In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

VISCOUS COMPRESSIBLE FLUID FLOW UNDER THE INFLUENCE OF A RESONANT ACOUSTIC FIELD IN A CIRCULAR TUBE

A THESIS

Presented to

The Faculty of the Graduate Division

by

Donald Arthur Willoughby

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

March, 1963

3b
12p.

VISCOUS COMPRESSIBLE FLUID FLOW UNDER THE INFLUENCE OF A RESONANT ACOUSTIC FIELD IN A CIRCULAR TUBE

Approved:

Date approved by Chairman: May 6, 1963

ACKNOWLEDGEMENTS

I wish to thank my advisor, Professor Kenneth R. Purdy, who suggested the problem, for his direction, advice, encouragement, and assistance. I also wish to thank Dr. Charles W. Gorton and Dr. Henderson C. Ward for reading the thesis. I am grateful to the Engineering Experiment Station of the Georgia Institute of Technology and to the Air Force for partially supporting my study under Contract AF-33(616)-8396.

My wife Barbara is in no small way responsible, through encouragement and understanding, for making the year of graduate study which led to this thesis successful. In addition, she has been of immense help in writing this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

The mathematical model of the problem consists of a viscous compressible fluid flowing laminarly in a circular tube. A resonant acoustic field is imposed on the fluid in the tube. The flow is assumed to be fully developed.

The equations of motion of a compressible fluid for constant viscosity are written in cylindrical coordinates and simplified by the condition of axial symmetry. The general method of solution of the resulting equations consists of (1) separating the velocities into time-dependent and time-average components, (2) simplifying the equations of motion by an order of magnitude analysis and by making simplifying assumptions about the sound field, (3) taking the time-average of the continuity and momentum equations, and (4) solving the simplified differential equations. Approximate solutions are obtained for the time-dependent and the time-average velocities.

A time-average stream function was obtained which predicts the steady vortex flow which has been observed experimentally for a Kundt's tube undergoing resonant acoustic vibrations. The maximum vortex thickness, the distance from the tube wall divided by the tube radius, is shown to be to a first approximation solely a function of the parameter $M_s^2/\overline{M}$. M_s is the acoustic Mach number which is based on the maximum amplitude of the time-dependent axial velocity, and $\overline{M}$ is the throughflow Mach number, which is based on the average axial velocity. Qualitatively, the time-average flow is quite similar to that

of channel flow as analytically determined by Purdy*. However, the predicted vortex thickness is greater for channel flow than for tube flow.

---

*K. R. Purdy, Viscous Flow Under the Influence of a Resonant Acoustic Field, Ph. D. Thesis, Georgia Institute of Technology, (in preparation).

CHAPTER I

INTRODUCTION

Background

It is well known that a tube subjected to a resonant sound field exhibits phenomena due to the presence of the sound. Kundt's tube is a classical physics experiment in which a glass tube containing dust filled air is subjected to a resonant sound field; the dust particles collect in distinct piles at the velocity nodes, thus demonstrating the presence of the standing sound waves. Lord Rayleigh [1]*, by investigating certain special points in a Kundt's tube, was able to deduce that there is a general circulation of the fluid from loops to nodes near the wall and in the reverse direction near the center of the tube.

Westervelt [2] considered the theory of general circulation caused by a sound field. Later Spurlock [3] considered the effects of acoustic vibrations on air flowing in a tube, but he limited his analysis to incompressible fluids. Recently Purdy [4] developed a solution for the velocity field of a compressible fluid under the influence of a resonant acoustic field in a channel with through flow.

Statement of the Problem

The mathematical model of the problem under consideration in this research consists of a viscous compressible fluid with constant

*Numbers in brackets refer to references in the Bibliography at the end of the thesis.

dynamic viscosity undergoing fully developed laminar flow in a circular tube. A sketch of the tube is shown in Figure 1. A resonant acoustic field is imposed on the fluid in the tube. Provided that the length of the tube is an integral multiple of the sonic wavelength, a resonant sound field is formed when a steady periodic sound wave is generated at one end of the tube. Under these conditions standing sound waves are formed in the tube; these waves have greatly amplified sound levels because of the resonant conditions.

The general method of solution used by Purdy for channel flow will be used to obtain the solution for the geometry of a circular tube. The general method consists of (1) separating the velocities into time-dependent and time-average components, (2) simplifying the equations of motion for a compressible fluid by an order of magnitude analysis and by making simplifying assumptions about the sound field, (3) taking the time-average of the continuity and momentum equations, and (4) obtaining approximate solutions to the simplified differential equations.

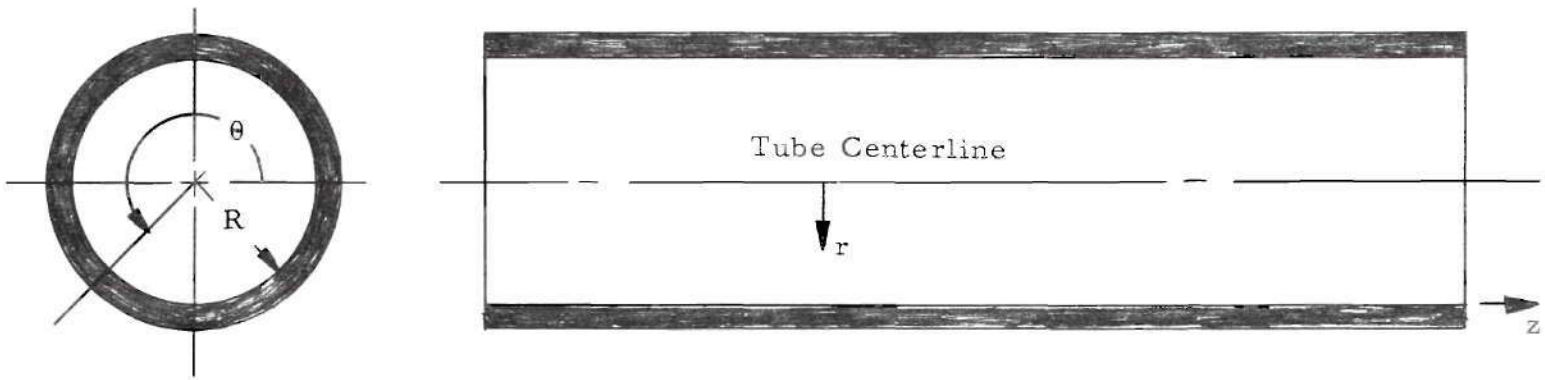


Figure 1. Flow System Model

CHAPTER II

DEVELOPMENT OF THE DIFFERENTIAL EQUATIONS

Fundamental Equations

The general vector equations which govern the motion of a compressible fluid with constant viscosity are

Continuity

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \bullet \vec{V}) + (\nabla \rho) \bullet \vec{V} = 0 \quad , \quad \text{and} \tag{1}$$

Momentum

$$\frac{\partial \vec{V}}{\partial t} + \nabla \frac{V^2}{2} - \vec{V} \times (\nabla \times \vec{V}) = W - \frac{\nabla P}{\rho} + \frac{\nu}{3} \nabla(\nabla \bullet \vec{V}) + \nu \nabla^2 \vec{V} \, . \tag{2}$$

If these equations are expanded in cylindrical coordinates as shown in Figure 1 and are simplified by the condition of axial symmetry, the following equations are obtained:

Continuity

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial r} + v \frac{\partial \rho}{\partial r} + \frac{\rho}{r} v + \rho \frac{\partial u}{\partial z} + u \frac{\partial \rho}{\partial z} = 0 \tag{3}$$

Momentum

$$\rho \left[\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right] = + Y - \frac{\partial P}{\partial r} \tag{4}$$

$$+ \mu \left[\frac{4}{3} \frac{\partial^2 v}{\partial r^2} + \frac{4}{3} \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{4}{3} \frac{v}{r^2} + \frac{1}{3} \frac{\partial^2 u}{\partial r \partial z} \right]$$

and

$$\rho \left[\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right] = + Z - \frac{\partial P}{\partial z} \tag{5}$$

$$+ \mu \left[\frac{1}{3} \frac{\partial^2 v}{\partial r \partial z} + \frac{1}{3r} \frac{\partial v}{\partial z} + \frac{4}{3} \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] .$$

Simplifying Assumptions

Additional assumptions are necessary in order to simplify the above equations. These assumptions are listed below.

1. The body forces are assumed to be negligible.

2. The pressure is assumed to consist of a time-mean component, $\overline{p}(r, z)$, and a time-dependent component, $p_1(z, t)$. The time-dependent component is assumed to be the same as the resonant acoustic vibrations in an inviscid fluid. These vibrations are independent of the r-direction and are periodic in both time, t, and position, z. Therefore,

$$p(r, z, t) = \overline{p}(r, z) + p_1(z, t) \quad . \tag{6}$$

3. It is assumed that spatial variations in density have only a small effect on the flow and that the time-mean density may be considered constant and equal to ρ_0. The periodic density component ρ_1 is assumed to be related to p_1 by the isentropic relationship. This gives

$$\rho(r, z, t) = \rho_0 + \rho_1(z, t) \tag{7}$$

The isentropic relationship is often used to relate periodic components of density and pressure.

4. It is assumed that the velocity components $u(r, z, t)$, and $v(r, z, t)$, may be written as the sum of a periodic component and a time-mean component, a technique that is attributable to Lin [5].

5. The assumed change of variables listed below can be used to write the continuity and momentum equations in non-dimensional form. They are

$$v' = v/U_0 = \overline{v}' + v_1' \tag{8a}$$

$$u' = u/U_0 = \overline{u}' + u_1' \tag{8b}$$

$$r' = r/\lambda \tag{8c}$$

$$z' = z/\lambda \tag{8d}$$

$$p' = p/\left(\rho_0 U_0^2\right) = \overline{p}' + p_1' \tag{8e}$$

$$\rho' = \rho/\rho_0 = 1 + \rho_1' \tag{8f}$$

$$\mu' = \mu/\mu_0 = \mu_0' = 1 \tag{8g}$$

$$t' = t/(\lambda/U_0) \tag{8h}$$

6. These dimensionless quantities are assumed to have the orders of magnitude listed below. The quantity δ may be defined as $\delta \ll 1$, or as $\delta = O(0.01)$

$$r' = O(\delta) \tag{9a}$$

$$z' = O(1) \tag{9b}$$

$$v_1' = O(\delta) \tag{9c}$$

$$\overline{v}' = O(\delta^2) \tag{9d}$$

$$u_1' = O(1) \tag{9e}$$

$$\overline{u}' = O(\delta) \tag{9f}$$

$$\rho_0' = O(1) \tag{9g}$$

$$\rho_1' = O(\delta) \tag{9h}$$

Resulting Differential Equations

Continuity

If the dimensionless continuity equation is written in terms of time-average and time-dependent quantities and if only the terms of order of magnitude one or greater are retained, the following dimensional equation results:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial r} + \frac{\rho_0 v_1}{r} + \rho_0 \frac{\partial u_1}{\partial z} = 0 \quad . \tag{10}$$

If the time-average over one cycle of the dimensionless continuity equation is taken, the following dimensional equation results after rearrangement:

$$\frac{\partial \overline{v}}{\partial r} + \frac{\overline{v}}{r} + \frac{\partial \overline{u}}{\partial z} = \tag{11}$$

$$- \frac{1}{\rho_0} \left[\overline{\rho_1 \frac{\partial v_1}{\partial r} + v_1 \frac{\partial \rho_1}{\partial r} + \frac{\rho_1 v_1}{r} + \rho_1 \frac{\partial u_1}{\partial z} + u_1 \frac{\partial \rho_1}{\partial z}} \right] .$$

Momentum

The pressure may be eliminated from the dimensionless momentum equation by noting that

$$\frac{\partial^2 P}{\partial r \partial z} = \frac{\partial^2 P}{\partial z \partial r}$$

and that

$$\frac{\partial}{\partial r}\ [\text{dimensionless (5)}] - \frac{\partial}{\partial z}\ [\text{dimensionless (4)}] \quad (12)$$

$$= \text{equation (12)}.$$

The Reynolds number, $\rho_0 U_0 \lambda / \mu_0$, is assumed to be of order $1/\delta^2$ or less. If only the terms that are order $1/\delta$ or larger are retained, equation (12) in dimensional terms becomes

$$\nu_0 \frac{\partial}{\partial r}\left[\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r}\right] = \frac{\partial}{\partial r}\left[\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial r} + u_1 \frac{\partial u_1}{\partial z}\right] . \quad (13)$$

By retaining only terms of order one or larger, the time-average of equation (12) gives the dimensional equation

$$\nu_0 \frac{\partial}{\partial r}\left[\frac{\partial^2 \overline{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}}{\partial r}\right] = \frac{\partial Q}{\partial r} \quad (14)$$

where

$$Q = \overline{v_1 \frac{\partial u_1}{\partial r} + u_1 \frac{\partial u_1}{\partial z} + \frac{\rho_1}{\rho_0}\frac{\partial u_1}{\partial t} + \frac{\rho_1}{\rho_0} v_1 \frac{\partial u_1}{\partial r} + \frac{\rho_1}{\rho_0} u_1 \frac{\partial u_1}{\partial z}} . \quad (15)$$

Boundary Conditions

Time-Dependent Velocities

Equations (10) and (13) must be solved to obtain the solution of the time-dependent velocities subject to the following three boundary conditions:

$$u_1 = v_1 = 0 \quad \text{at } r = R \quad (16a)$$

$$\frac{\partial u_1}{\partial r} = 0, \ v_1 = 0 \quad \text{at } r = 0 \quad (16b)$$

$$u_1 = - U_0 \cos(\omega t), \quad \text{at } z = 0, \ r = 0 \quad . \tag{16c}$$

The first and second conditions are the no-slip and the symmetry conditions respectively. The assumed acoustic field requires the third condition. This requirement is reasonable since Sanders [6] has shown that, except very near the wall, the velocity is the same in this case as that in an inviscid compressible fluid undergoing resonant acoustic vibrations.

Time-Average Velocities

Equations (11) and (14) must be solved to obtain the solution of the time-average velocities subject to the following boundary conditions:

$$\overline{u} = \overline{v} = 0 \quad \text{at } r = R \tag{17a}$$

$$\frac{\partial \overline{u}}{\partial r} = 0, \ \overline{v} = 0 \quad \text{at } r = 0 \tag{17b}$$

$$2 \pi \rho_0 \int_0^R r \overline{u} \, d r = \pi R^2 \rho_0 \overline{U} = \text{constant} \quad . \tag{17c}$$

The first and second boundary conditions are again the symmetry and the no-slip conditions respectively. The third condition requires a steady time-mean flow based on the assumption of a constant time-mean density.

CHAPTER III

SOLUTION OF THE TIME-DEPENDENT VELOCITIES

Successive Approximation Method

A method of successive approximations due to Schlichting [7] is used to obtain a solution to equations (10) , (13) , and (16) . The time dependent velocities $u_1(r, z, t)$ and $v_1(r, z, t)$ are assumed to consist of two terms. They are

$$u_1(r, z, t) = u_{10}(r, z, t) + u_{11}(r, z, t) \quad \text{and} \tag{18}$$

$$v_1(r, z, t) = v_{10}(r, z, t) + v_{11}(r, z, t) \tag{19}$$

where the first approximations, v_{10} and u_{10}, are the solution to

Continuity

$$\frac{\partial v_{10}}{\partial r} + \frac{v_{10}}{r} + \frac{\partial u_{10}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \rho_1}{\partial t} \quad , \quad \text{and} \tag{20}$$

Momentum

$$\nu_0 \left\{ \frac{\partial^2 u_{10}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{10}}{\partial r} \right\} = \frac{\partial u_{10}}{\partial t} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} \quad ; \tag{21}$$

and the second approximations, v_{11} and u_{11}, are the solution to

Continuity

$$\frac{\partial v_{11}}{\partial r} + \frac{v_{11}}{r} + \frac{\partial u_{11}}{\partial z} = 0 \quad , \quad \text{and} \tag{22}$$

Momentum

$$\nu_0 \left\{ \frac{\partial^2 u_{11}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{11}}{\partial r} \right\} = \tag{23}$$

$$+ \frac{\partial u_{11}}{\partial t} + v_{10} \frac{\partial u_{10}}{\partial r} + u_{10} \frac{\partial u_{10}}{\partial z}$$

with the boundary conditions:

$$u_{10} = u_{11} = v_{10} = v_{11} = 0 \quad \text{at } r = R \tag{24}$$

$$\frac{\partial u_{10}}{\partial r} = \frac{\partial u_{11}}{\partial r} = v_{10} = v_{11} = 0 \quad \text{at } r = 0 \tag{25}$$

$$u_{10} = - U_0 \cos(\omega t), \; u_{11} = 0 \quad \text{at } r = 0, \; z = 0 \quad . \tag{26}$$

Equation (20) is obtained from an order of magnitude analysis of equation (5) or from integrating equation (13) with respect to r.

In this analysis only the first approximations, u_{10} and v_{10}, are obtained. However, Purdy showed that, at least in the similar problem involving channel flow, the second approximations are small in relation to the first approximations.

Assumptions for First Approximations

Additional assumptions which are necessary in order to further simplify the problem are listed below.

1. The time-dependent component of pressure, $p_1(z, t)$, is assumed to be the same as that of a stationary inviscid fluid undergoing resonant vibrations. This pressure is given by Morse [8] as

$$p_1(z, t) = - i \rho_0 \, c U_0 \sin\left(\frac{\omega z}{c}\right) e^{-i\omega t} \quad . \tag{27}$$

2. It is also assumed that the first approximation to the z-velocity is of the form

$$u_{10}(r, z, t) = -U_0 \cos\left(\frac{\omega z}{c}\right) F_1(r)\, e^{-i\omega t} \tag{28}$$

where $F_1(r)$ is a yet to be determined function of r.

3. By neglecting pressure-density variations due to $\overline{p}(r, z)$ and $\overline{T}(r, z)$, by assuming that the fluid under consideration may be accurately approximated by a perfect gas with constant specific heats undergoing an isentropic compression-rarefaction process, and by assuming that $\left[\frac{P_1}{\overline{P}}\right] = O(\delta)$, it can be shown that

$$p_1 \approx c^2 \rho_1 \quad . \tag{29}$$

Resulting Velocity Equations

z-Velocity Component

Using the assumptions listed above, equation (21) becomes

$$\left\{F_1'' + \frac{F_1'}{r} + i\frac{\omega}{\nu_0} F_1 - i\frac{\omega}{\nu_0}\right\} U_0 \cos\left(\frac{\omega z}{c}\right) e^{-i\omega t} = 0 \quad . \tag{30}$$

If equation (30) is true for all z and t, then

$$F_1'' + \frac{F_1'}{r} + i\frac{\omega}{\nu_0} F_1 = +i\frac{\omega}{\nu_0} \quad . \tag{31}$$

If only the solutions that are finite at $r = 0$ are included, then the general solution to equation (31) is

$$F_1(r) = A\, J_0\left(i^{\frac{1}{2}} \beta r\right) + 1 \tag{32}$$

where A is a complex constant to be determined by the boundary conditions and where $\beta = \left[\frac{\omega}{\nu_0}\right]^{\frac{1}{2}}$. The function, $J_0\left(i^{\frac{1}{2}}\beta r\right)$, is a complex Bessel function of the first kind. If the appropriate boundary conditions are applied to the real part of equation (28), then the following result is obtained for the periodic z-velocity:

$$u_{10}(r, z, t) = U_0 \cos\left(\frac{\omega z}{c}\right)\left[\left\{\frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)}\ \mathrm{ber}(\beta r) + \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)}\ \mathrm{bei}(\beta r) - 1\right\}\cos(\omega t) + \left\{\frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)}\ \mathrm{ber}(\beta r) - \frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)}\ \mathrm{bei}(\beta r)\right\}\sin(\omega t)\right] . \tag{33}$$

A description of Kelvin's ber and bei functions is given by McLachlan [9]. The modulus, $M_0(\beta r)$, is defined by

$$M_0^2(\beta r) = \mathrm{ber}^2(\beta r) + \mathrm{bei}^2(\beta r) .$$

Boundary conditions (24) and (25) are satisfied exactly, and boundary condition (26) is satisfied approximately for u_{10} as given in equation (33).

Approximations of the Bessel functions for large βr are presented in Appendix D. After these approximations are substituted into equation (33), the following equation is obtained :

$$u_{10}(r, z, t) = -U_0 \cos\left(\frac{\omega z}{c}\right) \cdot \left\{\cos(\omega t) - \left[\sqrt{\frac{R}{r}}\, e^{-\frac{\beta R}{\sqrt{2}}\left(1 - \frac{r}{R}\right)}\right]\cos\left(\frac{\beta R}{\sqrt{2}}\left[1 - \frac{r}{R}\right] - \omega t\right)\right\} . \tag{34}$$

r-Velocity Component

After differentiation with respect to z, equation (33) may be substituted into equation (20). With equations (27) and (29) all of the terms in equation (20) are then known, and it may be integrated directly. Upon successive integration with respect to r and upon imposition of the requirement that v_{10} remain finite, the following result is obtained for the real part of v_{10}:

$$v_{10}(r, z, t) = + \frac{\omega}{\beta} \frac{U_0}{c} \sin\left(\frac{\omega z}{c}\right) \left[\left\{ \frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r) - \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r) \right\} \cos(\omega t) + \left\{ \frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r) + \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r) \right\} \sin(\omega t) \right] . \tag{35a}$$

Equation (35a) satisfies the symmetry boundary condition, equation (25), exactly; but the no-slip condition, equation (24), is not satisfied. However, the magnitude of the velocity at the wall, $v_{10}(R, z, t)$, is quite small in relation to U_0 or to the magnitude of the periodic z-velocity. The alternative to this arrangement is to satisfy the no-slip condition exactly and to allow a centerline r-velocity of infinite magnitude. The chosen arrangement was considered to be the most acceptable arrangement of the two possibilities.

After the approximations from Appendix D are substituted into equation (35a), the following result is obtained:

$$v_{10}(r, z, t) = + \frac{\omega}{\beta} \frac{U_0}{c} \sin\left(\frac{\omega z}{c}\right) \sqrt{\frac{R}{r}} \; e^{-\frac{\beta R}{\sqrt{2}}\left(1 - \frac{r}{R}\right)} \bullet \tag{35b}$$

$$\bullet \left\{ \sin\left(\frac{\beta R}{\sqrt{2}} \left[1 + \frac{r}{R}\right]\right) \cos(\omega t) \right.$$

$$\left. + \cos\left(\frac{\beta R}{\sqrt{2}} \left[1 - \frac{r}{R}\right] - \frac{\pi}{4}\right) \sin(\omega t) \right\} .$$

Dimensionless Velocity Components

Equations (33) and (34) for u_{10} and equations (35a) and (35b) for v_{10} may be written in dimensionless form by dividing the equations by the quantity U_0. A plot of the dimensionless time-dependent z-velocity, $u_{10}(r, z, t)$, is shown in Figure 2.

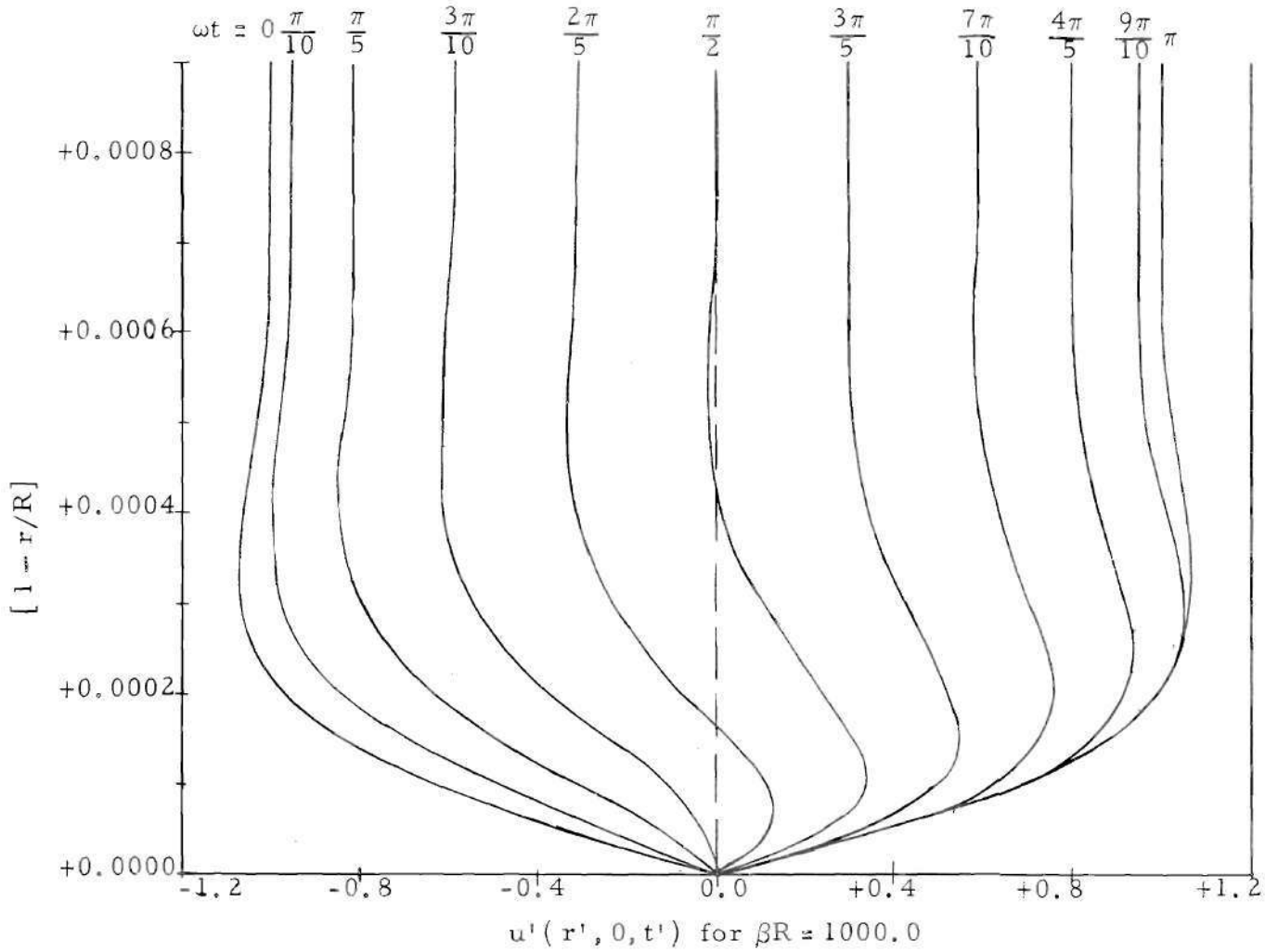


Figure 2. Time-Dependent Velocity Profiles

CHAPTER IV

SOLUTION OF THE TIME-AVERAGE VELOCITIES AND STREAM FUNCTION

Time-Average z-Velocity

A first approximation is now available for all of the quantities that appear in equation (15) for Q. After the indicated mathematical steps are taken including the time averaging process, equation (15) reduces to

$$Q = -\frac{\omega}{4}\,\frac{U_0}{c}\,\sin\left(\frac{2\omega z}{c}\right)\left\{\frac{M_0^2(\beta r)}{M_0^2(\beta R)} + 2 - 3\,\frac{\mathrm{ber}(\beta R)\,\mathrm{ber}(\beta r) + \mathrm{bei}(\beta R)\,\mathrm{bei}(\beta r)}{M_0^2(\beta R)}\right\} . \tag{36}$$

Substitution of equation (36) into equation (14), integration of equation (14) three times with respect to r, and imposition of the required boundary conditions gives the following result:

$$\bar{u}(r,z) = +\frac{1}{4}\,\frac{U_0^2}{c}\,\sin\left(\frac{2\omega z}{c}\right)\left\{-\frac{1}{2}\,\frac{M_0^2(\beta r)}{M_0^2(\beta R)} + 3\,\frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)}\,\mathrm{bei}(\beta r) - 3\,\frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)}\,\mathrm{ber}(\beta r) + \frac{1}{2} + \left(\frac{7\sqrt{2}}{\beta R} - 1\right)\left(1 - \frac{r^2}{R^2}\right)\right\} + 2\,\bar{U}\left(1 - \frac{r^2}{R^2}\right) . \tag{37a}$$

The integration which is required in order to obtain equation (37a) is straightforward, though somewhat tedious, except for one case which is discussed in Appendix C. McLachlan's Bessel Functions for Engineers [9] proved quite helpful in carrying out the above integration.

After the approximations of the ber and bei functions given in Appendix D are substituted into equation (37a), the following result is obtained:

$$\bar{u}(r,z) = 2\,\bar{U}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{37b}$$

$$- \frac{U_0^2}{8c}\sin\left(\frac{2\omega z}{c}\right)\left\{2\left[1 - \frac{7\sqrt{2}}{\beta R}\right]\left[1 - \left(\frac{r}{R}\right)^2\right] - 1\right.$$

$$+ 6\sqrt{\frac{R}{r}}\, e^{-\frac{\beta R}{\sqrt{2}}\left(1 - \frac{r}{R}\right)} \sin\left(\frac{\beta R}{\sqrt{2}}\left[1 - \frac{r}{R}\right]\right)$$

$$\left. + \frac{R}{r}\, e^{-\frac{2\beta R}{\sqrt{2}}\left(1 - \frac{r}{R}\right)}\right\} .$$

Time-Average r-Velocity

The time-average r-velocity can be obtained by solving equation (11) subject to boundary conditions (17). A first approximation to all of the terms in equation (11) is now known within a single differentiation, and these terms may be substituted into equation (11). Integration of this equation with respect to r and application of boundary conditions (17) gives the equation

$$\overline{v}(r,z) = + \frac{\omega}{4\beta} \frac{U_0^2}{c^2} \left\{ \frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r) + \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r) \right\} \quad (38a)$$

$$+ \frac{\omega}{4\beta} \frac{U_0^2}{c^2} \cos\left(\frac{2\omega z}{c}\right) \left\{ 6 \frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r) + 6 \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r) \right.$$

$$+ \frac{\mathrm{ber}(\beta r)\,\mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\,\mathrm{ber}'(\beta r)}{M_0^2(\beta R)} - \frac{\beta r}{2}$$

$$\left. - \beta r \left(\frac{7\sqrt{2}}{\beta R} - 1 \right) \left(1 - \frac{r^2}{2R^2} \right) \right\} .$$

Equation (38a) may be simplified by neglecting terms of order of magnitude $\frac{1}{\beta} << 1$. If this is done, the following result is obtained:

$$\overline{v}(r,z) = + \frac{\omega}{8} \frac{U_0^2}{c^2} \cos\left(\frac{2\omega z}{c}\right) \left[r \left(1 - \frac{r^2}{R^2} \right) \right] . \quad (38b)$$

Equations (38) satisfy the symmetry boundary condition, equation (17b), exactly; and they satisfy the no-slip condition, equation (17a), to a very close approximation.

Dimensionless Velocity Components

Equations (37) and (38) for $\overline{u}(r,z)$ and $\overline{v}(r,z)$ respectively may be written in dimensionless form by dividing the equations by the quantity U_0. This is exactly the step that was taken in order to express the time-dependent velocity components in dimensionless terms. A plot of the dimensionless time-average z-velocity is shown in Figure 3.

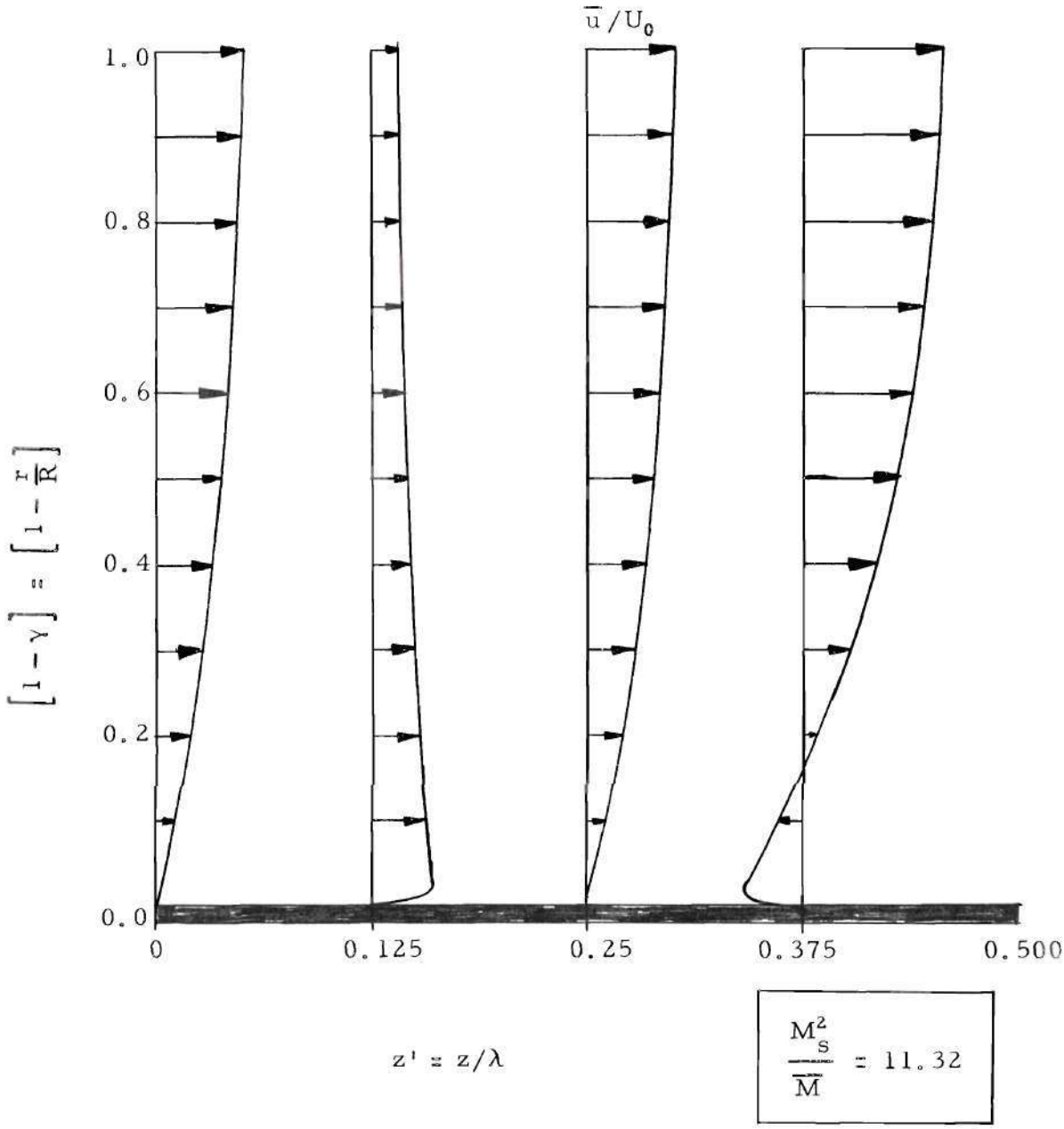


Figure 3. Time-Mean Velocity Profiles

Time-Average Stream Function

Stream Function Development

If spatial variations in the time-mean density are neglected, a stream function may be defined as

$$\psi = \psi_0 + 2\pi\rho_0 \int_0^r r\,\overline{u}(r, z)\,dr \tag{39}$$

where ψ_0 is a constant of integration such that $\psi = 0$ at $r = 0$. Equation (37) for $\overline{u}(r, z)$ may be substituted into equation (39). Upon integration this becomes

$$\psi = \psi_0 + \frac{\pi\rho_0}{4}\,\frac{U_0^2}{c}\,\sin\left(\frac{2\omega z}{c}\right)\left[\frac{-r\left\{\mathrm{ber}(\beta r)\,\mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\,\mathrm{ber}'(\beta r)\right\}}{\beta\,M_0^2(\beta R)}\right. \tag{40}$$

$$- 6\,\frac{r}{\beta}\,\frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)}\,\mathrm{ber}'(\beta r) - 6\,\frac{r}{\beta}\,\frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)}\,\mathrm{bei}'(\beta r)$$

$$\left. + \left\{\frac{7\sqrt{2}}{\beta R} - 1\right\}\left\{r^2 - \frac{r^4}{2R^2}\right\} + \frac{r^2}{2}\right]$$

$$+ 2\pi\rho_0\,\overline{U}\left\{r^2 - \frac{r^4}{2R^2}\right\} .$$

The requirement that $\psi = 0$ at $r = 0$ yields the result that $\psi_0 = 0$ since $\mathrm{ber}'(0) = 0$ and $\mathrm{bei}'(0) = 0$.

Dimensionless Stream Function

Equation (40) for the stream function may be written in dimensionless form if it is divided by $\rho_0\,\pi R^2\,\overline{U}$. Also let

$r/R = \gamma$

$M_s = U_0/c$ (sonic Mach number), and

$\overline{M} = \overline{U}/c$ (through-flow Mach number) .

If the above substitutions are made and if the approximations for the ber and bei functions for $\beta r >> 1$ given in Appendix D are used, the following equation can be obtained from equation (40):

$$\overline{\psi} = 2\left[\gamma^2 - \frac{1}{2}\gamma^4\right] + \frac{1}{4}\frac{M_s^2}{\overline{M}}\sin\left(\frac{2\omega z}{c}\right) \tag{41}$$

$$\cdot\left[-\frac{\sqrt{2}}{2\beta R}e^{-\sqrt{2}\beta R[1-\gamma]} - \frac{6}{\beta R}\sqrt{\gamma}\,e^{-\frac{\beta R}{\sqrt{2}}(1-\gamma)}\left\{\cos\left(\frac{\beta R}{\sqrt{2}}[1-\gamma] - \frac{\pi}{4}\right)\right\}\right.$$

$$\left. + \left[1 - \frac{7\sqrt{2}}{\beta R}\right]\left[\frac{1}{2}\gamma^4 - \gamma^2\right] + \frac{1}{2}\gamma^2\right]$$

By neglecting terms of order $\frac{1}{\beta R} << 1$, equation (41) becomes

$$\overline{\psi} = 2\gamma^2\left\{1 - \frac{1}{2}\gamma^2 - \frac{1}{16}\frac{M_s^2}{\overline{M}}\sin\left(\frac{2\omega z}{c}\right)[1-\gamma^2]\right\} . \tag{42}$$

Maximum Vortex Thickness

An equation for the dimensionless streamline $\overline{\psi} = 1$ can be obtained by substituting the value 1 for $\overline{\psi}$ in equation (42) and solving the resulting equation for γ_s. This gives

$$\gamma_s = \left[\frac{+\frac{1}{8}\frac{M_s^2}{\overline{M}}\sin\left(\frac{2\omega z}{c}\right) - 2 \pm \frac{1}{8}\frac{M_s^2}{\overline{M}}\sin\left(\frac{2\omega z}{c}\right)}{+\frac{1}{4}\frac{M_s^2}{\overline{M}}\sin\left(\frac{2\omega z}{c}\right) - 2}\right]^{\frac{1}{2}} . \tag{43}$$

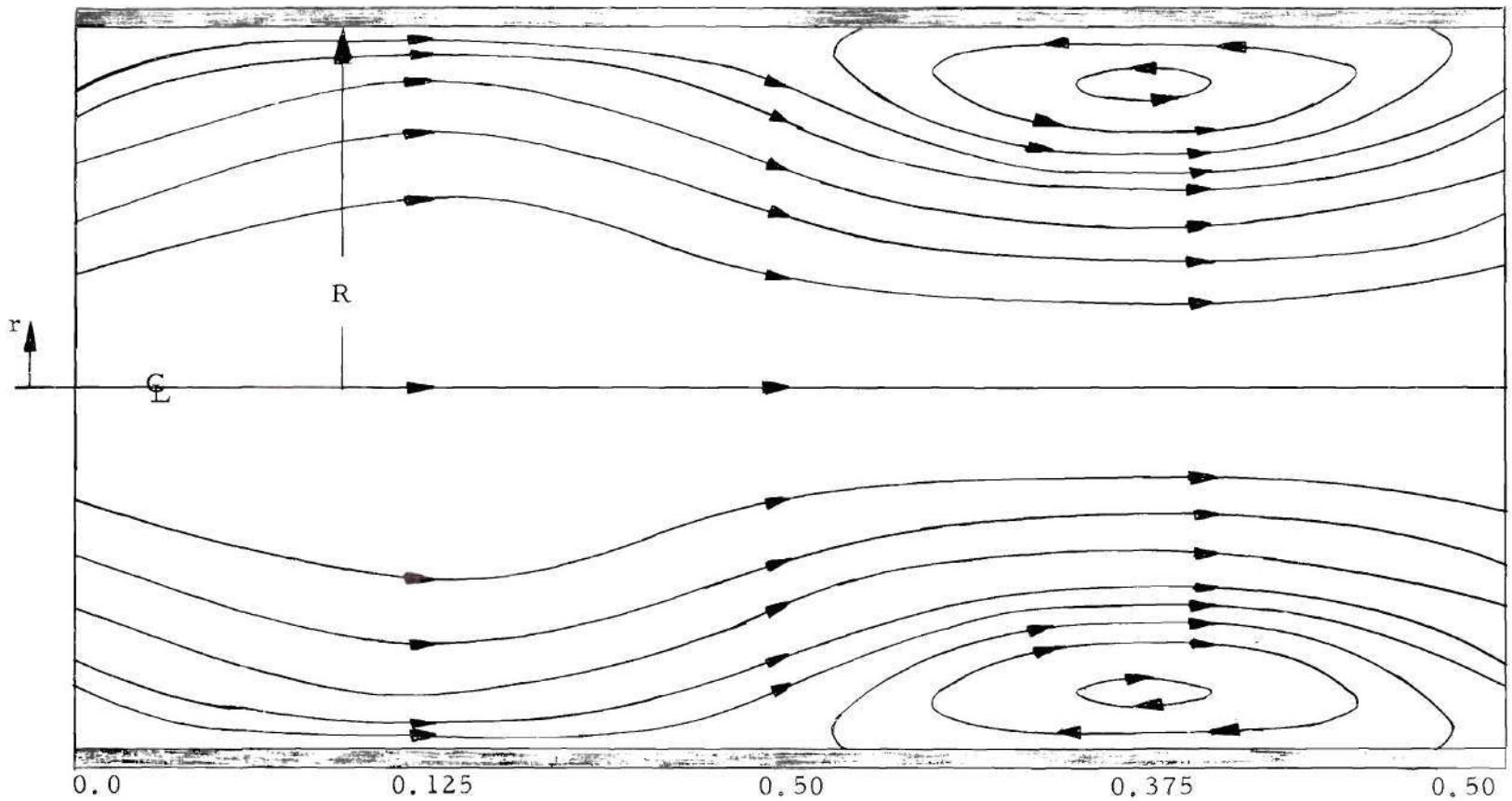
If the trivial solution of $\gamma = r/R = 1$ is eliminated, equation (43) gives

$$\gamma_s = \left[\frac{1}{1 - \frac{1}{8} \frac{M_s^2}{\overline{M}} \sin\left(\frac{2\omega z}{c}\right)} \right]^{\frac{1}{2}} . \tag{44}$$

The minimum values of γ_s can then be obtained by setting $\sin(2\omega z/c) = -1$. Then equation (44) becomes

$$\left(\gamma_s\right)_{min} = \left[\frac{1}{1 + \frac{1}{8} \frac{M_s^2}{\overline{M}}} \right]^{\frac{1}{2}} . \tag{45}$$

A plot of equation (41) for a typical example is shown in Figure 4. Also, a plot of $\left\{1 - \left(\gamma_s\right)_{min}\right\}$, the dimensionless distance from the tube wall, versus $M_s^2/\overline{M}$ is given in Figure 5. For comparative purposes a plot of the results obtained by Purdy for channel flow is included in Figure 5. Several experimental data points for flow in a circular tube which were obtained by Purdy are also included in this Figure.



$z' = z/\lambda$

$$\frac{M_s^2}{\overline{M}} = 11.32$$

Figure 4. Graphical Streamline Representation

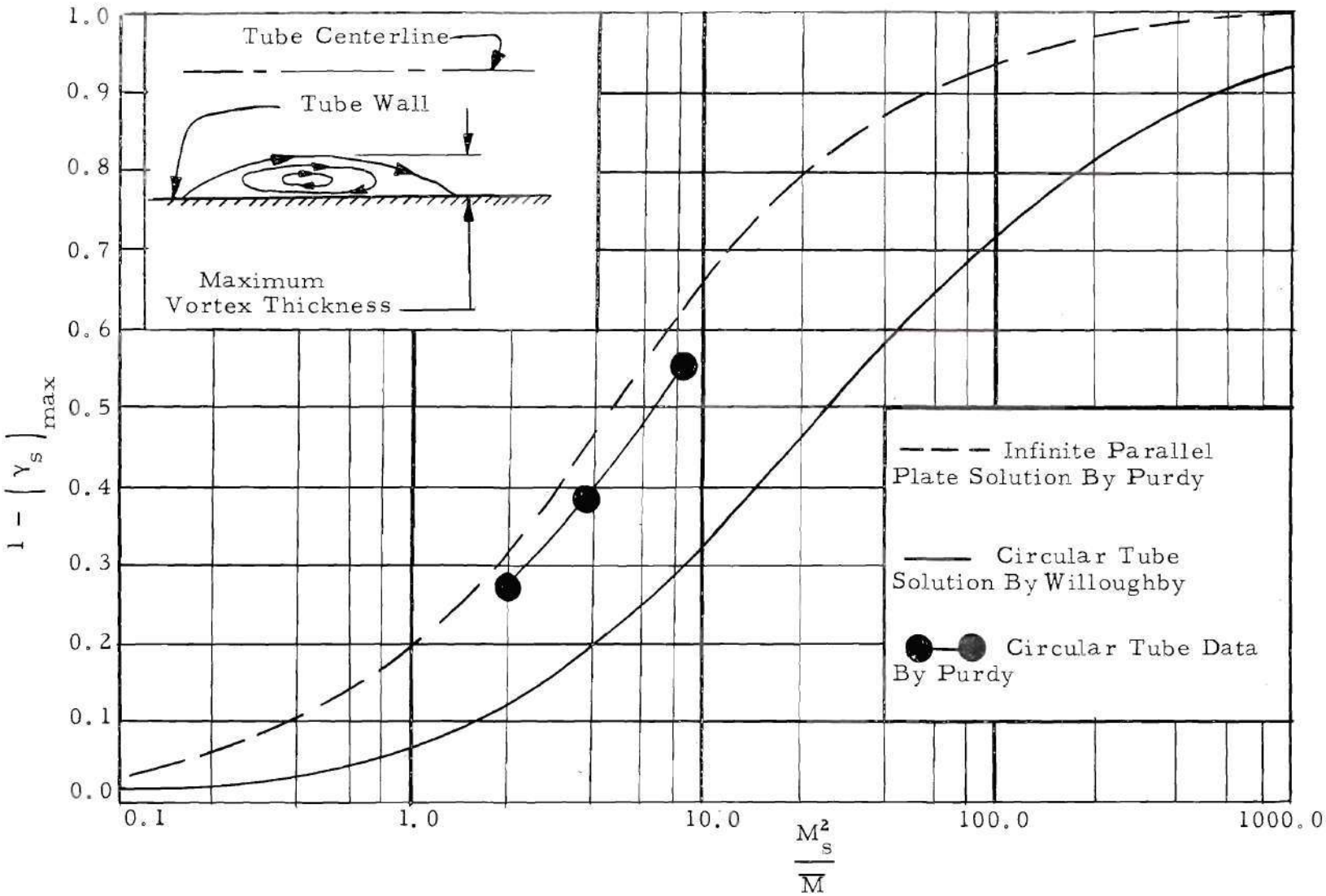


Figure 5. Maximum Vortex Thickness Versus Mach Number Parameter

CHAPTER V

DISCUSSION OF RESULTS

Periodic Velocities

Comments on $u_{10}(r, z, t)$

By substitution of equation (32) for $F_1(r)$ in equation (28) and by evaluation of the constant A appearing in equation (32), the following complex result is obtained for the first approximation to the periodic z-velocity:

$$u_{10}(r, z, t) = U_0 \cos(\omega z/c)\left[\frac{J_0\left(i^{\frac{1}{2}}\beta r\right)}{J_0\left(i^{\frac{1}{2}}\beta R\right)} - 1\right] e^{-i\omega t} \quad . \tag{46}$$

An interesting comparison can be made with regard to equation (46). The flow of a viscous incompressible fluid undergoing pulsating flow through a circular pipe was studied by Uchida [10]. Uchida obtained a solution for the z-velocity of the form

$$u(r, t) = U_0\left[\frac{J_0\left(i^{\frac{1}{2}}\beta r\right)}{J_0\left(i^{\frac{1}{2}}\beta R\right)} - 1\right] e^{-i\omega t} \tag{47}$$

which is quite similar to equation (46) except for the dependence on the z-coordinate due to the cosine term.

Spurlock [3] considered the effects of acoustic vibrations on air in a resonant tube. He limited his mathematical model to that of an incompressible fluid with a pressure distribution given by an

inviscid compressible fluid as was done by this writer. The result obtained by Spurlock was in agreement with that of equation (46).

The quantity in brackets in equation (46) is complex and may be written as

$$\begin{bmatrix}\text{brackets from}\\ \text{equation (46)}\end{bmatrix} = \Big[\text{Real}\Big]_R + i\Big[\text{Imaginary}\Big]_I \quad . \tag{48}$$

The factor $e^{-i\omega t}$ is also complex and is given by

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \quad . \tag{49}$$

Only the real part of equation (46) is of interest, and it can be obtained by multiplication of the real part of the product of equations (48) and (49) by the quantity $U_0 \cos(\omega z/c)$. This result is expressed in equation (33) and is stated here in abbreviated form as

$$u_{10}(r, z, t) = U_0 \cos(\omega z/c) \left\{ [\ ]_R \cos(\omega t) + [\ ]_I \sin(\omega t) \right\} \quad . \tag{50}$$

Spurlock considered only the real part of equations (48) and (49) and not the real part of their product. He obtained for the real part of equation (46) the equation

$$u(r, z, t) = U_0 \cos(\omega z/c) \left\{ [\ ]_R \cos(\omega t) \right\} \quad . \tag{51}$$

It should be pointed out that both Spurlock and Uchida assumed a time-dependency function of the form $e^{+i\omega t}$ rather than of the form $e^{-i\omega t}$ as was assumed by this writer and as was discussed above. This difference is purely a matter of preference since an identical result can be obtained for the real part of the periodic

z-velocity by use of either assumption. There is only a difference in sign in the discarded imaginary part of the periodic z-velocity.

Comments on $v_{10}(r, z, t)$

The result which was obtained for the real part of the periodic r-velocity before evaluation of the constant of integration can be written in abbreviated form as

$$v_{10}(r, z, t) = \frac{\omega}{\beta} \frac{U_0}{c} \sin(\omega z/c) \quad [\quad] + \frac{B}{r} \tag{52}$$

where B is the constant of integration.

The symmetry boundary condition requires that the periodic r-velocity at the tube centerline, $v_{10}(0, z, t)$, equal zero. The no-slip boundary condition requires that the velocity at the tube wall, $v_{10}(R, z, t)$, equal zero. Obviously only one of these conditions can be satisfied. Satisfaction of the no-slip boundary condition results in an infinite centerline velocity; satisfaction of the symmetry boundary condition results in a finite velocity at the tube wall. While neither of these possibilities is fully acceptable, the infinite velocity at the tube centerline is considered to be the most objectionable. Satisfaction of the symmetry boundary condition results in the expression for the r-velocity given in equation (35).

On the other hand if the no-slip boundary condition is selected to be satisfied, the following equation is obtained:

$$v(r, z, t) = + \frac{\omega}{\beta} \frac{U_0}{c} \sin\left(\frac{\omega z}{c}\right)\left[\left\{\frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r)\right.\right. \tag{53}$$

$$\left. - \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r)\right\} \cos(\omega t)$$

$$\left. + \left\{\frac{\mathrm{ber}(\beta R)}{M_0^2(\beta R)} \mathrm{ber}'(\beta r) + \frac{\mathrm{bei}(\beta R)}{M_0^2(\beta R)} \mathrm{bei}'(\beta r)\right\} \sin(\omega t)\right]$$

$$- \frac{R}{r} \frac{\sqrt{2}}{2} \frac{\omega}{\beta} \frac{U_0}{c} \sin\left(\frac{\omega z}{c}\right)\left[\cos(\omega t) + \sin(\omega t)\right] .$$

An attempt to solve for the time-average velocities by use of the alternate periodic r-velocity given by equation (53) proved fruitless.

Time-Average Velocities

Probably one of the more important results of this work is the demonstration and description of the time-average flow which is excited by the resonant sound field. As an example of its importance, the time-average flow is believed to be responsible for the increased heat transfer rates which have been observed in localized areas of a Kundt's tube undergoing a small laminar throughflow. Further work involving the time-average flow field in circular tubes may lead to a better understanding of the complex flow and energy transport phenomena which are involved.

Stream Function

The stream function given in equation (40) is based on the assumption of a constant time-mean density. This supposition in essence is equivalent to assuming that time-average pressure and temperature differences between any two points in the flow are negligibly small. This assumption is felt to be reasonable since minor variations in the time-average pressure and temperature are not expected to change the main course of the time-average flow.

For a fluid flow situation with constant density the stream function may be used to obtain both the r and z velocities for a problem of this type. The z-velocity can be obtained from the stream function, equation (40), by the relation

$$\bar{u}(r, z) = + \frac{1}{2\pi r \rho_0} \frac{\partial \psi}{\partial r} \quad ; \tag{54}$$

and the r-velocity can be obtained by the relation

$$\bar{v}(r, z) = - \frac{1}{2\pi r \rho_0} \frac{\partial \psi}{\partial z} \quad . \tag{55}$$

Use of equations (54) and (55) as a check on the compatibility of the time-average r and z velocities indicated that they are compatible.

It is of interest to compare the results of Purdy's analytical work for channel flow and his experimental data for pipe flow with the analytical results obtained for pipe flow by this writer. A plot showing the maximum dimensionless vortex thickness measured from the channel or tube wall is given in Figure 5. From inspection of the figure it can be seen that the analytical result for channel flow obtained

by Purdy predicts a vortex thickness that is greater than that predicted for pipe flow.

An important observation which can be made is that the character of the time average flow is determined by the parameters M_s, $\overline{M}$, and the wavelength of the vibrations. The vortex thickness depends only on M_s and $\overline{M}$ and is otherwise independent of frequency.

Order of Magnitude Restrictions

The dimensionless form of the periodic z-velocity, $u'(r, z, t)$, which can be obtained by dividing equation (33) by the quantity U_0 is of order of magnitude one. This order of magnitude is in agreement with the original assumption given in equation (9e). The dimensionless form of the periodic r-velocity, $v'(r, z, t)$, can likewise be obtained from equation (35). This velocity has an order of magnitude given as

$$v'(r, z, t) = O\left[\frac{(\omega \nu_0)^{\frac{1}{2}}}{c}\right] .$$

It is true that

$$\omega = \frac{2\pi c}{\lambda} ,$$

therefore

$$v'(r, z, t) = O\left[\left[\frac{2\pi \nu_0}{\lambda c}\right]^{\frac{1}{2}}\right]$$

which is of order (δ) for gases at normal atmospheric pressures and temperatures unless λ is very small. Thus equation (9d) which states

that $v'(r, z, t) = O(\delta)$ will be satisfied for all except very high frequency accoustic vibrations.

The dimensionless form of the time-average z-velocity, $\overline{u}(r, z)$, is of order of magnitude $\left[\frac{M_s}{4} + \frac{2\overline{M}}{M_s}\right]$. The original order of magnitude assumption for $\overline{u}(r, z)$ given in equation (9f) states that $\overline{u}(r, z) = O(\delta)$. The velocity $\overline{u}(r, z)$ is of $O(\delta)$ if $M_s = O(\delta)$ and if $\overline{M} = O(\delta^2)$ or less. These two restrictions are necessary in order to assure the validity of the solution. The dimensionless form of the time-average r-velocity, $\overline{v}(r, z)$, has an order of magnitude given by $O\left|\frac{\omega r}{8c} M_s\right| = O\left|\frac{2\pi}{8} \frac{r}{\lambda} M_s\right|$. If $r/\lambda = O(\delta)$ as has been previously required, then $\overline{v}(r, z)$ will be of $O(\delta^2)$ as assumed in equation (9d). These order of magnitude restrictions are in agreement with the original assumptions if the specified conditions are met; these restrictions illuminate the range of validity of the solution. The parameter $M_s^2/\overline{M}$ must be of order of magnitude one or larger for the validity of the solution to be assured.

CHAPTER VI

CONCLUSIONS

The solution which was obtained for the time-average velocities and stream function gives an analytical prediction of the time-average flow and vortex formation which have been observed experimentally for a Kundt's tube undergoing resonant acoustic vibrations. The maximum vortex thickness, the distance from the tube wall divided by the tube radius, is shown to be to a first approximation solely a function of the parameter $M_s^2/\overline{M}$. M_s is the acoustic Mach number which is based on the maximum amplitude of the time-dependent axial velocity, and $\overline{M}$ is the throughflow Mach number which is based on the average axial velocity. Qualitatively, the time-average flow is quite similar to that of channel flow as analytically determined by Purdy. However, the predicted vortex thickness is greater for channel flow than for tube flow.

CHAPTER VII

RECOMMENDATIONS

It is recommended that future work be directed toward obtaining the solution of the periodic velocities to a second approximation. Then the time-average flow should be obtained from the more complete periodic solution. It is further recommended that the time-average flow be used as part of a solution of the energy equation for flow in a resonant Kundt's tube.

APPENDICES

APPENDIX A

NOMENCLATURE

c	Isentropic velocity of sound	ft/sec
i	Square root of -1	Dimensionless
L	Length of tube	ft
$\overline{M}$	Average Mach number based on $\overline{U}$	Dimensionless
M_s	Acoustic Mach number based on U_0	Dimensionless
O	Order of magnitude	Dimensionless
p	Pressure of a fluid	lb_f/ft^2
$\overline{p}$	Time-mean pressure	lb_f/ft^2
p_1	Time-dependent pressure	lb_f/ft^2
p_0	Average value of $\overline{p}$ (assumed constant)	lb_f/ft^2
R	Radius of the tube	ft
r	Space coordinate	ft
T	Absolute temperature	°R
t	Time	sec
u	Total z component of velocity	ft/sec
$\overline{u}$	Time-mean component of z-velocity	ft/sec
u_1	Time-dependent component of z-velocity	ft/sec
u_{10}	First approximation to u_1	ft/sec
u_{11}	Second approximation to u_1	ft/sec
U_0	Maximum amplitude of the time-dependent z-velocity (u_1)	ft/sec
$\overline{U}$	Average time-mean z velocity based on the total volume rate of flow	ft/sec

v	Total r-component of velocity	ft/sec
$\bar{v}$	Time-mean component of r-velocity	ft/sec
v_1	Time-dependent component of r-velocity	ft/sec
v_{10}	First approximation to v_1	ft/sec
v_{10}	Second approximation to v_1	ft/sec
W	Body force vector	lb_f/ft^3
Y	Component of body force in r direction	lb_f/ft^3
Z	Component of body force in z direction	lb_f/ft^3
z	Space coordinate	lb_f/ft^3

Greek Letters

β	$\left[\frac{\omega}{\nu_0}\right]^{\frac{1}{2}}$	ft^{-1}
$[1-\gamma]$	$\left[\frac{R-r}{R}\right]$	Dimensionless
$[1-\gamma_s]$	Width of largest vortex	Dimensionless
$[1-\gamma_S]$	Maximum width of largest vortex	Dimensionless
δ	Magnitude ($\delta \ll 1$)	Dimensionless
θ	Space coordinate	radians
λ	Wavelength of acoustic vibrations	ft
μ	Dynamic viscosity of a fluid	lb_m/ft sec
ν_0	Average kinematic viscosity of a fluid	ft^2/sec
ρ	Density of a fluid	lb_m/ft^3
ρ_0	Time-mean component of density	lb_m/ft^3
ρ_1	Time-dependent component of density	lb_m/ft^3
ψ	Stream function (based on the time-mean velocity)	lb_m/sec

$\overline{\Psi}$	Dimensionless stream function $\dfrac{\Psi}{\pi R^2 \rho_0 \overline{U}}$	Dimensionless
ω	Frequency of the acoustic vibrations	rad/sec

APPENDIX B

INTEGRALS AND FORMULAS

The following integrals and formulas which were taken from McLachlan's Bessel Functions for Engineers are useful in carrying out the mathematical development which is given in the previous chapters:

$$\int r \, \mathrm{ber}(\beta r) \, dr = + \frac{r}{\beta} \, \mathrm{bei}'(\beta r) \quad , \tag{56}$$

$$\int r \, \mathrm{bei}(\beta r) \, dr = - \frac{r}{\beta} \, \mathrm{ber}'(\beta r) \quad , \tag{57}$$

$$\int r \left[\mathrm{ber}^2(\beta r) + \mathrm{bei}^2(\beta r) \right] dr = + \frac{r}{\beta} \left[\mathrm{ber}(\beta r) \, \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r) \, \mathrm{ber}'(\beta r) \right] \quad , \tag{58}$$

$$\int r \left[\mathrm{ber}^2(\beta r) - \mathrm{bei}^2(\beta r) \right] dr = + \frac{\beta^2 r^2}{2} \left[\mathrm{ber}^2(\beta r) - \mathrm{bei}^2(\beta r) - 2 \, \mathrm{ber}'(\beta r) \, \mathrm{bei}'(\beta r) \right] \quad , \tag{59}$$

$$\theta_n = \arctan \left[\frac{\mathrm{bei}_n(\beta r)}{\mathrm{ber}_n(\beta r)} \right] \quad , \quad \text{and} \tag{60}$$

$$M_n^2(\beta r) = \mathrm{ber}_n^2(\beta r) + \mathrm{bei}_n^2(\beta r) \quad . \tag{61}$$

The following two integrals are obvious from the definition of $\mathrm{ber}'(\beta r)$ and $\mathrm{bei}'(\beta r)$:

$$\int \mathrm{ber}'(\beta r)\, dr = \frac{\mathrm{ber}(\beta r)}{\beta} \quad , \quad \text{and} \tag{62}$$

$$\int \mathrm{bei}'(\beta r)\, dr = \frac{\mathrm{bei}(\beta r)}{\beta} \quad . \tag{63}$$

It can be seen that

$$\frac{d}{dr}\left[r\, \mathrm{ber}'(\beta r)\right] = + \beta r\, \mathrm{ber}''(\beta r) + \mathrm{ber}'(\beta r), \quad \text{and that} \tag{64}$$

$$\frac{d}{dr}\left[r\, \mathrm{bei}'(\beta r)\right] = + \beta r\, \mathrm{bei}''(\beta r) + \mathrm{bei}'(\beta r) \quad . \tag{65}$$

Therefore it follows that

$$\int \left[\beta r\, \mathrm{ber}''(\beta r) + \mathrm{ber}'(\beta r)\right] dr = r\, \mathrm{ber}'(\beta r), \quad \text{and that} \tag{66}$$

$$\int \left[\beta r\, \mathrm{bei}''(\beta r) + \mathrm{bei}'(\beta r)\right] dr = r\, \mathrm{bei}'(\beta r) \quad . \tag{67}$$

APPENDIX C

APPROXIMATE EVALUATION OF AN INTEGRAL

The integral given below comes about in the solution of the time-average z-velocity:

$$\frac{1}{M_0^2(\beta r)} \int \left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r) \right] dr \quad . \tag{68}$$

By use of several approximations, it is possible to obtain an accurate evaluation of the integral. The following formulas taken from McLachlan's <u>Bessel Functions for Engineers</u> are essential to the development of the integration:

$$\theta_0 = \arctan \left[\mathrm{bei}(\beta r) / \mathrm{ber}(\beta r) \right] \quad , \tag{69}$$

$$\theta_1 = \arctan \left[\mathrm{bei}_1(\beta r) / \mathrm{ber}_1(\beta r) \right] \quad , \tag{70}$$

$$M_0^2(\beta r) = \mathrm{ber}^2(\beta r) + \mathrm{bei}^2(\beta r) \quad , \tag{71}$$

$$M_1^2(\beta r) = \mathrm{ber}_1^2(\beta r) + \mathrm{bei}_1^2(\beta r) \quad , \tag{72}$$

$$\left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r) \right] = \tag{73}$$

$$M_0(\beta r)\ M_1(\beta r)\ \sin\left(\theta_1 - \theta_0 - \frac{\pi}{4} \right) \quad .$$

For $\beta r >> 1$ which is the case in this problem except where $r \approx 0$, the following approximations taken from MaLachlan are useful:

$$\theta_0 \approx \left[\frac{\beta r}{\sqrt{2}} - \frac{\pi}{8} \right] \quad , \tag{74}$$

$$\theta_1 \simeq \left[\frac{\beta r}{\sqrt{2}} + \frac{3\pi}{8}\right] \quad . \tag{75}$$

From equations (74) and (75) it follows for $\beta r >> 1$ that

$$\left[\theta_1 - \theta_0\right] \simeq \frac{\pi}{2} \quad . \tag{76}$$

From equation (76) it can be concluded that

$$\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \simeq + \frac{\sqrt{2}}{2} \quad , \quad \text{and that} \tag{77}$$

$$\cos\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \simeq + \frac{\sqrt{2}}{2} \tag{78}$$

if $\beta r >> 1$.

By use of equation (73) and by use of the inequality $\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \leq 1$, integral (68) may be written as

$$\frac{1}{M_0^2(\beta R)} \int \left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r)\right] dr \leq \tag{79}$$

$$\frac{1}{\beta M_0^2(\beta R)} \int M_0(\beta r)\ M_1(\beta r)\ d(\beta r) \quad .$$

Likewise for $\beta r >> 1$ by use of equations (73) and (77), integral (68) may be written

$$\frac{1}{M_0^2(\beta R)} \int \left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r)\right] dr = \tag{80}$$

$$\frac{1}{\beta\ M_0^2(\beta R)} \int M_0(\beta r)\ M_1(\beta r)\ \frac{\sqrt{2}}{2}\ d(\beta r) \quad .$$

Therefore it can be seen that integral (68) may be closely approximated

by equation (80) if $\beta r >> 1$. Integral (68) may be approximated to an order of magnitude by equation (80) for any value of βr since it is true that $O\left(\frac{\sqrt{2}}{2}\right) = O(1)$.

At this point the following identity which is taken from reference [11] will prove useful:

$$\frac{d}{d(\beta r)}\left[M_0(\beta r)\right] = M_1(\beta r)\cos\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \quad . \tag{81}$$

After substitution of equation (77) into equation (81) and after substitution of the resulting equation into equation (80), the following result was obtained for $\beta r >> 1$:

$$\frac{1}{M_0^2(\beta R)} \int\left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r)\right] dr = \tag{82}$$

$$\frac{1}{\beta M_0^2(\beta R)} \int M_0(\beta r)\ d\left[M_0(\beta r)\right] \quad .$$

Equation (82) may be easily integrated to give

$$\frac{1}{M_0^2(\beta R)} \int\left[\mathrm{ber}(\beta r)\ \mathrm{bei}'(\beta r) - \mathrm{bei}(\beta r)\ \mathrm{ber}'(\beta r)\right] dr = \tag{83}$$

$$\frac{1}{2\beta}\ \frac{M_0^2(\beta r)}{M_0^2(\beta R)} \quad .$$

Equation (83) also gives an estimate correct to an order of magnitude of integral (68) for $\beta r \approx 0$.

The accuracy of equation (83) can best be discussed by comparing the value of integral (68) evaluated between the limits 0 and βr

to the value of the integral evaluated between the limits 0 and βR. A typical value of βR was approximately 1000; the asymptotic expansions used in obtaining equation (83) are accurate to about four figures for $\beta r > 50$ with increasing accuracy as βr increases. For the above typical values, integral (68) evaluated between $\beta r = 0$ to $\beta r = 0.95\,\beta R$ has a value much less than one millionth of the value of integral (68) evaluated between $\beta r = 0$ to $\beta r = \beta R$. For this reason any error associated with obtaining a value of integral (68) accurate to only an order of magnitude for $\beta r \approx 0$ is certainly negligible when compared to the value of integral (68) evaluated between the limits 0 and βR.

APPENDIX D

AIDS TO NUMERICAL EVALUATION

The following formulas taken from McLachlan's Bessel Functions for Engineers are essential to obtaining numerical results from the equations appearing in the body of the thesis. These approximations apply only for $\beta r >> 1$:

$$\mathrm{ber}(\beta r) \simeq \frac{e^{\beta r/\sqrt{2}}}{\sqrt{2\pi\beta r}} \cos\left(\frac{\beta r}{\sqrt{2}} - \frac{\pi}{8}\right) . \tag{84}$$

$$\mathrm{bei}(\beta r) \simeq \frac{e^{\beta r/\sqrt{2}}}{\sqrt{2\pi\beta r}} \sin\left(\frac{\beta r}{\sqrt{2}} - \frac{\pi}{8}\right) . \tag{85}$$

$$\mathrm{ber}'(\beta r) \simeq \frac{e^{\beta r/\sqrt{2}}}{\sqrt{2\pi\beta r}} \cos\left(\frac{\beta r}{\sqrt{2}} + \frac{\pi}{8}\right) . \tag{86}$$

$$\mathrm{bei}'(\beta r) \simeq \frac{e^{\beta r/\sqrt{2}}}{\sqrt{2\pi\beta r}} \sin\left(\frac{\beta r}{\sqrt{2}} + \frac{\pi}{8}\right) . \tag{87}$$

$$M_0^2(\beta r) \simeq \frac{e^{2\beta r/\sqrt{2}}}{2\pi\beta r} . \tag{88}$$

$$M_1^2(\beta r) \simeq M_0^2(\beta r) . \tag{89}$$

BIBLIOGRAPHY

LITERATURE CITED

1. Lord Rayleigh, "On the Circulation of Air Observed in Kundt's Tubes and on Some Allied Acoustical Problems", Philosophical Transactions of the Royal Society of London, 175, Part I, (1884).

2. P.J. Westervelt, "The Theory of Steady Rotational Flow Generated by a Sound Field", Journal of the Acoustical Society of America, 25, (1953), pp. 60-67.

3. J.M. Spurlock, The Effects of Acoustic Vibrations on Air in a Resonant Horizontal Tube With and Without Throughflow, Ph.D. Thesis, Georgia Institute of Technology, (1961).

4. K.R. Purdy, Viscous Flow Under the Influence of a Resonant Acoustic Field, Ph.D. Thesis, Georgia Institute of Technology, (in preparation).

5. C.C. Lin, "Motion in the Boundary Layer with a Rapidly Oscillating External Flow", Ninth International Congress of Applied Mechanics, University of Brussels, Belguim, 4, (1956), p. 155.

6. J.V. Sanders, "A Photomultiplier-Schlieren For Acoustic Measurements and Some Investigations of the Kundt's Tube", Dissertation Abstracts, 22, 8, (1962).

7. H. Schlichting, "Berechung ebener periodischer Grenzschichtstromungen", Zeitschrift fuer Physik, 33, (1932), p. 327.

8. P.M. Morse, Vibration and Sound, 2nd ed., New York: McGraw-Hill Book Co., Inc., (1948).

9. N.W. McLachlan, Bessel Functions for Engineers, 2nd ed., Oxford: Oxford University Press, (1961).

10. S. Uchida, "The Pulsating Viscous Flow Superposed on the Steady Laminar Motion of Incompressible Fluid in a Circular Pipe", Zeitschrift fuer Angewandte Physik, VII, (1956), pp. 403-422.

11. N.W. McLachlan and A.L. Meyers, "Integrals Involving Bessel and Struve Functions", Philosophical Magazine, 21, (1936), p. 437.

12. H. Schlichting, Boundary Layer Theory, 4th ed., New York: McGraw-Hill Book Co., Inc., (1960), pp. 207-237.