\xi$$

in which $\overline{u}_z$ is a unit vector pointing in the direction of the positive z axis. Thus, $\overline{A}$ consists of a single component:

$$A_z(\rho, z) = \frac{\mu}{4\pi}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{I(\xi, t - \frac{r}{v})}{r}\frac{d\theta}{2\pi}\, d\xi \tag{77}$$

Likewise, the scalar potential is given by

$$\phi(\rho, z) = \frac{1}{4\pi\epsilon}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{\rho(\xi, t - \frac{r}{v})}{r}\frac{d\theta}{2\pi}\, d\xi \tag{78}$$

where $\rho(\xi, t - \frac{r}{v})$ denotes the linear surface charge distribution in the axial direction of the antenna. The scalar potential (78) may be expressed in terms of $I(\xi, t - \frac{r}{v})$ if use is made of the equation of continuity

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial (t - \frac{r}{v})} = 0$$

which, by virtue of the fact that $\partial/\partial t = \partial/\partial(t - \frac{r}{v})$, may be expressed as

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0 \tag{79}$$

Thus,

$$\rho = - \int^{t} \frac{\partial I(\xi, t - \frac{r}{v})}{\partial \xi} \, dt \tag{80}$$

and

$$\phi(\rho, z) = \frac{1}{4\pi\epsilon} \int_{z_1}^{z_2} \int_{0}^{2\pi} - \frac{1}{r} \left[\int^{t} \frac{\partial I(\xi, t - \frac{r}{v})}{\partial \xi} \, dt \right] \frac{d\theta}{2\pi} \, d\xi \tag{81}$$

The field intensities may be found using the expressions

$$\overline{E} = - \nabla \phi - \frac{\partial \overline{A}}{\partial t}$$

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A}$$

In cylindrical coordinates

$$\nabla \phi = \overline{u}_{\rho} \frac{\partial \phi}{\partial \rho} + \overline{u}_{\theta} \frac{\partial \phi}{\rho \partial \theta} + \overline{u}_{z} \frac{\partial \phi}{\partial z} \tag{82}$$

Therefore,

$$\overline{E} = -\,\overline{u}_\rho \left[\frac{1}{4\pi\epsilon} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{-1}{r} \left(\int^t \frac{\partial I}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right]$$

$$-\,\overline{u}_z \left[\frac{1}{4\pi\epsilon} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial z} \left\{ \frac{-1}{r} \left(\int^t \frac{\partial J}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right]$$

$$-\,\overline{u}_z \left[\frac{\mu}{4\pi} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial t} \left(\frac{I}{r} \right) \frac{d\theta}{2\pi}\, d\xi \right] \tag{83}$$

Now

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A} = \frac{1}{\mu} \left(-\,\overline{u}_\theta \frac{\partial Az}{\partial\rho} \right)$$

$$= -\,\overline{u}_\theta \left[\frac{1}{4\pi} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left(\frac{I}{r} \right) \frac{d\theta}{2\pi}\, d\xi \right] \tag{84}$$

so that

$$32\pi^2\epsilon \left(\frac{1}{2}\epsilon|\overline{E}|^2 - \frac{1}{2}\mu|\overline{H}|^2 \right) = \left[\int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \int^t \frac{\partial I}{\partial\xi}\, dt \right\} \frac{d\theta}{2\pi}\, d\xi \right]^2 \tag{85}$$

$$+ \left[\int_{z_1}^{z_2} \int_0^{2\pi} \left(\frac{\partial}{\partial z} \left\{ \frac{-1}{r} \int^t \frac{\partial I}{\partial\xi}\, dt \right\} + \mu\epsilon \frac{\partial}{\partial t} \left(\frac{I}{r} \right) \right) \frac{d\theta}{2\pi}\, d\xi \right]^2$$

$$-\,\mu\epsilon \left[\int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left(\frac{I}{r} \right) \frac{d\theta}{2\pi}\, d\xi \right]^2$$

Let $f(\rho, z, t)$ denote the right-hand member of (85). The variational principle assets that the integral

$$S[I] = \iiint_R f(\rho, z, t)\, d\rho\, dz\, dt \tag{86}$$

regarded as a functional of I, satisfies the first necessary condition for the existence of an extremum:

$$\delta S[I] = 0 \tag{87}$$

The letter R in (86) denotes a closed region in the ρ, z, t space. Let $I_0(\xi, t - \frac{r}{v})$ be the actual current distribution for which equation (87) is satisfied, and choose any arbitrary function $\eta(\xi, t - \frac{r}{v})$ which is continuously differentiable with respect to its arguments and which, furthermore, vanishes at points to be specified presently. If α is a constant, the function $I_0(\xi, t - \frac{r}{v}) + \alpha\eta(\xi, t - \frac{r}{v})$, when substituted for $I(\xi, t - \frac{r}{v})$ in (85), makes the integral (86) a function of α, once I_0 and η are assigned, and this integral takes on its minimum value when $\alpha = 0$. This fact is expressed mathematically by the condition

$$\left.\frac{dS(\alpha)}{d\alpha}\right|_{\alpha=0} = 0 \tag{88}$$

The procedure involved in the evaluation of (88) will now be illustrated in detail starting with the treatment of the first term in the right-hand member of (85). Thus,

$$S_1[I] = \iiint_R \left[\int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \left(\int^t \frac{\partial I(\xi,\ t - \frac{r}{v})}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right]^2 d\rho\ dz\ dt \tag{89}$$

and

$$S_1(\alpha) = \iiint_R \left[\int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \left(\int^t \frac{\partial\{I_0(\xi,\ t - \frac{r}{v}) + \alpha\eta(\xi,\ t - \frac{r}{v})\}}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right]^2 d\rho dz dt \tag{90}$$

Let

$$T_1(\rho,\ z,\ t) = \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \left(\int^t \frac{\partial I_0(\xi,\ t - \frac{r}{v})}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \tag{91}$$

Then, assuming that the proper requirements for the following differentiation are satisfied, it follows that

$$\left.\frac{dS_1(\alpha)}{d\alpha}\right|_{\alpha=0} = \iiint_R \left[2\ T_1(\rho,z,t) \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \left(\int^t \frac{\partial\eta(\xi,\ t - \frac{r}{v})}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right] d\rho dz dt \tag{92}$$

By virtue of its definition, T_1 is a functional of I_0. However, if I_0 does indeed exist then T_1 is a function of only the independent variables ρ, z, t, as indicated, and therefore (92) may be written as follows:

$$\left.\frac{dS_1(\alpha)}{d\alpha}\right|_{\alpha=0} = 2\iiint_R \left[\int_{z_1}^{z_2} \int_0^{2\pi} T_1(\rho,z,t) \frac{\partial}{\partial\rho} \left\{ \frac{1}{r} \left(\int^t \frac{\partial\eta(\xi,\ t - \frac{r}{v})}{\partial\xi}\, dt \right) \right\} \frac{d\theta}{2\pi}\, d\xi \right] d\rho dz dt \tag{93}$$

Suppose now that $I(\xi,\ t - \frac{r}{v})$ and, consequently, $\eta(\xi,\ t - \frac{r}{v})$ contain time only as a factor $e^{j\omega(t - \frac{r}{v})}$. Then

$$T_1(\rho,\ z,\ t) = \frac{e^{j\omega t}}{j\omega} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\partial}{\partial\rho} \left\{ \frac{e^{-j\beta r}}{r} \ \frac{\partial I_0(\xi)}{\partial\xi} \right\} \frac{d\theta}{2\pi}\, d\xi \qquad (94)$$

where

$$\beta = \frac{\omega}{v} = \omega \sqrt{\mu\epsilon} \qquad (95)$$

Observe that I_0 is a function of a single argument, namely ξ. Set

$$\psi(\rho,\ \theta,\ \xi - z) = \frac{e^{-j\beta r}}{r} \qquad (96)$$

and

$$G(\rho,\ \xi - z) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\rho,\ \theta,\ \xi - z)\, d\theta \qquad (97)$$

Then

$$T_1(\rho,\ z,\ t) = \frac{e^{j\omega t}}{j\omega} \int_{z_1}^{z_2} \frac{\partial G(\rho,\xi - z)}{\partial\rho} \ \frac{\partial I_0(\xi)}{\partial\xi}\, d\xi \qquad (98)$$

and

$$\left.\frac{dS_1(\alpha)}{d\alpha}\right|_{\alpha=0} = 2\iiint_R \left[\int_{z_1}^{z_2} T_1(\rho,z,t)\ \frac{e^{j\omega t}}{j\omega}\ \frac{\partial}{\partial\rho} \left\{ G(\rho,\ \xi - z)\ \frac{\partial\eta(\xi)}{\partial\xi} \right\} d\xi \right] d\rho\, dz\, dt \qquad (99)$$

Suppose now that $\eta(\xi)$ is chosen so that it vanishes at $\xi = z_1$ and $\xi = z_2$. Integration by parts puts (99) in the form

$$\left.\frac{dS_1(\alpha)}{d\alpha}\right|_{\alpha=0} = 2\iiint_R \left[-\int_{z_1}^{z_2} T_1(\rho,z,t)\,\frac{e^{j\omega t}}{j\omega}\,\frac{\partial^2 G(\rho,\ \xi - z)}{\partial\xi\ \partial\rho}\,\eta(\xi)\,d\xi\right] d\rho\ dz\ dt \quad (100)$$

The treatment of the remaining two terms in the right-hand side of (85) parallels that of the first, step by step. For sinusoidal time variations the results are:

$$T_2(\rho,z,t) = \frac{e^{j\omega t}}{j\omega}\int_{z_1}^{z_2}\left[-\frac{\partial G(\rho,\ \xi - z)}{\partial z}\,\frac{\partial I_0(\xi)}{\partial\xi} - \beta^2 G(\rho,\ \xi - z) I_0(\xi)\right] d\xi \quad (101)$$

$$\left.\frac{dS_2(\alpha)}{d\alpha}\right|_{\alpha=0} = 2\iiint_R \left[-\int_{z_1}^{z_2} T_2(\rho,z,t)\,\frac{e^{j\omega t}}{j\omega}\left\{\frac{-\partial^2 G(\rho,\ \xi - z)}{\partial\xi\partial z} + \beta^2 G(\rho,\ \xi - z)\right\}\eta(\xi)\,d\xi\right] d\rho dz dt \quad (102)$$

$$T_3(\rho,\ z,\ t) = -\mu\epsilon\, e^{j\omega t}\int_{z_1}^{z_2}\frac{\partial G(\rho,\ \xi - z)}{\partial\rho}\, I_0(\xi)\, d\xi \quad (103)$$

$$\left.\frac{dS_3(\alpha)}{d\alpha}\right|_{\alpha=0} = 2\iiint_R \left[\int_{z_1}^{z_2} T_3(\rho,\ z,\ t)\, e^{j\omega t}\,\frac{\partial G(\rho,\ \xi - z)}{\partial\rho}\,\eta(\xi)\, d\xi\right] d\rho\ dz\ dt \quad (104)$$

Equation (88) requires that

$$\left.\frac{d(S_1(\alpha) + S_2(\alpha) + S_3(\alpha))}{d\alpha}\right|_{\alpha=0} = 0 \quad (105)$$

That is, the sum of (100), (102) and (104) must vanish. If in these expressions the integrations with respect to the independent variables ρ, z, t involve integrands which are continuous functions of these coordinates, there must exist small regions in which the integrands do not change sign. Consequently, if equation (105) is to be satisfied for arbitrary regions R it is necessary that the sum of the integrands be identically zero; that is,

$$\int_{z_1}^{z_2} \Big\{ - T_1(\rho, z, t) \frac{e^{j\omega t}}{j\omega} \frac{\partial^2 G(\rho, \xi - z)}{\partial\xi \partial\rho} \tag{106}$$

$$- T_2(\rho, z, t) \frac{e^{j\omega t}}{j\omega} \left[\frac{-\partial^2 G(\rho, \xi - z)}{\partial\xi \partial z} + \beta^2 G(\rho, \xi - z) \right]$$

$$+ T_3(\rho, z, t)\, e^{j\omega t} \frac{\partial G(\rho, \xi - z)}{\partial\rho} \Big\} \eta(\xi)\, d\xi = 0$$

By virtue of the Fundamental Lemma of the calculus of variations, the coefficient of $\eta(\xi)$ in the integrand of (106) is identically zero. Thus,

$$- \frac{1}{j\omega} T_1(\rho, z, t) \frac{\partial^2 G(\rho, \xi - z)}{\partial\xi \partial\rho} \tag{107}$$

$$- \frac{1}{j\omega} T_2(\rho, z, t) \left\{ - \frac{\partial^2 G(\rho, \xi - z)}{\partial\xi \partial z} + \beta^2 G(\rho, \xi - z) \right\}$$

$$+ T_3(\rho, z, t) \frac{\partial G(\rho, \xi - z)}{\partial\rho} = 0$$

The factor $e^{j\omega t}$, appearing in the expressions of T_1, T_2, and T_3 may be divided out when the latter are substituted in (107):

$$\frac{\partial^2 G(\rho,\xi-z)}{\partial\xi\partial\rho}\int_{z_1}^{z_2}\frac{\partial G(\rho,\xi-z)}{\partial\rho}\;\frac{\partial I_0(\xi)}{\partial\xi}\,d\xi \tag{108}$$

$$+\left\{-\frac{\partial^2 G(\rho,\xi-z)}{\partial\xi\partial z}+\beta^2 G(\rho,\xi-z)\right\}\int_{z_1}^{z_2}\left\{-\frac{\partial G(\rho,\xi-z)}{\partial z}\;\frac{\partial I_0(\xi)}{\partial\xi}-\beta^2 G(\rho,\xi-z)I_0(\xi)\right\}d\xi$$

$$-\beta^2\,\frac{\partial G(\rho,\xi-z)}{\partial\rho}\int_{z_1}^{z_2}\frac{\partial G(\rho,\xi-z)}{\partial\rho}\,I_0(\xi)\,d\xi=0$$

Since $\frac{\partial G}{\partial z}=-\frac{\partial G}{\partial\xi}$ and $I_0(z_1)=I_0(z_2)=0$, integration of the terms involving $\frac{\partial I_0(\xi)}{\partial\xi}$ by parts puts equation (108) in the form

$$\frac{\partial^2 G(\rho,\xi-z)}{\partial z\partial\rho}\int_{z_1}^{z_2}\frac{\partial^2 G(\rho,\xi-z)}{\partial z\partial\rho}\,I_0(\xi)\,d\xi \tag{109}$$

$$+\left\{\frac{\partial^2 G(\rho,\xi-z)}{\partial z^2}+\beta^2 G(\rho,\xi-z)\right\}\int_{z_1}^{z_2}\left\{\frac{\partial^2 G(\rho,\xi-z)}{\partial z^2}+\beta^2 G(\rho,\xi-z)\right\}I_0(\xi)\,d\xi$$

$$+\beta^2\,\frac{\partial G(\rho,\xi-z)}{\partial\rho}\int_{z_1}^{z_2}\frac{\partial G(\rho,\xi-z)}{\partial\rho}\,I_0(\xi)\,d\xi=0$$

The next objective in the development is to show that equation (109) can be derived by a much shorter route, namely, by specifying the time variation from the outset and by applying the variational principle at a convenient point in free space. Thus, assuming time variations in the form $e^{j\omega(t - \frac{r}{v})}$, equation (77) may be written as follows:

$$A_z(\rho, z) = \frac{\mu}{4\pi} \int_{z_1}^{z_2} G(\rho, \xi - z)\, I_0(\xi)\, d\xi \tag{110}$$

The Lorentz condition

$$\nabla \cdot \overline{A} + \mu\epsilon \frac{\partial \phi}{\partial t} = 0$$

implies that

$$\phi = - \frac{1}{j\omega\mu\epsilon} \frac{\partial A_z}{\partial z} \tag{111}$$

Now, since

$$E = - \nabla\phi - j\omega\overline{A}$$

it follows that

$$\overline{E} = - \left(\overline{u}_\rho \frac{\partial \phi}{\partial \rho} + \overline{u}_\theta \frac{\partial \phi}{\rho \partial \theta} + \overline{u}_z \frac{\partial \phi}{\partial z} \right) - j\omega \, (\overline{u}_z A_z) \tag{112}$$

$$= \overline{u}_\rho \frac{1}{j\, 4\pi\omega\epsilon} \int_{z_1}^{z_2} \frac{\partial^2 G(\rho, \xi - z)}{\partial\rho \partial z} I_0(\xi)\, d\xi$$

$$+ \overline{u}_z \frac{1}{j\, 4\pi\omega\epsilon} \int_{z_1}^{z_2} \left\{ \frac{\partial^2 G(\rho, \xi - z)}{\partial z^2} + \beta^2 G(\rho, \xi - z) \right\} I_0(\xi)\, d\xi$$

Also

$$\bar{H} = \bar{u}_\theta H_\theta = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -\frac{1}{4\pi}\int_{z_1}^{z_2} \frac{\partial G(\rho,\xi-z)}{\partial \rho} I_0(\xi)\, d\xi \tag{113}$$

Expressions (112) and (113) may now be used in the evaluation of

$$32\pi^2\omega^2\epsilon\left(\frac{1}{2}\epsilon|\bar{E}|^2 - \frac{1}{2}\mu|\bar{H}|^2\right) = -\left[\int_{z_1}^{z_2} \frac{\partial^2 G(\rho,\xi-z)}{\partial\rho\partial z} I_0(\xi)\, d\xi\right]^2 \tag{114}$$

$$-\left[\int_{z_1}^{z_2}\left\{\frac{\partial^2 G(\rho,\xi-z)}{\partial z^2} + \beta^2 G(\rho,\xi-z)\right\} I_0(\xi)\, d\xi\right]^2$$

$$-\beta^2\left[\int_{z_1}^{z_2} \frac{\partial G(\rho,\xi-z)}{\partial \rho} I_0(\xi)\, d\xi\right]^2$$

Since (114) involves only the unknown function $I_0(\xi)$ the corresponding Euler equation is

$$\frac{\partial^2 G(\rho,\xi-z)}{\partial\rho\partial z}\int_{z_1}^{z_2} \frac{\partial^2 G(\rho,\xi-z)}{\partial\rho\partial z} I_0(\xi)\, d\xi \tag{115}$$

$$+\left\{\frac{\partial^2 G(\rho,\xi-z)}{\partial z^2} + \beta^2 G(\rho,\xi-z)\right\}\int_{z_1}^{z_2}\left\{\frac{\partial^2 G(\rho,\xi-z)}{\partial z^2} + \beta^2 G(\rho,\xi-z)\right\} I_0(\xi)\, d\xi$$

$$+\beta^2 \frac{\partial G(\rho,\xi-z)}{\partial \rho}\int_{z_1}^{z_2} \frac{\partial G(\rho,\xi-z)}{\partial \rho} I_0(\xi)\, d\xi = 0$$

The function $G(\rho, \xi\text{-}z)$ is continuously differentiable and, therefore, the order of differentiation with respect to its arguments is immaterial. Consequently equations (109) and (115) are identical.

Now the partial derivatives of $G(\rho, \xi\text{-}z)$ with respect to its arguments may be computed from its defining expression (97). Thus,

$$\frac{\partial G(\rho, \xi - z)}{\partial \rho} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \rho}\left(\frac{e^{-j\beta r}}{r} \right) d\theta \tag{116}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r} \right) \frac{dr}{d\rho}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\partial}{\partial r}\left(\frac{e^{-j\ r}}{r} \right) \right\} \frac{\rho - a \cos\theta}{r}\, d\theta$$

Inspection of the definition of r in (76) shows that when $\rho = 0$ both r and $\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r} \right)$ are independent of θ. Hence

$$\frac{\partial G(0, -z)}{\partial \rho} = \frac{1}{2\pi} \left[\frac{1}{r} \frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r} \right) \right]_{\rho=0} \int_0^{2\pi} (-\ a \cos\theta)\, d\theta \tag{117}$$

The value of the integral in (117) is identically zero; so

$$\frac{\partial G(0, \xi - z)}{\partial \rho} \equiv 0 \tag{118}$$

A similar procedure leads to the following partial derivative

$$\frac{\partial G(\rho,\xi-z)}{\partial z} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r}\right)\right\}\frac{-(\xi-z)}{r}\,d\theta \tag{119}$$

$$= \frac{z-\xi}{2\pi}\int_0^{2\pi} N(r)\,d\theta$$

where

$$N(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r}\right) \tag{120}$$

Therefore, the mixed partial derivative

$$\frac{\partial^2 G(\rho,\xi-z)}{\partial\rho\partial z} = \frac{z-\xi}{2\pi}\int_0^{2\pi}\frac{N(r)}{r}\ \frac{\rho-a\cos\theta}{r}\,d\theta \tag{121}$$

evaluated along the z axis vanishes identically:

$$\frac{\partial^2 G(0,\xi-z)}{\partial\rho\partial z} \equiv 0 \tag{122}$$

By virtue of the identities (118) and (122) equation (115) holds along the z-axis if and only if

$$\int_{z_1}^{z_2}\left\{\frac{\partial^2 G(0,\xi-z)}{\partial z^2} + \beta^2 G(0,\xi-z)\right\} I_0(\xi)\,d\xi = 0 \tag{123}$$

or

$$\frac{d^2}{dz^2}\int_{z_1}^{z_2} G(0,\xi-z)\,I_0(\xi)\,d\xi + \beta^2\int_{z_1}^{z_2} G(0,\xi-z)\,I_0(\xi)\,d\xi = 0 \tag{124}$$

where

$$G(0,\xi-z) = \frac{e^{-j\beta r}}{r} \; ; \qquad\qquad r = \left[(\xi-z)^2 + a^2 \right]^{1/2} \qquad (125)$$

Equation (124) may be regarded as a linear differential equation with constant coefficients the unknown, of course, being the integral

$$\int_{z_1}^{z_2} G(0,\xi-z)\; I_0(\xi)\; d\xi \qquad (126)$$

Thus, in the final analysis, $I_0(\xi)$ is the solution of the integral equation

$$\int_{z_1}^{z_2} G(0,\xi-z)\; I_0(\xi)\; d\xi = C\; e^{j\beta z} + D\; e^{-j\beta z} \qquad (127)$$

in which the arbitrary constants C and D may be determined from the boundary conditions $I_0(z_1) = I_0(z_2) = 0$, as specified by Schelkunoff (16). The exactness of these boundary conditions is demanded by the equation of continuity. Thus, from equation (111) it follows that the z-component of the electric field intensity may be expressed as

$$E_z = -\frac{\partial \phi}{\partial z} - j\omega\; A_z = \frac{1}{j\omega\mu} \left[\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z \right] \qquad (128)$$

and, therefore, for a distribution of current parallel to the z-axis,

$$E_z = \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} \left\{ \frac{\partial^2 G(\rho, \xi - z)}{\partial z^2} + \beta^2 G(\rho, \xi - z) \right\} I(\xi)\, d\xi \qquad (129)$$

On the other hand, for sinusoidal variations in time, the continuity equation (79) requires that

$$\rho(\xi) = - \frac{1}{j\omega} \nabla\cdot\bar{J} = - \frac{1}{j\omega} \frac{dI(\xi)}{d\xi} \qquad (130)$$

Therefore, the scalar potential, as defined by (78), becomes

$$\phi(\rho,\ z) = \frac{1}{4\pi\epsilon} \int_{z_1}^{z_2} \int_{0}^{2\pi} \rho(\xi)\ \psi(\rho,\ \theta,\ \xi - z)\ \frac{d\theta}{2\pi}\, d\xi \qquad (131)$$

$$= - \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} \frac{dI(\xi)}{d\xi}\ \ \psi(\rho,\ \theta,\ \xi - z)\ \frac{d\theta}{2\pi}\, d\xi$$

Hence,

$$\frac{\partial\phi(\rho,\ z)}{\partial z} = - \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} \int_{0}^{2\pi} \frac{dI(\xi)}{d\xi}\ \frac{\partial\psi(\rho,\ \theta,\ \xi - z)}{\partial z}\ \frac{d\theta}{2\pi}\, d\xi \qquad (132)$$

Integration by parts results in

$$\frac{\partial\phi(\rho,\ z)}{\partial z} = - \frac{1}{4\pi j\omega\epsilon} \left[\ I(\xi)\ \frac{\partial^2 G(\rho,\ \xi - z)}{\partial\xi\partial z}\ \right]_{z_1}^{z_2} \qquad (133)$$

$$+ \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} I(\xi)\ \frac{\partial^2 G(\rho,\ \xi - z)}{\partial\xi\partial z}\ \ d\xi$$

Since $\frac{\partial \psi}{\partial z} = - \frac{\partial \psi}{\partial \xi}$, it follows that

$$E_z = - \frac{\partial \phi}{\partial z} - j\omega A_z \tag{134}$$

$$= \frac{1}{4\pi j\omega\epsilon} \left[I(\xi) \frac{\partial^2 G(\rho, \xi - z)}{\partial \xi \partial z} \right]_{z_1}^{z_2} + \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} I(\xi) \frac{\partial^2 G(\rho, \xi - z)}{\partial \xi \partial z} d\xi$$

$$- j\omega \frac{\mu}{4\pi} \int_{z_1}^{z_2} I(\xi)\, G(\rho, \xi - z)\, d\xi$$

$$= \frac{1}{4\pi j\omega\epsilon} \left[I(\xi) \frac{\partial^2 G(\rho, \xi - z)}{\partial \xi \partial z} \right]_{z_1}^{z_2}$$

$$+ \frac{1}{4\pi j\omega\epsilon} \int_{z_1}^{z_2} \left\{ \frac{\partial^2 G(\rho, \xi - z)}{\partial z^2} + \beta^2 G(\rho, \xi - z) \right\} I(\xi)\, d\xi$$

Comparison of (134) with (129) shows that the two expressions are identical, as they should be, if and only if in (133) the term

$$\frac{1}{4\pi j\omega\epsilon} \left[I(\xi) \frac{\partial^2 G(\rho, \xi - z)}{\partial \xi \partial z} \right]_{z_1}^{z_2}$$

is identically zero. In general this condition is met only if

$$I(z_2) = I(z_1) = 0 \tag{135}$$

which was to be proved.

Obviously, $I_0(\xi)$ can be determined as the solution to equation (124) only to within a multiplicative constant; this constant, however, may be obtained from the driving conditions of the problem, that is from the requirement that $I_0(\xi)$ have a prescribed value at the driving point of the antenna. In usual theoretical analyses the coordinate system is so oriented, and the antenna so driven that perfect symmetry prevails and $I(-\xi) = I(\xi)$. Then (135) together with the driving conditions specifies two boundary conditions, thus permitting the evaluation of the constants C and D.

Comparison of Results.--The derivation of the integral equation obtained by the method of Hallén (4) may be outlined as follows: If variations in time are sinusoidal the vector potential, as a function of position, is

$$A_z(\rho,\ z) = \frac{\mu}{4\pi} \int_{z_1}^{z_2} G(\rho,\ \xi-z)\ I_0(\xi)\ d\xi \tag{110}$$

For purposes of theoretical analysis the actual situation is idealized to the extent that the antenna is considered as a thin-walled tube, radius a, of infinite conductivity, and is driven by a generator connected electrically to a very narrow gap in the center of the antenna. Thus, the applied electric field is assumed to be zero everywhere except at the driving point.

Now from the relations

$$\overline{E} = -\ \nabla\phi\ -\ j\omega\,\overline{A} \tag{136}$$

$$\nabla\cdot\overline{A} + j\omega\mu\epsilon\phi = 0 \tag{137}$$

it follows that the z-component of the electric field intensity is

$$E_z = -\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2 A_z}{\partial z} + \beta^2 A_z\right) \tag{138}$$

in which β is expressed by (95). Therefore, on the surface of the antenna

$$E^i = -\frac{j}{\omega\mu\epsilon}\left(\frac{\partial^2 A_z}{\partial z} + \beta^2 A_z\right) \tag{139}$$

where E^i denotes the impressed electric field, being finite at the driving point but zero everywhere else. In other words, E^i is an impulse type function with respect to the space variable z and, therefore, the complete solution of the differential equation (139) consists only of the complementary function

$$A_z = C\, e^{j\beta z} + D\, e^{-j\beta z} \tag{140}$$

On the surface of the antenna $\rho = a$; hence, (96) and (97) give

$$\psi(a,\ \theta,\ \xi - z) = \frac{e^{-j\beta r}}{r}$$

and

$$G(a,\ \xi - z) = \frac{1}{2\pi}\int_0^{2\pi} \psi(a,\ \theta,\ \xi - z)\, d\theta \tag{141}$$

in which

$$r = \sqrt{(\xi - z)^2 + a^2 - 2a^2\cos\theta + a^2} = \sqrt{(\xi - z)^2 + 4a^2\sin^2\frac{\theta}{2}} \tag{142}$$

Therefore, combining equations (110), (140), and (141) results in the integral equation

$$\int_{z_1}^{z_2} G(a,\ \xi-z)\ I_0(\xi)\ d\xi = C\ e^{j\beta z} + D\ e^{-j\beta z} \tag{143}$$

Equation (143) is Hallén's integral equation for the cylindrical antenna problem. Comparison of equations (143) and (127) now shows that one difference between them lies in the kernel G. In particular, the difference is in the expressions of r given by (125) and (142). However, solutions of equation (143) are usually based on the assumption that either $r = [(\xi-z)^2 + a^2]^{1/2}$ or, in the extreme, a = 0 and, therefore, $r = |\xi-z|$, a result which is deduced easily from both (125) and (142).

A second difference between equations (143) and (127) is due to the range of the variable z. Thus, whereas in (143) z is restricted to the range $z_1 \leq z \leq z_2$, in equation (127) z may take on any value between minus and plus infinity.

Insofar as the known solution of (143) are concerned neither of the differences cited is of any significant consequence. It is to be noted, however, that equation (143) is based on the assumption that the antenna comprises an infinitely conducting structure and is, therefore, not to be regarded as an exact mathematical model of a physical device. In view of this, it is claimed that equation (127) is exact whereas equation (143) is approximate.

A Critique on the Mechanics of Application of the Principle.--Although equation (109) was derived in accordance with well-established procedures of the calculus of variations, its solution is mathematically possible only if the point of observation is so chosen that only one term appears in its left-hand side. The reason for this is that the dummy variable ξ appears outside the integrals, thus making it impossible to solve equation (109) in the most general case. In order to illustrate the point, consider the problem of attempting to find a function f, continuously differentiable in the interval $0 \leq x \leq \pi$, which minimizes the difference

$$\left[\int_0^{\pi} f(x) \sin x \, dx \right]^2 - \left[\int_0^{\pi} f(x) \cos x \, dx \right]^2 \tag{144}$$

subject to the boundary conditions

$$f(0) = 1 \tag{145}$$

$$f(\pi) = -1 \tag{146}$$

On a trial-and-error basis it is found that expression (144) vanishes identically if

$$f(x) = \sin x + \cos x \tag{147}$$

Thus,

$$\left[\int_0^{\pi} (\sin x + \cos x) \sin x \, dx \right]^2 - \left[\int_0^{\pi} (\sin x + \cos x) \cos x \, dx \right]^2 \tag{148}$$

$$= \left[\int_0^{\pi} (\sin^2 x + \sin x \cos x) \, dx \right]^2 - \left[\int_0^{\pi} (\sin x \cos x + \cos^2 x) \, dx \right]^2$$

$$= \left[\int_0^{\pi} \left(\frac{1 - \cos 2x}{2} + \frac{\sin 2x}{2} \right) dx \right]^2 - \left[\int_0^{\pi} \frac{\sin 2x}{2} + \frac{1 + \cos 2x}{2} \right]^2$$

$$= \left[\left(\frac{x}{2} - \frac{\sin 2x}{4} - \frac{\cos 2x}{4} \right)_0^{\pi} \right]^2 - \left[\left(\frac{- \cos 2x}{4} + \frac{x}{2} + \frac{\sin 2x}{4} \right)_0^{\pi} \right]^2$$

$$= \left(\frac{\pi}{2} \right)^2 - \left(\frac{\pi}{2} \right)^2 \equiv 0$$

Hence, expression (147) is a solution to the problem since it also satisfies the boundary conditions (145) and (146).

The attack of the same problem by the variational method gives

$$2 \int_0^{\pi} f(x) \sin x \, dx \int_0^{\pi} \delta f(x) \sin x \, dx - 2 \int_0^{\pi} f(x) \cos x dx \int_0^{\pi} \delta f(x) \cos x \, dx = 0 \qquad (149)$$

Since the integrals $\left(\int_0^{\pi} f(x) \sin x \, dx \right)$ and $\left(\int_0^{\pi} f(x) \cos x \, dx \right)$ are constants they may be taken inside the second integrals, by which they appear multiplied in (149):

$$\int_0^{\pi} \left[\sin x \int_0^{\pi} f(x) \sin x \, dx - \cos x \int_0^{\pi} f(x) \cos x \, dx \right] \delta f(x) \, dx = 0 \qquad (150)$$

Hence, by the Fundamental Lemma of the calculus of variations

$$\sin x \int_0^{\pi} f(x) \sin x \, dx - \cos x \int_0^{\pi} f(x) \cos x \, dx = 0 \qquad (151)$$

Since the functions sin x and cos x are linearly independent, equation (151) has a solution for all x such that $0 \leq x \leq \pi$ if and only if

$$\int_0^{\pi} f(x) \sin x \, dx = 0 \tag{152}$$

$$\int_0^{\pi} f(x) \cos x \, dx = 0 \tag{153}$$

which is the trivial case. Obviously, the problem cannot be solved in the foregoing manner. A little reflection upon the preceding development shows that the only questionable step might be in considering the integrals $\left(\int_0^{\pi} f(x) \sin x \, dx\right)$ and $\left(\int_0^{\pi} f(x) \cos x \, dx\right)$ as constants, despite the fact that $f(x)$ is a function yet to be determined. Perhaps this is the source of the trouble.

Even though this particular point did not play an important role in the treatment of the cylindrical antenna problem there may be cases in which this hurdle must be overcome, and, therefore, other methods of solution must be known. One possible method which merits particular consideration is the so-called Ritz method (17), which is a procedure in which the function to be determined is approximated as a linear combination of n suitably chosen functions, in the form

$$y \approx C_1 W_1(x) + C_2 W_2(x) + \ldots\ldots + C_n W_n(x) \tag{154}$$

where C_1, C_2, ... C_n represent constants to be evaluated by the variational method. Each function $W_k(x)$ is chosen so that it satisfies the boundary conditions of the problem on an individual basis. When (154) is substituted in the integral to be made stationary and the necessary integrations performed, the quantity to be extremized changes from an integral to a polynomial in the C's. Setting each derivative of this polynomial with respect to the C's equal to zero provides as many algebraic equations as there are C's, and, therefore, their simultaneous solution yields the required values of these unknowns.

In applying the Ritz method to the preceding example the first trial must be of the form

$$f(x) = C_1(\sin x + \cos x) \tag{155}$$

because, first of all, the function $W_1(x) = \sin x + \cos x$ satisfies the prescribed boundary conditions (145) and (146), and, secondly, because the integrands in (144) contain circular functions as factors. Expressions (155) and (147) differ only in the multiplicative constant C_1 which, nevertheless, has no bearing on the fact that (144) vanishes identically when f(x) is specified by (155).

Next, let

$$W_1(x) = \sin x + \cos x \tag{156}$$

$$W_2(x) = \sin 3x + \cos 3x \tag{157}$$

Then

$$f(x) = C_1(\sin x + \cos x) + C_2(\sin 3x + \cos 3x) \tag{158}$$

Adding the function $W_2(x)$ makes no difference in the value of (144), which again is equal to zero, because $W_2(x)$ introduces circular functions of arguments $(2x)$ and $(3x)$ which, when integrated over the interval $[0, \pi]$ obviously vanish. Similar considerations show that if

$$W_1(x) = \sin x + \cos x \tag{159}$$

$$W_2(x) = \sin 3x + \cos 3x \tag{160}$$

$$W_n(x) = \sin(2n + 1)\, x + \cos(2n + 1)\, x \tag{161}$$

so that

$$f(x) = \sum_{k=0}^{n} C_k [\sin(2k + 1)\, x + \cos(3k + 1)\, x] \tag{162}$$

the difference (144) vanishes identically for all n, and, therefore, (162) is the general solution of the problem.

CHAPTER VI

THE APERTURE PROBLEM

Boundary conditions, in the electromagnetic theory sense, are meaningless unless the boundary separates two media of different electromagnetic properties. Therefore, the formulation of an aperture problem in the usual manner is impossible. The new approach presented in this work makes it possible to formulate the problem for the first time on a rigorous mathematical basis.

Consider, for example, an aperture antenna such as a radiating horn. In the case of any secondary source both electric and magnetic currents must be taken into account. However, since Maxwell's equations are satisfied where these currents are considered separately, as shown by Jordan (18), the mathematical expression of the variational principle

$$\delta \iiiint \left(\frac{1}{2}\,\epsilon|\overline{E}|^2 - \frac{1}{2}\,\mu|\overline{H}|^2\right)dx_o\,dy_o\,dz_o\,dt = 0 \qquad (163)$$

is satisfied by both the part of the total field generated by electric currents and that generated by magnetic currents. It is therefore permissible to consider a simple configuration, as in Figure 5, wherein a linear current density $\overline{J}(x)$, directed along the y axis and defined at all points of a two-dimensional region $R(-\frac{a}{2} \leq x \leq \frac{a}{2},\ -\frac{b}{2} \leq y \leq \frac{b}{2})$ with fixed magnitude at some given point in R, is to be determined such that equation (163) is satisfied. By virtue of the nature of the assumed

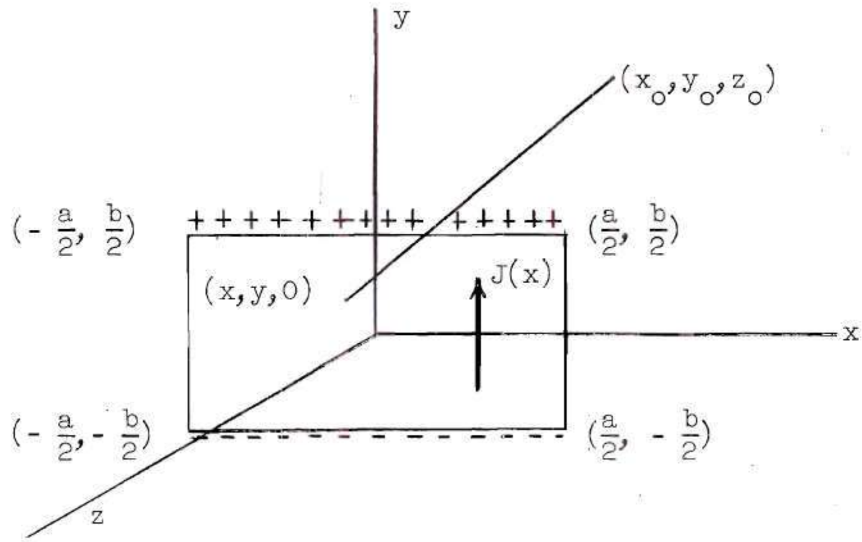

Figure 5. A Rectangular Aperture

current density vector $\overline{J}$, positive and negative line charge densities accumulate, respectively, at the boundaries $y = \frac{b}{2}$ and $y = -\frac{b}{2}$ of the rectangular source.

The vector potential at some point $P(x_o, y_o, z_o)$ of observation is calculated in the usual manner. Assuming sinusoidal time variations, it is given by

$$\overline{A}\,(x_o, y_o, z_o) = \overline{J}\,\frac{\mu}{4\pi} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} J(x)\,\psi(r)\,dy\,dx \qquad (164)$$

in which

$$\psi(r) = \frac{e^{-j\beta r}}{r} \qquad (165)$$

and

$$r = \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z)^2} \qquad (166)$$

Now

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t} = -\nabla\phi - j\omega\bar{A} \tag{167}$$

in which the differentiations denoted by the operator ∇ are with respect to the variables x_o, y_o, z_o. The Lorentz condition in the form

$$\phi = -\frac{1}{j\omega\mu\epsilon}\nabla\cdot\bar{A} \tag{168}$$

allows (167) to be put in the form

$$\bar{E} = -\frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\bar{A}) - j\omega\bar{A} \tag{169}$$

Let A denote the magnitude of the vector $\bar{A}$. Then

$$\nabla\cdot\bar{A} = \frac{\partial A}{\partial y_o} \tag{170}$$

and, therefore,

$$\nabla(\nabla\cdot\bar{A}) = \frac{\partial^2 A}{\partial x_o \partial y_o}\bar{i} + \frac{\partial^2 A}{\partial y_o^2}\bar{j} + \frac{\partial^2 A}{\partial z_o \partial y_o}\bar{k} \tag{171}$$

Since

$$\frac{\partial\psi}{\partial x_o} = -\frac{x - x_o}{r}\frac{d\psi}{dr} \tag{172}$$

with similar relations being satisfied by the other two variables, the third term in the right-hand member of (171) vanishes when $z_o = 0$. Further, assuming that $\bar{J}$ is an even function of x, substitution of (164) in (171) renders the integrand of the x-component of $\nabla(\nabla\cdot\overline{A})$ an odd function of x, so that

$$\frac{\partial^2 A}{\partial x_o \partial y_o} = 0; \qquad x_o = z_o = 0 \tag{173}$$

The preceding thoughts, when compiled, result in

$$\overline{E} = - \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2 A}{\partial y_o^2} + \beta^2 A\right]\overline{j}; \qquad x_o = z_o = 0 \tag{174}$$

In a like manner, it is found that

$$\overline{H} \equiv 0; \qquad x_o = z_o = 0 \tag{175}$$

Now let

$$G(x, y_o) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \psi(r)\, dy \tag{176}$$

where

$$r = \sqrt{(x)^2 + (y - y_o)^2} \tag{177}$$

If the point of observation lies on the y axis, then

$$A(y_o) = \frac{\mu}{4\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} G(x, y_o)\, J(x)\, dx \tag{178}$$

and

$$E(y_o) = - \frac{j}{\omega\mu\epsilon} \left[\frac{d^2 A}{d{y_o}^2} + \beta^2 A \right] \tag{179}$$

Equation (175) expresses the vanishing character of the magnetic field intensity of all points on the y axis. Therefore, $W_M = 0$ and the quantity $(W_E - W_M)$ has its least possible value certainly if the right-hand member of (179) is identically zero:

$$\frac{d^2}{dy_o} \int_{-\frac{a}{2}}^{\frac{a}{2}} G(x,\ y_o)\ J(x)\ dx + \beta^2 \int_{-\frac{a}{2}}^{\frac{a}{2}} G(x,\ y_o)\ J(x)\ dx = 0 \tag{180}$$

The vanishing of E at points where H = 0 is a condition imposed by Maxwell's equations upon time varying fields. The complete solution of equation (180) is obtained through an elementary application of the theory of differential equations. The result is

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} G(x,\ y_o)\ J(x)\ dx = D_1\ e^{j\beta y_o} + D_2\ e^{-j\beta y_o} \tag{181}$$

Equation (181) bears a striking resemblance to the integral equation describing the behavior of the cylindrical antenna whose solution, as pointed out earlier, is not a trivial matter. An interesting method of solving integral equations, such as (181), is the one suggested by Storm (4), and is carried out in the following manner.

Without loss in precision let $\psi(r) = \cos \beta\, r/r$ replace its complex counterpart $\psi(r) = e^{-j\beta r}/r$. Then the function G is real and equation (181) may be replaced by

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} G(x,\, y_o)\; J(x)\; dx = C_1 \cos \beta\, y_o + C_2 \sin \beta\, y_o \tag{182}$$

The Storm method is a procedure in which the unknown function is approximated as a linear combination of a dominant term and a trigonometric series, in the form

$$J(x) = B \sin \beta \left(\frac{a}{2} - |x|\right) + \sum_{n=0}^{N} F_n \cos \frac{(2n + 1)\pi}{a}\, x \tag{183}$$

Substitution of (183) into (182) yields

$$M_o(y_o) B + \sum_{n=0}^{N} S_n(y_o)\; F_n = C_1 \cos \beta\, y_o + C_2 \sin \beta\, y_o \tag{184}$$

where

$$M_o(y_o) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin \beta \left(\frac{a}{2} - |x|\right) G(x,\, y_o)\; dx \tag{185}$$

$$S_n(y_o) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \frac{(2n + 1)\pi}{a}\, x\; G(x,\, y_o)\; dx \tag{186}$$

Following Storm, if the evaluation of the integrals (185) and (186) can be carried out and, subsequently, equation (184) satisfied explicitly at $(N + 2)$ points there will result $(N + 2)$ equations in $(N + 2)$ unknowns, which are B and the $(N + 1)$ of the F_n's. There are two more constants to be evaluated, C_1 and C_2. Since the function $J(x)$ is presumed to be even, the boundary conditions $J(\frac{a}{2}) = J(-\frac{a}{2}) = 0$ furnish enough information for one additional equation. The last equation is obtained from the driving conditions, that is, from the prescribed value of the current density at the driving point. Simultaneous solution of the $(N + 4)$ equations will yield the values of the constants B and F_n which may then be substituted in (183), thus completing the solution of the problem.

CHAPTER VII

CURRENT DISTRIBUTIONS IN LINEAR ARRAYS

The purpose of this chapter is to examine the distribution of current in linear antenna arrays, subject to conditions that are to be specified presently, by a method of attack which is based upon the variational principle the general treatment of which is the subject of this study.

Three-Element Array.--Let an array of three short radiators, spaced equally along a straight line, as shown in Figure 6, be so excited that the current in the center element is $I_0\underline{/0}$ and the current in the outer two radiators $I_1\underline{/\alpha}$. Let k denote the ratio I_1/I_0. Inspection of

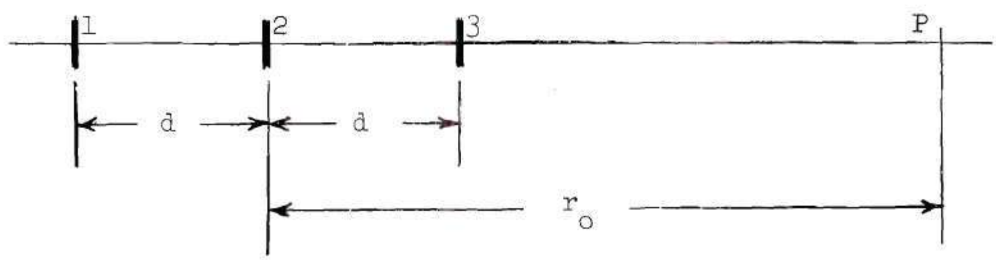

Figure 6. A Three-Element Array

equations (75) shows that at every point P on the axis of the array the radial component of the electric field generated by each of the radiators is zero, and so the electric field vector is parallel to the radiators; the magnetic field vector is normal to the plane of the paper. If the point P is sufficiently remote from the array, then, insofar as the magnitudes of the fields are concerned, the distance from the point P to each of the elements may be approximated by r_o. Under the foregoing assumptions the fields generated by the elements of the array are specified by the expressions

$$\begin{aligned} E_1 &= k\,E\,e^{j(\alpha - \beta d)} \\ E_2 &= E \\ E_3 &= k\,E\,e^{j(\alpha + \beta d)} \end{aligned} \tag{187}$$

and

$$\begin{aligned} H_1 &= k\,H\,e^{j(\alpha - \beta d)} \\ H_2 &= H \\ H_3 &= k\,H\,e^{j(\alpha + \beta d)} \end{aligned} \tag{188}$$

where the subscripts 1, 2, 3 refer to the respective elements of the array, and E and H are obtained from the set (75) with $\theta = 90°$ and $I = I_0$. The total field is specified by the sums

$$E_1 + E_2 + E_3 = \left[\, k\,e^{j(\alpha - \beta d)} + 1 + k\,e^{j(\alpha + \beta d)} \,\right] E \tag{189}$$

$$H_1 + H_2 + H_3 = \left[\, k\,e^{j(\alpha - \beta d)} + 1 + k\,e^{j(\alpha + \beta d)} \,\right] H \tag{190}$$

If the current I_0 is included in the bracketed expression appearing in equations (189) and (190), the relative magnitude of

$$W_E - W_M = \frac{1}{2}\,\epsilon\,|\overline{E}|^2 - \frac{1}{2}\,\mu\,|\overline{H}|^2 = \frac{\epsilon}{2}\left[\, |\overline{E}|^2 - Z_0{}^2\,|\overline{H}|^2 \,\right] \tag{191}$$

depends strictly upon the factor

$$k\ I_0\ e^{j(\alpha\ -\ \beta d)} + I_0 + k\ I_0\ e^{j(\alpha\ +\ \beta d)} \tag{192}$$

the absolute magnitude of which is equal to

$$I_0^{\ 2} + 4\ I_0\ I_1 \cos\alpha \cos\beta d + 4\ I_1^{\ 2} \cos^2\beta d \tag{193}$$

The development down to this point has been of an introductory nature. Now the problem may be stated thus: If only the center element is driven, what is the magnitude and phase of the current induced in the parasitic elements? What is, in other words, the complex current $I_1\underline{/\alpha}$ induced in the outer elements if the center element is driven with a current $I_o\underline{/0}$?

The solution proceeds as follows. In the first place, observing that the difference (191) must have the least possible value reduces the problem to one of finding the extremals of (193) subject to the constraint

$$I_0 = \text{constant} \tag{194}$$

Therefore, setting the derivatives of (193) with respect to I_1 and α each equal to zero results in the two algebraic equations

$$4I_0 \cos\alpha \cos\beta d + 8I_1 \cos^2\beta d = 0 \tag{195}$$

$$-\ 4I_0\ I_1 \sin\alpha \cos\beta d = 0 \tag{196}$$

which may be solved simultaneously for I_1 and α. For nonvanishing values of $\cos\beta d$, equation (196) demands that

$$\sin \alpha = 0$$

and, therefore,

$$\alpha = n\pi \qquad (n = 0, 1, 2, \ldots)$$

Equation (195) gives

$$I_1 = -\frac{I_0}{2}\frac{\cos \alpha}{\cos \beta d}$$

Now, when n is odd, $\cos \alpha = -1$ and

$$I_1 = \frac{I_0}{2 \cos \beta d} \tag{197}$$

When n is even, $\cos \alpha = 1$ and

$$I_1 = -\frac{I_0}{2 \cos \beta d} \tag{198}$$

Hence, for a fixed value of $\cos \beta d$ equations (197) and (198) are completely equivalent to the statement that

$$I_1 = \frac{I_0}{2 \cos \beta d}$$

and that $\alpha = 180°$ when $\cos \beta d$ is positive, or $\alpha = 0°$ when $\cos \beta d$ is negative. Figure 7 depicts graphically the functional relation between I_1 and (βd).

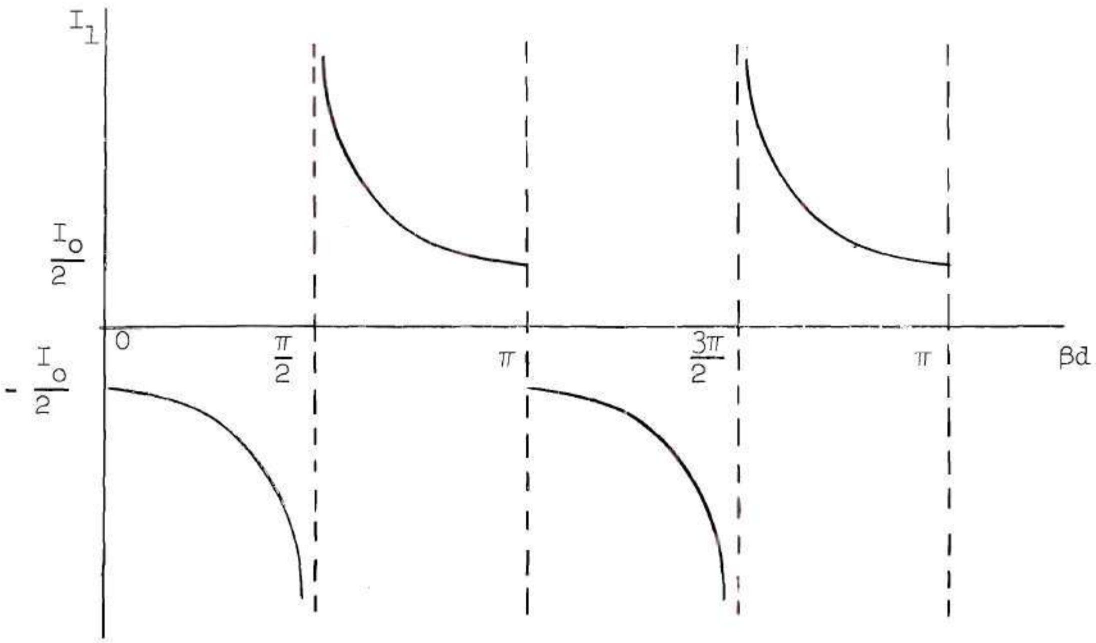

Figure 7. Variation of the Current Induced in the Parasitic Elements of a Three-Element Antenna Array with Changes in Spacing Between Elements

Under the foregoing conditions expression (193) is identically zero, and so it is concluded that the natural behavior of the three-element array is characterized by a vanishing field at distant points along the axis of the array.

Five-Element Array.--The treatment of an array with five elements follows much along the same lines with the exception that the development has to be modified in order to establish one additional constraint corresponding to the additional degree of freedom possessed by a five-element array as compared to the three-element array assuming, of course, that both arrays are center-driven. The work which follows shows that the search for this additional constraint leads first to an impasse, but finally ends after an analysis of a four-element array.

Figure 8 shows the pertinent geometry of a five-element array. As before, P is some distant point along the axis of the array. The expressions which correspond to (192) and (193) are respectively

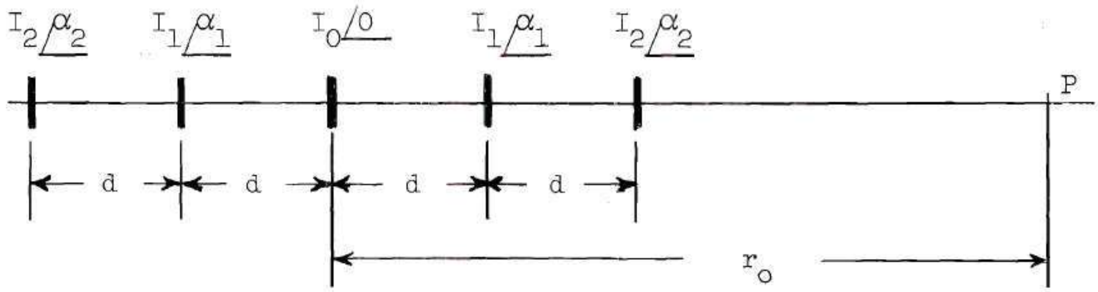

Figure 8. A Five-Element Array

$$I_2 e^{j(\alpha_2 - 2\beta d)} + I_1 e^{j(\alpha_1 - \beta d)} + I_0 + I_1 e^{j(\alpha_1 + \beta d)} + I_2 e^{j(\alpha_2 + 2\beta d)} \tag{199}$$

and

$$I_0^2 + 4I_2^2 \cos^2 2\beta d + 4I_1^2 \cos^2 \beta d + 8I_2 I_1 \cos(\alpha_2 - \alpha_1) \cos \beta d \cos 2\beta d \tag{200}$$

$$+ 4I_0 I_2 \cos \alpha_2 \cos 2\beta d + 4I_0 I_1 \cos \alpha_1 \cos \beta d$$

Suppose

$$I_0 = \text{constant} \tag{201}$$

As before, the first step is to set the derivatives of (200) with respect to the unknowns equal to zero:

$$8I_2\cos^2 2\beta d + 4I_0\cos\alpha_2\cos 2\beta d + 8I_1\cos(\alpha_2 - \alpha_1)\cos\beta d\cos 2\beta d = 0$$

$$8I_1\cos^2\beta d + 4I_0\cos\alpha_1\cos\beta d + 8I_2\cos(\alpha_2 - \alpha_1)\cos\beta d\cos 2\beta d = 0$$

$$- 8I_2I_1\sin(\alpha_2 - \alpha_1)\cos\beta d\cos 2\beta d - 4I_0I_2\sin\alpha_2\cos 2\beta d = 0 \qquad (202)$$

$$8I_2I_1\sin(\alpha_2 - \alpha_1)\cos\beta d\cos 2\beta d - 4I_0I_1\sin\alpha_1\cos\beta d = 0$$

Assuming $\cos\beta d \neq 0$, $\cos 2\beta d \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$ the preceding set may be simplified and put in the form

$$2I_2\cos 2\beta d + I_0\cos\alpha_2 + 2I_1\cos(\alpha_2 - \alpha_1)\cos\beta d = 0 \qquad (203)$$

$$2I_1\cos\beta d + I_0\cos\alpha_1 + 2I_2\cos(\alpha_2 - \alpha_1)\cos 2\beta d = 0 \qquad (204)$$

$$2I_1\sin(\alpha_2 - \alpha_1)\cos\beta d + I_0\sin\alpha_2 = 0 \qquad (205)$$

$$2I_2\sin(\alpha_2 - \alpha_1)\cos 2\beta d - I_0\sin\alpha_1 = 0 \qquad (206)$$

Equations (205) and (206) give the relations

$$I_1 = - I_0 \frac{\sin\alpha_2}{2\sin(\alpha_2 - \alpha_1)\cos\beta d} \qquad (207)$$

$$I_2 = I_0 \frac{\sin\alpha_1}{2\sin(\alpha_2 - \alpha_1)\cos 2\beta d} \qquad (208)$$

Substitution of (207) and (208) in equations (203) and (204) shows that the latter are satisfied identically. Therefore, unless

$$\sin(\alpha_2 - \alpha_1) = 0 \qquad (209)$$

the set (203) through (206) possesses no unique solution. In the event that condition (209) is true, then in conjunction with equations (205) and (206) it requires that

$$\alpha_1 = m\pi \qquad (m = 0, 1, 2, 3, \ldots) \tag{210}$$

$$\alpha_2 = n\pi \qquad (n = 0, 1, 2, 3, \ldots) \tag{211}$$

which means that the current I_1 and I_2 are either in phase or 180° out of phase. If they are in phase, equations (203) and (204) become

$$2I_2 \cos 2\,\beta d + I_0 \cos \alpha_2 + 2I_1 \cos \beta d = 0 \tag{212}$$

$$2I_1 \cos \beta d + I_0 \cos \alpha_1 + 2I_2 \cos 2\,\beta d = 0 \tag{213}$$

and, therefore, since $\cos \alpha_1 = \cos \alpha_2$, equations (212) and (213) are not independent. Either of them states that

$$(I_1 \cos \beta d + I_2 \cos 2\beta d) = -\frac{I_0}{2} \cos \alpha_1 \tag{214}$$

Observe, that upon setting $I_2 = 0$, equation (214) reduces to equation (197).

According to equation (214) the problem of a five-element array, subject to the condition (201), has an infinite number of solutions, a fact which can be predicted from the outset since the five-element array has two additional degrees of freedom when compared with the three-element array, and yet the two systems have identical constraints. Picking either I_1 or I_2 at random reduces the problem to the class of the preceding problem, namely that of the three-element array which, as already shown, has a unique solution once I_0 is fixed. Similar comments apply if I_1 and I_2 are 180° out of phase.

Evidently, what is needed here is a reduction in the number of degrees of freedom of the system. Among the several possibilities imaginable one is to fix the angles α_1 and α_2. Then the currents I_1 and I_2 are constrained by equations (203) and (204) which may now be written thus:

$$I_1 \cos \beta d \cos (\alpha_2 - \alpha_1) + I_2 \cos 2 \beta d = - \frac{I_0}{2} \cos \alpha_2 \tag{203a}$$

$$I_1 \cos \beta d + I_2 \cos 2 \beta d \cos (\alpha_2 - \alpha_1) = - \frac{I_0}{2} \cos \alpha_1 \tag{204a}$$

When solved simultaneously, equations (203a) and (204a) yield

$$I_1 = - \frac{I_0}{2} \frac{\sin \alpha_2}{\cos \beta d \sin (\alpha_2 - \alpha_1)} \tag{215}$$

$$I_2 = \frac{I_0}{2} \frac{\sin \alpha_1}{\cos 2 \beta d \sin (\alpha_2 - \alpha_1)} \tag{216}$$

Comparison shows that equations (215) and (216) are identical with equations (207) and (208), respectively. In fact, these same relations are obtained upon fixing I_1 and I_2 rather than α_1 and α_2, because only two of the four equations (203) through (206) are idependent. So, again equation (214) applies and, as before, it is impossible to find unique values for I_1 and I_2.

Observe, however, that an attempt has been made to solve a problem involving coupled systems when the coupling coefficients are unknown. What is lacking here is the type of information implied by, for instance, equations (197) and (198) regarding the three-element array. Specifically, the ratio

$$\left|\frac{I_1}{I_0}\right| = \frac{1}{2\cos\beta d} \tag{217}$$

is a current transfer ratio which conveys information as it relates to the system much like a transfer, or mutual, impedance does in circuit analysis. Therefore, the next obvious step in the development is to determine the current transfer ratio I_2/I_1 of the four-element array obtained when the center element in Figure 8 is deleted. Accordingly, upon setting $I_0 = 0$ expression (200) becomes

$$4I_2^{\,2}\cos^2 2\beta d + 4I_1^{\,2}\cos^2\beta d + 8I_2I_1\cos(\alpha_2 - \alpha_1)\cos\beta d\cos 2\beta d \tag{218}$$

Without loss in generality α_1 may be taken zero, and, therefore, expression (218), after division through by four, may be put in the form

$$I_2^{\,2}\cos^2 2\beta d + I_1^{\,2}\cos^2\beta d + 2I_2I_1\cos\alpha_2\cos\beta d\cos 2\beta d \tag{219}$$

Differentiating (219) first with respect to I_2 and then with respect to α_2, and setting each result equal to zero gives

$$2I_2\cos^2 2\beta d + 2I_1\cos\alpha_2\cos\beta d\cos 2\beta d = 0 \tag{220}$$

$$- 2 I_2 I_1 \sin \alpha_2 \cos\beta d \cos 2\beta d = 0 \tag{221}$$

Therefore,

$$I_2 = - I_1 \frac{\cos \alpha_2 \cos\beta d}{\cos 2 \beta d} \tag{222}$$

$$\sin \alpha_2 = 0 \tag{223}$$

It is clear now that, depending on the signs of cosβd and cos2βd, the current I_2 is either in phase or 180° out of phase with I_1. The current transfer ratio may be obtained easily from equation (222):

$$\frac{I_2}{I_1} = \frac{\cos\beta d}{\cos 2\beta d} \tag{224}$$

Now the relation (224) may be substituted in equation (214), with the latter modified to take into account the 180° phase difference. The result is

$$I_1 = \frac{I_0}{4 \cos\beta d} \tag{225}$$

$$I_2 = \frac{I_0}{4 \cos 2\beta d} \tag{226}$$

with $\alpha_1 = 180°$, $\alpha_2 = 180°$. Under these conditions expression (210) is identically zero.

The preceding analysis may be extended easily to linear, symmetrical arrays consisting of any odd number of short radiators.

CHAPTER VIII

FIELD THEORY, CIRCUIT RELATIONS, AND THE DYNAMICAL METHOD

The Connection Between Field and Circuit Theories.--The relation between circuit concepts and field theory has been the subject of extensive treatments in the literature of the past (19) (20). Thus, it has been shown that Kirchhoff's current law expresses the principle of conservation of charge, and that Kirchhoff's voltage law is a mathematical extension of the integral form of Maxwell's emf equation, that is, Faraday's law, on the premise that the current existing in a given closed circuit distributes itself uniformly throughout the conductors' cross section and has the same magnitude in all parts of the circuit. Such an assumption is of extreme importance in that it demands the circuit inductance to be a function of the circuit geometry alone and independent of the current; further, it neglects the effects of retardation, namely radiation, thus implying that circuit dimensions are negligible compared to wavelength; finally, it neglects the internal reactance of the conductors. These approximations are good only at low frequencies and, therefore, Kirchhoff's voltage law breaks down at higher frequencies unless provisions are made to incorporate properly selected lumped elements in the equivalent network. Thus, the two main laws of the electric circuit are just special cases of Maxwell's equations.

Circuit Aspects of the Variational Principle.--Inasmuch as the variational principle, as already shown, is based upon Maxwell's equations, a reasonable question that might be asked is, can this principle be used in place of Kirchhoff's voltage law? The answer is yes.

By way of illustration, consider first a simple series LC circuit and let q be the instantaneous charge on the plates of the condenser. Let $\dot{q}$ denote the time rate of change of q, i.e., $\dot{q} = dq/dt$. (In the language of dynamics q is the generalized coordinate of the problem.) Then

$$W_E = \frac{q^2}{2C} \tag{227}$$

is the instantaneous energy stored in the capacitance, and

$$W_M = \frac{1}{2} L(\dot{q})^2 \tag{228}$$

is the instantaneous energy stored in the inductance. Now form the integral

$$\int_{t_1}^{t_2} (W_E - W_M)dt = \int_{t_1}^{t_2} \left[\frac{q^2}{2C} - \frac{L}{2} (\dot{q})^2 \right] dt \tag{229}$$

and inquire as to what function $q(t)$ extremizes the value of (229). Thus posed, the variational problem implies the following Euler equation:

$$\frac{\partial}{\partial q} \left[\frac{q^2}{2C} - \frac{L}{2}(\dot{q})^2 \right] - \frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}} \left[\frac{q^2}{2C} - \frac{L}{2}(\dot{q})^2 \right] \right\} = 0 \tag{230}$$

which, when simplified, becomes

$$\ddot{q} + \frac{1}{LC} q = 0 \tag{231}$$

an equation that can be written down easily through an elementary application of Kirchhoff's voltage law.

The preceding example illustrates the application of the variational principle in the case of a conservative system. The treatment of the general electric circuit problem, by analogy with Hamilton's principle in its most general form as given by Hildebrand (21), may be based upon the equation

$$\delta \int_{t_1}^{t_2} W_M \, dt + \int_{t_1}^{t_2} (V_1 \, \delta q_1 + V_2 \, \delta q_2 + \ldots . + V_n \, \delta q_n) dt = 0 \tag{232}$$

in which the q's are the generalized coordinates, and the V_i's the generalized forces of the system corresponding to the q_i's. Equation (232) implies as many simultaneous equations as there are independent q_i's. Its application may be illustrated by considering a series RLC circuit excited by a source of E volts. In this case the generalized coordinate q is the charge on the condenser plates,

$$W_M = \frac{1}{2} L \dot{q}^2 \tag{233}$$

and the generalized force

$$V = E - R\dot{q} - \frac{q}{C} \tag{234}$$

Since equation (232) implies the following system of so-called Lagrange's equations

$$\frac{d}{dt}\left[\frac{\partial W_M}{\partial \dot{q}_i} \right] - \frac{\partial W_M}{\partial q_i} = V_i \qquad (i = 1, 2, \ldots n) \tag{235}$$

substitution of (233) and (234) in (235) yields the familiar equilibrium equation of the circuit

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = E \tag{236}$$

Equation (235) may be put in a more recognizable form upon letting

$$W_E = \frac{1}{2}\frac{q^2}{C} \tag{237}$$

$$V = E \tag{238}$$

$$P = \frac{1}{2} R\dot{q}^2 \tag{239}$$

Then a system with a single degree of freedom, such as the single-loop circuit being considered, is characterized by the condition

$$\delta \int_{t_1}^{t_2} (W_M - W_E)\, dt + \int_{t_1}^{t_2} \left(E - \frac{\partial P}{\partial \dot{q}}\right) \delta q\, dt = 0 \tag{240}$$

and the corresponding Lagrange equation becomes

$$\frac{d}{dt}\left[\frac{\partial(W_M - W_E)}{\partial \dot{q}}\right] - \frac{\partial(W_M - W_E)}{\partial q} + \frac{\partial P}{\partial \dot{q}} = E \tag{241}$$

In the past, several investigators have shown that the electric circuit, considered as a dynamic system, may be analyzed by means of Lagrange's equation. Since a treatment of the same topic here would seem redundant, the interested reader is referred to the existing literature (22)(23)(24). It is to be shown, however, that equation (232) may be used to derive the theorem of constant flux linkages. In this regard, the point of departure is an example which illustrates an application of the theorem.

Thus, consider the circuit of Figure 9, and let it be required to find the current in the circuit as a function of time after the switch

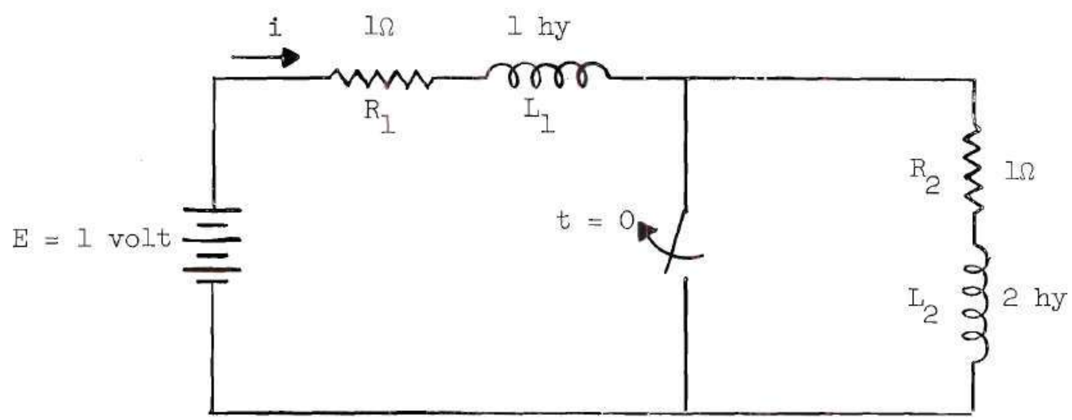

Figure 9. An Electric Circuit Undergoing a Transient

is opened at $t = 0$. The differential equation describing the network's behavior for $t > 0$ is

$$2i + 3\,\frac{di}{dt} = 1 \tag{242}$$

The complementary function of the differential equation (242) is $De^{-\frac{2}{3}t}$, where D is a constant which is to be evaluated, and its particular integral is $\frac{1}{2}$. Thus, the complete solution is

$$i(t) = \frac{1}{2} + De^{-\frac{2}{3}t} \tag{243}$$

Before the switch is opened, $i = 1$ ampere and the current through L_2 is zero. Now the question is, what $i(0)$ should be used to evaluate the constant D? Should $i(0)$ be zero or one? The answer to this question is furnished by the theorem of constant flux linkages which demands that the flux linkages of the circuit be maintained constant. Thus, evaluating the flux linkages for $t > 0$

$$L_1 \; i + L_2 \; i = (1)(i) + (2)(i) = 3i \tag{244}$$

and for $t < 0$

$$L_1 \; (1) + L_2 \; (0) = (1)(1) = 1 \tag{245}$$

and equating the numerical results (244) and (245) yields

$$i(0) = \frac{1}{3} \tag{246}$$

Evaluation of D in equation (24) is now a simple matter, and the complete solution becomes

$$i = \frac{1}{2} - \frac{1}{6} \, e^{-\frac{2}{3} t} \; ; \qquad t > 0 \tag{247}$$

In his book, Bewley (25) gives credit to Doherty (26) for being the first to formulate the theorem of constant flux linkages and to use it extensively in the study of machine transients. However, since Doherty's derivation of the theorem tacitly implies that circuits must contain no capacitances and no applied voltages but, aside from inductances, only resistances which are negligibly small, the application of the theorem appears to be restricted to a class of circuits which does not include the type of circuit considered earlier.

These restrictions may be eliminated as follows: As before, let it be postulated that magnetic energy in electrical circuits is analogous to kinetic energy in mechanical systems, and consider a circuit having a generalized force V_i, self inductance L_{ii} and mutual inductances M_{ik} with neighboring circuits. Then the magnetic energy stored in the circuit is

$$W_{M_i} = \frac{1}{2} L_i \dot{q}_i^{\,2} + \sum_k \frac{1}{2} M_{ik} \dot{q}_k^{\,2} \qquad (248)$$

When (248) is substituted in equation (232) and the indicated manipulations performed, it is found that

$$\frac{d}{dt}\left[L_{ii} q_i + \sum_k M_{ik} \dot{q}_k^{\,2} \right] = V_i \qquad (249)$$

Integration of equation (249) over the time interval $[t_o, t]$ gives

$$\left[L_{ii} \dot{q}_i + \sum_k M_{ik} \dot{q}_k \right]_{t_o}^{t} = \int_{t_o}^{t} V_i \, dt \qquad (250)$$

Unless the generalized force V_i is an impulse centered at t_o the right-hand member of equation (250) vanishes as $t \to t_o$ and, therefore,

$$\lim_{t \to t_o} \left[L_{ii} \dot{q}_i + \sum_k M_{ik} \dot{q}_k \right]_{t_o}^{t} = 0 \qquad (251)$$

which is a mathematical statement of the fact that the flux linkages of the circuit must be maintained constant. This proves the theorem.

It is important to note that V_i in equation (250) can include terms due to resistance, capacitance and source voltage. If, in particular, the source voltage happens to be an impulse, then equation (250) may be used to derive the initial conditions.

Suppose, for example, that a series RLC circuit, initially at rest, is excited by a unit impulse centered at $t_o = 0$. Then equation (250) gives

$$L\dot{q}(0) = 1$$

or the well-known initial condition

$$i(0) = \frac{1}{L}$$

In the most general case, it is possible to arrive at the initial conditions by solving simultaneously the set of algebraic equations implied by equation (250). If the circuit is characterized by N independent q_i's, there will accordingly be N independent equations to be solved simultaneously. The circuit does not necessarily have to be initially at rest, for, if any of the q_i's have non-zero values prior to $t = t_o$, the corresponding equations will be affected only to the extent that their right-hand members will contain additional terms, but they will still remain linear in form just as they would if all of the q_i's are zero prior to $t = t_o$.

CHAPTER IX

CONCLUSIONS

The analytical problem treated in this work demonstrates conclusively that certain problems in electromagnetic theory may be solved by integrating the point source solution of Maxwell's equations subject to the condition that in free space the difference in electric and magnetic energy contents has the minimum value allowed by the boundary conditions. The application of the method to the time-honored cylindrical antenna problem leads to an integral equation which is an improved version of the equation obtained when the same problem is treated by the method of Hallen. The aperture problem, formulated here for the first time on a rigorous basis, is found to lead to an integral equation which is similar in form to the equation describing the behavior of the cylindrical antenna. The natural behavior of linear arrays, consisting of an odd number of equally-spaced short antennas, is characterized by the fact that the far field along the axis of the array vanishes identically, provided that only the center element is driven. By observing that both Kirchhoff's equations and the equation expressing the variational principle are different manifestations of Maxwell's equations, it is not surprising to find that electric networks may be treated as dynamical systems. In this connection, a general proof of the theorem of constant flux linkages is found to eliminate certain ambiguities which restrict the applications of the theorem.

Since the complete development of the new approach to the electromagnetic problem is a complicated task, the scope of this work was accordingly restricted to the treatment of a few simple antenna configurations. The new method is, to be sure, no less complicated than the classical method, but it can be used in formulating some problems where the old method fails to do so.

APPENDIX

BASIC CONCEPTS OF THE CALCULUS OF VARIATIONS

<u>The Simplest Problem of the Calculus of Variations</u>.--Consider the integral

$$I = \int_{x_1}^{x_2} f(x,\ y,\ y')dx \tag{252}$$

in which the integrand f is a continuous function of its arguments, namely the variable x, the function y(x), and its derivative y'(x), and it has continuous second derivatives, either mixed or unmixed, with respect to all its arguments for all y' and for x and y in the region defined by

$$x_1 \leq x \leq x_2$$

$$y_1(x) < y(x) < y_2(x)$$

The problem is to determine y(x) such that the integral (252) has a relative extremum, either maximum or minimum, in this region subject to the boundary conditions

$$y(x_1) = K_1 \qquad\qquad y(x_2) = K_2$$

where K_1, K_2 are given constants. For the purposes of this discussion, let it be required that the so-called "basic" integral I is to be minimized. Then the question is, among all the so-called "admissible comparison functions" y(x) is there a particular function $y_o(x)$ such that

$$I[y_o] \leq I[y] \tag{253}$$

for all $y(x)$? The brackets in (253) are used to signify the fact that I is a "functional" of y.

Suppose $y_o(x)$ is, indeed, the minimizing function. Then

$$\int_{x_1}^{x_2} f(x,\ y_o,\ y_o')\ dx \leq \int_{x_1}^{x_2} f(x,\ y,\ y')\ dx \tag{254}$$

for every admissible comparison function $y(x)$. Let $\eta(x)$ be a given function which has continuous second partial derivatives on $[x_1,\ x_2]$, and which vanishses at x_1 and x_2. Then, for any constant α, the function $y_o(x) + \alpha\eta(x)$ is an admissible comparison function because, first of all, it has continuous second partial derivatives, being the sum of two functions satisfying this condition; secondly, $y_o(x_1) + \alpha\eta(x_1) = K_1$ and $y_o(x_2) + \alpha\eta(x_2) = K_2$; and, finally, $y_1(x) < y_o(x) + \alpha\eta(x) < y_2(x)$ when α is chosen conveniently small. As a result,

$$\int_{x_1}^{x_2} f(x,\ y_o,\ y_o')\ dx \leq \int_{x_1}^{x_2} f(x,\ y_o + \alpha\eta,\ y_o' + \alpha\eta')dx \tag{255}$$

The right-hand member of (255) is a function of α, once y and η are assigned, and has a minimum when $\alpha = 0$. Thus, letting $I(0)$ and $I(\alpha)$ denote respectively the left- and right-hand members of (255) the latter becomes

$$I(0) \leq I(\alpha) \tag{256}$$

and, therefore,

$$\frac{dI(\alpha)}{d\alpha} = 0 \qquad \text{when } \alpha = 0 \tag{257}$$

That is,

$$\frac{dI(0)}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f(x,y_o,y_o')}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f(x,y_o,y_o')}{\partial y_o}\frac{d(y_o+\alpha\eta)}{d\alpha} + \frac{\partial f(x,y_o,y_o')}{\partial y_o'}\frac{d(y_o+\alpha\eta')}{d\alpha}\right] dx \tag{258}$$

$$= \int_{x_1}^{x_2} [\eta \frac{\partial}{\partial y_o} f(x,y_o,y_o') + \eta' \frac{\partial}{\partial y_o'} f(x,y_o,y_o')]\, dx = 0$$

Integration by parts now gives

$$\int_{x_1}^{x_2} \eta' \frac{\partial}{\partial y_o'} f(x,y_o,y_o')dx = \eta \frac{\partial}{\partial y_o'} f(x,y_o,y_o')\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left[\frac{\partial}{\partial y_o'} f(x,y_o,y_o')\right] dx \tag{259}$$

But since, by assumption, $\eta(x_1) = \eta(x_2) = 0$, the first term in the right-hand member of (259) vanishes, so that equation (258) may now be written as

$$\frac{dI(0)}{d\alpha} = \int_{x_1}^{x_2} \eta \left\{\frac{\partial}{\partial y_o} f(x,y_o,y_o') - \frac{d}{dx}\left[\frac{\partial}{\partial y_o'} f(x,y_o,y_o')\right]\right\} dx = 0 \tag{260}$$

The next step in the development involves an application of the Fundamental Lemma of the calculus of variations which may be stated thus: Let x_1 and x_2 be real constants with $x_1 < x_2$. If $F(x)$ is continuous on

$[x_1, x_2]$ and if $\int_{x_1}^{x_2} \eta(x)\, F(x)\, dx = 0$ for every function $\eta(x)$ which has continuous second partial derivatives on $[x_1, x_2]$, and for which $\eta(x_1) = \eta(x_2) = 0$, then the function $F(x)$ is identically zero on $[x_1, x_2]$. Accordingly, the integrand in equation (260) must vanish identically:

$$\frac{\partial}{\partial y_o} f(x,y_o,y_o') - \frac{d}{dx}\left[\frac{\partial}{\partial y_o'} f(x,y_o,y_o')\right] = 0 \qquad (261)$$

which means that, if a y_o satisfying (254) exists, then it is a solution of the differential equation

$$\frac{\partial}{\partial y} f(x,y,y') - \frac{d}{dx}\left[\frac{\partial}{\partial y'} f(x,y,y')\right] = 0 \qquad (262)$$

which is usually called "Euler's equation."

An illuminating example of the manner in which the foregoing principles may be applied is furnished by the classical problem of showing mathematically that the shortest distance between two points in a plane is a straight line. Thus, let (x_1, y_1) and (x_2, y_2) be two points in the xy plane and let $y(x)$ represent the admissible comparison functions. The integral to be "extremized" is

$$I[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \qquad (263)$$

and the corresponding Euler's equation is

$$\frac{d}{dx}\left[\frac{\partial}{\partial y'} \sqrt{1 + y'^2}\right] = 0 \qquad (264)$$

Obviously, equation (264) gives

$$y' = C_1 \tag{265}$$

and, therefore,

$$y = C_1 x + C_2 \tag{266}$$

which is clearly the equation of a straight line. The constants of integration C_1, C_2 may be evaluated from the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

The Variational Notation.--The term $\alpha\eta(x)$, introduced in the preceding development, changes a given function $y(x)$ into a new function $y(x) + \alpha\eta(x)$. By convention

$$\delta y = \alpha\eta(x) \tag{267}$$

is called the "variation" of $y(x)$. Corresponding to this change in $y(x)$, a given function $f(x, y, y')$ varies by an amount which, neglecting higher order terms, is defined to be the variation of f and is, accordingly, expressed by

$$\delta f = \frac{\partial f}{\partial y}\alpha\eta + \frac{\partial f}{\partial y'}\alpha\eta' \tag{268}$$

Evidently, the integrand in equation (258) is, to within the multiplicative constant α, equal to the variation of $f(x, y, y')$. And in view of this, the preceding development of the simplest problem of the calculus of variations may be stated thus: A necessary condition that the integral

(252) be "stationary" is that its variation vanish:

$$\delta I = \int_{x_1}^{x_2} \delta f(x, y, y') \, dx = 0 \tag{269}$$

<u>More General Variational Problems</u>.--If the basic integral involves m independent variables x, y, z ... and n dependent variables y, u, v ..., a development similar to the one outlined earlier leads to one Euler equation for each of the n dependent variables. The interested reader is referred to the literature (8).

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