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PROPERTIES WHICH ARE INVARIANT UNDER CERTAIN MAPPINGS

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CHAPTER I

INTRODUCTION

The purpose of this study is to determine what properties are invariant under certain types of mappings. An attempt is made to find the weakest kind of function which will insure that some property of the domain is inherited by the range space. Often examples are given to show that the results obtained cannot be improved to a great extent.

To read this thesis, a basic knowledge of topology is needed, since many basic topological results are used without explanation throughout the thesis. Terms used in this study may be assumed to be defined as in Kelley [3] unless other definitions are stated.

In Chapter II, invariances of compactness properties are studied. A knowledge of this subject will be found useful in showing other results throughout the exposition. The properties studied in this chapter include compactness, countable compactness, sequential compactness, B-compactness, local compactness, and semi-compactness. The Lindelöf property is taken up here because of its close relationship to compactness. Some other compactness properties -- paracompactness, metacompactness, and precompactness -- are considered in Chapter V.

In Chapter III, separation properties are studied. The spaces considered include spaces which are T_0, T_1, Hausdorff, T_3, T_4, regular, normal, completely regular, and Tychonoff. It is necessary to discuss first and second countability and also dense subsets and separability in order to show that the property of being a compact metric space is an invariant.

In Chapter IV, connectedness properties are considered. Spaces which are connected, locally connected, and connected _im kleinen_ are studied. Quasi-components are taken up briefly; then a few theorems are given leading up to showing that arcwise connectedness is invariant under continuous functions whenever the range is Hausdorff. Next compact connected spaces, that is continua, including decomposable, indecomposable, and unicoherent continua, are considered. In addition, the property that all connected subsets are locally connected is studied. A characterization of monotone mappings is obtained.

In Chapter V, some interesting results are brought together which do not fit well into any of the first three chapters. These topics are unrelated; therefore the topics may be read in any order. The section on uniform spaces assumes considerable knowledge of this subject on the part of the reader. The main result of this section is the invariance of total boundedness under uniformly continuous functions. In the other sections, the subjects studied include cut points, homogeneous spaces, upper and lower semi-continuous decompositions, non-separated collections, saturated collections, paracompactness and metacompactness, and dimension.

This study is far from being a complete treatment of the subject of invariances. This is a large subject on which much work has been and is being done. It is hoped, however, that this study will provide an adequate introduction to the subject.

CHAPTER II

COMPACTNESS PROPERTIES

Definition 2.1. A topological space is said to be compact if every collection of open sets which covers the space has a finite subcollection which also covers the space. That is, every open cover has a finite subcover.

Definition 2.2. A function f from a space X onto a space Y is said to be continuous provided that for each open set U in Y, $f^{-1}(U)$ is open in X.

Definition 2.3. The notation $\{x\colon$ proposition about $x\}$ will be used to designate the set of all x such that the proposition about x is correct.

Theorem 2.4. Let f be a continuous function from a compact space X onto a space Y. Then Y is compact.

Proof. Let $\mathcal{U}$ be an open cover of Y. Let $f^{-1}(\mathcal{U}) = \{O \subset X\colon O = f^{-1}(U)$ for some U in $\mathcal{U}\}$. Then $f^{-1}(\mathcal{U})$ is an open cover of X. Choose a finite subset $\mathcal{F}$ of $f^{-1}(\mathcal{U})$ which covers X. Then $f(\mathcal{F}) = \{f(O)\colon O$ is in $\mathcal{F}\}$ is a finite subcover of $\mathcal{U}$. Thus Y is compact. $\square$

Definition 2.5. A topological space is said to be countably compact if every countable open cover has a finite subcover.

Theorem 2.6. Let f be a continuous function from a countably compact space onto a space Y. Then Y is also countably compact.

Proof. A proof may be obtained by making obvious alterations in the proof of Theorem 2.4. □

Definition 2.7. A topological space is said to be sequentially compact if every sequence in the space has a subsequence which converges to a point of the space.

Theorem 2.8. Let f be a continuous function from a sequentially compact space X onto a space Y. Then Y is also sequentially compact.

Proof. Let $\{y_i\}_{i=1}^{\infty}$ be a sequence of points in Y. For each i choose a point x_i in X so that $f(x_i) = y_i$. The sequence $\{x_i\}$ in X has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to some point, say p. But this implies that the subsequence $\{y_{n_i}\}_{i=1}^{\infty}$ in Y converges to f(p), for if U is any open set containing f(p), then $f^{-1}(U)$ is an open set about p and thus contains all but a finite number of members of the sequence $\{x_{n_i}\}_{i=1}^{\infty}$. This implies that U contains all but a finite number of members of $\{y_{n_i}\}_{i=1}^{\infty}$; so the subsequence converges. Thus Y is sequentially compact. □

Definition 2.9. A topological space is said to be B-compact (Bolzano compact)[1] if every infinite subset has an accumulation point.

Definition 2.10. The discrete topology for a set is the topology in which all sets are open.

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[1]This terminology is not standard. To the best of my knowledge, it was first used in Kneece [4].

B-compactness is not invariant under continuous functions, as shown by the following example.

Example 2.11. A continuous function which maps a B-compact space onto a space which is not B-compact.

Let the domain X be given by $X = \{0,1,2, . . .\}$, with a base for the topology consisting of sets of the form $\{2n, 2n + 1\}$. The range Y consists of the same set of points with the discrete topology. The function f maps each point n into the greatest integer in n/2. For this function the inverse of each point is just a member of the base of the topology of X. Clearly Y is not B-compact, while X is B-compact, for each odd integer is an accumulation point of any set containing the next lower even integer and each even integer is an accumulation point of any set containing the next larger odd integer.

Definition 2.12. A topological space is said to be Lindelöf if every open cover has a countable subcover.

Theorem 2.13. Let f be a continuous function from a Lindelöf space X onto a space Y. Then Y is Lindelöf.

Proof. A proof may be obtained by replacing the word "finite" with the word "countable" everywhere in the proof of Theorem 2.4. ▯

Definition 2.14. A topological space is said to be semi-compact if it is the union of a countable number of compact subsets.

A semi-compact space is Lindelöf, but the converse is not true.

<u>Example 2.15</u>. A Lindelöf space which is not semi-compact.

Let X be an uncountable set with open sets being the complements of countable sets. It is easy to see that the space is Lindelöff since each open set contains all but a countable number of points. However, since the only compact subsets are finite, the space is not semi-compact.

<u>Theorem 2.16</u>. Let f be a continuous function from a semi-compact space X onto Y. Then Y is semi-compact.

<u>Proof</u>. Let $X = A_1 \cup A_2 \cup \ldots$, where each A_i is compact. Then $Y = f(A_1) \cup f(A_2) \cup \ldots$, where each $f(A_i)$ is compact by the continuity of f. Thus Y is semi-compact. $\square$

<u>Definition 2.17</u>. A topological space is said to be <u>first countable</u> if for every point p of the space there exists a countable collection of open sets containing p, so that if U is any open set containing p, there is a member of the countable collection contained in U. That is, a topological space is first countable if the neighborhood system of every point has a countable base.

<u>Definition 2.18</u>. A topological space is said to be <u>second countable</u> if it has a countable base.

Next are stated some interesting results involving compactness which can be used to extend the results already obtained. Many of these results will be used later.

Compact spaces are always countably compact, and countably compact spaces are always B-compact. Countably compact spaces which are second countable are always compact. B-compact spaces which are T_1 are countably compact; if they are both T_1 and the first countable, they are sequentially compact. Sequentially compact spaces are countably compact and B-compact. Proofs of statements made in this paragraph may be found in Kneece [4].

Examples which show that the results of the preceding paragraph cannot be improved will not be given in all cases. However, some examples are in order. The domain space in Example 2.11 is B-compact but not compact, countably compact, or sequentially compact. The following example is also interesting.

Example 2.19. The first uncountable ordinal with the order topology, a space which is countably compact but not compact.

Let X be the first uncountable ordinal with the order topology. That is, a base for the topology of X consists of all sets of one of the following three forms:

(1) $\{x\colon\ x < x < b$ where a and b are members of $X\}$

(2) $\{x\colon\ a < x$ for some a in $X\}$

(3) $\{x\colon\ x < a$ for some a in $X\}$.

Here < is the usual well-ordering for the members of X. Note that X is an uncountable set, but that each member of X has only a countable number of predecessors. To see that X is not compact, consider the open covering $\mathcal{U}$ which consists of all sets of the form $\{x\colon\ x < a\}$ for a in X. Suppose there is a finite subcovering $U_1, U_2, \ldots, U_n$ where $U_k = \{x: x < a_k\}$, $1 \leq k \leq n$. Then let a be the largest a_k. There are only a countable number of elements that precede a, and X is uncountable. Thus there is a point b in X such that $a < b$. This b is not in any U_k. This is a contradiction. Hence X is not compact.

Next it is shown that X is B-compact. Let S be an infinite subset of X. Let S' be an arbitrary countably infinite subset of S. It will be shown that S' has an accumulation point in X. Since S' is a subset of an ordinal, it inherits the ordering of the ordinal. Let $S' = \{x_1, x_2, \ldots, x_n, \ldots\}$. The set of all points less than x_k is always countable for all $k = 1, 2, \ldots$. Thus the set $A = \{x: x < x_k$ for some $k = 1, 2, \ldots\}$ is the union of a countable number of countable sets and is thus countable. But X is uncountable; so there is a point p in the complement of A. By the definition of A, p is an upper bound for the set S'. Since S' has an upper bound, it has a least upper bound, by the properties of well-orderings. Let z be the least upper bound of S'. Inside any open set containing z an open set of the form $\{x: c < x < d\}$ which contains z can be chosen. Consider such an open set. Since $c < z$, c is not an upper bound for the set S'. Thus there is an x_r such that $c < x_r$. But $x_r \leq z < d$ so $c < x_r < d$. Thus x_r is a member of the open set. Thus any open set which contains z contains a member of S'. Thus z is an accumulation point of S' and hence of S. Thus X is

B-compact. Since $\{x: x < q\}$ and $\{x: q < x\}$ are both open sets, the complements of points are open. Hence X is T_1. Since B-compact spaces which are T_1 are countably compact, X is countably compact.

Theorem 2.20. A closed subset of a compact space is compact. (For a proof, see Hocking and Young [1], p. 20.)

Theorem 2.21. A compact subset of a compact Hausdorff space is closed. (See Hocking and Young [1], p. 38.)

Definition 2.22. A function is said to be closed if the image of closed sets are closed in the range.

Theorem 2.23. If f is a continuous function from a compact space X onto a Hausdorff space Y, then f is closed.

Proof. Let C be a closed subset of X. Since X is compact, so is C. Thus f(C) is compact. Since Y is Hausdorff, f(C) is closed. Thus f is a closed mapping. ▯

Definition 2.24. A function is said to be compact if the inverse of compact sets in the range is compact in the domain.

Theorem 2.25. If f is a continuous function from a compact space X onto a Hausdorff space Y, then the inverse of compact sets is compact. That is, f is a compact mapping.

Proof. Let C be a compact subset of Y. Since Y is Hausdorff, C is closed. Thus, $f^{-1}(C)$ is closed by the continuity of f. But X is compact; so $f^{-1}(C)$ is compact, since it is closed. This is the desired result. ▯

Theorem 2.26. The property that compact sets are always closed is invariant under functions which are both closed and compact.

Proof. Let f be a closed and compact function from X onto Y. Let A be a compact set in Y. It will be shown that A is closed. Since the function is compact, $f^{-1}(A)$ is compact. Thus $f^{-1}(A)$ is closed. But since f is closed, $f[f^{-1}(A)] = A$ is closed. This is the desired result. ▯

The reader familiar with the notion of "quasi-compact mapping" will notice from the proof that "closed" can be replaced by "quasi-compact" in Theorem 2.26.

It is well known that Hausdorff spaces have the property in the above theorem. The converse is not true. Example 2.15 is an example of a space which is not Hausdorff and each of whose compact sets is closed.

Definition 2.27. A topological space is said to be locally compact if each point has a compact neighborhood.

Definition 2.28. A function is said to be open if the image of open sets is always open in the range.

Theorem 2.29. Local compactness is invariant under open continuous functions.

Proof. Let f be a continuous open function from the space X onto the space Y. Let p be any point of Y. Choose a point q in X such that f(q) = p. Choose a compact neighborhood, N of q. Since f is open, f(N) is a neighborhood of p. It is compact since f is continuous. ▯

Some authors define locally compact to mean that each point of the space has a closed compact neighborhood. Under this definition the last theorem is not true, as shown by the following example, constructed by P. S. Schnare in [5].

Example 2.30. Let $A_n = \{(n, 1), (n, 2), \ldots, (n, n)\}$. Define $X = \bigcup_{n=1}^{\infty} A_n$. A base for the topology of X will be the collection of all sets which are contained in A_n for some n and which contain (n, 1). Since each A_n is open, closed, and compact, X is locally compact. Clearly X is T_o but not T_1.

Let Y be the set of positive integers with a topology which includes only the empty set and all sets containing 1. Clearly Y is T_o but not T_1, and not locally compact, if we require closed compact neighborhoods of each point. This follows since the closure of the set consisting of the single point 1 is Y.

Define f from X onto Y by $f(n, m) = m$. Then f is both continuous and open. In fact, f is a local homeomorphism from X onto Y.

CHAPTER III

SEPARATION PROPERTIES

Definition 3.1. Let f be a function from X into Y. The degree of f is the maximum number of points in any $f^{-1}(y)$ where y is a point in Y, if this is finite. If f is not of finite degree, then f is said to be of infinite degree. If $f^{-1}(y)$ is finite for all y in Y, then f is said to be finite-to-one.

Definition 3.2. A topological space is said to be T_1 if, and only if, for every two distinct points x and y there are two open sets, one of which contains x but not y, while the other contains y but not x. In other words, a set which consists of a single point is closed.

Theorem 3.3. The property of being T_1 is invariant under closed functions.

Proof. Since points are closed in the domain, their images are closed in the range. Thus the range is T_1. □

Theorem 3.4. The property of being T_1 is invariant under open finite-to-one functions. The functions need not be continuous.

Proof. Let f be an open finite-to-one function from a T_1 space X onto Y. Let p and q be any two distinct points of Y. Let $S = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ be the set of all points in X which map into p or q, with $f(x_i) = p$ if $1 \leq i \leq k$ and $f(x_j) = q$ if $k < j \leq n$. For each i, $q \leq i \leq n$, choose an open set O_{1i} which contains x_1 but does not contain x_i. Let O_1 be the intersection of all the sets O_{1i}. Then O_1 is an open set containing x_1, but no other x_i. Choose a set O_n which contains x_n but no other x_i in a similar manner. Then $f(O_1)$ is an open set containing p but not q, and $f(O_n)$ is an open set containing q but not p. Thus Y is T_1, $\square$

Definition 3.5. A topological space is said to be T_o if for every pair of distinct points of the space there is an open set which contains one point but not the other.

An analogous result may be obtained for T_o spaces, but first the following lemma is needed.

Lemma 3.6. If $x_1, x_2, \ldots, x_n$ is any finite set of distinct points in a T_o space X, then there is an open set about one of the points which does not contain any of the others.

Proof. The proof is by induction. The result is clearly true for $n = 2$. Assume the result is true for $n = k$, and let $x_1, x_2, \ldots, x_k, x_{k+1}$ be any k+1 distinct points of X. Consider only the first k of these points. About one of them, say x_i, there is an open set O that does not contain any other x_j for $1 \leq j \leq k$ and $j \neq i$. If x_{k+1} is not in O, the proof is completed. If x_{k+1} is in O, then an open set O' may be found about x_i or x_{k+1} which does not contain the other. In either case $O \cap O'$ is the desired open set. $\square$

Theorem 3.7. The property of being T_o is invariant under open finite-to-one functions.

Proof. Let f be an open finite-to-one function from a T_o space X onto Y. Let p and q be any two distinct points of Y. Let $x_1, x_2, \ldots, x_n$ be the set of all points in X which map onto p or q. By our lemma choose a point x_i and an open set O_i containing x_i, but not containing x_j for $1 \leq j \leq n$ and $j \neq i$. Then $f(O_i)$ is an open set containing p or q but not both. Thus Y is T_o. ▯

Definition 3.8. A topological space is said to be Hausdorff if for every pair of distinct points x and y of the space there are disjoint open sets, one of which contains x and the other y.

Theorem 3.9. The image of a Hausdorff space under an open one-to-one function is Hausdorff.

Proof. Let f be an open one-to-one function from a Hausdorff space X onto Y. Let p and q be any two distinct points of Y. Then $f^{-1}(p)$ and $f^{-1}(q)$ are distinct points in X. Choose disjoint open sets O_p and O_q in X with $f^{-1}(p)$ in O_p and $f^{-1}(q)$ in O_q. Then p is in $f(O_p)$, and q is in $f(O_q)$. Also $f(O_p)$ and $f(O_q)$ are disjoint, since O_p and O_q are disjoint and f is one-to-one. Thus Y is Hausdorff. ▯

This theorem may be restated in another form which is often useful.

Theorem 3.10. If f is a one-to-one continuous function from X onto Y and Y is Hausdorff, then X is Hausdorff.

Proof. For a proof apply the previous theorem to f^{-1}. ▯

If the requirement that f be one-to-one is dropped, Theorem 3.9 is false, as shown by the following example.

Example 3.11. An open discontinuous function of degree two which maps a Hausdorff space onto a space which is not Hausdorff.

The domain space X consists of the two distinct closed unit intervals U, from (0, 1) to (1, 1), and L from (0, 0) to (1, 0) in the Euclidean plane. The topology is the relativised topology of the plane. The range space Y consists of the unit interval L' from (0, 0) to (1, 0) and the unit interval U' from (0, 1) to (1, 1) with all the rational points omitted. Rational points are points with both rational coordinates when the space is thought of as a subset of the Euclidean plane. Other points are said to be irrational. A base for the topology of Y consists of all rational points in any open (relative to the plane) interval of L' together with an arbitrary collection of irrational points within that interval and the corresponding interval in U'.

The function f takes points of L into corresponding points of L'. Corresponding points are points which lie on a vertical line when the space is thought of as a subset of the Euclidean plane. The function f takes irrational points of U into corresponding points of U' and rational points of U into corresponding points of L'. Although the function f is very discontinuous, it is open, for any open interval in U or

L is carried into a member of the base for the topology of Y.

The space Y is T_1, but not Hausdorff, since two corresponding irrational points, one in U' and the other in L', cannot be separated by disjoint open sets.

Thus f is an open function which maps a compact metric space into a space that is T_1 but not Hausdorff. The range is not compact, nor even Lindelöf.

Definition 3.12. A topological space is said to be regular if, for each closed set A and each point p not in A, there are two disjoint open sets, one of which contains A and the other p. A T_1 regular space is called a T_3 space.

It is easily seen that a space is regular if, and only if, for each point p and each open set U containing p, there is an open set V containing p with $\overline{V}$ contained in U. That is, the set of all closed neighborhoods of a point is a base for the neighborhood system of the point. To construct a proof of this, let the complement of U be the closed set A in the definition of regularity.

Theorem 3.13. If f is a continuous closed finite-to-one function which maps a regular space X onto a space Y, then Y is regular.

Proof. Let A be any closed subset of Y, and let p be any point not in A. Now $f^{-1}(A)$ is a closed subset of X which does not contain any member of $f^{-1}(p)$. Let $f^{-1}(p) = \{q_1, q_2, \ldots, q_n\}$. Choose for each i between 1 and k disjoint open sets U_i and V_i with q_i in U_i and $f^{-1}(A)$ contained in V_i. Let

$U = \bigcup_{i=1}^{k} U_i$ and $V = \bigcap_{i=1}^{k} V_i$. Then U is an open set containing $f^{-1}(p)$, and V is an open set containing $f^{-1}(A)$. Furthermore, U and V are disjoint. Define $U' = Y - f(X-U)$ and $V' = Y - f(X-V)$. Since $X - U$ is closed, $f(X-U)$ is also closed, and thus $Y - f(X-U)$ is open. Similarly $Y - f(X-V)$ is open. Since $f^{-1}(p)$ is contained in U, p is in $Y - f(X-U)$. In the same way, A is contained in $Y - f(X-V)$. Furthermore, $Y - f(X-U)$ is disjoint from $Y - f(X-V)$, for if z is in both sets, then $f^{-1}(z)$ is contained entirely in both U and V, which is impossible. Thus Y is regular. ▯

Notice that a slight generalization of the proof of Theorem 3.13 will yield the following stronger result.

Theorem 3.14. If f is a continuous closed function with compact point inverses, then if the domain is regular so is the range.

Corollary 3.15. If f is a continuous finite-to-one closed function which maps a T_3 space X onto a space Y, then Y is T_3.

Proof. From Theorem 3.13, Y is regular. Since f is closed, Y is also T_1. Thus Y is T_3. ▯

Definition 3.16. A topological space is said to be normal if for every two disjoint closed subsets A and B there are two disjoint open subsets, one of which contains A and the other B. A normal space which is also T_1 is called a T_4 space.

Theorem 3.17. If f is a closed continuous function which maps a normal space X onto a space Y, then Y is normal.

Proof. Let A and B be two disjoint closed subsets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X. Choose disjoint open sets U and V with $f^{-1}(A)$ contained in U and $f^{-1}(B)$ contained in V. Let $U' = Y - f(X-U)$ and $V' = Y - f(X-V)$. Then A is contained in U', and B is contained in V', and U' and V' are both open and disjoint. Thus Y is normal. ▯

Corollary 3.18. If f is a closed continuous function which maps a T_4 space X onto a space Y, then Y is T_4.

Proof. Clearly Y is both normal and T_1 and hence T_4. ▯

Definition 3.19. A topological space X is said to be completely regular if, and only if, for every point p of X and every open set U containing p, there is a continuous real valued function which maps X onto [0, 1] so that $f(p) = 0$ and f is identically 1 on the complement of U. A completely regular space which is T_1 is called a Tychonoff space.

A completely regular space X is regular, for if p is any point, U is any open set containing p, and f is a continuous function from X into [0, 1]; then the inverse under f of [0, 1/2) is an open set containing p whose closure is contained in U. A regular T_1 space is Hausdorff, and hence a Tychonoff space is Hausdorff.

Definition 3.20. A one-to-one continuous function whose inverse is also a continuous function is called a homeomorphism. That is, homeomorphisms are one-to-one open continuous functions.

Theorem 3.21. If f is a homeomorphism which maps a completely regular space X onto a space Y, then Y is completely regular.

Proof. Let p be any point of Y, and let U be any open set containing p. Then $f^{-1}(U)$ is an open set containing $f^{-1}(p)$. Thus there is a real-valued function g which maps X into [0, 1] with $g[f^{-1}(p)] = 0$ and $g(x) = 1$ for all x in $X - f^{-1}(U)$. The function h defined by $h(y) = g[f^{-1}(y)]$ maps Y into [0, 1] and is continuous, since it is the composite of two continuous functions. Clearly $h(p) = g[f^{-1}(p)] = 0$ and $h(y) = g[f^{-1}(y)] = 1$ for all y not in U. Thus Y is completely regular. □

Definition 3.22. A function is exactly k to 1 if the inverse of each point in the range contains exactly k points.

Theorem 3.23. If f is an open, continuous, exactly k to 1 function which maps a Tychonoff space X onto a space Y, then Y is also a Tychonoff space.

Proof. Let p be any point of Y and let U be any open set containing p. Let $f^{-1}(p) = \{q_1, q_2, \ldots, q_k\}$. Then $f^{-1}(U)$ is an open set containing $f^{-1}(p)$ and hence each q_i. For each i, $1 \leq i \leq k$, choose a function g_i which maps X onto [0, 1] with $g_i(q_i) = 0$ and g_i identically 1 on the complement of $f^{-1}(U)$. Define g by $g(x) = \min\{g_1(x), g_2(x), \ldots, g_k(x)\}$. It is easy to see that g is a continuous function, for, given any point z and $\varepsilon > 0$, there exist open sets $O_1, O_2, \ldots, O_k$ containing z so that $|g_i(z) - g_i(x)| < \varepsilon$ for all x in O_i. Let $O = \bigcap_{i=1}^{k} O_i$. It is easy to see that $|g(z) - g(x)| < \varepsilon$ for all x in O. Hence, g is a continuous function. Clearly $g[f^{-1}(p)] = 0$ and g is identically equal to 1 on $X - f^{-1}(U)$.

Define a function F from Y onto [0, 1] by $F(x) = \min\{g(z):$ z is in $f^{-1}(x)\}$. Then $F(p) = 0$, since g maps each member of $f^{-1}(p)$ into 0, and F is identically equal to 1 on Y - U, since g is identically 1 on $X - f^{-1}(U)$. To see that F is continuous at a point x, let $\varepsilon > 0$ be given. Let $f^{-1}(x) = \{x_1, x_2, \ldots, x_k\}$. About each x_i, $1 \le i \le k$, choose an open set V_i so that, if y is in V_i, then $|g(x_i) - g(y)| < \varepsilon$. Since Tychonoff spaces are Hausdorff, there is no loss in generality in assuming that the V_i be disjoint sets. Let $V = \bigcap_{i=1}^{k} f(V_i)$. Since f is open, V is the intersection of a finite number of open sets; so V is open. If w is any point in V, then let $f^{-1}(w) = \{w_1, w_2, \ldots w_k\}$ where w_i is in V_i for $1 \le i \le k$. Thus $|g(x_i) - g(w_i)| < \varepsilon$ for all i, $1 \le i \le k$. But it can be shown that this implies $|\min_{1\le i\le k}\{g(x_i)\} - \min_{1\le i\le k}\{g(w_i)\}| < \varepsilon$; so $|F(x) - F(w)| < \varepsilon$. Thus F is a continuous function. Since $F(p) = 0$ and F is identically 1 on Y - U, Y is completely regular. Clearly Y is T_1 since f is open and of finite degree. Thus Y is a Tychonoff space. □

Theorem 3.24. Let f be a continuous open function from a first countable space X onto a space Y; then Y is first countable.

Proof. Let p be any point of Y. Choose a point q in X so that $f(q) = p$. Choose a countable base for q, say U_1, U_2, U_3, ... Then $f(U_1)$, $f(U_2)$, . . . is a countable collection of open sets containing p. To show that this is a countable base for the neighborhood system of p, let O be any open set containing p. Then $f^{-1}(O)$ is an open set containing q. Choose a U_k contained in $f^{-1}(O)$. Then $f(U_k)$ is contained in O. Thus $f(U_1)$, $f(U_2)$, $f(U_3)$, . . . is a countable base for the neighborhood system of p. Thus Y is first countable. $\square$

<u>Theorem 3.25</u>. Let f be a continuous open mapping from a space X onto a space Y. Let $\mathcal{B}$ be a base for the topology of X. Then $f(\mathcal{B}) = \{O \subset Y\colon\ f(V) = O$ for some V in $\mathcal{B}\}$ is a base for the topology of Y.

<u>Proof</u>. Let p be any point of Y, and let U be any open set containing p. Let q be a point in X so that $f(q) = p$. Then $f^{-1}(U)$ is an open set in X which contains q. Choose an open set V in $\mathcal{B}$ which contains q and is contained in $f^{-1}(U)$. Then $f(V)$ is a member of $f(\mathcal{B})$ that contains p and is contained in U. Thus $f(\mathcal{B})$ is a base for the topology of Y. $\square$

<u>Theorem 3.26</u>. Let f be a continuous open mapping from a second countable space X onto a space Y. Then Y is second countable.

<u>Proof</u>. Let $\mathcal{B} = \{U_1, U_2, . . .\}$ be a countable base for X. Then $f(\mathcal{B}) = \{f(U_1), f(U_2), . . .\}$ is a countable base for Y. Thus Y is second countable. $\square$

<u>Definition 3.27</u>. A subset A of a topological space X is said to be <u>dense</u> in X if the closure of A is X. Equivalently A is dense in X if every non-empty open subset of X contains at least one member of A.

<u>Definition 3.28</u>. A space is said to be <u>separable</u> if it contains a countable dense subset.

<u>Theorem 3.29</u>. If A is a dense subset of a topological

space X and f is a mapping from X onto Y, then f(A) is dense in Y.

Proof. Let O be any open subset of Y. Then $f^{-1}(O)$ is open in X and thus contains a point of A. Hence, O contains a point of f(A) and f(A) is dense in Y. $\square$

From this theorem, the following invariance follows immediately.

Theorem 3.30. Let f be a continuous function from a separable space X onto Y. Then Y is separable.

Theorem 3.31. Compact Hausdorff spaces are normal. (For a proof, see Kelley [3], p. 141.)

Theorem 3.32. In a metric space, being second countable and being separable are equivalent conditions.

Proof. First assume that X is a separable metric space with a countable dense subset A. The collection of all sets of the form $\{x: d(x, a) < r$ for r, a positive rational number, and a in $A\}$ form a countable base for X, where d is the metric for X. Thus X is second countable.

If X is second countable, then a countable dense subset may be found by taking a point out of each member of a countable base for X. Thus X is separable. $\square$

Theorem 3.33. Compact metric spaces are separable and second countable.

Proof. For each positive integer n we can choose a finite open cover all of whose members have diameter less than 1/n. The collection of all sets obtained in this way, for all n, is a countable base. Thus the space is second countable and hence separable. ▯

Theorem 3.34(Urysohn). A regular, second countable, T_1 space is metrizable. (For a proof, see Kelley [3], p. 125.)

Theorem 3.35. Let f be a continuous mapping from a compact metric space X onto a Hausdorff space Y. Then Y is also a compact metric space.

Proof. Since f is continuous, Y is compact. But Theorem 3.31 implies that compact Hausdorff spaces are normal. Thus Y is normal and thus regular. From the theorem by Urysohn, Theorem 3.34, a regular, T_1, second countable space is metrizable. Thus it is sufficient to show that Y is second countable. Compact metric spaces are second countable by Theorem 3.33; so X is second countable. Let $U_1, U_2, \ldots, U_n, \ldots$ be a countable base for X. Now f is a continuous function from a compact space to a Hausdorff space; so f is a closed mapping by Theorem 2.23. Thus $f(X - \bigcup_{k\varepsilon\mathcal{F}} U_k)$ is closed in Y where $\mathcal{F}$ is any finite subset of the positive integers. It is to be shown that the collection $\mathcal{U}$ of all sets of the form $Y - f(X - \bigcup_{k\varepsilon\mathcal{F}} U_k)$ is a countable base for Y. Certainly these sets are open. First it is shown that there is only a countable number of such sets, or equivalently, that there is only a countable number of possible choices for $\mathcal{F}$. For any integer n there is only a countable number of choices of $\mathcal{F}$ that contain exactly n members. Thus, since there are only countably many choices of n, there are only countably many choices for $\mathcal{F}$.

Thus there are only countably many open sets of the form $Y - f(X - \bigcup_{k \in \mathcal{F}} U_k)$.

Let p be any point in Y, and let O be any open set containing p. Clearly $f^{-1}(O)$ is an open set containing the closed set $f^{-1}(p)$. Since X is compact, $f^{-1}(p)$ is compact. The collection of all open sets U_i which are contained in $f^{-1}(O)$ is an open cover for $f^{-1}(p)$. Choose a finite subcollection U_1', U_2', . . ., U_k' of these which covers $f^{-1}(p)$. Then $f^{-1}(p)$ is contained in $\bigcup_{i=1}^{k} U_i'$. Hence, $f(X - \bigcup_{i=1}^{k} U_i')$ is an open set of the desired form which contains p. Since each U_i' is contained in $f^{-1}(O)$, it is easy to see that $Y - f(X - \bigcup_{i=1}^{k} U_i')$ is contained in O. Thus all sets of the form $Y - f(X - \bigcup_{i=1}^{k} U_i')$ form a countable base for Y. So Y is regular, T_1, and second countable, and thus metrizable, as desired. $\square$

It appears as if the last theorem can be extended to semi-compact metric spaces. This is not true, as shown by the following example.

Example 3.36. A one-to-one continuous function which maps a semi-compact, separable metric onto a Hausdorff space which is not metrizable.

Let X be the Euclidean plane E_2 with the usual topology together with a disjoint point w_1 which is both open and closed. Let Y be the Euclidean plane with the usual topology together with a disjoint point w_2, where open sets containing w_2 are complements of closed subsets of finite two dimensional Lebesgue measure in E_2. The proof that this is a topology for

Y, although tedious, is not difficult; so it is omitted. Let f be the function from X onto Y, which is the identity function on E_2 and maps w_1 onto w_2. It is easy to see that f is continuous.

A metric for the space X may be obtained by using the metric,

$$d(x,y) = \frac{|x-y|}{1+|x-y|},$$

for points in E_2 and by letting $d(x,w_1) = 1$ for all x in E_2. Any compact subset of E_2 together with w_1 is compact; so X is semi-compact. To show that Y is not metric, we must use w_2. Assume that Y is metric with metric d. Since $\{w_2\}$ is not open, there are points in Y - $\{w_2\}$ which are arbitrarily close to w_2. Choose a sequence of points $\{x_i\}_{i=1}^{\infty}$ so that $d(x_i, w_2) < 1/i$ for all i. Clearly the sequence $\{x_i\}_{i=1}^{\infty}$ converges to w_2. But $Y - \bigcup_{i=1}^{\infty} \{x_i\}$ is an open set containing w_2 but none of the points x_i. This is a contradiction. Thus Y is not metrizable. Notice that Y is Hausdorff and regular, but not second countable. The proof that Y is not second countable is similar to the proof that Y is not metric.

Example 3.37. A continuous open function which maps a compact metric space into a space with the trivial topology.

Let X be the unit interval [0, 1] with the usual topology. For each point x in X define $x' = \{y \in [0, 1]: |x - y|$ is rational$\}$. It can be easily shown that $x_1' = x_2'$ whenever $|x_1 - x_2|$ is rational. It is also easily seen that every open interval contained in [0, 1] contains members of each x'. By the structure theorem for open sets in E_1, each non-empty open set is a union of open intervals. Thus every open set contains members of each x'. Let f be the mapping defined by $f(x) = x'$ for all x in [0, 1], where the range Y is the set of all x' and has the trivial topology, the topology whose only open sets are Y and the empty set. Clearly f is continuous, since the inverse of Y is X and the inverse of the empty set is empty. Since each non-empty open set in X contains members of each x' and is thus mapped into all of Y, the mapping is open. Thus f maps the unit interval [0, 1] into a space with the trivial topology which is not even T_o. The degree of f is infinity.

CHAPTER IV

CONNECTEDNESS PROPERTIES

Definition 4.1. Let X be a topological space. Two subsets A and B are said to be separated if $A \cap \overline{B}$ and $\overline{A} \cap B$ are both void. Equivalently two disjoint sets A and B are separated if neither set contains an accumulation point of the other.

Definition 4.2. A topological space X is connected if it is not the union of two separated non-empty sets. Equivalently, X is connected if the only subsets which are both open and closed are the entire space X and the empty set. A subset of X is connected if it is connected in the relative topology.

Definition 4.3. Let X be a topological space. If $X = A \cup B$ where A and B are non-empty separated subsets of X, then we say $A \cup B$ is a separation for X.

Theorem 4.4. Let f be a continuous function from a connected space X onto a space Y. Then Y is also connected.

Proof. Suppose that Y is not connected and that $Y = A \cup B$ is a separation. That is, A and B are both open and closed, non-empty, disjoint subsets of Y. Then $X = f^{-1}(A) \cup f^{-1}(B)$, which is a separation of X. This is a contradiction. Thus Y is connected. $\square$

Definition 4.5. A topological space is said to be locally connected at a point x if for every open set U containing x there is a connected open set V which contains x and is contained in U. A topological space is <u>locally connected</u> if it is locally connected at each point.

<u>Definition 4.6.</u> Let A be a subset of a topological space X. The <u>component</u> of A containing a point p in A is the maximal connected subset of A that contains p.

<u>Theorem 4.7.</u> If f is a continuous function from a space X onto a space Y and A is a subset of X with a component C, then f(C) is contained in a component of f(A).

<u>Proof.</u> The proof follows from the fact that f(C) is connected. □

<u>Corollary 4.8.</u> If f is a continuous function from a space X onto Y and K is a component of a subset B of Y, then $f^{-1}(K)$ is a union of components of $f^{-1}(B)$.

<u>Proof.</u> Otherwise some component of $f^{-1}(B)$ maps into more than one component of B. □

<u>Theorem 4.9.</u> A topological space X is locally connected if, and only if, for every point p contained in X and every open set O containing p, the component of O that contains p is open.

<u>Proof.</u> If all components of open sets are open, it is clear that the space is locally connected. Let p be any point in X and let O be any open set containing p. Let O_p be the component of O containing p. Assume that O_p is not open. Let q be a point in O_p and also in the closure of $X - O_p$. Now O is an open set containing q and the space is also lo-

cally connected at q. Choose U to be a connected open subset of O which contains q. Then U is contained entirely in the component of O that contains q, namely O_p. But q is a limit point of X - O_p and hence of X - U; so U is not open. This is a contradiction. Thus, the component of O that contains p is open. □

The results of this theorem are needed in the next proof.

Theorem 4.10. Let f be a continuous function from a compact locally connected space X onto a Hausdorff space Y. Then Y is compact and locally connected.

Proof. From the fact that f is continuous, it follows that Y is compact. It is to be shown that Y is locally connected. Let p be any point of Y, and let O be any open set containing p. It will be shown that the component of O which contains p, say O_p, is open. By Corollary 4.8 it follows that $f^{-1}(O_p)$ is a union of components of $f^{-1}(O)$. Thus by the local connectedness of X, it follows that $f^{-1}(O_p)$ is open. Thus $X - f^{-1}(O_p)$ is closed. But since Y is Hausdorff, f is a closed mapping. Thus $f[X - f^{-1}(O_p)] = Y - O_p$ is closed. Thus O_p is open. Hence Y is locally connected. □

If the assumption that the domain space is compact is dropped, Theorem 4.10 is not true. (For an example showing this, see Hocking and Young [1], p. 124.)

Definition 4.11. A topological space is connected im kleinen at a point x if for each neighborhood N of x the component of N that contains x is a neighborhood of x. (Recall that N is a neighborhood of x if N contains an open set containing x.) A topological space is connected im kleinen if it is connected im kleinen at each point.

If a space is locally connected at a point, then it is certainly connected im kleinen at that point. The converse of this is not true. (For an example showing this, see Hocking and Young [1], p. 113.) However, for spaces the following result is true.

Theorem 4.12. A topological space X is connected im kleinen if, and only if, it is locally connected.

Proof. If X is locally connected, then clearly it is connected im kleinen. If it is connected im kleinen, then let p be any point of X, and let U be any open set containing p. Consider the component U_p of U that contains p. Suppose this is not open. Choose a point q in both U_p and the boundary of U_p. Then the component of U that contains q is U_p, and U_p is not a neighborhood of q. This is a contradiction. Thus U_p is open, and so the space is locally connected. ▯

In view of the last theorem, any invariance of local connectedness which applies to an entire space also applies to connectedness im kleinen.

Definition 4.13. A function f is said to be quasi-compact if the image of each closed inverse set is closed, or equivalently if the image of each open inverse set is open.

Theorem 4.14. Local connectedness is invariant under quasi-compact mappings.

Proof. Let f be a quasi-compact mapping from the locally connected space X onto Y. Let p be any point in Y, and let O be any open set in Y containing p. Let C be the component of O which contains p. Then $f^{-1}(O)$ is open, and $f^{-1}(C)$ is an open inverse set. Thus $C = f[f^{-1}(C)]$ is open in Y, and Y is locally connected. $\square$

Definition 4.15. A subset Q of a space X is a quasi-component of X provided that for any separation of X, say $X = A \cup B$ where A and B are non-empty separated sets, Q is contained in either A or B; but Q is not a proper subset of another set with this property. If S is a subset of a topological space X, then quasi-components of S are quasi-components in the relative topology.

Theorem 4.16. Let f be a continuous function from a topological space X onto a space Y. Let S be a subset of X with a quasi-component Q. Then $f(Q)$ is contained in a quasi-component of $f(S)$.

Proof. Let $f(S) = A \cup B$ be a separation of $f(S)$. Suppose $f(Q)$ is not contained entirely in A or B. Then choose p in $A \cap f(Q)$ and q in $B \cap f(Q)$. Clearly $S = [S \cap f^{-1}(A)] \cup [S \cap f^{-1}(B)]$ is a separation of S, since $f^{-1}(A)$ and $f^{-1}(B)$ are open in the relative topology. Choose points p_1 and q_1 in Q so that $f(p_1) = p$ and $f(q_1) = q$. Then p_1 is in $S \cap f^{-1}(A)$ and q_1 is in $S \cap f^{-1}(B)$. Thus Q has at least one point in each member of the separation for S. This is a contradiction.

Thus $f(Q)$ is contained in one quasi-component of $f(S)$. ▯

Definition 4.17. An arc is a homeomorphic image of the unit interval. A space is arcwise connected if there is an arc connecting any two distinct points.

Theorem 4.18. The continuous image of a closed interval in E_1 of finite length onto a Hausdorff space Y is compact, connected, locally connected, and metric.

Proof. The finite closed interval is connected, compact locally connected, and compact metric. All of these are invariant under continuous functions when the range is Hausdorff. Thus, Y is compact, connected, locally connected, and metric, as required. ▯

Theorem 4.19. A compact, connected, locally connected, metric space is arcwise connected. (For a proof, see Whyburn [6], p. 36.)

Theorem 4.20. Let f be a continuous function from an arcwise connected space X onto a Hausdorff space Y. Then Y is arcwise connected.

Proof. Let p and q be any two points of Y. Choose p' and q' in X so that $f(p') = p$ and $f(q') = q$. Choose an arc A between p' and q'. Now A is the homeomorphic image of the unit interval I, and thus a continuous image of the unit interval. Let g be a continuous function from the unit interval onto A. Then $f(A) = f[g(I)]$; so $f(A)$ is the continuous image of the unit image. Hence $f(A)$ is compact, connected, locally connected, and metric, by Theorem 4.18. Thus $f(A)$ is arcwise connected by Theorem 4.19. Thus there is an arc joining p and q that is contained in f(A) and thus in Y. Thus Y is arcwise connected. ▯

Definition 4.21. A topological space is said to be a continuum if it is both compact and connected. A subset of a space is a continuum if it is a continuum in the relative topology.

Theorem 4.22. The property of being a continuum is invariant under continuous functions.

Proof. The result is clear since both compactness and connectedness are invariant under continuous functions. ▯

Theorem 4.23. Let f be a closed function from a topological space X onto a space Y. Let A be a subset of X which is an inverse set. Let g be f restricted to A. Then g is a closed function from A onto f(A).

Proof. Let C be a closed subset of A. Then $C = C_o \cap A$ where C_o is closed in X. Thus $g(C) = g(C_o \cap A) = f(C_o \cap A) = f(C_o) \cap f(A)$. Since $f(C_o)$ is closed in Y, g(C) is closed in f(A). Thus g is a closed function. ▯

Definition 4.24. Let f be a function from a space X onto a space Y. If $f^{-1}(y)$ is a continuum for all y in Y, then f is said to be monotone.

Theorem 4.25. Let f be a continuous function from a compact space X onto a Hausdorff space Y. Then f is monotone if, and only if, the inverse of connected sets is connected.

Proof. If the inverse of connected sets is connected, then the inverse of points are connected. They are also closed and thus compact. Thus, f is monotone.

Now assume that f is monotone. Clearly f is closed since X is compact and Y is Hausdorff. Let A be a connected subset of Y. Suppose $f^{-1}(A)$ is not connected. Let $f^{-1}(A) = A_1 \cup A_2$ where A_1 and A_2 are disjoint, non-empty and open in $f^{-1}(A)$. Since f is closed, and hence closed if restricted to the inverse set $f^{-1}(A)$, $f(A_1)$ and $f(A_2)$ are closed in A. Since $f(A_1)$ and $f(A_2)$ are non-empty, they must not be disjoint, since otherwise they form a separation of A. Let p be a point in both $f(A_1)$ and $f(A_2)$. Choose q_1 in both $f^{-1}(p)$ and A_1, and q_2 in $f^{-1}(p)$ and A_2. But $f^{-1}(p)$ is contained in $f^{-1}(A)$ and thus in $A_1 \cup A_2$. Hence, $[A_1 \cap f^{-1}(p)] \cup [A_2 \cap f^{-1}(p)]$ is a separation of $f^{-1}(p)$. This is a contradiction. Thus $f^{-1}(A)$ is connected. □

<u>Corollary 4.26</u>. Let f be a compact mapping from a space X onto a Hausdorff space Y. Then f is monotone if, and only if, the inverse of connected sets is connected.

<u>Proof</u>. If the inverse of each connected set is connected, then the inverse of each point is connected. The inverse of each point is compact since f is compact. Thus f is monotone. The other part of the proof is the same as in Theorem 4.25. □

<u>Theorem 4.27</u>. Let X be a compact space with the property that every connected subset is locally connected. Let f be a continuous, monotone mapping from X onto a Hausdorff

space Y. Then every connected subset of Y is locally connected.

Proof. Let A be a connected subspace of Y. Then $f^{-1}(A)$ is connected since the mapping is monotone. Thus $f^{-1}(A)$ is locally connected. But f is closed, since it is a monotone mapping from a compact space to a Hausdorff space. But being locally connected is invariant under a closed mapping. Thus A is locally connected. This is the desired result. $\square$

Theorem 4.28. If f is a compact monotone mapping of a topological space X onto a Hausdorff continuum Y, then X is also a continuum.

Proof. Since the mapping is compact, we have X is compact. Assume X is not connected, and let $X = A \cup B$ where $A \cup B$ is a separation for X. Since X is compact, both A and B are compact. Thus f(A) and f(B) are compact sets. They are disjoint since the mapping is monotone. Also f(A) and f(B) are closed, and hence open, since they are compact and Y is Hausdorff. Thus $Y = f(A) \cup f(B)$ where $f(A) \cup f(B)$ is a separation of Y. This is a contradiction. Thus X is connected and hence a continuum. $\square$

By a slight modification of the proof of Theorem 4.28, the following corollary may be established.

Corollary 4.29. Let f be a compact monotone mapping from a topological space X onto a Hausdorff space Y. Let D be a continuum contained in Y. Then $f^{-1}(D)$ is a continuum in X.

<u>Definition 4.30</u>. A continuum is said to be <u>decomposable</u> if it is the union of two proper subsets which are also compact and connected. A continuum which does not have this property is said to be <u>indecomposable</u>.

<u>Theorem 4.31</u>. Being decomposable is invariant under continuous one-to-one functions.

<u>Proof</u>. Let f be a continuous one-to-one function from a decomposable continuum X onto a space Y. Let $X = A \cup B$ where A and B are proper subcontinua. Then $Y = f(A) \cup f(B)$ where f(A) and f(B) are continua by the continuity of f. Since f(A) and f(B) are proper by the one-to-one property of f, it follows that Y is decomposable. □

<u>Theorem 4.32</u>. Being an indecomposable continuum is invariant under compact monotone mappings if the range is Hausdorff.

<u>Proof</u>. Let f be a compact monotone mapping from an indecomposable continuum X onto a Hausdorff space Y. Clearly Y is a continuum, since f is continuous. Suppose Y is decomposable into proper subcontinua A and B. Then $f^{-1}(A)$ and $f^{-1}(B)$ are also proper subcontinua of X, and X equals their union. Thus X is decomposable. This is a contradiction. Thus Y is also indecomposable. □

<u>Definition 4.33</u>. A continuum C is <u>unicoherent</u> if whenever C equals $A \cup B$ with A and B continua, then $A \cap B$ is a continuum.

<u>Theorem 4.34</u>. Unicoherence is invariant under monotone

mappings if the range is Hausdorff.

Proof. Let f be a monotone mapping from a unicoherent continuum C onto a Hausdorff space Y. Since f is continuous it follows that f(C) is a continuum. Let $f(C) = A \cup B$ where A and B are continua. Since f is monotone it follows that $f^{-1}(A)$ and $f^{-1}(B)$ are connected. Since the range is Hausdorff, A and B are closed. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are closed subsets of a compact set C. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are continua, and $C = f^{-1}(A) \cup f^{-1}(B)$. So $f^{-1}(A) \cap f^{-1}(B)$ is a continuum. But $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$; so $f[f^{-1}(A \cap B)] = A \cap B$ is a continuum. Thus f(C) is unicoherent. $\square$

CHAPTER V

MISCELLANEOUS RESULTS

Cut Points and Homogeneous Spaces

Definition 5.1. A point x of a component C is a cut point of C if $C-\{x\}$ is separated.

Theorem 5.2. The image of a cut point under a homeomorphism is a cut point.

Proof. Let X be a connected topological space, and let x be a cut point of X. Let $X - \{x\} = A \cup B$ where A and B are both open and closed in $X - \{x\}$. Let f be a homeomorphism of X onto Y. Then $Y - \{f(x)\} = f(A) \cup f(B)$ where f(A) and f(B) are both open and closed in $Y - \{f(x)\}$. Thus $Y - \{f(x)\}$ is separated. But $Y = f(X)$ is connected, since X is connected. Thus f(x) is a cut point in Y. ▯

Definition 5.3. A topological space X is said to be homogeneous if for any two points x and y in X there is a homeomorphism from X onto itself that carries x onto y.

Theorem 5.4. The property of being homogeneous is invariant under homeomorphisms.

Proof. Let h be a homeomorphism from a homogeneous space X onto a space Y. Let p and q be any two points in Y. Then $h^{-1}(p)$ and $h^{-1}(q)$ are two points in X. Choose a homeomorphism f from X onto itself with $f[h^{-1}(p)] = h^{-1}(q)$. Thus $h\{f[h^{-1}(p)]\} = q$. Thus the composite function $h \circ f \circ h^{-1}$ is a homeomorphism of Y onto itself which maps p onto q. Thus Y is homogeneous. □

Before giving examples showing some cases where being homogeneous or being not homogeneous is not invariant, the following theorem is needed.

Theorem 5.5. Let X be a topological space with at least one cut point and at least one non-cut point. Then X is not homogeneous.

Proof. The result is clear, since a homeomorphism can not map a cut point onto a non-cut point. □

Example 5.6. A continuous, open, and closed function of degree two which maps a homogeneous space onto a space which is not homogeneous.

Let the domain X be the unit circle centered at the origin with the relativised topology of the plane. Any circle is homogeneous, since any point can be mapped into any other point by a rotation; and rotations are homeomorphisms. The range Y is the closed interval [-1, 1], with the relativised topology of the plane. The interior points of Y are cut points while the end points of Y are not cut points. Thus Y is not homogeneous. Let f be the function which maps the point (x, y) on the unit circle onto the point x in the closed interval [-1, 1]. That is, f(x, y) = x. It is easy to see that f is continuous, open, closed, and of degree two.

Example 5.7. A continuous, open, and closed function of degree two which maps a space which is not homogeneous onto a space which is homogeneous.

Let the domain X be two unit circles centered at (1, 0) and (-1, 0). The circles are tangent at the origin, and the origin is a cut point. However, all other points are not cut points. Thus X is not homogeneous. Let Y be the unit circle centered at the point (1, 0). Any circle is homogeneous; so Y is homogeneous. Define the function f from X onto Y by $f(x, y) = (|x|, y)$. It is easy to see that f is continuous, open, closed, and of degree two.

Decompositions

Definition 5.8. A disjoint collection of subsets which covers a space X is said to be a decomposition of X.

Definition 5.9. A decomposition of a space X into a collection $\mathcal{D}$ of disjoint closed sets is upper semi-continuous provided the union of all elements of $\mathcal{D}$ intersecting any closed set is closed. Equivalently the union of all members of the decomposition contained in any open set is open.

Theorem 5.10. If X is a topological space, $\mathcal{D}$ is an upper semi-continuous decomposition of X, and f is a homeomorphism from X onto a space Y, then $f(\mathcal{D})$ is an upper semi-continuous decomposition of Y.

Proof. Since f is a homeomorphism, the members of $f(\mathcal{D})$ are closed and disjoint. Let C be any closed subset of Y. Then $f^{-1}(C)$ is a closed subset of X. Let A be the union of all members of $\mathcal{D}$ that intersect $f^{-1}(C)$. Then A is closed. Thus

$f(A)$ is closed. But $f(A)$ is the union of all members of $f(\mathcal{D})$ that intersect C. Thus $f(\mathcal{D})$ is an upper semi-continuous decomposition of Y. □

<u>Definition 5.11</u>. A decomposition $\mathcal{D}$ is <u>lower semi-continuous</u> provided that each member of $\mathcal{D}$ is closed and the union of all elements of $\mathcal{D}$ intersecting an open set in X is open. Equivalently the union of all members contained in any closed set is closed.

<u>Theorem 5.12</u>. If h is a homeomorphism from X onto Y and if $\mathcal{D}$ is a lower semi-continuous decomposition of X, then $h(\mathcal{D})$ is a lower semi-continuous decomposition of Y.

<u>Proof</u>. Let U be an open set in Y. Then $h^{-1}(U)$ is open in X. Let A be the union of all members of $\mathcal{D}$ that intersect $h^{-1}(U)$. Then A is open. But $h(A)$ is open and the union of all members of $h(\mathcal{D})$ that intersect U. Thus $h(\mathcal{D})$ is lower semi-continuous. □

<u>Theorem 5.13</u>. Let f be a continuous closed function from X onto Y. Let $\mathcal{D}$ be an upper semi-continuous decomposition of Y. Then the decomposition $\mathcal{D}'$ of X generated by the inverse under f of members of $\mathcal{D}$ is an upper semi-continuous decomposition of X.

<u>Proof</u>. The members of $\mathcal{D}'$ are closed, disjoint, and cover X. Thus they form a decomposition of X. Let C be any closed set in X. Then $f(C)$ is closed in Y. Let V be the union of all members of $\mathcal{D}$ that intersect $f(C)$. This is a

closed set, and so $f^{-1}(V)$ is closed. Clearly $f^{-1}(V)$ is the union of all members of $\mathcal{D}'$ that intersect C. Thus $\mathcal{D}'$ is upper semi-continuous. □

Theorem 5.14. If f is an open mapping from X onto Y and $\mathcal{D}$ is a lower semi-continuous decomposition of Y, then the decomposition $\mathcal{D}'$ of X obtained from the inverse under f of members of $\mathcal{D}$ is also lower semi-continuous.

Proof. The members of $\mathcal{D}'$ are disjoint, closed, and cover X. Thus they form a decomposition of X. Let U be any open subset of X. Then f(U) is an open subset of Y. Thus the union of all members of $\mathcal{D}$ which intersect U is open. Call this union A. Then $f^{-1}(A)$ is open. Also $f^{-1}(A)$ is the union of all members of $\mathcal{D}'$ that intersect U. Thus $\mathcal{D}'$ is lower semi-continuous. □

Theorem 5.15. Let f be a quasi-compact mapping from X onto Y. Let $\mathcal{D}$ be a decomposition of Y. Let $\mathcal{D}'$ be the decomposition of X obtained from the inverse under f of members of $\mathcal{D}$. If $\mathcal{D}'$ is upper semi-continuous, then $\mathcal{D}$ is also upper semi-continuous.

Proof. Let C be any closed subset of Y. Let B be the union of all members of $\mathcal{D}$ that intersect C. Clearly, $f^{-1}(C)$ is closed. The union, A, of all members of $\mathcal{D}'$ that intersect $f^{-1}(C)$ is closed. Furthermore, $A = f^{-1}(B)$; so A is a closed inverse set. Thus $f(A) = B$ is closed. Thus $\mathcal{D}$ is upper semi-continuous. □

Theorem 5.16. Let f be a quasi-compact mapping from X onto Y. Let $\mathcal{D}$ be a decomposition of Y. Let $\mathcal{D}'$ be the decomposition of X obtained from the inverse under f of members of $\mathcal{D}$. If $\mathcal{D}'$ is lower semi-continuous, then $\mathcal{D}$ is also lower semi-continuous.

Proof. Let C be any closed set in Y. Let B be the union of all members of $\mathcal{D}$ that are contained in C. Since $f^{-1}(C)$ is closed, the union A of all members of $\mathcal{D}'$ that are contained in $f^{-1}(C)$ is closed. Furthermore, $A = f^{-1}(B)$; so A is a closed inverse set. Thus $f(A) = B$ is closed. Thus $\mathcal{D}$ is lower semi-continuous. $\square$

Results in Uniform Spaces

In this section some results in uniform spaces are proved. It is assumed that the reader has a basic knowledge of uniformities and uniform spaces. (The terminology used is the same as that of Kelley [3], Chapter 6.)

Definition 5.17. Let X be a topological space with the topology determined by the uniformity $\mathcal{U}$. Let Y be a space with the topology determined by the uniformity $\mathcal{V}$. Let f be a function from X into Y. Then f is a uniformly continuous function relative to $\mathcal{U}$ and $\mathcal{V}$ if for each member V of $\mathcal{V}$ the set $\{(x, y): (f(x), f(y)) \in V\}$ is in the uniformity $\mathcal{U}$.

Definition 5.18. Let X be a topological space with the topology determined by the uniformity $\mathcal{U}$. The gage of $\mathcal{U}$ is the family of all pseudo-metrics which are uniformly continuous on $X \times X$ relative to the product uniformity determined by $\mathcal{U}$.

<u>Definition 5.19</u>. A uniform space X with uniformity $\mathcal{U}$ is said to be <u>totally bounded</u> if for each positive r and each pseudo-metric d in the gage of $\mathcal{U}$ there is a finite number of sets with d-diameter less than r that covers X. The term <u>pre-compact</u> is often used to mean totally bounded.

Total boundedness is invariant under uniformly continuous functions. To prove this the following lemma is needed.

<u>Lemma 5.20</u>. Let X be a uniform space with uniformity $\mathcal{U}$, and Y be a uniform space with uniformity $\mathcal{V}$. Let $\mathcal{D}_x$ be the gage of $\mathcal{U}$ and $\mathcal{D}_y$ be the gage of $\mathcal{V}$. Then a function f from X onto Y is uniformly continuous if, and only if, for each positive number r and pseudo-metric d_2 in the gage $\mathcal{D}_y$ there is a positive number t and a pseudo-metric d_1 in $\mathcal{D}_x$ such that if $d_1(p, q) < t$ then $d_2(f(p), f(q)) < r$.

<u>Proof</u>. The proof follows quickly from the fact that the collection of all sets of the form $V_{dr} = \{(x, y): d(x, y) < r\}$ for r positive and d in $\mathcal{D}_x$ forms a base for $\mathcal{U}$ and a similar collection forms a base for $\mathcal{V}$. $\square$

<u>Theorem 5.21</u>. Let X be a uniform space with uniformity $\mathcal{U}$ and Y be a uniform space with uniformity $\mathcal{V}$. Assume that X is totally bounded and that f is a uniformly continuous function from X onto Y. Then Y is totally bounded.

<u>Proof</u>. Let $\mathcal{D}_x$ be the gage for $\mathcal{U}$ and $\mathcal{D}_y$ be the gage for $\mathcal{V}$. Let d_2 in the gage $\mathcal{D}_y$ and a positive number r be given. It is to be shown that Y can be covered by a finite collection of sets of d_2-diameter less than r. By Lemma 5.20,

choose d_1 in $\mathcal{D}_X$ and a positive number t so that $d_1(p, q) < t$ implies that $d_2(f(p), f(q)) < r$. Cover X with a finite number of sets with d_1-diameter less than t. The images of these sets under f cover Y and each has a d_2-diameter less than r. Thus Y is totally bounded. □

Definition 5.22. Let X be a uniform space with the uniformity $\mathcal{U}$. Then a sequence $\{x_n\}$ is said to be a Cauchy sequence if for each member U of the uniformity $\mathcal{U}$ there is an integer N so that wherever n and m are greater than N, we have that (x_n, x_m) is in U.

Theorem 5.23. Let $\{x_n\}$ be a Cauchy sequence in a uniform space X with uniformity $\mathcal{U}$. Let f be a uniformly continuous function from X onto the uniform space Y with uniformity $\mathcal{V}$. Then the sequence $\{f(x_n)\}$ is a Cauchy sequence in Y.

Proof. Let V be a member of $\mathcal{V}$. Let $Z = \{(x, y): (f(x), f(y)) \varepsilon V\}$. Then Z is a member of $\mathcal{U}$. Choose N so that $m > N$ and $n > N$ implies that (x_n, x_m) is in Z. Then $m > N$ and $n > N$ also implies that $(f(x_n), f(x_m))$ is in V. Thus $\{f(x_n)\}$ is a Cauchy sequence in Y. □

Theorem 5.23 is still true if we replace the Cauchy sequence by a Cauchy net. The proof is completely analogous and is thus omitted.

It should be pointed out that being a Cauchy sequence is not invariant under homeomorphisms, even in metric spaces.

Example 5.24. A homeomorphism which maps a Cauchy sequence into a sequence that is not a Cauchy sequence.

Let X be the half open interval $(0, 1]$ with the usual topology. Let Y be the interval $[1, \infty)$ also with the usual topology. Define $f(x) = 1/x$ for all x in X. It can be shown that f is a homeomorphism. The Cauchy sequence $\{1/n\}$ is mapped into the sequence $\{n\}$ which is not a Cauchy sequence.

Definition 5.25. A metric, or uniform, space is said to be complete if each Cauchy sequence has a limit.

Completeness is another property which is not invariant under homeomorphisms.

Example 5.26. A homeomorphism which maps a complete metric space onto a metric space which is not complete.

Let X be the interval $[1, \infty)$ with the usual topology and Y be the interval $(0, 1]$ with the usual topology. Define $f(x) = 1/x$ for all x in X. It can be shown that f is a homeomorphism from X onto Y. It is clear that X is complete, but Y is not complete, as shown by the sequence $\{1/n\}$.

Non-separated and Saturated Collections

Definition 5.27. Let x and y be two points of a topological space X. Then a subset A of X is said to separate x and y in X if there are disjoint sets U_1 and U_2 both open in $X - A$ such that $X - A = U_1 \cup U_2$ and x is in U_1 while y is in U_2. That is, $U_1 \cup U_2$ is a separation of $X - A$, with x in U_1 and y in U_2.

Definition 5.28. Let $\mathcal{U}$ be a disjoint collection of subsets of a topological space X. Then $\mathcal{U}$ is said to be non-separated in X if no two points in one member of $\mathcal{Y}$ is separated in X by any other member of $\mathcal{Y}$.

Theorem 5.29. Let $\mathcal{Y}$ be a non-separated collection in a space X. Let f be a one-to-one continuous function from X onto Y. Then the collection $f(\mathcal{Y})$ of all images of members of $\mathcal{Y}$ is non-separated in Y.

Proof. Suppose the theorem is not true. Let A and B be members of $\mathcal{Y}$ such that f(A) separates two points f(x) and f(y) both in f(B). Let $Y - f(A) = A_1 \cup A_2$ where f(x) is in A_1, f(y) is in A_2, and $A_1 \cup A_2$ is a separation of Y - f(A). Clearly $X-A = f^{-1}(A_1) \cup f^{-1}(A_2)$ where x is in $f^{-1}(A_1)$ and y is in $f^{-1}(A_2)$. To show that $f^{-1}(A_1) \cup f^{-1}(A_2)$ is a separation of X - A, it only needs to be shown that $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are open in X - A. Let $A_1 = U_1 \cap (Y - f(A))$ and $A_2 = U_2 \cap (Y - f(A))$ where both U_1 and U_2 are open in Y. Then $f^{-1}(A_1) = f^{-1}(U_1) \cap (X - A)$ and $f^{-1}(A_2) = f^{-1}(U_2) \cap (X - A)$. Since $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open in X, it follows that $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are open in X - A. Thus $f^{-1}(A_1) \cup f^{-1}(A_2)$ is a separation of X - A with x in $f^{-1}(A_1)$ and y in $f^{-1}(A_2)$. This is a contradiction. Thus $f(\mathcal{Y})$ is non-separated in Y. $\square$

Definition 5.30. Let f be a function from a topological space X onto a space Y. Then f is said to be weakly monotone if the inverse of each point of Y is connected.

Theorem 5.31. Let $\mathcal{Y}$ be a non-separated collection of subsets of a space Y. Let f be an open weakly monotone function, not necessarily continuous, which maps a topological space X onto the space Y. Then the collection $f^{-1}(\mathcal{U})$ of inverse sets of members of $\mathcal{U}$ are non-separated in X.

<u>Proof</u>. Assume that the theorem is false. Let $f^{-1}(A)$ separate two points p and q of $f^{-1}(B)$ where A and B are in $\mathcal{U}$. Let $X - f^{-1}(A) = A_1 \cup A_2$ where A_1 and A_2 are disjoint and both open in $X - f^{-1}(A)$ with p in A_1 and q in A_2. Then $Y - A = f(A_1) \cup f(A_2)$. Since A_1 is open in $X - f^{-1}(A)$, $A_1 = O_1 \cap [X - f^{-1}(A)]$. Thus

$$f(A_1) = f[O_1 \cap (X - f^{-1}(A))] = f(O_1) \cap (Y - A).$$

The latter equality may be proved using the fact that $X - f^{-1}(A)$ is an inverse set. Since $f(O_1)$ is open in Y, it follows that $f(A_1)$ is open in Y - A. In an analogous way it may be shown that $f(A_2)$ is open in Y - A.

Clearly f(p) is in $f(A_1)$ and f(q) is in $f(A_2)$ with both f(p) and f(q) in B. It must be shown that $f(A_1)$ and $f(A_2)$ are disjoint. Suppose z is a point in both $f(A_1)$ and $f(A_2)$. Then $f^{-1}(z)$ is contained in $A_1 \cup A_2$. But this is impossible since z is in both $f(A_1)$ and $f(A_2)$. Thus A separates two points of B in Y. This is a contradiction. Thus $f^{-1}(\mathcal{U})$ is non-separated in X. $\square$

<u>Definition 5.32</u>. A collection of subsets $\mathcal{U}$ of a space X is said to be <u>saturated</u> if A is in $\mathcal{U}$ and p is in X but p is not in A; then there is a member B of $\mathcal{U}$ that separates A from p.

<u>Theorem 5.33</u>. If $\mathcal{U}$ is a saturated collection of subsets of a topological space X and f is a one-to-one open

function from X onto Y, then $f(\mathcal{Y})$ is saturated in Y. Notice that f need not be continuous.

Proof. Let p be any point in Y which is not in a member f(A) of $f(\mathcal{Y})$. Then $f^{-1}(p)$ is not in A. Let B be a member of $\mathcal{Y}$ which separates $f^{-1}(p)$ and A. Let $X - B = A_1 \cup A_2$ be a separation of X - B with $f^{-1}(p)$ in A_1 and A contained in A_2. Then $A_1 = U \cap (X - B)$ for some open set U. Since

$$f(A_1) = f[U \cap (X - B)] = f(U) \cap f(X - B),$$

it follows that $f(A_1)$ is open in Y - f(B). Similarly $f(A_2)$ is open in Y - f(B). Also $Y - f(B) = f(A_1) \cup f(A_2)$. Since $f(A_1)$ and $f(A_2)$ are disjoint, it follows that $f(A_1) \cup f(A_2)$ is a separation of Y - f(B). Clearly p is in $f(A_1)$ and f(A) is contained in $f(A_2)$. Thus f(B) separates p and f(A). Thus the collection $f(\mathcal{Y})$ is saturated in Y. ▯

Since for one-to-one functions, being open and being closed are equivalent, the requirement that f be open may be replaced with the requirement that f be closed in Theorem 5.33.

Paracompactness and Metacompactness

Definition 5.34. Let $\mathcal{U}$ be a cover of a topological space X. A cover $\mathcal{V}$ of the space is said to be a refinement of the cover $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$.

Definition 5.35. Let $\mathcal{U}$ be a family of subsets of a topological space X. Then $\mathcal{U}$ is said to be locally finite if for each point of the space there is an open set which intersects at most a finite number of members of $\mathcal{U}$.

Definition 5.36. A topological space is said to be paracompact if it is regular and every open cover has an open locally finite refinement.

Theorem 5.37. Being paracompact and T_1 is invariant under open mappings that are exactly k to 1.

Proof. Let f be an open mapping which is exactly k to 1 from a paracompact T_1 space X onto Y. It follows that Y is regular and T_1 by a previous theorem. Let $\mathcal{U}$ be any open cover of Y. Then X is covered by the set of all inverses of members of $\mathcal{U}$, that is, $f^{-1}(\mathcal{U})$. Choose a locally finite refinement $\mathcal{V}$ of $f^{-1}(\mathcal{U})$. Then $f(\mathcal{V})$, the collection of images under f of members of $\mathcal{V}$, is an open refinement of $\mathcal{U}$.

It will be shown that $f(\mathcal{V})$ is locally finite. Let p be any point of Y. There are exactly k points $q_1, q_2, \ldots, q_k$ in X that map onto p. About each of these q_i's choose an open set O_i that intersects only a finite number of members of $\mathcal{V}$. Since X is Hausdorff, there is no loss in generality in taking the O_i to be disjoint. Let O be the intersection of all the $f(O_i)$ as i varies from 1 to k. Then O is an open set in Y that contains p. It will be shown that O intersects only a finite number of members of $f(\mathcal{V})$. If V is in $\mathcal{V}$ and intersects O_i for some i, then it is possible for f(V) to intersect O. But there are only a finite number of such V, for there are only k of the sets O_i, and only a finite number of members of $\mathcal{V}$ intersect each one. It is sufficient to show that no other member of $f(\mathcal{V})$

can intersect O. Let V' be a member of $\mathcal{V}$ which does not intersect any O_i. Suppose $f(V')$ intersects O at a point, say y. Then $f^{-1}(y)$ contains at least $k + 1$ points, one in each O_i and one in V'. This is a contradiction. Thus O intersects only a finite number of members of $f(\mathcal{V})$. Thus $f(\mathcal{V})$ is locally finite. Thus Y is paracompact, and the theorem is proved. $\square$

<u>Definition 5.38</u>. Let $\mathcal{U}$ be a family of subsets of a topological space X. Then $\mathcal{U}$ is said to be <u>point finite</u> if each point of X is contained in at most a finite number of members of .

<u>Definition 5.39</u>. A topological space X is said to be <u>metacompact</u> if each open cover has an open, point finite refinement.

<u>Theorem 5.40</u>. Being metacompact is invariant under open finite-to-one mappings.

<u>Proof</u>. Let f be an open finite-to-one mapping which maps a metacompact space X onto a space Y. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U})$ is an open cover of X. Choose an open point finite refinement $\mathcal{V}$ of $f^{-1}(\mathcal{U})$. Then $f(\mathcal{V})$ is an open refinement of $\mathcal{U}$. Let p be any point of Y. Let $q_1, q_2, \ldots, q_k$ be the finite collection of all points, such that $f(q_i) = p$. Only a finite number of members of $\mathcal{V}$ intersect any q_i. But the images of these members of $\mathcal{V}$ are the only members of $f(\mathcal{V})$ which can contain p. Thus, only a finite number of members of $f(\mathcal{V})$ contain p. Thus $f(\mathcal{V})$ is a point finite

refinement of $\mathcal{U}$. Thus Y is metacompact. ▯

Dimension

Definition 5.41. Let A be a subset of a topological space X. Then p is a boundary point of A if every open set containing p intersects both A and the complement of A. The boundary of A is the set of all boundary points of A.

Theorem 5.42. Let f be a homeomorphism from a topological space X onto a space Y. Let A be a subset of X. Then the boundary of f(A) is the image of the boundary of A under f.

Proof. Let p be any point in the boundary of A. Let U be any open set containing f(p). Then $f^{-1}(U)$ is an open set containing p. Thus $f^{-1}(U)$ intersects both A and X - A. Hence, U intersects both f(A) and Y - f(A). Thus f(p) is in the boundary of f(A).

Let q be any point in the boundary of f(A). Let V be any open set containing $f^{-1}(q)$. Then f(V) is an open set containing q. Thus f(V) intersects both f(A) and Y - f(A). Hence V intersects both a and X - A. Thus $f^{-1}(q)$ is in the boundary of A. ▯

Definition 5.43. The dimension of the empty set is -1. No other space has -1 for its dimension.

Definition 5.44. Let p be a point in a topological space X. Then X is said to have dimension $\leq$ n at p if for every neighborhood of U of p there is a neighborhood V containing p and contained in U whose boundary has dimension $\leq$

$n - 1$. Here it is assumed that n is non-negative. The dimension of X at p is said to be n if the dimension of X at p is $\leq n$, but it is false that the dimension of X at p is $\leq n - 1$.

Definition 5.45. A topological space X has dimension n if the dimension of X at each of its points is $\leq n$ and there is at least one point of X at which the dimension is n. The space X is said to have dimension ∞ if the dimension of X is not n for any finite n.

Theorem 5.46. The dimension of a space is invariant under a local homeomorphism.

Proof. The proof will be by induction. The result is clearly true for dimension equal to -1. For dimension equal to 0 the result follows, since sets with empty boundaries, that is, both open and closed sets, are mapped into sets which are both open and closed by local homeomorphisms.

Assume that the result is true for all dimensions less than or equal to $n - 1$. The result is to be shown for dimension n. Let h be a local homeomorphism from an n dimensional topological space X onto a space Y. Let y be any point in Y. Let U be a neighborhood of y. Choose a point x in X so that $h(x) = y$. Then $h^{-1}(U)$ is an open set containing x. Choose an open set O containing x so that h restricted to O is a homeomorphism. Choose a neighborhood V of x and contained in both O and $h^{-1}(U)$, which has a boundary with dimension less than n. Then $h(V)$ is a neighborhood of y which is contained in U. The boundary of $h(V)$ is the image of the boundary of V under h, since h restricted to an open set containing V is a homeomorphism. But the dimension of the boundary of V is less than n. Thus by our inductive assumption, the dimension of $h(V)$ is less than n. Thus the dimension of Y is less than or equal to n.

To show that the dimension is equal to n, choose a point x in X at which X has dimension n. There is a neighborhood, say N, of x such that any neighborhood of x that is contained in N has a boundary with dimension $n - 1$ or greater. There is no loss in generality in choosing N so that h restricted to N is a homeomorphism. Clearly $h(N)$ is a neighborhood of $h(x)$. Let U be any neighborhood of $h(x)$ which is contained in $h(N)$. Let g be the inverse function for h restricted to N. Then g is a homeomorphism from $h(N)$ onto N. Assume that the dimension of the boundary of U is less than $n - 1$. Then $g(U)$ is a neighborhood of x contained in N whose boundary has dimension less than $n - 1$. This is a contradiction. Thus the dimension of the boundary of U is $n - 1$ or more. Hence the dimension of Y at $h(x)$ is greater than or equal to n. With what has already been shown, it is clear that the dimension of Y is exactly n. $\square$

Suppose that f is a function from a space X onto a space Y. Suppose the dimension of X is n. It is sometimes possible to obtain bounds on the dimension of Y even though the dimension of Y may not be known exactly. Next two such results are stated without proof concerning closed mappings. The abbreviation dim X for the dimension of the space X is used.

Theorem 5.47. Let f be a closed continuous function from a separable metric space X onto a separable metric space Y. Suppose dim X - dim Y = k where k is a positive integer. Then there is a point y in Y so that $f^{-1}(y)$ has dimension at least k. (See Hurewicz and Wallman [2], pp. 91-93.)

In this theorem the dimension of the range is lower than the dimension of the domain. An analogous result holds if the dimension of the range has a higher dimension than the range.

Theorem 5.48. Let f be a closed mapping from a separable metric space X onto a separable metric space Y. Suppose that dim Y - dim X = k where k is positive. Then f is of degree at least k + 1. That is, $f^{-1}(y)$ contains at least k + 1 points for some y in Y. (See Hurewicz and Wallman [2], p. 93.)

Definition 5.49. A topological space has covering dimension n if each open cover has an open refinement in which at most n + 1 sets intersect at a point and there is at least one open cover for which at least n + 1 sets intersect at some point in each open refinement.

Definition 5.50. A topological space has countable covering dimension n if each countable open cover has a countable open refinement in which at most n + 1 sets intersect at a point, and there is at least one open cover for which at least n + 1 sets intersect at some point in each countable open refinement.

A definition of finite covering dimension may be obtained by replacing the word countable by the word finite everywhere in Definition 5.50.

Theorem 5.51. If f is an open mapping of degree k from a topological space X of covering dimension n onto a space Y, then the covering dimension of Y is at most [(n+1)k-1].

Proof. Let $\mathcal{U}$ be an arbitrary open cover of Y. Then $f^{-1}(\mathcal{U})$ is an open cover of X. Choose an open refinement $\mathcal{V}$ of $f^{-1}(\mathcal{U})$ such that at most n + 1 members of $\mathcal{V}$ intersect. Now $f(\mathcal{V})$ is an open refinement of $\mathcal{U}$ in Y. Let p be a point in Y. Then $f^{-1}(p)$ contains at most k different points. At each of these at most (n + 1) members of $\mathcal{V}$ can intersect. Hence at p at most (n + 1)k members of $f(\mathcal{V})$ can intersect. Thus an open refinement of the arbitrary open cover $\mathcal{U}$ of Y in which at most (n + 1)k members intersect is found. Thus the covering dimension of Y is at most [(n+1)k-1]. □

In Theorem 5.51, covering dimension may be replaced by countable covering dimension or by finite covering dimension without affecting the correctness of the theorem. The proof is essentially the same as before and is omitted.

In separable metric spaces it is well known that finite covering dimension is equivalent to the dimension previously defined. (See Hurewicz and Wallman [2], p. 67.) Thus, for separable metric spaces, the following theorem may be used with either definition of dimension.

Since the concept of dimension is so closely related to boundary points, the following results are included here.

Theorem 5.52. Let f be a function which is both open and closed, but not necessarily continuous, from a topological space X onto a topological space Y. Let A be a subset of X. Then every boundary point of f(A) is of the form f(p) where p is in the boundary of A.

Proof. Since $f(\overline{A})$ is a closed set containing f(A), all boundary points of f(A) are of the form f(p) where p is in the boundary of A or in the interior of A. But if p is in the interior of A, it follows that f(p) is not a boundary point of f(A) by the fact that f is open. Thus the result follows. ▯

Corollary 5.53. Let f be a function which is both open and closed from a topological space X onto a topological space Y. Let A be a subset of X. Then the cardinality of the set of boundary points of f(A) is less than or equal to the cardinality of the set of boundary points of A.

Proof. The result is clear since each boundary point of f(A) is the image of a boundary point of A. ▯

Theorem 5.54. Let B be a subset of a topological space X such that each point p of B has arbitrarily small open sets in X whose boundaries are finite. That is, inside of each open set containing p there are open sets with finite boundaries. If f is an open, closed, continuous function from X onto Y, then each point of f(B) also has arbitrarily small open sets with finite boundaries.

<u>Proof</u>. Let q be any point of f(B). Let U be any open set containing q. Then $f^{-1}(U)$ is an open set about some point p in B with f(p) = q. Choose an open set V containing p and contained inside $f^{-1}(U)$ with a finite boundary. Then f(V) is an open set containing q which has a finite boundary and is contained in U. This is the desired result. $\Box$

If the word <u>finite</u> is replaced by the word <u>countable</u> everywhere in the preceding theorem and proof, another correct theorem is obtained.

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