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OFFICE OF RESEARCH ADMINISTRATION

RESEARCH PROJECT INITIATION

Date: February 17, 1975

Project Title: Unsteady Viscous Flow

Project No: E-16-656

Principal Investigator Dr. J. C. Wu

Sponsor: Office of Naval Research; Arlington, Va.

Agreement Period: From 1/1/75 Until 12/31/~~76~~ 82

Type Agreement: Contract No. N00014-75-C-0249

Amount: 40,000

Reports Required: Progress Reports; Computer Software; Final Report

Sponsor Contact Person (s):

Technical Matters
(Scientific Officer)
Director, Fluid Dynamics Program
Mathematical & Information Sciences Div.
Office of Naval Research
800 North Quincy Street
Arlington, Virginia 22217

Contractual Matters
thru ORA
Mr. R. J. Whitcomb
ACO
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GEORGIA INSTITUTE OF TECHNOLOGY OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT TERMINATION SHEET

Date 2/21/83

Project Title: Unsteady Viscous Flow

Project No: E-16-656 - CONTINUED AS E-16-625 (per Dwight Allen)

Project Director: Dr. Wu

Sponsor: ONR

Effective Termination Date: 12/31/82

Clearance of Accounting Charges: 12/31/82

Grant/Contract Closeout Actions Remaining:

- [x] Final Invoice and Closing Documents
- [ ] Final Fiscal Report
- [x] Final Report of Inventions
- [x] Govt. Property Inventory & Related Certificate
- [ ] Classified Material Certificate
- [ ] Other

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ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

First Quarterly Progress Report

for the period

January 1 to March 31, 1975

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu

Professor of Aerospace Engineering

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quicy Street

Arlington, Virginia 22217

During the reporting period, the oscillating airfoil problem was formulated in a coordinate system attached to the airfoil using the vorticity and the velocity vector as dependent variables. The kinetic part of the problem was expressed as a new vorticity transport equation containing a new term which is absent in the equation for steady state problems. The kinematic part of the problem was expressed as an integral representation of the velocity vector. This integral representation is presented in a paper (Ref. 1) authored by the principal investigator during the reporting period.

A new method for treating the farfield velocity boundary condition and the condition of the "non-velocity" flow variable on the solid surface (i.e., the extraneous boundary condition) was developed and utilized to solve selected test problems. This method was shown to satisfy the integral law for the total vorticity in the flowfield and to remove the major difficulties and uncertainties encountered in other methods. The method yielded stable and accurate results for all test problems treated (Ref. 1).

It was shown that the determination of the initial vorticity distribution on the airfoil surface after it is set into translational and oscillatory motions may be treated as a special case of the computation of the extraneous boundary vorticity distribution. A general computer program was prepared for computing the boundary vorticity distribution for various types of airfoils. Test computations were performed for the NACA 0009 airfoil section. It was demonstrated that the initial vorticity distribution obtained here are in excellent agreement with those deduced from the potential flow solutions given in NASA TR-824.

With previous finite-difference or finite-element methods, the kinematic part of the computation required up to 90% of the total computer time for the type of problem under consideration. Work completed thus far indicates that the integro-differential method led to a drastic improvement in the kinematic part of computation. As a result, the kinematic computation now constitutes of only a small part of the total computational effort. Accordingly, the kinetic part of the computation now deserves additional attention. During the reporting period, special efforts were made to select an optimum method for the solution of the new vorticity transport equation. The alternating direction implicit (ADI) method was experimented with and shown to be subjected to severe stability restriction for the initial time steps, immediately after the airfoil is set into motion, when the predominant mechanism for vorticity transport is diffusion. A finite element system of equations was developed to replace the differential equations for vorticity transport using Galerkin's approach. Rectangular elements were used for regions away from the airfoil in order to facilitate the use of the flowfield segmentation technique. A computer program was prepared for the numerical solution of the system of equations. Concurrently, several coordinate transformation techniques are being examined in detail for possible adaptation to the oscillating airfoil problem. These efforts related to the kinetic part of the computation are expected to continue well into the third quarter of this research program.

Reference 1. J. C. Wu, "Velocity and Extraneous Boundary Conditions of Viscous Flow Problems," AIAA Paper No. 75-47, presented at the AIAA 13th Aerospace Sciences Meeting, Jan. 1975.

ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

Second Quarterly Progress Report

for the period

April 1 to June 30, 1975

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu

Professor of Aerospace Engineering

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the second quarter of the present research, progress was made in using Galerkin's approach to solve the vorticity transport equation. A hybrid finite-difference, finite-element approach was developed for this kinetic part of the computation.

The flowfield is divided into two regions for the 9% thick symmetric Joukowski airfoil. Region 1 encloses the airfoil and contains 208 nodes, including the nodes on the surface of the airfoil and those on the common boundary of regions 1 and 2. Linear interpolation functions and triangular elements are used for this region. Galerkin's process is used to establish integral relation from the vorticity equation. A finite element method was developed to obtain the nodal vorticity values at a new time level from known nodal values of velocity and vorticity at an old time level.

Region 2 encloses region 1 and contains approximately 2500 non-zero vorticity grid points in a rectangular finite-difference grid. The grid spacings are $\Delta x = 0.05$ and $\Delta y = 0.025$. The use of the hybrid approach offers the following distinct advantages:

(1) The inherent flexibility of the finite element method in spacing the nodal points permits the boundary of the airfoil to be mapped accurately. That is, the nodes are made to coincide with the airfoil boundary and no interpolation is needed for boundary values. The inherent flexibility also permits a denser concentration of nodes in regions where gardients of field variables are expected to be large.

(2) The use of Dufort-Frankel scheme in region 2 eliminates the need of inverting large matrices and, at the same time, permits the use of a large number of field points to describe the wake in sufficient detail.

The kinematics of the problem is treated using the integral representation of the velocity vector presented in Ref. 1. The local generation of vorticity on the airfoil surface is included in the kinematic computations (Ref. 2).

A computer program incorporating the above features has been developed to analyze two-dimensional unsteady viscous flows over arbitrarily shaped bodies undergoing arbitrarily defined motions. The input data to this program are:

(1) the Reynolds number for the problem,

(2) the time history of the solid motion,

(3) the grid system for region 2,

(4) the finite element nodes for region 1.

This program has been calibrated by treating the following test problems:

(1) Flow over an impulsively started circular cylinder at a Reynolds number of 1000.

Results obtained are in good agreement with finite-difference results. At the first few time steps, spurious vorticity values are obtained, in regions far from the body where the vorticity is expected to be negligible. These spurious values, however, tend to disappear as the solution progresses in time. The computation was terminated after five time steps.

(2) Flow over a NACA0012 airfoil at zero angle of attack and a number of 1000.

Results are obtained for the first few time steps. They show the correct trend.

(3) Flow over the Joukowski airfoil oscillating about zero degree angle of attack with a reduced frequency of 0.3 and a 15° amplitude, with the axis of rotation located 1/4 chord from the leading edge. Thus far, this computation was performed for only two time steps.

The program is being further calibrated by solving the problem of the Joukowski airfoil set into motion impulsively at an angle of attack of 15% and a Reynolds number of 1000. The flowfield has been computed for the first thirty time steps (dimensional time of 0.288). The following features are observed:

(1) The local generation of vorticity on the airfoil surface is accurately predicted by the kinematic computation. For example, at $t = 0^{+}$, the computed vorticity distribution on the airfoil surface gives -2.2035 compared to the theoretical value of -2.2655.

(2) A vortex of very large strength appears at the trailing edge at $t = o^{+}$. As time progresses, this vortex strength decreases, accompanied by the motion of the rearward stagnation point towards the trailing edge. The forward stagnation point moves towards the leading edge during the same time period.

(3) The vortex strength on the surface continuously decreases everywhere. The surface vorticity distributions plotted at various times are in good agreement with the results of Ref. 3.

REFERENCES

1. Wu, J. C. and Thompson, J. F., "Numerical Solutions of Time-Dependent Incompressible Navier-Stokes Equations using an Integro-Differential Formulation," Computers and Fluids, Vol. 1, pp. 197-215, 1975.
2. Wu, J. C., "Velocity and Extraneous Boundary Conditions of Viscous Flow Problems," AIAA Paper No. 75-47, 1975.
3. Metha, U. B., "Starting Vortex, Separation Bubbles, and Stall - A Numerical Study of Laminar Unsteady Flow Around an Airfoil," Ph.D. Thesis, Illinois Institute of Technology, 1972.

E-16-656

ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

Third Quarterly Progress Report

for the period

July 1 to September 30, 1975

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu

Professor of Aerospace Engineering

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the third quarter of the present research, the problem of an impulsively started Joukowski airfoil at a 15° angle of attack was further studied in order to calibrate and update the computer program developed earlier. It was found that the use of explicit methods (both the simple explicit and the Dufort-Frankel) for the solution of the vorticity transport equation in the region covered by finite elements led to spurious oscillatory velocity and vorticity values, especially in surface vorticity values. The amplitude of the oscillation could be kept low only with the use of very small time steps. This difficulty has been noted previously by Bratnow and Ecer (AIAA Journal, Vol. 12, No. 11). It has been resolved in the present work by the use of the Crank-Nicholson method for the vorticity transport equation in the finite-element region.

In the kinematic part, efforts were devoted to improving the accuracy of the surface vorticity computation, particularly for the leading edge region. Earlier methods of computing the surface vorticity value were based on the approach described in AIAA paper 75-47 by the principal investigator. The approach required that the total contribution to the normal velocity of the vorticity on the solid surface S, the vorticity away from S, and the freestream to be such that the normal velocity boundary condition is satisfied on S. Computationally, one has

$$[a_{i_j}]\,\zeta_j = \{b_i\} \qquad i = 1, 2, \ldots, N$$

where $[a_{i_j}]$ and $\{b_i\}$ are known, N is the number of data points on S and ζ_j are vorticity values.

It has been shown that the rank of the coefficient matrix $[a_{ij}]$ is N - 1. The additional equation needed to obtain a unique set of values of ζ_j is supplied by the principle of total vorticity conservation. For symmetric flows such as the flow past a circular cylinder with no circulation and the flow past an airfoil at zero angle of attack, the symmetric condition can be used to replace the principle of total vorticity conservation and accurate results were obtained near the leading edge. For the airfoil at an angle of attack, a new formulation which explicitly computes the effect of vorticity distribution in the fluid domain on the surface distribution of vorticity has been developed. This new method is highly accurate for all parts of the airfoil and shall be discussed in detail in a future article.

In anticipation of the fact that at high Reynolds numbers the boundary layer is very thin near the leading edge, the grid work near the leading edge is being modified so that the spacing between data point is smaller. Simultaneously, other possible approaches, such as considering the boundary layer near the leading edge to be simply represented by the vortex sheet, are being examined. This aspect of the effort is expected to continue into the fifth quarter of this research.

ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

Fourth Quarterly Progress Report

for the period

October 1 to December 31, 1975

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

and

N. L. Sankar, Research Assistant

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting period, numerical results were obtained for two problems: (1) starting flow over a 9% symmetric Joukowski airfoil at 15° angle of attack, and (2) starting flow over a 12% symmetric Joukowski airfoil at zero angle of attack.

In the previous quarterly progress report, the problem of accurate determination of the vorticity distribution on the surface of the airfoils was described. During the fourth quarter, the previously mentioned explicit method of computing the surface vorticity distribution was implemented and utilized in computing flows described above. The explicit method is further generalized to include the effect of rotation of the airfoil. A computer program subroutine will be prepared during the next quarter utilizing the explicit method in oscillating airfoil computations.

An additional source of inaccuracy was uncovered during the fourth quarter of this research. At relatively large time levels, spurious numerical oscillations of significant magnitude were observed. The cause of these oscillations was traced back to the convection term in the vorticity transport equation. To eliminate these oscillations, either the cell Reynolds number must be restricted or the approximation of the convection term must be compromised. The latter alternative was chosen in the present effort. The spurious numerical oscillation was eliminated by a suitable modification of the coefficient matrix for the convective term.

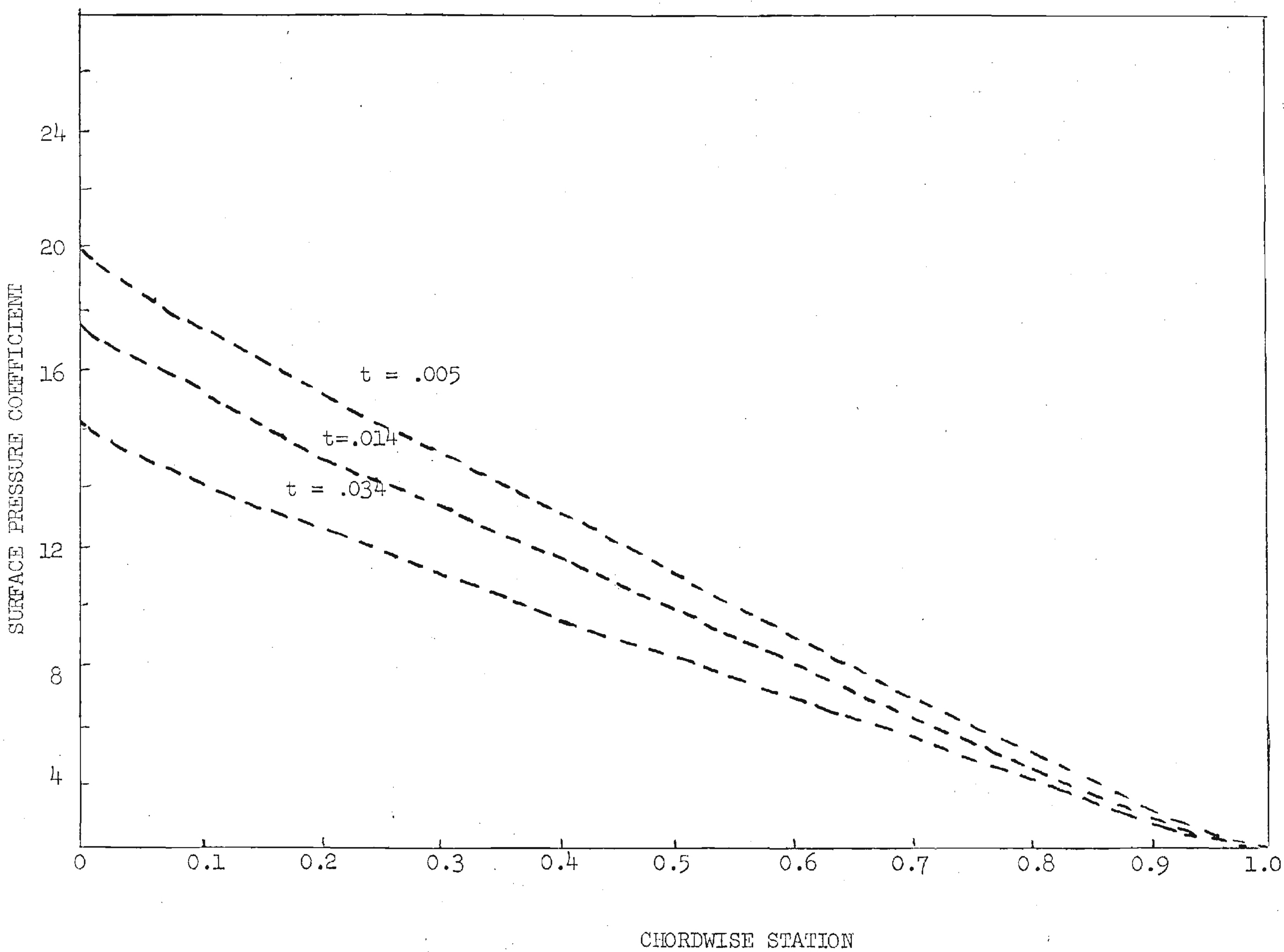
Additional nodes and elements were introduced to improve the solution accuracy for the 12% airfoil case. (The finite element region now contains 354 elements instead of 306 elements used previously.) In the finite element region, it was found that the use of Galerkin formulation for the kinematics of the problem yielded more accurate overall solution than the integral representations formulation.

The starting flow over the 12 ~~9~~% Joukowski airfoil was computed up to a dimensionless time level of 0.754, the reference time being the chord length divided by the freestream velocity. The pressure and vorticity distribution on the airfoil are shown on the attached figures for several time levels. Variation of the drag coefficient with time is presented in the attached table. 75 time steps were needed to reach the time level 0.754. The solution at this time level will be used as a starting solution for the oscillating airfoil computations. Programming for the oscillating airfoil case is being completed.

Time Level	Pressure Drag C_{D_P}	Viscous Drag C_{D_f}	Total Drag, C_D
.005	1.626	.426	2.052
.014	1.414	.418	1.832
.030	1.152	.403	1.555
.058	.820	.378	1.198
.098	.636	.358	.994
.138	.537	.345	.882
.218	.454	.333	.787
.306	.400	.323	.723
.386	.356	.314	.670
.466	.312	.302	.614
.546	.290	.291	.581
.626	.278	.282	.560
.706	.286	.276	.562

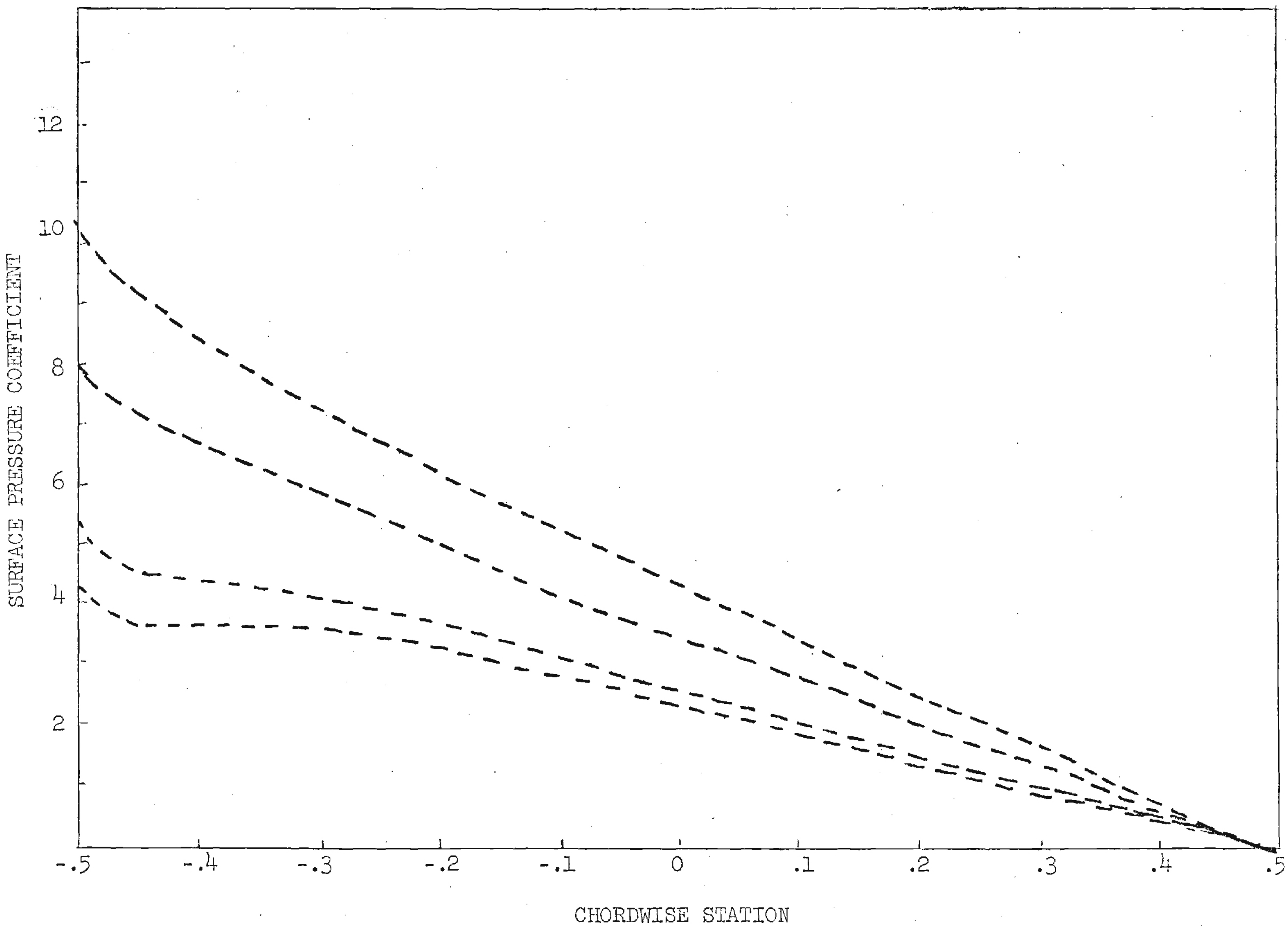
SURFACE PRESSURE DISTRIBUTION OVER

12% Joukowski Aerofoil at Zero Angle of Attack



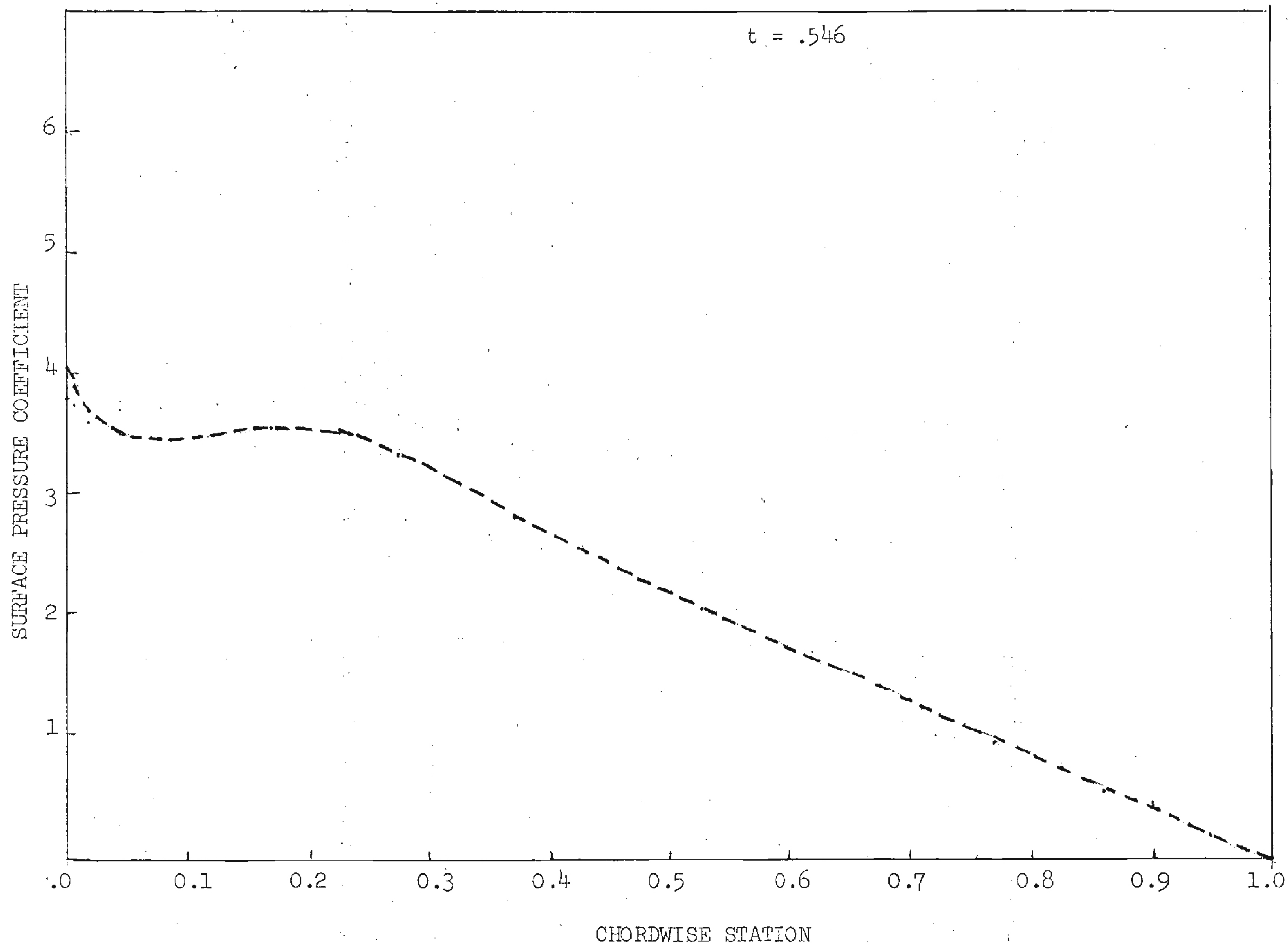
12% Joukowski Airfoil - $\alpha = 0^{\circ}$, $R_e = 1000$

SURFACE PRESSURE DISTRIBUTION

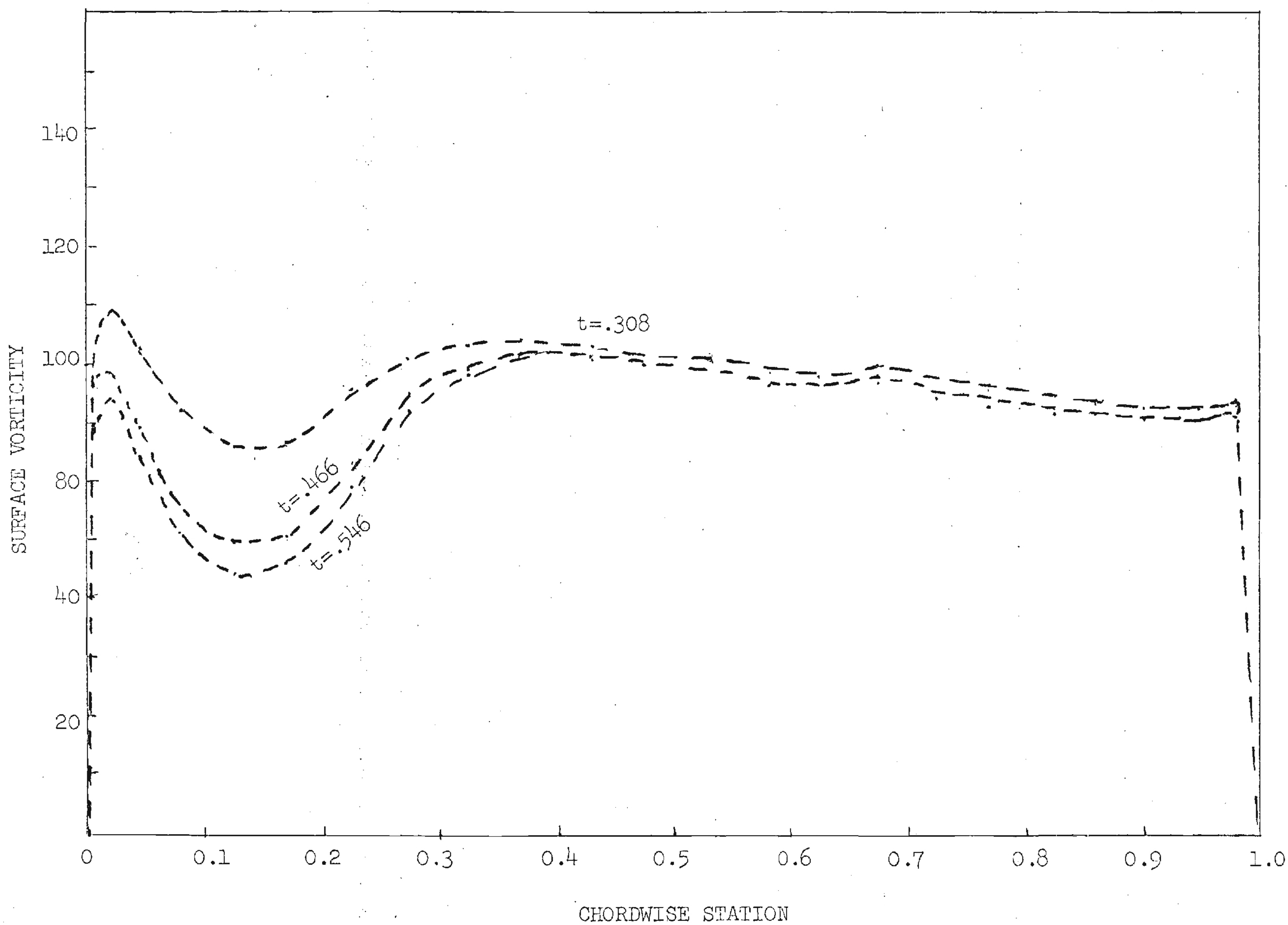


12% Joukowski Aerofoil - $\alpha = 0^{\circ}$, $R_e = 1000$.

SURFACE PRESSURE DISTRIBUTION

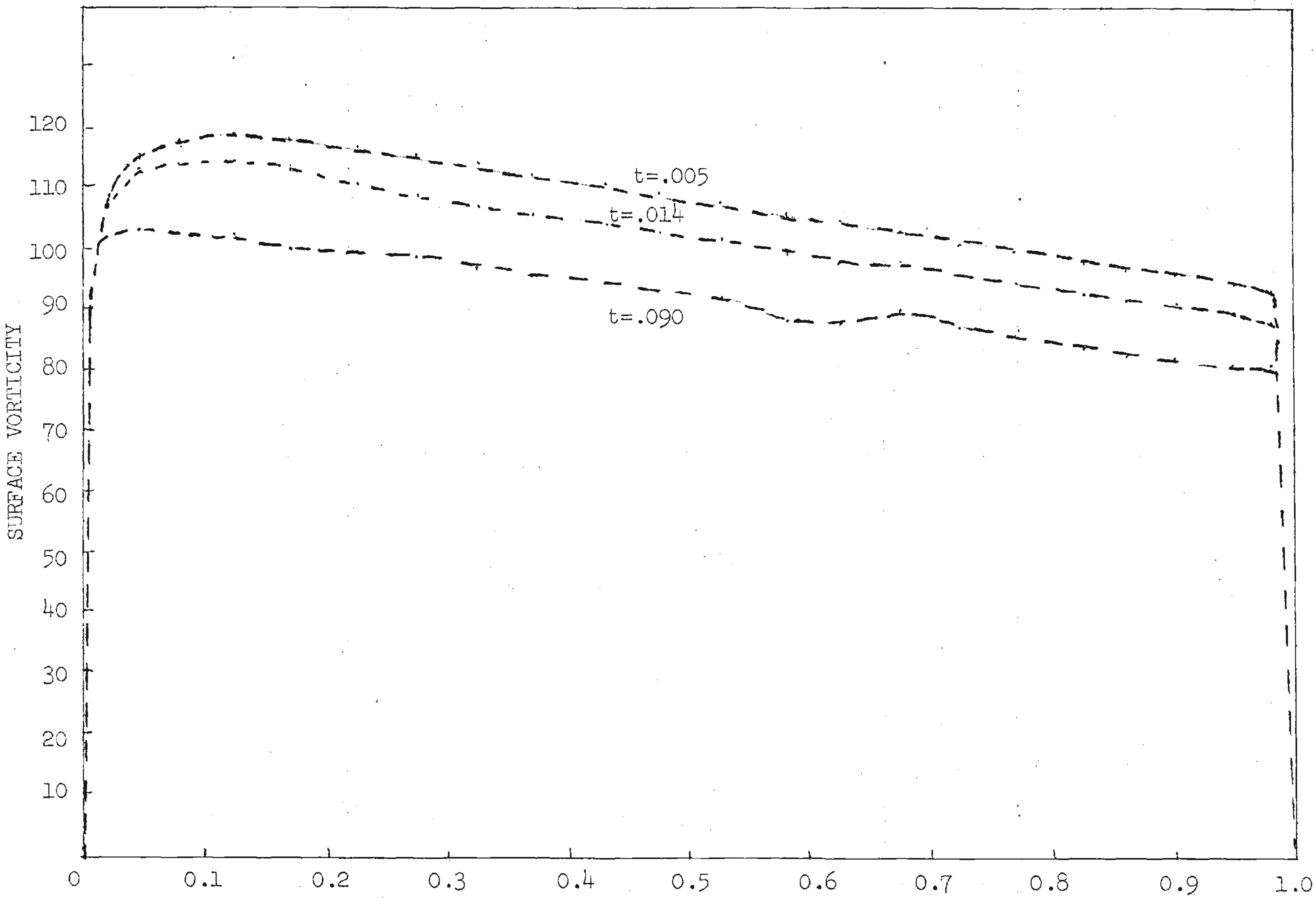


12% Joukowski Aerofoil $\alpha = 0$ $R_e = 1000$



12% Joukowski Aerofoil $\alpha = 0^{\circ}$ $R_e = 1000$

SURFACE VORTICITY DISTRIBUTION



ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

Fifth Quarterly Progress Report

for the period

January 1 to March 31, 1976

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

and

N. L. Sankar, Research Assistant

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting period, the computation of the starting flow over a 12% symmetric Joukowski airfoil at zero angle of attack, initiated during the previous quarter, was continued to a dimensionless time level of 2.488, the reference time being the chord length divided by the freestream velocity. At this time level the rate of change of flow parameters was very slow and the computer results provided a reasonable estimate of steady state values.

Using the steady state solution for the zero angle of attack case as the starting solution, the angle of attack was increased impulsively to 3°. The computation of flowfield surrounding the airfoil was performed up to a time level of 2.052.

A program for computing flows about an oscillating airfoil has been prepared. The 12% Joukowski airfoil was started impulsively from rest with a time dependent angle of attack described by $\alpha = 6^{\circ} \sin(0.6T)$. The computation was performed up to a time level of $T = 1.786$.

In computations for airfoils at non-zero angles of attack, surface pressure distributions were determined from computed vorticity gradient at the airfoil surface. It was suspected that the accuracy of the surface pressure distribution determined in this manner may be questionable. Consequently, a viscous flow past a finite flat plate at a Reynolds number of 1000 was treated to resolve this question. This finite flat plate problem was solved previously using the finite-difference integro-differential method and detailed results were available for comparison with the present solution. Vorticity distributions obtained using the present procedure were found to compared reasonably with the previous results. The new results, however, indicated that modifications of the solution procedures are desirable for the computation of surface pressure distributions. As a consequence of the finite flat-plate study, the following improvements are being incorporated into computer programs for the oscillating airfoil:

(a) a more refined procedure for computing the vorticity distribution on the airfoil surface.

(b) a higher order method of computing velocity values at nodes due to vorticities distributed in the finite elements surrounding the nodes.

During this quarter, two written articles were authored by the principal investigator. A paper entitled "Numerical Boundary Conditions for Viscous Flow Problem", was submitted to the AIAA Journal. The paper has been accepted by the Journal for publication and is now in print. A second paper entitled "Finite Element Solution of Flow Problems Using Integral Representations" is scheduled for presentation and is being printed in the Proceedings of the Second International Symposium on Finite Element Methods in Flow Problems.

ONR Contract No. N00014-75-C-0249

Identifying Number NR 061-226

Sixth Quarterly Progress Report

for the period

April 1 to June 30, 1976

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

and

N. L. Sankar, Research Assistant

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting period, the computation of the starting flow over a 12% thick airfoil at zero angle of attack and a Reynolds number of 1000 was completed. It was found that the computed surface pressure distribution differs substantially from that predicted by potential flow analysis. A remarkable behavior of the pressure distribution is that downstream of the mid-chord station the computed surface pressure remains nearly constant even though no flow separation occurs. An estimate of the boundary layer displacement thickness along the airfoil surface was made using the computed velocity profiles. It was shown that near the trailing edge this displacement thickness is comparable to one half the airfoil thickness, i.e., it is nearly 6% of the chord length. Thus the observed behavior of the surface pressure is not unreasonable. For the 12% thick airfoil, the effect of displacement thickness on the pressure distribution is expected to be significant up to Reynolds numbers of several hundred thousands, although for the higher Reynolds numbers this effect is not expected to be substantial. For thinner airfoils, the effect is even more important.

Although the pressure distribution over the airfoil is not adequately described by the potential flow theory in the present case, the boundary layer simplications, i.e., the neglect of diffusion in the tangential direction and the omission of the normal component of the momentum equation, is still valid in the viscous region of the flow. Also, when a region of flow separation exists and the pressure imposed on the attached region of the viscous flow cannot be predicted by potential theory, the boundary layer simplications are again valid in the attached region. The integro-differential approach used in the present research does not require a matching of the potential flow with the viscous flow. The computation procedure based on the integro-differential approach does not require a determination of the pressure "imposed" on the boundary layer. Therefore, it is possible to utilize the boundary layer simplications in the attached region of the viscous flow, even though the imposed pressure is not obtainable from a potential flow analysis. A preliminary study of this possibility has been initiated. The initial goal is to determine whether substantial reduction in computing effort is realizable for an airfoil oscillating at relatively high frequency and small amplitude about a low angle of attack.

Additional efforts have been directed towards the establishment of a highly accurate method of predicting the surface pressure and shear stress from the computed vorticity distribution. Bulk computation for oscillating airfoils will begin after the completion of this important area of study.

An article entitled "Finite Element Solution of Flow Problems Using Integral Representations" has been presented at the Second International Symposium on Finite Element Methods in Flow Problems. The article has been published by the International Centre for Computer Aided Design, Conference Series No. 2/76, June, 1976, pp. 203-216.

A second article entitled "Explicit Finite Element Solution of the Viscous Flow Problem" has been prepared for the 1976 International Conference on Finite Element Methods in Engineering.

ONR Contract No. NOOO14-75-C-0249

Identifying Number NR 061-226

Seventh Quarterly Progress Report

for the period

July 1 to September 30, 1976

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

and

N. L. Sankar, Research Assistant

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

Efforts during the reporting period include (1) the development of general formulas relating the aerodynamic forces and moments acting on an airfoil to the time variation of vorticity field, (2) the computation of flow fields about a 12% symmetric Joukowski airfoil oscillating in pitch aobut a mean angle of attack of 3^{o}, and (3) the study of flowfield segmentation technique.

As mentioned earlier in previous progress reports, surface pressure distributions determined from computed vorticity gradients at the airfoil surface may be sometimes questionable. In an effort to provide alternative methods of computing the aerodynamic forces and moments acting on airfoils in time-dependent motions (with steady state as a limiting case), the classical circulation theory for thin airfoils was critically reviewed and subsequently extended. General formulas for the lift, the drag, and the moment acting on airfoils were established without relying on the concept of "bound vortices" or "bound vorticity" or assuming the fluid to be inviscid. The formulas are applicable to thick airofils as well as thin ones. They not only remove the limitations of previous theories in calculating the lift and the moment, but also offer a simple method for the calculation of drag forces due to unsteady airfoil motion.

The computation of flowfields about the oscillating airfoil is progressing at a reasonable pace. The above described general formulas are being programmed as a part of the solution procedure.

A study of a flow past a finite flat plate at a finite angle of attack was carried out in which the flowfield segmentation technique was examined. This study demonstrated that the flow field segmentation technique provides more than a factor of two reducion in computer time, with no adverse effect on solution accuracy. It was also established that the optimum operation count using the segmentation technique compares favorably with that using the ADI method.

ONR Contract No. NOOO14-75-C-0249

Identifying Number NR 061-226

Eighth Quarterly Progress Report

for the period

October 1 to December 31, 1976

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

and

N. L. Sankar, Research Assistant

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

The eighth quarterly report for the subject research is expanded to present a review of the research initiated on January 1, 1975. During the time period from January 1, 1975 to December 31, 1976, three articles were authored by the principal investigator and his co-worker as a result of this research. These articles are listed at the end of this quarterly report. Reprints of these articles will be distributed as technical reports in accordance with the distribution list provided by ONR for this project.

The work statement for the current project includes the following specific tasks:

1. The optimization of the flowfield segmentation technique for the vorticity computation.
2. The establishment of a computer program applicable to incompressible flow past airfoils of arbitrary shape executing arbitrary unsteady motions using the integro-differential method and incorporating the segmentation technique.
3. The prediction of the time-dependent laminar incompressible flow fields associated with an NACA 0012 airfoil oscillating in pitch for specified values of Reynolds number, mean angle of attack, amplitude, frequency, and pitch axis location of oscillation.
4. Exploration of the extensions of the integro-differential method to turbulent and to compressible flows.

Tasks 1, 2, and 4 have been completed. Task 3 has been carried out for a 12% thick Joukowski airfoil for several of the reduced frequencies and mean angles of attack originally suggested. The remaining cases are expected to be completed very shortly.

Under Task 1, a study of a flow past a finite flat plate at a finite angle of attack was carried out by A. H. Spring, a Ph.D. candidate working on his thesis in absentia. This study demonstrated that this segmentation technique provides at least a factor of two reduction in computer time with no adverse effect on solution accuracy. In addition, it has been shown that the optimum operation count using the segmentation method compares favorably with that using the ADI method. The successful development of the segmentation technique in conjunction with the integro-differential method, which permits the confinement of the solution field to the vortical region of the flow, ensures the superiority of the method in computational efficiency.

In the kinematic part of the computation, the use of the integro-differential method allows the velocity field to be computed by evaluating an integral containing only the vorticity distribution and spatial coordinates in its integrand. A numerical quadrature procedure has been developed which is similar to that used in finite-element methods. The region of integration is divided into elements of various shapes. Interpolation functions are used in each element and expressed in terms of nodal values of vorticity. Analytic integrations then yield the velocity value at each node as the sum of products of vorticity values and geometric functions associated with each node. This procedure allows an inherent flexibility in the selection of data node locations and the accomodation of complex boundary geometries.

Under Task 2, a program for the prediction of time-dependent laminar incompressible flows past a 12% thick Joukowski airfoil has been prepared, calibrated, and refined. This program can be easily modified and used for airfoils of other types. With previous finite-difference or finite-element methods, the kinematic part of the computation required up to 80% of the total computer time for the type of problems under consideration. The integro-differential method led to a drastic improvement in the kinematic part of computation. As a result, the kinematic computation now constitutes only a small part of the total computing effort. Since the kinetic part of computation now becomes the bottleneck in the overall solution procedure, further improvements in the kinematic computation cannot lead to further drastic improvements in the overall computational efficiency. Considerable effort has been consequently devoted to the development of more efficient numerical methods for solving the kinetic equation of vorticity transport.

Finite-element methods are inherently mroe flexible than finite-difference methods in the seleciton of data node locations and the accomodation of complex boundary geometries. However, finite-difference expressions are somewhat easier to formulate and the coefficient matrices involved in the finite-difference equations are somewhat less cumbersome than those of finite-element methods. Also, within either the finite-difference or the finite-element methods, explicit techniques of solution generally require much less computation per time step than do implicit techniques However, since explicit techniques generally are subject to very stringent stability restrictions on the maximum time step, the use of implicit method is generally preferred on an overall basis. These relative advantages and disadvantages of the finite-difference methods vs. the finite element methods and of the explicit techniques vs. the implicit techniques are well known. No successful attempt has been made, however, by previous investigators to devise a method that possesses the advantages mentioned above and alleviates the disadvantages. Under the curent project, such a method has been established by N. L. Sankar as part of his Ph.D. thesis in preparation.

Sankar has successfully developed a hybrid method in which the solution field, already confined to the vortical region, is divided into an inner region and an outer region. In the inner region, a finite-element method based on Galerkin's procedure is used to set up a system of algebraic equations containing the unknown vorticity values at a new time level. This inner region envelops the airfoil. The airfoil boundary is accurately represented by the finite-element nodes. This region is kept small so that it contains a relatively few closely spaced nodes. For example, for the 12% thick Joukowski airfoil, the inner region is assigned 232 nodes while the total number of nodes in the solution field is about 2500. (Note that for an impulsively started airfoil, as the vortical region grows with time, the total number of nodes increases). The close spacing of nodes in this region is required by the solution resolution since the vorticity gradient is relatively large near the airfoil, particularly in the boundary layer. The relatively small number of nodes in the inner region allows an implicit solution procedure to be employed without excessive computing efforts.

The outermost layer of elements in the inner region is made to overlap with a finite-difference grid used for the outer region. The finite-difference grid lines are arranged to coincide with the coordinate lines. Relatively large grid line spacings are adequate in the outer region since the vorticity gradient is relatively small far form the airfoil. The Dufont-Frankel finite-difference technique, which is explicit and is not subject to stringent stability restrictions on the time step in regions of large grid spacings, is used for the outer region.

The hybrid method has been calibrated by treating the test problem of a circular cylinder. It has been, and is being, used to obtain solutions for time-dependent laminar incompressible flows associated with an oscillating 12% thick airfoil. The method has been described in Article 1, which presents sample results for the airfoil. Additional numerical results, detailed mathematical analyses, numerical procedures, computer programs, and discussions of the major difficulties encountered and resolved, are being included in Sankar's Ph.D. thesis and in a final report of the current project.

Under task 4, a systematic study of formulation of integral representations for elliptic and parabolic differential equations has been completed. Application of the integro-differential method to various types of flow problems--potential and viscous, time-dependent and steady, incompressible and compressible, laminar and turbulent --has been considered. The results of this study are presented in a recent article (Article 2). The availability of integro-differential formulations under very general circumstances is expected to be a significant factor in future solutions of flow problems of all types. Some initial work has been performed in using a two-equation turbulence model in conjunction with the integro-differential method for the solution of turbulent flow problems. Very encouraging results have been obtained for several relatively simple test problems.

In addition to the specific tasks reviewed above, general formulas relating the lift, the drag, and the moment acting on an airfoil to the time variation of vorticity field has been examined.

A particularly difficult aspect of previous numerical methods for the solution of viscous flow problems has been the establishment of "extraneous boundary conditions." It was shown earlier by the principal investigator that the use of the integro-differential method eliminates this difficulty. Under the current project, further work has been carried out to formalize a kinematic approach for establishing the extraneous vorticity boundary conditions. It has been shown that this approach leads to an integral equation containing the boundary vorticity value as the unknown function. The necessary and sufficient condition under which this integral equation yields a unique solution has been studied. It has been found that under certain circumstances, an integral equation, similar to that encountered in the study of the extraneous boundary condition problem, can be established for the study of motions surrounding an unsteady airfoil. Thus, the techniques established for the extraneous boundary condition problem are useful in the development of an unsteady airfoil theory.

PUBLISHED ARTICLES

1. Wu, J. C. and Sankar, N. L., "Explicit Finite Element Solution of the Viscous Flow Problem," Proceedings of the 1976 International Conference on Finite Element Methods in Engineering, 1976.

2. Wu, J. C., "Finite Element Solution of Flow Problems Using Integral Representations," Proceedings of Second International Symposium on Finite Element Methods in Flow Problems, International Centre for Computer Aided Design, Conference Series No. 2/76, pp. 203-216, 1976.

3. Wu, J. C., "Numerical Boundary Conditions for Viscous Flow Problems", AIAA Journal, Vol. 14, No. 8, pp. 1042-1049, August, 1976.

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Ninth Quarterly Progress Report

for the period

January 1 to March 31, 1977

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

Numerical computation of the time-dependent flow fields associated with a 12% symmetric Joukowski airfoil oscillating sinusoidally about a pitching axis located ¼ chord from the leading edge, with a mean angle of attack of 3^{o}, and amplitude of 1^{o}, and a reduced frequency of 0.3, has begun. The main purpose of this computation is to provide numerical results for a case where the flow does not separate over a large region on the upper side of the airfoil. It is planned to utilize these results and other results for different angles of attack and reduced frequencies as standards against which various simplified numerical procedures as well as improved and extended airfoil theories for steady and unsteady flows can be tested. A number of modifications to the previously developed computer program for the oscillating airfoil problem are being considered. They include the possibilities of performing the kinetic computation for several time steps before recomputing a new velocity field (kinematic computation), of using an implicit procedure for the kinetic computation in both the inner finite-element region and the outer finite-difference region, and the use of a Fourier series in the computation of the surface vorticity. These modifications are directed towards further reductions of the needed computer time and improvements in solution accuracy, which are highly important for the intended purpose of the present numerical results.

A study has been initiated to extend the general relations between the time-variation of the vorticity field and the aerodynamic forces and moments to three-dimensional problems. The two-dimensional versions of these general relations have been presented in the renewal proposal for this project, dated November 5, 1976. A thorough literature review has been made for the three-dimensional problems. Certain fundamental theorems deemed necessary for the derivation of three-dimensional relations have been obtained. It is hoped that these fundamental theorems will permit a rigorous derivation of the three-dimensional relations.

During this Quarter, the principal investigator of this project visited Ankara and Cairo, the former at the invitation of the Middle East Technical University to contribute to a NATO project studying the oscillating airfoil problem. He gave seminars in both cities on the general topic of numerical methods for the solution of the Navier-Stokes equations.

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Tenth Quarterly Progress Report

for the period

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UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

A major emphasis during the reporting quarter has been the continued computation of the time-dependent flow fields associated with a 12% symmetric Joukowski airfoil oscillating sinusoidally about a pitching axis located ¼ chord from the leading edge, with a mean angle of attack of 3^{o}, an amplitude of 1^{o}, and a reduced frequency of 0.3. Numerical solutions have been carried out up to a dimensionless time level of 8, which corresponds to the movement of the airfoil relative to the freestream through a distance of eight chord lengths. It is anticipated that this computation will be completed during the subsequent quarter of research. Additional numerical results will be obtained for the same airfoil oscillating at a higher reduced frequency of 3, and also for a higher angle of attack so that the phenomena of dynamic stall occurs. An article is being prepared summarizing the numerical methods developed as well as numerical results for the airfoil problem, obtained previously and currently, under this project. This article is scheduled for presentation at the AGARD Fluid Dynamics Symposium on Unsteady Aerodynamics to be held in Ottawa, Canada, on 26-28 of September, 1977. The article is entitled "A Numerical Study of Unsteady Viscous Flows around Airfoils." It will be published in the Proceedings of the Symposium by AGARD.

The development of three-dimensional relations between the time-variation of the vorticity field and the aerodynamic forces and moments has progressed more rapidly than anticipated. The significances of these results in practical applications are being studied. It is anticipated that a report summarizing the basic concepts of a new airfoil and wing theory based on these relations will be prepared during the next quarter (July to September, 1977).

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Eleventh Quarterly Progress Report

for the period

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UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting quarter, computations of flows past a 12% symmetric airfoil were completed. Five sets of computer solutions are now available for the following five flows past the airfoil at a Reynolds number of 1000: (a) impulsively started motion, zero angle of attack; (b) impulsively started motion, 3° angle of attack; (c) oscillation in pitch with an amplitude of $\pm 1^\circ$ about a mean angle of attack of 3°, reduced frequency 0.3; (d) same as case "c" except that the reduced frequency is 3; (e) oscillation in pitch with an amplitude of $\pm 6^\circ$ about a mean angle of attack of 9°, reduced frequency of 3. The pitching axis for all oscillating airfoil cases is at 1/4 chord from the leading edge.

Selected results for some of the above cases are included in a paper presented at the AGARD Fluid Dynamics Symposium on Unsteady Aerodynamics held in Ottawa, Canada, on September 26-28 of 1977. The written version of this paper, entitled "A Numerical Study of Viscous Flows Around Airfoils", is being published in the Proceedings of the Symposium by AGARD.

The development of three-dimensional relations between the time-variation of the vorticity field and the aerodynamic forces and moments have been completed. A report summarizing the major results of a new airfoil and wing theory is being prepared.

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Twelfth Quarterly Progress Report

for the period

October 1 to December 31, 1977

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

The following paragraphs are modified version of a section of a renewal proposal recently submitted by the principal investigator to ONR. They summarize the work accomplished during the calendar year 1977. A major emphasis during the reporting quarter (October to December, 1977) has been the analysis of the considerable amounts of numerical results generated, the development of a better understanding of the interplay between the numerical approaches and the physical aspects of the general viscous flow problem, and the planning for additional efforts in studying the oscillating airfoil problem.

The research program proposed in November of 1976 is a long termed program requiring three years of research effort. It contains the following specific tasks:

1. The systematic development, calibration, and application of improvements and extensions of presently available steady and unsteady airfoil theories for calculating aerodynamic forces and moments.
2. The establishment of simplified numerical procedures for the solution of viscous flow equations.
3. The prediction of time-dependent laminar incompressible flow-fields associated with several types of unsteady airfoil motion utilizing the existing and the simplified numerical procedures.
4. The exploration of extensions of the general relations between the time-variation of vorticity field and the aerodynamic forces and moment to three-dimensional problems.

Two-dimensional formulas relating the forces and moments acting on an airfoil section to the time-variation of vorticity-moment integrals were obtained as extensions of the circulation theory of airfoils and presented in the proposal. During 1977, general formulas that are valid for two- and three-dimensional viscous flows past one or more finite solid bodies of arbitrarily prescribed shapes and executing arbitrarily prescribed steady or time-dependent motions have been obtained. The derivation of these general formulas has proceeded on a formal mathematical basis. Several theorems for viscous flows, utilized in the derivation of the general formulas, have been proved. These theorems are of significance by themselves. Derivations of the general formulas and proofs of the theorems are included in a ONR report to be released. Task 4 described above is essentially completed.

Under Task 3, it was proposed that a 12% thick Joukowski airfoil be studied. The unsteady motions considered were to be principally oscillating motions. However, impulsively started motions would also be studied. Cases where the boundary layers are attached all the way to the trailing edge were to be emphasized initially because of the direct usefulness of these cases in assessing the range of validity of the current unsteady airfoil theories and of the new theory. During 1977, five sets of computer solutions were obtained for the following five flows about a 12% thick Joukowski airfoil at a Reynolds number of 1000: (a) impulsively started motion, zero angle of attack; (b) impulsively started motion, 3° angle of attack; (c) oscillation in pitch with an amplitude of $\pm 1^{\circ}$ about a mean angle of attack of 3°, reduced frequency 0.3; (e) oscillation in pitch with an amplitude of $\pm 6^{\circ}$ about a mean angle of attack of 9°, reduced frequency of

3. The pitching axis for all oscillating airfoil cases is at 1/4 chord from the leading edge.

The results for the above cases are presented in a Ph.D. thesis by N.L. Sankar. Aside from these computations performed under ONR support, results were also obtained for a 9% thick impulsively started airfoil at 15^{o} angle of attack. The substantial amounts of numerical solution obtained during the past twelve months indicate that the integro-differential approach has progressed to the stage where it is feasible to develop, for two-dimensional laminar flows, packages of computer codes that are efficient, reasonably universal, sufficiently accurate, and relatively simple to utilize. In fact, for steady internal flows, such a package of computer codes that are efficient, reasonably universal, sufficiently accurate, and relatively simple to utilize. In fact, for steady internal flows, such a package of computer code was recently prepared by Dr. M. M. Wahbah, a member of our research team at Georgia Tech, under a separate project. This code is being utilized in a research program in which various turbulence models for separated steady flows are tested. For external laminar flows, existing computer programs require about 10 minutes of CDC-6600 CPU time to advance the numerical solution by one dimensionless unit of physical time, i.e., the time interval during which the airfoil advances by one chord length relative to the freestream. Consequently, drastic further improvements in solution efficiency is no longer a critical factor in the development of a general-purpose user-oriented package of computer code for external laminar flows in two-dimensions. Thus far, however, efforts at Georgia Tech have continued in developing simplified and more efficient numerical procedures, principally in anticipation of future applications involving turbulent and three-dimensional flows that are known to require several hundred times the computation required for two-dimensional laminar flows. In this regard, under Task 2, an integral equation approach which is expected to eliminate much of the computation in the boundary-layer region of the flowfield has been examined. It was pointed out in the proposal that for two-dimensional flows containing no appreciable separation region, the near vortical system can by approximated by a vortex sheet enveloping the airfoil. The vortex strength of this sheet can be obtained by solving an integral equation. It has been shown recently that in the more general case where an appreciable separation region exists, the boundary layer portion of the flow can still be approximated by a vortex sheet. The strength of this vortex sheet can be obtained by solving a modified integral equation.

Under Task 1, several possible extensions and improvements of the existing airfoil theory were pointed out in the proposal. Detailed studies of these possibilities have been initiated. Previous "inviscid" airfoil theories do not permit the calculation of the drag. The present theory relates the diffusion of vorticity in the vortical wake to a drag force. Furthermore, even if the vortical wake is considered inviscid, the present theory predicts a non-zero drag due to the rotational and unsteady translational motions of an airfoil. The present theory offers a simple method for quantitative determination of this drag. In particular, for two-dimensional problems, an extremely simple procedure has been developed for the force experienced by a solid body undergoing unsteady translation.

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Thirteeth Quarterly Progress Report

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Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

It was previously reported that, during 1977, five sets of computer solutions were obtained for the following five flows past the airfoil at a Reynolds number of 1000: (a) impulsively started motion, zero angle of attack; (b) impulsively started motion, 3^{o} angle of attack; (c) oscillation in pitch with an amplitude of $\pm 1^{o}$ about a mean angle of attack of 3^{o}, reduced frequency 0.3; (d) same as case "c" except that the reduced frequency is 3; (e) oscillation in pitch with an amplitude of $\pm 6^{o}$ about a mean angle of attack of 9^{o}, reduced frequency of 3. The pitching axis for all oscillating airfoil cases is at 1/4 chord from the leading edge. These results are presented and discussed by N. L. Sankar in his Ph.D. thesis entitled "Numerical Study of Laminar Unsteady Flow over Airfoils", Georgia Institute of Technology.

It was also previously reported that work has been in progress to develop a new theory for aerodynamic forces and moments. This theory, described in our renewal proposal for this project dated November 1977, relates the aerodynamic forces and moments to the time-variation of vorticity-moment integrals. The development of this theory for both two-dimensional and three-dimensional flows was completed during the reporting quarter. The theory is now established rigorously on a formal mathematical basis. A number of ambiguities in the understanding of previous aerodynamic theories has been removed by the present theory, which permits a clearer interpretation of various aerodynamic phenomena.

The major emphasis of this research during the reporting period has been the development of a better understanding of the interplay between the numerical and physical aspects of the viscous flow problem. This understanding has been brought into focus by the simultaneous availability, for the first time, of detailed numerical results for several unsteady airfoil problems and of a theory which relates flow details to the forces and moments on the airfoil. Considerable amounts of information have been generated through an in-depth examination of the numerical results with the help of the new aerodynamics theory.

The theory has been applied also in an analytical study of the two-dimensional unsteady flow problem. It has been shown that the theory yields an extremely simple procedure for the determination of the apparent mass and the apparent moment of inertia of a solid body of any arbitrarily prescribed shape undergoing any arbitrarily prescribed unsteady translational or rotational motions.

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Fourteenth Quarterly Progress Report

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James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

Several manuscripts describing various aspects of the present research were prepared during the reporting quarter. The manuscripts are:

1. "Viscous Flow around Oscillating Airfoil --- A Numerical Study", by N. L. Sankar and J.C. Wu, AIAA Paper 78-1225, presented at the AIAA 11th Fluid and Plasma Dynamics Conference, July 1978.
2. "Integral-Representation Approach for Time-Dependent Viscous Flows", by J. C. Wu and Y. M. Rizk, Proceedings of the Sixth International Conference on Numerical Methods in Fluid Dynamics, 1978, in print.
3. "Numerical Solution of Unsteady Flow Problems using Integro-Differential Approach", by J. C. Wu, M. M. Wahbah, and A. Sugavanam, Proceedings of the Symposium on Nonsteady Fluid Dynamics of the 1978 Winter Annual Meeting of the ASME, in print.
4. "A Theory for Aerodynamic Forces and Moments", by J. C. Wu, Georgia Institute of Technology Report, June 1978, in print.

In addition to the above manuscripts, another report entitled "A Modern Look at Apparent Mass and Apparent Moment of Inertia of Solids Undergoing Unsteady Motions", by J. C. Wu and N. L. Sankar, has been written. This manuscript for report is being revised before final typing. The preparation of five full-length manuscripts during the reporting quarter taxed the attention of the investigators. It is felt, however, that the substantial amount of information generated recently under the present project should be documented and made available in the open Literature in a timely manner. The preparation of these manuscripts has offered the principal investigator an opportunity to review comprehensively the research work accomplished during the past several quarters. As a result of this review, a decision has been made to emphasize, during the next two quarters, the following two tasks: (a) the development of efficient numerical procedures based on the formulation of the viscous flow problem entirely as integral representations and (b) the investigation of the possibility of treating the boundary layer region separately from the recirculation and wake regions of the flow. Preliminary results obtained thus far indicate that both approaches just mentioned are well suited for high Reynolds number flows. Detailed studies will be made to assess in quantitative as well as qualitative terms the advantages of these new approaches.

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Fifteenth Quarterly Progress Report, July 1, 1978 through August 31, 1978

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Sixteenth Quarterly Progress Report

for the period

September 1 to December 31, 1978

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

A generalized wake-integral theory relating the drag force acting on a three-dimensional lifting body to properties of the vortical wake downstream of the body was further developed during the reporting quarter. This theory is a generalization of the well known theory of Betz, who expressed profile drag acting on a solid body as a wake integral, i.e., an integral over only the vortical wake region downstream of the solid body. The experimental determination of the profile drag using Betz's formula is convenient and efficient, since measurements need to be made only in the vortical wake region of a single downstream plane. The determination of the induced drag in Betz's formula requires measurements of velocity components perpendicular to the free stream direction over a large region and is not limited to the vortical wake. These transverse velocity components are usually too small in regions away from the vortical wake for measurements with good accuracy. For these reasons, the experimental determination of the induced drag presented serious difficulties in cost as well as in accuracy.

The present wake-integral theory is an offshoot of the new theory for aerodynamic forces and moments developed as a part of the present research program. With the present theory, the induced drag was expressed as the sum of two integrals representing separately the contributions of the axial and transverse components of the wake vorticity. It was shown that under quite general circumstances the transverse vorticity integral is negligible compared with the axial vorticity integral. In consequence, the induced drag is accurately determined by axial vorticity measurements in the vortical wake only. This means that the experimental determination of the induce drag using the present theory is efficient and accurate. Together with the theory of Betz, the present theory offered considerable insight into the interplay between the drag components and the wake characteristics.

Exploratory wind-tunnel results obtained by J. E. Hackett and his co-workers at the Lockheed-Georgia Company were utilized to verify the validity of some of the theoretical conclusions. It is worthy of note that these experimental results lend substantial support to the expectation that the profile drag and the induced drag can be separately measured through wake integrals and such measurements can provide accurate quantified knowledge about the relative importance of various flow features to drag.

In addition to the development of the wake-integral theory for drag, some attention was devoted to the development of a numerical procedure in which the attached boundary layer region is solved separately from the remainder of the viscous flow. It is expected that more definitive conclusions regarding this procedure will be reached during the next two quarters.

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Seventeenth Quarterly Progress Report

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Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Director, Fluid Dynamics Program

Mathematical and Information Sciences Division

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting period, the development of the wake-integral theory for drag measurement, described in the preceeding (September to December, 1978) quarterly progress report for this project, reached a reasonable stage of completion. The theory was shown to offer considerable insight to the interplay between the profile and the induced drag components. Many theoretical conclusions based on this theory were verified experimentally. In particular, it was shown conclusively that the induced drag, like the profile drag, can be accurately determined by measurements in the vortical wake of a single downstream plane. Additional research efforts beyond the present research are being planned. A paper, entitled "A Generalized Wake-Integral Approach for Drag Determination in Three-Dimensional Flows", was jointly authored by the principal investigator and J. E. Hackett and D. E. Lilley of the Lockheed-Georgia Company. The latter two authors provided the experimental data and data reduction efforts for comparison with the theory developed at Georgia Tech. This paper was published by AIAA as a preprint, AIAA Paper No. 79-0279, and was presented at the 17th Aerospace Sciences Meeting of AIAA in New Orleans, La.

A focal point of research during the reporting quarter was the development of a numerical method for the separate treatment of attached and detached flow regions in general viscous flows. In such flows, there exist usually two distinct regions, one attached and the other detached, in which viscous effects are important. In the attached region, the flow is generally adequately described by boundary-layer equations. In the detached flow, simplifications of the Navier-Stokes are only occasionally justifiable. The numerical solution of boundary layer equations is substantially simpler than that of Navier-Stokes equations. Prevailing numerical methods for the general viscous problem, however, do not treat the attached and detached regions separately — the Navier-Stokes equations are solved in both regions. In consequence, highly efficient and accurate computer codes presently available for boundary layers cannot be utilized in studies of general viscous flows.

The description of the boundary-layer includes a specification of the velocity at the outer edge of the boundary-layer, or, equivalently, the pressure gradient along the boundary layer. In a general viscous flow past a finite solid body, the presence of "strong viscous-inviscid interaction" makes it unacceptable to determine this boundary-layer-edge velocity through a potential flow calculation based on the solid body shape, even when boundary-layer displacement effects are accounted for. This fact represents a major obstacle to a separate treatment of the attached and detached regions. Earlier proposals for removing this obstacle suggested repeated computations of the attached and detached flow regions individually. The influence of one region on the other is then determined through an iterative matching procedure. Such a procedure requires very large amounts of computation. In contrast, the present method permits the separate computation of the attached and detached flow regions without using an iterative matching procedure. In consequence, the method offers several important advantages in computation.

A research paper presenting the basic concepts of this new method is scheduled for presentation at the AIAA 4th Computational Fluid Dynamics Conferences in July of 1979. This paper is entitled "Separate Treatment of Attached and Detached Flow Regions in General Viscous Flows", and is co-authored by J. C. Wu and U. Gulcat. Because of the great potential of this method for future three-dimensional flow computations, it is planned to carry on this specific topic of research for the remainder of this year and beyond.

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Progress Report

for the period

January 1 to March 31, 1982

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Fluid Dynamics Program

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting quarter, several manuscripts describing various aspects of the present research were presented. The manuscripts are:

1. "Principal Solutions and Finite-Element Procedures," by J. C. Wu, Proceedings of the 4th International Symposium on Finite Element Methods in Flow Problems, in print.

2. "Sources, Sinks, Vortices and Flow Computations," by J.C. Wu, Proceedings of International Conference on Finite Element Methods, in print.

Copies of these two manuscripts are attached to the Quarterly Report. In addition to these manuscripts, a report entitled "Unsteady Aerodynamics Forces Associated with a Vortex Flowing past a Lifting Body" has been prepared. The manuscript for this report is being revised before final typing. The preparation of these three full-length manuscripts during the reporting quarter required substantial attention by the principal investigator. It is felt, however, that the substantial amount of information generated recently under the present research should now be documented and made available in the open literature. Of particular significance is the emergence, through the present research, of an insight into the interplay between the physical and the numerical aspects of the unsteady viscous flow problem. This insight, continually cultivated since the inception of this project, has enabled the principal investigators to develop not only highly efficient numerical methods for the solution of viscous flow problems, including those at high Reynolds numbers, but also new and general theories for aeordynamic forces in unsteady viscous flows. The cross-fertilization between the theoretical and computational branches of fluid dynamics has been remarkably fruitful. It is planned to further investigate several promising approaches for the study of unsteady aerodynamics forces on the basis of the theories and numerical methods developed under this project.

4th International Symposium on Finite Element Methods in Flow Problems, July, 1982

PRINCIPAL SOLUTIONS AND FINITE-ELEMENT PROCEDURES

J.C. Wu

Georgia Institute of Technology
Atlanta, Georgia
U.S.A.

INTRODUCTION

Until relatively recently, the world community of fluid dynamicists has emphasized the development of the finite-difference approach for flow problems. The application of the finite-difference approach to certain types of flows, such as viscous flows in contact with solids with complex shapes and containing appreciable regions of separation, however, encountered serious difficulties. The need for alternatives to the finite-difference approach has led to a rapid growth of interest in the use of the finite-element approach for the solution of flow problems. The development of finite-element procedures for fluid-dynamics applications has been strongly influenced by the earlier successes of the finite-element approach in solid mechanics applications. In extending the application of the finite-element approach to the more general realm of continuum mechanics, however, many difficulties encountered earlier in the development of the finite-difference approach for flow problems have reappeared.

During the past decade, this author and his co-workers have carried out a research program with the objective of establishing an alternative numerical approach for the solution of flow problems. A focal point of this program has been the development of a finite-element procedure which utilizes integral representations for the flow variables obtained through the concept of principles solutions of differential equations. This approach yields algebraic equations that are drastically different from the more familiar finite-element and finite-difference equations and offers several decisive advantages. In a forthcoming publication[1], this approach will be reviewed and summarized. The purpose of the present article is twofold: (1) to discuss the role of the principal solutions in the formulation of flow problems for finite-element computations, and (2) to present selected results that illustrate the essential features of this approach.

PRINCIPLE SOLUTIONS

The concept of principal solutions is fundamental to the method of solving flow equations described in this paper. The principal solution $P(\vec{r};\vec{r}_o)$ of the elliptic equation is defined by

$$P(\vec{r};\vec{r}_o) = \begin{cases} -\dfrac{1}{2\pi}\,\ell n\,\dfrac{1}{r'} & \text{in two-dimensional problems} \\[2ex] -\dfrac{1}{4\pi r'} & \text{in three-dimensional problems} \end{cases} \tag{1}$$

where $\vec{r}$ is a position vector and r' is the magnitude of $\vec{r}\,'$, defined by

$$\vec{r}\,' = \vec{r}_o - \vec{r} \tag{2}$$

P satisfies Laplace's equation everywhere except at the point $\vec{r} = \vec{r}_o$ where it is singular. The singularity of P at $\vec{r} = \vec{r}_o$ is such that

$$\int_R \nabla^2 P \; dR \;=\; \begin{cases} 1 & \text{if R contains the point } \vec{r} = \vec{r}_o \\ 0 & \text{if R does not contain the point } \vec{r} = \vec{r}_o \end{cases} \tag{3}$$

The principal solution $Q(\vec{r}, t; \vec{r}_o;, t_o)$ of the parabolic equation is defined by

$$Q(\vec{r}, t; \vec{r}_o, t_o) = \frac{1}{\left[4\pi a(t - t_o)\right]^{d/2}} \exp -\left\{\frac{r'^2}{4\nu(t - t_o)}\right\} \tag{4}$$

where t is the time coordinate, $t > t_o$, d is the dimensionality of the problem, i.e., d=2 for a problem involving two spatial dimensions and d=3 for a problem involving three spatial dimensions, and ν is the diffusion coefficient. Q satisfies the homogeneous diffusion equation $(\frac{\partial Q}{\partial t} = \nu \nabla^2 Q = 0)$ and possess the following integral property:

$$\int_R QdR = \begin{cases} 1 & \text{if R contains the point } \vec{r} = \vec{r}_o \\ 0 & \text{if R does not contain the point } \vec{r} = \vec{r}_o \end{cases} \tag{5}$$

The physical significance of the principal solutions P and Q are well-known. If there exists a unit source at the point $\vec{r}_o$, then the scalar potential associated with this unit source at the point $\vec{r}$ in an infinite unbounded region is given by P. If, at the time t_o there exists in an infinite unbounded region a unit of some physical quantity at the point $\vec{r} = \vec{r}_o$, then Q represents the distribution of this physical quantity in space at a later time $t > t_o$ as a result of the diffusion process.

VISCOUS FLOW EQUATIONS

The time dependent motion of an incompressible viscous fluid is governed by the law of mass conservation and Newton's laws of motion. The mathematical statements of these laws are familiarly expressed in terms of the velocity vector $\vec{v}$ and the pressure p. It is convenient, however, to introduce the concept of the vorticity vector $\vec{\omega}$ and re-express the problem as follows:

$$\nabla \cdot \vec{v} = 0 \tag{6}$$

$$\nabla \times \vec{v} = \vec{\omega} \tag{7}$$

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) + \nu \nabla^2 \vec{\omega} \tag{8}$$

Equation (6) is the continuity equation for an incompressible fluid. Equation (7) is the definition of the vorticity vector. These two equations together describe the kinematic aspect of the viscous flow problem[1]. That is, they express the relationship between the vorticity field at any given instant of time and the velocity field at the same instant. The kinetic aspect of the problem, equation (8), expresses the transport of the vorticity by the processes of convection, vorticity stretching, and viscous diffusion.

In the solution of a time dependent viscous flow problem, it is convenient to follow the kinetic development of the vorticity field in the fluid. A numerical procedure can be established in which the solution is advanced from an old time level to a new time level through a computation loop consisting of a kinetic step and a kinematic step. In the kinetic step, with vorticity and velocity values known at the old time level, equation (8) is solved to obtain vorticity values at the new time level. In the kinematic step, velocity values corresponding to the vorticity values at the new time level are obtained by solving equations (6) and (7).

In order to carry out the computational loop described above repeatedly and advance the solution, boundary values of the vorticity vector at each new time level must be determined. Since the boundary vorticity values are not ordinarily known directly from the physics of the problem, the problem of determining the vorticity boundary condition has been called the "extraneous boundary condition problem"[2]. As it turns out, this extraneous boundary condition problem is a part of the kinematics of the overall problem. The integral representation of the kinematic aspect of the problem provides a convenient way to compute the boundary vorticity values accurately[2].

INTEGRAL REPRESENTATIONS

Equations (6) and (7) form an elliptic set. Using Green's theorem, equations (6) and (7) have been recast into the following integral representation[3]:

$$v = -\int_R \vec{\omega}_o \times \nabla_o P \, d R_o + \oint_B (\vec{v}_o \cdot \vec{n}_o + \vec{v}_o \times \vec{n}_o \times) \nabla_o P \, d B_o \tag{9}$$

where the subscript "o" indicates that a variable, a differentiation, or an integration is in the $\vec{r}_o$ space, $\vec{n}_o$ is the unit normal vector directed outward from the region R, and B is the boundary of R.

The velocity and vorticity fields in Equation (9) are for the same instant of time. The integral representation is composed of an integral over the region R and an integral over the boundary B. Both integrals contain the gradient of P in their integrands. Physically, the quantity $-\vec{\omega}_o \times \nabla_o P \, d R_o$ gives the velocity field associated with the vorticity in the elemental region $d R_o$ in accordance with the Biot-Savart Law. The region integral gives therefore the "Biot-Savart" contribution of the vorticity present in R to the velocity field in R. The boundary integral represents the contribution of the velocity boundary condition on B to the velocity field in the region R. Alternatively, the boundary integral may be viewed as representing the contribution of the vorticity field outside the region R to the velocity field in the region R.

The vorticity transport equation, equation (8), is a vector parabolic differential equation. An integral representation[4,5] of the vorticity vector that satisfies equation (8) everywhere in the region R is

$$\vec{\omega}(\vec{r}, t) = \int_R (\vec{\omega}_o Q)_{t=o} \, dR_o + \int_o^t dt_o \int_R (\vec{v}_o \times \vec{\omega}_o) \times \nabla_o Q \, dR_o + \int_o^t dt_o \oint_B \left[\omega_o (\nabla_o Q \cdot \vec{n}_o) - Q_o (\vec{n}_o \cdot \vec{v}) \vec{\omega}_o \right] dB_o \tag{10}$$

The principal solution Q gives the vorticity distribution in an infinite unlimited stationary fluid at the time level t as a result of diffusion of a unit amount of vorticity located at $\vec{r}_o$ at a preceding time level t_o. The first integral in equation (10) therefore represents the effect of an initial (t=0) vorticity distribution. In actual flows, a stationary fluid cannot co-exist with a non-zero vorticity field. The vorticity distribution changes as a result of not only the diffusion process but also the convection and the vortex stretching processes. The cumulative effects of these additional processes on the vorticity distribution at the time level t are represented by the second integral. The third integral gives the effect of the boundary values of $\vec{\omega}$ on the vorticity distribution in R at the time level t. This integral may be viewed as representing the effect of diffusion of the vorticity field outside the region R on the vorticity field in the region R.

SOLUTION PROCEDURE

Equations (9) and (10) express the entirety of the time-dependent incompressible viscous flow problem mathematically as integral representations. The use of the familiar finite-element methodology leads to a set of algebraic equations approximating equation (9) and (10). In essence, the region R is divided into subregions (elements). The boundary B is divided into boundary elements. The integrals in equation (9) are replaced by sums of element integrals over individual elements. Element interpolation functions are introduced, the field variables, $\vec{v}$ and $\vec{\omega}$, expressed in terms of their nodal values, and the element integrals evaluated analytically. Equation (9) then yields a set of algebraic equations in the form:

$$\vec{v}_m = \sum_{n=1}^{N} \vec{A}_{mn} \times \vec{\omega}_n + \sum_{b=1}^{B} (B_{mb} \ \vec{v}_b + \vec{C}_{mb} \times \vec{v}_b) \tag{11}$$

where the subscripts m, n, and b designate respectively velocity nodes in R, vorticity nodes in R, and velocity nodes on B; $\vec{A}_{mn}$, B_{mb}, and $\vec{C}_{mb}$ are geometric coefficients depending only upon the relative positions of the nodes m and n, or m and b; N and B are respectively the total number of nodes in R and on B. There exists a large number of options in the selection of specific element types and interpolation-functions.

For equation (10), the use of the finite-element methodology in the numerical quadratures with respect to time and space yields algebraic equations approximating equation (10). There exist again a large number of options in the selection of specific element types and interpolation functions, with respect to both time and space. The algebraic equations utilized by the present author and his co-workers are described in the references given[6]. In the remainder of this section, the distinguishing features resulting from the use of equations (9) and (10) in the numerical procedures are outlined.

As described earlier, the computation loop that advances the solution from an old time level to a new time level is composed of a kinetic step and a kinematic step. In these steps, equations (9) and (10) may be used in place of equations (6), (7) and (8). In the kinetic step, the distinguishing feature of the integral representation, equation (10), is that it expresses the several kinetic processes that redistribute the vorticity in the fluid as separate integrals. This feature offers the possibility of using different numerical quadrature procedures for the evaluation of the different kinetic processes. In this manner, the drastically different characteristic time scales of these kinetic processes can be individually accommodated. Also, as noted before, the contribution of the initial vorticity distribution in R to the distribution at a later time is identical to the contribution of the diffusion process in an infinite unbounded region. In evaluating the first integral in equation (10), therefore, one needs to solve only the homogeneous diffusion equation in an infinite unbounded region. The form of the principal solution Q, which appears in each of the three integrals of equation (10), indicates that interpolation functions other than the familiar polynomials should be used for both the time-element and the space elements. It appears that simple, efficient and accurate procedures can be developed on the basis of these attributes.

Equations (6) and (7) form an elliptic system. Conventional finite-element and finite-difference equations approximating these equations are implicit. The integral representation (9), however, yields explicit algebraic equations of the form (11). This unique attribute of the integral representation permits the velocity values to be computed in any selected region of the flowfield. It is obvious from the vorticity transport equation (8) that information about the velocity field is needed only in the region of non-zero vorticity in order to determine the effect of convection. This region of non-zero vorticity is the only region in which viscous effects are important. In many viscous flow problems, including those involving massive regions of separation, the viscous region occupies only a small portion of the total flowfield. The integral representation permits the confinement of the computation to the viscous region. In consequence, drastic reductions in the number of data points and in the amount of computation are achieved.

The integral representation permits the confined solution field to be segmented and the computation within each segment performed independent of those in other segments[7]. It also permits the boundary layer region of the flow to be treated separately from the detached viscous region[8]. Furthermore, it is simple to form hybrid approaches[9] in which the integral representation approach is utilized advantageously in some parts of the flow.

ILLUSTRATIVE RESULTS

It has been found that the integral-representation approach is well-suited for a wide range of viscous flow problems involving flow separation. In recent articles, this author and his co-workers have presented numerical results for turbulent flows[10], for compressible flows[11], and for internal steady flows[6]. In these recent works, integral representations are utilized in the kinematic part of the computation. Rizk[5], however, obtained results for viscous flows using integral representations in both the kinetic and kinematic parts of the computation. Selected results obtained by Rizk are presented here to illustrate the application of the approach.

The flow past a circular cylinder at a Reynolds number based on the cylinder diameter of 40 is computed twice, once using the integral representations in the entire flowfield and once using a hybrid approach. In both computations, the cylinder is set into motion from rest and the time-dependent solution is advanced to a steady state. With the hybrid approach, the flowfield is divided into a small inner region and an outer region. The integral representations approach is used in the inner region and finite-difference procedures are used in the outer region. As shown earlier, the integral-representation approach leads to a drastic reduction in the number of data points needed in the computation. However, as is evident from equation (11), the use of algebraic equations obtained from the integral-representations leads to a relatively high operation counts, of the order P, for the computation of each field variable value. By using the integral representation in only the inner region, which contains a small number of nodes, the operation count for the computation of each field variable value in the inner region is reduced. The operation count in the outer region is determined by the specific finite-difference procedure used and is not large. As a result, the hybrid procedure is more efficient than the straight integral-representation procedure. The results of the hybrid computation is in comparison with those obtained using the straight integral-representation and also with finite-difference results of Jain and Roa[12] below. It is seen that the hybrid results and the integral-representation results are in excellent agreement with each other. Furthermore, these results are in good agreement with finite-difference results.

	Finite Different	Integral Representation	Hybrid
Drag Coefficient	1.55	1.57	1.57
Pressure drag-Total drag ratio	0.66	0.67	0.67
Separation Point (Angle from Front Stagnation Point)	127.3°	125.6°	125.8°

The accuracy of the integral-representation and hybrid procedures are further attested to by a comparison of the computed pressure distributions over the circular cylinder with the experimental data of Grove. The integral-representation and hybrid results are graphically indistinguishable and are in good agreement with the experimental data.

The hybrid procedure is used in the calculation of a flow past a 9% thick symmetric Joukowski airfoil at an angle of attack of 15° and at a Reynolds number based on the chord length of 1000. The airfoil is set into motion from rest and the solution is carried to a large time level. Unlike in the cases of the circular cylinder, the flowfield around the airfoil does not approach an asymptotic steady state. Rather, a cyclic shedding of vortices from the separated region near the airfoil occurs. This cyclic shedding of vortices is similar to the well-known Karman vortex shedding behind a circular cylinder. A typical streamline plot about the airfoil during the shedding of a vortex is shown in Figure 2.

CONCLUDING REMARKS

The finite-element procedure that has been developed on the basis of principal solutions represents a major departure from previous finite-difference and finite-element methods. The use of the integral representation containing the principal solutions offer a number of decisive advantages. Highly efficient computer codes have been prepared for the computation of two-dimensional laminar and turbulent, steady and time-dependent, incompressible and compressible viscous flows involving large separation regions in two-dimensions. Current efforts are directed towards the full utilization of the advantageous attributes of the integral representation for the kinetic aspect of viscous flows and the development of highly efficient three-dimensional algorithms.

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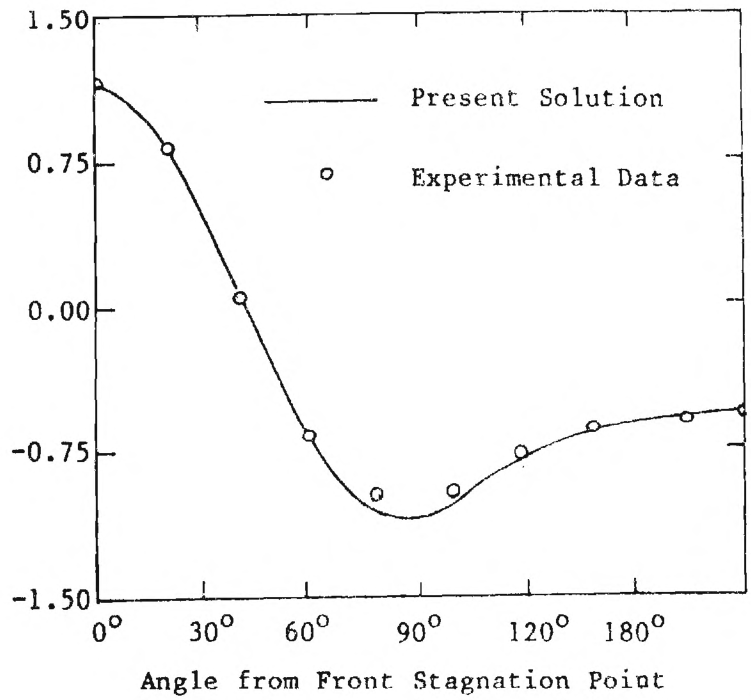

Figure 1. Pressure Distribution on Circular Cylinder

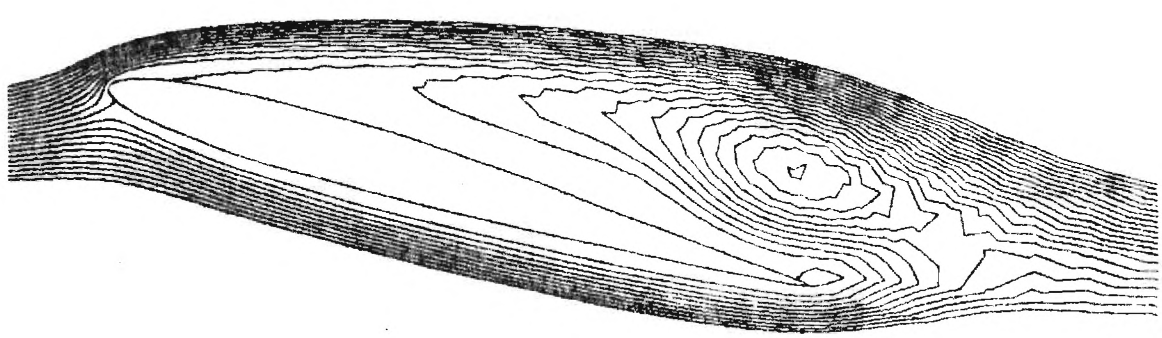

Figure 2. Streamlines About an Airfoil

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SOURCES, SINKS, VORTICES AND FLOW COMPUTATIONS

J. C. Wu
Professor of Aerospace Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332
U. S. A.

Summary

A finite-element method using integral representation of field variables is described. The integral representations are linked to the concept of sources, sinks and vortices. The interplay between the physical and the numerical aspects of the method is discussed. The advantages offered by the method and the scope of its application are outlined.

1. Introduction

The use of the finite-element approach in the solution of flow problems is a relatively recent development. During the past few years, however, research articles dealing with various aspects of this subject have proliferated. Extensive efforts are now in progress at a number of institutions at various points of the world to develop finite-element methods for flow applications. This extensive effort is motivated by the need for alternatives to the finite-difference approach which has occupied the center stage in the fluid dynamics community for several decades. The impetus for the present intensive activities in the finite-element approach was provided by the success of the approach in solid mechanics applications. It should be emphasized even at the risk of appearing trivial that, although the fluid medium and the solid medium are both continua, many physical processes of importance in flow problems are absent or unimportant in solids. The success of the finite-element approach in solid applications therefore does not assure its success also in flow applications. Indeed, because of the diverse physical features and processes present, numerical procedures that transcends the previously observed difficulties must be tailored to fit the important flow characteristics.

In this paper, a finite-element approach for the computation of flow problems based on the notion of flow elements - sources, sinks, and vortices - is described. The development of this approach, called the integral representation approach, has been underway at the Georgia Institute of Technology since the early 1970s. Flow problems computed using this approach include steady and time-dependent, incompressible and compressible, laminar and turbulent, two-dimensional and three-dimensional, potential and viscous, boundary-layer and separated types. Through analyses and numerical illustration, this alternative approach has been shown to overcome many difficulties experienced by finite-difference and other finite-element methods.

The establishment of this approach has undergone several stages of development. In each stage, selected attributes of this approach have been incorporated into the numerical procedure and substantial improvements in solution efficiency and accuracy have resulted. The overall result is an ability to compute a variety of flows accurately and economically on a routine basis. In a series of previous articles and doctoral theses at Georgia Tech,[1-10] the detailed mathematical and numerical analyses related to this approach have been presented. In the present paper, the interplay between the physical and numerical aspects of the present approach is discussed. The linkage between the principal solutions of differential equations and the flow elements is emphasized. It is well-known that the notion of sources, sinks, and vortices are indispensable in the interpretation of many important aerodynamic phenomena. It is clear that these flow element can also play an important role in the development of computational methods for flow problems.

Because of length limitation, this paper discusses mainly the incompressible flow problem. Compressible flows are only mentioned briefly here and are discussed extensively elsewhere.[7,8]

2. Vorticity and Dilatation Fields

The vorticity and the dilatation are defined respectively as the curl and the divergence of the velocity field. In some flows, the magnitude of the vorticity field in the neighborhood of a surface in the fluid is much larger than that elsewhere. Such a concentration of the vorticity field occurs, for example, in the boundary layer region adjacent to a solid surface. For some applications, it is justifiable to approximate a concentration of vorticity by a vortex sheet. A vortex sheet is mathematically a surface of discontinuity of the velocity component tangential to the surface. Conceptual and numerical difficulties may arise in connection with the use of the notion of the vortex sheet. These difficulties however, are not difficult to resolve if the vortex sheet is viewed as an approximation to a continuous distribution of vorticity.

Vortex filaments or lines are useful approximations of concentrated vorticity along lines in the fluid. Such concentrations of vorticity exist, for example, in tip vortices trailing a lifting wing.

In analogy to vortex sheets, source sheets can be used to approximate concentrated dilatation in the neighborhood of a surface. Source and sink sheets are mathematically surfaces of discontinuity of the velocity component normal to the surface. They should be viewed as approximations to real distributions of dilatation in the fluid so as to avoid apparent conceptual and numerical difficulties. In the case of a shock wave in a supersonic flow, for example, although the shock wave is justifiably approximated by a sink sheet in many applications, the fact that the shock does possess a finite structure should be remembered.

3. Dynamics of Vortices, Sources and Sinks

The law of conservation of mass states that the dilatation field vanishes everywhere in an incompressible flow. Sources and sinks therefore do not represent physical reality in such flows. They nevertheless provide mathematical conveniences in computations. In contrast, vortex sheets and lines do represent real distributions of vorticity and, for this reason, are more preferable in numerical procedures for incompressible flows.

The differential equations describing the incompressible flow are familiarly expressed as the continuity and Navier-Stokes equations. Taking the curl of each term in the Navier-Stokes equation yields the vorticity transport equation

$$\frac{\partial\vec{\omega}}{\partial t} = (\vec{\omega}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla})\vec{\omega} + \nu\,\nabla^2\vec{\omega}$$

where $\vec{v}$ and $\vec{\omega}$ are respective the velocity vector and the vorticity vector and ν is the kinematic viscosity of the fluid. The right-hand side terms in equation (1) represent the amplification and rotation of vorticity by the strain rate, the convection of vorticity with the fluid in motion, and the diffusion of vorticity through viscous action. If the fluid were inviscid, then the vorticity flux associated with each material element moving with the fluid would remain a constant for all times. This well-known theorem of Helmholtz, a proof of which is available in many textbooks, is modified in the case of a real fluid by the process of viscous diffusion. Since the process of diffusion alters the distribution of vorticity without creating or destroying it, vorticity flux can neither be created nor be destroyed in the interior of the real fluid.

Consider a solid body immersed in an incompressible fluid occupying an infinite region. The solid body is initially at rest in the fluid also at rest. A subsequent motion of the solid induces a corresponding motion of the fluid. At large time levels after the motion's onset, if the solid motion is uniform relative to the freestream, then the possibility of an asymptotic steady flow exists. Alternatively, the possibility of a time-dependent flow involving periodic shedding of vortices, as evidenced by the well-known Karman vortex sheet, also exists. The vorticity is obviously everywhere zero in the fluid prior to the motion's onset. Consequently, immediately after the onset of the motion, the vorticity is everywhere zero in the fluid except at the boundary in contact with the solid. That is, the fluid motion immediately after the onset of the motion has a non-zero tangential velocity relative to the solid. The discontinuity in the tangential velocity constitutes a vortex sheet at the boundary. At subsequent time levels, the vorticity in the vortex sheet spreads into the interior of the fluid domain by diffusion and, once there, is transported away from the boundary by both convection and diffusion. At the same time, the no-slip condition provides a mechanism for the continual generation of vorticity at the boundary. The general pattern of flow development therefore consists of the continual generation of vorticity at the solid boundary, the diffusion of the vorticity from the solid boundary into the fluid, and the subsequent transport of the vorticity away from the solid by convection and diffusion.

The transport of vorticity by convection is a finite rate process. The fundamental solution of the diffusion equation decays exponentially with the square of the distance from the source location. Consequently, the effective rate of diffusive transport of vorticity is also finite. Since vorticity must originate at the solid boundary, the vortical region of the flow is of finite extent at any finite time level after the initiation of the solid motion.

If the coefficient of viscosity ν is very small and the flow Reynolds number is very large, as is the case in most applications, then the effective rate of diffusion is much slower than the rate of convection. A large region of the flow, ahead and to the side of the solid, is then essentially free of vorticity and therefore inviscid.

For the case where the flow does not separate from the solid surface, as, for example, in flows past a thin airfoil at a small angle of attack sketched in Figure 1, the vorticity is concentrated in thin boundary layers adjacent to the solid surface where the vorticity originates. In addition, as the boundary layer fluid leaves the solid, carrying vorticity with it, a thin vortical wake trailing the solid is formed. In potential flow computations, the vorticity fields are approximated by vortex sheets. The simplifications resulting from this approximation are remarkable, as discussed in Section 5 of this paper.

If the detailed flow structures within the vortex layer are of interest, then the boundary layer equations need to be solved. The use of the vorticity and the notions of vortex offers an alternative to the familiar solution methods, as described in Section 6.

For the case where massive regions of flow separation are present near the solid, as sketched in Figure 2 for an airfoil at a high angle of attack, the Navier-Stokes equations enter the solution procedure. The concept of vortex is again well-suited for the computation of this type of flow, known as the general viscous flow, as discussed in Section 6.

For compressible flows, the law of conservation of mass does not require the dilatation field to be everywhere zero in the flowfield. Indeed, the dilatation field is important over a much greater region than the vortical region. A differential equation describing the transport of dilatation has been derived and presented earlier.[7]

The dilatation transport equation has been recently utilized together with the vorticity transport equation in the computation of compressible general viscous flows. The computation procedure that has been developed is ultimately related to the concept of sources and sinks.

4. Kinematics of Flows

In a numerical solution procedure for incompressible flows, it is convenient to follow the kinetic development of the vorticity field. With the velocity and vorticity fields known at an old time level, equation (1) can be solved numerically to establish a new vorticity distribution at a subsequent time level for the interior of the fluid domain. The solution of equation (1) can be carried out by using a finite-different approach or a finite-element approach. Alternatively, the integral-representation approach can be used. With each approach, a variety of options exist regarding the specific numerical schemes to use. In recent articles, the advantages of using hybrid schemes to accommodate the important physical features present in the flows have been discussed.[10]

Having established the new vorticity distribution in the fluid domain, the solution can be advanced further by again solving equation (1), provided that the velocity field and the boundary values of the vorticity are determined for the new time level. The new velocity field and the new vorticity boundary values are related to the new vorticity field in the fluid domain through the following kinematic equations:

$$\nabla \cdot \vec{v} = 0$$

$$\nabla \times \vec{v} = \vec{\omega}$$

Equations (2) and (3) are respectively the continuity equation and the definition of the vorticity field. They relate the velocity field at a given instant of time to the vorticity field at the same instant.

By using the principal solution of the elliptic equations and the Green's theorem, equations (2) and (3) can be recast into an integral representation for the velocity vector[11]

$$v = -\frac{1}{2(d-1)\pi}\left\{ \int_R \frac{\vec{\omega}_o \times (\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^d}\, dR_o - \oint_B \frac{\left[(\vec{v}_o \cdot \vec{n}_o) - (\vec{v}_o \times \vec{n}_o)\times\right](\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^d}\, dB_o \right\} \quad (4)$$

where $\vec{r}$ is a position vector, d is the dimensionality of the problem i.e., d=2 for two-dimensional flows and d=3 for three-dimensional flows, $\vec{n}$ is the unit normal vector directed outward from R, and the subscript "o" designates a variable or an integration in the $\vec{r}_o$ space. The vectors $\vec{v}$ and $\vec{\omega}$ are for the same instant of time.

The first integral in equation (4) represents the Biot-Savart contribution to the velocity field. For example, the familiar Biot-Savart law for an infinite straight vortex line in an infinite unlimited fluid states that the magnitude of the velocity associated with it is inversely proportional to the distance from the vortex line to the point where the velocity is measured and the direction of the velocity vector is perpendicular to the vortex line. This statement may be viewed as a result of the first integral of equation (4) as applied to an infinite vortex tube of infinitesimal cross-section in an infinite fluid.

The second integral in equation (4) gives a potential velocity field, say $\vec{v}_p$, in the region R. $\vec{v}_p$ represents the contribution of the velocity boundary values. Alternatively, $\vec{v}_p$ may be considered the contribution of a vorticity distribution which is present outside R on the velocity field in R. Equation (4) is therefore a generalized version of the familiar law of Biot-Savart and is intimately connected to the notion of vortex.

Finite-difference and familiar finite-element equations approximating the elliptic system of equations (2) and (3) are implicit. With equation (4), however, velocity values can be computed explicitly, point by point, using known distribution of vorticity in R and known boundary velocity values on B. Furthermore, if equation (4) is used to compute the velocity values on the boundary of a subregion in R, then the information necessary for further computation of velocity values within the subregion becomes entirely contained in the subregion and its boundary. In other words, the computation of velocity values in the subregion no longer depends upon information outside the subregion.

The ability to compute the velocity values explicitly and the ability to treat each subregion of the flow independently of the remainder of the flow field are two unique and most important features of the integral representation.

5. Potential Flow

In the case of an attached two-dimensional steady flow, the thin wake trailing the finite solid body is composed of two layers of vortices of equal magnitude and opposite sense. The effect of this double layer may be small. The strength of the vortex sheet representing the boundary layers can then be determined through the use of the integral representation (4). To illustrate the concept involved, consider a steady flow past an airfoil with a freestream velocity $\vec{v}_\infty$. The application of equation (4) to the boundary B on the airfoil yields,

$$\frac{1}{2\pi} \int_{B^+} \frac{\vec{\gamma}_o \times (\vec{r}_o - \vec{r}_b)}{|\vec{r}_o - \vec{r}_b|^2} \, dB_o = \vec{v}_\infty \qquad (5)$$

where $\vec{\gamma}_o$ is the strength of the vortex sheet representing the boundary layer, $\vec{r}_b$ is a point on B, and B^+ is the surface enveloping B and at an infinitesimal distance from it.

In equation (5), a distinction is made between the suface B^+, where the vortex sheet is located, and the surface B, which is a part of the airfoil and on which the velocity is given by the prescribed motion of the airfoil. This distinction is consistent with the physics of the problem since the boundary layer is located outside the solid surface.

For the two-dimensional flow problem, the vorticity vector $\vec{\omega}$ is directed perpendicular to the plane of the flow. In consequence, the unknown function $\vec{\gamma}$ in equation (5) has only one component. Equation (5) is a vector Fredholm integral equation with two scalar components. The solution of either of the two component equations, however, satisfies the other. It is noted that the solution to either component of equation (5) contains an arbitrary constant. This arbitrary constant can be determined uniquely by the Kutta condition. It is noted in passing that instead of applying equation (4) to B, one may apply it to the surface B^{++} which envelops B^+ and is at an infinitesimal distance from B^+. The resulting integral equation differs from equation (5) only by the amount $-\gamma\vec{e}_t$, where $\vec{e}_t$ is the unit vector tangent to B, added to the right side. This added term accounts for the discontinuity across the vortex sheet of the tangential velocity component. By removing the vortex sheet from the region of concern, possible conceptual confusions and mathematical difficulties are removed.

The most widely used numerical procedures for computing incompressible potential flows today are based on the concept of fictitious sources and sinks. Source and sink elements are placed inside the solid body in such a way that the normal velocity boundary conditions are satisfied on the surface of the solid. This method of sources and sinks leads to an equation identical to the tangential component of equation (5). It is worthy of note, however, that the vortex conception of equation (5) is based upon real phenomena. For this reason, it offers important advantages in its generalization for viscous flow applications.

With the strength of the vortex sheet determined, the velocity field everywhere can be immediately established through the generalized Biot-Savart law. In this regard, it is worthy of note that the vortex method leads to a particularly simple expression for the pressure coefficient acting on the solid surface. Since the vortex sheet strength represents the magnitude of the discontinuity in the tangential velocity and the velocity is zero on the solid surface, the velocity magnitude on the fluid side of the vortex sheet is simply γ. Physically, then, γ gives the boundary layer edge velocity. The pressure coefficient on the fluid side is simply $1 - (\gamma / v_\infty)^2$. Since the pressure across a boundary layer is constant, and the vortex sheet is an approximation of the boundary layer, the pressure coefficient acting on the solid surface is also given by $1 - (\gamma / v_\infty)^2$.

For some problems, the structure of the thin wake is important. One such structure, illustrated in Figure 1, shows the roll-up of the wake behind a non-lifting flat plate. The flow pattern computed using the vortex concept agrees remarkably well with the flow pattern observed experimentally.

For lifting bodies, a starting vortex is always present in the fluid. Even though this starting vortex may be thought of as located at an infinite distance from the solid, its presence is being felt in the boundary layer by the requirement that the total vorticity in the boundary layer, i.e., the circulation, must be equal in magnitude and opposite in sign to the total vorticity in the starting vortex. For lifting bodies with trailing edges that are not sharp, the Kutta condition is ambiguous. The circulation can be determined by the total vorticity in the starting vortex.[12] For three-dimensional flows, because of the solenoidal nature of the vorticity field and because the vorticity in the fluid must originate from the solid surface, the starting vortex is connected to the solid surface by trailing vortices. For time-dependent flows, trailing vortices are also present. In some applications, the effects of viscous diffusion is unimportant in the trailing vortices. The trailing vortices may therefore be approximated by vortex sheets and vortex filaments. If the position of the trailing vortices are known, then an equation similar to equation (5) can be written for the entire vortex system. For this situation, the kinetic aspect of the problem does not enter the solution procedure. The dimensionality of the problem is reduced by one. For example, the two-dimensional airfoil problem discussed earlier reduces to the problem of determining γ on the boundary by solving the integral equation (5), which is one-dimensional. The vortex approach is therefore extremely attractive. In many applications, however, the position of the trailing vortices must be computed by treating the kinetic equation.

6. Boundary Layers and Separated Flows

The concept of sources, sinks and vortices offers decisive advantages in the computation of boundary layers and separated flows. Some of these advantages are outlined below.

6.1 Computation of Vorticity Boundary Condition

It is simple to show that, in the case of a flow where the vorticity is not confined to the immediate vicinity of solid surfaces, equation (5) can be generalized by adding a term to its right side representing the Biot-Savart contribution of equation (4) to the boundary velocity. This generalized integral equation gives a boundary vortex strength accurately simulating the process of vortex generation on the solid surface.[13]

6.2 Boundary Layer Computations

As discussed earlier, the boundary layer edge velocity is equal to the strength of the vortex sheet in a steady flow. Thus the determination of the vortex sheet strength provides the needed boundary condition for boundary layer computation without a need to compute the potential flow details. Similar advantages are available for the computation of time-dependent boundary layers.[14]

6.3 Confinement of Solution Field

From equation (1), it is obvious that the convection of vorticity is important only where the vorticity is not negligibly small. Thus, in solving equation (1), information about the velocity is needed only in the viscous region. Since, with the generalized Biot-Savart law, equation (4), velocity values can be computed point by point explicitly, it is possible to compute the velocity values only for the viscous region. In consequence, the large potential region of the flow, which surrounds the relatively small viscous region in a general viscous flow, does not need to be computed. The drastic reduction in the number of computation points resulting from the confinement of the computation to the viscous region leads to a drastic reduction in the amount of computation required.[11]

6.4 Separate Treatment of Boundary Layer and Separated Parts of General Viscous Flow

The viscous part of the general viscous flow, where the computation is confined to, contains the boundary layer region and the separated flow region. The characteristic length scale of the boundary layer is much smaller than that of the separated flow region. Futhermore, the physical characteristics of the two regions are different. The use of equation (4) permits the two viscous regions to be separately treated. The result is a procedure that is tailored to fit the physical features of the flow. The procedure is extremely efficient and accurate. It should be noted that as the flow Reynolds number increases, the ratio of the boundary layer length-scale to the separated-flow length-scale decreases and the problem of simultaneouly accommodating the two diverse length scales becomes more severe. The familiar finite-difference and finite-element methods usually encounter a "Reynolds number limit" beyond which excessive computer time is required to solve a given problem. The method based on the generalized Biot-Savart law, with the two confined viscous regions separately treated, transcends this limitation and is insensitive to the flow Reynolds number.[14]

In many applications, it is desirable to combine the unique attributes of the present approach with other, more conventional approaches so as to better fit the physics of the flow problem. It has been found that the present approach is extremely flexible and is well suited for hybridization with other approaches.

There are a number of other advantageous attributes as well as a large amount of analyses and computed results available. At the present stage of development, the present approach has been thoroughly calibrated. The approach permits two-dimensional high Reynolds number flows, including turbulent flows, to be computed routinely and economically. Some three-dimensional results have been obtained for flows past solids with simple shapes. A research program aimed at developing this approach for computing three-dimensional flows past solids with complex shapes has been initiated.

7. Acknowledgement

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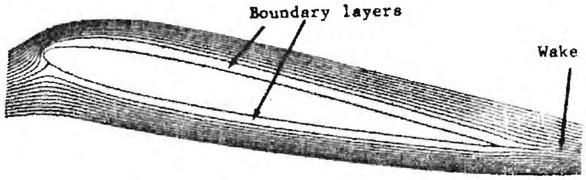

Figure 1. Attached Flow Past an Airfoil

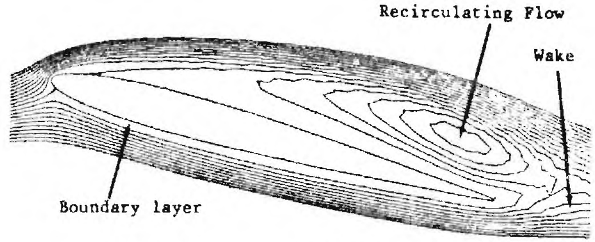

Figure 2. Separate Flow Past an Airfoil

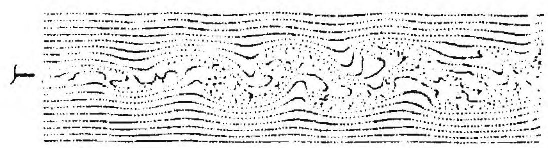

Figure 3. Wake Roll-up Behind a Flat Plate

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UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Fluid Dynamics Program

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting quarter, the general theory for aerodynamic forces and moments in time-dependent viscous flows has been further developed and the application of this theory to several types of unsteady viscous flows has been studied. This theory, described last year in a journal article (AIAA Journal, April 1981), relates the rates of change of vorticity moments in the combined fluid-solid domain to the aerodynamic forces and moments acting on the solids. The mathematical derivation of the equations forming this theory is based on the viscous flow equations and is rigorous. No simplifying assumptions other than those contained in the Navier-Stokes equations have been introduced. The general theory is now complete for incompressible flows and is valid for both two-dimensional and three-dimensional problems. Progress has been made in extending this theory for compressible flows by analyzing the kinematics and the kinetics of the dilatation field, which is not everywhere zero in a compressible flow.

The application of the present theory to several problems of practical importance has been studied. For applications where the vorticity in the viscous flow exists only in narrow bands such as the boundary layers, the use of the general theory is particularly simple. The apparent mass properties of solid bodies, for example, are related, through the general theory, to the process of vorticity generation on the surfaces of the solids. For two- and three-dimensional bodies, the apparently mass properties are determined in a simple manner without the need of involvement with the concepts of impulse and kinetic energy. For three-dimensional configurations, previous approaches to the question of apparent mass properties are overly complex to yield meaningful information for all except extremely simple configurations. The general theory gives accurate results in a simple manner for most complex configurations.

During previous quarters, the development of the integro-differential approach for the solution of unsteady viscous flow problems involving massive flow separation have produced two powerful methods. One of the two methods deals with the kinetics (as well as the kinematics) of the problem using integral representations. The other method treats the attached and separated flows independently. These methods have been established for two-dimensional problems. Over the past several years, the integro-differential formulation has undergone a series of improvements and refinements, such as the "flow-field segmentation" techniques and the hybrid "boundary layer/Navier-Stokes" techniques. Each improvement has led to a large reduction in the computer resources required to solve complex, separated flow fields. In the following table, the progressive reduction in the computer time required to solve a separated flow problem using the integro-differential approach is shown. A peculiar feature appearing in this table is that the calculations at Reynolds numbers 1000 and 40,000 require less time than the calculations at a Reynolds number of 40. This is because at the lower Reynolds number boundary layer assumptions are not valid, and the hybrid boundary layer/Navier-Stokes technique is not applicable.

Computer Time Requirements in Minutes of CDC 6600
for the Circular Cylinder Problem

	Year			
Reynolds Number	1970	1975	1980	1985
40	95	40	25	(20)[a]
1000	(300)	80	20	(6)
40000	(1200)	(250)	20	(6)
3,600,000	-	-	80	(15)

[a]Numbers in parentheses are estimated values.

The efficiency of this approach makes it feasible to further develop it for three-dimensional and turbulent separated flow problems. The present method is reviewed in the folllowing article which appeared during the reporting quarter:

J. C. Wu: Problems of General Viscous Flow," Chapter 4 of Developments in Boundary Element Methods - 2, Applied Sciences Publishers Ltd., 1982.

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Chapter 4

PROBLEMS OF GENERAL VISCOUS FLOW

J. C. WU

School of Aerospace Engineering, Georgia Institute of Technology, USA

SUMMARY

The kinetics of the time-dependent incompressible viscous flow problem, including both the kinematic aspect and the kinetic aspect, are formulated as integral representations of field variables. Numerical procedures utilising the integral representations are outlined for the potential flow problem and the general viscous flow problem. The role of the boundary element method in the solution procedure is discussed. Distinguishing features and advantages of the approach are described. Selected results are presented to illustrate the application of this approach.

4.1 INTRODUCTION

During the past two decades there has been a steady increase in the willingness of fluid dynamicists to accept the computer as a valuable, perhaps indispensable, tool in engineering efforts. This willingness is a consequence of the impressive progress made in the routine and accurate solution of differential equations describing certain types of flow on the computer. For other types of flow, a capability for the routine and accurate computation has yet to be established, and the willingness to accept computer solution has spurred extensive research activities. It has long been recognised, of course, that the mathematical difficulties attendant to the analytical solution of flow equations are often formidable. Numerical approach offers the only promise, for the foreseeable future, of accurate quantitative solutions under reasonably general circumstances.

Until relatively recently, computational fluid dynamicists have emphasised the development of the finite difference approach for viscous flow problems. Progress made in this regard during the past two decades for boundary layer flows has been extensive. For general viscous flows which contain appreciable regions of separation, however, the application of the finite difference approach encounters serious difficulties. The need for alternatives to the finite difference approach has been recognised by many researchers and has led, in the past few years, to a rapid growth of interest in the application of the finite element method to flow problems. An impetus for this interest was provided by the impressive success of the finite element approach in many solids mechanics applications. Many researchers have suggested that the application of the finite element approach to flow problems is a natural extension of its application to structure and elasticity problems. Such an extension, however, has not been straightforward. Many of the major difficulties experienced in the application of the finite difference approach have reappeared in connection with the use of the finite element approach for flow problems.

In restrospect, the disparity of success of the finite different approach as applied to different types of viscous flow problem is not unexpected. The starting point of the finite difference approach is the discretisation of the familiar differential equations of flow. These differential equations, formulated long ago with classical mathematical analyses in the background, do not always represent the most suitable, or even a reasonable, formulation for numerical procedures. Regarding the finite element approach, it should be emphasised, even at the risk of appearing trivial, that although the fluid and the solid are both continuous media, many important physical processes in fluid dynamics are absent or unimportant in solids mechanics. The success of the finite element approach in solids mechanics therefore does not assure its success in fluid dynamics. Also, it is known that if the finite element nodes are arranged in a uniformly spaced rectangular array, then the algebraic equations obtained through the concept of the variational principle or that of residuals are often identical to those obtained through a finite difference procedure using the same grid system. In consequence, the familiar finite element approach is not as fundamentally different from the finite difference approach as some researchers have claimed it to be.

In this chapter an approach which utilises the finite element methodology but which does not rely on the concept of the variational principle, or

that of the residuals such as the Galerkin's procedure, is described. With this approach, the differential equations of motion are recast into the form of integral representations. Each of the integral representations is composed of an integral over the fluid domain and an integral over the boundary of this domain. During the past decade, several researchers have shown, by analyses and by numerical illustrations, that the solution procedure for flow problems based on the integral representation concept offers a number of decisive advantages. In particular, it has been found that the integral representation approach removes several major obstacles experienced in the prevailing finite difference and finite element solutions of the general viscous flow problem. In this chapter, the integral representation approach is reviewed and summarised. Important features of this approach are described. In particular, the distinguishing role of the boundary integral in the integral representation approach is examined. One purpose of the present paper is to bring into focus the understanding, acquired during the past decade of research into the integral representation approach, of the interplay between the numerical and the physical aspects of the general viscous flow problem. This interplay often dictates the utilisation of the integral representation formulation and hence the boundary element method for superior solution efficiency and solution accuracy.

The discussions of this chapter are centred upon time-dependent laminar incompressible general viscous flow past the exterior of solid bodies. This type of flows possesses the essential features of interest and serves to bring into focus the most important concepts associated with the integral representation approach. It has been found that this approach is well suited for a wide range of flow problems. In several recent articles, for example, this author and his co-workers have presented integral representation results for turbulent flows (Wu *et al.*, 1977; Wu & Sugavanam, 1978; Sugavanam, 1979), for compressible flows (Wu, 1974; El-Refaee, 1981), and for internal and steady flows (Wu & Wahbah, 1976; Wahbah, 1978). While the focal point of this chapter is the development of the integral representation approach to the solution of general viscous flows, the approach is also useful in solving boundary layer and potential flows. In fact, for potential flows, the integral representations reduce to a single boundary integral equation. The boundary integral method, developed on the basis of fictitious singularities, such as source and sink elements, is today widely used in computing complex potential flows (e.g. Hess, 1980). In this chapter, a different viewpoint is brought forth by a derivation of a boundary integral equation as a special case of a general viscous flow.

4.2 DIFFERENTIAL FORMULATION

The time-dependent motion of an incompressible viscous fluid is governed by the law of mass conservation and Newton's laws of motion. The mathematical statements of these two laws are familiarly expressed in the differential form and known as the continuity and Navier–Stokes equations, respectively. Expressed in right-handed Cartesian tensor notations, these equations are

$$\frac{\partial v_j}{\partial x_j} = 0 \tag{1}$$

and

$$\frac{\partial v_i}{\partial t} + \frac{\partial (v_i v_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{2}$$

where v_i is the ith component of the velocity vector, p is the static pressure, ρ is the density and ν is the kinematic viscosity, considered to be uniform for simplicity.

Equations (1) and (2) are in principle sufficient for the determination of p and v_i, known as the primitive variables of the problem, provided that appropriate initial and boundary conditions for the velocity vector are prescribed. It is, however, advantageous to introduce the concept of the vorticity vector ω_i defined by

$$\omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} \tag{3}$$

where ε_{ijk} is the alternating unit tensor.

The vorticity ω_i is a derived variable. With it, the formulation of the problem is conveniently partitioned into its kinematic and kinetic aspects. The kinematic aspect of the problem is described by eqns. (1) and (3). This aspect expresses the relationship between the vorticity field at any given instant of time and the velocity field at the same instant. The kinetic aspect of the problem is described by the vorticity transport equation:

$$\frac{\partial \omega_i}{\partial t} = \omega_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial \omega_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \tag{4}$$

Equation (4) is obtained from eqn. (2) by taking the curl of each term in that equation and using eqns. (1) and (3). The three right-hand side terms in eqn. (4) represent, respectively, the amplification and rotation of vorticity by the strain rate, the convection of vorticity with the fluid and the diffusion of the vorticity through viscous action. These kinetic processes redistribute the vorticity in the fluid. In the numerical solution procedure, it is convenient to follow the development of the vorticity field in the fluid. A numerical procedure therefore can be established in which the solution is advanced from an old time level to a new time level through a computation loop consisting of the following three steps:

(a) With known vorticity and velocity values at the old level, eqn. (4) is solved to obtain vorticity values in the interior of the fluid domain at the new level.

(b) New boundary values of the vorticity are computed using the 'no slip' condition.

(c) New velocity values corresponding to the new vorticity values are obtained by solving eqns. (1) and (3).

Step (a) is the kinetic part of the computation loop. Since eqn. (4) is parabolic in its time–space relation, step (a) requires the solution of an initial value–boundary value problem. The process of vorticity generation on the solid boundary is not described by the kinetic process of vorticity diffusion and convection (Wu, 1976). Boundary values of vorticity, however, are necessary to advance the solution further in time. This computation of the boundary values of vorticity, step (b) in the loop, is critical to the accuracy of the time-dependent solution and is discussed later in this chapter.

Step (b) completes the computation of vorticity values at the new time level. Step (c) utilises the just-computed vorticity values to establish a set of new velocity values. This step is the kinematic part of the loop. It requires the solution of a boundary value problem since eqns. (1) and (3) constitute an elliptic system of differential equations.

The general solution procedure just outlined enables the time development of the vorticity field in the fluid to be simulated computationally. The velocity is in fact considered to be an auxiliary variable in the loop. The computation of the velocity values is being carried out to provide the needed 'coefficients' for the kinetic processes of vorticity transport.

With the general procedure just outlined, if the prescribed velocity boundary conditions are independent of time, then the possibility of an asymptotic steady flow exists in the limit of large time. Alternatively, the possibility of a time-dependent flow involving periodic vortex shedding, as evidenced by the well-known Karman vortex sheet behind a circular cylinder, also exists. If the prescribed velocity boundary conditions are time dependent, then the flow is necessarily time dependent.

4.3 BOUNDARY CONDITIONS

The kinematics and kinetics of the flow can be thought of as two interlaced problems in a numerical solution procedure. The kinetic computation of new vorticity values and the kinematic computation of new velocity values are obviously interlaced, since the kinetic computation of the convective effects in eqn. (4) requires a knowledge of the velocity field and the kinematic computation of new velocity values requires a knowledge of the vorticity field. In addition, the kinematics and kinetics of the problem are linked through the boundary conditions needed for these two aspects.

The differential equations describing the kinematics of the flow, eqns. (1) and (3), are linear. Equation (1) implies the existence of a vector potential, the curl of which is the velocity vector. Placing this definition of the vector potential into eqn. (3) gives an elliptic partial differential equation for the vector potential. The kinematic part of the solution procedure therefore requires the solution of a boundary value problem.

The appropriate boundary conditions for the kinematics of the problem are as follows. If the vorticity field is known in a simply connected region R, then either one of the following velocity conditions on the closed boundary B of the region R is sufficient to determine the velocity field in R uniquely:

$$v_i n_i = f \qquad \text{on } B \tag{5}$$

or

$$\epsilon_{ijk} v_j n_k = g_i \qquad \text{on } B \tag{6}$$

where n_i is the unit normal vector on B directed outward from the region R, and f and g_i are known functions of the time and the position vector on the boundary B.

If the region R is multiply connected, then the value of circulation in the several independent circuits of the region must be specified together with the condition (eqn. (5)). In the following discussions, references to the boundary condition (eqn. (5)) also imply a knowledge about the circulation values wherever needed, although specific mentions of this need may be omitted at times.

To prove the above statements regarding the boundary condition, let there be two velocity fields v_{1i} and v_{2i}, each satisfying eqns. (1) and (3). Define a third velocity field in R given by

$$v_{3i} = v_{1i} - v_{2i} \tag{7}$$

It is obvious that v_{3i} is solenoidal and irrotational. In consequence, v_{3i} possesses a scalar potential ϕ which satisfies a Laplace equation. If v_{1i} and

v_{2i} have the same tangential component on B, then the tangential component of v_{3i} is zero on B. This means that the tangential derivative of the scalar potential of v_{3i} is zero on B. Therefore the scalar potential is a constant on B. By virtue of the principle of extremum for the Laplace equation, the scalar potential is then a constant in R. In consequence, $v_{3i} = 0$ in R and $v_{1i} = v_{2i}$. In other words, the condition (6) gives a unique solution of eqns. (1) and (3). Alternatively, if v_{1i} and v_{2i} have the same normal component on B, then the normal component of v_{3i} is zero on B. This means that the normal derivative of the scalar potential is zero on B and the scalar potential is a constant in R. Therefore $v_{3i} = 0$ in R and $v_{1i} = v_{2i}$. In other words, the condition (5) gives a unique solution of eqns. (1) and (3).

Consider a viscous flow past the exterior of a finite solid body. The fluid boundary B is composed of the solid surface S and a surface, S_∞, infinitely far from S. If the freestream velocity and the motion of the solid are prescribed, then, with the no-slip condition on S, both conditions (5) and (6) are known on S and on S_∞ at each time level. If the vorticity distribution is known completely in the fluid region R, then the kinematics of the flow problem is overspecified. For example, suppose one uses the known condition (5) to solve for the velocity field in R, then the tangential velocity on the boundary B, determined as a part of the solution, is in general different from the known condition (6) on B. From the above consideration, it is obvious that the vorticity distribution in the region R is subject to a kinematic restriction. This restriction is imposed by the velocity boundary condition given by the physics of the problem.

The vorticity transport equation, eqn. (4), is parabolic in its time–space relation. The spatial differential operator in eqn. (4) is elliptic. The kinetic part of the computation therefore requires the solution of an initial value–boundary value problem. The initial condition for the vorticity vector is provided by the physics of the problem. That is, with any known initial distribution of the velocity field, the initial vorticity distribution is determined by eqn. (3). The boundary condition for the vorticity vector is needed to proceed with the solution. This boundary condition is determined by the kinematic restriction discussed in the preceding paragraph. The computation of the boundary vorticity values can be accomplished through the use of a boundary integral equation, as described by Wu (1976). This procedure is discussed later. In fact, it is shown later that the normal gradient of vorticity, which is related to the tangential pressure gradient on the solid boundary, is similarly determined by a kinetic restriction.

4.4 VORTICITY DYNAMICS

The vorticity transport equation is non-linear since the first two terms on its right-hand side involve the product of v_i and ω_i, and v_i is kinematically a function of ω_i. Because of this non-linearity, the mathematical analysis of the kinetic aspect of the general viscous flow presents greater difficulties than that presented by the kinematic aspect. It is, however, possible to establish a substantial amount of understanding about the kinetic processes involved in the flow without detailed mathematical analyses.

Consider a finite solid body immersed in an infinite incompressible fluid with uniform viscosity and density. The solid body is initially at rest. Subsequent prescribed motion of the solid body induces a corresponding motion of the fluid.

If the fluid is inviscid, then the last term in eqn. (4) vanishes. The vorticity is then convected with the fluid in the sense that the vorticity flux $\omega_j ds_j$ associated with each material element ds_j moving with the fluid remains a constant for all times. This well-known theorem of Helmholtz, a proof of which is available in many text books (e.g. Sommerfeld (1950)), must be modified in the case of a real fluid because of the presence of vorticity diffusion. According to eqn. (4), changes in the vorticity flux $\omega_j ds_j$ take place only because of viscous diffusion described by the last term of this equation. Since the process of diffusion merely redistributes the vorticity within a fluid, vorticity is not created, or destroyed, in the interior of a fluid domain. More detailed discussions of this point are presented by Wu (1981).

For the problem under consideration, the vorticity is obviously everywhere zero in the fluid prior to the initiation of the solid motion. Consequently, immediately after the onset of the motion, the vorticity is everywhere zero in the fluid except at the boundary in contact with the solid. That is, the fluid motion immediately after the onset of the motion has a non-zero tangential velocity relative to the solid at the solid boundary. The discontinuity in the tangential velocity constitutes a sheet of concentrated vorticity (vortex sheet) at the boundary. At subsequent time levels, this vorticity spreads into the interior of the fluid domain by diffusion and, once there, is transported away from the boundary by both convection and diffusion. At the same time, the no-slip condition provides a mechanism for the continual generation of vorticity at the boundary. The general pattern of flow development therefore consists of the continual generation of vorticity at the solid boundary, the diffusion of the vorticity from the solid boundary into the fluid, and the subsequent transport of the vorticity away from the solid by convection and diffusion.

It is obvious that the transport of vorticity by convection is a finite rate process. The process of diffusion possesses an infinite signal speed. It is well known, however, that the fundamental solution of the diffusion equation decays exponentially with the square of the distance from the source location (see Section 4.5). Consequently, the effective rate of diffusive transport of vorticity is also finite. Since vorticity must originate at the solid boundary, the vortical region of the flow is of finite extent at any finite time level after the initiation of the solid motion. The vorticity is zero on S_∞. Outside the vortical region, the flow is essentially free of vorticity and therefore irrotational. If the flow Reynolds number is not small, then the effective rate of diffusion is much smaller than that of convection. Therefore, a large region of the fluid, ahead and to the side of the solid, is essentially free of vorticity. The last term in eqn. (4) may be rewritten as $-\nu\epsilon_{ijk}(\partial\omega_k/\partial x_j)$. Therefore the flow in this large region, where the vorticity and its derivatives are zero, is inviscid or potential.

The vortical region of the general viscous flow is composed of three distinct flow components: the boundary layer, the recirculating flow, and the wake (see Fig. 7d). The boundary layer is an attached flow region. In the boundary layer, the flow direction is nearly tangential to the solid surface. Since the effective rate of diffusion is much smaller than that of convection, the vorticity generated on the solid boundary cannot penetrate far into the fluid domain by diffusion in the boundary layer. In consequence, the boundary layer is thin compared to the characteristic length of the solid body. In terms of non-dimensional length, with the characteristic solid length as the reference length, the boundary layer thickness scale is of the order $Re^{-1/2}$, where Re is the Reynolds number based on the free-stream velocity and the characteristic solid length. In comparison, the length scales of the recirculating flow and the wake are of the order of one. In the inviscid region of the flow, the dimensionless length scale is of the order infinity since the spatial derivatives of the vorticity, which is the dependent variable of concern, is zero. With a finite difference procedure or a finite element procedure based on the variational principle or the concept of residuals, the entire flow field, inclusive of the viscous components and the potential region, must be computed. The existence of the diverse length scales then presents substantial difficulties in the computation of high Reynolds number general viscous flows. In particular, it is difficult to design a computation grid that provides a sufficient solution resolution in the boundary layer and yet does not contain an excessively large number of grid points in the potential flow region. With increasing Reynolds number, the thickness of the boundary layer decreases. The number of grid points, and hence also the computer time required, increases rapidly with increasing Reynolds number. For this reason, successful finite difference and finite element solutions of the general viscous flow problem are often 'Reynolds number limited' and are available only for two-dimensional flows involving relatively simple geometries. The currently popular procedure for alleviating this difficulty is the use of expanding grids, i.e. grids with increasing grid spacing as the distance from the solid (and from the viscous region) increases. It is known, however, that expanding grids do give rise to grid-associated errors which can be large for rapidly expanding grids needed for high Reynolds number flows.

It is possible to remove the potential flow region from the solution procedure by using an integral representation formulation for the kinematics of the problem, as shown by Wu and Thompson (1973). This formulation therefore removes the difficulties caused by the disparity of the viscous and potential length scales. Furthermore, it is possible to compute the boundary layer component separately from the detached viscous flow components, as shown by Wu & Gulcat (1981) and by Gulcat (1981). In so doing, not only are the difficulties caused by the disparity of length scales of the several viscous components removed, but also the amount of computation required is further drastically reduced.

4.5 INTEGRAL REPRESENTATIONS

The general solution procedure for the time-dependent flow described earlier is composed of repetitive solutions of a parabolic equation and an elliptic equation.

An alternative to the more familiar finite difference and finite element procedures is the integral representation procedure. With this procedure, the differential equations describing the flow are recast into integral representations. The solution of the flow problem is then based on the numerical quadrature of the integrals. The mathematical foundation of the integral representation formulation is the principal solution, also referred to as the fundamental solution, of differential equations. In this section, derivations of integral representations for the elliptic and the parabolic equations are presented.

4.5.1 Scalar Elliptic Equations

Consider a Poisson's equation

$$\frac{\partial^2 \xi}{\partial x_j \partial x_j} = g \tag{8}$$

The Green's theorem states that

$$\int_{R'} \left(P \frac{\partial^2 \xi}{\partial x_j \partial x_j} - \xi \frac{\partial^2 P}{\partial x_j \partial x_j} \right) \mathrm{d}R = \int_{B'} \left(P \frac{\partial \xi}{\partial x_j} - \xi \frac{\partial P}{\partial x_j} \right) n_j \, \mathrm{d}B \tag{9}$$

where R' is a closed region bounded by B', n_j is the unit outward vector normal to B', ξ and P are finite and continuous and possess continuous first and second partial derivatives in R'.

Let P be the principal solution of the elliptic differential equation defined by

$$P = \begin{cases} -\dfrac{1}{2\pi} \ln \dfrac{r_0}{r'} & \text{for two-dimensional problems} \\[2ex] -\dfrac{1}{4\pi r'} & \text{for three-dimensional problems} \end{cases} \tag{10}$$

where r_0 is the magnitude of the vector x_{0i} and r' is the magnitude of the vector x_i' defined by

$$x_i' = x_{0i} - x_i \tag{11}$$

The principal solution P satisfies the Laplace equation everywhere except at the point $x_i = x_{0i}$, where it is singular. This singular point is excluded from the region R' by considering B' to be composed of two parts, B and C, C being a small closed surface surrounding the point $x_i = x_{0i}$ and bounding R internally. One then has, in the region R',

$$\frac{\partial P}{\partial x_i} = -\frac{x_i'}{2(d-1)\pi r'^d} \tag{12}$$

where d is the dimensionality of the problem, i.e. $d = 2$ for a two-dimensional problem and $d = 3$ for a three-dimensional problem

One also has, in R',

$$\frac{\partial^2 P}{\partial x_j \partial x_j} = 0 \tag{13}$$

Using eqns. (8) and (13), eqn. (9) is rewritten as

$$\int_{R'} Pg \, \mathrm{d}R = \int_B \left(P \frac{\partial \xi}{\partial x_j} - \xi \frac{\partial P}{\partial x_j} \right) n_j \, \mathrm{d}B + \int_C \left(P \frac{\partial \xi}{\partial x_j} - \xi \frac{\partial P}{\partial x_j} \right) n_j \, \mathrm{d}B \tag{14}$$

For a three-dimensional problem, let C be a sphere of radius ε centred at the point x_{0i}. On C, the principal solution and its gradient are, according to eqns. (10) and (12), $-1/(4\pi\varepsilon)$ and $-n_j/(4\pi\varepsilon^2)$, respectively. As $\varepsilon \to 0$, the surface area of C approaches zero as ε^2. The part of the last integral in eqn. (14) involving P therefore does not contribute to the integral. The remaining part of the last integral in eqn. (14) gives simply the value of ξ at the point x_{0i}. It can be similarly shown, by letting C be a circle of radius ε centred at the point x_{0i}, that the last integral in eqn. (14) gives simply the value of ξ at the point x_{0i} for a two-dimensional problem. One therefore obtains the following integral representation for ξ:

$$\xi(x_{0i}) = \int_R Pg\,\mathrm{d}R - \int_B \left(P\frac{\partial \xi}{\partial x_j} - \xi\frac{\partial P}{\partial x_j}\right) n_j\,\mathrm{d}B \tag{15}$$

In eqn. (15), the region of integration R' has been replaced by R, which is bounded by B. This replacement is permissible since the integral over the interior of S tends to zero as $\varepsilon \to 0$.

The concept of the principal solution is essential to the integral representation formulation. If there exists a unit source at the point x_{0i}, then the scalar potential associated with this unit source at the point x_i in an infinite unbounded region is given by the principal solution P. It is worthy of note that the derivation of eqn. (15) is straightforward if the following integral property of the principal solution is recognised:

$$\int_{R'} \frac{\partial^2 P}{\partial x_j \partial x_j}\,\mathrm{d}R = \begin{cases} 1 & \text{if } R' \text{ contains the point } x_i = x_{0i} \\ 0 & \text{if } R' \text{ does not contain the point } x_i = x_{0i} \end{cases} \tag{16}$$

With eqn. (16), one has

$$\int_R \xi \frac{\partial^2 P}{\partial x_j \partial x_j}\,\mathrm{d}R = \xi(x_{0i}) \qquad \text{if } R \text{ contains the point } x_i = x_{0i} \tag{17}$$

In eqn. (9), if one replaces R and B' by R and B, respectively, and uses eqns. (8) and (17), one obtains eqn. (15) immediately. Interchanging the independent variables x_{0i} and x_i, one then has

$$\xi(x_i) = \int_R Pg_0\,\mathrm{d}R_0 - \int_B \left(P\frac{\partial \xi_0}{\partial x_{0j}} - \xi_0\frac{\partial P}{\partial x_{0j}}\right) n_{0j}\,\mathrm{d}B_0 \tag{18}$$

where the subscript 0 indicates a variable, or a differentiation, or an integration evaluated in the x_{0i} space, i.e. $g_0 = g(x_{0i})$. The integral representation for ξ, eqn. (18), is completely equivalent to the differential equation (eqn. (8)).

4.5.2 Vector Elliptic Equations

Consider a vector η_i whose divergence and curl are b and c_i, respectively:

$$\frac{\partial \eta_j}{\partial x_j} = b \tag{19}$$

and

$$\varepsilon_{ijk}\frac{\partial \eta_k}{\partial x_j} = c_i \tag{20}$$

Equations (19) and (20) form an elliptic set. They give

$$\frac{\partial^2 \eta_i}{\partial x_j \partial x_j} = \frac{\partial b}{\partial x_i} - \varepsilon_{ijk}\frac{\partial c_k}{\partial x_j} \tag{21}$$

Since each component of η_i satisfies a Poisson's equation, an integral representation for η_i is obtainable from eqn. (18) by substituting ξ by η_i and g by the right-hand side terms of eqn. (21). The resulting integral representation for η_i contains derivatives of b and c_i in the integrand for the integral over R and a derivative of η_i in the boundary integral.

An integral representation for η_i in which the derivatives of b, c_k, and η_i are absent is obtainable by decomposing η_i into its irrotational and solenoidal parts.

As is well known, the irrotational part of a vector can be expressed as the gradient of a scalar potential and the solenoid part of the vector can be expressed as the curl of a vector potential. One writes therefore

$$\eta_i = \frac{\partial \phi}{\partial x_i} + \varepsilon_{ijk}\frac{\partial \psi_k}{\partial x_j} \tag{22}$$

where ϕ and ψ_k are respectively the scalar potential and the vector potential of η_i. From eqns. (19) and (22) one obtains a Poisson's equation for ϕ:

$$\frac{\partial^2 \phi}{\partial x_j \partial x_j} = b \tag{23}$$

From eqns. (20) and (22) one obtains by requiring ψ_i to be solenoidal,

$$\frac{\partial^2 \psi_i}{\partial x_j \partial x_j} = -c_i \tag{24}$$

Integral representations for ϕ and ψ_i are obtainable immediately from eqn. (18). By placing these integral representations into eqn. (22) and noting that $\partial P/\partial x_i = -(\partial P/\partial x_{0i})$, one obtains

$$\eta_i = -\int_R \left(b_0 \frac{\partial P}{\partial x_{0i}} + \varepsilon_{ijk} c_{0j} \frac{\partial P}{\partial x_{0k}} \right) \mathrm{d}R_0 + \int_B \left(\eta_{0j} n_{0j} \frac{\partial P}{\partial x_{0i}} - \varepsilon_{ijk} \varepsilon_{jmn} \eta_{0m} n_{0n} \frac{\partial P}{\partial x_{0k}} \right) \mathrm{d}B_0 \tag{25}$$

Because of the absence of the derivative terms in the integrands, the integral representation (eqn. (25)) is better suited for numerical procedures than the one obtained using eqn. (21).

4.5.3 Parabolic Equations

Consider the inhomogeneous diffusion equation

$$\frac{\partial f}{\partial t} - a \frac{\partial^2 f}{\partial x_j \partial x_j} = k \tag{26}$$

The principal solution Q of the diffusion equation is

$$Q = \frac{1}{(4\pi a(t - t_0))^{d/2}} \exp\left\{ -\frac{r'^2}{4a(t - t_0)} \right\} \tag{27}$$

where d is the dimensionality of the problem and $t \geq t_0$.

The principal solution (eqn. (27)) possesses the following properties:

$$\frac{\partial Q}{\partial t} - a \frac{\partial^2 Q}{\partial x_j \partial x_j} = 0 \qquad \text{for all } t \text{ values} \tag{28}$$

$$Q = 0 \qquad \text{if } t = t_0 \text{ and } x_i \neq x_{0i} \tag{29}$$

$$\int_R Q \, \mathrm{d}R = \begin{cases} 0 & \text{if } t = t_0 \text{ and } R \text{ does not contain the point } x_{0i} \\ 1 & \text{if } t = t_0 \text{ and } R \text{ contains the point } x_{0i} \end{cases} \tag{30}$$

It is simple to show that eqn. (28) is correct by carrying out the needed differentiations with respect to time and space. Equation (29) is obviously correct. The first part of eqn. (30) follows directly from eqn. (29). The second part of eqn. (30) can be shown to be correct by changing the variable of integration from x_i to ζ_i, with

$$\zeta_i = \frac{x_i'}{(4a(t - t_0))^{1/2}} \tag{31}$$

and perform the integration over an infinite unbounded region.

From eqn. (30), one concludes that, for $t = t_0$

$$\int_R Q k_0 \, \mathrm{d}R_0 = k(x_i) \qquad \text{if } R \text{ contains the point } x_{0i} \tag{32}$$

It is simple to show that

$$\frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial t_0} \qquad \text{and} \qquad \frac{\partial^2 Q}{\partial x_i \partial x_i} = \frac{\partial^2 Q}{\partial x_{0i} \partial x_{0i}}$$

Therefore, from eqn. (28), one has

$$\frac{\partial Q}{\partial t_0} + \frac{\partial^2 Q}{\partial x_{0i} \partial x_{0i}} = 0 \tag{33}$$

Consider the identity

$$Q\left(\frac{\partial f_0}{\partial t_0} - a \frac{\partial^2 f_0}{\partial x_{0j} \partial x_{0j}} \right) + f_0 \left(\frac{\partial Q}{\partial t_0} + a \frac{\partial^2 Q}{\partial x_{0j} \partial x_{0j}} \right) = \frac{\partial}{\partial t_0}(f_0 Q) + a \frac{\partial}{\partial x_{0j}} \left(f_0 \frac{\partial Q}{\partial x_{0j}} - Q \frac{\partial f_0}{\partial x_{0j}} \right) \tag{34}$$

Integrating each term in eqn. (34) over the region R' and over the time interval $0 < t_0 < t$ and using eqns. (26) and (33), one obtains

$$\int_0^t \mathrm{d}t_0 \int_R Q k_0 \, \mathrm{d}R_0 = \int_R [(f_0 Q)_{t_0 = t} - (f_0 Q)_{t_0 = 0}] \, \mathrm{d}R_0 + a \int_0^t \mathrm{d}t_0 \int_B \left(f_0 \frac{\partial Q_0}{\partial x_{0j}} - Q \frac{\partial f_0}{\partial x_{0j}} \right) n_{0j} \, \mathrm{d}B_0 \tag{35}$$

In obtaining the last integrals in eqn. (35), the divergence theorem has been used.

Placing eqn. (32) into eqn. (35), one obtains

$$f = \int_R (f_0 Q)_{t_0 = 0} \, \mathrm{d}R_0 + \int_0^t \mathrm{d}t_0 \int_R Q k_0 \, \mathrm{d}R_0 - a \int_0^t \mathrm{d}t_0 \int_B \left(f_0 \frac{\partial Q_0}{\partial x_{0j}} - Q \frac{\partial f_0}{\partial x_{0j}} \right) n_{0j} \, \mathrm{d}B_0 \tag{36}$$

Equation (36) is an integral representation for the function f which satisfies the inhomogeneous parabolic differential equation (eqn. (26)).

4.6 BOUNDARY INTEGRAL AND POTENTIAL FLOW

The kinematics of the viscous flow problem described in terms of differential equations are eqns. (1) and (3). Using eqns. (19), (20) and (25), one writes

$$v_i = -\int_R \varepsilon_{ijk}\omega_{0j}\frac{\partial P}{\partial x_{0k}}\,\mathrm{d}R_0 + \int_B \left(v_{0j}n_{0j}\frac{\partial P}{\partial x_{0i}} + \varepsilon_{ijk}\varepsilon_{jmn}v_{0m}n_{0n}\frac{\partial P}{\partial x_{0k}}\right)\mathrm{d}B_0 \tag{37}$$

Equation (37) is an integral representation for the velocity vector. It is simple to show that this integral representation satisfies eqns. (1) and (3). The physical meanings of the two integrals in eqn. (37) are as follows. The integral over the region R represents the contribution of the vorticity field to the velocity field. This integral, in fact, is a generalised statement of the well-known law of Biot–Savart. It describes the velocity field associated with distribution of vorticity in an infinite unbounded region. The boundary integral gives a potential flow in the region R.

Consider a finite solid body immersed in an infinite incompressible fluid and moving with a uniform velocity. In a reference frame attached to the solid body, the velocity on the solid boundary S is zero. The surface S therefore does not contribute to the boundary integral in eqn. (37). On the surface S_∞, infinitely far from S, the velocity is $v_{\infty i}$, the free-stream velocity. Let S be a large spherical surface of radius R in a three-dimensional problem which encloses S in its interior and which is centred at the origin. As R approaches infinity, according to eqn. (12), $\partial P/\partial x_{0i}$ approaches $n_{0i}\,(4\pi R^2)$. In consequence, the integrand of the boundary integral in eqn. (37) approaches

$$(v_{\infty j}n_{0j}n_{0i} - \varepsilon_{ijk}\varepsilon_{jmn}v_{\infty m}n_{0n}n_{0k})\,4\pi R^2$$

The quantity inside the parentheses gives $v_{\infty i}$. Therefore the boundary integral yields $v_{\infty i}$ and eqn. (37) reduces to

$$v_i = -\int_R \varepsilon_{ijk}\omega_{0j}\frac{\partial P}{\partial x_{0k}}\,\mathrm{d}R_0 + v_{\infty i} \tag{38}$$

For a two-dimensional flow, it can be shown, by considering S_∞ to be a large circle which encloses S in its interior, that the boundary integral in eqn. (37) yields $v_{\infty i}$. Equation (38) is therefore valid for both the two-dimensional problem and the three-dimensional problem.

It should be emphasised that the boundary integral in eqn. (37) contains

both the tangential and the normal components of the boundary velocity. Indeed, the derivation of eqn. (38), which is convenient to use in a numerical procedure, utilises a knowledge of both boundary velocity components on B. The prescription of both the tangential and the normal components of the boundary velocity, as shown in Section 4.3, over-specifies the problem. Nevertheless, if the prescribed normal component and tangential component of the boundary velocity are compatible to one another, then the use of both these components in the computation of the velocity field in the region R is legitimate. For example, if a velocity field v_i is determined on the basis of the prescribed tangential component of the boundary velocity and this velocity field possesses on the boundary a normal component which is identical to the prescribed value, then the two prescribed boundary velocity components are compatible to one another. The requirement that the velocity boundary conditions be compatible places a restriction on the vorticity distribution in R.

Consider for the moment a two-dimensional potential flow past the exterior of a finite solid body. It is generally thought that a potential flow is an inviscid flow. In reality, it is more appropriate to think of the potential flow as an approximation of a high Reynolds number viscous flow involving no appreciable region of flow separation (Wu, 1981). In such a flow, the region of non-negligible vorticity, and hence also the effects of viscosity, is confined to the thin boundary layer surrounding the solid and a thin wake trailing the body. The thin wake can be represented kinematically by a pair of vortex sheets of opposite senses located close to one another. For certain types of flow, the effects of this vortex wake on the flow near the solid body may be neglected. The boundary layer surrounding the solid body can be represented kinematically by a vortex sheet which, in the case of a lifting body, possesses a total circulation. Physically, a vortex sheet is a concentrated layer of vorticity. If only the vortex sheet representing the boundary layer is important to the flow field near the solid, then eqn. (38) can be written as

$$v_i = -\int_{S^+} \varepsilon_{ijk}\gamma_{0j}\frac{\partial P}{\partial x_{0k}}\,\mathrm{d}B_0 + v_{\infty i} \tag{39}$$

where γ_i is the strength of the vortex sheet representing the boundary layer, and S^+ is the surface which encloses the solid surface S and is separated from S by an infinitesimal distance.

If γ_i is known, then the velocity field everywhere in the flow field can be computed using eqn. (39). The restriction on the vorticity field, imposed by the compatibility of the velocity boundary conditions, suggests a method for determining the distribution of γ_i on the surface S^+. Since eqn. (39) is valid everywhere in the closed region R bounded by S, it can be used to compute the value of v_i on S. The value of v_i on S, however, is known. Indeed, it is simply zero for the problem under consideration. One therefore has

$$0 = -\int_{S^+} \varepsilon_{ijk}\gamma_{0j} \frac{\partial P_s}{\partial x_{0k}} \, \mathrm{d}B_0 + v_{\infty i} \tag{40}$$

where P_s is the principal solution of the elliptic equation defined by eqn. (10), with x_{0i} on the surface S^+ and x_i on the solid boundary S.

Equation (40) is a vector equation and can be written as two component scalar equations for the present two-dimensional problems. In particular, one has for the tangential component

$$\frac{1}{2\pi} \int_{S^+} \frac{\gamma_0(x_{0j} - x_j)n_{0j}}{r'^2} \, \mathrm{d}B_0 = -v_{\infty t} \tag{41}$$

where $v_{\infty t}$ is the tangential component of the free-stream velocity on S. x_{0j} and x_j are respectively the position vectors on S^+ and S, and γ is the strength of the vortex sheet.

The left-hand side of eqn. (41) gives the velocity field due to the vortex sheet on the boundary S^+. Equation (41) states that this velocity field, when combined with the free-stream velocity, satisfies the condition that the tangential velocity is zero on the solid boundary.

The normal component of eqn. (40) can be similarly written. There exist, therefore, two scalar integral equations with γ as the unknown function. It shall be shown that the solution obtained from each equation satisfies the other. In consequence, the two prescribed components of the boundary velocity are compatible to one another.

Consider the region R_s occupied by the solid body. Kinematically, the velocity field and the vorticity field in this region are related by the continuity equation (eqn. (1)) and the vorticity transport equation (eqn. (3)). For the problem under consideration, the solid body is not undergoing a rotation and ω_i is zero in the solid. A scalar potential ϕ therefore exists in R_s. The continuity equation further states that this scalar potential satisfies the Laplace equation and is harmonic. A solution for ϕ obtained from eqn. (41) obviously satisfies the condition that the tangential velocity component, or the tangential derivative of ϕ, is zero on the boundary S. Therefore ϕ is a constant on S. The principle of extremum for a harmonic function then requires ϕ to be a constant throughout R.

Therefore the normal derivative of ϕ, which is equal to the normal velocity component, is zero on the boundary S. In other words, a solution obtained from eqn. (41) ensures that the normal component of the boundary velocity is also zero. Similarly, it can be shown that a solution for the velocity field obtained from the normal component of eqn. (40) satisfies the condition that the normal velocity component is zero on the boundary S. This velocity field also possesses a zero tangential component of the boundary velocity.

It should be pointed out that the solution of eqn. (40) contains an arbitrary constant since the homogeneous counterpart of eqn. (40) possesses a complementary solution. This arbitrary constant is determined by the value of the circulation Γ about the solid body defined by

$$\Gamma = \int_{S^+} \gamma \, \mathrm{d}B \tag{42}$$

The value of Γ can often be established through a consideration of the physics of the problem, e.g. the use of the well-known Kutta condition in connection with a lifting body with a sharp trailing edge.

The computation of a potential flow field using the integral representation (eqn. (38)) is composed of two major steps. First, the boundary integral equation (eqn. (41)), or a similar equation which represents the normal component of eqn. (40), subject to the auxiliary condition (eqn. (42)), is solved to determine the function γ on S^+. Then the function γ is placed into eqn. (39), which represents an approximation of eqn. (38), to compute the velocity field.

The computation of γ can be accomplished by the use of a finite element technique. The boundary S^+ is represented by line elements with associated nodes. The boundary integral is replaced by a sum of integrals over individual elements. Element interpolation functions are introduced and the element integrals are evaluated analytically for each velocity node, yielding a set of algebraic equations of the form

$$\sum_{n=1}^{N} G_{mn}\gamma_n = -v_{\infty tm} \tag{43}$$

where m and n designate respectively nodes on the boundaries S and S^+, γ_n is the strength of the vortex sheet at the node n, $v_{\infty tm}$ is the tangential component of the free stream velocity at the node m, and G_{mn} is the geometric coefficient determined from the element integrals and is dependent only on the relative location of the nodes m and n, and N is the total number of vortex nodes on S^+.

It can be shown that the rank of the coefficient matrix G_{mn} is $N-1$. The auxiliary condition (eqn. (42)) yields

$$\sum_{n=1}^{N} A_n \gamma_n = \Gamma \tag{44}$$

Equations (43) and (44) form a set of linear algebraic equations containing γ_n as unknowns. The size of the coefficient matrix is not large and a variety of numerical techniques are available for the computation of γ_n.

In the computation of the velocity field, a finite element technique again can be utilised. The finite element analogue of eqn. (39) is of the form

$$v_{qi} = \sum_{n=1}^{N} H_{qn}\gamma_n + v_{\infty i} \tag{45}$$

where v_{qi} is the ith component of the velocity vector at the node q and H_{qn} is the geometric coefficient dependent on the relative position of the nodes q and n.

Equation (45) is an explicit equation for the computation of the velocity values. That is, each algebraic equation represented by eqn. (45) contains only one unknown velocity value. With known values of γ_n, the values of v_i are computed one node at a time. The computation of the v_i value at each node in the flow field is accomplished independently of the v_i value at other nodes. In contrast, with a finite difference method or a finite element method based on the variational principle or the residual concept, one obtains a set of simultaneous algebraic equations. Each algebraic equation contains more than one nodal value of the velocity (or of the scalar potential). The value of the velocity at each node depends on the values of the velocity at the neighbouring nodes.

Equation (45) may be viewed as an inverted equation corresponding to the matrix equation obtained using a finite difference or a conventional finite element discretisation procedure. The distinguishing ability of the boundary integral formulation for the explicit computation of the velocity values offers great advantages in flow computation. For example, in an aerodynamic computation of the pressure distribution on the surface of an airfoil, one only needs to determine the velocity immediately adjacent to the airfoil. The pressure distribution is then immediately obtained from the well known Bernoulli's equation. With the boundary integral approach just described, the computations of the velocity values can be confined to a surface S^{++} surrounding and immediately adjacent to the surface S^+. In fact, it is easy to see that the vortex sheet in this case represents a discontinuity of the tangential velocity component across the sheet. Since the velocity on S is zero, the magnitude of the velocity on the surface S^{++} is equal to the local strength of the vortex sheet. In any event, with the boundary integral method the computation of the present two-dimensional potential flow is reduced to a one-dimensional procedure. Similarly, the computation of a three-dimensional potential flow is reduced to a two-dimensional procedure. This unique feature is not available with a finite difference or a conventional finite element procedure.

In the following section, the generalisation of the concepts just described for potential flow computation is described.

4.7 BOUNDARY INTEGRAL AND VORTICITY BOUNDARY CONDITION

In a potential flow, the vorticity field is negligible outside the thin boundary layer and there is no need in the computational procedure to be concerned with the kinetic aspect of the flow problem dealing with vorticity transport. The solution procedure for the potential flow therefore deals only with eqn. (39), which is a simplified version of eqn. (38).

In a general viscous flow, the kinetic aspect of the flow problem does enter into the computational procedure. In addition, the full integral representation for the kinematics of the flow (eqn. (37) or eqn. (38)), must be treated in place of the simplified equation (eqn. (39)). Many of the concepts and features discussed in Section 4.6 in connection with the potential flow, however, remain applicable.

Unlike the potential flow, the presence of the vorticity in regions outside the thin boundary layer has a significant effect on the flow near the solid. Consider the time-dependent two-dimensional flow of a viscous fluid past the exterior of a finite solid body. The numerical procedure outlined in Section 4.2 can be utilised to simulate the flow computationally. Suppose step (a) of the computation loop described in Section 4.2 is completed. The computation of the boundary vorticity values in step (b) can be accomplished by using a boundary integral procedure similar to that described in

Section 4.6 for the computation of the vortex sheet strength γ. This boundary integral procedure is described in detail by Wu (1976). An alternative procedure for the explicit computation of the boundary vorticity values is described below.

Let the region R be mapped into finite elements with associated nodes. Values of vorticity at nodes not located on the solid boundary S are known, while the values of the vorticity at the boundary nodes are to be computed. Element interpolation functions are introduced and the element integrals are evaluated analytically for each velocity node. A detailed description of the derivations using polynomials as interpolation functions is given by Wahbah (1978). The resulting algebraic equations are explicit and permit the node by node computation of the velocity values in the flow field. For example, for a two-dimensional flow, the algebraic equations are of the form

$$v_{qi} = \sum_{p=1}^{P} K_{qp}\omega_p + v_{\infty i} \tag{46}$$

where K_{qp} is a geometric coefficient dependent on the relative position of the nodes q and p, ω_p is the value of the vorticity at the node p, and P is the number of nodes in the flow field where the vorticity is non-zero.

It is simple to see that eqn. (46) contains the following subset of equations

$$0 = \sum_{p=1}^{P} K_{mp}\omega_p + v_{\infty i} \tag{47}$$

where the subscript m designates the node m located on S.

Equation (47) can be written as

$$0 = \sum_{S=1}^{N} K_{ms}\omega_s + \sum_{n=N+1}^{P} K_{mn}\omega_n + v_{\infty i} \tag{48}$$

where the subscript s designates a node on S and the subscript n designates a node not on S. The dimension of the coefficient matrix K_{ms} is $N \times N$, where N is the number of nodes located on S. Since this dimension is in general not

large, the matrix can be easily inverted, yielding an expression of the form

$$\omega_s = \sum_{n=N+1}^{P} M_{mn}\omega_n + f_s \tag{49}$$

where M_{mn} is a geometric coefficient and f_s is known.

Equation (49) permits the boundary vorticity values to be computed explicitly. Placing eqn. (49) into eqn. (47) gives, after rearrangement of terms

$$v_{qi} = \sum_{n=N+1}^{P} N_{qr}\omega_r + g_{qi} \tag{50}$$

Equation (50) permits the velocity values to be computed using vorticity values at nodes not located on the boundary.

In the above discussions, eqn. (38) is utilised to outline a procedure for computing the velocity values in an external viscous flow past a finite solid body. For other problems involving other types of flow, e.g. an internal flow, the no-slip and the free stream boundary conditions may not be everywhere applicable and eqn. (37) may have to be used. As already discussed, the boundary integral in eqn. (37) gives a potential flow in the region R. With the tangential and normal velocity boundary conditions known, this boundary integral can be evaluated for each node in the flow field. One therefore obtains algebraic equations of a form identical to eqn. (46), except that the term $v_{\infty i}$ is replaced by the term u_{qi} representing the contribution of the boundary integral to the velocity field at the node q. Except for the added computation of the values of u_{qi}, the procedure for computing the velocity field is identical to that outlined for eqn. (46).

For the two-dimensional flow under consideration, the vorticity vector possesses only one component. In consequence, eqn. (46) is relatively simple. For three-dimensional flows, eqn. (37) or eqn. (38) yields algebraic equations that are lengthier than eqn. (46) because the vorticity vector in a three-dimensional flow possesses in general three components. For three-dimensional flows, an integral representation for the vector potential can be used in place of the integral representation for the velocity vector. For two-dimensional flows, the vector potential reduces to the stream function (Sampath, 1977). The various integral representations all possess a common feature—they permit the flow field to be computed explicitly, node by node. This distinguishing feature offers a number of important advantages. It should be emphasised that with a finite difference or a usual finite element procedure, each of the algebraic equations obtained contains more than one unknown value of the velocity (or the vector potential). The solution of these equations requires an implicit procedure.

4.8 INTEGRAL REPRESENTATION AND VORTICITY TRANSPORT

The kinetics of the viscous flow problem is described by the vorticity transport equation (eqn. (4)). Using eqns. (26) and (36), one writes

$$\omega_i = \int_R (\omega_{0i} Q)_{t_0=0}\, \mathrm{d}R_0 + \int_0^t \mathrm{d}t_0 \int_R \left(\omega_{0j} \frac{\partial v_{0i}}{\partial x_{0j}} - v_{0j} \frac{\partial \omega_{0i}}{\partial x_{0j}} \right) Q\, \mathrm{d}R_0$$
$$+ \nu \int_0^t \mathrm{d}t_0 \int_B \left(\omega_{0i} \frac{\partial Q}{\partial x_{0j}} - Q \frac{\partial \omega_{0i}}{\partial x_{0j}} \right) n_{0j}\, \mathrm{d}B_0 \qquad (51)$$

where Q is the principal solution of the diffusion equation and is defined by eqn. (27), with a in that equation replaced by the coefficient of viscosity ν.

Equation (51) expresses the entirety of the kinetics of a viscous flow in the form of an integral representation for the vorticity vector. The physical meaning of the fundamental solution Q is well known. If at the time $t = t_0$ there exists in an infinite unlimited region R a unit of some physical quantity, say f, at the point $x_i = x_{0i}$, then Q represents the distribution of f at the time level t, with $t > t_0$, in the region R due to the process of diffusion. If f is non-zero at the time level $t = t_0$ only in an elemental region $\mathrm{d}R_0$ located at $x_i = x_{0i}$, and the value of f in $\mathrm{d}R_0$ is f_0, then the distribution of f at the time level t is $Qf_0\,\mathrm{d}R_0$. If, at the time level t_0, the distribution of f is known in the infinite unlimited region R_∞, then the distribution of f at the time level t is expressible as

$$f(x_i, t) = \int_{R_\infty} Q f(x_{0i}, t_0)\, \mathrm{d}R_0 \qquad (52)$$

Equation (52) is valid in an infinite unlimited region. From eqn. (52), it is clear that the first integral in eqn. (51) represents the contribution of the initial vorticity distribution, through the process of diffusion, to the distribution of vorticity at the subsequent time level. The second and third integrals in eqn. (51) are due to the fact that in the present problem the kinetic processes present are not limited to that of diffusion and the region R is not unlimited.

The second integral in eqn. (51) is present because the diffusion equation, eqn. (4), satisfied by the vorticity field is inhomogeneous. The inhomogeneous term represents the effects of vorticity stretching and of convection. The first term in the integrand of the second integral of eqn. (51) gives the effect of vorticity stretching which is a three-dimensional phenomenon and is absent in two-dimensional flows. The two terms of the integrand of the second integral can be rewritten as

$$Q\varepsilon_{ims} \frac{\partial}{\partial x_{0m}} (\varepsilon_{sjk} \omega_{0j} v_{0k}) = \varepsilon_{ims} \frac{\partial}{\partial x_{0m}} (\varepsilon_{sjk} \omega_{0j} v_{0k} Q) - \varepsilon_{ijk} \varepsilon_{jms} v_m \omega_s \frac{\partial Q}{\partial x_{0k}} \qquad (53)$$

The use of Gauss' theorem then permits the second integral of eqn. (51) to be rewritten as

$$\int_0^t \mathrm{d}t_0 \int_B Q \varepsilon_{ijk} \varepsilon_{jms} \omega_{0m} v_{0s} n_{0k}\, \mathrm{d}B_0 - \int_0^t \mathrm{d}t_0 \int_R \varepsilon_{ijk} \varepsilon_{jms} v_m \omega_s \frac{\partial Q}{\partial x_{0k}}\, \mathrm{d}R_0$$

For the external flow problem under consideration, the boundary B is composed of the solid surface S and the surface S_∞ infinitely far from S. Since $v_i = 0$ on S and $\omega_i = 0$ on S_∞, the integral over B in the above expression vanishes. One therefore obtains from eqn. (51)

$$\omega_i = \int_R (\omega_{0i} Q)_{t_0=0}\, \mathrm{d}R_0 + \int_0^t \mathrm{d}t_0 \int_R \varepsilon_{ijk} \varepsilon_{jms} v_m \omega_s \frac{\partial Q}{\partial x_{0k}}\, \mathrm{d}R_0$$
$$+ \nu \int_0^t \mathrm{d}t_0 \int_S \left(\omega_{0i} \frac{\partial Q}{\partial x_{0j}} - Q \frac{\partial \omega_{0i}}{\partial x_{0j}} \right) n_{0j}\, \mathrm{d}B_0 \qquad (54)$$

In eqn. (51), the integrand of the second integral contains a derivative of the vorticity vector. In eqn. (52), this derivative term is absent. Equation (54) is more convenient to use in computation (Wu & Rizk, 1978).

The third integral in eqn. (51), or eqn. (54), gives the effects of the presence of the boundary B on the vorticity in the region R. Since the vorticity values are zero on the surface S_∞, the integration for the third integral needs to be performed only over the solid surface S. On S, vorticity is being generated continually and is being diffused from the boundary into the fluid domain. The evaluation of the third integral in eqn. (54) requires a knowledge of both the value and the normal derivative of the vorticity vector on the boundary B.

Consider step (a) of the computation loop described in Section 4.2. Let the vorticity and the velocity values at an old time level, say $t - \Delta t$, be known at every node of interest in R and on B. The computation of the vorticity values at the new time level t can be accomplished by using eqn. (54) by letting $t - \Delta t$ be the initial time level $t_0 = 0$. The finite element analogue of eqn. (54), for a two-dimensional flow, is therefore

$$\omega^t_q = \sum_{p=1}^{P} B_{qpj}(v_j\omega)^t_p + \sum_{n=1}^{N} C_{qn}\omega^t_n + \sum_{n=1}^{N} D_{qn}\left(\frac{\partial\omega}{\partial x_j} n_j\right)^t_n + h_q \tag{55}$$

where the superscript t denotes the time level at which a variable is evaluated, the subscript q denotes a node in R at which a new vorticity value is to be computed, the subscript p denotes a point in R, the subscript n denotes a point on the solid boundary S. B_{qp} is a geometric coefficient dependent upon the relative position of the nodes q and p, C_{qn} and D_{qn} are geometric coefficients dependent upon the relative position of the nodes q and n, and h_q is dependent upon the vorticity and velocity values at the old time level $t - \Delta t$ and is known.

Equation (55) contains the vorticity and velocity values in R as well as the values of the normal derivative of vorticity on S as unknowns. This equation contains a subset of equations:

$$\omega^t_m = \sum_{p=1}^{P} B_{mpj}(v_j\omega)^t_p + \sum_{n=1}^{N} C_{mn}\omega^t_n + \sum_{n=1}^{N} D_{mn}\left(\frac{\partial\omega}{\partial x_j} n_j\right)^t_n + h_m \tag{56}$$

where m is a node on the boundary S.

The coefficient matrix D_{mn}, an $N \times N$ matrix, can be easily inverted and used to obtain a set of explicit expressions for $[(\partial\omega/\partial x_i)n_j]^t_n$ in terms of h_m, ω^t_m, and $(v_i\omega)^t_p$. Placing this expression for $[(\partial\omega/\partial x_i)n_j]^t_n$ and eqn. (49) into eqn. (55) and using the prescribed boundary values of v_j, one obtains, after rearrangement of terms, the following expression for ω^t_q:

$$\omega^t_q = \sum_{r=N+1}^{P} E_{qr}\omega^t_r + \sum_{r=N+1}^{P} F_{qrj}(v_j\omega)^t_r + k_q \tag{57}$$

where E_{qr} and F_{qr} are geometric coefficients dependent only upon the relative position of the nodes q and r, and k_q is dependent upon the vorticity values at the old time level and is known.

Equation (57) is implicit and contains unknown values of ω^t_n and v^t_n. This equation is non-linear in the sense that v^t_{ip} is a function of ω^t_p kinematically and the product $(v_j\omega)^t_p$ is present in the equation.

4.9 SOLUTION PROCEDURE

The use of integral representations in the numerical solution of flow problems represents a major departure from previous finite difference and finite element methods. This integral-representation approach offers a number of major advantages and, at the same time, requires the use of drastically new numerical procedures. In the following, a basic procedure for computing time-dependent incompressible general viscous flows using integral representations is described. Many alternative procedures can be utilised in place of a part or all of the procedures described here. In many applications, it is advantageous to combine the unique attributes of the integral representation approach with various features of other more conventional approaches.

Equations (37) and (54) express the entirety of the time-dependent incompressible general viscous flow mathematically as integral representations. The three steps forming the computation loop given in Section 4.2 can be carried out as follows for the external flow problem:

(a) With known values of $v^{t-\Delta t}_{ip}$ and $\omega^{t-\Delta t}_p$ in R, the term k_q in eqn. (57) is evaluated. Equation (57) is then solved to determine values of vorticity ω^t_q at all nodes not on S for the new time level t. Since the dimension of the coefficient matrices E_{qr} and F_{qr} is $P \times P$, and P, being the total number of non-boundary nodes, is usually very large, the solution of eqn. (57) requires an iterative procedure. With a sufficiently small time interval Δt, the values of v^t_{ir} which appear in eqn. (57) may be approximated by the values of $v^{t-\Delta t}_{ir}$ during the iterations. If the time interval Δt is large, then values of v^t_{ir} need to be evaluated repeatedly during the iteration.

(b) With non-boundary values of vorticity determined for the time level t, corresponding nodal values of vorticity ω^t_s on the boundary S are computed using the explicit eqn. (49).

(c) The velocity values v^t_q for the time level t at the non-boundary nodes are computed using an explicit expression such as eqn. (46). Boundary values of the velocity v^t_s are prescribed (zero for the present problem) and need not be computed.

The distinguishing feature of the integral representation approach for elliptic systems is that it permits explicit numerical procedures. This feature offers several highly significant attributes. These attributes are summarised below. During the past few years, the advantages of these attributes have been demonstrated conclusively by extensive analytical and numerical investigation. Considerable experience has been accumulated regarding the application of this approach. A series of Ph.D. theses, research papers and computer codes have been made available. Highly complicated general viscous flow problems have been solved using this approach. The discussions below are brief. Suitable references containing complete analyses and extensive numerical illustrations are given for each of the attributes discussed.

4.9.1 Confinement of Solution Field (Wu & Thompson, 1973)

The unique ability of the integral representation approach for the explicit evaluation of velocity values enables the solution field for the kinematic computation to be confined to any selected region of the flow field. It is obvious that each term in eqn. (57) vanishes wherever vorticity vanishes. Therefore, in the kinetic part of the computation, it is only necessary to compute new vorticity values in the viscous region where the vorticity and its derivatives are not zero. In this computation, velocity values need to be known only in the viscous region. Therefore, with the integral representation approach, one confines the kinematic computation of velocity values to the viscous region. As discussed earlier, the viscous region usually occupies only a small portion of the total flow field. The use of the integral representation approach therefore requires a drastically smaller number of data nodes than prevailing finite difference and finite element methods requiring the computation of velocity values in the potential region as well as the viscous region. The resulting reduction in computing effort is also drastic (Wu & Sampath, 1976; Sampath, 1977). The difficulties caused by the fact that the potential flow length scale is incompatible with the viscous flow scale are removed.

4.9.2 Segmentation of Solution Field (Wu *et al.*, 1974)

With eqn. (46), the computation of each velocity value requires multiplication, where Q is the number of non-boundary nodes where the vorticity values are not negligible. If the number of nodes Q is large, then the total amount of computation required for all the values of velocity is proportional to Q^2 and is large. For such cases, a technique of segmentation of flow field can be used to drastically reduce the amount of computation required. For example, by dividing the solution field into two segments, each containing approximately $Q/2$ nodes, computing the velocity values on the boundary of each region using eqn. (46) and then using explicit algebraic equations obtained from eqn. (37) to compute velocity values in each segment separately, the number of multiplications required to compute

each velocity value is reduced to about $Q/2$. If Q is a very large number, then successive segmentation of the flow field is advantageous. Each level of segmentation reduces the amount of computation substantially without adversely influencing solution accuracy.

4.9.3 Computation of Vorticity Boundary Condition (Wu, 1976)

Previous methods for computing the boundary vorticity values are based upon extrapolation procedures and often experience stability and accuracy difficulties. The integral representation approach for computing boundary vorticity values has been shown to produce stable and accurate results. This approach simulates the physical process of vorticity generation on solid boundaries and is discussed fully by Wu (1976).

4.9.4 Separate Treatment of Attached and Detached Flow Regions (Wu & Gulcat, 1981; Gulcat, 1981)

The use of the integral representation approach permits the attached component (boundary layer) and the detached components (recirculating and wake flows) of the viscous flow to be separately treated. The detailed procedure for this method is presented by Wu and Gulcat (1981) and Gulcat (1981). By using boundary layer equations rather than the full Navier–Stokes equations in regions where the boundary layer simplifications are justified, not only is the amount of computation substantially reduced, but also several major difficulties experienced in previous numerical solutions of the general viscous flow problem are eliminated. These difficulties arise because of the existence of the diverse length scales associated with the several flow components and because of the excessive demands on the numerical procedure arising from the retention of negligibly small terms while computing the boundary layer flow. By treating the boundary layer flow separately from the detached flow, these difficulties are removed. The computer time required by this approach is relatively small and is insensitive to the flow Reynolds number (Wu & Gulcat, 1981).

4.9.5 Hybrid and Other Techniques (Wu & Sankar, 1978; Sampath, 1977)

In many applications, it is advantageous to combine the unique attributes of the integral representation approach with various features of other, more conventional, approaches. As an example, in the kinematic part of the computation of an external flow, it is permissible to use eqn. (49) to compute the velocity values at the outer boundary of the vortical region. Once this is accomplished, a finite difference or a finite element method with superior solution efficiency, not necessarily one based on an integral representation, can be used to compute velocity values inside this boundary (Sampath, 1977). As another example, the solution field can be divided into an inner region near the solid body and an outer region far from the solid body. Velocity values on the demarcation boundary of the two regions can be computed using eqn. (49). Finite element methods, including the integral representation method, can then be used to compute values of velocity in the inner region so as to accommodate the complex solid boundary shapes. In the outer region, finite difference equations, with comparatively simpler coefficient matrices, can be used (Wu & Sankar, 1978; Sankar, 1977).

It has been found that the use of the integral representation approach for at least a part of the kinematic computation offers great flexibility in the solution procedure. For example, a finite Fourier series method has been developed in conjunction with the integral representation to allow very rapid computation of velocity values in a conformally transformed plane (Wu & Sugavanam, 1978; Sugavanam, 1979).

For the parabolic system, the distinguishing feature of the integral representation is that it expresses the several kinetic processes that redistribute the vorticity in the fluid as separate integrals. Thus, in eqn. (54), the three integrals express respectively the contributions of the initial condition, the inhomogeneous term (vorticity stretching and convection) and the diffusion of vorticity generated on the boundary. This fact suggests that, with an integral representation formulation, different numerical quadrature procedures can be used for the three different physical processes. For example, the time step for the diffusion process can be different from the time step for the convection process so as to differentiate between the different characteristic time scales of the two processes. For high Reynolds number flows, the effective diffusion speed is much slower than the convection speed. In a numerical procedure, the time interval Δt is usually selected on the basis of the convection speed. Within the time interval thus selected, the effect of diffusion from the solid boundary S is felt only at nodes located near S. This fact has been utilised by Rizk (1980) to design a highly efficient procedure for the kinetic part of the computation. It is also of interest to note that the contribution of the initial condition, represented by the first integral in eqn. (54), is identical to the contribution of the diffusion process in an infinite unbounded region. In evaluating this integral, therefore, one needs to solve only the homogeneous diffusion equation in an infinite unlimited region. It appears likely that simple, efficient and accurate procedures can be developed to take advantage of this observation.

The expression of the entirety of the general viscous flow problem in the form of integral representations offers an inherent flexibility in accommodation of complex boundary shapes and in the spacing of the nodes in accordance with the local length scales of the problem. This flexibility results from the fact that, like other finite element procedures, the computations required are those of numerical integration rather than numerical differentiation. The integral representation procedure, however, offers a number of advantages not available with the usual finite element procedures.

4.10 ILLUSTRATIVE PROBLEMS

The usefulness of the integral representation approach in the solution of the general viscous flow problem has been demonstrated by application of this approach to flows around finite flat plates, circular cylinders and airfoils at high angles of attack. Various techniques described in Section 4.9 have been used to obtain numerical solutions for each of these problems. In the following, only one set of these results is presented for each problem.

In all the results presented, the solid body is considered to be initially at rest in a fluid also at rest. At a given time level, say $t = 0$, the solid body is set into motion and thereafter kept moving at a uniform and constant velocity. The time-dependent flow induced around the solid body is then simulated numerically. The computations are carried to a sufficiently large time level so that either a steady state solution or a periodic solution is obtained.

4.10.1 Flat Plates

The flow past a finite flat plate at a zero angle of attack provides a good test case for the present approach. Although the flow around the plate remains attached, there exists a well-known boundary layer solution of Blasius against which the asymptotic steady state numerical results can be calibrated. Figure 1 shows a steady state velocity profile at the mid-plate obtained by Sankar (1977) for a Reynolds number, based on the flat plate length and the free stream velocity, of 1000. Sankar's results are in excellent agreement with results obtained by others using various techniques described in Section 4.9. The agreement between Sankar's results and Blasius' solution, also shown in Fig. 1, is good. Sankar's results show an overshoot in the velocity profile, i.e. the velocity values are higher than the free stream velocity at some points in the boundary layer, which is absent in Blasius' solution. Since the favourable pressure gradient caused by the

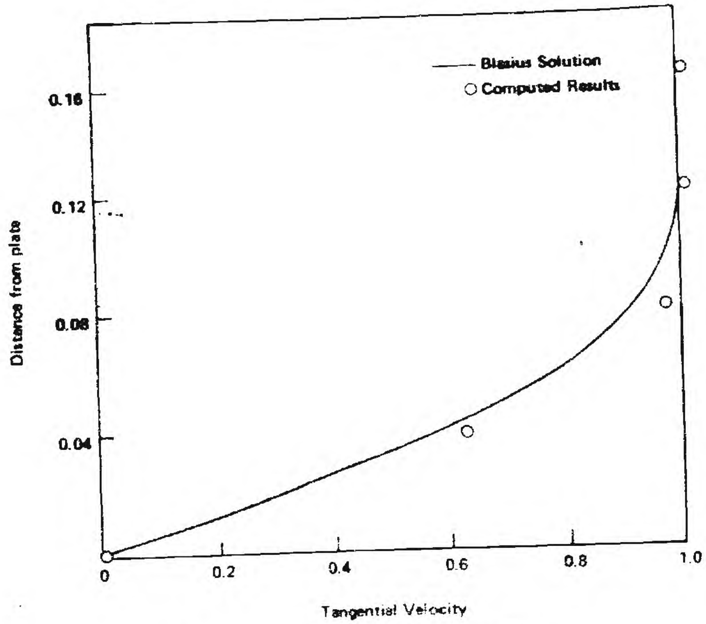

FIG. 1. Mid-plate velocity profile

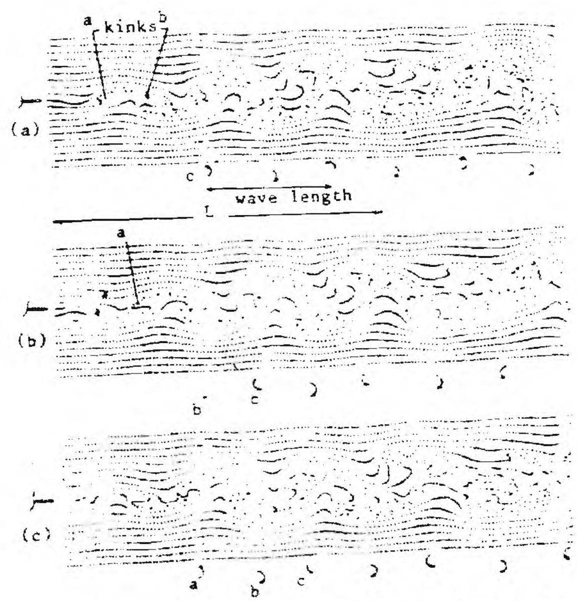

FIG. 2. Roll-up of vortices in the wake of a flat plate

boundary layer displacement effect is significant at the Reynolds number of 1000, this overshoot is expected in all solutions based on the Navier–Stokes equations.

At higher Reynolds numbers, the wake flow behind the flat plate becomes unstable. Figure 2 shows computed filament lines in the wake of a flat plate at a Reynolds number of 14 000 obtained by Gulcat (1981). The roll-up of wake vortices observed is found to be in remarkable agreement with experimental results of Taneda (1958).

4.10.2 Circular Cylinders

Figure 3 shows a comparison between the computed (Sampath, 1977) and experimentally determined (Grove *et al.*, 1964) pressure distributions on a circular cylinder at a Reynolds number of 40 based on the circular cylinder diameter and the free stream velocity. Figure 4 shows a similar comparison between the computed (Gulcat, 1981) and experimental (Linke, 1931) results for the case of a Reynolds number of 40 000. The agreement between

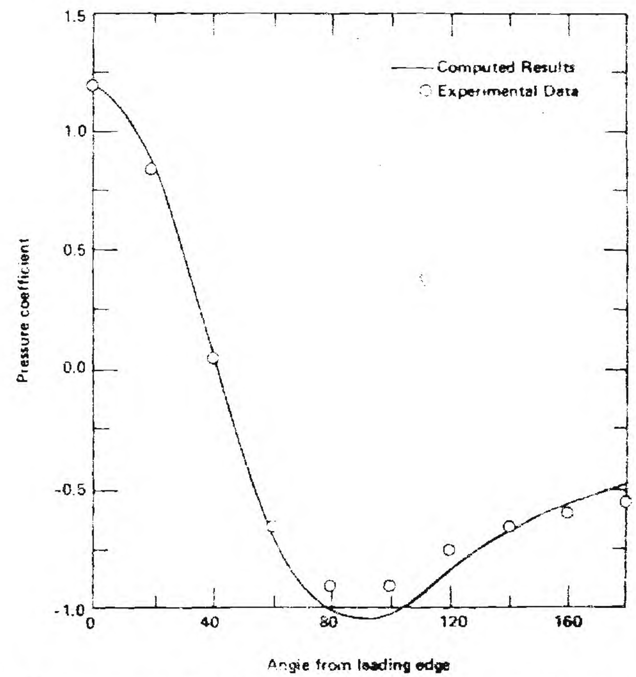

FIG. 3. Surface pressure distribution over a circular cylinder at a Reynolds number of 40.

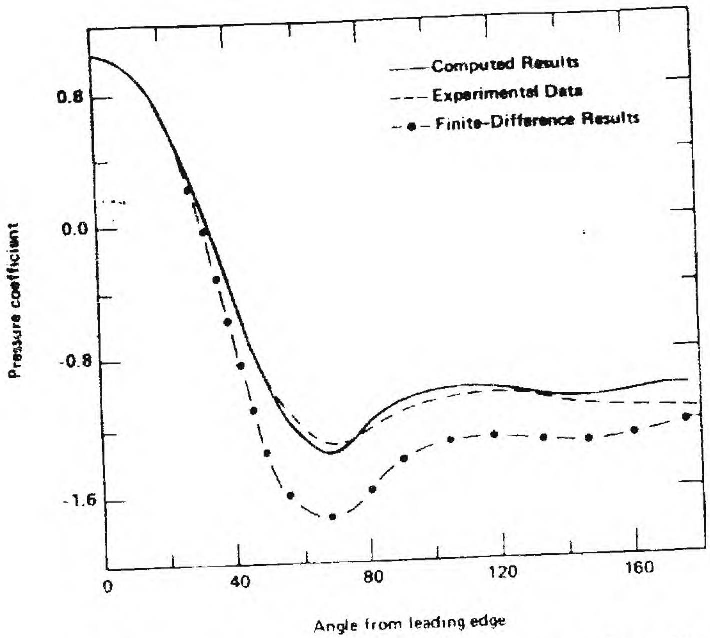

FIG. 4. Surface pressure distribution on a circular cylinder at a Reynolds number of 40 000.

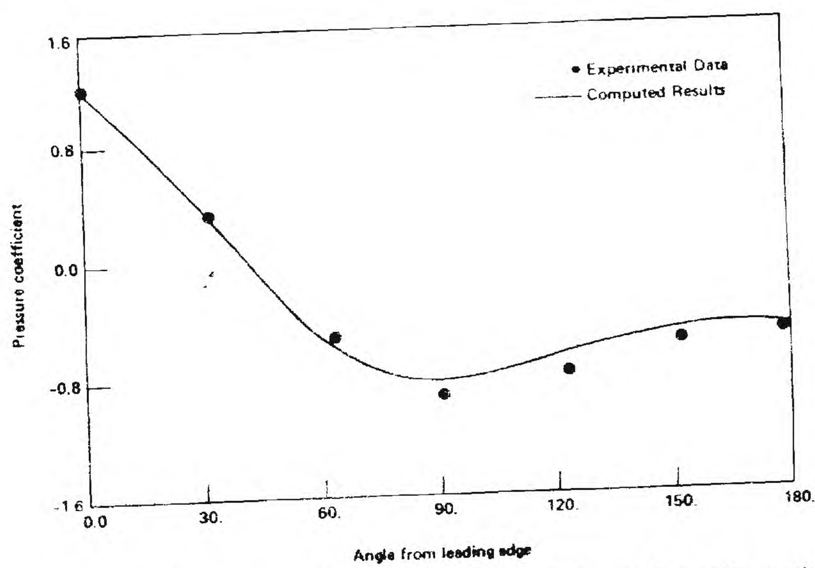

FIG. 5. Surface pressure distribution on a circular cylinder at a Reynolds number of $3{\cdot}6 \times 10^6$.

Gulcat's results and the experimental data is remarkably good. The small disagreement in the region near the rear stagnation point is expected since, in Gulcat's computation, no provision is made to model flow turbulence occurring in the recirculating component and in the wake component of the flow.

Figure 5 shows a comparison between the computed (Sugavanam, 1980) and experimental (Achenbach, 1968) pressure distributions on a circular cylinder at a Reynolds number of $3{\cdot}6 \times 10^6$. A two-equation differential K–ε model (Launder & Spalding, 1974) is used to simulate flow turbulence.

In Fig. 6 a comparison between the computed drag coefficients and the experimental data (Schlichting, 1968) over a range of Reynolds numbers is shown. The good agreement between the data is encouraging. Equally

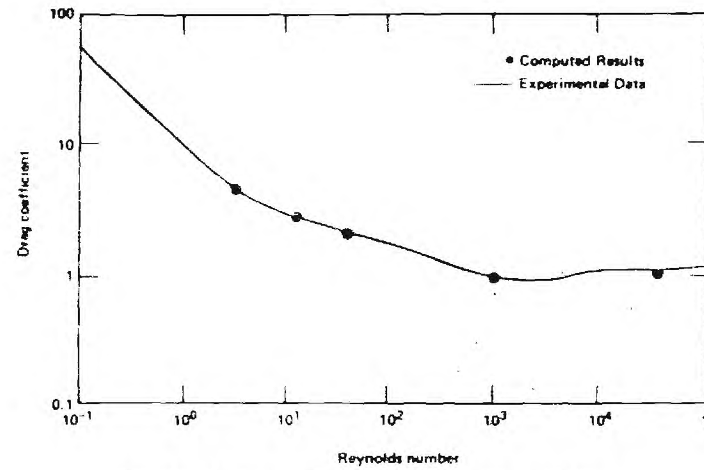

FIG. 6. Drag coefficient for circular cylinder.

assuring is the fact that, by incorporating an increasing number of special techniques made possible by the use of integral representations, each stage of development of the integral representation approach has led to a substantial reduction in the computer time and data storage requirements. In Table 1 are summarised the computer time requirements for computing the steady flow around a circular cylinder at various Reynolds numbers. A peculiar feature appearing in this table is that, at the present, the computation of the flow at the higher Reynolds numbers of 1000 and 40 000 actually requires less computer time than that required at the Reynolds number of 40. This is because at the low Reynolds number of 40, boundary

TABLE I

COMPUTER TIME REQUIREMENT IN MINUTES OF CDC-6600 CPU FOR THE CIRCULAR CYLINDER PROBLEM

Reynolds number	*1970*	*Year* *1975*	*1980*	*1985*
40	95	40	25	(20)[a]
1 000	(300)	80	20	(6)
40 000	(1 200)	(250)	20	(6)
3 600 000	—	—	80	(15)

[a] Numbers in parentheses are estimated values.

layer simplifications are not justifiable in the attached region and the hybridisation of the boundary layer and Navier–Stokes approaches is not useful.

4.10.3 Airfoils

In the sample problems described above, integral representations are utilised for the kinematic part of the computation. The kinetic part of the problem is kept in its differential form and computed using either a finite difference method, a conventional finite element method or a hybrid method. This approach is named the integral-differential approach. In computing flows about airfoils at high angles of attack, Sankar (1977), Sampath (1977), Sugavanam (1980), Gulcat (1981), and El Refaee (1981) used the integral–differential approach. Rizk (1980) used integral representations for both the kinetic part and the kinematic part of the computation.

Figure 7 shows a sequence of flow patterns obtained by Rizk (1980) at different time levels after the impulsive start of the airfoil motion. The airfoil is a 9% thick symmetric Joukowski airfoil at an angle of attack of 15° relative to the free stream. The flow Reynolds number, based on the chord length and the free stream velocity, is 1000. The non-dimensional time level measured from the onset of the airfoil motion is T. The reference time is the chord length divided by the free stream velocity. The stream function values are non-dimensionalised. The reference stream function values are the product of the freestream velocity and the chord length. The streamlines shown are spaced 0·02 apart.

Immediately after the impulsive start, the vorticity is non-zero only at the airfoil surface. The flow in the remainder of the fluid domain is inviscid. The

streamlines shown in Fig. 7(a) are identical to those for a potential flow with zero circulation around the airfoil. The rear stagnation point for this flow is located upstream of the trailing edge on the upper surface of the airfoil.

After the onset of the motion, the rear stagnation point moves rapidly towards the trailing edge. This movement is accompanied by a curling up of the vorticity near the trailing edge of the airfoil. A starting vortex is then formed from the curled-up vorticity and is shed from the airfoil as shown in Fig. 8. The lines in Fig. 8 are equivorticity contours. The starting vortex moves downstream and diffuses rapidly. During the formation and shedding of the starting vortex, the vorticity diffuses from the airfoil boundary into the fluid. Boundary layers are formed around the airfoil. The presence of the boundary layers is evident from the presence of the vorticity around the airfoil shown in Fig. 8 as well as from the displacement of streamlines away from the airfoil surface shown in Fig. 7(b).

The streamline pattern shown in Fig. 7(c) indicates the appearance of a small clockwise rotating separation bubble on the upper surface of the airfoil. This bubble grows in size until the reattachment point reaches the trailing edge, as is shown in Fig. 7(d).

The next stage of development begins when the separation bubble bursts and the reattachment point of the bubble lifts off from the airfoil, as shown in Fig. 7(e). The bursting of the bubble is followed by the shedding of vortices from the vicinity of the airfoil. While the starting vortex is counterclockwise, the vortices shed after the bursting of the bubble are clockwise.

Following the bursting of the clockwise bubble, small counterclockwise bubbles appear. The subsequent lift-off of the counterclockwise bubbles is indicative of the shedding of counterclockwise vortices as shown in Fig. 7(f). The behaviour of the lift force acting on the airfoil is consistent with the observed shedding of vortices during this period. That is, when clockwise vortices are shed, the decrease in the clockwise circulation around the airfoil leads to a decrease in the lift force. Similarly, the increase in the clockwise circulation associated with the shedding of counterclockwise vortices is accompanied by an increase in the lift force.

In Fig. 7(g) a streamline pattern which resembles that shown in Fig. 7(d) is shown. A closed clockwise separation bubble exists on the upper surface of the airfoil. The reattachment point of the bubble is at the trailing edge. The flow has undergone a sequence of events, during which vortices of opposite senses have been shed, between the time levels for Figs. 7(e) and 7(g).

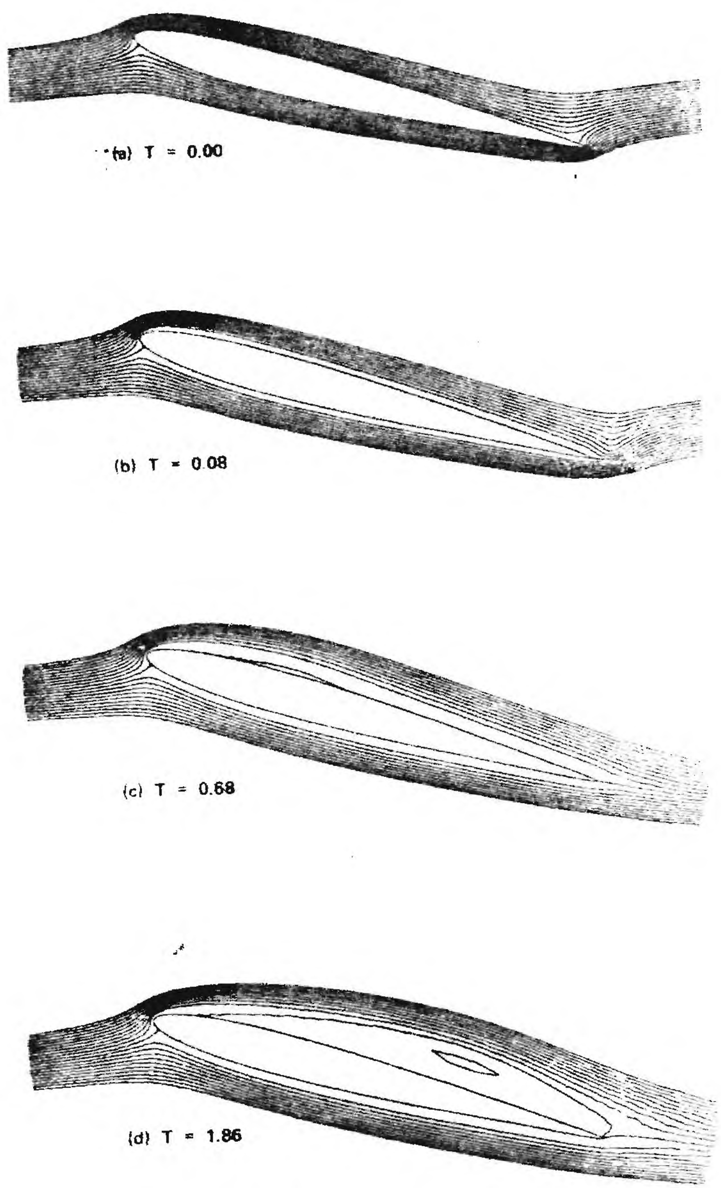

FIG. 7. Streamlines around an airfoil. (a) $T = 0{\cdot}00$; (b) $T = 0{\cdot}08$; (c) $T = 0{\cdot}68$; (d) $T = 1{\cdot}86$; (e) $T = 2{\cdot}54$; (f) $T = 6{\cdot}26$; (g) $T = 6{\cdot}60$.

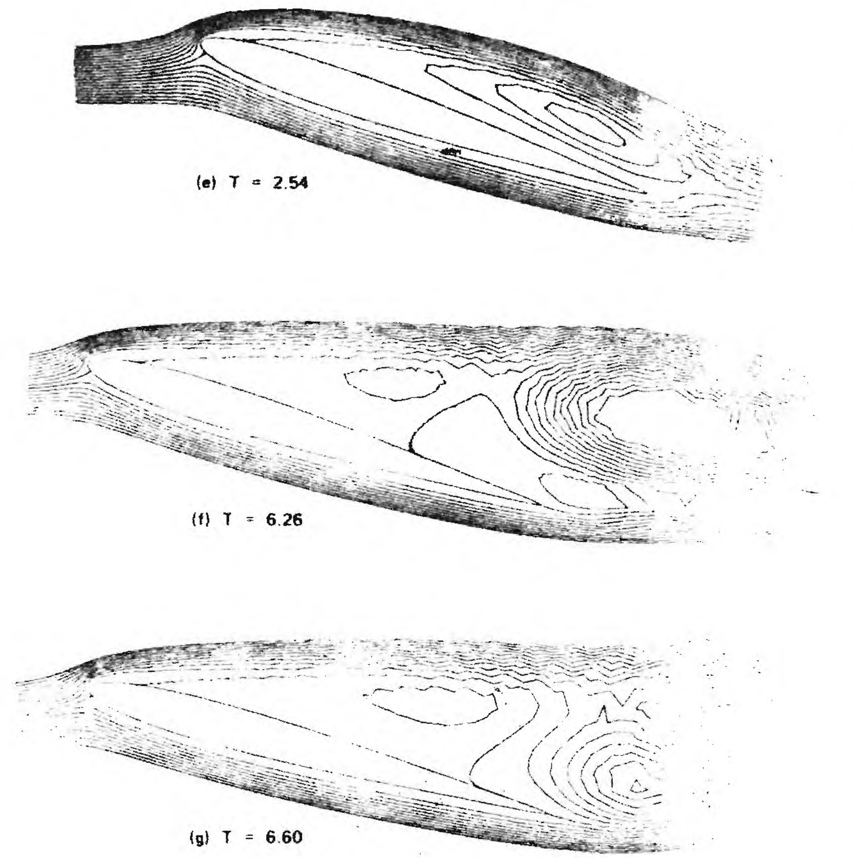

FIG. 7.—*contd.*

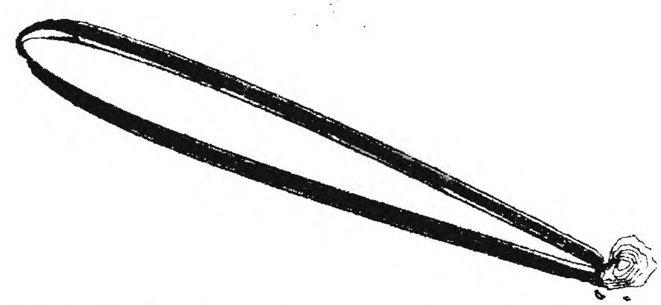

FIG. 8. Equi-vorticity contour around an airfoil at $T = 0{\cdot}08$.

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Progress Report

for the period

July 1 to Sept. 30, 1982

UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Fluid Dynamics Program

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

During the reporting quarter, efforts have continued in the further development of the general theory for time-dependent aerodynamic forces and moments in viscous flows. Several problems involving vorticities confined to narrow bands have been analyzed. The problem of unsteady shedding of vortices near the trailing edge of airfoils, that is, the problem of unsteady Kutta condition, has been scrutinized. Various numerical procedures for this problem are being developed.

Two articles prepared earlier have appeared in the published literature. These are:

1. J. C. Wu, "Principal Solutions and Finite-Element Procedures," Proceedings of the 4th International Symposium on Finite Element Methods in Flow Problems, University of Tokyo Press, 1982, pp. 1063-1070.

2. J. C. Wu, "Sources, Sinks, Vortices and Flow Computations", Proceedings of the International Conference on Finite Element Methods, Gordon and Breach, Science Publishers, Inc., 1982, pp. 540-545.

Manuscripts of these two articles were submitted earlier to ONR.

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Progress Report

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UNSTEADY VISCOUS FLOW

Prepared by

James C. Wu, Professor

School of Aerospace Engineering

Georgia Institute of Technology

for

The Fluid Dynamics Program

Office of Naval Research

800 North Quincy Street

Arlington, Virginia 22217

The attached manuscript, prepared at the invitation of the 3rd International Symposium on Numerical Methods for Engineering, summarizes the efforts that were carried out under the present program. This paper will be presented in March of 1983. The most significant accomplishment during the present quarter is the clarification of the role of boundary vorticity on the unsteady development of flows. The manuscript discusses this problem in detail.

The principal investigator has been asked by the editors of the International Journal for Numerical Methods in Fluids to prepare an extended manuscript for the Journal. Some efforts have been devoted to the initial writing of a paper reviewing the recent progress made under the present project.