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ON THE GENERATION OF GAUSSIAN RANDOM PROCESSES IN A POSITION PARAMETER

A THESIS

Presented to

The Faculty of the Graduate Division

by

William Alton Yates

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Electrical Engineering

Georgia Institute of Technology

June, 1966

ON THE GENERATION OF GAUSSIAN RANDOM PROCESSES

IN A POSITION PARAMETER

Approved:

Chairman

Date approved by Chairman: May 10, 1966

ACKNOWLEDGMENTS

This research project is part of a larger study sponsored by the Simulation Branch of the Computation Laboratory at the George C. Marshall Space Flight Center in Huntsville, Alabama, under Contract No. NAS8-2473 to the Georgia Institute of Technology.

It is a pleasure to express my sincere appreciation to my thesis advisor, Dr. David L. Finn, for his assistance and encouragement in the development of this thesis. I also wish to express my gratitude to Drs. Joseph L. Hammond, Jr. and T. M. White for their service as members of the reading committee.

Special appreciation is given to the Hughes Aircraft Company, Ground Systems Group in Fullerton, California, for the Staff Doctoral Fellowship which provided the opportunity to pursue this research.

I wish to thank Mrs. Betty Sims for her services in typing this thesis.

Finally, I wish to convey special thanks to my wife, Jeanie, who provided the understanding, encouragement and inspiration which made this work possible.

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SUMMARY

The objective of this research is to develop and investigate a procedure for synthesizing slowly time-varying filters which generate Gaussian random processes for use in simulation studies. The procedure has application in many physical problems which require laboratory simulation in lieu of expensive live tests. The class of stochastic processes for which the synthesis procedure is valid are those in which only the first and second statistical moments are of interest. It is known that the first two moments of any stationary or nonstationary random process can be represented as the output of a linear filter whose input is stationary Gaussian white noise. The synthesis procedure provides a technique for approximating prescribed stationary Gaussian random processes that depend only on a position parameter. The position parameter dependence on time allows the generation of the random process in the time domain. The slowly time-varying filters are implemented by use of conventional analog computer components, e.g., amplifiers, integrators, multipliers, resolvers, etc.

The Gaussian random process $g(x)$ to be simulated is specified by its first two statistical moments, the mean M_g and the autocovariance function r_g. These statistics are specified in the position parameter x. In addition, the position parameter time dependence $x(t)$ is assumed known. Application of the procedure accomplishes the shaping of the output of a stationary Gaussian white noise source with an appropriate analog computer network. The prescribed statistics and the position

parameter time dependence are used in the procedure to determine the structure and the inherent parameters of the shaping network. Since the required random process is stationary in the parameter x, an appropriate network transfer function is obtained by factoring the power spectral density function of the process. The parameters of the network are modified by the position time dependency to provide a time-varying filter whose output is in general nonstationary in time. The resulting output random process is an approximation to the composite random process g(x(t)). The position time dependency used in the procedure is the velocity v(t) and is related to the position parameter x(t) by its first time derivative.

The computer network has two inputs; the first is stationary Gaussian white noise, and the second is the velocity profile v(t). The use of the velocity as a controller or auxiliary input is advantageous since the effects of the random process on a vehicle translating through it may be observed in simulation studies with many different vehicle velocity profiles without redesigning the basic network. The external generator is all that need be modified. The output of the computer network is a Gaussian random process that is stationary in the position parameter x and, in general, is nonstationary in the time parameter t. The output is an approximation of the composite random process g(x(t)) and hence its statistics are approximations of the prescribed random process statistics.

In some cases the synthesis procedure results in a computer network whose output is an exact representation of the composite random process. In general, however, the output random process is an approxi-

mation. When the velocity, $\dot{x}(t) = v(t)$, is constant the synthesis procedure yields an exact result. The general restriction on the class of velocity profiles in order that the procedure yield a close approximation to $g(x(t))$ is that the first time derivative, $\ddot{x}(t) = \dot{v}(t)$, have a small value. Experimental and analytical examples of both exact and approximate representations of $g(x(t))$ are presented.

In an application of the synthesis procedure, a simulation study is to be made of a random process known to be an active random disturbance in a particular region. The statistics of the process are stationary in the spatial coordinate x of the region. A vehicle translates through the region in time and experiences the effects of the random disturbance. The synthesis procedure produces a mechanization system whose output simulates the effects of the prescribed random disturbance on some sensing element on the vehicle body whose variable location is specified by the position parameter x. The prescribed random process is $g(x)$. Since the position of the sensing element is specified by $x(t)$, the instantaneous effect of the random disturbance g at a particular time t_o is given by $g(x(t_o))$, which is recognized as a composite function. The random disturbance for all time t is given by $g(x(t))$. Thus the mechanization system generates the composite process $g(x(t))$ when one input is stationary Gaussian white noise and the second input is $v(t) = \frac{dx(t)}{dt} = \dot{x}(t)$. When $\ddot{x}(t)$ is constrained to have a small value the procedure yields a good approximation to the composite function $g(x(t))$.

The approach used admits constraints on the class of prescribed random processes which can be generated.

1. The prescribed random process is stationary in the position

parameter x, hence the statistics are independent of the origin of the parameter x. The random process may become nonstationary as position x of the sensing element translates in time.

2. The random process is specified by its autocovariance function $r_g(x_1,x_2)$ and its mean value M_g. Without loss of generality the procedure presented is concerned with the realization of random processes g(x) with mean zero. Random processes having nonzero mean can be realized as the sum of the random process generated by this technique and the output of a source having output numerically equal to M_g. For random processes with mean zero, it is sufficient to specify the autocorrelation function $R_g(x_1,x_2)$ since it is equal to the autocovariance for this condition. Nonzero mean random processes are first translated to zero mean processes and the procedure is continued as before. The nonzero mean is then added back to the output random process generated by the procedure.

3. The random process is stationary in the position parameter x, hence the power spectral density of the random process is found by the direct Fourier transform of the autocorrelation function. The power spectral density is constrained to be real, non-negative, an even function of frequency, and expressible as a ratio of polynomials in frequency ω.

4. In all cases the synthesis procedure yields an exact representation of the composite process when the second time derivative of position x(t) is zero. In some cases the synthesis procedure yields an exact representation of the composite process when x(t) is arbitrary except that $\dot{x}(t)$ is not allowed to change sign. In general, however, the second time derivative of position x(t) must be restricted to small

values for the synthesis procedure to yield a good approximation of the composite random process.

As an outgrowth of the study, a more general synthesis procedure established by Webb, Hammond, et al., was extended to provide an exact representation of the composite random process $g(x(t))$. This state variable approach extension yields an analog computer mechanization system whose output is exactly the composite random process with the constraints (1), (2), and (3). The position $x(t)$ is arbitrary except that $\dot{x}(t)$ is not allowed to change sign. The extension of the general synthesis procedure is called "the parallel procedure" throughout this study. It is used to provide the exact nth order differential equations which are used to perform a comparative analysis with the nth order differential equations derived by the approximate synthesis procedure.

The parallel procedure is much more involved mathematically and leads to a more complex analog computer mechanization than the approximate synthesis procedure. First and second order analytical examples are presented with the comparative analysis made between the two procedures where significant. The complexity of the parallel approach is certainly offset when precise representation of $g(x(t))$ is required for a simulation study.

The measurement system which outputs the experimental data is presented. The measurement system and the mathematical calculations are not unlike those made by Bendat and Bryan. Several velocity profiles are used in both the first order and second order examples. The theoretical and experimental data for the autocovariance function are illustrated for each example. The process mean and standard deviation

with their respective confidence level error boundaries are illustrated for each example. The analytical and experimental examples present the salient features of the approximate synthesis procedure.

The main advantage of the approximate synthesis procedure is its simplicity with its principal disadvantage being its restriction to stationary processes in the position parameter x. On the other hand the parallel procedure is attractive in that it applies to stationary or nonstationary processes in which the autocovariance function $r_g(x_1,x_2)$ can be expressed as a finite sequence of terms which are separable in the x_1 and x_2 variables, i.e.,

$$r_g(x_1,x_2) = \sum_{i=1}^{n} \phi_i(x_1)\gamma_i(x_2).$$

It has the main disadvantage of being quite complex mathematically and leads to an involved analog computer mechanization system.

CHAPTER I

INTRODUCTION

In this chapter the objectives of the research are stated. The need for analog computer simulation studies of certain physical problems which occur frequently in flight tests, environmental tests, quality control, etc., is presented. An approach for the synthesis of time-varying filters adaptable for mechanization on an analog computer which can perform laboratory simulation of the physical problem is stated and some of the many possible areas of application are indicated. Previous and current work by other authors that is related to the research project is cited and summarized.

Statement of the Problem

The objective of the research is to develop and investigate a procedure for synthesizing slowly time-varying filters which generate nonstationary Gaussian stochastic processes. The time-varying filters are mechanized on analog computers using conventional analog computer elements such as multipliers, resolvers, amplifiers, integrators, etc. The procedure has application in many physical problems which require laboratory simulation studies in lieu of expensive live tests.

In a certain class of physical problems a vehicle experiences a random disturbance g as a function of its position parameter x. The statistics of the random disturbance in the parameter x are known. The translation of the vehicle through the random disturbance is given by

the position time relationship x(t). However, the position time relationship is not specified *a priori*. The simulation task is to generate a nonstationary Gaussian random process in the time domain which characterizes the random disturbance g(x) as x varies with time, a variation not previously specified. In the general problem the random disturbance g is a stochastic process that may depend on several position parameters as well as time given by g(x,y,z;t). In the problem of this research g is taken to depend only on x. The random process g(x), specified by its autocovariance function $r_g(x_1,x_2)$ and its mean value M_g, represents the random disturbance on the vehicle at each position x. The composite function g(x(t)) then represents the random disturbance on the vehicle at each instant of time t. Since the composite function g(x(t)) depends only on the time parameter t it can be simulated by an analog computer directly in the time domain.

The lack of *a priori* knowledge about x(t) requires that the analog computer mechanization have two inputs. The first is stationary Gaussian white noise w(t). The second, the auxiliary controller, is dependent on x(t). For the procedure presented in this research the dependency used for the auxiliary controller is the velocity function, related to x(t) by

$$v(t) = \frac{dx(t)}{d(t)} \qquad (1\text{-}1)$$

or by

$$x(t) = \int_{t_o}^{t} v(u)du + x(t_o) \ . \qquad (1\text{-}2)$$

This choice of dependency has direct utility in the physical problems of interest since a velocity sensor output from a vehicle translating through a random disturbance is perhaps more likely to be available than position information. Figure 1-1 illustrates a system block diagram of the problem. Figure 1-2 illustrates a typical sample function of the random process g(x). It is not unlike a sample function from more familiar random processes in the time domain.

The procedure is restricted to random processes g that are stationary in the position parameter x, but g(x(t)) may become nonstationary in the time parameter t depending on the nature of the position time relationship. The procedure yields in general a random process which is an approximation of the composite random process g(x(t)). In some cases, however, the procedure yields an exact result, i.e., identically g(x(t)). Analytical examples of both approximate and exact results are presented in Chapter III.

Applications

The need for simulation of nonstationary random processes arises quite naturally since in nature most physical systems experience nonstationary disturbing phenomena. Laboratory simulation of various random processes provides a means of eliminating the expensive live tests when trying to prove a system response in a random disturbance environment.

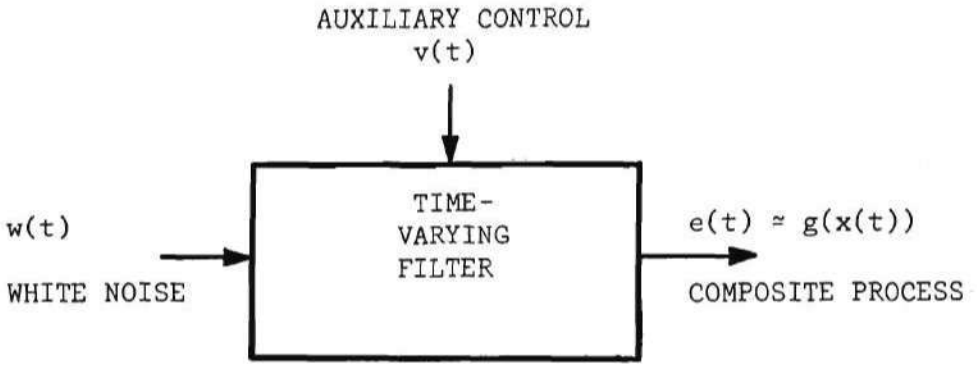


Figure 1-1. System Block Diagram.

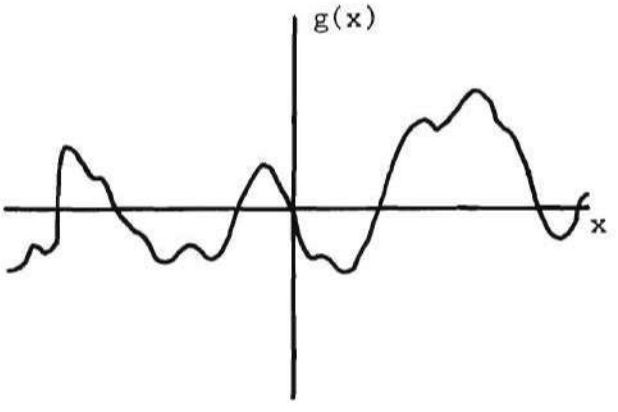


Figure 1-2. Typical Sample Function of the Random Process g(x).

The precise simulation of a specified random process is sometimes mathematically involved and requires availability of extensive analog equipment. A simplified procedure for simulating approximately a specified random process lends itself toward a reduction of analog equipment required for the simulation.

A study of possible applications of the procedure indicates that the mechanization procedure has substantial potential usefulness. Many different physical processes conceptually might be represented as random processes depending on a position parameter. In all of the suggested areas, the sample functions of the random process represent some type of random irregularity whose value is dependent only upon the position with respect to a spatial coordinate system. Two areas of application are suggested with a brief explanation of each.

Vibration Studies

The sample functions might represent the random surface variations on an airport runway, a superhighway, or a railroad roadbed. The airplane, automobile, or train would experience the random disturbance as it translated over the surface. For a wide range of velocity profiles, the effects of the random surface irregularity upon the translating vehicle could be studied in a laboratory simulation of the random process by once designing the time-varying filters and simply changing the velocity profile generator.

Quality Control Studies

In quality control studies, the sample functions might represent:

1. random surface irregularities in a channel or tube that is guiding the flow or movement of a liquid or gaseous substance;

2. random variations in the density or hardness of a solid material being processed by a machine tool, i.e., lathe, mill, saw, drill press, etc.; or

3. random variations in the width or thickness of a long ribbon of material subject to a rolling operation in an industrial plant, i.e., textiles, steel, paper, etc.

In all of the cases mentioned, the sample functions represent some type of random irregularity or disturbance whose value is dependent only upon the position of some sensor with respect to a spatial coordinate system. The effects of the disturbance could be studied as a function of the rate of movement of the material or substance.

Related Work

The related work is reported in four subdivisions of the field and a reasonably extensive bibliography is included at the end of this research presentation. The research project presented is part of a larger study that has been sponsored by the Simulation Branch of the Computation Laboratory at the George C. Marshall Space Flight Center in Huntsville, Alabama, under Contract No. NAS8-2473 to the Georgia Institute of Technology.

Several technical reports related to the general problem of the generation of nonstationary Gaussian random processes are now briefly summarized.

Simulation

The first area of effort in the field of stochastic processes and the one primarily related to this research is the field of simula-

tion. The references of current import follow: The work of Webb, et al. (1), presents a synthesis method for the generation of nonstationary time parameter random processes having prescribed first and second statistical moments. The autocovariance function is expanded into a finite series from which the coefficients of a differential equation can be found to characterize a time-varying filter to generate the process. This work was also presented by Webb (2). This particular approach is extended in Chapter IV of this presentation to include the composite function and to provide an exact solution for the analytical examples investigated by the approximate procedure.

The work of Finn (3) presents a special class of stochastic processes designated as partially stationary stochastic processes. The class is potentially useful in the development of methods of generating nonstationary stochastic processes for simulating the random wind disturbances which affect a rocket during flight. The work of Finn and Yates (4) is a prelude to the comprehensive details of this research material, which presents the problem of the generation of random processes in a position parameter. The work of Johnson (5) and Johnson and Loftin (6) is also related to the specific problem of generation of random processes for simulation purposes. Other authors in this category are Korn and Korn (7, 8, and 9) and Bendat (10).

Analysis and Measurement

The second area of effort in the field of stochastic processes is the area of statistical analysis and measurement of random processes. This category includes techniques and data processing schemes both analog and digital used to analyze and measure random processes, i.e., to

determine process statistics, to determine stationarity and nonstationarity, to apply parametric curve fitting techniques, etc. These techniques have been applied to many types of random processes whose sources are quite varied, i.e., flight dynamics, acoustic research, seismology recordings, structural vibration data, etc. Directly related to the techniques used in this research for the measurement and analysis of the generated random processes is the work of Bryan (11). The Measurement Analysis Corporation of Los Angeles is also very active in this area. Some of the current work by Bendat, et al. (12, 13, 14, and 15), is related to the analysis techniques of this research.

Stochastic Control Systems

The third area of effort in the field of stochastic processes is the area of stochastic control system theory. This is a relatively new approach to design and analysis of control systems using state variable techniques in the time domain rather than in the conventional frequency domain. Some of the authors that have contributed in this area are Laning and Battin (16), Zadeh and DeSoer (17), Leondes (18), Solodovnikov (19), and Webb (2). A current contribution that presents the application of state variable theory to analog computer systems is authored by Hammond (20). This particular work backgrounds the derivation and presentation in Chapter IV.

Filter Optimization Techniques

The fourth area of effort in the field of stochastic processes is the area of filter optimization theory and techniques. This work largely presents techniques and considerations which extend the basic optimum filter theory of Wiener (21) to include filter systems with

nonstationary inputs. Authors mentioned in this category are Zadeh (22), Davis (23), Lampard (24), Urkowitz (25), and many others included in the "other references" section of the Bibliography.

General Approach to Problem

The general approach to the study and investigation of the research problem previously stated is as follows:

First, the relationship between the specified random process statistics and the approximating physical analog mechanization system establishes the synthesis procedure.

Second, several applicable examples are investigated analytically and experimentally to determine the validity of the mechanization system employed.

Third, a parallel approach is developed and extended which provides an exact representation of the specified random process statistics and the analog computer representation for tnis approach is determined.

Fourth, a comparative analysis is considered which presents the approximate procedure in comparison with the exact parallel procedure.

CHAPTER II

PHYSICAL PROBLEM FORMULATION

In this chapter a general physical description and a summary mathematical description of the physical problem are given. The physical restrictions and mathematical assumptions are stated. The synthesis procedure is outlined, and the class of problems for which it is applicable and useful is indicated. The relationship of the procedure to another more involved approach is discussed and some of the limitations of each approach are presented.

Mathematical Statement and Assumptions

The concept of the problem is made quite precise by identifying all of the constraints, assumptions, and definitions that pertain to it. To show specifically the nature of the problem, one of the areas of application presented in Chapter I is selected and detailed. The physical problem that frames the remainder of this research presentation is that of a vibration disturbance. The Gaussian random process g affects a vehicle in the position parameter x as the vehicle translates across the region where the random process is active.

Description

A stochastic process (or random process) $g(x)$, defined for $-\infty < x < \infty$, is an indexed collection of random variables. The random process may be characterized as an ensemble of sample functions with an associated probability measure. The sample functions of the random

process g(x) are dependent on the position parameter x. Sample functions of the vibration disturbance random process under consideration might typically have an appearance as illustrated in Figure 2-1. The sample functions represent conceptually the value of the random disturbance as a function of the position x. A particular point on the vehicle body houses a velocity sensing element. As the vehicle translates across the random process g(x) the position of the sensing element is represented by the position coordinate x. The sensing element experiences the irregularity of the random process g(x) at each position x. The sensor position time variations can be described as a deterministic function of time x(t), i.e., the instantaneous location of the sensing element is x(t). Hence, the effect of the random process g on the sensing element at the instant of time t is completely described by the composite random process g(x(t)). A typical sample function for the random process g(x) is shown in Figure 2-2. An assumed position time relationship x(t) for the sensing element is also illustrated. The position parameter process g(x) is then transformed into a time parameter process g(x(t)) by the composition of x by g. The effect of the random disturbance at position x' is given by g(x') and the instantaneous effect of the disturbance at the time t' is given by g(x(t')).

The fact that the composite process g(x(t)) is dependent only upon the time parameter t provides the potentiality that the effect of the random process g(x) on the sensing element having position x(t) may be simulated by an analog computer system operating directly in the time domain.

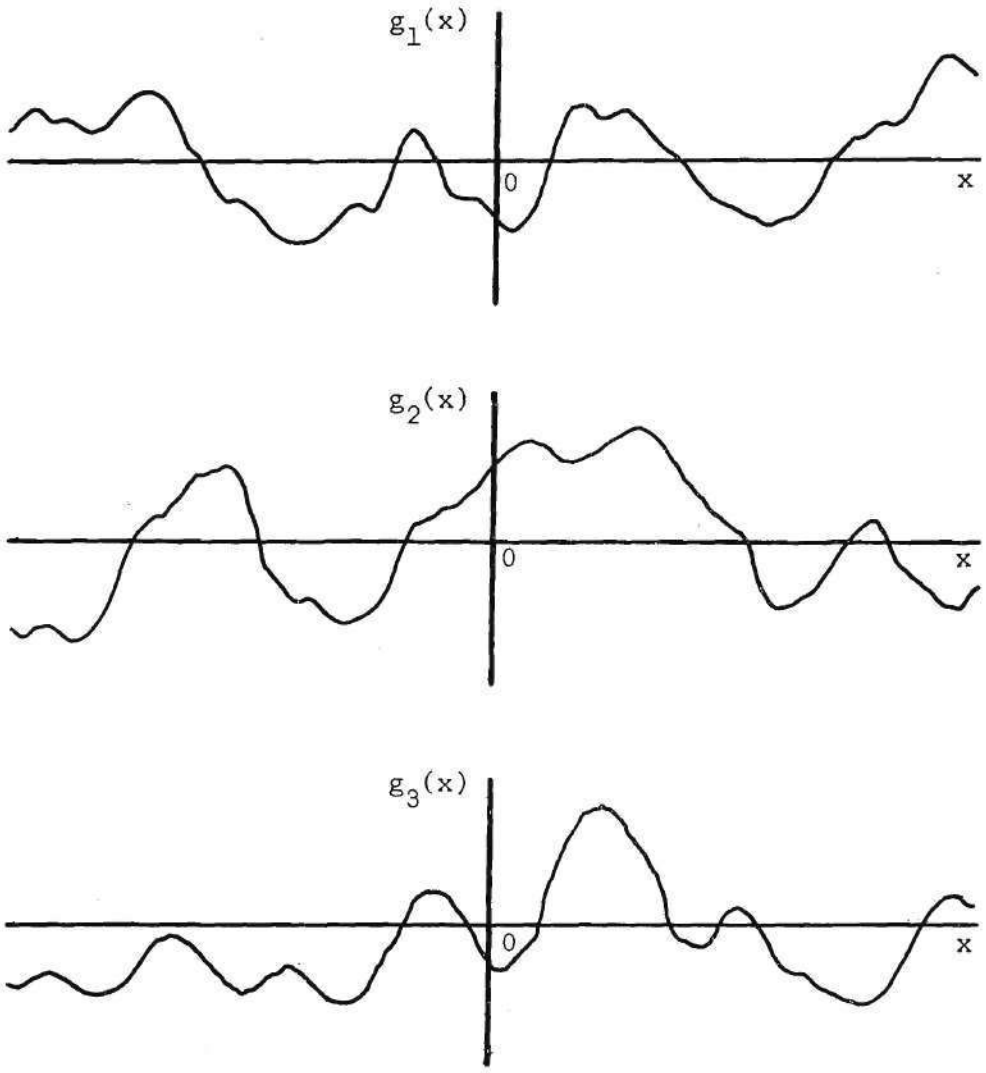


Figure 2-1. Sample Functions of a Random Process Depending on a Position Parameter.

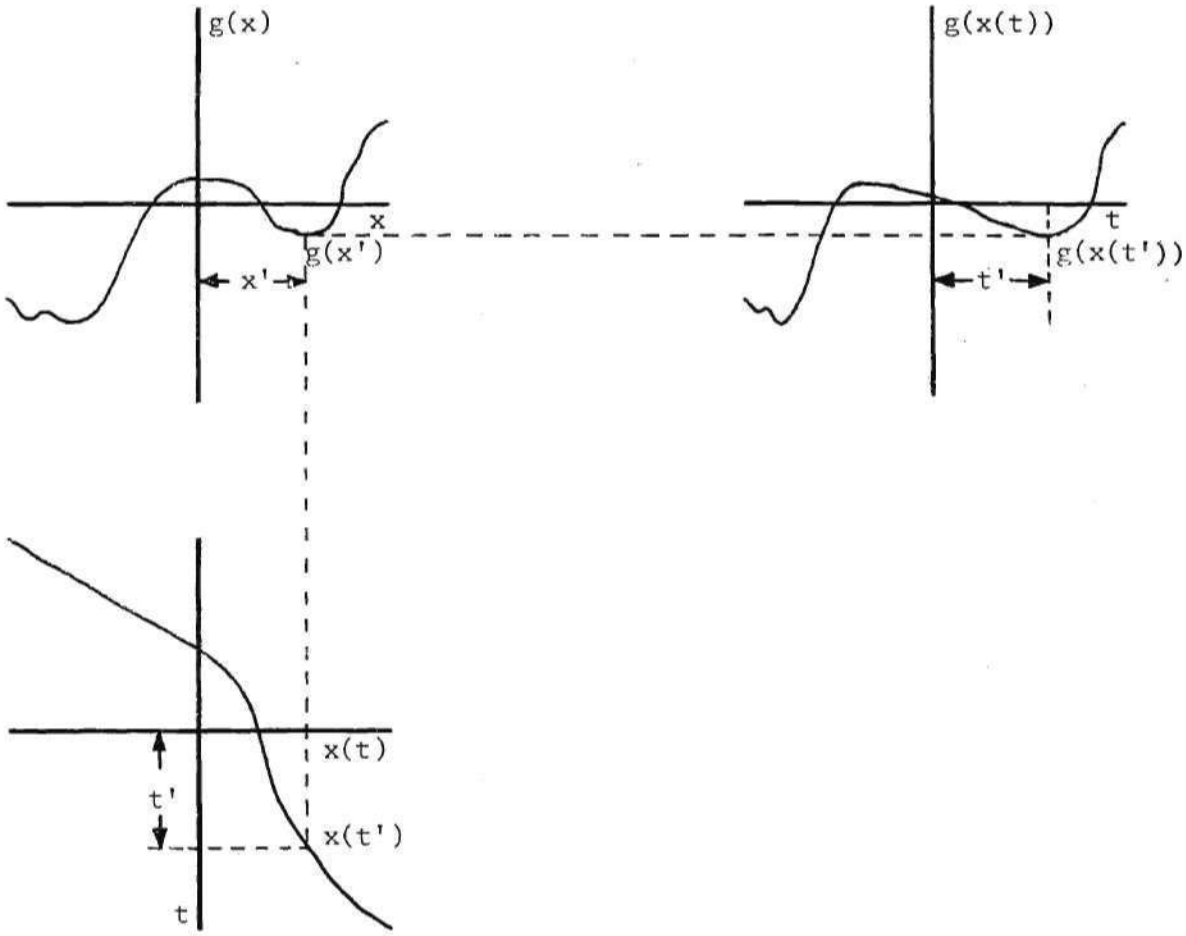


Figure 2-2. Transformation of a Position Parameter Process g(x) into a Time Parameter Process g(x(t)).

Characterization and Constraints

The generation of a stationary Gaussian random process $g(x)$, specified by its mean M_g and its autocovariance function $r_g(x_1,x_2)$, is of interest. The mean M_g of the random process $g(x)$ is defined as the expected value of the random variable generated by the random process g for each fixed position x and is written as

$$M_g = E\{g(x)\} , \tag{2-1}$$

where $E\{\cdot\}$ denotes the expected value operation. The autocovariance function $r_g(x_1,x_2)$ of the random process $g(x)$ is defined as the expected value of the product of the random variables generated by the random process g about its mean at two fixed positions x_1 and x_2, written as

$$r_g(x_1,x_2) = E\{[g(x_1) - E\{g(x_1)\}][g^*(x_2) - E^*\{g(x_2)\}]\} . \tag{2-2}$$

The asterisk denotes the complex conjugate (see Davenport and Root (26, page 59)); however, only real random processes are of interest so Equation 2-2 is written as

$$r_g(x_1,x_2) = E\{[g(x_1) - E\{g(x_1)\}][g(x_2) - E\{g(x_2)\}]\} . \tag{2-3}$$

Equation 2-3 reduces to the following expression

$$r_g(x_1,x_2) = E\{g(x_1)g(x_2)\} - E\{g(x_1)\}E\{g(x_2)\} . \tag{2-4}$$

The autocorrelation function of g(x) is given by

$$R_g(x_1,x_2) = E\{g(x_1)g(x_2)\} , \tag{2-5}$$

which is the first term on the right hand side of Equation 2-4, hence the autocovariance is written

$$r_g(x_1,x_2) = R_g(x_1,x_2) - E\{g(x_1)\}E\{g(x_2)\} . \tag{2-6}$$

The random processes of interest are wide-sense stationary in the position parameter x, hence the first two statistical moments, the mean and the autocovariance function, are independent of the origin of the parameter x (see Davenport and Root (26, page 159)). For the wide-sense stationary process g(x) the mean value is constant, i.e.,

$$M_g = E\{g(x)\} = \text{constant}, \tag{2-7}$$

and the autocovariance function depends only on the difference of the two positions x_1 and x_2, i.e.,

$$r_g(x_1,x_2) = r_g(\tau) = R_g(\tau) - M_g^2 , \tag{2-8}$$

where $x_1 - x_2 = \tau$.

There is no loss of generality when the constant of Equation 2-7 is zero. If a process with a nonzero mean is specified a new variable $\bar{g}(x)$ is defined where,

$$\bar{g}(x) = g(x) - M_g \quad . \tag{2-9}$$

The random process $\bar{g}(x)$ is generated and the output of a source whose value is equal to the mean M_g is added back into the random process $\bar{g}(x)$, thus yielding the original random process $g(x)$. For zero mean processes the autocovariance function is equal to the autocorrelation function. For nonzero mean processes the autocorrelation function is determined from Equation 2-8.

A Gaussian random process is uniquely determined by its mean and its autocorrelation function (see Davenport and Root (26, Chapter 8)). Hence, the generation of a Gaussian random process with mean M_g and autocorrelation function R_g is sufficient for the purposes of this research.

Since $g(x)$ is wide-sense stationary, the power spectral density $S_g(\omega)$ can be related to the autocorrelation function $R_g(\tau)$ by the direct and inverse Fourier transform relationships

$$S_g(\omega) = \int_{-\infty}^{\infty} R_g(\tau)\varepsilon^{-j\omega\tau}\, d\tau \quad , \tag{2-10}$$

and

$$R_g(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega)\varepsilon^{j\omega\tau}\, d\omega \quad , \tag{2-11}$$

respectively (see Davenport and Root (26, page 104), and Papoulis (27, page 338)).

Of subsequent interest is the reduction of the power spectral density $S_g(\omega)$ to the following form

$$S_g(\omega) = H_g(p)H_g^*(p) \ , \ p = j\omega \ . \qquad (2\text{-}12)$$

This is accomplished by spectral factorization. The transfer function $H_g(p)$ that results is restricted to be a ratio of polynomials. To satisfy this requirement the power spectral density is constrained to be real, non-negative, an even function of frequency ω, and expressible as a ratio of polynomials in frequency ω. Any autocorrelation function whose direct Fourier transform is a power spectral density meeting these constraints is an admissible autocorrelation function. Some of the allowable autocorrelation functions for which the procedure is applicable are given in Chapter III.

Another constraint that arises in Chapter III is a direct result of the technique used for mechanizing the time-varying filter. This constraint is that the position time relationship $x(t)$ be restricted to have a non-negative first time derivative and a small second time derivative.

A summary of the constraints that have developed in the characterization of the problem are:

First, the random process g is wide-sense stationary in the position parameter x.

Second, a specified random process with nonzero mean is translated to a new random process with mean zero. The nonzero mean is added to the new random process generated by the procedure.

Third, the admissible autocorrelation functions are those whose direct Fourier transforms are even, real, non-negative, and expressible as a ratio of polynomials in frequency ω.

Fourth, the position time relationship of the vehicle translating through the disturbance g must have a non-negative first time derivative and a small second time derivative.

Mechanization Procedure Outline

The steps involved in using the approximating procedure are outlined. Details of each step with assumptions, and limitations are discussed in Chapter III. The information available from which the mechanization system is developed includes the autocorrelation function $R_g(\tau)$ (determined from the autocovariance function given by Equation 2-8), the mean M_g, and the set of velocity profiles $v(t)$.

Step One: The power spectral density $S_g(\omega)$ is determined from the autocorrelation function $R_g(\tau)$ by the direct Fourier transform.

Step Two: The power spectral density $S_g(\omega)$ is factored by spectral factorization to determine the transfer function of a fixed parameter linear filter $H_g(p)$.

Step Three: The transfer function $H_g(p)$ is translated in gain and in bandwidth to a translated transfer function $H_v(p)$. The translated transfer function $H_v(p)$ is expressed as a ratio of polynomials.

Step Four: $H_v(p)$ is considered as the ratio of the Laplace transform of the output $e(t)$ to the Laplace transform of the input $w(t)$ of a fixed parameter linear filter. The differential operator $p = \frac{d}{dt}$ is applied to the ratio of polynomials to determine in general an nth order differential equation.

Step Five: The nth order differential equation is reduced to n first order differential equations.

Step Six: The equation(s) are implemented by a generalized analog computer mechanization wherein the v of Step Three is used as an auxiliary controlling input and is allowed to vary with time. The set of v(t) are then generated by function generators and applied as the second input to the filter. When $v(t) = \frac{dx(t)}{dt}$ the output e(t) approximates the composite function g(x(t)).

A Parallel Approach

In parallel with the approximating procedure, an exact approach is developed in order to provide a means of comparison and evaluation from the viewpoints of mathematical complexity, equipment usage, and utility. The exact approach is an extension of the work of Webb (2) and of Webb, et al. (1), to include the composite function g(x(t)).

The parallel approach is more general than the approximate approach and indeed the approximate approach could be considered a special case of the parallel approach. The parallel approach is mechanized to provide the same flexibility with respect to having no *a priori* knowledge about v(t) as with the approximate approach. The parallel approach applies to the class of physical problems discussed in Chapter I.

The principal restriction of the parallel procedure is that the autocovariance function $r_g(x_1,x_2)$ must be expressible as a finite series of terms

$$r_g(x_1,x_2) = \sum_{i=1}^{n} \phi_i(x_1)\gamma_i(x_2) \ . \qquad (2\text{-}13)$$

The approach is more general in that the restriction of stationarity in the position parameter x is not required.

The parallel approach has some inherent disadvantages in that it is quite involved mathematically and requires extensive use of analog equipment, but it maintains the advantage of exactness. This approach is developed in Chapter IV.

CHAPTER III

THE SYNTHESIS PROCEDURE AND MECHANIZATION SYSTEM

In this chapter the details of the synthesis procedure are established. One mechanization system on the analog computer of the time-varying filter is discussed and illustrated.

First and second order analytical results obtained by applying the synthesis procedure are presented. The limitations of the results, stability, and errors are mentioned.

The Procedure Details

An outline of the procedure was presented in Chapter II. The steps listed are now expanded and the necessary constraints are included in the appropriate steps.

The specified process statistics are assumed given as the mean M_g and the autocovariance function $r_g(x_1,x_2)$. When a nonzero mean is specified, the random process is translated by the assignments of Equation 2-9 with a resulting mean of zero and autocorrelation function determined by Equation 2-8. Steps One through Six of the procedure are followed, generating the translated random process. The nonzero mean is added to the output random process yielding the specified random process.

The general class of velocity profiles v(t) is assumed known; however, the individual functions need not be specified *a priori*. The procedure leads to a generalized mechanization scheme whereby the composite function g(x(t)) can be simulated.

Step One: The power spectral density $S_g(\omega)$ is determined from the autocorrelation function $R_g(\tau)$ by the direct Fourier transform

$$S_g(\omega) = \int_{-\infty}^{\infty} R_g(\tau)\varepsilon^{-j\omega\tau}\, d\tau \ . \tag{3-1}$$

The admissible autocorrelation functions are those whose power spectral density is real, even, non-negative, and expressible as a ratio of polynomials in frequency ω.

A further restriction for the autocorrelation function to represent a real physical process is that the value of the function at $\tau = 0$ must be greater than or equal to the value of the function for all other values of τ, i.e.,

$$R_g(0) \geq R_g(\tau) \ . \tag{3-2}$$

Examples of the admissible autocorrelation functions are

$$A^2\varepsilon^{-\alpha|\tau|} \ , \tag{3-3}$$

$$A^2\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|} \ ,$$

$$A^2\varepsilon^{-\alpha|\tau|}\cos(\alpha|\tau| + \theta) \ , \text{ and}$$

$$A^2\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|}\cos(\alpha|\tau| + \theta) \ .$$

Appendix A gives the derivation of these basic terms as valid auto-

correlation functions.

Step Two: The power spectral density $S_g(\omega)$, restricted as in Step One, is factored by spectral factorization techniques to determine the transfer function of a fixed parameter linear filter $H_g(p)$ (see Davenport and Root (26, pages 171 and 227)), where

$$S_g(\omega) = |H_g(p)|^2 = H_g(p)H_g^*(p), \; p = j\omega \; . \tag{3-4}$$

The resulting transfer function represents a fixed parameter linear filter whose output has power spectral density $S_g(\omega)$ when the input to the filter is white noise with power spectral density of unity. The transfer function of the time-invariant linear filter is expressed as a ratio of two polynomials as

$$H_g(p) = \frac{a_o + a_1 p + a_2 p^2 + \ldots + a_m p^m}{b_o + b_1 p + b_2 p^2 + \ldots + b_n p^n} \; . \tag{3-5}$$

A technique for determining the a_i's and the b_i's in the transfer function $H_g(p)$ is discussed by Davenport and Root (26, page 233). The expression for $H_g(p)$ is evaluated for two examples in this chapter.

Step Three: $H_g(p)$ is translated in gain and in bandwidth to the new fixed parameter filter $H_v(p)$ by the relationship

$$H_v(p) = \frac{1}{\sqrt{|v|}} \; H_g(p/v) \; . \tag{3-6}$$

The translated transfer function, representing a fixed parameter linear filter, is expressed as a ratio of polynomials by

$$H_v(p) = \frac{1}{\sqrt{|v|}} \frac{a_o + a_1 p/v + a_2 (p/v)^2 + \ldots + a_m (p/v)^m}{b_o + b_1 p/v + b_2 (p/v)^2 + \ldots + b_n (p/v)^n} . \qquad (3\text{-}7)$$

Now if white noise of unity power spectral density is an input to a fixed parameter linear filter with transfer function $H_v(p)$ the output has a spectral density of

$$S_v(\omega) = |H_v(p)|^2 = \left| \frac{1}{\sqrt{|v|}} H_g(p/v) \right|^2 = \frac{1}{|v|} S_g(\omega/v) \ , \ p = j\omega . \qquad (3\text{-}8)$$

Step Four: The differential operator $p = \frac{d}{dt}$ is applied to $H_v(p)$ in order to determine in general an nth order differential equation written in operator form as

$$L_t \, e(t) = N_t \, w(t) \qquad (3\text{-}9)$$

where e(t) and w(t) are the output and input, respectively, of a fixed parameter linear filter. Expanding the operators L_t and N_t, the differential equation is written

$$\sqrt{|v|} \left(b_o v^n e(t) + b_1 v^{n-1} \frac{de(t)}{dt} + \ldots + \frac{d^n e(t)}{dt^n} \right) \qquad (3\text{-}10)$$

$$= a_o v^n w(t) + a_1 v^{n-1} \frac{dw(t)}{dt} + \ldots + a_{n-1} v \frac{d^{n-1}w(t)}{dt^{n-1}} .$$

Here it is assumed, without loss of generality, that m = n - 1 and that $b_n = 1$.

Step Five: The nth order differential equation in Step Four is reduced to n first order differential equations. A set of equations which is equivalent to Equation 3-10 for fixed positive values of v is

$$e_1(t) = e(t) \tag{3-11}$$

$$\frac{de_1(t)}{dt} = \sqrt{|v|}\,(e_2(t) + a_{n-1}w(t) - b_{n-1}\sqrt{|v|}\,e_1(t))$$

$$\frac{de_2(t)}{dt} = v(e_3(t) + a_{n-2}w(t) - b_{n-2}\sqrt{|v|}\,e_1(t))$$

$$\frac{de_3(t)}{dt} = v(e_4(t) + a_{n-3}w(t) - b_{n-3}\sqrt{|v|}\,e_1(t))$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{de_{n-1}(t)}{dt} = v(e_n(t) + a_1 w(t) - b_1\sqrt{|v|}\,e_1(t))$$

$$\frac{de_n(t)}{dt} = v(a_o w(t) - b_o\sqrt{|v|}\,e_1(t)) .$$

Step Six: The n first order equations are implemented by a generalized mechanization scheme discussed in the next section of this chapter. The set of v(t) are developed by function generators and become the auxiliary controlling input for the filter described by $H_v(p)$. When the velocity profile is related to position by

$$v(t) = \frac{dx(t)}{dt} \tag{3-12}$$

or

$$x(t) = \int_{t_o}^{t} v(u)du + x(t_o) \tag{3-13}$$

the output e(t) of the filter closely approximates the composite function g(x(t)). The representation is valid for all positive velocity profiles and the quality of the approximation in most cases is dependent on the value of the second time derivative of x(t). The representation is exact for velocity profiles whose first derivative is identically zero and it is expected to be a good approximation for those velocity profiles whose first derivative is small numerically.

The Mechanization System

The generalized mechanization system to implement the n first order differential equations is presented. The development that led to the general system that approximates g(x(t)) is described. In general the function of time x(t) should be constrained to have a small second derivative with respect to time in order for the output e(t) of the mechanization system to provide a close approximation to the composite

random process g(x(t)). This is to say physically that the vehicle or sensing element should have a small acceleration.

To develop the mechanization system, consider the stationary Gaussian random process g(x) where x is replaced by t. The time dependent process g(t) can be realized by the application of Gaussian white noise of unity power spectral density to a fixed parameter linear filter whose transfer function magnitude squared is

$$S_g(\omega) = |H_g(p)|^2 , \quad p = j\omega . \tag{3-14}$$

Such a transfer function is expressed by Equation 3-5. It may be realized in a variety of ways by the use of conventional analog computer elements (references 7, 8, and 28). Specifically, $H_g(p)$ may be realized by any appropriate mechanization of the differential equation shown in Equation 3-15

$$b_o\, e(t) + b_1 \frac{de(t)}{dt} + b_2 \frac{d^2e(t)}{dt^2} + \ldots + b_n \frac{d^ne(t)}{dt^n} \tag{3-15}$$

$$= a_o\, w(t) + a_1 \frac{dw(t)}{dt} + a_2 \frac{d^2w(t)}{dt^2} + \ldots + a_m \frac{d^mw(t)}{dt^m} .$$

The output of a linear filter having the transfer function $H_g(p)$ with a Gaussian white noise random process input w(t) having a unity power spectral density is a Gaussian random process e(t) having power spectral density $S_g(\omega)$ given by Equation 3-14. The corresponding

autocorrelation function

$$R_g(t_1,t_2) = R_g(t_1 - t_2) = R_g(\tau) \tag{3-16}$$

is given by the inverse Fourier transform of $S_g(\omega)$

$$R_g(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega)\varepsilon^{+j\omega\tau}\, d\omega \ . \tag{3-17}$$

Thus the filter output e(t) is a realization of the random process g(x) where x is replaced by t. It is clear that e(t) thus represents a realization of the composite random process g(x(t)) where the position of the sensing element is described by the function x(t) = t. Figure 3-1 depicts the mechanization of the transfer function $H_g(p)$.

This approach is now extended to include the mechanization of g(x(t)) when x(t) is given by x(t) = vt + x(0) . The quantities v and x(0) are assumed to be arbitrary constants. It is noted that v is the first derivative of x(t) with respect to time. Again the mechanization is accomplished in a straightforward manner by applying Gaussian white noise w(t) with a unity power spectral density to a fixed parameter linear filter having an output e(t). The output is to be given by

$$e(t) = g(x(t)) = g(vt + x(0)) \ . \tag{3-18}$$

The autocorrelation function of the output is $R_v(t_1,t_2)$, but since the

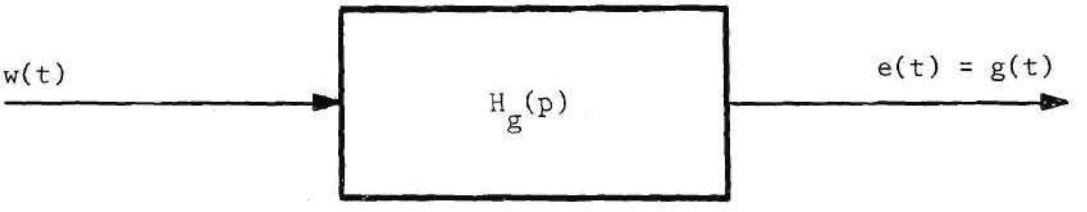


Figure 3-1. Mechanization of the Composite Process g(x(t)) with x(t) = t.

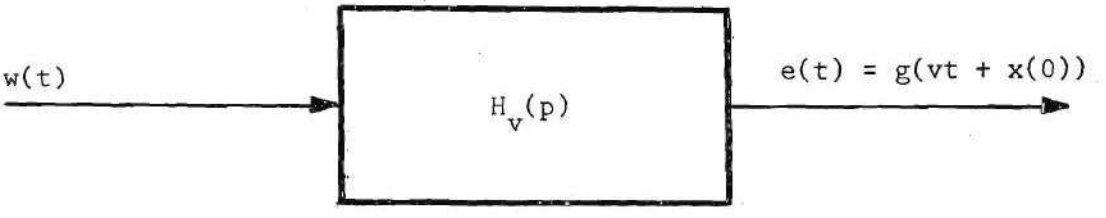


Figure 3-2. Mechanization of the Composite Process g(x(t)) with x(t) = vt + x(0).

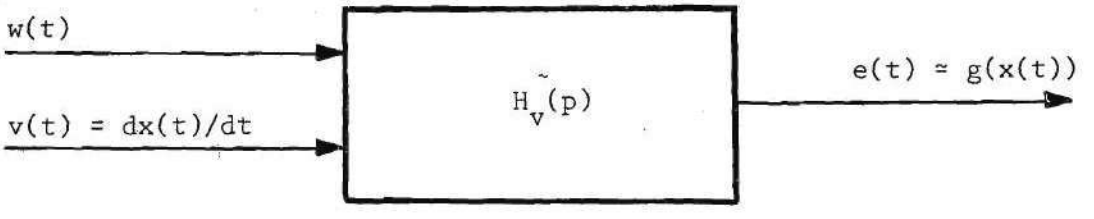


Figure 3-3. Approximate Mechanization of the Composite Process g(x(t)) with x(t) Arbitrary.

random process is stationary, it is expressed as

$$R_v(t_1 - t_2) = E\{g(vt_1 + x(0))g(vt_2 + x(0))\} \qquad (3\text{-}19)$$

$$= R_g(vt_1 + x(0) - vt_2 - x(0)) = R_g(v(t_1 - t_2)) = R_g(v\tau) \ .$$

Hence, a filter whose output e(t) has autocorrelation function

$$R_v(\tau) = R_g(v\tau) \qquad (3\text{-}20)$$

accomplishes the mechanization of g(x(t)) where x(t) = vt + x(0). The power spectral density of e(t) is given by the direct Fourier transform of $R_v(\tau)$ expressed as

$$S_v(\omega) = \int_{-\infty}^{\infty} R_v(\tau)\varepsilon^{-j\omega\tau}\, d\tau \ . \qquad (3\text{-}21)$$

Substituting for $R_v(\tau)$ by Equation 3-20 the power spectral density becomes

$$S_v(\omega) = \int_{-\infty}^{\infty} R_g(v\tau)\varepsilon^{-j\omega\tau}\, d\tau \ . \qquad (3\text{-}22)$$

Letting $\gamma = v\tau$ and $d\gamma = vd\tau$, Equation 3-22 is

$$S_v(\omega) = \frac{1}{|v|} \int_{-\infty}^{\infty} R_g(\gamma)\epsilon^{-j(\omega/v)\gamma}\, d\gamma \ . \qquad (3\text{-}23)$$

Comparing Equation 3-23 with the standard form of the Fourier transform, it reduces to

$$S_v(\omega) = \frac{1}{|v|} S_g(\omega/v) \ . \qquad (3\text{-}24)$$

This mechanization may be accomplished by applying a Gaussian white noise random process w(t) having power spectral density of unity, $S_w(\omega) = 1$, to a linear filter having a transfer function $H_v(p)$, such that

$$|H_v(p)|^2 = S_v(\omega) = \frac{1}{|v|} S_g(\omega/v) = \frac{1}{|v|} |H_g(p/v)|^2 \ , \ p = j\omega \ . \qquad (3\text{-}25)$$

Applying Equation 3-25 to the expression of Equation 3-5, a satisfactory transfer function to accomplish the mechanization is obtained as

$$H_v(p) = \frac{1}{\sqrt{|v|}} H_g(p/v) = \frac{1}{\sqrt{|v|}} \frac{a_o + a_1 p/v + \ldots + a_m (p/v)^m}{b_o + b_1 p/v + \ldots + b_n (p/v)^n} \ . \qquad (3\text{-}26)$$

$H_v(p)$ may be realized in a variety of ways by the use of conventional analog computer elements (see references 7, 8, and 28). Specifically, $H_v(p)$ may be realized by an appropriate mechanization of the following differential equation

$$\sqrt{|v|}\,(b_o v^n\, e(t) + b_1 v^{n-1}\,\frac{de(t)}{dt} + \ldots + \frac{d^n e(t)}{dt^n}) \qquad (3\text{-}27)$$

$$= a_o v^n\, w(t) + a_1 v^{n-1}\,\frac{dw(t)}{dt} + \ldots + a_{n-1} v\,\frac{d^{n-1} w(t)}{dt^{n-1}} \quad .$$

It is assumed with no loss of generality that $m = n - 1$ and $b_n = 1$. Figure 3-2 illustrates this mechanization. It is clear then that the output $e(t)$ is representative of the composite process $g(x(t))$ when $x(t) = vt + x(0)$. It is noted that $H_v(p)$ is realized as a fixed parameter linear filter which requires that a different filter be used for each value of v.

Figure 3-3 depicts an extended approach to the mechanization of the filter which generates the composite random process $g(x(t))$ which does not need to be redesigned for various values of v. In fact, v is used as an auxiliary input to the filter $\tilde{H}_v(p)$. $\tilde{H}_v(p)$ is related to the original $H_v(p)$ in that the a_i's and b_i's of both filters are equal. Consideration of the system of n first order equations given by Equation 3-11 shows a possible means for synthesizing a system that generates a random process that approximates the composite random process $g(x(t))$ when $x(t)$ is an arbitrary function of time having a small second derivative. If v in Equation 3-11 is redefined as

$$v(t) = \frac{dx(t)}{dt} \qquad (3\text{-}28)$$

or

$$x(t) = \int_0^t v(u)du + x(0) \quad , \qquad (3\text{-}29)$$

the system of equations becomes

$$e_1(t) = e(t) \tag{3-30}$$

$$\frac{de_1(t)}{dt} = \sqrt{|v(t)|}\,(e_2(t) + a_{n-1}w(t) - b_{n-1}\sqrt{|v(t)|}\,e_1(t))$$

$$\frac{de_2(t)}{dt} = v(t)(e_3(t) + a_{n-2}w(t) - b_{n-2}\sqrt{|v(t)|}\,e_1(t))$$

$$\frac{de_3(t)}{dt} = v(t)(e_4(t) + a_{n-3}w(t) - b_{n-3}\sqrt{|v(t)|}\,e_1(t))$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{de_{n-1}(t)}{dt} = v(t)(e_n(t) + a_1w(t) - b_1\sqrt{|v(t)|}\,e_1(t))$$

$$\frac{de_n(t)}{dt} = v(t)(a_ow(t) - b_o\sqrt{|v(t)|}\,e_1(t)) \,.$$

This representation provides a direct approach to modify the mechanization of Figure 3-3 so that v(t), the time derivative of the position parameter x(t), may be applied as the position function input to the system. A mechanization for this set of equations is shown in Figure 3-4. The differential equations of Equation 3-30 may be mechanized by the use of standard analog computer components (see

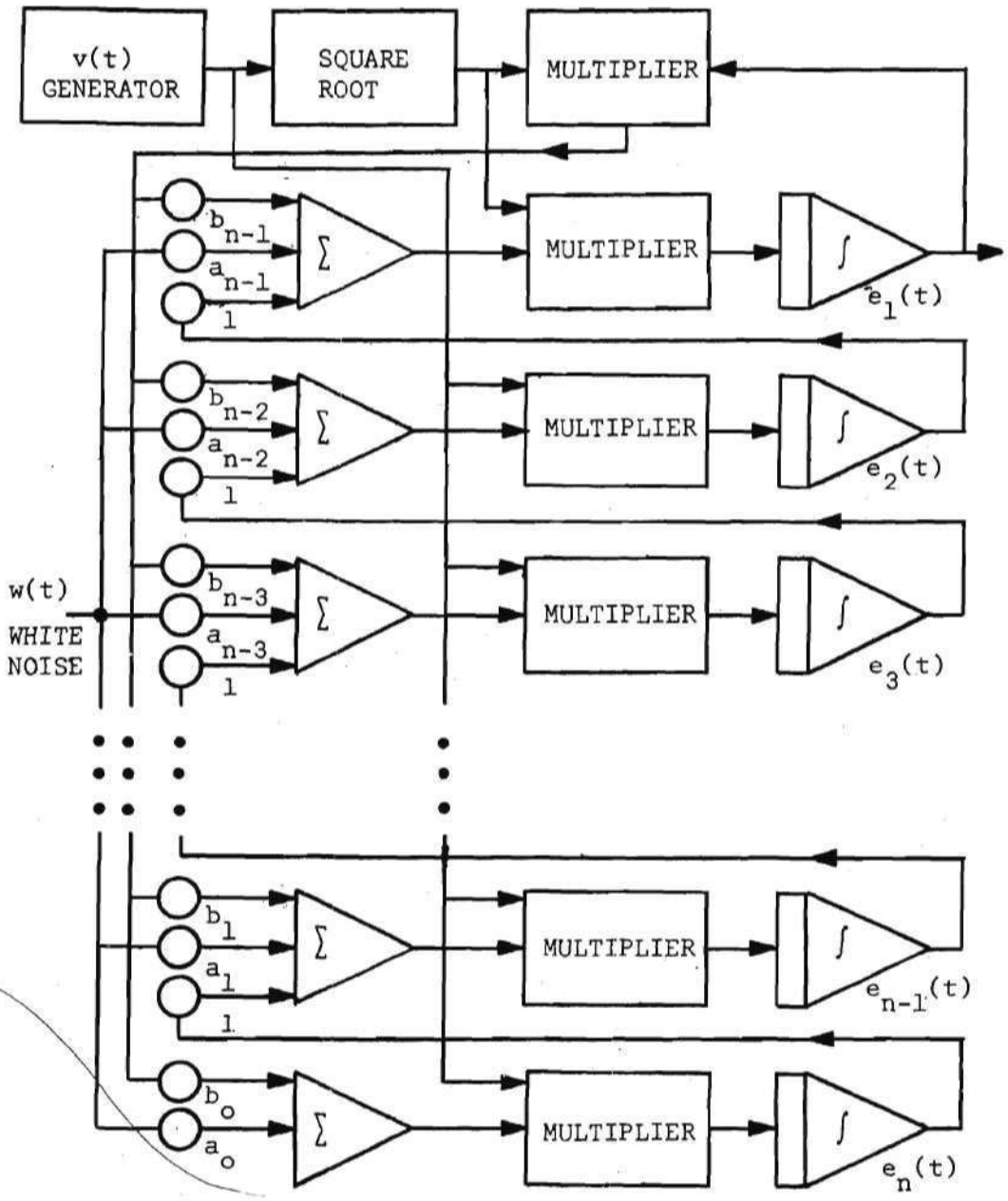


Figure 3-4. Mechanization System for the Composite Process g(x(t)).

references 7, 8, and 28) such as integrators, amplifiers, multipliers, resolvers, etc. The use of the v(t) function generator allows the implementation of the composite process g(x(t)) without redesigning the filter for variations of x(t).

The mechanization system presented in Figure 3-4 provides an exact realization of g(x(t)) for any constant value of v(t); hence, it is expected to provide a close approximation to an exact realization of g(x(t)) for slowly varying functions of time v(t) related to x(t) by Equations 3-28 and 3-29. In order to realize a good approximation, the second derivative of x(t) is to be maintained at a small value.

It is emphasized that in general any analog computer mechanization of the transfer function $H_v(p)$ given by Equation 3-26 provides a possibility of adaptation for approximating the composite random process g(x(t)). The mechanization selected allows by the addition of multipliers and other components, the versatility gained with v(t) appearing as an input to the filter.

Adjustment of the White Noise Generator

The preceding development was based on a white noise source with unity power spectral density, i.e.,

$$S_w(\omega) = 1 \ . \qquad (3\text{-}31)$$

This is not normally the case and hence the experimental results are normalized so that a comparison can be made between the theoretical and experimental results.

However, for the cases of interest in this research the stationarity of the random process g(x) provides a method of adjusting the white noise input, the forward gain of the mechanization system, or the gain of the sampling system to provide an output random process that need not be normalized to compare with the theoretical results (see Finn and Yates (4, page 13)). The statistical mean squared value $E\{g(x(t))^2\}$ of the composite random process g(x(t)) is independent of the waveform x(t). To accomplish the adjustment, the input v(t) is adjusted to any convenient fixed value. The output of the system is then a stationary Gaussian random process in the time parameter t. The timewise average of the squared output is then equal to the statistical average of the squared output. Thus, an rms meter may be used to determine $E\{g(x(t))^2\}$. The level of the white noise input may be varied until the rms meter at the output indicates the correct mean squared value for the composite random process g(x(t)).

Analytical Results

In this section analytical results for two specified autocorrelation functions are presented. The synthesis procedure yields differential equations which when implemented on an analog computer generate the composite process g(x(t)) associated with a stationary Gaussian process g(x) having the specified autocorrelation function.

Analytical Example One

For the first analytical example the autocorrelation function for g(x) is assumed to be

$$R_g(x_1,x_2) = A^2\varepsilon^{-\alpha|x_1 - x_2|} , \qquad (3\text{-}32)$$

and the mean value of g(x) is assumed to be

$$M_g = 0 .$$

Assigning the difference $x_1 - x_2 = \tau$ the autocorrelation function is written as

$$R_g(\tau) = A^2\varepsilon^{-\alpha|\tau|} . \qquad (3\text{-}33)$$

The power spectral density $S_g(\omega)$ for g(x) is found by use of Equation 3-1 to be

$$S_g(\omega) = \frac{2A^2\alpha}{\omega^2 + \alpha^2} . \qquad (3\text{-}34)$$

By use of Equation 3-4 $H_g(p)$ is determined to be

$$H_g(p) = \frac{\sqrt{2A^2\alpha}}{p + \alpha} . \qquad (3\text{-}35)$$

The translated transfer function $H_v(p)$ defined by Equation 3-6 for the example under consideration is

$$H_v(p) = \sqrt{\frac{2A^2\alpha}{|v|}} \cdot \frac{1}{p/v + \alpha} . \qquad (3\text{-}36)$$

$H_v(p)$ may be realized by an analog computer mechanization of the following first order differential equation,

$$\dot{e}(t) + \alpha v e(t) = \sqrt{2A^2\alpha|v|}\, w(t) . \qquad (3\text{-}37)$$

As previously suggested the output e(t) of an analog computer mechanization of Equation 3-37 is expected to approximate the composite process g(x(t)) when the position parameter is given as

$$x(t) = \int_{t_o}^{t} v(u)du + x(t_o) \qquad (3\text{-}38)$$

and x(t) has a small second derivative.

For this example the quality of the approximation may be investigated by obtaining the analytical solution to the differential Equation 3-37.

The general solution (see Coddington (29, page 43)) of a first order differential equation in the form of Equation 3-37 is

$$e(t) = \qquad (3\text{-}39)$$

$$\varepsilon^{-\int_{t_o}^{t} \alpha v(u)du} \int_{t_o}^{t} \varepsilon^{\int_{t_o}^{u} \alpha v(s)ds} \sqrt{2A^2\alpha|v(u)|}\, w(u)du + C\varepsilon^{-\int_{t_o}^{t} \alpha v(u)du} .$$

The v(t) is restricted to be non-negative and $C = e(t_o)$.

The general solution cannot be further simplified since the integrand of the first term contains the representation of a Gaussian white noise random process w(t). However, the autocorrelation function for e(t) can be found. If the white noise random process is assumed to have a unity power spectral density, $S_w(\omega) = 1$, then the autocorrelation function reduces[†] to

$$E\{e(t_1)e(t_2)\} = A^2\varepsilon^{-\alpha\left|\int_{t_o}^{t_1} v(u)du - \int_{t_o}^{t_2} v(u)du\right|}, \qquad (3\text{-}40)$$

which for $t_1 \geq t_2$ becomes

$$E\{e(t_1)e(t_2)\} = A^2\varepsilon^{-\alpha\left|\int_{t_2}^{t_1} v(u)du\right|} = A^2\varepsilon^{-\alpha|x(t_1) - x(t_2)|}. \qquad (3\text{-}41)$$

By comparing Equations 3-32 and 3-41 it is obvious that the synthesis procedure has realized exactly the autocorrelation function $R_g(x_1,x_2)$ where $x_1 = x(t_1)$ and $x_2 = x(t_2)$. In this example an exact realization of the composite process g(x(t)) is achieved when the first derivative of x(t), $\dot{x}(t) = v(t)$, is restricted to be non-negative, but the second derivative of x(t), $\ddot{x}(t) = \dot{v}(t)$, need not be restricted to small values. The first order example is implemented by the general mechanization system. The experimental results are presented in Chapter V.

[†]The details of this derivation are presented in Appendix C.

Analytical Example Two

For the second analytical example the autocorrelation function for g(x) is assumed to be

$$R_g(x_1,x_2) = A^2\varepsilon^{-\alpha|x_1 - x_2|} + B^2\varepsilon^{-\beta|x_1 - x_2|} , \qquad (3\text{-}42)$$

and the mean value of g(x) is assumed to be

$$M_g = 0 .$$

Assigning the difference $x_1 - x_2 = \tau$ the autocorrelation function is written as

$$R_g(\tau) = A^2\varepsilon^{-\alpha|\tau|} + B^2\varepsilon^{-\beta|\tau|} . \qquad (3\text{-}43)$$

The power spectral density $S_g(\omega)$ for g(x) is found by use of Equation 3-1 to be

$$S_g(\omega) = \frac{(2A^2\alpha + 2B^2\beta)\omega^2 + 2\alpha\beta(A^2\beta + B^2\alpha)}{\omega^4 + 2\alpha^2\beta^2\omega^2 + (\alpha^2\beta^2)^2} . \qquad (3\text{-}44)$$

By use of Equation 3-4 $H_g(p)$ is determined to be

$$H_g(p) = \frac{\sqrt{2A^2\alpha + 2B^2\beta}\left(p + \sqrt{\frac{2\alpha\beta(A^2\beta + B^2\alpha)}{2A^2\alpha + 2B^2\beta}}\right)}{(p + \alpha)(p + \beta)} . \qquad (3\text{-}45)$$

The translated transfer function $H_v(p)$ defined by Equation 3-6 for this example is

$$H_v(p) = \sqrt{\frac{2A^2\alpha + 2B^2\beta}{|v|}} \cdot \frac{\left(\frac{p}{v} + \sqrt{\frac{2\alpha\beta(A^2\beta + B^2\alpha)}{2A^2\alpha + 2B^2\beta}}\right)}{(\frac{p}{v} + \alpha)(\frac{p}{v} + \beta)} . \qquad (3\text{-}46)$$

By making the following assignments

$$a_1 = \sqrt{2A^2\alpha + 2B^2\beta} \qquad (3\text{-}47)$$

and

$$a_o = \sqrt{2\alpha\beta(A^2\beta + B^2\alpha)} ,$$

Equation 3-46 reduces to

$$H_v(p) = \frac{a_1}{\sqrt{|v|}} \cdot \frac{(p/v + a_o/a_1)}{(p/v + \alpha)(p/v + \beta)} . \qquad (3\text{-}48)$$

$H_v(p)$ may be realized by an analog computer mechanization of the following second order differential equation

$$\sqrt{|v|}\,(\ddot{e}(t) + (\alpha + \beta)v\dot{e}(t) + \alpha\beta v^2 e(t)) = a_1 v\dot{w}(t) + a_o v^2 w(t) . \qquad (3\text{-}49)$$

With the following additional assignments

$$b_1 = \alpha + \beta \tag{3-50}$$

and $$b_o = \alpha\beta \ ,$$

the second order differential Equation 3-49 can be reduced by the assignments of Equations 3-50 to two first order differential equations

$$\dot{e}_1(t) = \sqrt{|v|}\,(e_2(t) + a_1 w(t) - b_1\sqrt{|v|}\,e_1(t)) \tag{3-51}$$

and $$\dot{e}_2(t) = v(a_o w(t) - b_o\sqrt{|v|}\,e_1(t)) \ .$$

In the analog computer mechanization of Equation 3-51 the output e(t) is given by

$$e(t) = e_1(t) \ . \tag{3-52}$$

The output of the analog computer mechanization is expected to approximate the composite process g(x(t)) when the position parameter x is given as

$$x(t) = \int_{t_o}^{t} v(u)du + x(t_o) \ , \tag{3-53}$$

if x(t) has a small second derivative and the first derivative $\dot{x}(t)$ is

non-negative. For this example the solution of the second order differential equation cannot be found for general values of v(t); hence, the autocorrelation function cannot be investigated analytically. One approach to determine the validity of the approximation to the composite process is to implement the time-varying filter and complete a statistical analysis of the output ensemble. This approach is presented in Chapter V. Another approach to determine the validity of the approximation is to compare term by term the coefficients of the second order differential equations represented by the mechanization system and the ones obtained by applying the parallel procedure discussed in Chapter IV. The results of this comparison are presented in Chapter IV.

The second order example is implemented by the general mechanization system and the experimental results are presented in Chapter V.

Error Considerations

There are two basic ways to consider the errors that result from the approximate synthesis procedure:

1. Data error analysis.
2. Theoretical error analysis.

The statistical data that result from the experimental measurements provide the information to satisfy (1). An analysis is made in Chapter V of the experimental data which applies confidence level error boundaries to both the mean and the standard deviation statistics of the resulting output random process. The experimental data include errors from several sources, e.g., estimate bias errors, measurement system errors, ensemble truncation errors, mechanization errors, and errors resulting from the approximate nature of the synthesis procedure.

An unbiased estimator is used to determine the process statistics which eliminates the estimate bias error. The errors produced in the measurement system as a result of sampling and timing errors are shown by Bryan (11, page 15) to be negligible when compared with the ensemble truncation errors. The inherent errors in the analog components, multipliers, resolvers, amplifiers, and integrators classified as mechanization errors are considered negligible by proper zeroing, and alignment of the analog equipment. The remaining two sources of errors, ensemble truncation errors, and approximate procedure errors, are not separable from an examination of the experimental data. However, the cumulative error from all of the itemized sources is bounded by the confidence level error boundaries as shown in the data presentation in Chapter V. This fact certainly lends credibility to the "goodness" of the approximate procedure.

A technique for determining theoretically the error resulting from the approximate procedure is not apparent at this time. The difference between the solution of the system of equations and the desired composite function expressed by

$$\varepsilon = e(t) - g(x(t)) \qquad (3\text{-}54)$$

has not been determined analytically. However, a comparison approach has been considered which provides a comparison of coefficients of the nth order differential equations resulting from the approximate procedure and the parallel exact procedure. This approach is applied to the first and second order examples; the results are presented in

Chapter IV. The logic of this approach is simply that if the coefficients of two differential equations are correspondingly essentially the same then the solutions of the two equations are essentially the same. Furthermore, implication on the quality of the similarity of the resulting statistics is even better since the statistics are themselves averages and would not tend to show the "fine grain" differences prevalent between the similar solutions.

Stability

Some heuristic comments are made concerning the stability of the time-varying filters in lieu of an extensive study of stability since that subject was not the objective of the present research.

Two viewpoints are in order: consideration of the mechanization of the differential equations, and consideration of the translated transfer functions. To present the discussion the equations are again written.

Example One. The translated transfer function is

$$H_v(p) = \frac{\sqrt{2A^2\alpha|v|}}{p + \alpha v}, \qquad (3\text{-}55)$$

and the first order differential equation is

$$\dot{e}(t) = \sqrt{|v|}\,(\sqrt{2A^2\alpha}\,w(t) - \alpha\sqrt{|v|}\,e(t)) . \qquad (3\text{-}56)$$

Example Two. The translated transfer function is

$$H_v(p) = \frac{a_1\sqrt{|v|}\,(p + \frac{a_o}{a_1} v)}{(p + \alpha v)(p + \beta v)}, \qquad (3\text{-}57)$$

and the two first order differential equations are

$$\dot{e}_1(t) = \sqrt{|v|}\,(e_2(t) + a_1 w(t) - b_1\sqrt{|v|}\,e_1(t)) \qquad (3\text{-}58)$$

and $$\dot{e}_2(t) = v(a_o w(t) - b_o\sqrt{|v|}\,e_1(t)) .$$

Note that the transfer functions are in the normal break frequency format; the differential equations are in the form as is mechanized on the analog computer.

Stability is assured for all positive fixed values of v for the filters represented by Equations 3-55 and 3-57, since they are the results of spectral factorization of a power spectral density $S_v(\omega)$ that represents a real random process (see Davenport and Root (26, page 106)) where

$$S_v(\omega) = H_v(p)H_v^*(p) = |H_v(p)|^2 . \qquad (3\text{-}59)$$

The filter is represented by $H_v(p)$, a transfer function all of whose poles are in the left half of the p-plane, hence stable for all values of v greater than zero. If v becomes zero the mechanization system remains stable since clearly the right-hand side of Equations 3-56 and 3-58 is zero.

CHAPTER IV

A PARALLEL PROCEDURE

In this chapter another procedure for synthesizing the time-varying filters is presented. Its relationship to the physical problem is stated. The parallel procedure is an extension of results obtained by Webb, et al. (1), by an appropriate consideration of the composite function. His results are modified to apply to the physical problem. The parallel procedure yields exact results for the first and second order analytical examples presented in Chapter III. The exact results are used to provide a goodness comparison of the approximate synthesis procedure.

Relation to Physical Problem

The parallel procedure is applicable to the physical problems presented in Chapter I but it also has application to a larger class of physical problems. Spectral factorization techniques are not employed in the parallel procedure. The procedure techniques make it useful in generating stochastic processes which may be nonstationary in both the position parameter and the time parameter. The parallel procedure will realize any random process, insofar as its first two statistical moments are concerned, as the output of a linear time-varying network excited by stationary white noise. The parallel procedure involves expanding the autocovariance function of the desired process into a finite series. The finite series expansion is then used to determine

the coefficients of a linear differential equation whose analog realization is the required network.

The parallel procedure, without loss of generality considers the realization of random processes with mean zero. As before the nonzero mean processes can be realized as the sum of the random process generated by this technique and a deterministic function equal to the required mean.

The Composite Modification

The sequence of the parallel procedure is presented in this section. The notation used in this presentation is similar to that of Webb (1 and 2) so that cross-reference may be accomplished with minimum difficulty.

The autocovariance function for the composite random process g(x(t)) with mean zero is given by

$$r(t',t) = E\{g(x(t_1))g(x(t_2))\} , \qquad (4\text{-}1)$$

where E is the expected value operator and

$$t' = \text{larger of } t_1 \text{ and } t_2$$
$$t = \text{smaller of } t_1 \text{ and } t_2 .$$

The representation of the autocovariance function that is most useful in developing the required analog networks is in the following form

$$r(t',t) = \sum_{i=1}^{n} \phi_i(x(t'))\gamma_i(x(t)) \quad . \tag{4-2}$$

The form is a finite sum of separable functions in the parameters x(t') and x(t). Many physical processes (see Webb (1, page 4)) can be represented in this format; all of the admissible functions given in Appendix A for the approximate procedure can be expressed as finite sums of separable functions in the parameters x(t') and x(t). Recall that the position parameter's dependency of interest is the velocity function given by

$$x(t) = \int_{o}^{t} v(u)du \tag{4-3}$$

or

$$v(t) = \frac{dx(t)}{dt} \quad .$$

Thus the autocovariance expansion is written

$$r(t',t) = \sum_{i=1}^{n} \phi_i\left(\int_{o}^{t'} v(u)du\right)\gamma_i\left(\int_{o}^{t} v(u)du\right) \quad . \tag{4-4}$$

Using now the parallel of Webb's procedure, a set of first order differential equations can be derived which when mechanized on an analog computer will generate the prescribed random process. The input to the system is stationary Gaussian white noise w(t) and the output e(t) is a random process that is equal to the composite random process g(x(t)).

The system is described by the nth order differential equation

of the form

$$e^{(n)}(t) + p_{n-1}(t)e^{(n-1)}(t) + \ldots + p_1(t)e^{(1)}(t) + p_o(t)e(t) \qquad (4\text{-}5)$$

$$= q_m(t)w^{(m)}(t) + \ldots + q_1(t)w^{(1)}(t) + q_o(t)w(t) ,$$

which can be written in operator form as

$$L_t\, e(t) = N_t\, w(t) , \qquad (4\text{-}6)$$

where

$$L_t = \sum_{i=0}^{n} p_i(t) \frac{d^i}{dt^i} , \quad p_n = 1 , \qquad (4\text{-}7)$$

and

$$N_t = \sum_{i=0}^{m} q_i(t) \frac{d^i}{dt^i} . \qquad (4\text{-}8)$$

The homogeneous equation is written

$$L_t\, e(t) = 0 . \qquad (4\text{-}9)$$

Assuming without loss of generality that m = n - 1, the nth order

differential equation is converted to a set of n first order differential equations by the following identifications

$$e_1(t) = e(t) \tag{4-10}$$

$$\dot{e}_1(t) = e_2(t) - a_{n-1}(t)e_1(t) + b_m(t)w(t)$$

$$\dot{e}_2(t) = e_3(t) - a_{n-2}(t)e_1(t) + b_{m-1}(t)w(t)$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{e}_{n-1}(t) = e_n(t) - a_1(t)e_1(t) + b_1(t)w(t)$$

$$\dot{e}_n(t) = - a_o(t)e_1(t) + b_o(t)w(t) \ .$$

In matrix notation this set can be written more concisely as

$$\underline{\dot{e}}(t) = A(t)\underline{e}(t) + B(t)w(t) \ , \tag{4-11}$$

where $$e(t) = \underline{H}\underline{e}(t) \ ,$$

$$\underline{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix}, \qquad B(t) = \begin{bmatrix} b_m(t) \\ b_{m-1}(t) \\ \vdots \\ b_1(t) \\ b_o(t) \end{bmatrix},$$

$$H = [1\ 0\ 0\ \ldots\ 0],$$

and

$$A(t) = \begin{bmatrix} -a_{n-1}(t) & 1 & 0 & \cdots & 0 \\ -a_{n-2}(t) & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -a_1(t) & 0 & 0 & \cdots & 1 \\ -a_o(t) & 0 & 0 & \cdots & 0 \end{bmatrix}. \qquad (4\text{-}12)$$

The elements of the A(t) and B(t) matrices a_k and b_k are related to the coefficients p_k and q_k of Equation 4-5 by

$$p_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!(n-1-j-k)!} \; a_{n-1-j}^{(n-1-j-k)} \qquad (4\text{-}13)$$

and

$$q_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!(n-1-j-k)!} \; b_{n-1-j}^{(n-1-j-k)} \; . \qquad (4\text{-}14)$$

When the p_k, q_k are known, Equations 4-13 and 4-14 can be solved sequentially for the a_k and b_k.

The homogeneous vector differential equation associated with Equation 4-11 is

$$\dot{\underline{e}}(t) = A(t)\underline{e}(t) \; . \qquad (4\text{-}15)$$

Webb shows (1, pages 6-13) that the n ϕ_i in Equation 4-2 can be taken as the fundamental set of solutions to the homogeneous differential equation given by Equation 4-15. The operator L_t, and hence the coefficients p_k can be obtained by evaluating the following determinant

$$L_t\, e(t) = \det \begin{bmatrix} e(t) & \phi_1(x(t)) & \ldots . \phi_n(x(t)) \\ e^{(1)}(t) & \phi_1^{(1)}(x(t)) \ldots . \phi_n^{(1)}(x(t)) \\ e^{(2)}(t) & \phi_1^{(2)}(x(t)) \ldots . \phi_n^{(2)}(x(t)) \\ e^{(3)}(t) & \phi_1^{(3)}(x(t)) \ldots . \phi_n^{(3)}(x(t)) \\ \vdots & \vdots & \vdots \\ e^{(n)}(t) & \phi_1^{(n)}(x(t)) \ldots . \phi_n^{(n)}(x(t)) \end{bmatrix} = 0 \quad . \quad (4\text{-}16)$$

Note that the superscripts denote differentiation with respect to time, hence the chain rule must be used in the $\phi_i^{(n)}(x(t))$ evaluations.

The elements of the A(t) matrix can now be determined using Equation 4-13; again the chain rule must be used in the differentiations.

If the $\phi_i(x(t))$ are the n linearly independent solutions to Equation 4-9, then the fundamental matrix solution $\phi(x(t))$ for Equation 4-15 satisfying

$$\frac{d\phi(x(t))}{dt} = A(t)\phi(x(t)) \quad , \qquad (4\text{-}17)$$

is defined by

$$\phi(x(t)) = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{n2} \\ \vdots & \vdots & & \vdots \\ \phi_{1n} & \phi_{2n} & \cdots & \phi_{nn} \end{bmatrix} , \tag{4-18}$$

where

$$\phi_{ij} = \frac{d\phi_{ij-1}}{dt} + a_{n-j+1}\phi_{i1} \tag{4-19}$$

and

$$\phi_{k1} = \phi_k , \quad (k = 1,\ldots,n) . \tag{4-20}$$

The elements b_k of the B(t) matrix can now be determined. Webb (1, page 13) defines a D matrix whose elements are given as

$$d_{ij}(x(t)) = \frac{\gamma_i(x(t))}{\phi_i(x(t))} , \; i = j \tag{4-21}$$

and

$$d_{ij}(x(t)) = 0 , \quad i \neq j .$$

The autocovariance matrix can be written as

$$R(t',t) = \phi(x(t'))D(x(t))\phi^{T}(x(t)) \ , \qquad (4\text{-}22)$$

where $(\ )^{T}$ denotes matrix transpose. The element in the first row and first column of this matrix is just the scalar autocovariance $r(t',t)$. The elements b_k of the matrix $B(t)$ can be determined from the properties of the $R(t',t)$ matrix.

Define by $R^{*}(t',t)$ the extension of $R(t',t)$ when the sign of the difference t_1-t_2 changes. Then

$$R^{*}(t',t) = \phi(x(t))D(x(t'))\phi^{T}(x(t')) = R^{T}(t,t') \ . \qquad (4\text{-}23)$$

Denote $\Delta(t',t)$ as the difference

$$\Delta(t',t) = R(t',t) - R^{*}(t',t) \ . \qquad (4\text{-}24)$$

The elements b_k of $B(t)$ are evaluated from

$$b_{m-i+1} = \sqrt{-\delta_{ii}} \ , \qquad (4\text{-}25)$$

where δ_{ii} are the diagonal elements of

$$\left.\frac{\partial\Delta(t',t)}{\partial t'}\right|_{t' = t} \ . \qquad (4\text{-}26)$$

All of the elements of the differential equation

$$\dot{\underline{e}}(t) = A(t)\underline{e}(t) + B(t)w(t) \qquad (4\text{-}27)$$

are known. The equation can be realized by the system of Figure 4-1.

Analytical Examples

Two analytical examples are presented that will serve to clarify the parallel procedure and provide exact results to be used in the comparative analysis.

Example One

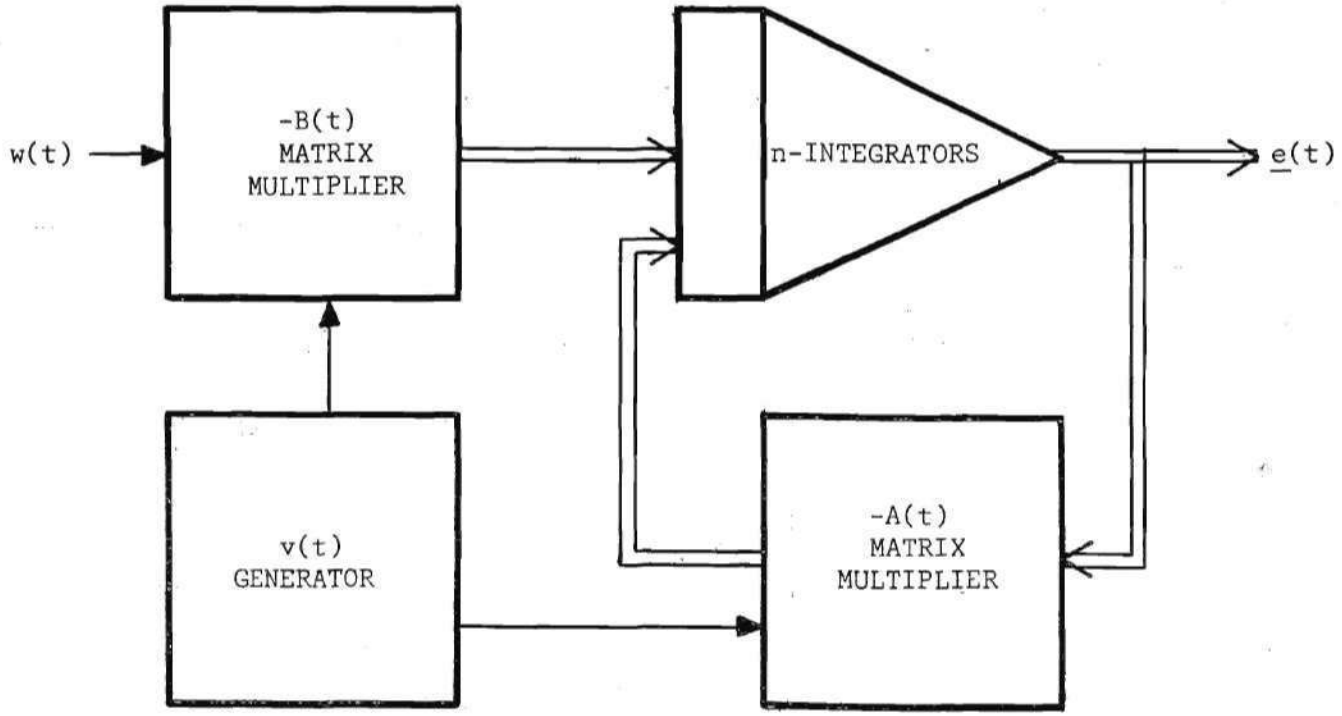
It is desired to generate a composite random process g(x(t)) with autocovariance

$$r(t',t) = A^2\varepsilon^{-\alpha|x(t') - x(t)|} \qquad (4\text{-}28)$$

and mean zero. Equation 4-28 can be expanded into a finite sum separable in the parameters x(t') and x(t) as follows, where $x(t') \geq x(t)$,

$$r(t',t) = A\varepsilon^{-\alpha x(t')}\ A\varepsilon^{\alpha x(t)} \ . \qquad (4\text{-}29)$$

By inspection the $\phi_i(x(t))$ and the $\gamma_i(x(t))$ are assigned as



NOTE: Double lines ═══ indicate multivariable signal flow.

Figure 4-1. Generalized Computer Implementation of Equation 4-27.

$$\phi_1(x(t)) = A\varepsilon^{-\alpha x(t)} \tag{4-30}$$

$$\gamma_1(x(t)) = A\varepsilon^{\alpha x(t)} \ .$$

The operator $L_t\ e(t)$ is given by

$$L_t\ e(t) = \det \begin{bmatrix} e(t) & A\varepsilon^{-\alpha \int_o^t v(u)du} \\ e^{(1)}(t) & -\alpha A v(t) \varepsilon^{-\alpha \int_o^t v(u)du} \end{bmatrix} = 0 \ , \tag{4-31}$$

where the chain rule is used to determine the time derivative of $\phi(x(t))$. The expansion of the determinant is a first order differential equation given by

$$\dot{e}(t) + \alpha v(t) e(t) = 0 \ . \tag{4-32}$$

The expression for p_o is given as

$$p_o = \alpha v(t) \ . \tag{4-33}$$

Using Equation 4-13 the a_k is determined, when $n = 1$ and $k = 0$, as $p_o = a_o = \alpha v(t)$. The elements of the ϕ matrix are determined using

Equations 4-19 and 4-20. The ϕ matrix has only one element in this example given by

$$\phi_{11} = \phi_1 = A\varepsilon^{-\alpha x(t)} \quad . \tag{4-34}$$

The single element of the D matrix is determined using Equation 4-21

$$d_{11} = \frac{A\varepsilon^{\alpha x(t)}}{A\varepsilon^{-\alpha x(t)}} = \varepsilon^{2\alpha x(t)} \quad . \tag{4-35}$$

The autocovariance matrix is determined using Equation 4-22 to be

$$R(t',t) = A\varepsilon^{-\alpha x(t')}\varepsilon^{2\alpha x(t)}A\varepsilon^{-\alpha x(t)} \quad , \tag{4-36}$$

which reduces to

$$R(t',t) = A^2\varepsilon^{-\alpha x(t')}\ \varepsilon^{\alpha x(t)} \quad .$$

The extension of $R(t',t)$ when $t \geq t'$ is determined to be

$$R^*(t',t) = A^2\varepsilon^{-\alpha x(t)}\ \varepsilon^{\alpha x(t')} \quad . \tag{4-37}$$

The difference matrix $\Delta(t',t)$ is given by

$$\Delta(t',t) = R - R^* = A^2[\varepsilon^{-\alpha(x(t') - x(t))} - \varepsilon^{-\alpha(x(t) - x(t'))}] \quad . \tag{4-38}$$

The single element of the B(t) matrix is determined from Equations 4-25 and 4-26, when m = n - 1 = 0 and i = 1 to be

$$b_o = \sqrt{-\delta_{11}} \ .$$

The value of δ_{11} is obtained from the single term of Equation 4-38 as

$$\left.\frac{\partial \Delta(t',t)}{\partial t'}\right|_{t' = t} = -2\alpha A^2 v(t) \ . \qquad (4\text{-}39)$$

The b_o term is then given by

$$b_o = (2\alpha A^2 v(t))^{1/2} \ . \qquad (4\text{-}40)$$

The elements of the differential equation

$$\dot{\underline{e}}(t) = A(t)\underline{e}(t) + B(t)w(t)$$

are single element matrices given as

$$A(t) = |-\alpha v(t)|$$

and

$$B(t) = |(2\alpha A^2 v(t))^{1/2}| \ ,$$

thus the first order differential equation is

$$\dot{e}(t) = - \alpha v(t) e(t) + (2\alpha A^2 v(t))^{1/2} w(t) \ . \qquad (4\text{-}41)$$

Comparing this result with Equation 3-37 it is found to be identical; thus the approximate procedure yields the same result as the parallel exact procedure for the first order example.

Example Two

The autocovariance function given for the second example leads to a second order system. It is

$$r(t',t) = A^2 \varepsilon^{-\alpha|x(t') - x(t)|} + B^2 \varepsilon^{-\beta|x(t') - x(t)|} \ . \qquad (4\text{-}42)$$

The parallel procedure is now used where the autocovariance is expanded by Equation 4-2 to be

$$r(t',t) = \phi_1(x(t'))\gamma_1(x(t)) + \phi_2(x(t'))\gamma_2(x(t)) \ , \qquad (4\text{-}43)$$

with $\phi_1 = A\varepsilon^{-\alpha x(t)}$ $\qquad \gamma_1 = A\varepsilon^{\alpha x(t)}$

$\phi_2 = B\varepsilon^{-\beta x(t)}$ $\qquad \gamma_2 = B\varepsilon^{\beta x(t)}$

and $x(t') \geq x(t)$.

In order to see the effects of the composite function the sequence of the procedure is followed in a general notation. The expressions for ϕ and γ are then substituted and finally the dependency of $v(t) = \frac{dx(t)}{dt}$ is substituted.

The set ϕ_1, ϕ_2 are taken as solutions to the homogeneous differential equation

$$L_t\, e(t) = \det \begin{bmatrix} e & \phi_1 & \phi_2 \\ \dot{e} & \dot{\phi}_1 & \dot{\phi}_2 \\ \ddot{e} & \ddot{\phi}_1 & \ddot{\phi}_2 \end{bmatrix} = 0 \ . \qquad (4\text{-}44)$$

Expanding the Equation 4-44 the differential equation becomes

$$\ddot{e} + \left(\frac{\ddot{\phi}_1\phi_2 - \phi_1\ddot{\phi}_2}{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}\right)\dot{e} + \left(\frac{\dot{\phi}_1\ddot{\phi}_2 - \ddot{\phi}_1\dot{\phi}_2}{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}\right)e = 0 \ . \qquad (4\text{-}45)$$

The expressions for p_1 and p_o are the coefficients of $\dot{e}$ and e, respectively. The coefficients in the alternate vector representation of Equation 4-45 are determined by Equation 4-13 to be

$$a_1 = \frac{\ddot{\phi}_1\phi_2 - \phi_1\ddot{\phi}_2}{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2} \qquad (4\text{-}46)$$

and

$$a_o = \frac{\phi_1\dddot{\phi}_2 + 2\dot{\phi}_1\ddot{\phi}_2 - 2\ddot{\phi}_1\dot{\phi}_2 - \dddot{\phi}_1\phi_2}{[\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2]} - \left(\frac{\ddot{\phi}_1\phi_2 - \phi_1\ddot{\phi}_2}{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}\right)^2 \ . \qquad (4\text{-}47)$$

The ϕ matrix is determined by Equations 4-19 and 4-20 to be

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 + a_1\phi_1 & \dot{\phi}_2 + a_1\phi_2 \end{bmatrix} . \qquad (4\text{-}48)$$

The elements of the D matrix are determined by Equation 4-21 to be

$$d_{12} = d_{21} = 0$$

$$d_{11} = \frac{\gamma_1}{\phi_1} , \qquad d_{22} = \frac{\gamma_2}{\phi_2} . \qquad (4\text{-}49)$$

The autocovariance matrix is determined by Equation 4-22 to be

$$R(t',t) = \tag{4-50}$$

$$\begin{bmatrix} \phi_1(t') & \phi_2(t') \\ \dot{\phi}_1(t') + a_1(t')\phi_1(t') & \dot{\phi}_2(t') + a_1(t')\phi_2(t') \end{bmatrix}$$

$$\cdot \begin{bmatrix} \dfrac{\gamma_1(t)}{\phi_1(t)} & 0 \\ 0 & \dfrac{\gamma_2(t)}{\phi_2(t)} \end{bmatrix} \cdot \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) + a_1(t)\phi_1(t) & \dot{\phi}_2(t) + a_1(t)\phi_2(t) \end{bmatrix} .$$

$$R(t',t) = \tag{4-51}$$

$$\begin{bmatrix} \phi_1(t')\gamma_1(t) + \phi_2(t')\gamma_2(t) & (*)^{\dagger} \\ & [\dot{\phi}_1(t') + a_1(t')\phi_1(t')][\gamma_1(t)\dot{\phi}_1(t)/\phi_1(t) + a_1(t)\gamma_1(t)] \\ (*)^{\dagger} & + [\dot{\phi}_2(t') + a_1(t')\phi_2(t')][\gamma_2(t)\dot{\phi}_2(t)/\phi_2(t) + a_1(t)\gamma_2(t)] \end{bmatrix} .$$

$^{\dagger}(*)$ These terms are not used further and are hence omitted here.

The expression for $R^{*}(t',t)$ can be written by inspection of Equation 4-51 by an interchange of t' and t. The terms of interest in the difference matrix $\Delta(t',t) = R(t',t) - R^{*}(t',t)$ are

$$\delta_{11} = \phi_1(t')\gamma_1(t) + \phi_2(t')\gamma_2(t) - \phi_1(t)\gamma_1(t') - \phi_2(t)\gamma_2(t') \quad (4\text{-}52)$$

δ_{12}, not used

δ_{21}, not used

$$\delta_{22} = [\dot{\phi}_1(t') + a_1(t')\phi_1(t')][\gamma_1(t)\dot{\phi}_1(t)/\phi_1(t) + a_1(t)\gamma_1(t)]$$

$$- [\dot{\phi}_1(t) + a_1(t)\phi_1(t)][\gamma_1(t')\dot{\phi}_1(t')/\phi_1(t') + a_1(t')\gamma_1(t')]$$

$$+ [\dot{\phi}_2(t') + a_1(t')\phi_2(t')][\gamma_2(t)\dot{\phi}_2(t)/\phi_2(t) + a_1(t)\gamma_2(t)]$$

$$- [\dot{\phi}_2(t) + a_1(t)\phi_2(t)][\gamma_2(t')\dot{\phi}_2(t')/\phi_2(t') + a_1(t')\gamma_2(t')] \; .$$

The elements of the B matrix are determined by

$$- b_1^2(t) = \frac{\partial}{\partial t'} [\delta_{11}] \Big|_{t' = t} \quad (4\text{-}53)$$

$$- b_o^2(t) = \frac{\partial}{\partial t'} [\delta_{22}] \Big|_{t' = t} \; .$$

Performing the indicated operations of Equation 4-53 the expressions for b_1 and b_o in the parameter t are

$$b_1^2(t) = \phi_1\dot{\gamma}_1 + \phi_2\dot{\gamma}_2 - \dot{\phi}_1\gamma_1 - \dot{\phi}_2\gamma_2$$

$$b_o^2(t) = -[\ddot{\phi}_1 + a_1\dot{\phi}_1 + \dot{a}_1\phi_1][\gamma_1\dot{\phi}_1/\phi_1 + \dot{a}_1\gamma_1] \qquad (4\text{-}54)$$

$$+ [\dot{\phi}_1 + a_1\phi_1][\frac{\phi_1(\dot{\gamma}_1\dot{\phi}_1 + \gamma_1\ddot{\phi}_1) - \gamma_1\dot{\phi}_1^2}{\phi_1^2} + a_1\dot{\gamma}_1 + \dot{a}_1\gamma_1]$$

$$- [\ddot{\phi}_2 + a_1\dot{\phi}_2 + \dot{a}_1\phi_2][\gamma_2\dot{\phi}_2/\phi_2 + a_1\gamma_2]$$

$$+ [\dot{\phi}_2 + a_1\phi_2][\frac{\phi_2(\gamma_2\ddot{\phi}_2 + \dot{\gamma}_2\dot{\phi}_2) - \gamma_2\dot{\phi}_2^2}{\phi_2^2} + a_1\dot{\gamma}_2 + \dot{a}_1\gamma_2] .$$

The components of the vector Equation 4-15 are now determined where

$$A(t) = \begin{bmatrix} -a_1(t) & 1 \\ -a_o(t) & 0 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} b_1(t) \\ b_o(t) \end{bmatrix} . \qquad (4\text{-}55)$$

The expressions for a_o, b_o, a_1, and b_1 are perfectly general for the functions ϕ_1, ϕ_2, γ_1, and γ_2. For the stated autocovariance function of

Equation 4-42 the functions and the required derivatives are assigned as follows.

$$\phi_1 = A\varepsilon^{-\alpha x(t)} \tag{4-56}$$

$$\dot{\phi}_1 = -\alpha A\varepsilon^{-\alpha x(t)}\dot{x}(t)$$

$$\ddot{\phi}_1 = \alpha^2 A\varepsilon^{-\alpha x(t)}\dot{x}(t)^2 - \alpha A\varepsilon^{-\alpha x(t)}\ddot{x}(t)$$

$$\dddot{\phi}_1 = -\alpha^3 A\varepsilon^{-\alpha x(t)}\dot{x}(t)^3 + 3\alpha^2 A\varepsilon^{-\alpha x(t)}\dot{x}(t)\ddot{x}(t) - \alpha A\varepsilon^{-\alpha x(t)}\dddot{x}(t)$$

$$\phi_2 = B\varepsilon^{-\beta x(t)}$$

$$\dot{\phi}_2 = -\beta B\varepsilon^{-\beta x(t)}\dot{x}(t)$$

$$\ddot{\phi}_2 = \beta^2 B\varepsilon^{-\beta x(t)}\dot{x}(t)^2 - \beta B\varepsilon^{-\beta x(t)}\ddot{x}(t)$$

$$\dddot{\phi}_2 = -\beta^3 B\varepsilon^{-\beta x(t)}\dot{x}(t)^3 + 3\beta^2 B\varepsilon^{-\beta x(t)}\dot{x}(t)\ddot{x}(t) - \beta B\varepsilon^{-\beta x(t)}\dddot{x}(t)$$

$$\gamma_1 = A\varepsilon^{\alpha x(t)}$$

$$\dot{\gamma}_1 = \alpha A\varepsilon^{\alpha x(t)}\dot{x}(t)$$

$$\gamma_2 = B\varepsilon^{\beta x(t)}$$

$$\dot{\gamma}_2 = \beta B\varepsilon^{\beta x(t)}\dot{x}(t) \ .$$

By substitution into Equations 4-46, 4-47, and 4-54 one obtains

$$a_1(t) = (\alpha + \beta)v(t) - \frac{\dot{v}(t)}{v(t)} \ , \qquad (4\text{-}57)$$

$$a_o(t) = \alpha\beta v(t)^2 - (\alpha + \beta)\dot{v}(t) + \frac{\ddot{v}(t)}{v(t)} - \left(\frac{\dot{v}(t)}{v(t)}\right)^2 \ ,$$

$$b_1(t) = ([2\alpha A^2 + 2\beta B^2]v(t))^{1/2} \ ,$$

and

$$b_o(t) = \{2v(t)[\alpha A^2(\beta v(t) - \frac{\dot{v}(t)}{v(t)})^2 + \beta B^2(\alpha v(t) - \frac{\dot{v}(t)}{v(t)})^2]\}^{1/2} \ .$$

The two first order differential equations found by using Equations 4-15 and 4-55 are

$$e_1 = e \qquad (4\text{-}58)$$

$$\dot{e}_1 = e_2 - ((\alpha + \beta)v - \dot{v}/v)e_1 + (v[2\alpha A^2 + 2\beta B^2])^{1/2} w$$

$$\dot{e}_2 = -(\alpha\beta v^2 - (\alpha + \beta)\dot{v} + \ddot{v}/v - (\dot{v}/v)^2)e_1$$

$$+ \{2v[\alpha A^2(\beta v - \dot{v}/v)^2 + \beta B^2(\alpha v - \dot{v}/v)^2]\}^{1/2} w .$$

These equations realize exactly the composite random process g(x(t)) when the autocovariance of the random process is given by Equation 4-42 with mean zero. This equation is compared in the next section of this chapter with the approximate equations derived using the simplified procedure. The system of Equations 4-58 can be mechanized by the generalized computer implementation shown in Figure 4-1 with v(t) as an auxiliary input as with the mechanization of the approximate procedure.

Comparative Analysis

In this section a comparative analysis is made of the second order example. The second order differential equations resulting from the parallel approach and the approximate approach are compared. The comparison is based on the similarity of coefficients of the two equations.

The second order differential equation arising from the parallel approach is

$$\ddot{e} + [(\alpha + \beta)v - \dot{v}/v]\dot{e} + \alpha\beta v^2 e = \{[2\alpha A^2 + 2\beta B^2]v\}^{1/2}\dot{w} \qquad (4\text{-}59)$$

$$+ [\{2v[\alpha A^2(\beta v - \dot{v}/v)^2 + \beta B^2(\alpha v - \dot{v}/v)^2]\}^{1/2} + [2\alpha A^2 + 2\beta B^2]^{1/2}\dot{v}/2v]w .$$

The second order differential equation arising from the approximate procedure is obtained from Equation 3-52 with the substitutions of Equations 3-51 and 3-47. It is written as

$$\ddot{e} + [(\alpha + \beta)v - \dot{v}/2v]\dot{e} + [\alpha\beta v^2 + 1/2\ (\alpha + \beta)\dot{v}]e \qquad (4\text{-}60)$$

$$= \{[2\alpha A^2 + 2\beta B^2]v\}^{1/2}\ \dot{w} + (2\alpha\beta[A^2\beta + B^2\alpha]v^3)^{1/2}\ w \ .$$

It can be shown that the coefficients of Equations 4-59 and 4-60 approach common expressions as the parameter $\dot{v}(t)$ approaches zero. The common coefficients are given in Equation 4-61 as

$$\ddot{e} + (\alpha + \beta)v\dot{e} + \alpha\beta v^2 e = [(2\alpha A^2 + 2\beta B^2)v]^{1/2}\ \dot{w} \qquad (4\text{-}61)$$

$$+ [2\alpha\beta(A^2\beta + B^2\alpha)v^3]^{1/2}\ w \ .$$

Indritz (30, page 354) proves that small changes in continuous coefficients of a second order differential equation cause only small changes in the solution of the differential equation. The theorem is now applied to Equations 4-59 and 4-60. The coefficients of both equations satisfy the continuity requirement for the parameters $v(t)$ and $\dot{v}(t)$ of interest. Hence, the solutions are continuous in those parameters. It can be shown that when $\dot{v}(t)$ is sufficiently close to zero the solutions of the two differential equations approach a common solution, hence they approach each other, i.e., the absolute value of the difference between the solutions to Equations 4-59 and 4-60 can be

made arbitrarily small. It can further be shown that two functions which are arbitrarily close have autocorrelation functions which are arbitrarily close.

In order to make a numerical comparison for the second order example the coefficients are identified as b_2, b_1, b_o, a_1, and a_o by the following equation

$$\sqrt{v}\,(b_2\ddot{e} + b_1 v\dot{e} + b_o v^2 e) = a_1 v\dot{w} + a_o v^2 w \ . \qquad (4\text{-}62)$$

The expressions for b_2, b_1, b_o, a_1, and a_o are now tabularized in Table 4-1. The quantities in the second column of the table are from Equation 4-59; the third column entries are from Equation 4-60. The substitutions $k = \beta/\alpha$, $A = B$, and $(\dot{v}(t)/v(t)^2) = m$ are made.

Table 4-1. Coefficient Comparison

Coefficient	Exact	Approximate
b_2	1	1
b_1	$\alpha(1 + k) - m$	$\alpha(1 + k) - m/2$
b_o	$k\alpha^2$	$k\alpha^2 + m\alpha(1 + k)/2$
a_1	$[2\alpha A^2(1 + k)]^{1/2}$	$[2\alpha A^2(1 + k)]^{1/2}$
a_o	$\{2\alpha A^2[\alpha^2 k(1 + k) - \alpha(4mk) + m^2(1 + k)]\}^{1/2} + (m/2)[2\alpha A^2(1 + k)]^{1/2}$	$[2A^2\alpha^3(1 + k)k]^{1/2}$

The curves of Figure 4-2 illustrate the coefficients plotted versus the ratio of β to α, i.e., $k = \beta/\alpha$. The curves of a_1 and b_2 are not shown since by the comparison of Table 4-1 they are equal for both the exact and approximate coefficients. The solid curves illustrate the range of the coefficients from the exact equation; the triangles (Δ) represent the data points calculated from the coefficients of the approximate equation. The numerical values ($A^2 = B^2 = 10$, and $m = 0.001$) selected for the illustration are the same as the ones used in the experimental results of Chapter V.

Figure 4-3 is a different presentation of the coefficients where their numerical values are plotted against the parameter $m = \dot{v}(t)/v(t)^2$. The values selected for the illustration are $A^2 = B^2 = 10$, $k = \beta/\alpha = 2$, $\alpha = 0.01$. The plots of b_2 and a_1 are again omitted since the coefficients are equal. The coefficients from the exact equation are plotted as solid curves; the values of the coefficients from the approximate equation are plotted as triangles (Δ) for $m > 0$, and as circles (o) for $m < 0$. The values of the coefficients from the approximate equation and the values of the coefficients from the exact equation are very close for $m \leq 0.001$, with the approximation becoming less valid as m gets larger.

These results then illustrate numerically the dependence of the approximate approach on the parameter $\dot{v}(t)$ and indicate that in the limit as $\dot{v}(t)$ approaches zero the approximate and exact coefficients approach a common value.

An extension of this comparison technique to higher order equations is deemed feasible. Tomovic indicates in his work (31, pages

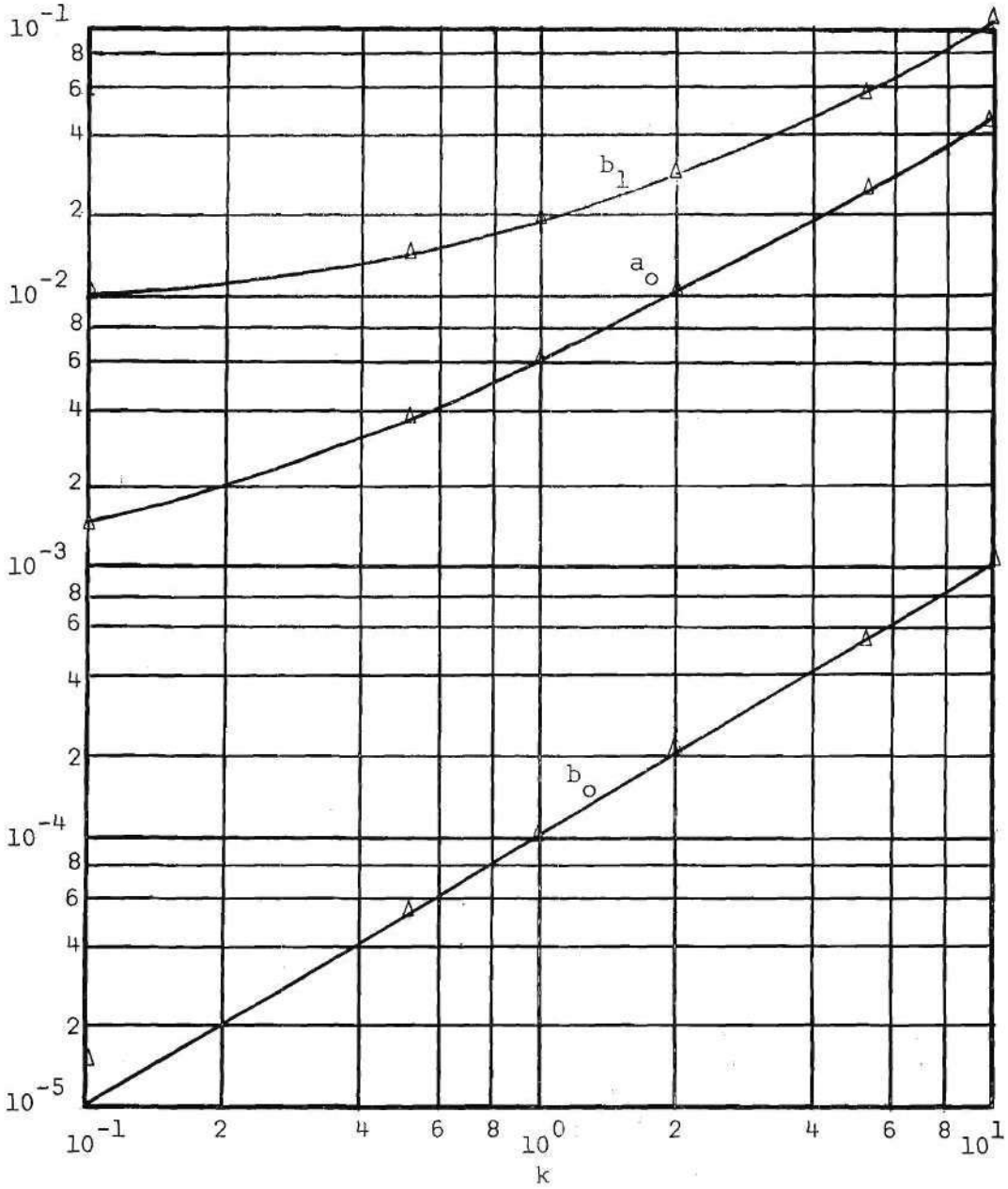


Figure 4-2. Magnitude of Coefficients versus $k = \beta/\alpha$.

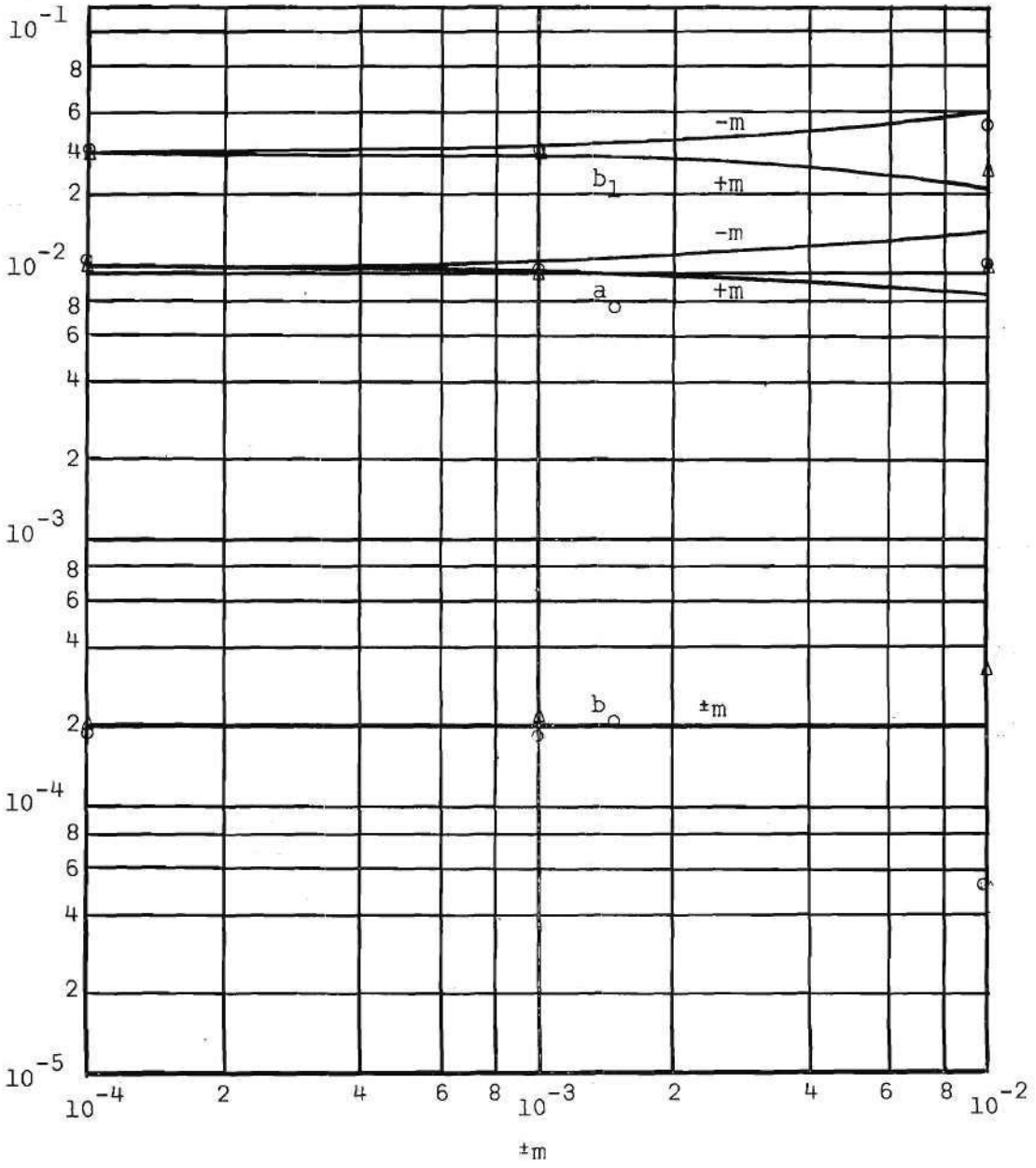


Figure 4-3. Magnitude of Coefficients versus $m = \dot{v}(t)/v(t)^2$.

25-28) that the order of the differential equation and the number of parameters effecting the changes in the continuous coefficients do not affect the generality of the conclusion, i.e., that the solutions are continuous in the parameters and hence small changes in the continuous coefficients cause only small changes in the solutions. Further substantiation of this remark requires a detailed study of the general problem of sensitivity, a study that is not one of the objectives of this research.

CHAPTER V

EXPERIMENTAL RESULTS

This chapter presents experimental results obtained in the course of the research. The two analytical results obtained in Chapter III are implemented by the general mechanization scheme on the analog computer and the output process statistics are illustrated for various velocity profiles. The measuring system used to obtain the sampled data and the digital processing scheme are discussed. A summary of the experimental results is presented at the end of the chapter.

Measuring System

The measuring system used to obtain the experimental results is a slightly modified version of a system developed by R. E. Bryan. Only the details of the measurement system that are necessary for clarity of this study are presented. Further information and design details are found in the work of Bryan (11).

The data collected from the measuring system are used to determine the first two statistical moments of the output stochastic process e(t), which is in general a nonstationary stochastic process. The statistical moments of interest are the process mean M(t) and the process covariance $R_c(t,t-\tau)$.[†] The nonstationary random process statistics are determined

[†]The notation used here means a specified autocovariance function $R_c(t_1,t_2)$. R_c is used to compute the theoretical curves, which are compared with the discrete time estimates $\hat{R_c}(t_1,t_2)$. The standard deviation is called the deviation and the autocovariance is called the covariance in this chapter.

from a sequence of sample functions from the ensemble which characterizes the random process e(t). Figure 5-1 illustrates the ensemble of sample functions which characterizes the nonstationary random process. The output process sample functions are sampled at each t_i time to collect the data points for the statistical analysis.

The quantities of interest are the mean

$$M(t_i) = E\{e_i\} \tag{5-1}$$

and the covariance

$$R_c(t_i,t_j) = E\{[e_i - E\{e_i\}][e_j - E\{e_j\}]\} , \tag{5-2}$$

which reduces to

$$R_c(t_i,t_j) = E\{e_i e_j\} - E\{e_i\}E\{e_j\} , \tag{5-3}$$

which further reduces for random processes e(t) with mean value zero, i.e., $E\{e_i\} = 0$, to

$$R_c(t_i,t_j) = E\{e_i e_j\} , \tag{5-4}$$

where e_i and e_j are the values of e(t) at $t = t_i$ and $t = t_j$, respectively. $E\{\cdot\}$ denotes the expected value operation. It is noted that for the problems of interest the covariance function reduces to the autocorrelation function represented by $E\{e_i e_j\}$.

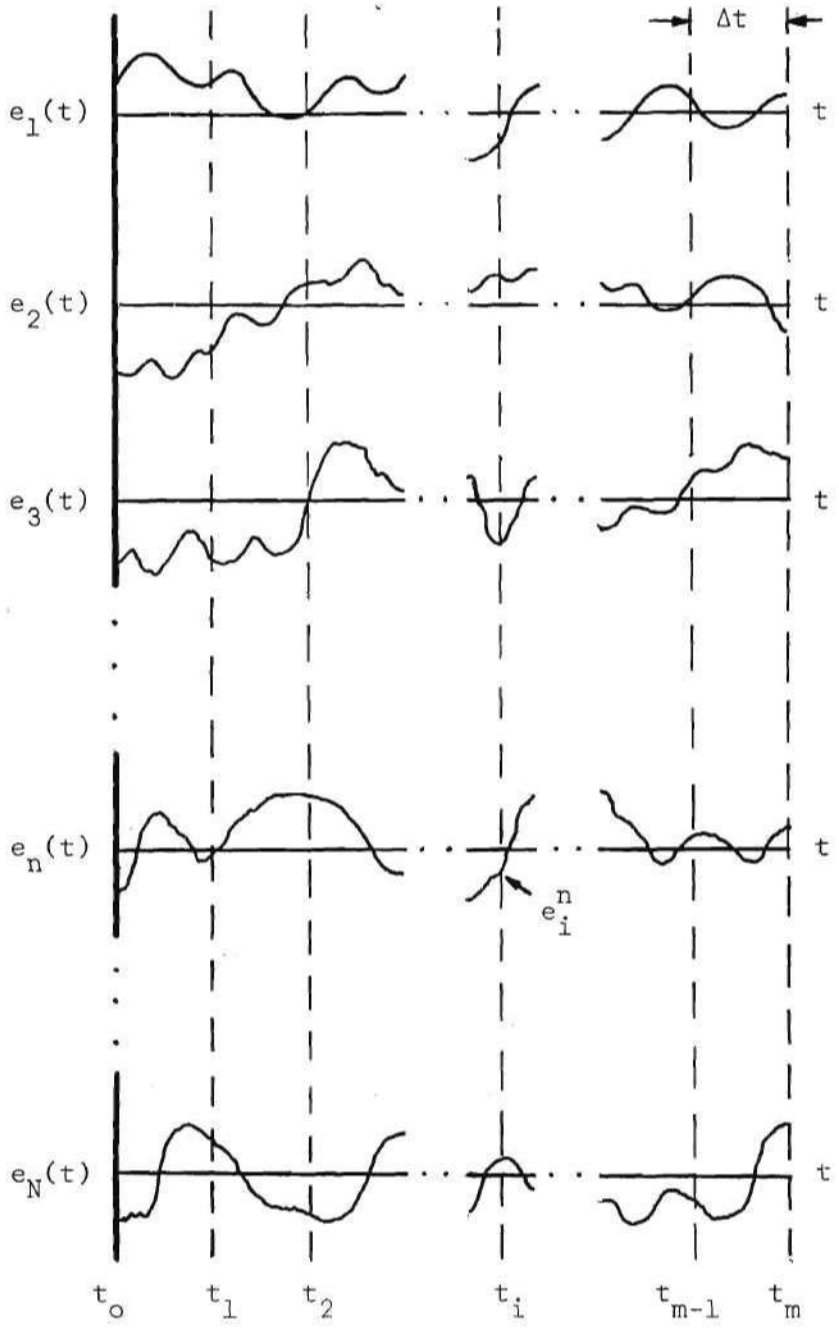


Figure 5-1. Ensemble of Sample Functions Representing the Nonstationary Stochastic Process e(t).

In all of the experimental data presentations the measuring system computes the discrete time equivalents $M(t_i)$ and $R_c(t_i,t_j)$ of the continuous time variables $M(t)$ and $R_c(t,t-\tau)$ where

$$t = t_i, \; i = 0, 1, \ldots, m \tag{5-5}$$

and

$$\tau = t_i - t_j, \quad j \leq i \; . \tag{5-6}$$

In the experiments $N = 200$, $\Delta t = 1$ second, and $m = 9$, and the calculations made are then estimates of the actual quantities. The estimate used for the mean is

$$\hat{M}(t_i) = \frac{1}{N} \sum_{n=1}^{N} e_i^n \; , \tag{5-7}$$

where the superscript n and subscript i indicate that e_i^n is the value of the nth sample function at $t = t_i$, and N is total number of sample functions used in the average. The estimate used for the covariance is

$$\hat{R_c}(t_i,t_j) = \frac{1}{N} \sum_{n=1}^{N} (e_i^n \, e_j^n) - \hat{M}(t_i)\hat{M}(t_j) \; . \tag{5-8}$$

The estimate used for the standard deviation is

$$\hat{\sigma}(t_i) = \sqrt{\hat{R_c}(t_i,t_i)} \; . \tag{5-9}$$

These estimates become better and better estimates of the true values

of the variables as N, the number of measurements, increases without limit (see Davenport and Root (26, page 79)). A summary discussion of the statistical error introduced by truncation of the ensemble of sample functions is included in Appendix B.

Figure 5-2 depicts the schematic diagram of the physical components that comprise the measurement system. The time-varying filter inputs are white noise w(t) and the velocity profile v(t). The output of the time-varying filter is e(t). The analog signals $e_n(t)$, n = 1, ...,N are sampled, digitized, and serialized for the paper tape punch. The analog system is synchronized in the compute and reset cycles by the time reference-counter controller component. The paper tape sampled data is translated to digital punched card format by an IBM tape to card converter. The digital cards are inserted as data cards in a computation routine for the B-5500 digital computer. The digital computer performs the ensemble statistical analysis and outputs the covariance function estimates $\hat{R_c}(t_i,t_j)$, the deviation function estimate $\hat{\sigma}(t_i)$, and the mean value function estimates $\hat{M}(t_i)$ given by Equations 5-7, 5-8, and 5-9.

In the experimental data comparisons that follow, the estimate $\hat{R_c}(t_1,t_2)$ is compared with the specified statistic $R_g(t_1,t_2)$. The format of $\hat{R_c}(t_1,t_2)$ as an output of the digital computer is illustrated in Figure 5-3. The values of t_1 and t_2 range from zero to nine seconds in one second intervals. The lower half of the diagram represents the covariance function where $t_1 < t_2$. Since the functions of interest are symmetric in the parameters t_1 and t_2, it is not necessary to include that data.

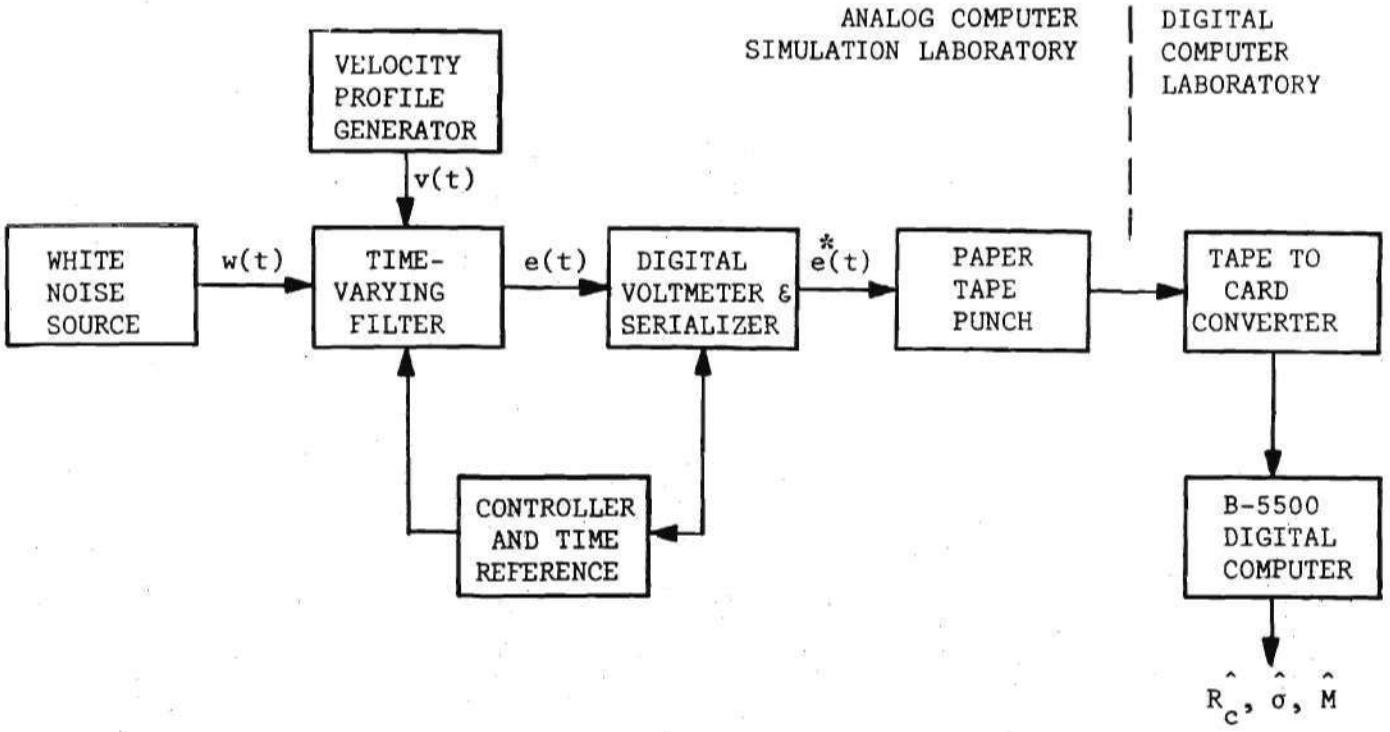


Figure 5-2. Measurement System.

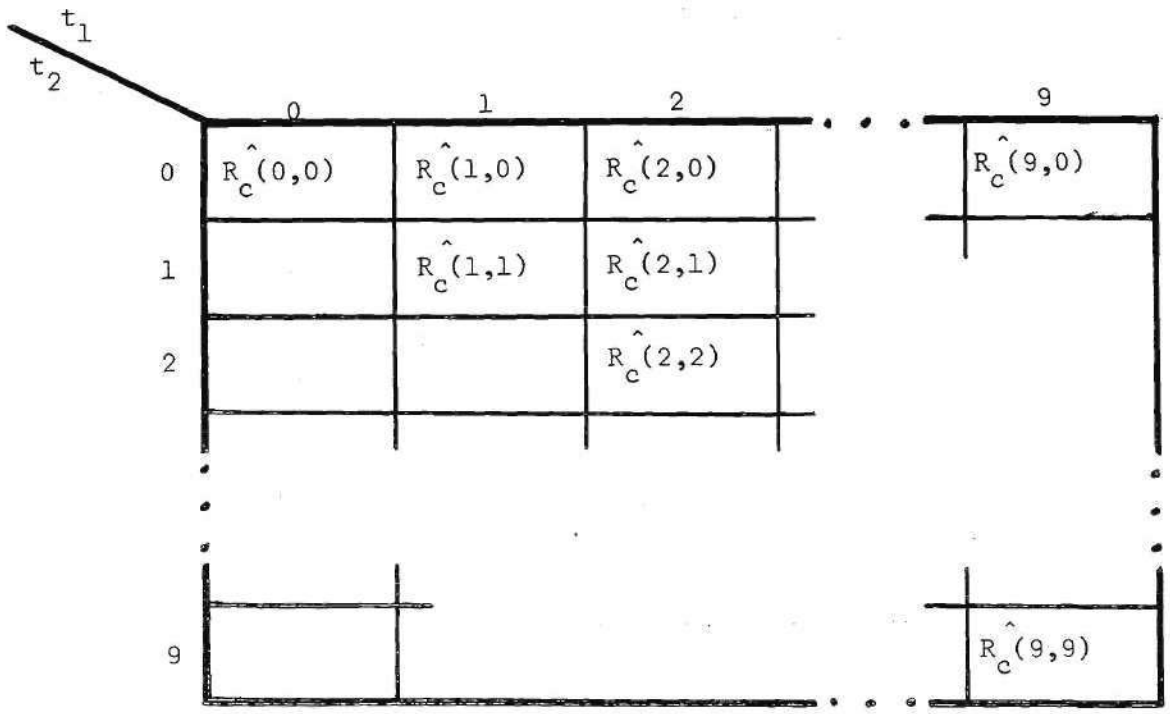


Figure 5-3. Digital Data Format for Covariance Estimates $\hat{R}_c(t_1,t_2)$.

Experimental Examples

The analytical results, Equations 3-37 and 3-51 are programmed on the analog computer using the mechanization system in Figure 3-4, and the measurement system illustrated in Figure 5-2. The theoretical curves and the measured data points are presented for the covariance function. The process mean and deviation with their respective confidence level error boundaries are shown for each of the cases considered.

Example One

The covariance function specified for the first experimental example is

$$R_g(x_1,x_2) = 1\varepsilon^{-0.005|x_1 - x_2|} , \qquad (5\text{-}10)$$

where

$$x(t) = \int_0^t v(u)du + x(0) . \qquad (5\text{-}11)$$

Using the synthesis procedure the corresponding differential equation related to Equation 3-37 where v is allowed to vary with time is given by

$$\dot{e}_1(t) = \sqrt{|v(t)|}\,(a_o w(t) - b_o\sqrt{|v(t)|}\,e_1(t)) . \qquad (5\text{-}12)$$

This equation is in the form required for the mechanization system. The analog computer mechanization of Equation 5-12 is shown by the simplified block diagram in Figure 5-4. The system of Figure 5-4 provides an exact realization of the composite random process g(x(t)), when x(t) and v(t) are related by Equation 5-11. For this example the coefficients

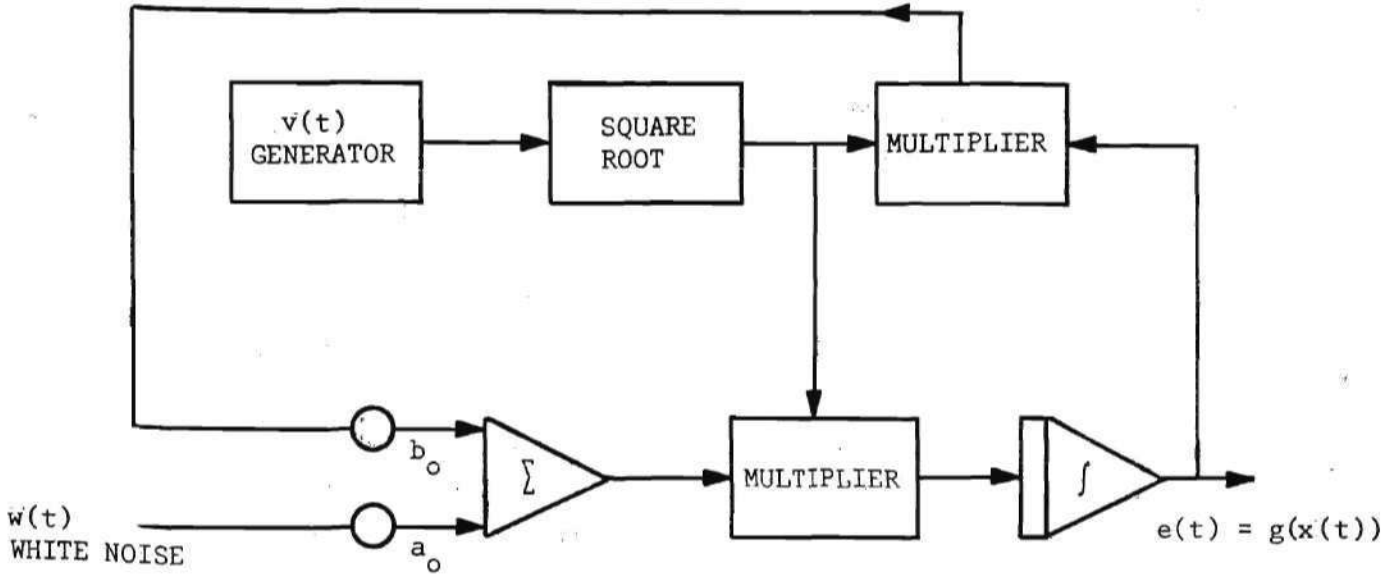


Figure 5-4. Mechanization System for Example One.

are $a_o = 1.0$ and $b_o = 0.005$.

Case One. Each of the following five cases uses different velocity profiles. Some lead to an output process that is stationary in time, others nonstationary in time. Figure 5-5 presents the first velocity profile.

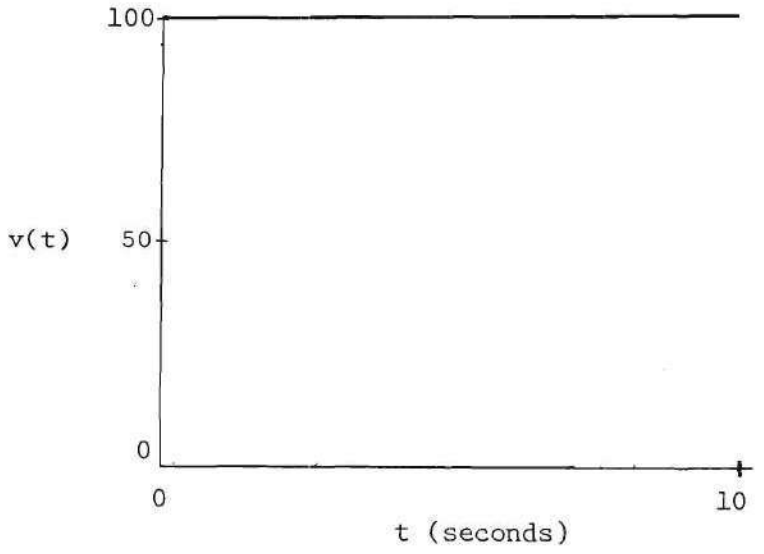


Figure 5-5. Example One--Velocity Profile One.

For Case One the position parameter x is given by

$$x(t) = 100t + x(0) \ . \qquad (5\text{-}13)$$

The mechanization system generates a random process that is stationary in both position x and time t, since the velocity v(t) is a constant. Figure 5-6 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for $t_2 = 1$, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure

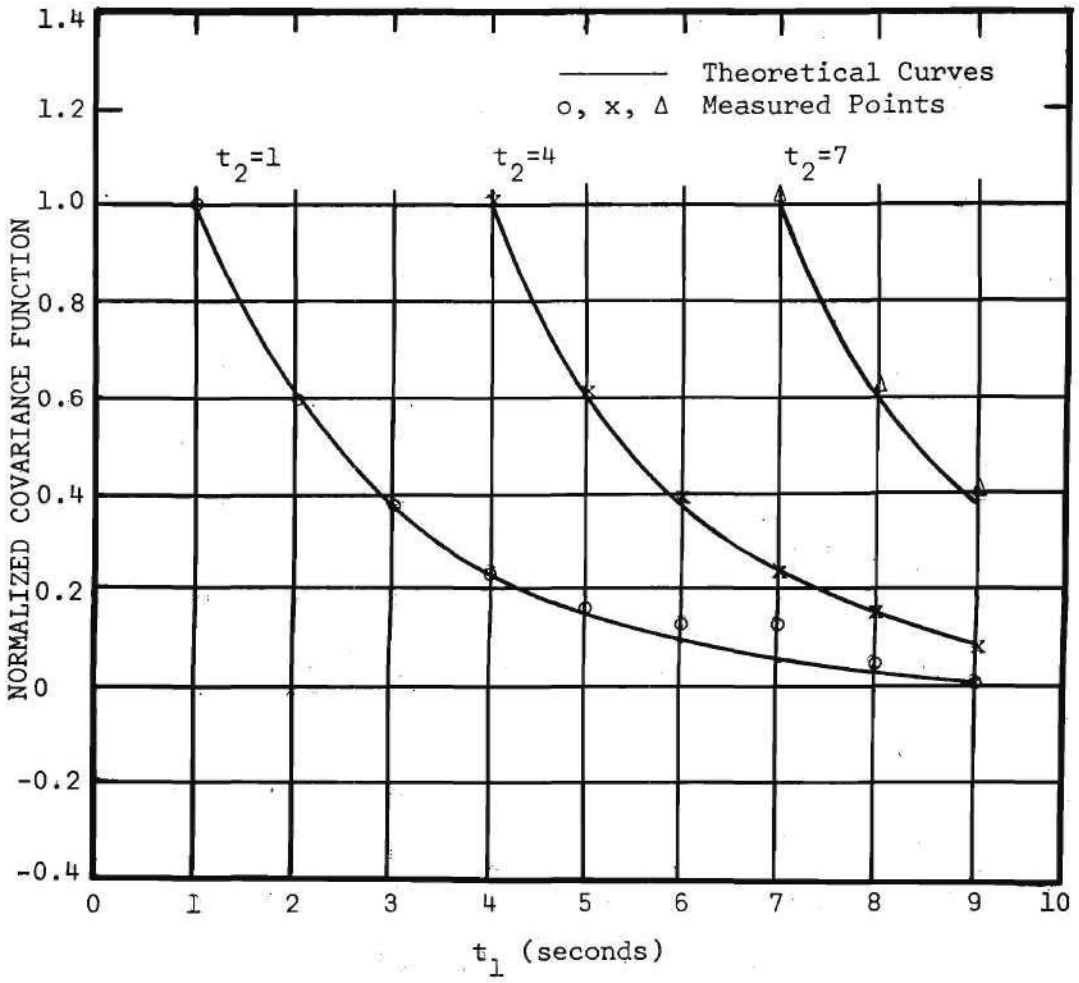


Figure 5-6. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example One--Velocity Profile One.

5-7 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

The theoretical curves are plots of Equation 5-10 and the experimental data points are derived from the estimator of the covariance function, Equation 5-8. In each case of the experimental examples the mean value is specified to be zero. However, the small value of $\hat{M}(t_i)$ noted on the data is not unexpected since the ensemble of e(t) is truncated at N = 200 sample functions and also the noise source w(t) cannot be adjusted to yield a mean value of exactly zero (see Bryan (11, page 22)).

Case Two. The velocity profile used in the second case of Example One is illustrated in Figure 5-8.

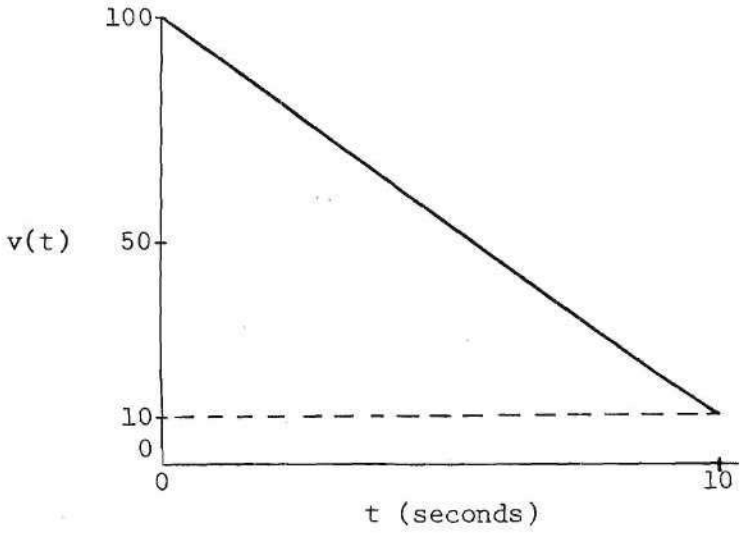


Figure 5-8. Example One--Velocity Profile Two.

The output random process in this case is stationary in position x but nonstationary in time t. The position parameter x is given by

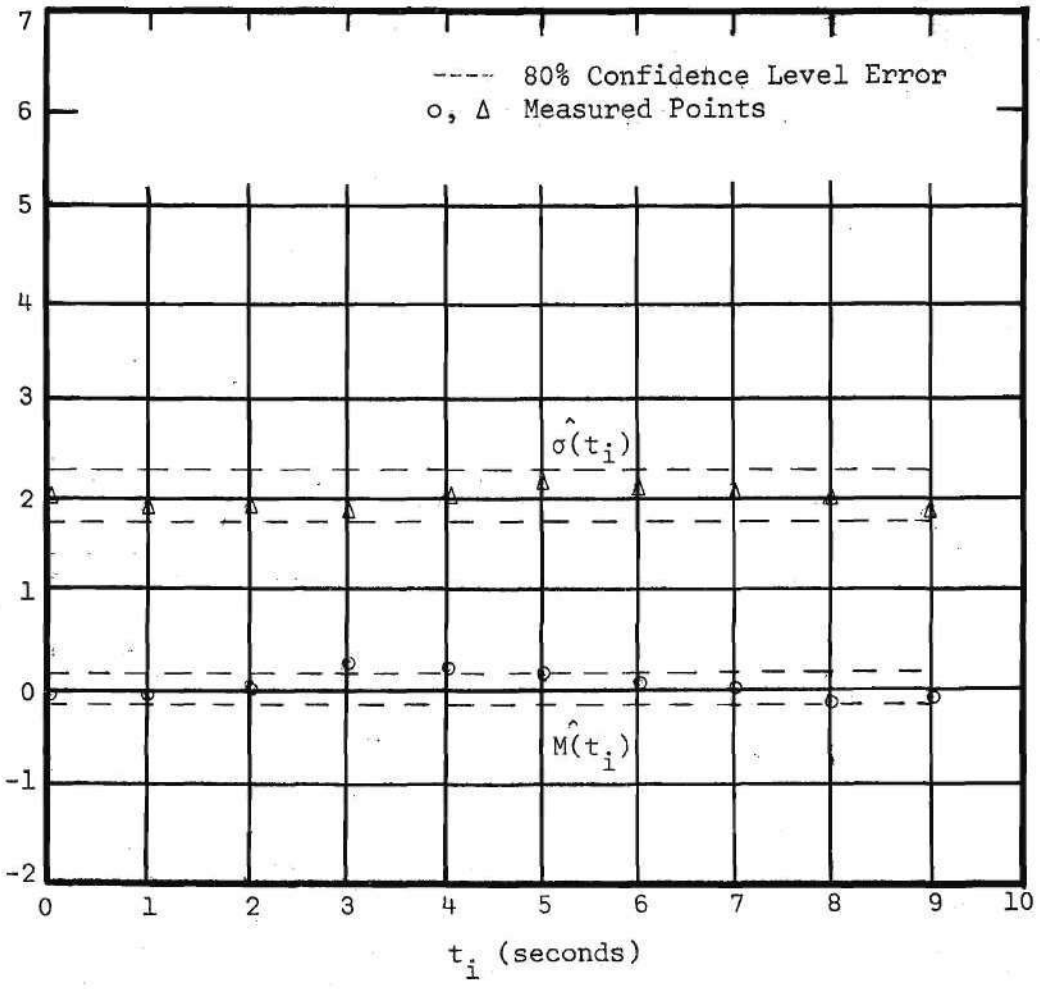


Figure 5-7. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example One--Velocity Profile One.

$$x(t) = -4.5t^2 + 100t + x(0) \ . \qquad (5\text{-}14)$$

Figure 5-9 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-10 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Three. The velocity profile used in the third case of Example One is illustrated in Figure 5-11.

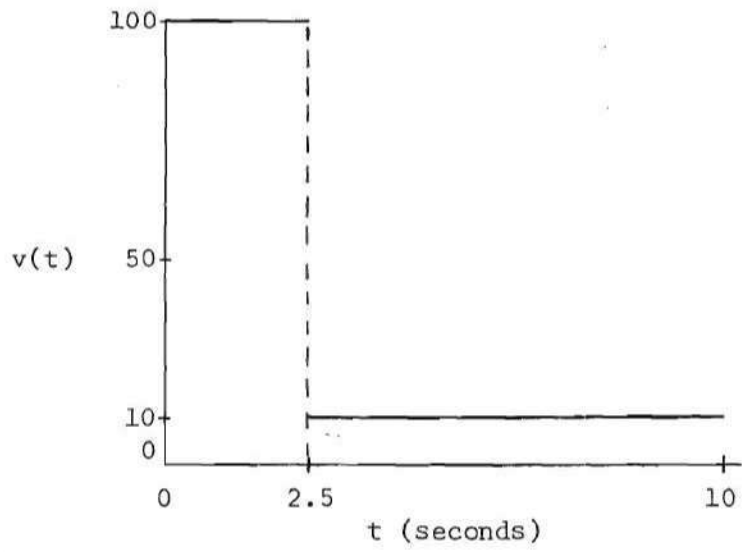


Figure 5-11. Example One--Velocity Profile Three.

The output random process in this case is again stationary in position x but nonstationary in time t. The position parameter x is given by

$$x(t) = \begin{cases} 100t + x(0) & , \ 0 \le t < 2.5 \\ 10t + 225 + x(0), & 2.5 \le t \le 10 \ . \end{cases} \qquad (5\text{-}15)$$

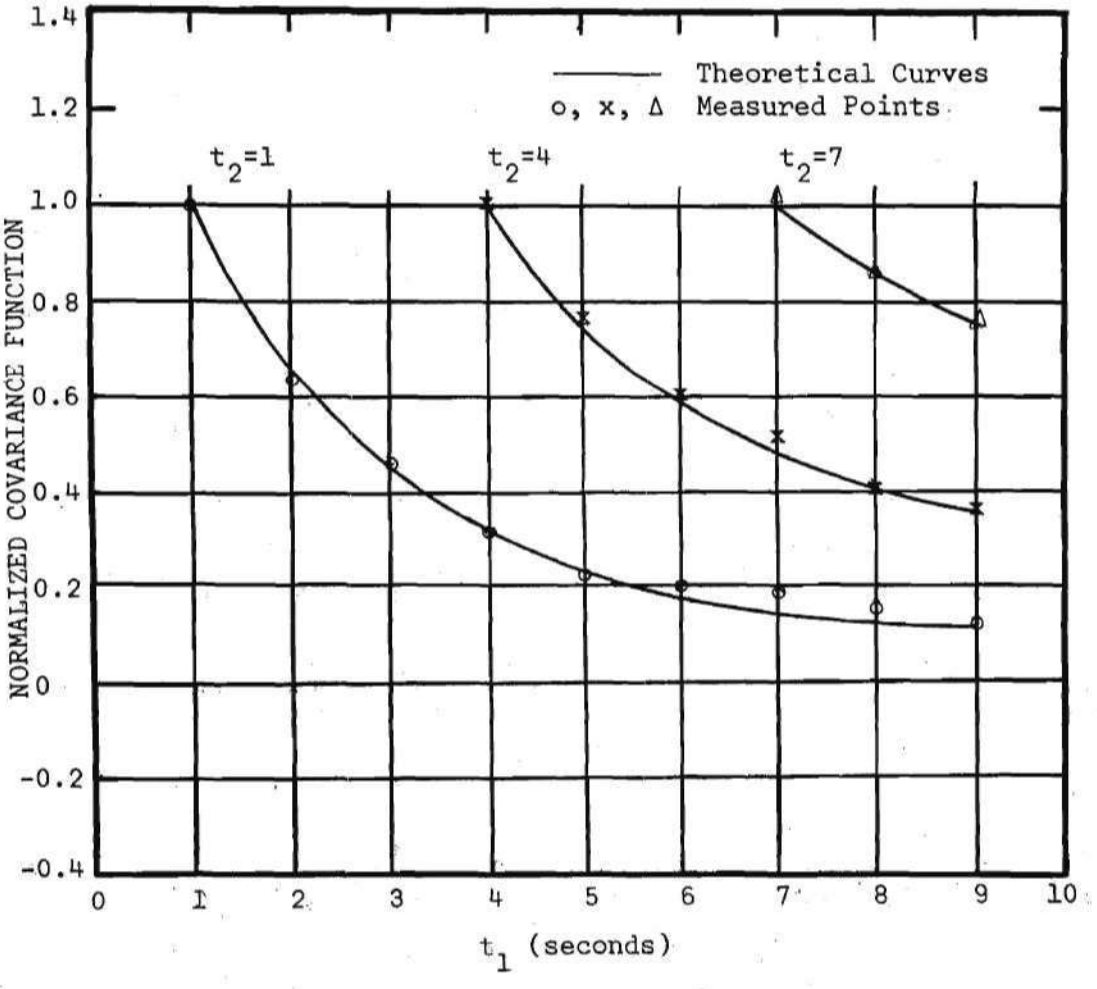


Figure 5-9. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example One--Velocity Profile Two.

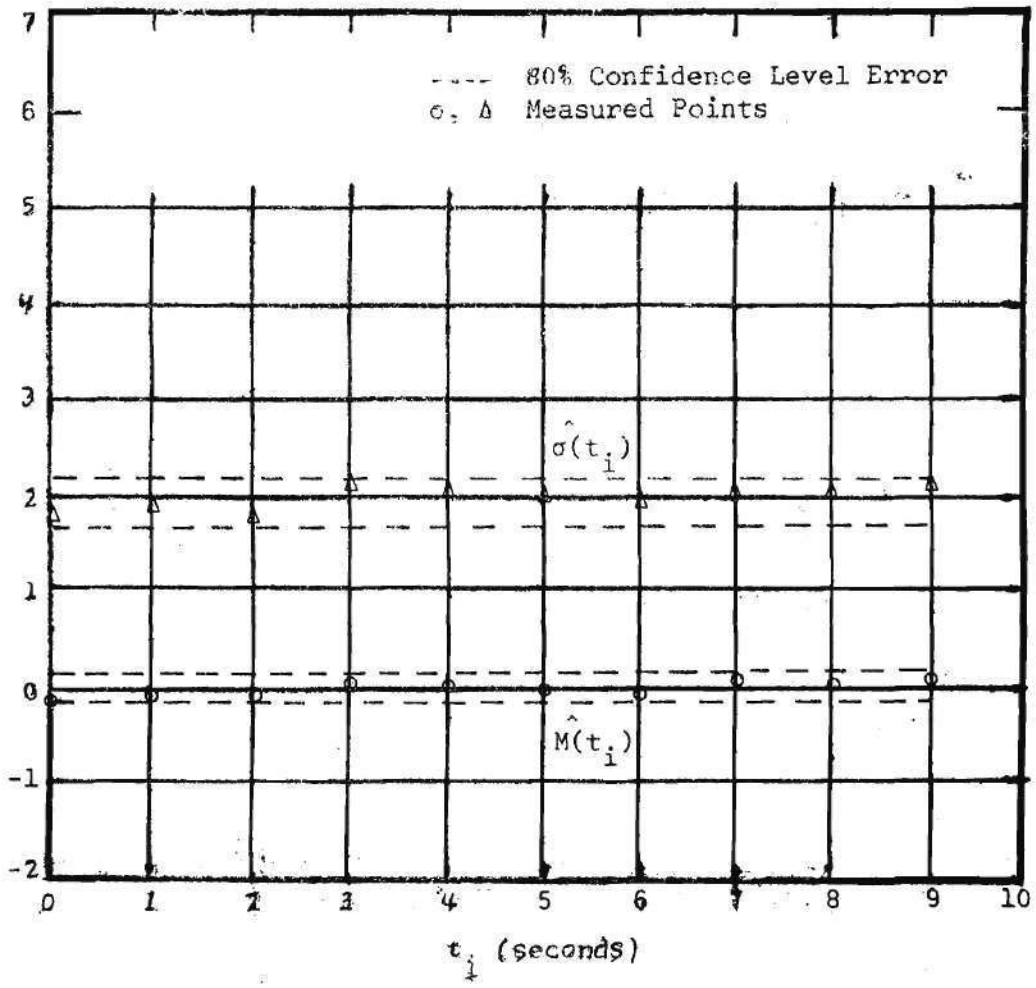


Figure 5-10. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example One--Velocity Profile Two.

Figure 5-12 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-13 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Four. The velocity profile used in the fourth case of Example One is illustrated in Figure 5-14.

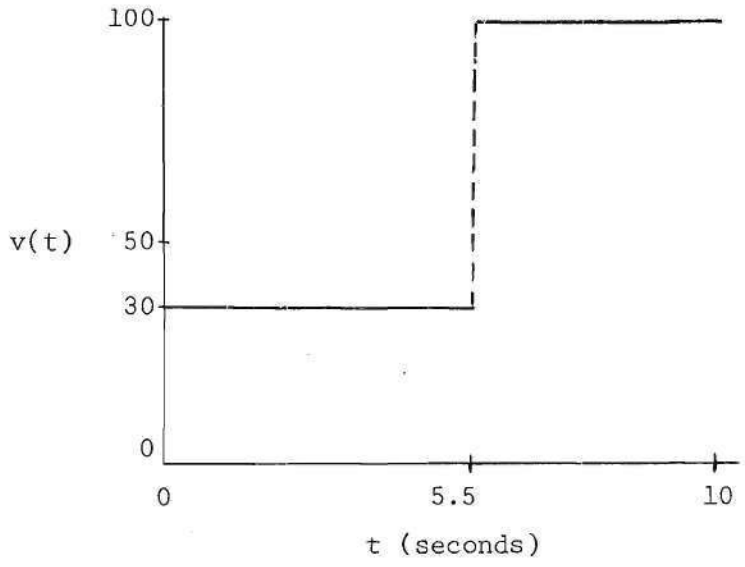


Figure 5-14. Example One--Velocity Profile Four.

The output random process in this case is again stationary in position x but nonstationary in time t. The position parameter x is given by

$$x(t) = \begin{cases} 30t + x(0) & , \ 0 \le t < 5.5 \\ 100t - 385 + x(0), & 5.5 \le t \le 10 \ . \end{cases} \qquad (5\text{-}16)$$

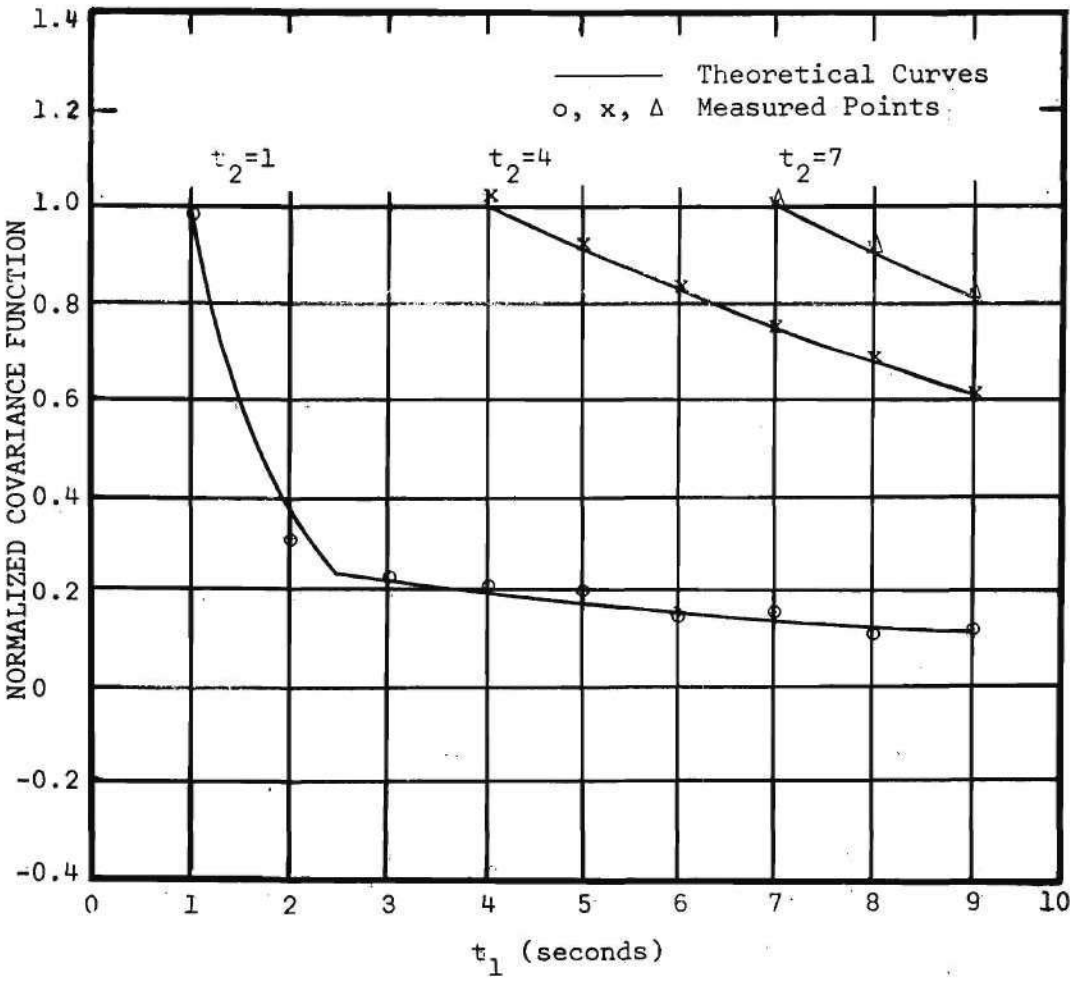


Figure 5-12. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example One--Velocity Profile Three.

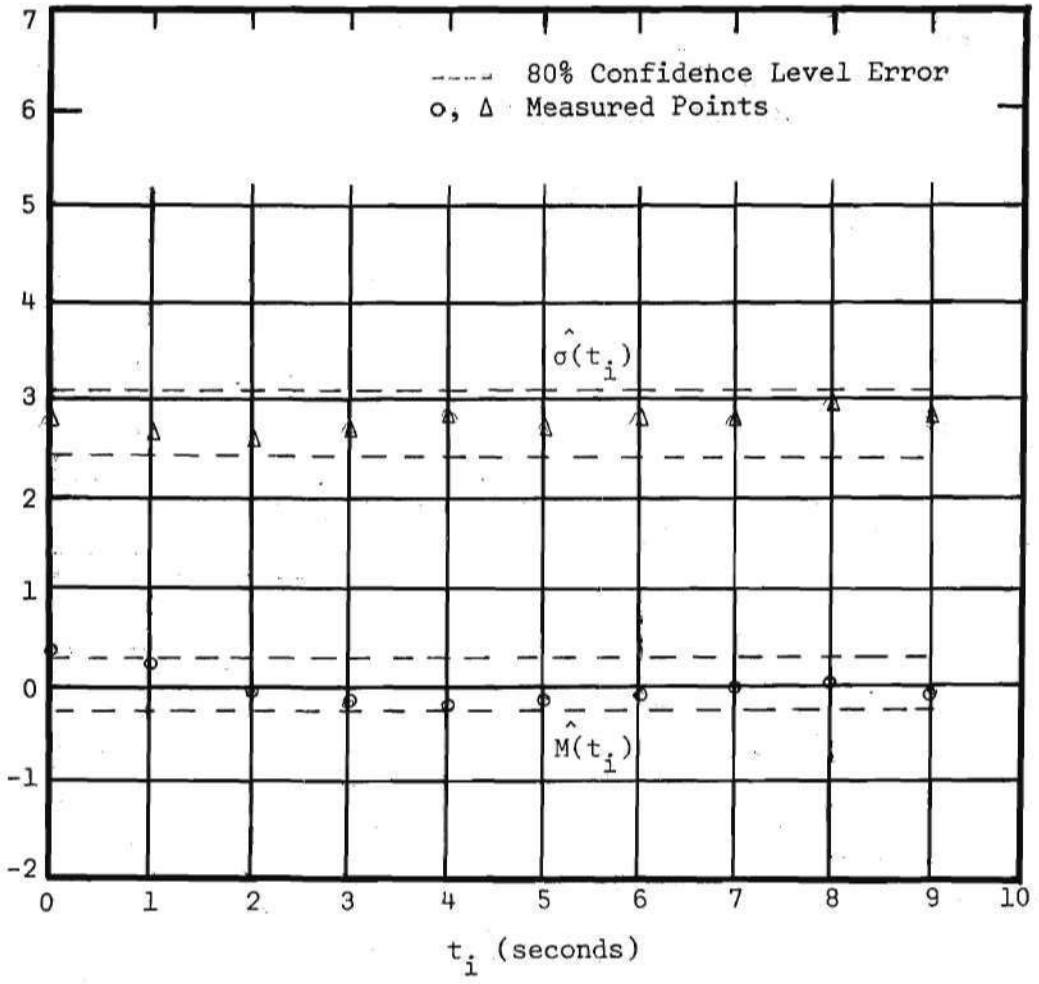


Figure 5-13. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example One--Velocity Profile Three.

Figure 5-15 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-16 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Five. The velocity profile used in the fifth case of Example One is illustrated in Figure 5-17.

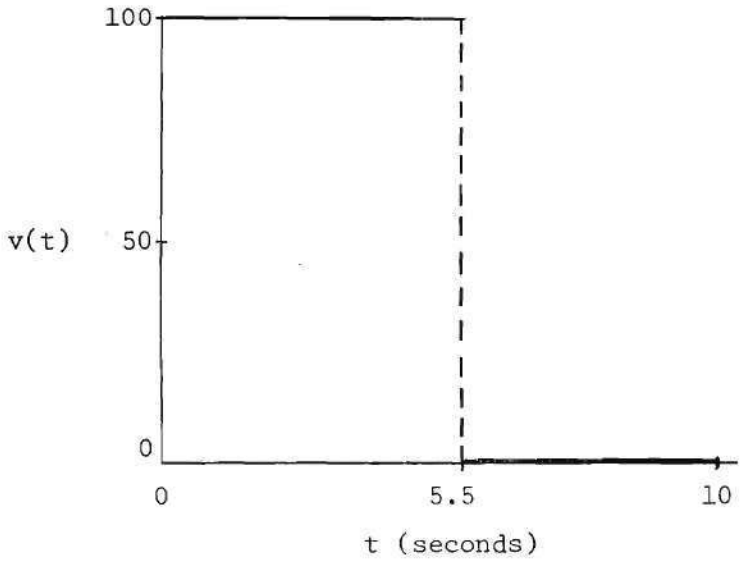


Figure 5-17. Example One--Velocity Profile Five.

The output random process in this case is again stationary in position x but nonstationary in time t. The position parameter x is given by

$$x(t) = \begin{cases} 100t + x(0), & 0 \le t < 5.5 \\ 550 + x(0), & 5.5 \le t \le 10 \end{cases} \quad (5\text{-}17)$$

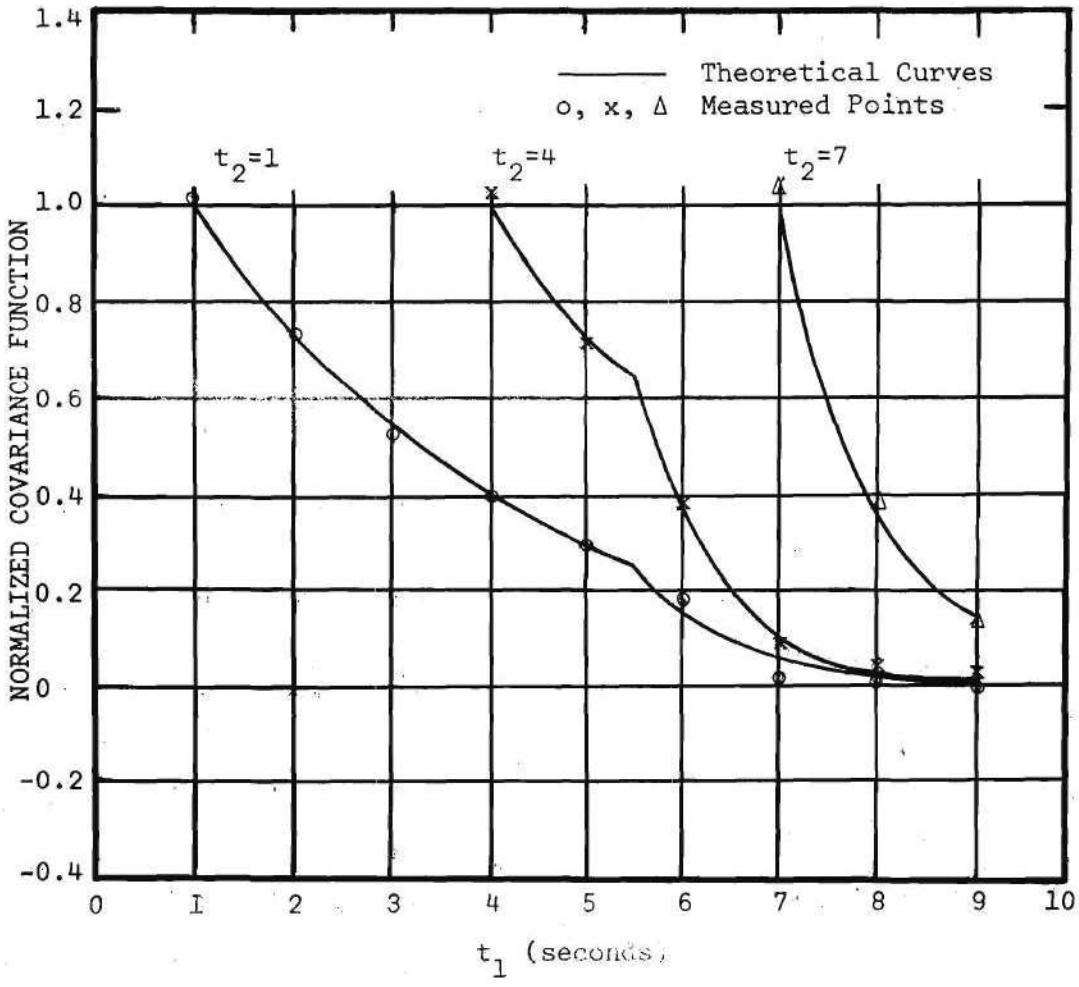


Figure 5-15. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example One--Velocity Profile Four.

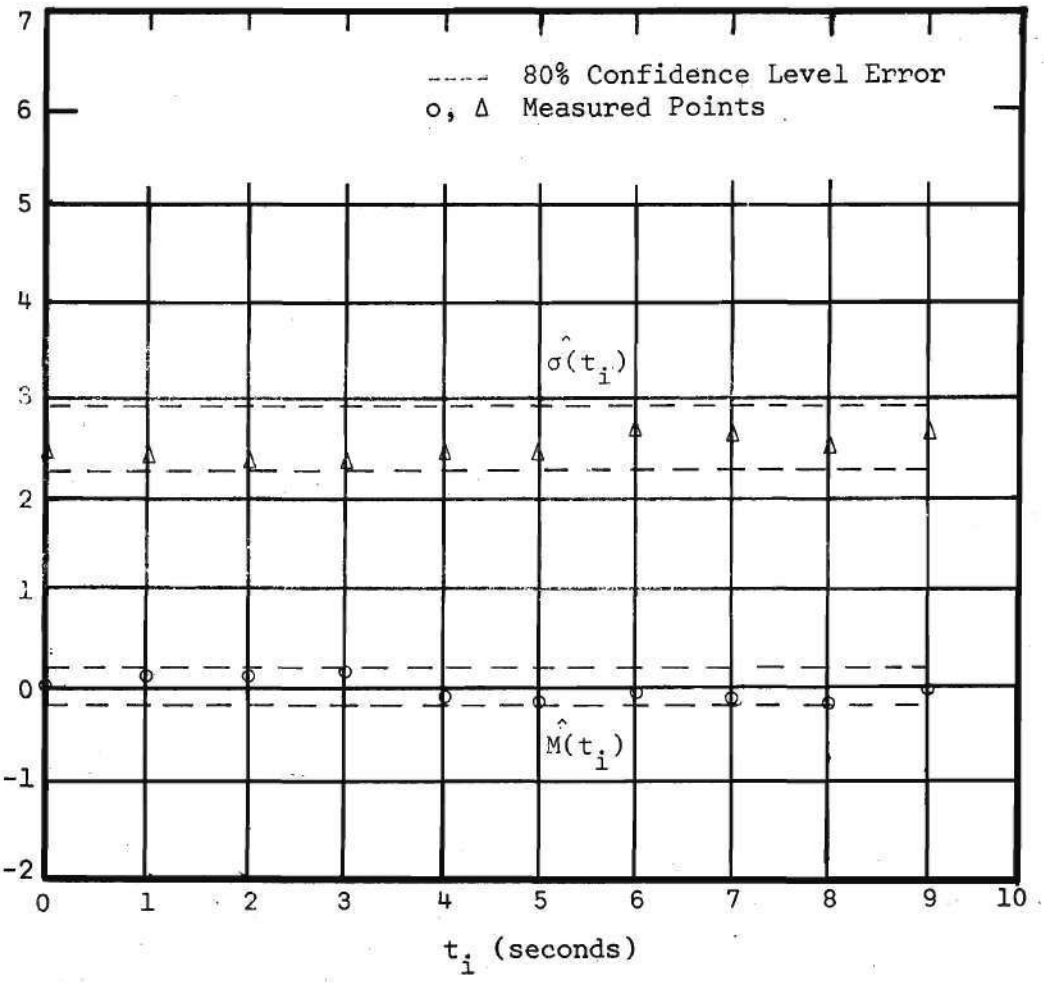


Figure 5-16. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example One--Velocity Profile Four.

Figure 5-18 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-19 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Example Two

The covariance function specified for the second experimental example is

$$R_g(x_1,x_2) = 10\varepsilon^{-0.01|x_1 - x_2|} + 10\varepsilon^{-0.02|x_1 - x_2|} , \quad (5\text{-}18)$$

where

$$x(t) = \int_0^t v(u)du + x(0) . \quad (5\text{-}19)$$

Using the synthesis procedure the corresponding system of first order differential equations related to Equation 3-51 when v is allowed to vary with time is given by

$$\dot{e}_1(t) = \sqrt{|v(t)|}\,(e_2(t) + a_1 w(t) - b_1\sqrt{|v(t)|}\,e_1(t)) \quad (5\text{-}20)$$

$$\dot{e}_2(t) = v(t)(a_o w(t) - b_o\sqrt{|v(t)|}\,e_1(t)) .$$

These equations are in the form required for the mechanization system. The analog computer mechanization system of Equation 5-20 is shown by

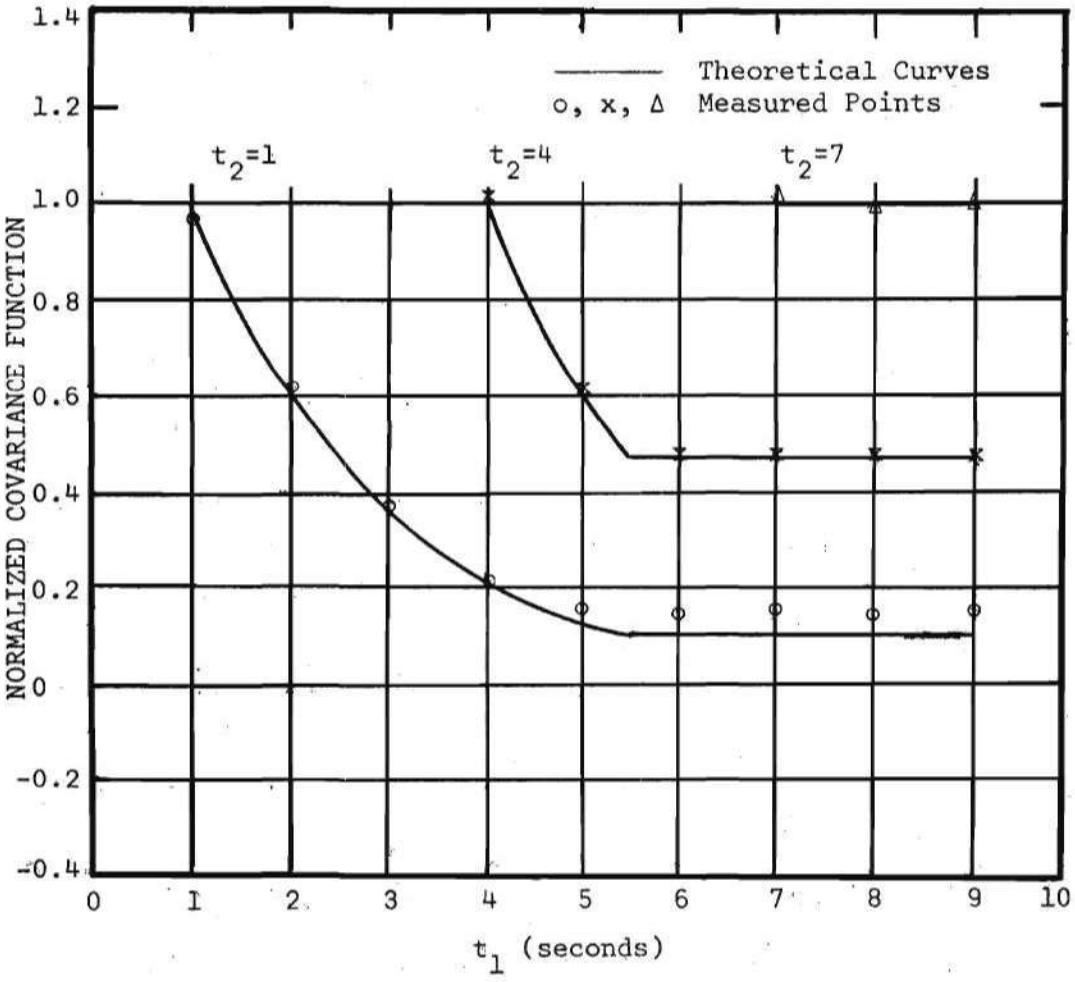


Figure 5-18. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example One--Velocity Profile Five.

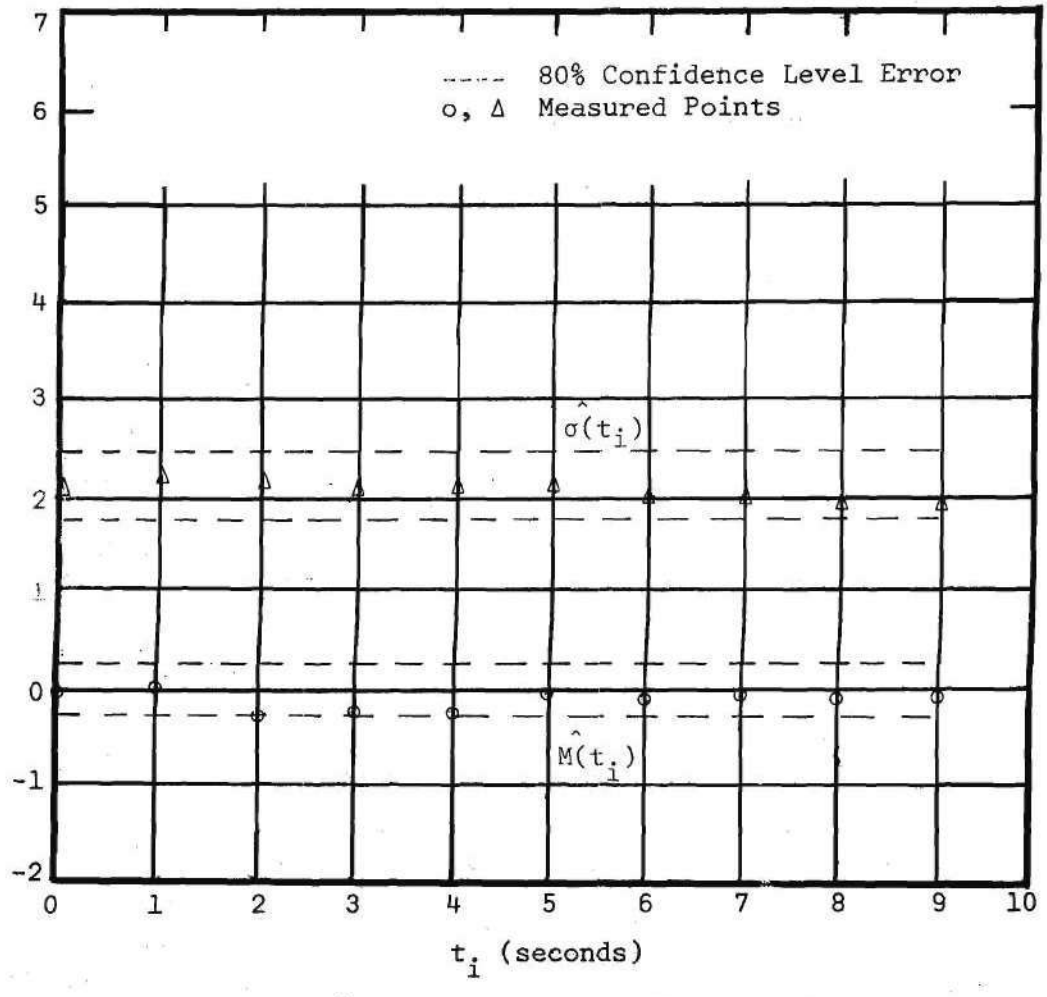


Figure 5-19. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example One--Velocity Profile Five.

the simplified block diagram in Figure 5-20. The mechanization system provides an exact realization of the composite random process g(x(t)) when the second derivative of x(t) is zero, i.e., v(t) is constant. The mechanization system is expected to provide a close approximation to the composite random process g(x(t)) when the second derivative of x(t) is small. The comparative analysis of the second order equations made in Chapter IV shows that the coefficients of the second order differential equations using the synthesis procedure and the ones from the parallel procedure are quite close in numerical value. The similarity of the differential equations indicates similarity of solutions e(t) which in turn implies similarity in the quantity $E\{e(t_1)e(t_2)\}$. Further since the operation of expected value is an averaging operation it is expected to smooth any effect of the small difference in the values of the coefficients. The data presented in the cases of Example Two substantiate the assertion. The relationship between x(t) and v(t) for each of the velocity profiles studied is given by Equation 5-19. For Example Two the coefficients are $a_o = 0.011$, $a_1 = 0.775$, $b_o = 0.0002$, and $b_1 = 0.03$.

Case One. Each of the following five cases uses different velocity profiles. Some of the cases lead to an output random process that is stationary in time, others nonstationary in time. For each case the random process is stationary in the position parameter x(t). Figure 5-21 presents the first velocity profile. For case one the position parameter x is given by

$$x(t) = 10t + x(0) . \qquad (5\text{-}21)$$

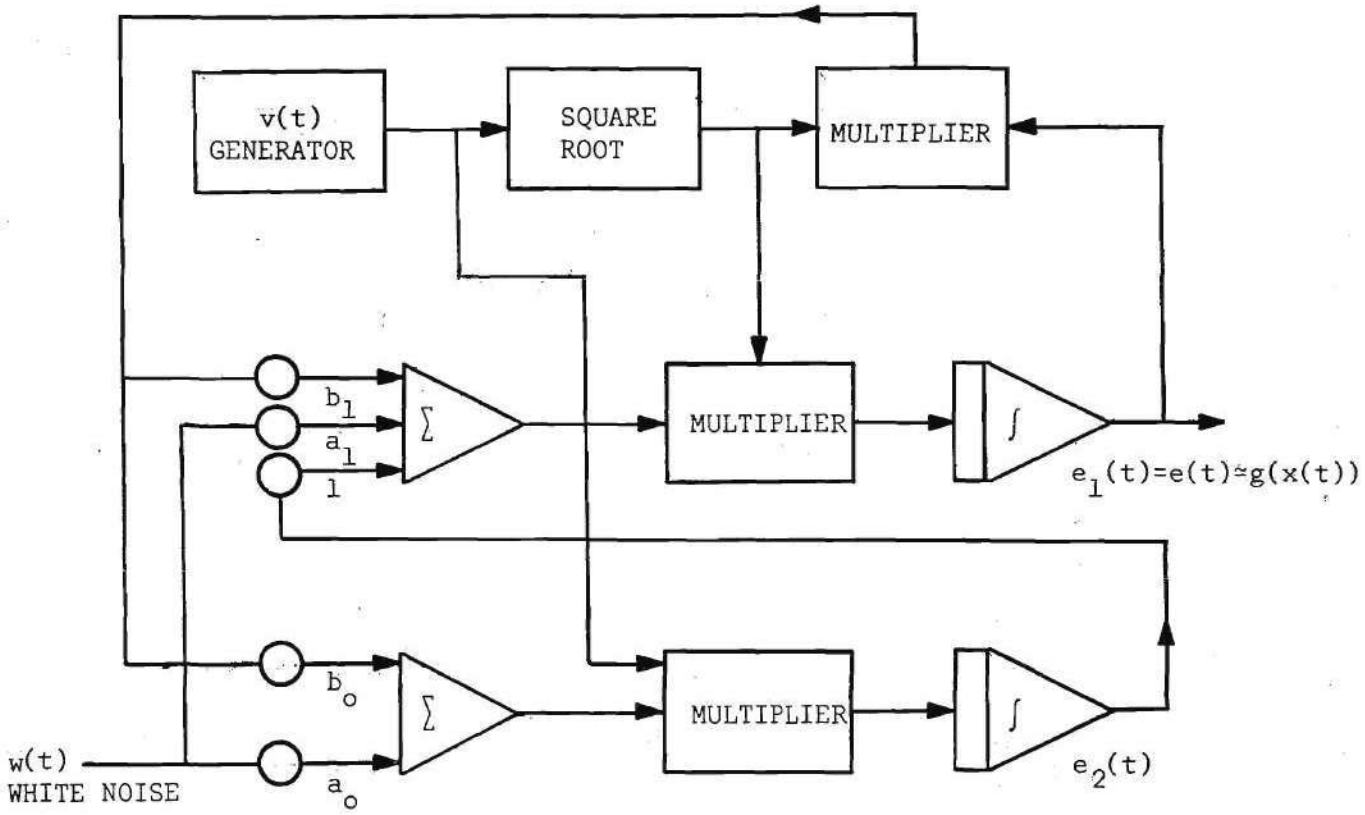


Figure 5-20. Mechanization System for Example Two.

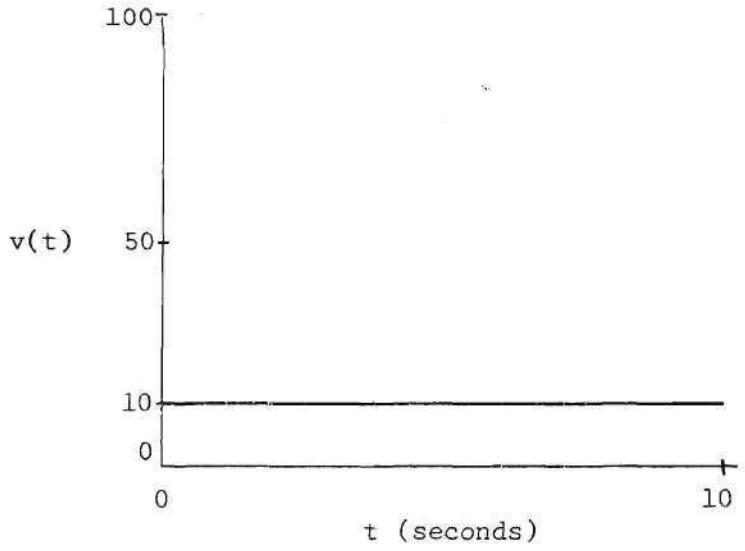


Figure 5-21. Example Two--Velocity Profile One.

The mechanization system generates a random process that is stationary in both position x and time t. Since the velocity v(t) is a constant, the output random process is given by

$$e(t) = g(x(t)) \ . \tag{5-22}$$

Figure 5-22 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-23 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Two. The velocity profile used in the second case of Example Two is illustrated in Figure 5-24. The output random process in this case is again stationary in both position x and time t. The position parameter x is given by

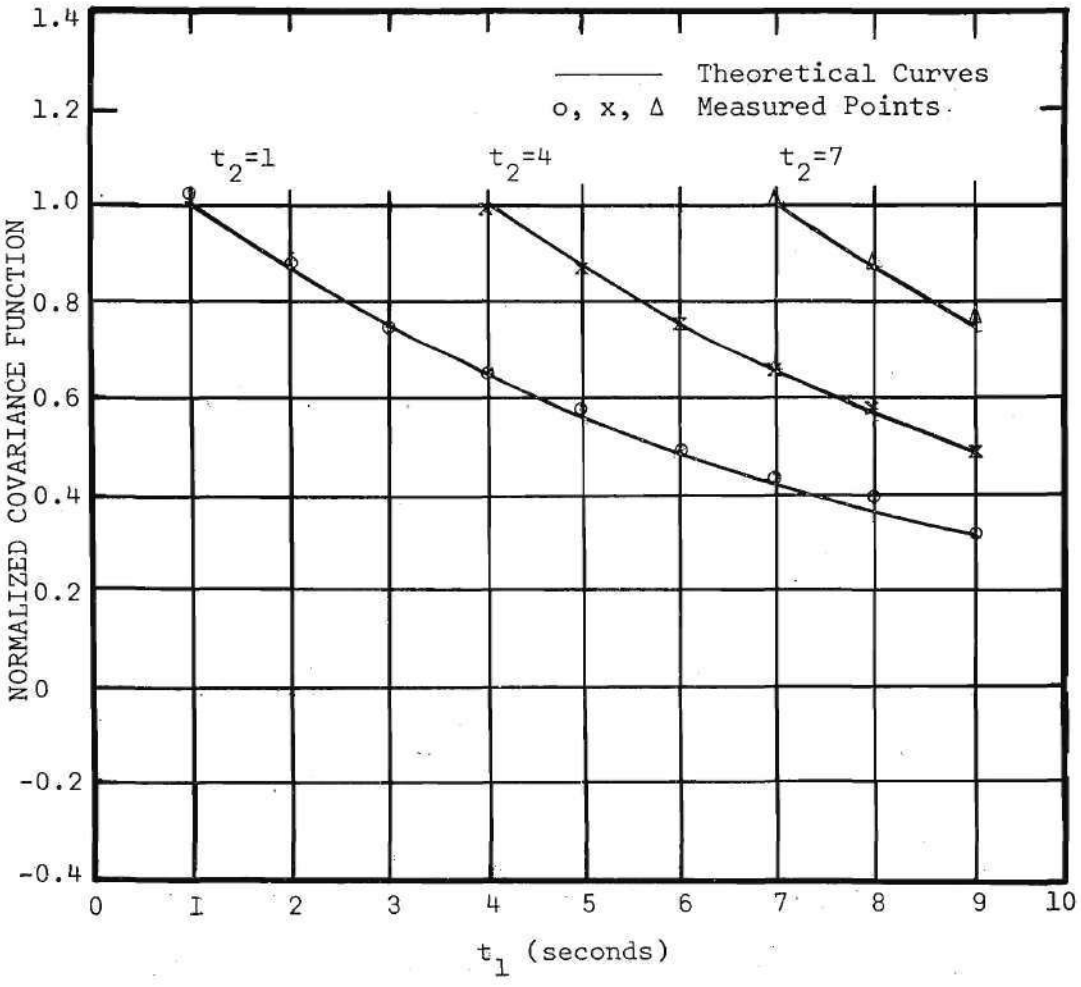


Figure 5-22. Normalized Covariance Function $\hat{R}_c(t_1,t_2)$ for Example Two--Velocity Profile One.

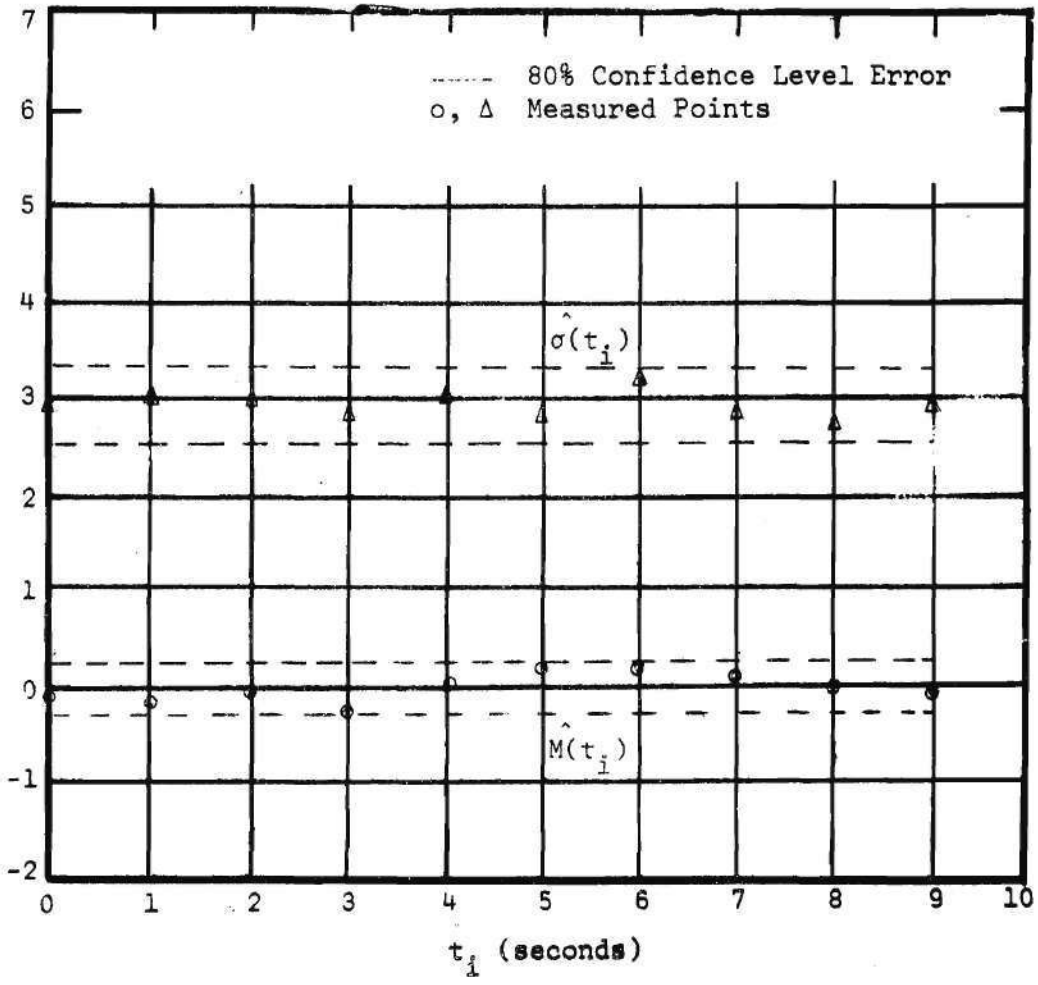


Figure 5-23. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example Two--Velocity Profile One.

$$x(t) = 50t + x(0) \ . \qquad (5\text{-}23)$$

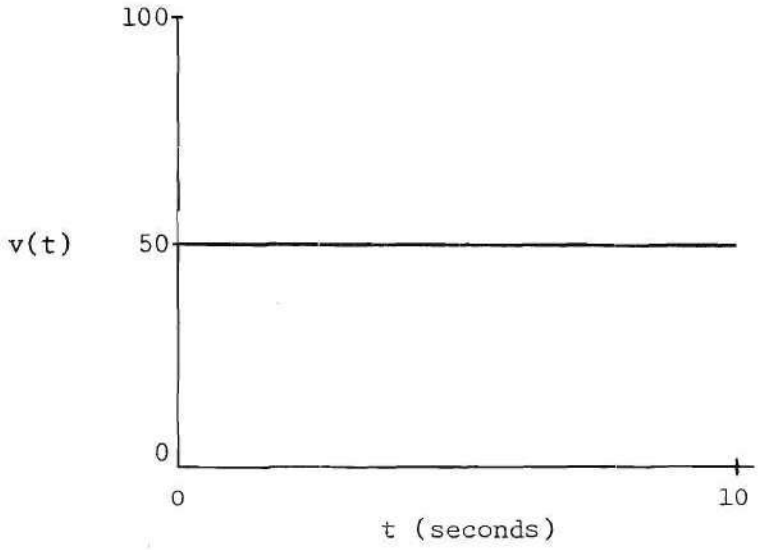


Figure 5-24. Example Two--Velocity Profile Two.

The mechanization system output is given by $e(t) = g(x(t))$. Equality is achieved since $v(t)$ is constant.

Figure 5-25 illustrates the normalized covariance function, $\hat{R_c}(t_1,t_2)$, for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-26 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Three. The velocity profile used in the third case of Example Two is illustrated in Figure 5-27. The output random process in this case is stationary in position x, but nonstationary in time t. The position parameter x is given by

$$x(t) = 0.5t^2 + 50t + x(0) \ . \qquad (5\text{-}24)$$

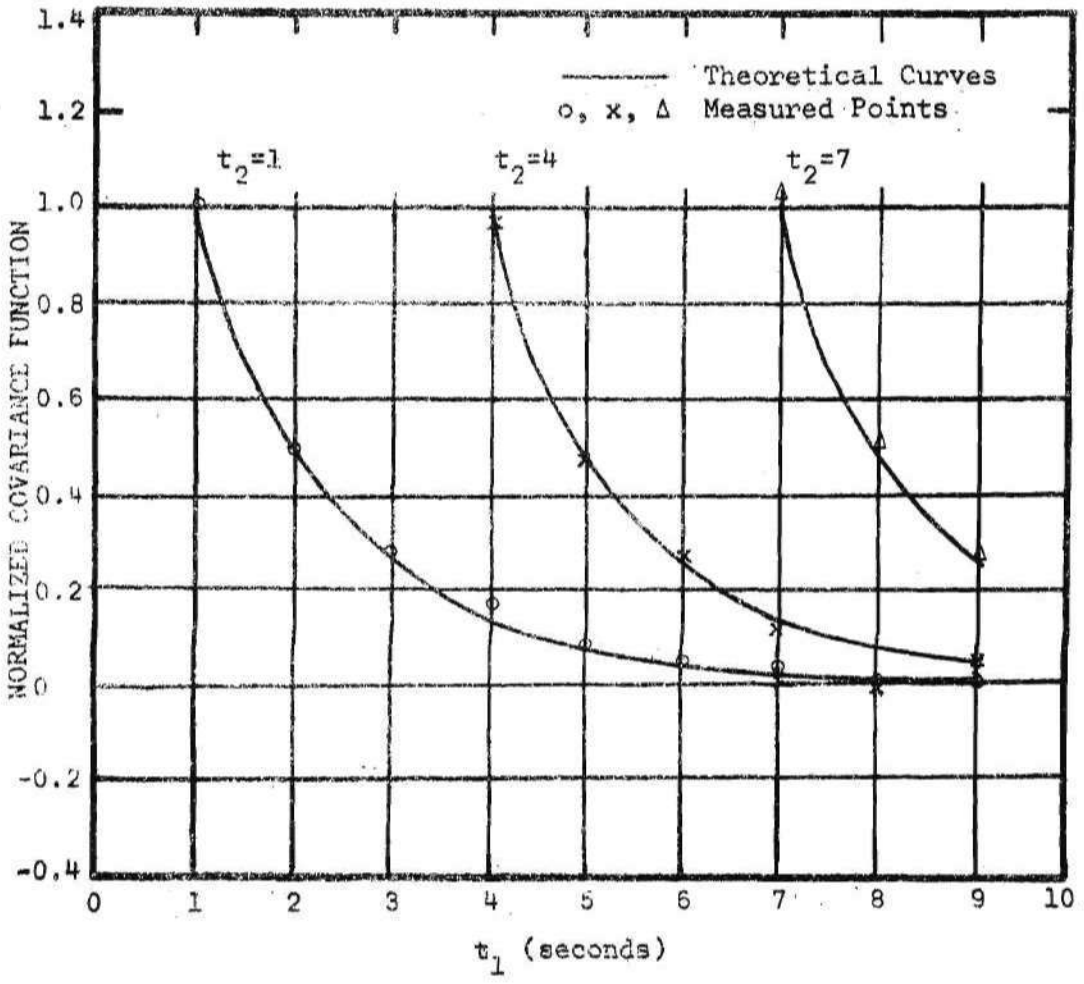


Figure 5-25. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example Two--Velocity Profile Two.

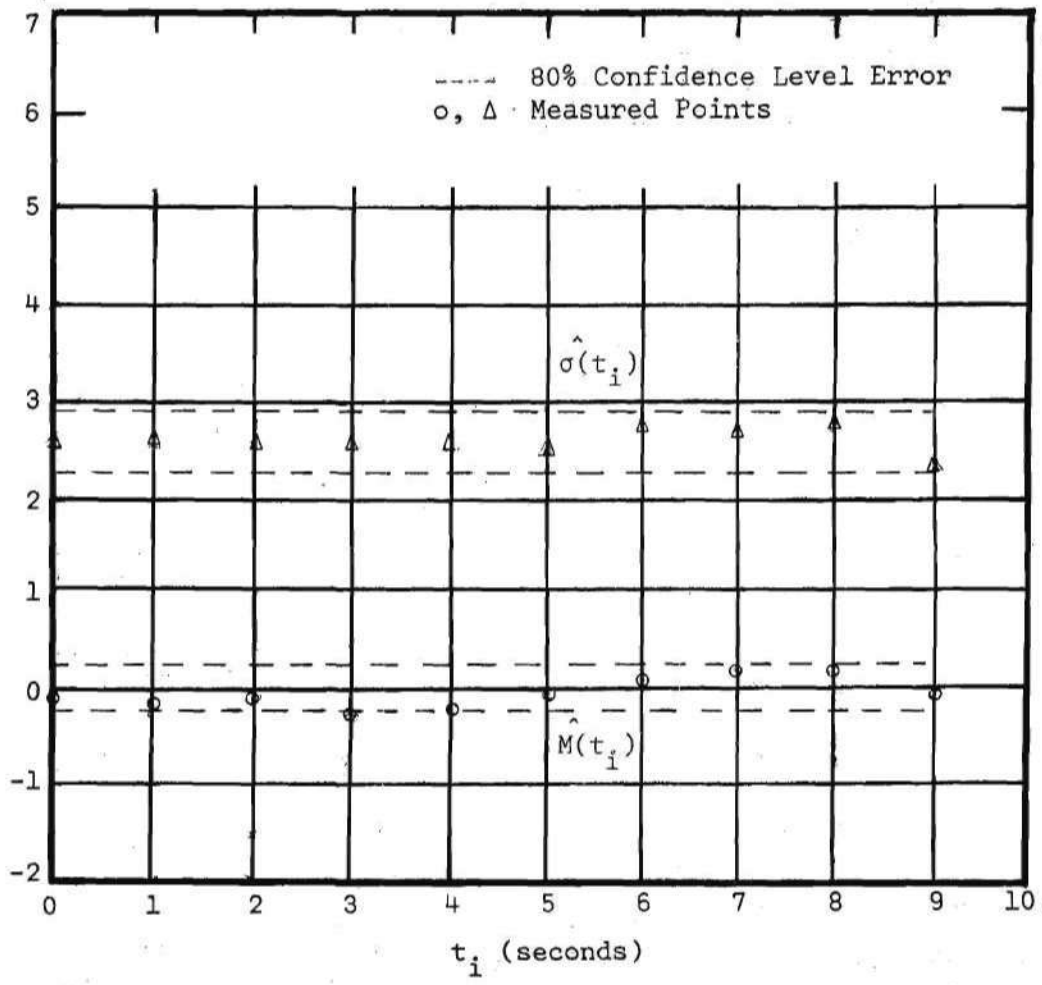


Figure 5-26. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example Two--Velocity Profile Two.

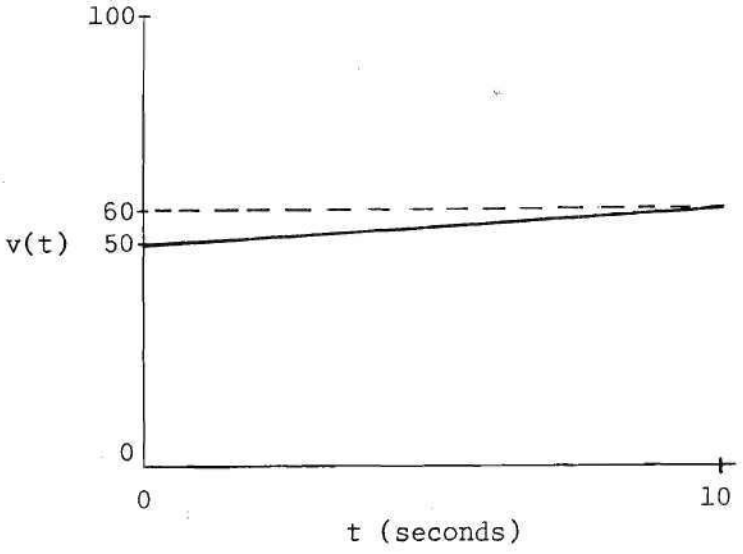


Figure 5-27. Example Two--Velocity Profile Three.

The mechanization system output in this case is a close approximation to the composite process written as

$$e(t) \simeq g(x(t)) \ . \qquad (5\text{-}25)$$

Figure 5-28 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-29 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Four. The velocity profile used in the fourth case of Example Two is illustrated in Figure 5-30. The output random process in this case is again stationary in position x but nonstationary in time t. The position parameter x is given by

$$x(t) = - 2.5t^2 + 50t + x(0) \ . \qquad (5\text{-}26)$$

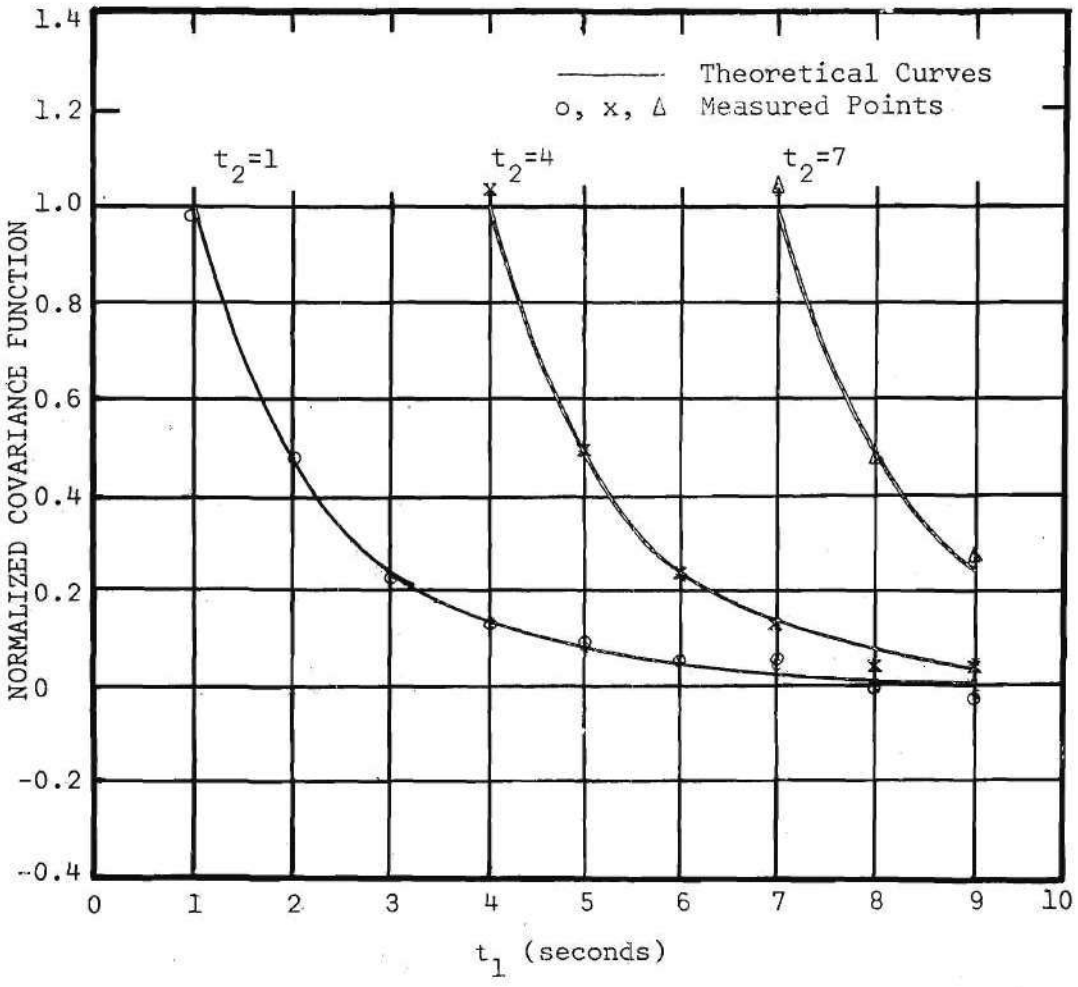


Figure 5-28. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example Two--Velocity Profile Three.

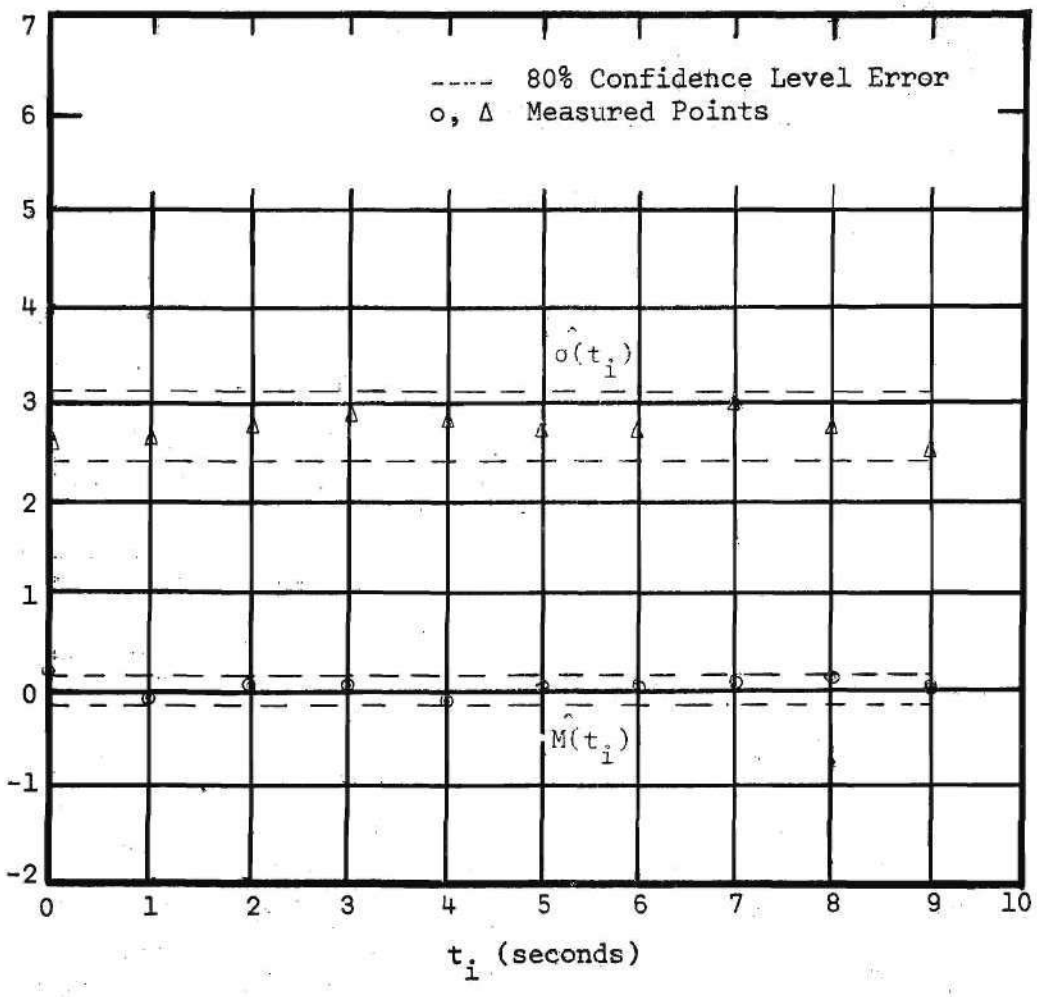


Figure 5-29. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example Two--Velocity Profile Three.

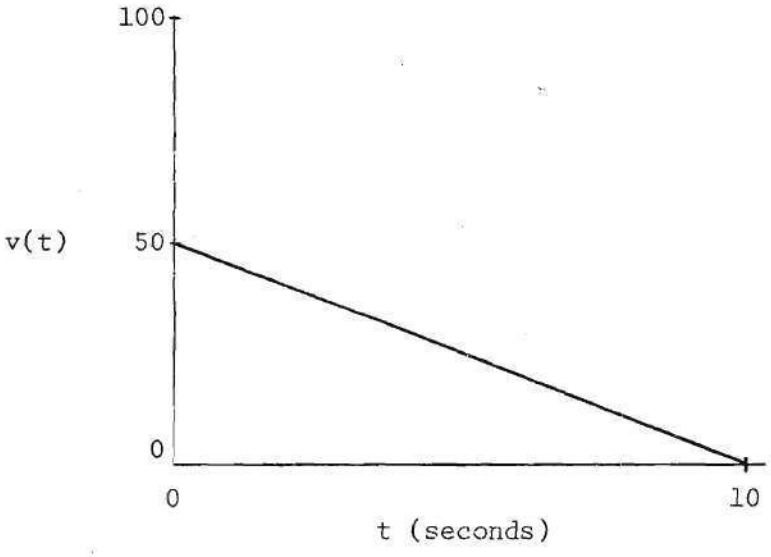


Figure 5-30. Example Two--Velocity Profile Four.

The mechanization system is expected to yield a close approximation to g(x(t)) during the initial part of the run, but as v(t) approaches zero the ratio of $\dot{v}(t)/v(t)^2$ becomes large and that parameter was required in the comparative analysis of Chapter IV to be small for a valid approximation. The smoothing effect of the expected value operation is demonstrated here since the experimental data is still quite close to the theoretical curves.

Figure 5-31 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-32 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

Case Five. The velocity profile used in the fifth case of Example Two is illustrated in Figure 5-33.

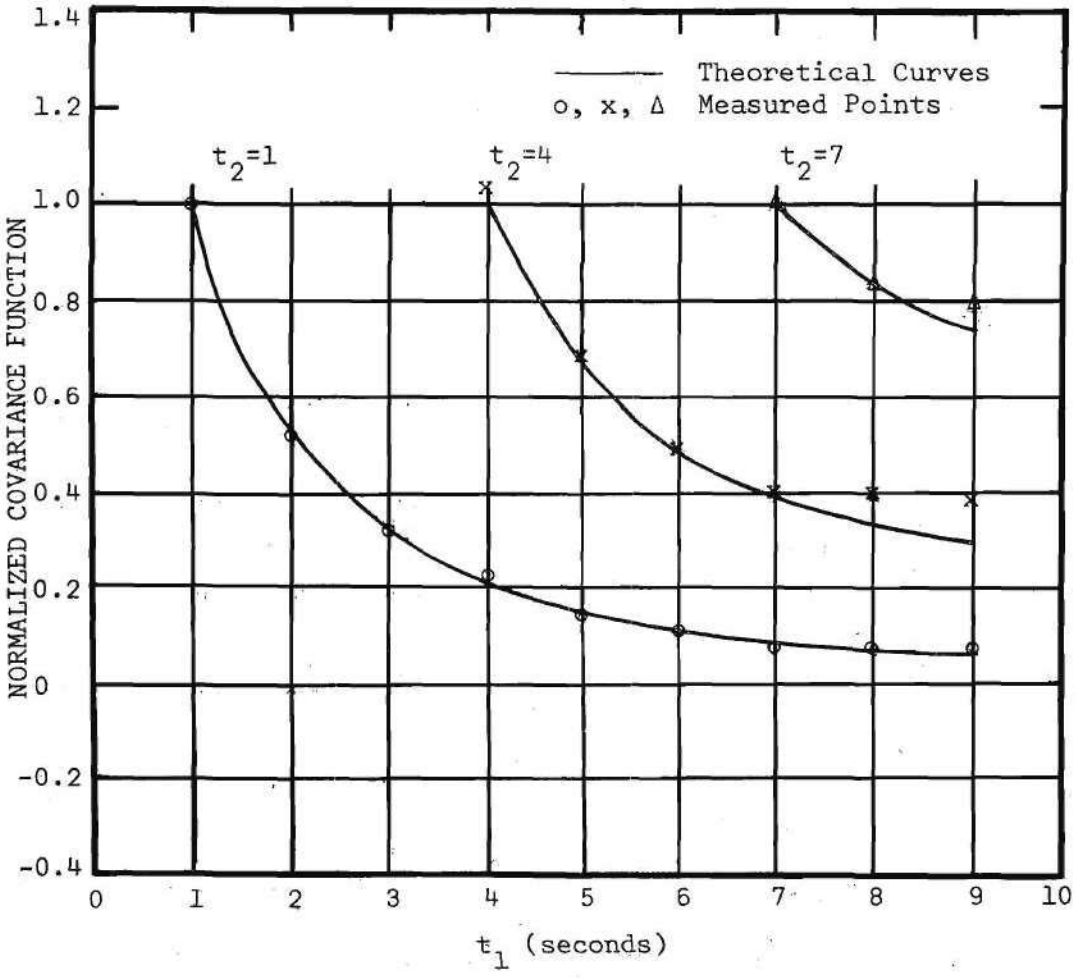


Figure 5-31. Normalized Covariance Function $\hat{R_c}(t_1,t_2)$ for Example Two--Velocity Profile Four.

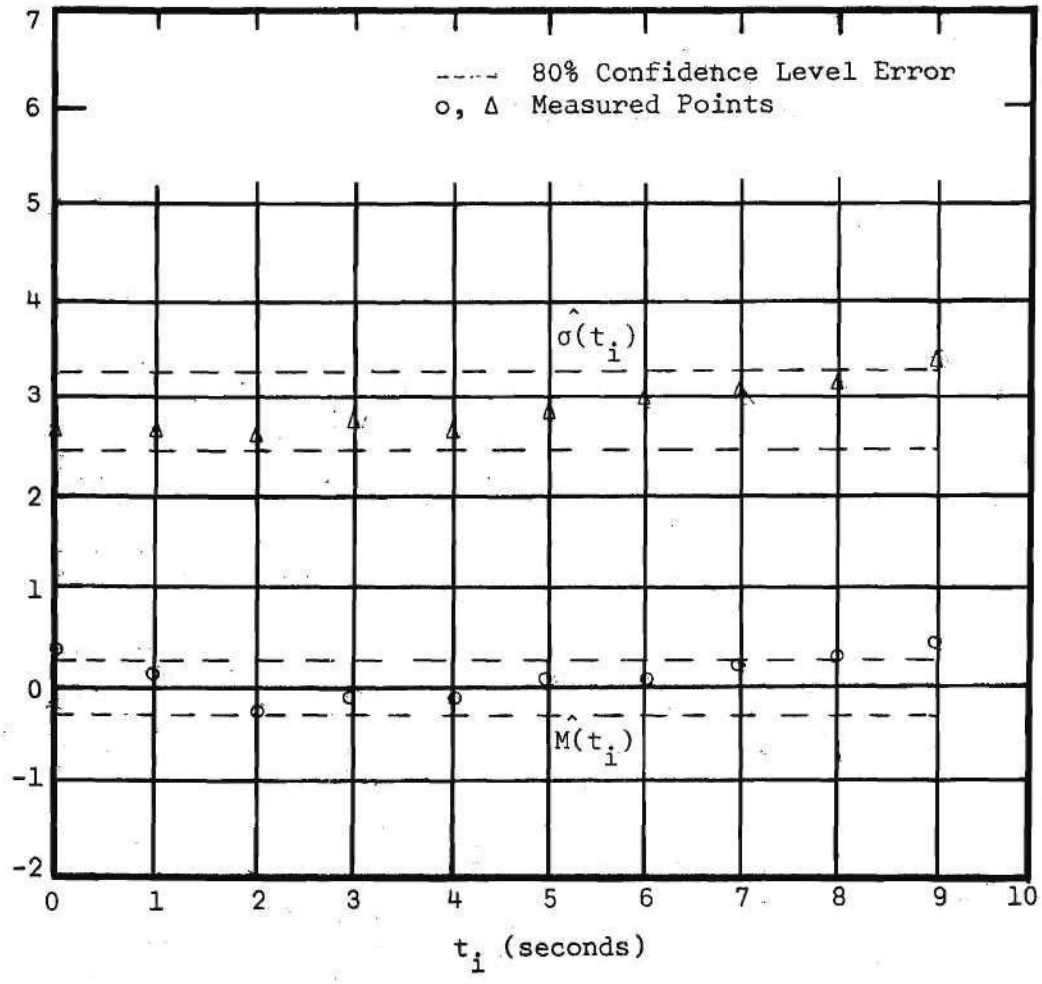


Figure 5-32. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example Two--Velocity Profile Four.

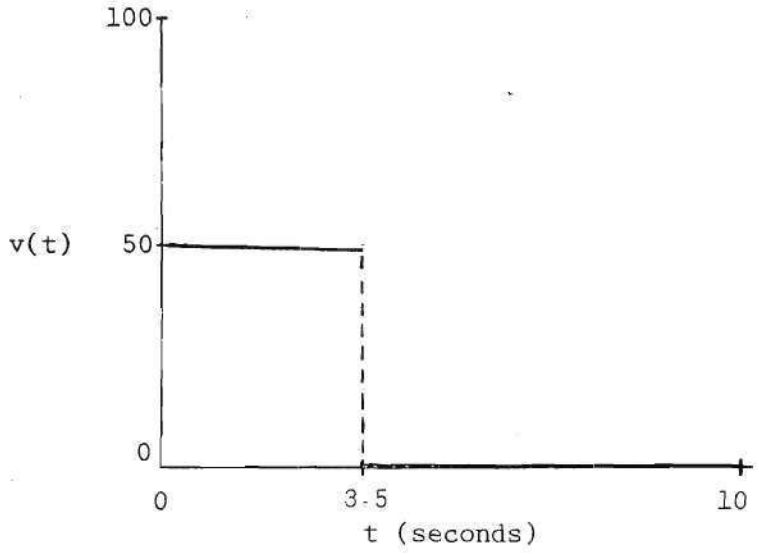


Figure 5-33. Example Two--Velocity Profile Five.

The output random process is stationary in position x but nonstationary in time t. The position parameter x is given by

$$x(t) = \begin{cases} 50t + x(0), & 0 \leq t < 3.5 \\ 175 + x(0), & 3.5 \leq t \leq 10 \end{cases} . \qquad (5\text{-}27)$$

The mechanization system is expected to yield a close approximation to g(x(t)) for this velocity profile.

Figure 5-34 illustrates the normalized covariance function $\hat{R_c}(t_1,t_2)$ for t_2 = 1, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds. Figure 5-35 illustrates the process mean $\hat{M}(t_i)$ and the deviation $\hat{\sigma}(t_i)$ with their corresponding confidence level error boundaries.

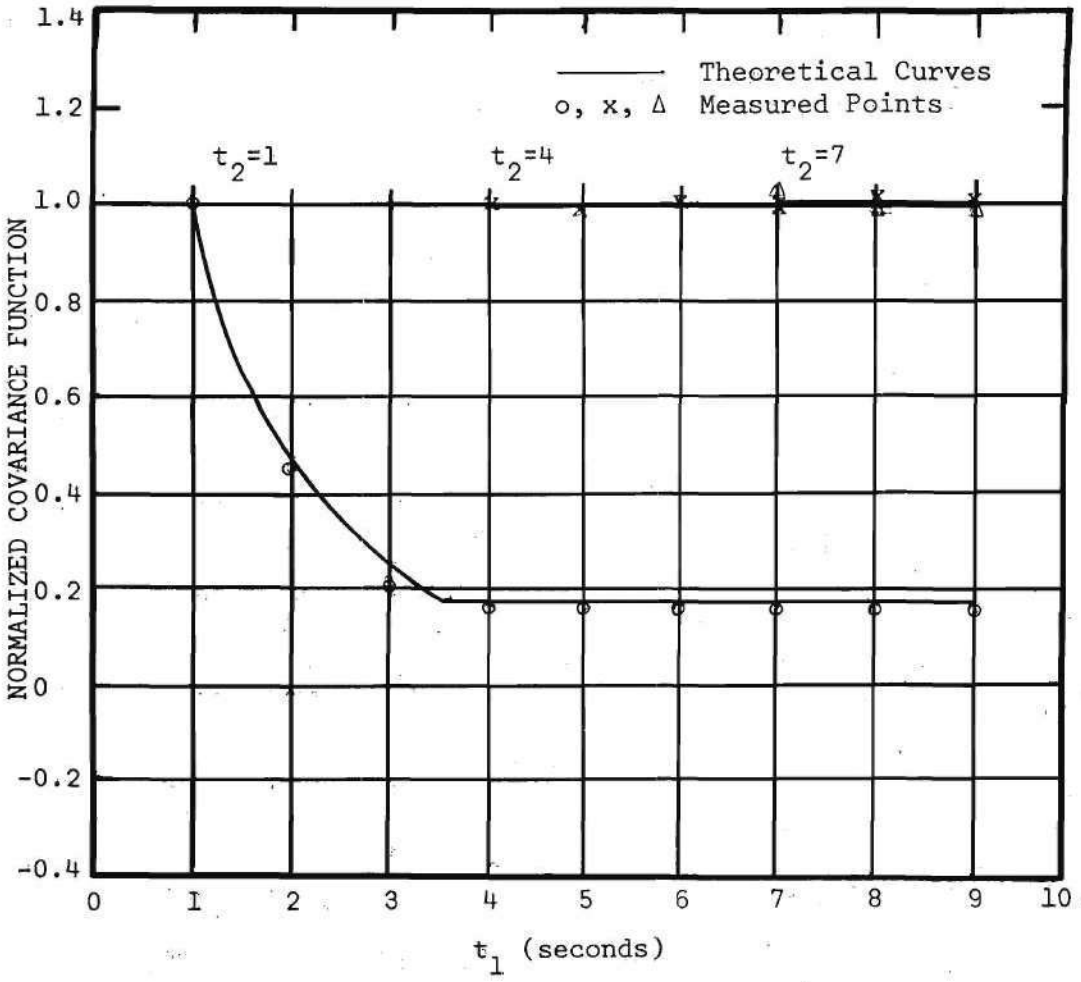


Figure 5-34. Normalized Covariance Function $\hat{R}_c(t_1,t_2)$ for Example Two--Velocity Profile Five.

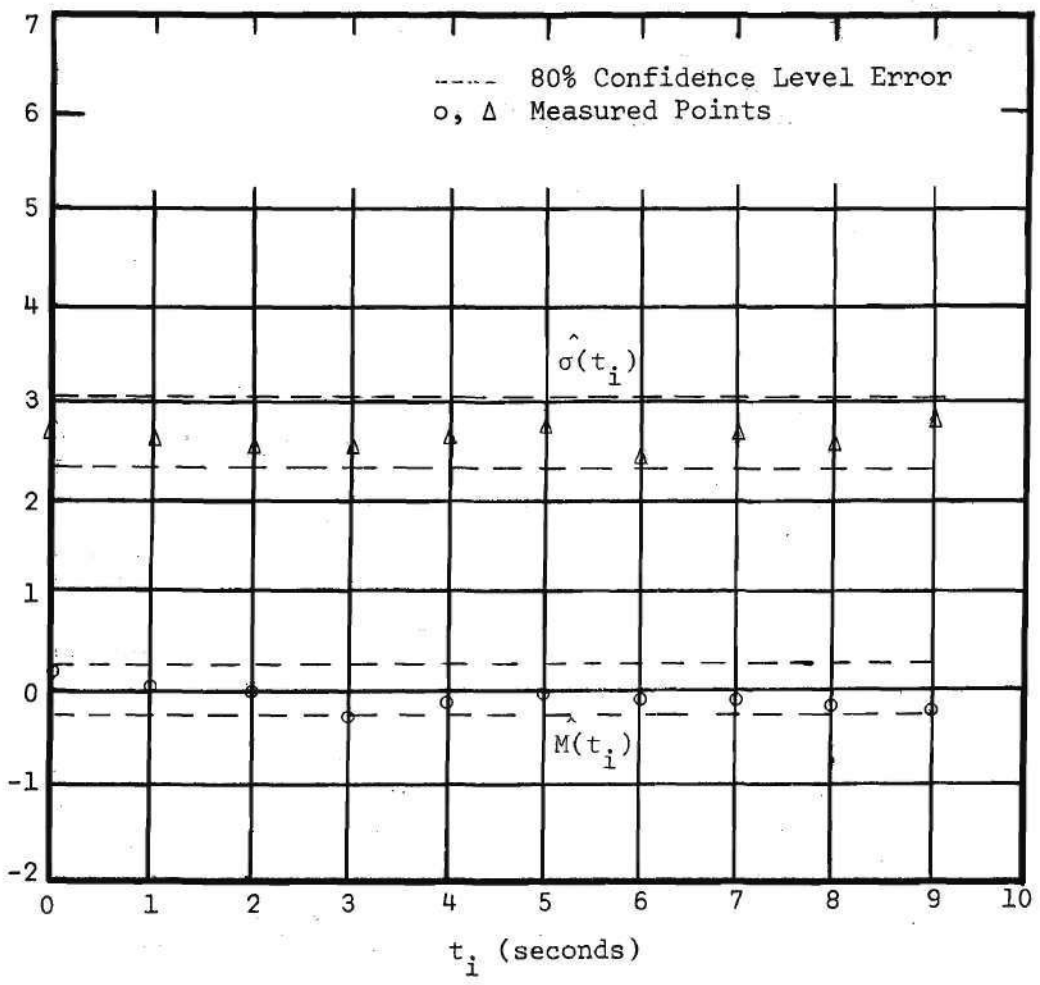


Figure 5-35. Mean $\hat{M}(t_i)$ and Deviation $\hat{\sigma}(t_i)$ for Example Two--Velocity Profile Five.

Summary of Experimental Results

For both of the examples, five velocity profiles were investigated. In all of the cases of the two examples, the specified random processes were stationary in the position parameter x(t). In several of the cases the random processes were nonstationary in the time parameter t.

For all of the cases in Example One, the mechanization system yields exactly the composite random process g(x(t)). The experimental data plotted versus the normalized covariance function are quite close to the desired statistics. The random process mean and deviation estimates conform to the 80 per cent confidence level error boundaries.

For the first two cases for Example Two, the mechanization system yields exactly the composite random process g(x(t)). The last three cases approximate closely the composite random process. The experimental data plotted versus the normalized covariance function are again quite close to the desired statistics. As in Example One the random process mean and deviation estimates conform to the 80 per cent confidence level error boundaries.

CHAPTER VI

CONCLUSIONS

In this chapter the problem statement and the general approach are summarized and compared in complexity and utility to the parallel approach. The experimental results are considered and some areas of new research and suggestions for extensions to the approach of this research are presented.

The problem of this research is to develop and investigate a procedure for synthesizing slowly time-varying filters which generate stationary Gaussian random process in a position parameter, but which may be nonstationary in the time parameter.

The general approach taken to solve the problem was as follows: First, to establish the relationship between the specified random process statistics (the mean and the autocovariance) and a realizable analog computer mechanization which approximates the random process. Second, to investigate analytically and experimentally several examples using the synthesis procedure to determine the quality of the approximate mechanization system. Third, to develop a parallel approach which realizes exactly the prescribed random process. This exactness provides a basis for a comparative analysis of the nth order differential equations derived from the approximate approach and the parallel approach.

The synthesis procedure generates a Gaussian random process which approximates the statistics of the composite random process $g(x(t))$.

The specified statistics used in the procedure are the mean and the autocovariance function stated in the position parameter x. The relationship of x to time t is not required *a priori* but the parameter x(t) must have a first time derivative given by the velocity v(t). The velocity v(t) is then used as an auxiliary input to the mechanization system. Application of the procedure is straightforward and leads, in general, to a system of n first order differential equations. The system of equations is realized by a mechanization system which generates a solution that approximates the composite random process. The approximate representation of g(x(t)) is actually exact for all cases where $\dot{x}(t)$ is constant; the approximation is expected to be good when $\dot{x}(t)$ is a slowly varying function of time.

The constraints that arise in the development of the procedure restrict the types of autocovariances that can be used in the approximation procedure. The general form of the autocovariance is established.

A parallel approach is developed that yields for all of the admissible autocovariance functions an exact result. It is an extension of the procedure developed by Webb, et al. (1), to include the composite function g(x(t)). The parallel approach is more general than the approximate approach, which could be considered a special case of the parallel approach. The principal restriction of the parallel procedure is that the autocovariance function must be expressable as a finite series of terms separable in the parameters x_1 and x_2 given as

$$r_g(x_1,x_2) = \sum_{i=1}^{n} \phi_i(x_1)\gamma_i(x_2) \ .$$

The parallel approach has some inherent disadvantages in that it is quite involved mathematically and requires extensive use of analog equipment. It has the distinct advantage of exactness, which is costly both mathematically and from the standpoint of equipment usage.

The experimental data substantiate the assertion that the approximate procedure yields an output random process that is close to the specified composite random process. The discrete data points are in agreement with the predicted theoretical curves for the normalized autocovariance function. The mean and the deviation of the random processes were observed to conform to the 80 per cent statistical confidence level error boundaries. Thus, the approximate procedure is useful for simulation problems where precise solutions are not required and especially in those problems where analog equipment limitations dictate the quality of the simulation.

One area for future investigation is the extension of the approximate approach through linearization techniques to position functions $x(t)$ which are not slowly time-varying functions. Since the mechanization system is exact for velocities $v(t)$ which are constant, the position functions $x(t)$ would be broken into straight line segments, which represent constant velocity segments. With appropriate bandwidth considerations it may be possible to develop a more exact approximate procedure for the segmented velocity profiles while still maintaining the simplicity of the approximate procedure.

Another area for future investigation is an error analysis of the approximate synthesis procedure.

APPENDIX

APPENDIX A

ADMISSIBLE AUTOCORRELATION FUNCTIONS

This appendix is included to provide the foundation for the discussions of Chapter III regarding the types of autocorrelation functions for which the synthesis procedure is valid. The constraints that $S(\omega)$, the direct Fourier transform of $R(\tau)$, be even, non-negative, real, and expressible as a ratio of polynomials in ω certainly becomes a restriction on $R(\tau)$. The implications of this power spectral density constraint on the autocorrelation function $R(\tau)$ and the admissible forms for that function are developed in this appendix. Also, the relationship of the constrained $S(\omega)$ and the transfer function $H(p)$ of a fixed parameter linear filter derived by spectral factorization techniques is presented. Several transform pairs are presented with their respective transfer functions.

Fundamental Considerations

The random processes of interest are real random processes; hence, the autocorrelation function is real and an even function (see Papoulis (27, page 337)), i.e.,

$$R(\tau) = R(-\tau) = R^{*}(\tau) \qquad \text{(A-1)}$$

and

$$R(0) \geq R(\tau) \; . \qquad \text{(A-2)}$$

Since $R(\tau)$ is real and even the power spectral density is real, even, and non-negative (see Papoulis (27, pages 338 and 347)), i.e.,

$$S(-\omega) = S(\omega) \geq 0 \qquad \text{(A-3)}$$

for all real ω. If the additional constraint on $S(\omega)$, that it be expressible as a ratio of polynomials in ω, i.e., that $S(\omega)$ is a *rational* function of ω, is imposed, all of the required constraints on $S(\omega)$ are present and the allowable forms for $R(\tau)$ may be considered.

It is of interest first to show the realization of the transfer function $H(p)$ at this point. With the above constraints the power spectral density can be expressed as follows,

$$S(\omega) = S(p/j) = A(-p^2)/B(-p^2) \ , \quad \omega = p/j \ , \qquad \text{(A-4)}$$

where $A(-p^2)$ and $B(-p^2)$ are polynomials in $-p^2$ with real coefficients. The degree of $A(-p^2)$ is smaller than the degree of $B(-p^2)$ when the $R(\tau)$ does not contain impulses.

If p_i is a root of the polynomial $A(-p^2)$, $-p_i$ is also a root; hence, the roots of $A(-p^2)$ are symmetrical with respect to the imaginary axis in the p-plane. The same is true for $B(-p^2)$. Thus, $S(p/j)$ can be factored as

$$S\left(\frac{p}{j}\right) = \frac{A(-p^2)}{B(-p^2)} = \frac{C(p)}{D(p)} \ \frac{C(-p)}{D(-p)} \ , \qquad \text{(A-5)}$$

where C(p) and D(p) contain all of the roots of $A(-p^2)$ and $B(-p^2)$, respectively, that lie in the left-half plane. Thus, the transfer function of a realizable fixed parameter linear filter can be taken as

$$H(p) = \frac{C(p)}{D(p)} . \qquad \text{(A-6)}$$

The transfer function H(p) of Equation A-6 can always be mechanized by the approximate synthesis procedure. It is known (see Papoulis (27, page 410)) that if $R(\tau)$ is a finite sum of exponentials then $S(\omega)$ is a *rational* function meeting all the requirements that enable the determination of H(p). Papoulis (27, page 410) points out that any $R(\tau)$ can be approximated sufficiently closely by a finite sum of exponentials. Hence, the approximate mechanization procedure can be used to simulate any desired $R(\tau)$ by expressing it as a finite sum of exponentials. The parallel approach that is developed in Chapter IV is also applicable for this set of $R(\tau)$.

Thus it is established that

$$R(\tau) = \sum_{i=1}^{n} a_i \, \varepsilon^{-\alpha_i |\tau|} \qquad \text{(A-7)}$$

is an admissible form for the mechanization procedure.

It is of interest to determine the most general form that the autocorrelation function may have.

The power spectral density $S(\omega)$ is an even, non-negative, ratio of polynomials in ω^2 written as

$$S(\omega) = \frac{A(\omega^2)}{B(\omega^2)} . \qquad \text{(A-8)}$$

The roots of the denominator $B(\omega^2)$ can be considered in two categories; distinct roots and repeated roots.

Consider first only distinct real roots of the polynomial $B(\omega^2)$. It can always be factored into the following type of expression

$$B(\omega^2) = b_n(\omega^2 + p_1^2)(\omega^2 + p_2^2) \ldots (\omega^2 + p_k^2) \ldots (\omega^2 + p_n^2) \qquad \text{(A-9)}$$

$$B(\omega^2) = b_n \prod_{k=1}^{n} (\omega^2 + p_k^2) ,$$

where the order of the $B(\omega^2)$ is 2n. Since $S(\omega)$ is non-negative the p_k^2 are greater than zero. The factor b_n is the coefficient of the ω^{2n} term. There is no loss of generality by assigning $b_n = 1$.

Assume for definiteness that p_k is greater than zero, then making the substitution $-p^2 = \omega^2$ Equation A-9 becomes with further factoring

$$B(-p^2) = (-p + p_1)(p + p_1)(-p + p_2)(p + p_2) \ldots (-p + p_n)(p + p_n) \qquad \text{(A-10)}$$

$$B(-p^2) = \prod_{k=1}^{n} (-p + p_k)(p + p_k) .$$

The factors in the expression are denominators in a partial fraction expansion of $\frac{A(-p^2)}{B(-p^2)}$ given by

$$\frac{A(-p^2)}{B(-p^2)} = \frac{K_1}{-p + p_1} + \frac{K_1}{p + p_1} + \frac{K_2}{-p + p_2} + \frac{K_2}{p + p_2} \qquad \text{(A-11)}$$

$$+ \ldots + \frac{K_k}{-p + p_k} + \frac{K_k}{p + p_k} + \ldots + \frac{K_n}{-p + p_n} + \frac{K_n}{p + p_n} .$$

The K_i's are determined by the method of Cheng (32, page 187). Considering the expansion in pairs

$$\frac{K_k}{p + p_k} + \frac{K_k}{-p + p_k} , \qquad \text{(A-12)}$$

the first term has no poles in the right-half plane; therefore, its inverse transform say c(t) is zero for $t < 0$. The second term is obtained from the first by replacing p with -p; hence, the inverse transform of the second term is c(-t) (see Papoulis (27, page 355)). Thus

$$R_k(\tau) = c(\tau) + c(-\tau) . \qquad \text{(A-13)}$$

The form of the transform is

$$c(t) = \frac{K_k}{2p_k} \varepsilon^{-p_k t} , \quad t \geq 0 , \qquad \text{(A-14)}$$

and

$$c(-t) = \frac{K_k}{2p_k} \varepsilon^{p_k t} , \quad t < 0 .$$

Thus one writes

$$R_k(\tau) = \frac{K_k}{2p_k} \varepsilon^{-p_k|\tau|} \quad \text{for all } \tau \ . \tag{A-15}$$

Hence, for each of the pairs of terms the inverse transform is of the form of Equation A-15; thus the autocorrelation function is a finite sum of exponentials given by

$$R(\tau) = \sum_{k=1}^{n} \frac{K_k}{2p_k} \varepsilon^{-p_k|\tau|} \ . \tag{A-16}$$

Now let one consider the case of repeated real roots. Consider Equation A-9 where the first term is a repeated factor of multiplicity m. It will suffice to illustrate the modification of the expansion where only one of the roots is repeated. Consider the following expression

$$B(\omega^2) = (\omega^2 + p_1^2)^m (\omega^2 + p_2^2) \ \ldots \ (\omega^2 + p_k^2) \ \ldots \ (\omega^2 + p_n^2) \ . \tag{A-17}$$

Substituting $-p^2 = \omega^2$, and further factoring yields

$$B(-p^2) = (-p + p_1)^m (p + p_1)^m (-p^2 + p_2^2) \tag{A-18}$$

$$\ldots \ (-p^2 + p_k^2) \ \ldots \ (-p^2 + p_n^2) \ .$$

The form of the partial fraction expansion of Equation A-18 is given as

$$\frac{A(-p^2)}{B(-p^2)} = \frac{K_{1m}}{(-p + p_1)^m} + \frac{K_{1m-1}}{(-p + p_1)^{m-1}} + \ldots + \frac{K_{12}}{(-p + p_1)^2} + \frac{K_{11}}{(-p + p_1)} \quad \text{(A-19)}$$

$$+ \frac{I_{1m}}{(p + p_1)^m} + \frac{I_{1m-1}}{(p + p_1)^m} + \ldots + \frac{I_{12}}{(p + p_1)^2} + \frac{I_{11}}{(p + p_1)}$$

$$+ \frac{K_2}{(-p^2 + p_2^2)} + \frac{K_3}{(-p^2 + p_3^2)} + \ldots + \frac{K_k}{(-p^2 + p_k^2)} + \ldots + \frac{K_n}{(-p^2 + p_n^2)} .$$

From before the terms with coefficients K_2, K_3, ..., K_n have inverse transforms which are sums of exponentials of the form of Equation A-15. The coefficients of the terms arising from the repeated root, i.e., K_{1m}, ..., K_{11} and I_{1m}, ..., I_{11} can be found using the techniques of Cheng (32, page 193). It can be shown that $K_{1k} = I_{1k}$, since if $p = -p_1$ is a root of $\frac{A(-p^2)}{B(-p^2)}$ so also is $p = p_1$ a root.

Thus Equation A-19 can now be considered term by term for inverse transforms. It is noted that the terms with K_{1k} coefficients have roots in the right-half plane and the I_{1k} terms have roots in the left-half plane. Using the technique of Equation A-14 and taking the terms of equal powers, i.e.,

$$\frac{I_{1k}}{(p + p_1)^k} + \frac{K_{1k}}{(-p + p_1)^k} = \frac{K_{1k}}{(p + p_1)^k} + \frac{K_{1k}}{(-p + p_1)^k} , \quad \text{(A-20)}$$

the inverse transform of the first term on the right-hand side is c(t) and is zero for $t < 0$; the inverse transform of the second term on the right-hand side is c(-t) and is zero for $t > 0$. The inverse transform of the first term on the right-hand side is given as

$$c(t) = L^{-1}\left[\frac{K_{1k}}{(p + p_1)^k}\right] = \frac{K_{1k}}{(k-1)!}\, t^{k-1}\, \varepsilon^{-p_1 t} , \qquad \text{(A-21)}$$

for $t \geq 0$, where L^{-1} denotes the inverse Laplace transform. The contribution to the autocorrelation function from the pair of terms is then

$$R_k(\tau) = c(\tau) + c(-\tau) , \qquad \text{(A-22)}$$

which becomes

$$R_k(\tau) = \frac{K_{1k}}{(k-1)!}\, |\tau|^{k-1}\, \varepsilon^{-p_1|\tau|} . \qquad \text{(A-23)}$$

Similarly each of the roots has this form of inverse transform; thus the autocorrelation function $R(\tau)$ is given by

$$R(\tau) = \sum_{k=1}^{m} \frac{K_{1k}}{(k-1)!}\, |\tau|^{k-1}\, \varepsilon^{-p_1|\tau|} \qquad \text{(A-24)}$$

$$+ \sum_{k=2}^{n} \frac{K_k}{2p_k}\, \varepsilon^{-p_k|\tau|} .$$

Thus we have established the general form for the admissible autocorrelation functions for implementation by the mechanization procedure when the roots are real, distinct and/or multiples.

An example is presented which illustrates the way in which Equation A-24 changes if the roots are complex. The complex roots occur in conjugate pairs and since if p_i and p_i^* are roots of $\frac{A(-p^2)}{B(-p^2)}$ so also is $-p_i$ and $-p_i^*$, hence the roots occur in symmetry about both the imaginary axis and the real axis in the p-plane.

Consider the following power spectral density

$$S(\omega) = \frac{16\omega^2}{\omega^4 - 6\omega^2 + 25} . \qquad \text{(A-25)}$$

Making the substitution $-p^2 = \omega^2$ the power spectral density can be expanded into the following four terms

$$S(p/j) = \frac{2 + j1}{p + 1 - j2} + \frac{2 - j1}{p + 1 + j2} \qquad \text{(A-26)}$$

$$+ \frac{2 + j1}{-p + 1 - j2} + \frac{2 - j1}{-p + 1 + j2} .$$

In this case $p_i = -(1 + j2)$, $p_i^* = -(1 - j2)$, $-p_i = -(1 + j2)$, and $-p_i^* = -(1 - j2)$ are all roots of $B(-p^2)$ illustrating the above comment. The first pair of terms of S(p/j) has all of its poles in the left-half plane; therefore, its inverse transform c(t) is zero for $t < 0$. The second pair of terms is obtained from the first by replacing p with -p;

hence the inverse transform of the second term is c(-t) (see Papoulis (27, page 355)). Thus the autocorrelation function is

$$R(\tau) = c(\tau) + c(-\tau) \ . \qquad \text{(A-27)}$$

for all τ.

The inverse transform of the first pair is

$$c(t) = (2 + j1)\varepsilon^{(-1 + j2)t} + (2 - j1)\varepsilon^{(-1 - j2)t} \qquad \text{(A-28)}$$

$$c(t) = 2\varepsilon^{-t}(2\cos 2t - \sin 2t)$$

for $t \geq 0$.

The transform for the second pair is obtained by evaluating c(-t) as

$$c(-t) = 2\varepsilon^{t}(2\cos 2(-t) - \sin 2(-t)) \qquad \text{(A-29)}$$

for $t < 0$.

Thus $R(\tau)$ can be written by use of absolute values to be

$$R(\tau) = 2\varepsilon^{-|\tau|}(2\cos 2\tau - \sin 2|\tau|) \ . \qquad \text{(A-30)}$$

If the roots of this example were repeated roots of multiplicity m the autocorrelation function has the same form as derived in Equation A-23.

$$R(\tau) = \sum_{k=1}^{m} \frac{2K_{1k}}{(k-1)!} |\tau|^{k-1} \varepsilon^{-|\tau|} (2\cos 2\tau - \sin 2|\tau|) . \quad \text{(A-31)}$$

The most general type of term in the class of admissible autocorrelation functions has the form of Equation A-31; however, some of the specified coefficients are results of the specific example chosen.

Particular Pairs

Example One

The first analytical result of Chapter III is the first example.

$$R(\tau) = A^2 \varepsilon^{-\alpha|\tau|} \quad \text{(A-32)}$$

$$S(\omega) = \frac{2A^2\alpha}{\omega^2 + \alpha^2}$$

$$H(p) = \frac{\sqrt{2A^2\alpha}}{p + \alpha} .$$

Example Two

The second example is a finite sum of exponentials given as the second analytical result of Chapter III.

$$R(\tau) = A^2 \varepsilon^{-\alpha|\tau|} + B^2 \varepsilon^{-\beta|\tau|} \quad \text{(A-33)}$$

$$S(\omega) = \frac{(2A^2\alpha + 2B^2\beta)\omega^2 + 2\alpha\beta(A^2\beta + B^2\alpha)}{\omega^4 + 2\alpha^2\beta^2\omega^2 + (\alpha^2\beta^2)^2}$$

$$H(p) = \frac{\sqrt{2A^2\alpha + 2B^2\beta}\left(p + \sqrt{\frac{2\alpha\beta(A^2\beta + B^2\alpha)}{2A^2\alpha + 2B^2\beta}}\right)}{(p + \alpha)(p + \beta)} .$$

Example Three

The third example is a finite sum. The second term is dominated by the first to maintain the inequality $R(0) \geq R(\tau)$.

$$R(\tau) = A^2\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|} \qquad \text{(A-34)}$$

$$S(\omega) = \frac{2A^2\alpha}{\omega^2 + \alpha^2} + \frac{2A^2\alpha(\alpha^2 - \omega^2)}{(\omega^2 + \alpha^2)^2} = \frac{4A^2\alpha^3}{(\omega^2 + \alpha^2)^2}$$

$$H(p) = \frac{\sqrt{2A^2\alpha^3}}{(p + \alpha)^2} .$$

Example Four

The fourth example is a finite sum. The coefficients are chosen to maintain the inequality $R(0) \geq R(\tau)$.

$$R(\tau) = A^2\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|} + \frac{A^2\alpha^2}{3}|\tau|^2\varepsilon^{-\alpha|\tau|} \qquad \text{(A-35)}$$

$$S(\omega) = \frac{4A^2\alpha^3}{(\omega^2 + \alpha^2)^2} + \frac{4A^2\alpha[\alpha^2 - 3\omega^2]}{(\omega^2 + \alpha^2)^3}$$

$$S(\omega) = \frac{16}{3} \frac{A^2\alpha^5}{(\omega^2 + \alpha^2)^3}$$

$$H(p) = \frac{\sqrt{16A^2\alpha^5/3}}{(p + \alpha)^3} \quad .$$

Example Five

Again the example is a finite sum.

$$R(\tau) = A^2\varepsilon^{-\alpha|\tau|} + A^2\alpha|\tau|\varepsilon^{-\alpha|\tau|} + \frac{A^2\alpha^2}{3}|\tau|^2\varepsilon^{-\alpha|\tau|} \qquad \text{(A-36)}$$

$$+ \frac{4A^2\alpha^3}{27}|\tau|^3\varepsilon^{-\alpha|\tau|}$$

$$S(\omega) = \frac{8A^2\alpha^3}{9} \frac{(7\alpha^4 + \omega^4)}{(\omega^2 + \alpha^2)^4} \quad .$$

Another term of the form

$$C|\tau|^n\varepsilon^{-\alpha|\tau|} \qquad \text{(A-37)}$$

can always be added to the autocorrelation functions in the forms of Examples Three, Four, and Five. The spectral density of the added term can be found from

$$S(\omega) = C2Re\left[\frac{n!}{(\alpha + j\omega)^{n+1}}\right] \quad , \tag{A-38}$$

where Re(•) means the real part of (•) . This term is then added to the previous collection of terms. The negative terms in the numerator can always be cancelled by proper choice of the constant C.

Example Six

The spectral density of the following form

$$S(\omega) = \frac{1}{(\omega^2 - \omega_o^2)^2 + \beta^2\omega^2} \quad , \tag{A-39}$$

has three basic autocorrelation functions associated with it (see Papoulis (27, page 355)). Define the value of $\omega_1 = \sqrt{\omega_o^2 - \beta^2/4}$ and the autocorrelation function is given by:

If $\beta^2 < 4\omega_o^2$, then

$$R(\tau) = \frac{1}{2\beta\omega_o^2} \varepsilon^{-\beta|\tau|/2} \left(\cos \omega_1\tau + \frac{\beta}{2\omega_1} \sin \omega_1|\tau|\right) . \tag{A-40}$$

If $\beta^2 = 4\omega_o^2$, then

$$R(\tau) = \frac{1}{2\beta\omega_o^2} \varepsilon^{-\beta|\tau|/2} \left(1 + \frac{\beta}{2}|\tau|\right) . \tag{A-41}$$

If $\beta^2 > 4\omega_o^2$, then, with $\omega_2 = \sqrt{(\beta^2/4) - \omega_o^2}$

$$R(\tau) = \frac{1}{2\beta\omega_o^2} \varepsilon^{-\beta|\tau|/2} \left(\cosh \omega_2\tau + \frac{\beta}{2\omega_2}\sinh \omega_2|\tau|\right) . \qquad \text{(A-42)}$$

The transfer function representing the fixed parameter linear filter is given by

$$H(p) = \frac{1}{p^2 + \beta p + \omega_o^2} . \qquad \text{(A-43)}$$

Note that Equation A-41 reduces for $\beta^2 = 4\omega_o^2$ to the form of Example Three with the spectral density given by

$$S(\omega) = \frac{1}{(\omega^2 + \omega_o^2)^2} . \qquad \text{(A-44)}$$

Another form that reduces from the power spectral density of Equation A-39 is

$$S(\omega) = \frac{1}{\omega^4 + 1} , \qquad \text{(A-45)}$$

when $\beta = \sqrt{2}$, $\omega_o^2 = 1$, and $\omega_1 = \frac{1}{\sqrt{2}}$. For this combination with $\beta^2 < 4\omega_o^2$, Equation A-40 reduces to

$$R(\tau) = \frac{1}{2} \varepsilon^{-|\tau|/2} \cos(|\tau|/\sqrt{2} - \pi/4) . \qquad \text{(A-46)}$$

These examples are not intended to be exhaustive but they do illustrate the types of autocorrelation functions for which the approximate procedure can be used. The parallel procedure can be used with each of the above examples.

APPENDIX B

EXPERIMENTAL DATA PRESENTATION

This appendix is included to provide a better understanding of the experimental data presentation in Chapter V. The normalization procedure, the source of confidence limits, and the estimation equations are presented.

Theoretical Data

The theoretical data for the covariance functions are plotted as continuous functions in the parameters t_1 and t_2 for the five cases of each example. The theoretical data are normalized for comparison with the experimental discrete values. Equations 5-10 and 5-18 are normalized by dividing by 1 and 20, respectively. This achieves a maximum value of unity when the difference $t_1 - t_2 = 0$.

Experimental Data

The experimental data represent the discrete time equivalents of the continuous theoretical functions. The data are plotted as discrete points for $t_2 = 1$, 4, and 7 seconds as t_1 ranges from 0 to 9 seconds at each integer value of seconds. The data are obtained from the digital computer statistical calculations and normalized for the presentation. Recall the expression for the estimates of the covariance

$$\hat{R_c}(t_i,t_j) = \frac{1}{N} \sum_{n=1}^{N} (e_i^n e_j^n) - \hat{M}(t_i)\hat{M}(t_j) \ . \qquad \text{(B-1)}$$

The data points are normalized by dividing the Equation B-1 by the following expression

$$\frac{1}{10} \sum_{i=0}^{9} \hat{R_c}(t_i,t_i) \ , \qquad \text{(B-2)}$$

where $t_o = 0$, $t_1 = 1$, $t_2 = 2$, etc. This choice of normalization provides a uniform technique for all of the covariance data presentations. This method of normalization does not preclude an experimental value for the covariance function greater than unity as can be observed in several of the figures in Chapter V.

Confidence Level Error Boundaries

This part of the appendix is devoted to a summary of the statistical error associated with the estimates $\hat{M}(t_i)$ and $\hat{\sigma}(t_i)$ as defined by Equations 5-7 and 5-9. The error is formulated in terms of N, the number of sample functions used in the estimates, and the statistical nature of the random processes under consideration. The rms statistical error of the estimate $\hat{M}(t_i)$ is given by Bryan (11, page 16) to be

$$SD[\hat{M}(t_i)] = \frac{\sigma(t_i)}{\sqrt{N}} \ , \qquad \text{(B-3)}$$

where SD means standard deviation and $\sigma(t_i)$ is the square root of the process variance at $t = t_i$ time, i.e., $R(t_i,t_i) = \sigma^2(t_i)$. Bryan further shows that the estimate $\hat{M}(t_i)$ is an unbiased estimator and that the expected value of $\hat{M}(t_i)$ is $M(t_i)$. Using this information and the fact

that $\hat{M}(t_i)$ is Gaussian distributed, a band may be defined (in terms of $M(t_i)$, $\sigma(t_i)$, and N) into which a given per cent, say K per cent, of the estimates $\hat{M}(t_i)$ will fall. The band defines the K per cent confidence level. The 80 per cent confidence level bounds for the Gaussian probability distribution is 2.56 standard deviations wide and symmetrical with respect to the mean value. The boundaries for the 80 per cent confidence level for $\hat{M}(t_i)$ are given by

$$M(t_i) \pm 1.28 \frac{\sigma(t_i)}{\sqrt{N}} . \qquad \text{(B-4)}$$

Figure B-1 illustrates the 80 per cent confidence level error boundaries for the $\hat{M}(t_i)$ estimates as a function of the number of sample functions N.

The 80 per cent confidence level boundaries may be established for the random process standard deviation estimate $\hat{\sigma}(t_i)$. This random variable $\hat{\sigma}(t_i)$ is chi-distributed with N degrees of freedom and mean value $\sigma(t_i)$. An expression which defines the 80 per cent confidence level error boundaries for the standard deviation estimates is

$$\sigma(t_i) \begin{matrix} + K_1 \sigma(t_i) \\ \\ - K_2 \sigma(t_i) \end{matrix} \qquad \text{(B-5)}$$

where K_1 and K_2 as functions of N are defined by the curves in Figure B-2.

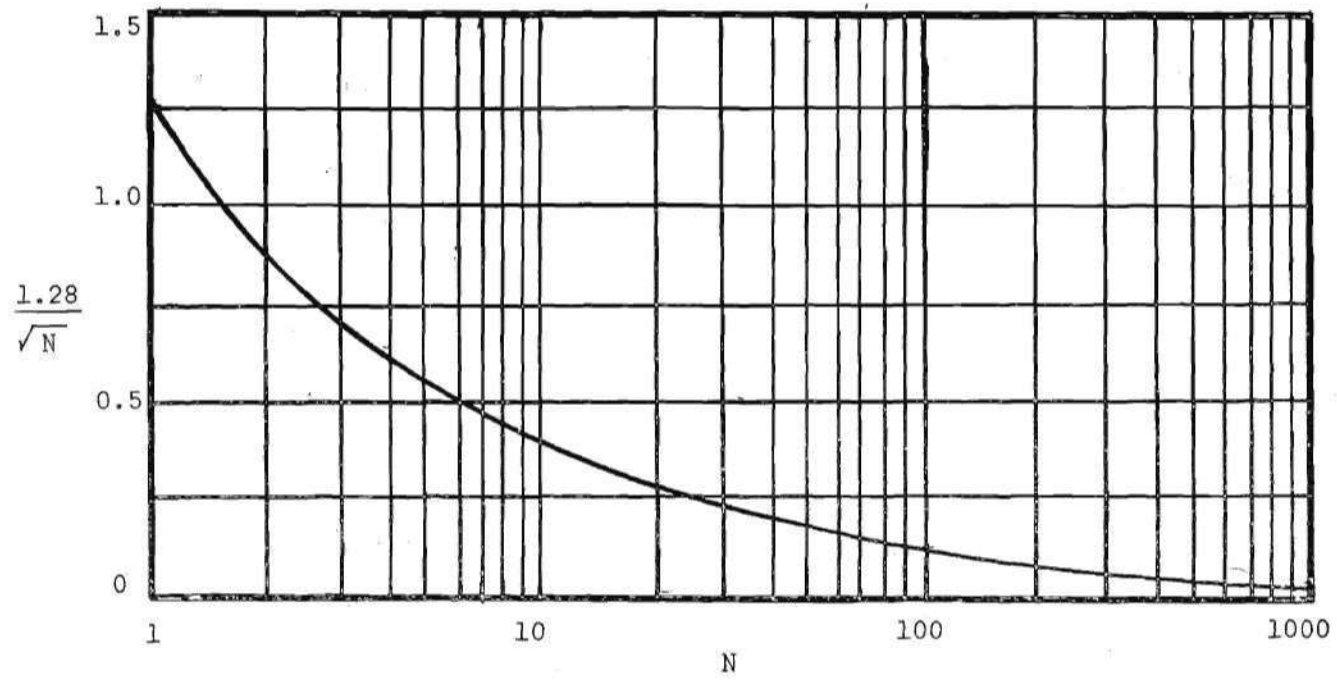


Figure B-1. 80 Per Cent Confidence Level Error Bounds for $\hat{M}(t_i)$ Estimates.

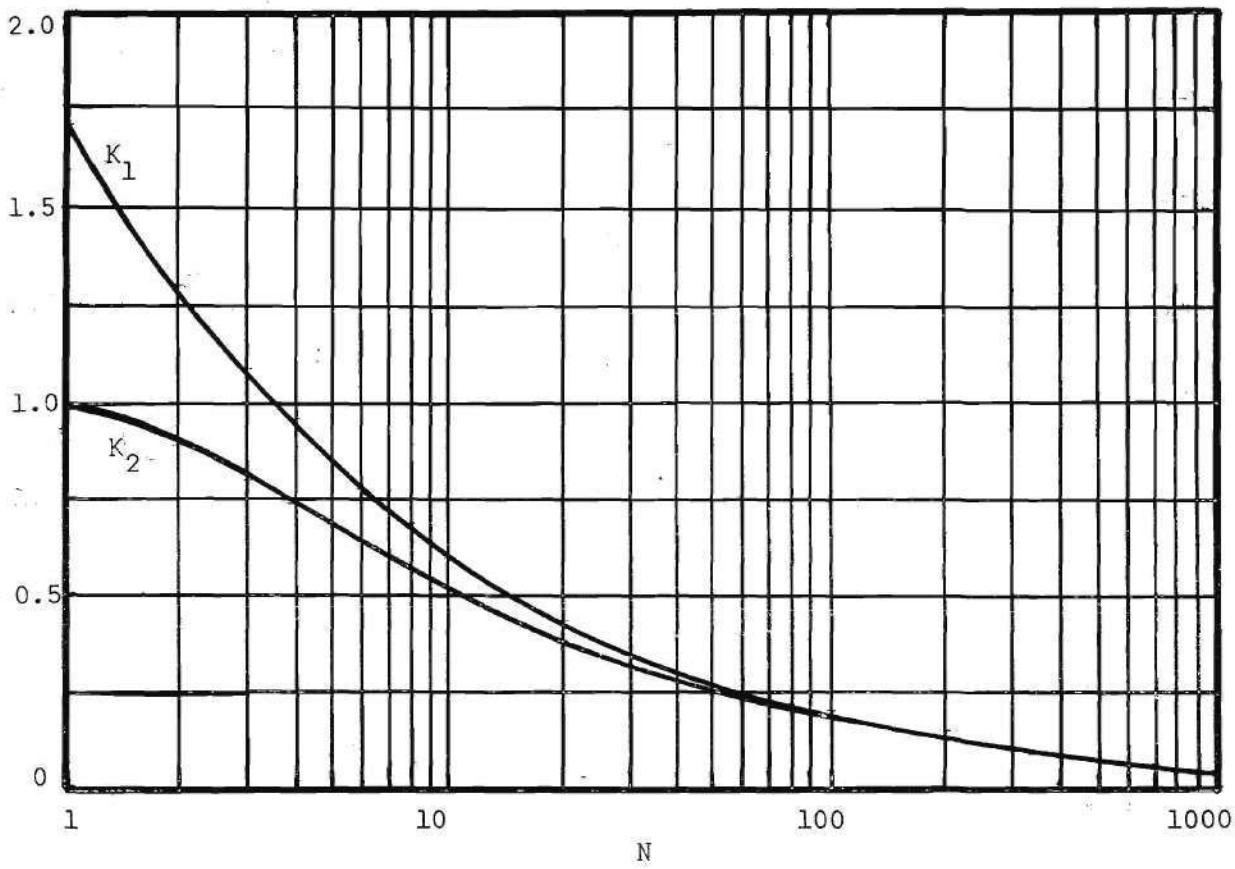


Figure B-2. 80 Per Cent Confidence Level Error Bounds for $\hat{\sigma}(t_i)$ Estimates.

Equations B-4 and B-5 and the curves of Figures B-1 and B-2 are used in the analysis of the examples presented in Chapter V.

APPENDIX C

DERIVATION OF EQUATION 3-40

This appendix is included to provide the detailed mathematics for the statement of the results in Equation 3-40. Some additional material on the nature of the solution to the first order differential equation is also included for completeness.

The equation of concern is written as

$$\dot{e}(t) + \alpha v(t) e(t) = \sqrt{2A^2 \alpha \, |v(t)|} \; w(t) \; . \qquad \text{(C-1)}$$

This equation is of the form

$$\dot{y}(x) + a(x) y(x) = b(x) \; . \qquad \text{(C-2)}$$

The homogeneous solution to Equation C-2 is

$$\phi_1(x) = \varepsilon^{-A(x)} \; , \qquad \text{(C-3)}$$

where $A(x)$ is a function such that $\dot{A}(x) = a(x)$. The particular solution to Equation C-2 is

$$\psi(x) = \varepsilon^{-A(x)} \int_{x_o}^{x} \varepsilon^{A(t)} \, b(t) dt \; . \qquad \text{(C-4)}$$

Then if C is any constant

$$\phi = \psi + C\phi_1 \qquad \text{(C-5)}$$

is a solution of Equation C-2 and every solution has this form (see Coddington (29, page 41)).

By direct substitution the homogeneous representation of Equation C-1 is

$$\dot{e}(t) + \alpha v(t)e(t) = 0 \ , \qquad \text{(C-6)}$$

which has a solution

$$\phi_1(t) = \varepsilon^{-A(t)} \ , \qquad \text{(C-7)}$$

where

$$A(t) = \int_{t_o}^{t} \alpha v(u)du \ . \qquad \text{(C-8)}$$

The particular solution for Equation C-1 is written

$$\psi(t) = \varepsilon^{-A(u)} \int_{t_o}^{t} \varepsilon^{A(u)} b(u)du \ , \qquad \text{(C-9)}$$

with

$$b(u) = \sqrt{2A^2\alpha|v(u)|}\; w(u) \quad . \tag{C-10}$$

The complete solution is then

$$e(t) = \psi(t) + C\phi_1(t) \quad , \tag{C-11}$$

which by substituting Equations C-7, C-8, C-9, and C-10 becomes

$$e(t) = \varepsilon^{-\int_{t_o}^{t} \alpha v(s)ds} \int_{t_o}^{t} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} \left(2A^2\alpha|v(s)|\right)^{1/2} w(s)ds + C\,\varepsilon^{-\int_{t_o}^{t} \alpha v(s)ds} \quad . \tag{C-12}$$

Evaluating Equation C-12 at $t = t_o$ time, the value for C can be determined as

$$C = e(t_o) \tag{C-13}$$

which is a random variable with mean zero.

The mean value of the solution e(t) is determined by the expected value of e(t) written as

$$M_e = E\{e(t)\} \quad . \tag{C-14}$$

By appropriate interchange of integration and expected value operation in Equation C-12 (see Davenport and Root (26, page 182)), the mean value of e(t) is determined to be zero when the expected value of the white noise random process w(t) is zero, i.e.,

$$E\{w(t)\} = 0 \ . \tag{C-15}$$

Of most interest is the autocorrelation function for the solution e(t) given by

$$R_e(t_1,t_2) = E\{e(t_1)e(t_2)\} \tag{C-16}$$

Expanding Equation C-16 and restricting v(t) to be non-negative

$$E\{e(t_1)e(t_2)\} = E\{MN(\int_{t_o}^{t_1} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} (2A^2\alpha v(s))^{1/2} w(s)ds) \tag{C-17}$$

$$\cdot (\int_{t_o}^{t_2} \varepsilon^{\int_{t_o}^{x} \alpha v(u)du} (2A^2\alpha v(x))^{1/2} w(x)dx)$$

$$+ MN(\int_{t_o}^{t_1} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} (2A^2\alpha v(s))^{1/2} w(s)ds)e(t_o)$$

$$+ MN\left(\int_{t_o}^{t_2} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} (2A^2 \alpha v(s))^{1/2} w(s)ds \right) e(t_o)$$

$$+ MNe(t_o)^2 \} ,$$

where
$$M = \varepsilon^{-\int_{t_o}^{t_1} \alpha v(u)du} , \tag{C-18}$$

and
$$N = \varepsilon^{-\int_{t_o}^{t_2} \alpha v(u)du} . \tag{C-19}$$

By interchanging the order of integration and expected value operation (see Davenport and Root (26, page 182)) Equation C-17 becomes

$$E\{e(t_1)e(t_2)\} = \tag{C-20}$$

$$MN\left[\left(\int_{t_o}^{t_1} \int_{t_o}^{t_2} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} \varepsilon^{\int_{t_o}^{x} \alpha v(u)du} 2A^2 \alpha (v(s)v(x))^{\frac{1}{2}} E\{w(s)w(x)\}dxds \right) \right.$$

$$\left. + \int_{t_o}^{t_1} \varepsilon^{\int_{t_o}^{x} \alpha v(u)du} (2A^2 \alpha v(x))^{1/2} E\{w(x)e(t_o)\}dx \right.$$

$$+ \int_{t_o}^{t_2} \varepsilon^{\int_{t_o}^{x} \alpha v(u)du} (2A^2 \alpha v(x))^{1/2} E\{w(x)e(t_o)\}dx$$

$$+ E\{e(t_o)^2\}] \ .$$

The two middle terms vanish for all values of x, since $E\{w(x)e(t_o)\} = 0$, except possibly for $x = t_o$. However, for this value of x the limits on the integral are from t_o to t_o which again contributes zero value. Hence, the expression reduces to

$$E\{e(t_1)e(t_2)\} = \tag{C-21}$$

$$MN[\int_{t_o}^{t_1} \int_{t_o}^{t_2} \varepsilon^{\int_{t_o}^{s} \alpha v(u)du} \varepsilon^{\int_{t_o}^{x} \alpha v(u)du} 2A^2 \alpha (v(s)v(x))^{1/2} E\{w(s)w(x)\}dxds$$

$$+ E\{e(t_o)^2\}] \ .$$

The integrand of the first term includes the autocorrelation function of the stationary white noise random process at times s and x,

$$E\{w(s)w(x)\} = R_w(s,x) \ . \tag{C-22}$$

But w(t) is a stationary random process whose autocorrelation function is the impulse function given by

$$R_w(s,x) = R_w(s - x) = \begin{cases} 1 , & x = s \\ 0 , & \text{otherwise} . \end{cases} \qquad \text{(C-23)}$$

Using the sifting properties of the impulse function Equation C-21 reduces to

$$E\{e(t_1)e(t_2)\} = \qquad \text{(C-24)}$$

$$MN\Big[\int_{t_o}^{t_2} \varepsilon^{2\int_{t_o}^{s} \alpha v(u)du} \; 2A^2 \alpha v(s)(1)ds + E\{e(t_o)^2\}\Big] ,$$

when $t_o < t_2 < t_1$. The only difference when $t_o < t_1 < t_2$ is the change of the upper limit on the integral from t_2 to t_1.

The first term is of the form

$$\int_a^b \varepsilon^u \, du = \varepsilon^u \Big|_{u=a}^{u=b} . \qquad \text{(C-25)}$$

Using Equation C-25 to evaluate Equation C-24, the autocorrelation function is written

$$E\{e(t_1)e(t_2)\} = MNA^2[\varepsilon^{2\int_{t_o}^{t_2} \alpha v(u)du} - 1] + MNE\{e(t_o)^2\} \quad . \qquad (C\text{-}26)$$

Substituting the values of M and N from Equations C-18 and C-19, Equation C-26 reduces to

$$E\{e(t_1)e(t_2)\} = A^2\varepsilon^{-\int_{t_2}^{t_1} \alpha v(u)du} + [E\{e(t_o)^2\} - A^2]MN \quad . \qquad (C\text{-}27)$$

The second of the two terms vanishes since the $E\{e(t_o)^2\}$ is equal to the variance of the random process at t_o time and A^2 is the value of the variance for all time t.

Hence, the autocorrelation function reduces to

$$E\{e(t_1)e(t_2)\} = A^2\varepsilon^{-\int_{t_2}^{t_1} \alpha v(u)du} \qquad (C\text{-}28)$$

for $t_o < t_2 < t_1$ and similarly

$$E\{e(t_1)e(t_2)\} = A^2\varepsilon^{-\int_{t_1}^{t_2} \alpha v(u)du} \qquad (C\text{-}29)$$

for $t_o < t_1 < t_2$.

This is expressed as

$$E\{e(t_1)e(t_2)\} = A^2 \varepsilon^{-\alpha\left|\int_{t_2}^{t_1} v(u)du\right|} \qquad \text{(C-30)}$$

for all t_2 and $t_1 > t_o$.

Recall that the originally specified autocorrelation function was

$$R_g(x_1,x_2) = A^2 \varepsilon^{-\alpha|x(t_1) - x(t_2)|} \qquad \text{(C-31)}$$

which is identical with the autocorrelation function of the output of the random process e(t), when

$$x_1 = x(t_1) = \int_{t_o}^{t_1} v(u)du + x(t_o) \qquad \text{(C-32)}$$

and

$$x_2 = x(t_2) = \int_{t_o}^{t_2} v(u)du + x(t_o) \; . \qquad \text{(C-33)}$$

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During his undergraduate work he was employed as a cooperative education student at Bendix Aviation Corporation in Mishawaka, Indiana. Upon completion of the B.E.E. degree he accepted a Master of Science Fellowship program with the Hughes Aircraft Company of Fullerton, California. He has held the positions of Member of Technical Staff, Group Head, and Staff Engineer during his seven years with the Hughes Aircraft Company.

He entered the graduate school at Georgia Institute of Technology in September, 1963, as a Howard Hughes Staff Doctoral Fellow and held the Fellowship until June, 1966.