3/17/65
b

THE DEVELOPMENT OF MATHEMATICAL MODELS TO DESCRIBE SEAT ALLOCATION IN STADIA

A THESIS

Presented to

The Faculty of the Graduate Division

by

Walter Lee Hughey

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Industrial Engineering

Georgia Institute of Technology

May 1968

THE DEVELOPMENT OF MATHEMATICAL MODELS

TO DESCRIBE SEAT ALLOCATION IN STADIA

Approved:

Chairman

Date approved by Chairman: April 8, 1968

ACKNOWLEDGMENTS

The author would like to express deep appreciation to Mr. Cecil G. Johnson, his thesis advisor, for his inspiration and guidance during the preparation of this thesis. The author would also like to express appreciation to Dr. I. E. Perlin and Dr. H. M. Wadsworth for their professional advice and assistance.

The interest and patience of the author's thesis committee provided the motivation to pursue this research.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (concluded)

SUMMARY

The topic of this research was the result of a project undertaken in 1960 by the Rich Electronic Computer Center to develop an automated technique for assigning seats to season ticket applicants in the stadium of Grant Field at the Georgia Institute of Technology. Whereas a practical solution to the Georgia Tech problem was sought, the purpose of this thesis was to explore the general problem of seat allocation in stadia.

The original objective of this thesis was to develop a mathematical model to describe the operating characteristics of seat allocation in stadia. Research into the characteristics of stadium seating revealed that developing specific models for specific situations was more feasible than developing a general model for all situations. This discovery led to the development of four models to fit the following four situations: new assignment; reassignment; selective assignment; and optimum allocation.

These four models were programmed for a digital computer and tested with sufficient data to verify the technique. The first model assigns each applicant, ranked in priority order, the best available seats where seats have not been held before. The second and third models offer applicants the option of improvement or choice of specific seats where there was history of previous assignments. Model four allocates applicants to sections of seats in an optimum manner based on qualifications for the seats.

The first three models have essentially the same basic assignment criteria. The second and third models are refinements of the first. The fourth model considers the effectiveness of assignment and produces a least cost solution.

CHAPTER I

INTRODUCTION

Problem

Seat Allocation

The seat allocation problem is concerned with the assignment of seats in a manner which satisfies the greatest number of applicants. Satisfaction criterion is based on an allocation policy which attempts to supply the best available (adjacent) seats to each applicant.

Stadium Seating

Seat allocation is part of a general process called "stadium seating." This process can be divided into three parts: determination of applicant priority, ranking of seats, and assignment of seats to applicants. Determination of applicant priority and seat ranking are independent of each other, but the assignment of seats is completely dependent upon the other two. It can therefore be stated that the determination of applicant priority and seat ranking are necessary inputs to seat assignment. Though all three parts of stadium seating are discussed, only the problem of seat assignment[1] is analyzed in great detail.

Background

Stadium seating (as a problem) must be examined in parts as described above. The factors which influence the determination of applicant priority and

---

1 For the purpose of this research, "assignment" and "allocation" are considered synonymous.

seat quality are highly subjective. Determination of applicant priority is often dependent upon the supply and demand relationship, and seat ranking is almost entirely dependent upon the physical seat arrangement.

Problem Refinement

To describe the operating characteristics of the seat allocation problem as outlined, it is necessary to study hypothetical stadia in detail. It is impossible to evaluate every conceivable configuration, but the task can be made feasible by judiciously choosing some representative characteristics. Combinations of the characteristics produce a number of possible systems, some of which can be immediately eliminated due to incompatible properties. The remaining sets provide the basis for an acceptable definition of the seat assignment problem.

Precise Definition. Given n applicants, in priority order, and m seats ranked according to desirability, the problem is to assign each seat to one and only one applicant in such a manner that at the time of assignment, each applicant receives the best available location. This definition is the description of assignment that applies to all seating situations, when the input restrictions are met.

Modeling Seat Allocation. According to the definition of seat assignment and the explanation of a solution objective to the problem of allocation, some form of a mathematical model is necessary to describe the general system. The overall objective of this investigation is therefore to develop a mathematical model to describe seat allocation in stadia, about which certain assumptions are made. The most important assumption concerns the satisfaction of the greatest number of applicants. Judging whether one seat is better than another ultimately determines how well satisfied the applicants are going to be. If the seat ranking procedure is poor, the allocation will not improve the results. If the process of ordering the seats is good, then the assumption that the model maximizes satisfaction is a reasonable one. There are other assumptions related to each model refinement which are presented in Chapter III, Model Development. Seat assignment, however, is independent of the environment and its characteristics are therefore generally describable.

Applicant Priority

When an applicant must be assigned a seat location according to his rank in the group, a reliable process for the determination of priority is needed. Such a process is usually based on a set of factors, properly weighted, to provide a realistic order within the group. The relative rank of an individual often differs from stadium to stadium because of the variance between the factors used to determine priority. It is difficult enough to obtain agreement on a method of classifying people in one situation, let alone more than one. The determination of applicant priority and the subsequent process of ranking is qualitative rather than quantitative. It is a function of the particular environment under study, and each stadium administration must ultimately decide what constitutes the basis of priority.

Seat Ranking

When an applicant must be assigned the best seat location available, a reliable process for determination of seat quality is needed. Unfortunately, rules for establishing the preference of one seat over another in stadia[2], cannot be generalized to fit all or even most environments. There simply is not enough

---

[2] Stadia, in this study, is meant to include any configuration of seats, whether it be called an arena, coliseum, theater, or stadium.

similarity because no two stadium configurations are exactly alike. The quality of the seats usually depends upon the field of vision from each location and such decisions are at best subjective and sometimes arbitrary.

Since the processes of establishing applicant priority and ranking seats are subjective, the stadium administration must develop these based on environment. Both the determination of applicant priority and seat ranking are therefore required inputs to seat allocation.

CHAPTER II

LITERATURE SURVEY

The literature contains a number of applications characteristically similar to the problem of seat allocation. Though no published material on the specific subject of stadium seating was found, the similarity of the problem to certain linear programming applications is notable.

During the course of the survey, a number of useful articles were discovered in The Journal of Industrial Engineering, The Journal of the Operations Research Society of America, and the Naval Research Logistics Quarterly. Other information came from books and papers, and a selected set of bibliographies (listed as "Other References"). All of the cited material is related to the topics of assignment or allocation. The following paragraphs provide a partial foundation for "model development" and generally fall into one of three categories, (1) descriptive material, (2) problem formulation, and (3) method of solution.

Descriptive Material

Because the seat allocation problem was originally envisioned as a type of allocation problem, the description in (23) is presented:

> Problems of allocation arise whenever there are a number of activities to perform, but limitations on either the amount of resources or the way they can be spent prevent us from performing each separate activity in the most effective way conceivable. In such situations we wish to allot the available resources to the activities in a way that will optimize the total effectiveness.

When the demand for seats exceeds the supply available, the previous description holds. For the case when demand exactly equals supply the definition in (9) of the "assignment" problem applies:

> In its simplest form, the assignment problem is stated as follows: Given n resources, n uses to which these resources can be put, and a measure of efficiency or lack of efficiency for the assignment of any resource to any use, we are to maximize the efficiency of the assignment. The resources could be supplies, men, machines, etc., and the corresponding uses to which they are to be put could be indicated as demands, jobs, and production programs in the respective cases.

In addition to defining the assignment problem, it is pointed out in this reference that assignment is a special case of the linear programming problem. Several techniques using linear programming as a basis have been developed to determine optimal assignments of personnel to jobs, man-hours to machines, plants to locations, tractors to trailers, and even electronic equipment to ships. These linear programming applications and the assignment problems found in (22) resemble the stadium seating problem in description.

An interesting analogy to the process of seat allocation in stadia is reflected in (23) by the statement that "the assignment problem is a type of allocation problem in which n items are distributed among n boxes, one item to a box, in such a way that the return resulting from the distribution is optimized." This description emphasizes a restriction that is important in seat assignment, that no more than one person may occupy any one seat.

Complications result in solving assignment problems when the demand for an item of supply is allowed to exceed one. It is pointed out in (9) that:

> More complex assignment problems arise if we consider cases in which the demands are for multiples of the basic commodities, and assume that supplies of the basic commodity are concentrated in distinct groups and that the costs of providing a unit demand for distinct concentrations differ. The complexity arises because of the additional number of units of the basic commodity, for although we may treat each unit separately and proceed as before, the order of the matrix involved becomes increasingly unwieldy as n increases.

There is an important point here concerning the demand for multiples of the basic commodity which is inherent in the seat allocation problem; one must consider the situation in which applicants request more than one seat.

For most allocation situations, the authors of (19) state that "one infallible procedure for determining the minimum-cost assignment would be to try all possibilities, calculate the cost of each, and select the cheapest." Though this procedure will work, it becomes difficult to execute when the number of applicants is large (e.g., 10,000) and the number of seats is large (e.g., 20,000). To consider all the possible assignments in a reasonable length of time is impossible, even if a computer is used.

Linear programming problems have been solved in numerous ways, but the two most used are the simplex (15) and transportation methods. Special cases of linear programming are not easily solved by these methods, but "one special case - the assignment technique - has many applications in the areas of allocation and scheduling," as indicated in (6). It is important to note that this same logic applies to the stadium seating problem, i.e., it is not solved easily (or perhaps, at all) by existing methods.

Many techniques have been implemented in the solution of various assignment problems as illustrated in (11), (12), (16), (17), (20), and (25). Some of the methods employed are rigorous in the sense that they provide an <u>optimum</u> solution based on measures of effectiveness. Other approaches are intuitive and require less time, i.e., they do not <u>guarantee</u> an optimum solution, but if properly used, provide an acceptable one. Whichever solution technique is employed, its basic foundation is a function of the initial categorization and subsequent formulation of the problem. This logic was followed in the development of the scheduling problem (8), (11), (12), (14), and (24); the network flow problem (4), (5), and (10); the quota problem (25); the personnel assignment problem (1), (13), and (21); the allocation of machines to locations (2), (3), (7), and (20); and the traveling-salesman problem (18).

Those applications of the assignment problem which resemble the seat allocation problem have been presented. Three of these do provide a basis for problem formulation and one provides a method of solution. The other material (already presented) is of interest because of the similarity to the seating problem. The references which have direct bearing on the problem are cited individually in the following order: the scheduling problem; the network-flow problem; and the quota problem.

Problem Formulation

Of the types of problems generally solved by linear programming techniques, the priority scheduling problem is formulated like the seat allocation problem. From a paper (12) authored by Gass, in which man-hours are distributed to project work requirements in priority order for an engineering research and manufacturing company, the general problem is stated as follows:

> The project number will be denoted by i ($i = 1, 2, \ldots, M$), and it is known that a total of A_i man-hours must be applied to project i during a given sequence of months in order to successfully complete the project. For a given month j, ($j = 1, 2, \ldots, N$), the project planning group knows how many total man-hours, B_j, are available for use by the projects active in month j. At least d_{ij} man-hours and no more than e_{ij} man-hours must be expended on project i in month j. If we define y_{ij} as the total number of man-hours assigned to project i in month j, the above verbal description can be expressed mathematically by the following constraints:
>
> (1) $$\sum_{j=1}^{n} y_{ij} = A_i \qquad i = 1, 2, \ldots, m$$
>
> (2) $$\sum_{i=1}^{m} y_{ij} = B_j \qquad j = 1, 2, \ldots, n$$
>
> (3) $$0 \le d_{ij} \le y_{ij} \le e_{ij}$$
>
> It is assumed that for any two projects i and k, project i has a higher priority in the assignment of available man-hours if $i < k$. It is also assumed that for any two months, p and q of a project, month p has a higher priority if $p < q$. In other words, the desire is to satisfy the requirements of project i before any assignment is made to project k ($i < k$) by allocating the available man-

> hours to month p before any man-hours are assigned in month q ($p < q$). The computational problem is then one of finding a set of y_{ij} that satisfy (1), (2), and (3) in a manner that also satisfies the priority restrictions.

This verbal and mathematical formulation describes the seat allocation problem except for the inclusion of minimum and maximum man-hour expenditures for each project in each month. By substituting applicants for projects and sections (sets of seats) for months, the mathematical formulation described above can be used to represent the seat allocation problem.

Method of Solution

Network Flow Problem

The second important application, the network-flow problem, provides the foundation for a special case of the seat allocation problem and suggests a method of solving it. A very simple algorithm for finding a maximal flow and minimal cut in a transportation network is described in (10). The network-flow problem to which the Ford-Fulkerson technique may be applied is stated in the following way:

> One is given a network of directed arcs and nodes with two distinguished nodes, called source and sink, respectively. All other nodes are called intermediate. Each directed arc in the network has associated with it, a non-negative integer, its flow capacity. Source arcs may be assumed to be directed away from the source, sink arcs into the sink. Subject to the conditions that the flow in an arc is in the direction of the arc and does not exceed its capacity, and that the total flow into any intermediate node is equal to the flow out of it, it is desired to find a maximal flow from source to sink in the network, i.e., a flow which maximizes the sum of the flows in source (or sink) arcs.

To extend the work described in the previous reference so that it conforms more specifically to the seat assignment problem, it is necessary to include a publication (5) in which additional parameters (constraints) are considered. This paper, which refers to an earlier one by the same principal author, includes and extends the original material. The extractions which follow are from both papers:

> In (4), networks that had a specified "capacity" as well as a "unit cost" associated with each direction of every edge or "link" were considered. The problem under consideration concerned the determination of maximum steady-state rate of flow that could be sustained between two specified vertices, and the determination of a specific family of minimum-cost flow patterns between these vertices - one pattern for every integral rate of flow from zero to the maximum.
>
> The method adopted in (4) was one of discovering a succession of path flows from the origin to the destination in such a way that each new path flow contributed one more unit of flow with the least possible increase in cost.
>
> A vital point in the procedure was the fact that a starting feasible flow pattern having minimum cost was available. (This was simply the pattern having zero flow in each link and consequently zero total cost.)
>
> An important extension of this problem arises if one wishes to "force" flow through one or more links. In this case, there may not be any feasible flow patterns if the constraints are too stringent.

Two additional aspects of the network flow problem have now been added to the concept in (10), that of a "unit cost" associated with each link and the desirability to "force flow" through one or more links. As a matter of interest, all of the fore-going literature extractions on network flows are applicable in model development for the explicit case of optimum seat allocation.

Quota Problem

A third type of problem, described in (25), is applicable in description and method of solution to the seat allocation problem: it is the "quota problem" of personnel assignment. The material follows a pattern of progressive refinement from the simple case to more complicated cases. It is this method of attack in formulating the "quota problem" that is useful in the development of seat allocation models. The problem is described in the following way:

> Consider a set of persons and a set of job categories, and suppose that for each job category there is a quota. Suppose further that for each person it is known with regard to each job category whether he is "qualified" or "not qualified." The quota problem can be stated as follows:

a. Does there exist an assignment of persons to jobs such that each person is qualified on the job to which he is assigned?

b. If the answer to (a) is yes, find such an assignment.

Here the specific reference to a person being "qualified" or "unqualified" is a significant characteristic similar to a specific case of the seat allocation problem.

Relevance of Literature

Based on the literature survey, the seat allocation problem was categorized as an assignment type of problem. The references cited were useful in descriptive support, formulation, or solution of the seat assignment problem. Three specialized linear programming applications aided in modeling and solving the problem. Two of these suggest how the major inputs, i.e., applicants and seats, might be handled relative to each other. The third application, in network flow theory, suggests an optimum method of solving the case where demand equals or exceeds supply, i.e., the number of requests for seats is equal to or greater than the number of seats available.

The assignment-type problem descriptions produced two important ideas, (1) if a method for determining an "optimum" solution to the seat assignment problem is to be developed, some measure of what constitutes optimality must be established, and (2) if such a measure cannot be found, then a noniterative procedure based on a valid interpretation of the priority restrictions[1] must be sought. Based on this premise the general seat assignment problem may be stated in the following way: Given m applicants in a predetermined order of

[1] "Priority restrictions" are derived from the imposed sequence of applicants from highest to lowest priority, and from the ranking of seats (or sections) from best to worse.

hierarchy and n seats ranked according to desirability, the problem is to assign each applicant in turn, the most desirable seats available which are adjacent and correspond in number to the request. For the special case when demand is equal to or greater than supply, the problem may also be formulated as follows: Consider i sections of a given number of seats with designated rank either better than, equal to, or worse than all other sections, and k applicants, each of which is specifically "qualified" or "unqualified" for one or more sections, and a cost is associated with assigning an applicant to a section in which he is unqualified, the problem is to assign applicants to sections in such a way as to minimize the total cost of assignment. An alternative criterion for the optimum would be to minimize the number of "misassignments," i.e., the number of applicants assigned seats in sections for which they are unqualified.

In the general formulation of the seat assignment problem, the solution may not be an optimum but is certainly an acceptable one since the results conform to the imposed priority restrictions. When the special case model is employed, the solution is an optimum, defined in terms of the costs (or measures) associated with misassignments.

CHAPTER III

MODEL DEVELOPMENT

Approach

This research produced the general observation that there is little commonality in the operating characteristics of stadium seating situations. There is, however, sufficient similarity in seat allocation itself to provide a basis for constructing models with different objectives in assignment. As a result, four models were developed for situations with specific objectives. The research was pursued in a problem oriented manner considering typical rather than hypothetical situations. Solutions for the four seat allocation problems were thus developed with the emphasis on applications instead of derivations.

The Models

Four models are presented in this thesis and are classified according to the type of situation.

Model I: New Assignment. The "New Assignment" Model allocates seats to applicants where no seats were previously held.

Model II: Reassignment. The "Reassignment" Model allocates seats in an existing stadium with a history of assignments and the option of improvement is given to applicants who held seats previously.

Model III: Selective Assignment. The "Selective Assignment" Model operates like the Reassignment Model, with the added option of the choice of specific seats by an applicant.

Model IV: Optimum Allocation. The "Optimum Allocation" Model minimizes total cost in a gross assignment of applicants to sections of seats. No options are permitted in this model and applicants are assigned to sections based on cost qualifications.

Rules of Assignment

A basic set of rules apply to the first three models and are listed as follows:

1. Each applicant must be assigned a unique number equivalent to his rank, so that no two applicants may have the same numerical priority.

2. Each seat must be assigned a unique number equivalent to its rank, so that no two seats may be equally desirable and create an ambiguous choice.

3. The models always seek the best available seat or set of seats, which are adjacent, in satisfying an applicant's request.

4. When an applicant has held seats previously and chooses the option of improvement, the models allocate no worse seats than those previously held.

The Optimum Allocation Model assigns applicants to sections based on predetermined costs associated with placing each applicant in every section. In contrast to the first three models, the total effectiveness of assignment is considered in allocating applicants to sections. The costs of assignment vary depending upon the relative qualification for the section. For each applicant, a choice of costs is made for assigning him to each section. Zero costs stipulate sections in which an applicant is qualified to sit. Other positive values are specified for the assignment of applicants to sections when not qualified. Procedures are suggested in the model description for scaling the assignment of costs, thus reducing the setup time for the model's operation.

Assumptions

There are a number of assumptions inherent in the models. For example, in the first three models it is assumed that an applicant wants seats located adjacent to one another. This assumption does not permit split assignments, i.e., assignment of seats in more than one location even if better seats are available but do not meet the requested amount. It is therefore possible for an applicant requesting a small number of seats to receive a better location than an applicant requesting a greater number of seats with a higher priority.

It is assumed that both applicants and seats have been ranked prior to assignment. In simple cases the solution may be found by observation, however, when numerous applicants and seats are involved the process becomes more complicated.

In the Optimum Allocation Model it is assumed that a reliable scale of costs can be developed to represent the relative penalty for assigning applicants to sections. In reality this could be extremely difficult to justify. The requirement is similar to ranking applicants and seats in the other three models.

Model I

General Description

Consider a stadium in which applicants have never been assigned seats before. In this situation, each applicant $A_1, A_2, A_3, \ldots, A_n$ is treated in priority order and each seat $S_1, S_2, S_3, \ldots, S_m$ is considered for assignment if it has not already been assigned, and if it is the best seat available. This is the first and simplest case of priority assignment in which no alternatives are offered to an applicant. Seats are assigned as a function of priority and availability.

Operational Logic

The model examines each applicant in priority order and scans the list of seats to find the best location for the assignment. The first step in the model's process is to examine the number of seats requested by the applicant. (This number acts as a tally in the assignment process and is decremented with the assignment of adjacent seats.) Next an inquiry is made to the list of seats in the order of desirability, i.e., seat S_1 is tested for occupancy. If the seat is vacant, $S_1 = 0$, it is available for tentative assignment. Each time a seat is found vacant and subsequently assigned, the number remaining in the request is reduced by one. If the request has not been filled, the model searches for the adjacent seat, determines its occupancy status, and proceeds according to the outcome. This process is continued until the request is satisfied or an exit condition is reached. If the seat is occupied, $S_1 = 1$, the model indexes to the next most desirable location. Conceivably the entire list of seats may not yield a sufficient number of adjacent seats to satisfy the request. In this case, the unfilled requirement is reported and the next request is considered for assignment.

Two conditions can terminate a tentative assignment. When an adjacent seat is found to be occupied the process must be incremented to a location where seats are unoccupied. When an "end-of-row" situation occurs and the applicant's request is still unfilled, the process must be incremented. This, of course, does not necessarily terminate the filling of a request.

Mathematical Relationships

The assignment problem can be stated mathematically as follows.

Given:

1. A vector S_i representing a seat location in a stadium where initially $\sum_i S_i = 0$. During the course of assignment S_i has the following properties,

$$S_i = 0, \text{ if the seat is vacant,}$$

$$S_i = 1, \text{ if the seat is occupied.}$$

The quality of each seat is predetermined and characterized by the following,

$i = 1$, the best seat,

$i = 2$, second best,

$i = 3$, third best, etc.

2. A vector A_j representing an applicant request for one or more seats (usually not to exceed a specified number). The sequence of processing requests depends upon the numerical values of priority,

$j = 1$, first (highest priority),

$j = 2$, second

$j = 3$, third, etc.

Find: A matrix $X = \| x_{ij} \|$, known as the assignment matrix, such that,

$$\sum_i x_{ij} = A_j \qquad \text{for } j = 1, 2, 3, \ldots, m$$

and

$$x_{ij} = \begin{cases} 1, & \text{if seat is assigned to applicant } j \\ 0, & \text{otherwise} \end{cases}$$

likewise

$$\sum_j x_{ij} = 1 \qquad \text{for assigned } i.$$

Solution: The assignment process is iterative. The following conditions must hold for each allocation.

1. An assignment is not complete until

$$\sum_i x_{ij} = A_j \qquad \text{for any } j.$$

2. Each x_{ij} assigned to j must be adjacent although i need not

be numerically consecutive, but

$$i^{(1)} < i^{(2)} < i^{(3)} < i^{(n)}$$

where

$i^{(n)}$ is the n^{th} seat assigned in the set.

Technique of Solution: In this model, the conditions are fairly simple and the technique of solution follows two terminal rules.

1. If the number of adjacent seats is not sufficient to complete an assignment, i.e., $\sum_i x_{ij} \neq A_j$, less desirable seats are assigned if available so that adjacency is maintained for a request. In other words, splitting a request, separating the seats assigned to an applicant, is not possible with the model.

2. If the end-of-row is reached before the request is filled, less desirable seats are assigned if available so that adjacency is maintained. An outline of the steps in this technique is described below and illustrated in Figure 1.

1. Set $i = 1$, $j = 1$.

2. If $S_i = 0$, assign S_i to A_j and record x_{ij} (in this case x_{11}) = 1. Also set $S_i = 1$.

3. Set $A_j = A_j - 1$. Mechanically, the original value of A_j is retained for subsequent assignments if the initial attempt fails to satisfy the conditions imposed above.

4. If $A_j > 0$, execute the following series of steps to locate the next adjacent seat. If $A_j = 0$, execute step 9.

5. Set $k = i$. k now becomes the variable used to find the seat adjacent to i.

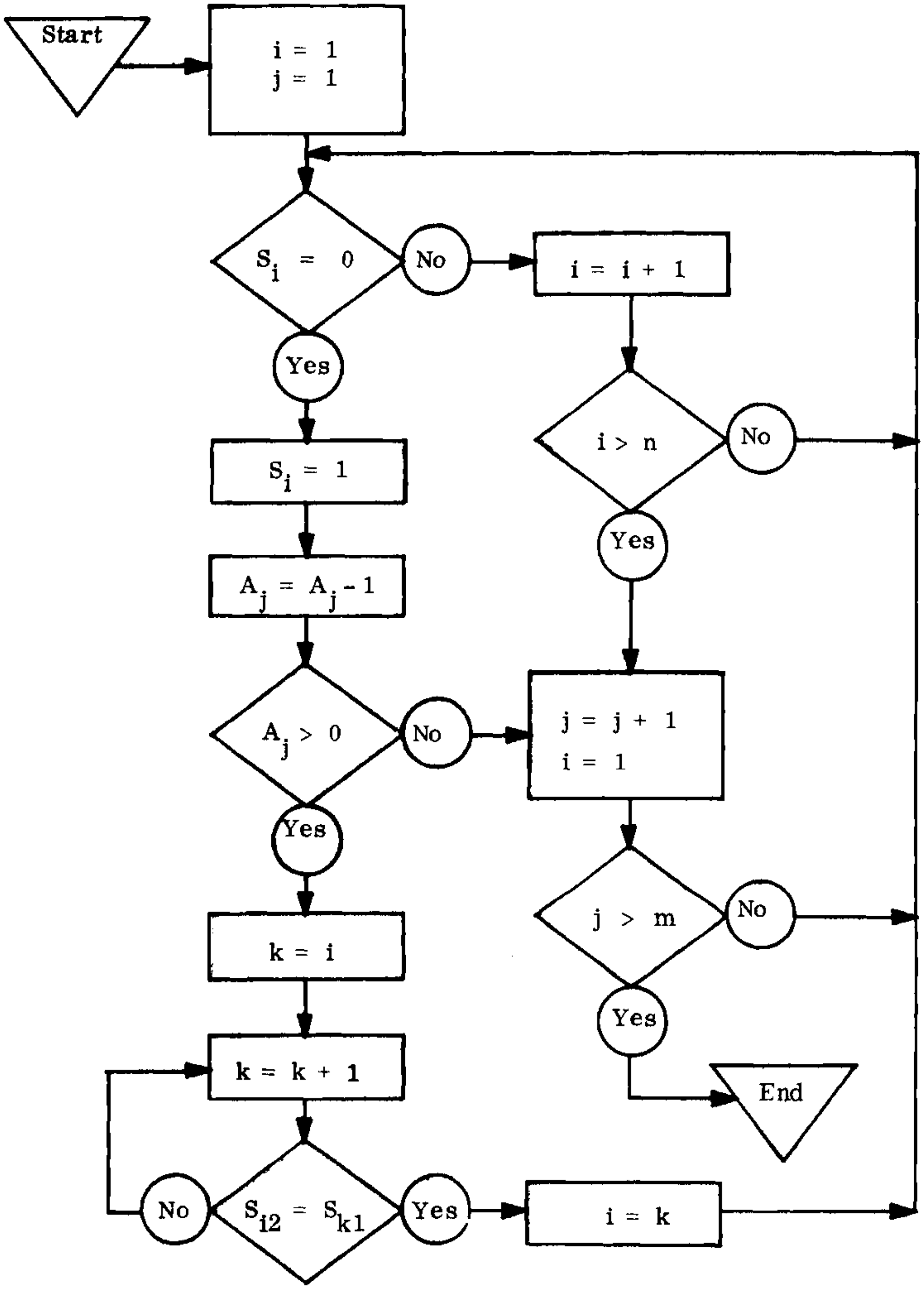


Figure 1. Model I Logic Diagram

6. Index $k = k + 1$.

7. Compare S_{i2} with S_{k1} where

S_{i2} = real seat adjacent to i, and

S_{k1} = real seat k.

8. If $S_{i2} = S_{k1}$, set $i = k$ and repeat the process beginning with step 2. If $S_{i2} \neq S_{k1}$, repeat steps 6 and 7 until $S_{i2} = S_{k1}$. If either $S_i = 1$ or $S_{i2} = 0$, which indicates S_i has been assigned or S_{i2} is the end of row, set $i = i + 1$, initialize all other variables, and repeat the process beginning with step 2.

9. Finalize the process by setting $S_i = 1$ and recording $x_{ij} = 1$ for each i assigned to j.

10. Index $j = j + 1$. Set $i = 1$ and index until $S_i = 0$. Repeat the process beginning with step 2 for all subsequent applicant requests until all applicants are assigned seats or all seats are assigned under the conditions imposed above.

Illustrative Example

Consider a stadium in which the seats are ranked uniquely in the order of their desirability. The quality of the seats decreases as the numerical value increases. Applicants are fixed in number, ranked in a priority order, and the magnitude of each request is given. In this example, demand exactly equals supply. Figure 2 represents a matrix of rows for seats and columns for applicants; initially the matrix is filled with zeros. There are four applicants and six seats and the requests breakdown as follows: $A_1 = 2$; $A_2 = 1$; $A_3 = 1$; and $A_4 = 2$. Applying the iterative steps described above produces the following results.

1. $i = 1, j = 1$.

2. $S_1 = 0$; therefore, $S_1 \subset A_1$, $S_1 = 1$, and $x_{11} = 1$ in the matrix.

Applicant j / Seat i	1	2	3	4	$\sum_j x_{ij}$
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
$\sum_i x_{ij}$	0	0	0	0	0

Figure 2. Initial State of Matrix (Model I)

3. $A_1 = 2 - 1 = 1$.

4. $A_1 > 0$.

5. $K = 1$.

6. $K = 1 + 1 = 2$.

7. $S_{12} = S_{21}$ since S_2 is adjacent to S_1.

8. $i = 2$.

9. $S_2 = 0$; therefore, $S_2 \subset A_1$, $S_2 = 1$, and $x_{21} = 1$ in the matrix.

10. $A_1 = 1 - 1 = 0$.

11. $A_1 = 0$; therefore execute steps 9 and 10.

12. Since $S_1 = S_2 = 1$ and $x_{11} = x_{21} = 1$, set $\sum_{j=2}^{4} x_{1j} = 0$ and $\sum_{j=2}^{4} x_{2j} = 0$. Fill the remainder of the row in the assignment matrix with zeros.

Repeating the process for A_2, A_3, and A_4, produces the assignment matrix shown in Figure 3.

Model II

General Description

Consider a stadium in which applicants have held seats previously and are being reassigned based on their priority and the location of the previous assignment. In this situation, each applicant $A_1, A_2, A_3, \ldots, A_n$ is treated in priority order and each seat $S_1, S_2, S_3, \ldots, S_m$ is considered for assignment, if it has not already been assigned, if it is the best one available, and better than the previous location. This is the second case of assignment in which the history of previous seat assignments is considered and the option of improvement is permitted. If an applicant does not choose the option of improvement, he will receive the same seats he held previously. If it is a new application, then it is processed in the same manner as described in Model I.

Seat i \ Applicant j	1	2	3	4	$\sum_j x_{ij}$
1	1	0	0	0	1
2	1	0	0	0	1
3	0	1	0	0	1
4	0	0	1	0	1
5	0	0	0	1	1
6	0	0	0	1	1
$\sum_i x_{ij}$	2	1	1	2	6

Figure 3. Final State of Matrix (Model I)

Operational Logic

The first step in this model's process is to determine which applicants held seats previously and of these, which are requesting the same seats. The model immediately assigns these applicants to their former locations and fills in the assignment matrix to indicate that these seats are no longer available. Next the model examines, again in priority order, the remainder of the applicants and attempts to find a better location for those requesting improvement. If no better seat is available the model assigns the location previously held. If improvement can be accomplished, the seats previously held are released to subsequent applicants.

A tentative assignment is terminated (as in Model I) when an occupied seat is found or an end-of-row situation is reached before a request is filled.

Mathematical Relationships

The assignment problem can be stated mathematically as follows.

Given:

1. A vector S_i representing a seat location in a stadium where initially $\sum_i S_i = 0$. During the course of assignment S_i has the following properties,

$$S_i = 0, \text{ if the seat is vacant,}$$

$$S_i = 1, \text{ if the seat is occupied.}$$

The quality of each seat is predetermined and characterized by the following,

$$i = 1, \text{ the best seat,}$$

$$i = 2, \text{ second best,}$$

$$i = 3, \text{ third best, etc.}$$

2. A vector A_j representing an applicant request for one or more seats. The sequence of processing requests depends upon the numerical

values of priority,

$$j = 1, \text{ first (highest priority)},$$
$$j = 2, \text{ second},$$
$$j = 3, \text{ third, etc.}$$

In this model, two additional variables must be given for applicants that previously held seats,

$$A_{j1} = \begin{cases} 1, & \text{if improvement is requested} \\ 0, & \text{if same seat(s) are requested} \end{cases}$$

and

$$A_{j2} = \text{location of first real seat previously held.}$$

Find: A matrix $X = \| x_{ij} \|$, known as the assignment matrix, such that,

$$\sum_i x_{ij} = A_j \qquad \text{for } j = 1, 2, 3, \ldots, m$$

and

$$x_{ij} = \begin{cases} 1, & \text{if seat is assigned to applicant } j \\ 0, & \text{otherwise} \end{cases}$$

likewise

$$\sum_j x_{ij} = 1 \quad \text{for assigned } i.$$

Solution: The assignment process is iterative. The following conditions must hold for each allocation.

1. An assignment is not complete until

$$\sum x_{ij} = A_j \qquad \text{for any j.}$$

2. Each x_{ij} assigned to j must be adjacent although i need not be numerically consecutive, but

$$i^{(1)} < i^{(2)} < i^{(3)} < i^{(n)}$$

where

$$i^{(n)} \text{ is the } n^{th} \text{ seat assigned in the set.}$$

3. The assignment matrix is completed first for each $A_{j1} = 0$ and $A_{j2} > 0$ (both conditions must hold), i.e., $x_{ij} = 1$ for each j meeting the conditions.

4. Remaining j are assigned according to the rules in Model I with the qualification that $i^{(1)} \leq i^{(2)}$, meaning assigned i must be at least as good as the location $i^{(j2)}$ -- which represents the quality of the best seat previously held by j. This rule insures that for $A_{j1} = 1$ and $A_{j2} > 0$ no worse i will be assigned than previous i (i^{j2}).

Technique of Solution: In this model, the conditions are more complex than in Model I, but the technique of assignment follows the same terminal rules.

1. If the number of adjacent seats is not sufficient to complete an assignment, i.e., $\sum_i x_{ij} \neq A_j$, less desirable seats are assigned if available and provided $i^{(1)} \leq i^{(j2)}$, so that adjacency is maintained.

2. If the end-of-row is reached before the request if filled, less desirable seats are assigned if available and provided $i^{(1)} \leq i^{(j2)}$, so that adjacency is maintained. For the option of improvement, assignment can be terminated if the best available seats are no better than the ones previously held. In this case

the original seats are reassigned to the applicant.

An outline of the steps in this technique is described below and illustrated in Figure 4.

1. Set $i = 1$, $j = 1$.

2. If $A_{j2} > 0$ and $A_{j1} = 0$, proceed until all applicants requesting same seats are processed.

a. Compare S_{i1} with A_{j2} where

S_{i1} = real seat of i, and

A_{j2} = real seat held by j.

b. If $S_{i1} = A_{j2}$, execute the steps beginning with c. If $S_{i1} \neq A_{j2}$, set $i = i + 1$ and repeat step a.

c. If $S_i = 0$, assign S_i to A_j and record $x_{ij} = 1$. Also set $S_i = 1$.

d. Set $A_j = A_j - 1$. Mechanically, the original value of A_j is retained for subsequent assignments if the initial attempt fails to satisfy the conditions imposed above.

e. If $A_j > 0$, execute the following series of steps to locate the next adjacent seat. If $A_j = 0$, execute step j.

f. Set $k = 1$. k now becomes the index used to find the seat adjacent to i.

g. Index $k = k + 1$.

h. Compare S_{i2} with S_{k1} where

S_{i2} = real seat adjacent to i, and

S_{k1} = real seat k.

i. If $S_{i2} = S_{k1}$, set $i = k$ and repeat the process beginning with step c. If $S_{i2} \neq S_{k1}$, repeat steps g. and h. until $S_{i2} = S_{k1}$. If either

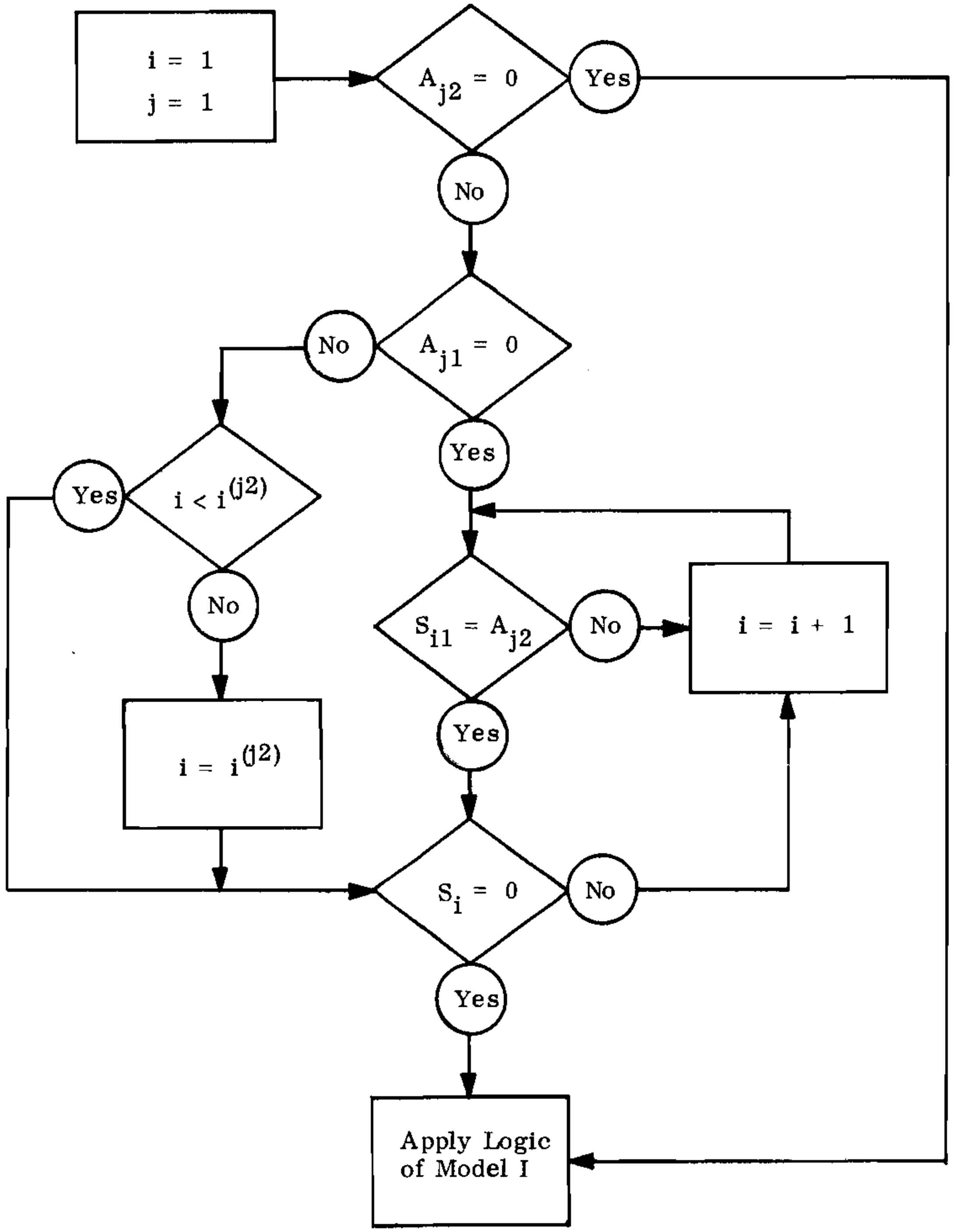


Figure 4. Model II Logic Diagram

$S_i = 1$ or $S_{i2} = 0$, which means S_i has been assigned or S_{i2} is the end-of-row, set $i = i + 1$, initialize all affected variables and repeat the process beginning with step c.

j. Finalize the process by setting $S_i = 1$ and recording $x_{ij} = 1$ for each i assigned to j.

k. Index $j = j + 1$. Set $i = 1$ and repeat the process beginning with step a. for all subsequent applicant requests meeting the conditions above, i.e., $A_{j2} > 0$ and $A_{j1} = 0$.

3. If $A_{j2} > 0$ and $A_{j1} = 1$, proceed until all applicants requesting improvement are processed.

a. If $S_i = 0$, compare S_{i1} with A_{j2} to determine if $i < i^{(j2)}$.

b. If $i < i^{(j2)}$, execute the steps in 2.c. through 2.j. above. Return to the step 3.d. below when complete.

c. If $i \geq i^{(j2)}$, set $i = i^{(j2)}$ and execute the steps in 2.c. through 2.j. above. This reassigns the applicant's original seats. Return to step 3.d. below when complete.

d. Index $j = j + 1$. Set $i = 1$ and repeat the process beginning with step a. for all subsequent applicant requests meeting the conditions above, i.e., $A_{j2} > 0$ and $A_{j1} = 1$.

4. If $A_{j2} = 0$, follow the steps exactly as described in Model I for all new applicants. In actual operation, the Reassignment Model tentatively assigns the seats originally held by applicants requesting improvement to insure receipt of no worse than his previous assignment. When these requests are processed for improvement, the seats held previously are released for subsequent applicants if improvement is achieved.

Illustrative Example

Consider a stadium in which the seats are ranked uniquely in the order of their desirability. The quality of these seats decreases as the numerical value increases. Applicants are fixed in number, ranked in a priority order, and separated into two groups by option (same seat or improvement). The location of previously assigned seats and the number of seats requested is included. Figure 5 illustrates the seating configuration for this example. The quality of the seat and the applicant previously holding the location is recorded in the row-seat box. The assignment matrix is originally filled with zeros as shown in the Model I example. Applicant data for the example is listed below in tabular form.

Magnitude of Request	Option[1]	Previous Seats
$A_1 = 2$	$A_{11} = 0$	$A_{12} = 1,2$
$A_2 = 2$	$A_{21} = 1$	$A_{22} = 4,5$
$A_3 = 4$	$A_{31} = 1$	$A_{32} = 8,9,10,15$
$A_4 = 1$	$A_{41} = 0$	$A_{42} = 6$
$A_5 = 1$	$A_{51} = 0$	$A_{52} = 11$
$A_6 = 1$	$A_{61} = 1$	$A_{62} = 12$
$A_7 = 3$	$A_{71} = 1$	$A_{72} = 13,14,19$
$A_8 = 2$	$A_{81} = 1$	$A_{82} = 18,20$

[1] $A_{j1} = 0$ means same seats; $A_{j1} = 1$ means improve seats.

Applying the iterative steps described above produces the following results.

1. $i = 1,\ j = 1$.
2. All $A_{j1} = 0$ are processed first, i.e., A_1, A_4, and A_5.
   a. $S_{11} = A_{12}$ since $i = 1$ and $i^{(j2)} = 1$.

Seat / Row	1	2	3	4
1	1 / 1	2 / 1	3 /	7 /
2	4 / 2	5 / 2	6 / 4	11 / 5
3	8 / 3	9 / 3	10 / 3	15 / 3
4	12 / 6	13 / 7	14 / 7	19 / 7
5	16 /	17 /	18 / 8	20 / 8

NOTE: The number above the hash is the quality (rank) of the seat and the number below the hash is the applicant (number) holding the seat. Blanks below the hash represent relinquished seats.

Figure 5. Seating Configuration: Model II

b. $S_1 = 0$.

c. $S_1 = 1$ and $x_{11} = 1$, therefore S_1 is now assigned to A_1.

d. $A_1 = 2 - 1 = 1$.

e. $A_1 > 0$.

f. $k = 1$.

g. $k = 1 + 1 = 2$.

h. $S_{12} = S_{21}$ since S_2 is adjacent to S_1.

i. $i = 2$.

j. $S_2 = 1$ and $x_{21} = 1$, therefore S_2 is now assigned to A_1.

k. Since $A_1 = 0$, assignment for A_1 is complete.

l. Since $S_1 = S_2 = 1$ and $x_{11} = x_{21} = 1$, set $\sum_{j=2}^{8} x_{1j} = 0$; and $\sum_{j=2}^{8} x_{2j} = 0$. Fill in the remainder of the row in the assignment matrix with zeros.

Repeating the process for A_4 and A_5 produces the partial assignment matrix shown in Figure 6.

3. All $A_{j1} = 1$ are processed next, i.e., A_2, A_3, A_6, A_7, and A_8.

a. $i = 1$, $j = 2$ and $S_1 = 1$; therefore, index $i = i + 1$ until $S_i = 0$ or S_3.

b. $3 < 4$ since $S_3 < S_4$, which is A_2 previous best seat.

c. $S_3 = 0$, therefore, set $S_3 = 1$ and $x_{32} = 1$, so that S_3 is now assigned to A_2.

d. $A_2 = 2 - 1 = 1$.

e. $A_2 > 0$.

f. $k = i = 3$.

g. $k = 3 + 1 = 4$.

h. $S_{32} \neq S_{41}$ since S_3 is not adjacent to S_4.

Applicant j / Seat i	1	2	3	4	5	6	7	8	$\sum_j x_{ij}$
1	1	0	0	0	0	0	0	0	1
2	1	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	1	0	0	0	0	1
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0
11	0	0	0	0	1	0	0	0	1
12	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0
$\sum_i x_{ij}$	2	0	0	1	1	0	0	0	4

Figure 6. State of Matrix: Partial Assignment (Model II)

i. $k = 4 + 1 = 5$, $S_{32} \neq S_{51}$; $k = 5 + 1 = 6$, $S_{32} \neq S_{61}$; $k = 6 + 1 = 7$, and now $S_{32} = S_{71}$; $i = 7$.

j. $S_7 = 0$, therefore set $S_7 = 1$ and $x_{72} = 1$, so that S_7 is now assigned to A_2.

k. Since $A_2 = 0$, assignment for A_2 is complete.

l. Since $S_3 = S_7 = 1$ and $x_{32} = x_{72} = 1$; set $\sum_{j=1, j=3}^{8} x_{3j} = 0$ and $\sum_{j=1, j=3}^{8} x_{7j} = 0$. Fill the remainder of the row in the assignment matrix with zeros.

Repeating the process for A_3, A_6, A_7, and A_8 produces the completed assignment matrix shown in Figure 7.

Model III

General Description

Consider a stadium in which some individuals have held seats previously and are being reassigned on the basis of two mutually exclusive, optional criteria.

1. Choice of specific seat(s).
2. Choice of improvement.

In this situation, each applicant A_1, A_2, A_3, ..., A_n is treated in priority order and each seat S_1, S_2, S_3, ..., S_m is considered for assignment if it is specifically selected or is the best available. This is the third case of assignment in which the history of previous seat assignments is considered and the option of a specific location or improvement is permitted. For an applicant that selects a particular location, the model searches the seat list and assigns them if available, then releases the applicant's previous holdings. If the selected seats are not available, either the best available or the previous seats are assigned. For an applicant that does not specify a particular location but does request improvement, the model finds the best available seat, compares it with the applicant's old seat

Seat i \ Applicant j	1	2	3	4	5	6	7	8	$\sum_j x_{ij}$
1	1	0	0	0	0	0	0	0	1
2	1	0	0	0	0	0	0	0	1
3	0	1	0	0	0	0	0	0	1
4	0	0	0	0	0	1	0	0	1
5	0	0	0	0	0	0	0	0	0
6	0	0	0	1	0	0	0	0	1
7	0	1	0	0	0	0	0	0	1
8	0	0	1	0	0	0	0	0	1
9	0	0	1	0	0	0	0	0	1
10	0	0	1	0	0	0	0	0	1
11	0	0	0	0	1	0	0	0	1
12	0	0	0	0	0	0	1	0	1
13	0	0	0	0	0	0	1	0	1
14	0	0	0	0	0	0	1	0	1
15	0	0	1	0	0	0	0	0	1
16	0	0	0	0	0	0	0	1	1
17	0	0	0	0	0	0	0	1	1
18	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0
$\sum_i x_{ij}$	2	2	4	1	1	1	3	2	16

Figure 7. Final State of Matrix (Model II)

location, and assigns the seats if better or returns the original seats if not. If a better seat is located and assigned, the previous seats are released for subsequent applicants.

Operational Logic

The model examines each applicant in priority order, determines if specific seats have been requested, and either assigns them if available or finds the next best location available. The first step in the model's process is to identify the choice an applicant has made. If a specific seat location is requested, the model immediately determines the occupancy status of this seat and the adjacent seats required to complete the assignment. As in Model II, if the applicant requests improvement, the model assigns no worse seats than those previously held. This guarantee however does not hold if the specific seats are requested. The process of assignment is essentially the same once the initial seat of a set to be assigned is located. Unfilled requests are treated the same as described for Model I. Tentative assignments are terminated the same as described for Models I and II, i.e., when an occupied seat is found or an end-of-row situation is reached before a request is filled.

Mathematical Relationships

The assignment problem can be stated mathematically as follows.

Given:

1. A vector S_i representing a seat location in a stadium where initially $\sum_i S_i = 0$. During the course of assignment S_i has the following properties,

$$S_i = 0, \text{ if the seat is vacant}$$

$$S_i = 1, \text{ if the seat is occupied.}$$

The quality of each seat is predetermined and characterized by the following,

$i = 1$, the best seat,

$i = 2$, second best,

$i = 3$, third best, etc.

2. A vector A_j representing an applicant request for one or more seats. The sequence of processing requests depends upon the numerical values of priority,

$j = 1$, first (highest priority),

$j = 2$, second,

$j = 3$, third, etc.

In this model, three additional variables must be given for applicants that previously held seats,

$$A_{j1} = \begin{cases} 1, \text{ if the specific seats are requested} \\ 0, \text{ if improvement is requested} \end{cases}$$

A_{j2} = location of first real seat previously held

A_{j3} = location of first real seat specifically requested.

Find: A matrix $X = \| x_{ij} \|$, known as the assignment matrix, such that,

$$\sum_i x_{ij} = A_j \qquad \text{for } j = 1, 2, 3, \ldots, m$$

and

$$x_{ij} = \begin{cases} 1, \text{ if seat is assigned to applicant } j \\ 0, \text{ otherwise} \end{cases}$$

likewise

$$\sum_j x_{ij} = 1 \qquad \text{for assigned } i.$$

Solution: The assignment process is iterative. The following conditions must hold for each allocation.

1. An assignment is not complete until

$$\sum_i x_{ij} = A_j \quad \text{for any j.}$$

2. Each x_{ij} assigned to j must be adjacent although i need not be numerically consecutive, but

$$i^{(1)} < i^{(2)} < i^{(3)} < i^{(n)}$$

where

$i^{(n)}$ is the n^{th} seat assigned in the set

3. The assignment matrix is completed first for each $A_{j1} = 1$ and $A_{j3} > 0$, i.e., $x_{ij} = 1$, if i is available to j; otherwise assignment is made as in 4.

4. The assignment matrix is then completed for each $A_{j1} = 0$ and $A_{j2} > 0$, or if A_{j3} is occupied ($x_{ij} = 1$), provided $i^{(1)} \leq i^{(j2)}$ -- meaning assigned i must be at least as good as the location $i^{(j2)}$, the quality of the best seat previously held by j.

Technique of Solution: In this model, the conditions are slightly more complicated than in Model II, but the technique of assignment follows the same terminal rules.

1. If the number of adjacent seats is not sufficient to complete an assignment, i.e., $\sum_i x_{ij} \neq A_j$, less desirable seats are assigned if available and provided $i^{(1)} \leq i^{(j2)}$, so that adjacency is maintained.

2. If the end-of-row is reached before the request is complete, less desirable seats are assigned if available and provided $i^{(1)} \leq i^{(j2)}$, so that

adjacency is maintained. For the option of improvement, assignment can be terminated if the best available seats are no better than the ones previously held. In this case the original seats are reassigned to the applicant.

An outline of the steps involved in this technique is described and illustrated in Figure 8.

1. Set $i = 1$, $j = 1$.

2. If $A_{j1} = 1$ and $A_{j3} > 0$, proceed with the following steps.

   a. Compare S_{i1} with A_{j3} where

      S_{i1} = real seat of i, and

      A_{j3} = real seat specifically requested by j.

   b. If $S_{i1} = A_{j3}$, execute the steps beginning with c. If $S_{i1} \neq A_{j3}$, set $i = i + 1$ and repeat step a.

   c. If $S_i = 1$, proceed to 3. below.

   d. If $S_i = 0$, assign S_i to A_j by setting $S_i = 1$ and recording $x_{ij} = 1$ in the matrix.

   e. Set $A_j = A_j - 1$. Mechanically, the original value of A_j is retained for subsequent assignments if the initial attempt fails to satisfy the conditions imposed above.

   f. If $A_j > 0$, execute the following series of steps to locate the next adjacent seat. If $A_j = 0$, execute step k.

   g. Set $k = i$. k now becomes the index used to find the seat adjacent to i.

   h. Index $k = k + 1$.

   i. Compare S_{i2} with S_{k1} where

      S_{i2} = real seat adjacent to i, and

      S_{k1} = real seat k.

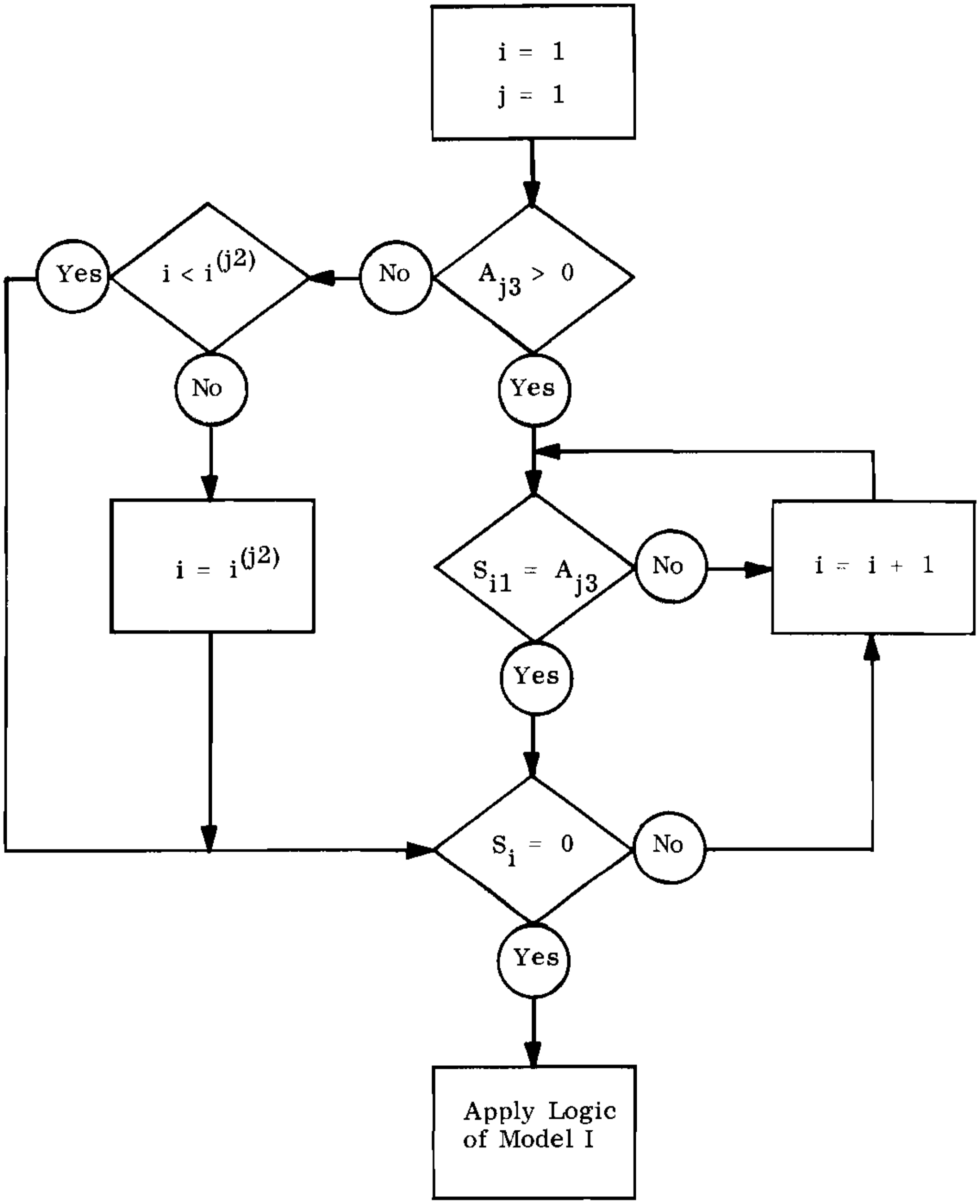


Figure 8. Model III Logic Diagram

j. If $S_i = S_{k1}$, set $i = k$ and repeat the process beginning with step c. If $S_{i2} \neq S_{k1}$, repeat steps h. and i. until $S_{i2} = S_{k1}$. If either $S_i = 1$ or $S_{i2} = 0$, which means S_i has been assigned or S_{i2} is the end-of-row, set $i = i + 1$, initialize all affected variables, and repeat the process beginning with step c.

k. Finalize the process by setting $S_i = 1$ and recording $x_{ij} = 1$ for each i assigned to j.

l. Index $j = j + 1$. Set $i = 1$ and index until $S_i = 0$. Repeat the process beginning with step a. for all subsequent applicant requests meeting the conditions above, i.e., $A_{j1} = 1$ and $A_{j3} > 0$.

3. If $A_{j1} = 0$ and $A_{j2} > 0$, proceed with the following steps.

a. If $S_i = 0$, compare S_{i1} with A_{j2} to determine if $i < i^{(j2)}$.

b. If $i < i^{(j2)}$, execute the steps in 2.c. through 2.k. above. Return to step 3.d. below when complete.

c. If $i \geq i^{(j2)}$, set $i = i^{(j2)}$ and execute the steps in 2.c. through 2.k. above. This reassigns the applicant's original seats. Return to step 3.d. below when complete.

d. Index $j = j + 1$. Set $i = 1$ and repeat the process beginning with step a. for all subsequent applicant requests meeting the conditions above, i.e., $A_{j2} > 0$ and $A_{j1} = 0$.

Illustrative Example

Consider a stadium in which the seats are ranked uniquely in the order of their desirability. The quality of the seats decreases as the numerical value increases. Applicants are fixed in number, ranked in priority order, and the option of specific seats or improvement is permitted. The location of previously assigned seats, the location of specific seats, and number of seats requested is included. Figure 9 illustrates the seating configuration for this example. The quality of each seat and the applicant previously holding the location is recorded in the row-seat box. The assignment matrix is originally filled with zeros as shown in the Model I example. Applicant data for the example is listed below in tabular form.

Magnitude of Request	Option[1]	Previous Seats	Specific Seats
$A_1 = 2$	$A_{11} = 0$	$A_{12} = 1,2$	
$A_2 = 2$	$A_{21} = 1$	$A_{22} = 3,4$	$A_{23} = 5,6$
$A_3 = 1$	$A_{31} = 1$	$A_{32} = 8$	$A_{33} = 7$
$A_4 = 4$	$A_{41} = 1$	$A_{42} = 11,12,17,18$	$A_{43} = 7,8,13,14$
$A_5 = 4$	$A_{51} = 0$	$A_{52} = 15,16,19,20$	
$A_6 = 2$	$A_{61} = 0$	$A_{62} = 9,10$	
$A_7 = 2$	$A_{71} = 0$	$A_{72} = 13,14$	

[1] $A_{j1} = 0$ means improve seats; $A_{j1} = 1$ means assign specific seats.

Applying the iterative steps described above produces the following results.

1. $i = 1$, $j = 1$.

2. Since $A_{11} = 0$, the steps for assignment when an applicant requests improvement are used.

   a. $S_1 = 0$.

   b. $S_1 = 1$ and $x_{11} = 1$, therefore, S_1 is now assigned to A_1.

   c. $A_1 = 2 - 1 = 1$.

   d. $A_1 > 0$.

Seat \ Row	1	2	3	4
1	1 / 1	2 / 1	5 /	6 /
2	3 / 2	4 / 2	9 / 6	10 / 6
3	7 /	8 / 3	13 / 7	14 / 7
4	11 / 4	12 / 4	17 / 4	18 / 4
5	15 / 5	16 / 5	19 / 5	20 / 5

NOTE: The number above the hash is the quality (rank) of the seat and the number below the hash is the applicant (number) holding the seat. Blanks below the hash represent relinquished seats.

Figure 9. Seat Configuration: Model III

e. $k = 1$.

f. $k = 1 + 1 = 2$.

g. $S_{12} = S_{21}$, since S_2 is adjacent to S_1.

h. $i = 2$.

i. $S_2 = 1$ and $x_{21} = 1$, therefore S_2 is now assigned to A_1.

j. Since $A_1 = 0$, assignment for A_1 is complete.

k. Since $S_i = S_2 = 1$ and $x_{11} = x_{21} = 1$; set $\sum_{j=2}^{7} x_{ij} = 0$ and $\sum_{j=2}^{7} x_{ij} = 0$. Fill the remainder of the row in the assigned matrix with zeros.

l. $j = j + 1 = 1 + 1 = 2$ and $i = 1$, proceed to 3.

3. Since $A_{21} = 1$, the steps for assignment when an applicant requests specific seats are used.

a. $i = 1$, $j = 2$ and $S_1 = 1$; therefore, index $i = i + 1$ until $S_1 = 0$ or S_3.

b. $S_{31} \neq A_{23}$, therefore $i = 3 + 1 = 4$; $S_{41} \neq A_{23}$, therefore $i = 4 + 1 = 5$; and now $S_{51} = A_{23}$; so $i = 5$.

c. $S_5 = 0$, therefore, set $S_5 = 1$ and $x_{52} = 1$, so that S_5 is now assigned to A_2.

d. $A_2 = 2 - 1 = 1$.

e. $A_2 > 0$.

f. $k = i = 5$.

g. $k = 5 + 1 = 6$.

h. $S_{52} = S_{61}$ since S_6 is adjacent to S_5.

i. $S_6 = 0$, therefore, set S_6 and $x_{62} = 1$, so that S_6 is now assigned to A_2.

j. Since $A_2 = 0$, assignment for A_2 is complete.

k. Since $S_5 = S_6 = 1$ and $x_{52} = x_{62} = 1$, set $\sum_{j=1, j=3}^{8} x_{5j} = 0$ and $\sum_{j=1, j=3}^{8} x_{6j} = 0$. Fill the remainder of the row in the assignment matrix with zeros.

Repeating the process for A_3 through A_7 produces the completed assignment matrix shown in Figure 10.

Model IV

General Description

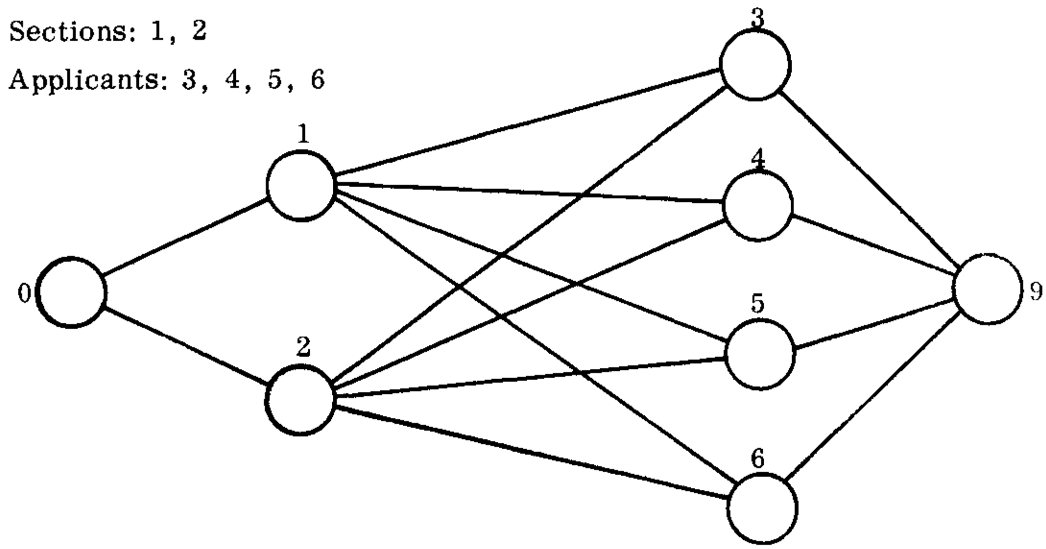
Consider a stadium in which finite costs are associated with assigning each applicant to each section. In this situation, applicants $A_1, A_2, A_3, \ldots, A_n$ are processed collectively and allocated to sections $S_1, S_2, S_3, \ldots, S_m$ in an optimum manner. The measure of effectiveness for the optimum solution is minimum cost. Neither priority nor seat quality is considered in this model and the solution is expressed by the cost of allocating applicants to sections of seats. The model technique of finding an optimum solution is based on network flow theory which is documented in the references discussed in Chapter II, Literature Survey.

Operational Logic

The model examines applicant requests in the order in which they are input and finds the minimum cost route through the network of links connecting each section with each applicant. An example network of nodes and links is depicted in Figure 11 for illustrative purposes. Applicants are given node numbers and a link is constructed to a "sink" or terminal point. The number of seats requested is expressed as an upper-bound flow which must be forced through the network to satisfy the requirement. Sections are given node numbers and a link is constructed to a "source" or origin point. The number of seats in the

Seat i \ Applicant j	1	2	3	4	5	6	7	$\sum_j x_{ij}$
1	1	0	0	0	0	0	0	1
2	1	0	0	0	0	0	0	1
3	0	0	0	0	0	1	0	1
4	0	0	0	0	0	1	0	1
5	0	1	0	0	0	0	0	1
6	0	1	0	0	0	0	0	1
7	0	0	1	0	0	0	0	1
8	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	1	1
10	0	0	0	0	0	0	1	1
11	0	0	0	1	0	0	0	1
12	0	0	0	1	0	0	0	1
13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
15	0	0	0	0	1	0	0	1
16	0	0	0	0	1	0	0	1
17	0	0	0	1	0	0	0	1
18	0	0	0	1	0	0	0	1
19	0	0	0	0	1	0	0	1
20	0	0	0	0	1	0	0	1
$\sum_i x_{ij}$	2	2	1	4	4	2	2	17

Figure 10. Final State of Matrix (Model III)



Link Information

From	To	Cost	Capacity	Type
0	1	0	10	Source
0	2	0	10	Source
1	3	2	5000	Connecting
1	4	0	5000	Connecting
1	5	0	5000	Connecting
1	6	4	5000	Connecting
2	3	1	5000	Connecting
2	4	1	5000	Connecting
2	5	0	5000	Connecting
2	6	0	5000	Connecting
3	9	0	6	Sink
4	9	0	4	Sink
5	9	0	4	Sink
6	9	0	6	Sink

Figure 11. Simple Network (Model IV)

section is expressed as an upper-bound flow which cannot be exceeded when satisfying requirements. Links joining applicant and section node points are also constructed, and costs are assigned reflecting the price of routing flow through the link, i.e., assigning the applicant to the section. Each time a chain flow (a series of links from source to sink) is developed the cost is compared with all other possible chain flows to insure a minimum. The result is a set of chain flows, satisfying as many requirements as possible, for which the total cost is a minimum. Though the allocation of seats may result in the physical separation of an applicant's request into two or more sections, the total assignment of seats can never exceed the upper-bound flow (number of seats) in a section. Likewise, an applicant can never receive more seats than requested since the upper-bound flow again limits the allocation mechanism.

The termination of the allocation process is predicated upon the criterion for determining an optimum solution. This condition is reached when the interchange of any two applicant assignments does not substantially alter (reduce) the original minimum-cost solution.

Mathematical Description

The allocation problem can be stated mathematically as follows: Consider an applicant-section network made up of origins (sections) and destinations (applicants) described as node points. At most one link joins a given pair of nodes, without ambiguity, so that the link joining nodes i and j can be denoted by the symbol (i, j). Each link has two parameters associated with it, which are nonnegative integers. One is cost and is denoted $c(i, j)$. The other is capacity, or upper-bound flow denoted $b(i, j)$. Each origin, i, has associated with it a positive integer, C_i, representing the maximum capacity in seats that can be allocated from the section at node i. Each destination, j, has associated with it a positive integer R_j representing the requirement, in number of seats, that must be satisfied for the applicant at node j.

To avoid mentioning a link twice, first as (i, j) and then as (j, i) an enumerating set S is introduced. This is a set of ordered pairs, one for each link, such that each link is represented by either (i, j) or (j, i) but not both. Flow in a link is an integer value with an orientation. If (i, j) S, the flow associated with (i, j) is denoted by $f(i, j)$ and $f(i, j) > 0$ indicates a flow from i to j where $f(i, j) < 0$ indicates a flow from j to i. In both cases $|f(i, j)|$ is the magnitude of flow.

<u>Node Input/Output.</u> If a node i is fixed and the expression

$$Y(i) = \sum_{S} f(i, j) - \sum_{S} f(j, i)$$

is considered, then $Y(i)$ represents the sum of link flows directed away from node i minus the sum of flows directed toward node i. $Y(i)$ is termed the net output at node i. Similarly, $-Y(i)$ is the net input at node i.

<u>Source and Sink.</u> The basic network of nodes and links is augmented by adding two hypothetical nodes called source (000) and sink (999). For each origin (section) i, add to S a link $(000, i)$ with parameters $b(000, i) = B_j$ and $c(000, i) = 0$. Similarly for each destination (applicant) j, add to S a link $(j, 999)$ with parameters $b(j, 999) = R_j$ and $c(j, 999) = 0$.

<u>Problem.</u> Consider now the following problem in an enlarged network. Find a set of link flows such that $Y(000)$ is maximized and the following conditions are satisfied.

1. $Y(i) = 0$ for all nodes except 000 and 999.
2. $|f(i, j)| < c(i, j)$ for every link (i, j). In addition the solution

must reflect total flow-cost that is as small as possible.

3. Total flow-cost = $\sum_{S} \left| f(i,j) \right| - c(i,j)$. The solution is subject to the restrictions that the net output from an origin, i, must not exceed B_i, the net input to a destination, j, must be at most R_j, input must equal output at all nodes, and no link capacity (upper-bound) may be exceeded.

Mathematical Algorithm. A computational algorithm, patterned in principle after certain "labeling" processes, was devised for network flow problems of the type described. This process is described below with appropriate symbolism. Links are arranged in a specific sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$. A table of link information is organized as follows.

Link Information

From	To	Cost	Capacity	Flow
i_1	j_1	$c(i_1,j_1)$	$b(i_1,j_1)$	0
:	:	:	:	:
i_k	j_k	$c(i_k,j_k)$	$b(i_k,j_k)$	0

The procedure will gradually replace the zero flows by appropriate nonzero flows, continuing the replacement until a point is reached such that the set of link flows in the table is a solution to 1., 2., and 3. A second table, related to node information, must also be set up.

Node Information

Node	Label	Approach Link
000	0	—
001	∞	—
:	:	:
999	∞	—

At the outset a "label" $V(i)$ is assigned to each node i in the following manner: $V(000) = 0$, and $V(i) = \infty$ for all other nodes. The "approach link" associated with a node has no significance at the start of the algorithm. The following procedures constitute the network analysis "labeling" process.

1. For each link (i_n, j_n) in turn, make the following replacements: If $V(i_n) + c(i_n, j_n) < V(j_n)$, and $f(i_n, j_n) < b(i_n, j_n)$, replace $V(j_n)$ by $V(i_n) + c(i_n, j_n)$ and list i_n, j_n as the approach link associated with node j_n. If $V(j_n)$ and list i_n, j_n as the approach link associated with node j_n. If $V(j_n) + c(i_n, j_n) < V(i_n)$, and $b(i_n, j_n) \neq 0$, replace $V(i_n)$ by $V(j_n) + c(i_n, j_n)$ and list (i_n, j_n) as the approach link associated with node i_n. If neither of the sets of conditions holds, make no replacements.

2. If a replacement is made for at least one link, repeat step 1. When a stage is reached such that no further replacements can be made, proceed to step 3.

3. If $V(999) = \infty$, then the set of flows currently listed in the table of link data constitutes a solution to the problem. If $V(999)$ represents the cost of the chain from 000 to 999 that consists entirely of unsaturated links, i.e., links whose flow, in the direction of the chain, is less than the link capacity. In the latter case, proceed to step 4.

4. Starting at node 999, "trace backward" along the approach link associated with 999 to its other node, say node i_1. From i_1 trace backward along the approach link associated with i_1 to a second node, say i_2. Continuing in this manner, at some stage s, $i_s = 000$ will be found, i.e., a chain that connects 000 and 999 will have been traced backward.

5. For each link (i,j) in the chain found in step 4. calculate $b(i,j) - f(i,j)$ if the orientation of this link in the chain agrees with that given in the link

table. Calculate $b(i,j) + f(i,j)$ if the orientations are opposite. Let M denote the minimum of the quantities calculated.

6. Adjust flows in the last column of the link table as follows: For each link in the chain found above, add or subtract M from the presently tabulated flow, according as the orientation of the link in this chain agrees with or differs from that given in the link table. If the adjusted flow in any link is now negative, change its sign and reverse the order of the two nodes in the table. (This is done merely to avoid the necessity of storing negative quantities in the table.)

7. Record the chain 000, i_{s-1}, i_{s-2}, ..., i_1, 999 found in this iteration of the procedure and also the associated flow M. The first and last nodes need not be recorded, since they are necessarily 000 and 999.

8. Restore the labels of all nodes to their original values, erase all approach links, and return to step 1.

After a finite number of chain flows has been found, a stage will be reached such that $V(999) = \infty$ in step 3. When this occurs, the link flows tabulated in the link table constitute a solution to the problem. The set of chain flows recorded in step 7 represents the decomposition of the solution into flows along specific links from origins to destinations.

CHAPTER IV

MODEL APPLICATION

General

The four models presented in Chapter III, Model Development, are programmed for operation on a digital computer. In the previous chapter the discussion was confined to the mathematical logic of the "inner loop" assignment process. This chapter presents the preparation of inputs, mechanics of operation, and interpretation of results for the models. In addition, some insight is obtained from the manipulation of the models and their results. The following paragraphs describe the results obtained from applications of the models to typical stadium situations.

Model I

The "New Assignment" Model is applicable to stadium seating situations in which applicants have not held seats or previous seat holdings are disregarded. Since this model was the first application of the basic assignment loop, extensive testing was conducted to perfect the mechanism. Situations were developed for conditions when supply exceeds demand and vice versa. Similarly, small and large seat configurations were used and relationships between seats were varied to insure the reliability of the assignment mechanism.

Model Mechanics

The computer model developed to simulate the assignment of applicants to seats in a new stadium is also equipped with methods for checking the accuracy of inputs. Requests that exceed specified maximums or row limitations are rejected. At the conclusion of the model's operation, assignments, unfilled requests, and unassigned seats are listed. Figure 12 illustrates the major inputs, functions, and outputs of Model I. The outputs are sufficiently detailed to permit manual adjustment of the results or, if necessary, revision of the inputs for subsequent reruns of the model.

The basic inputs to the model are applicant requests and seat records. Both applicant requests and seat records must be uniquely ranked and sorted in priority sequence. Requests out-of-priority-sequence are rejected along with those mentioned above. The application contains the priority (applicant number), applicant name, and number of seats requested. Each seat record contains the index (desirability indicator), real seat location, adjacent seat location, and occupancy indicator. All values are normally numerical although alphabetic symbols could be used for real seat locations if desired. The occupancy indicator is the only value changed during assignment. The indicator is zero if the seat is unoccupied, but once assigned the value is changed to one, indicating that the seat is now occupied.

Model Results

Two sets of results are discussed for Model I. The first set is based on an applicant list with the total number of seats requested less than the number available, i.e., demand less than supply. The second set is based on an applicant list with the total number of seats requested greater than the number available, i.e., demand exceeds supply. For both runs the seat configuration shown in Figure 13 was used. The desirability of each seat is given by the value in the square and the real location is defined by the corresponding row and seat coordinates. The applicant list given in Figure 14 provides the priority of and seats requested by each applicant. Applicants with asterisks were added to the

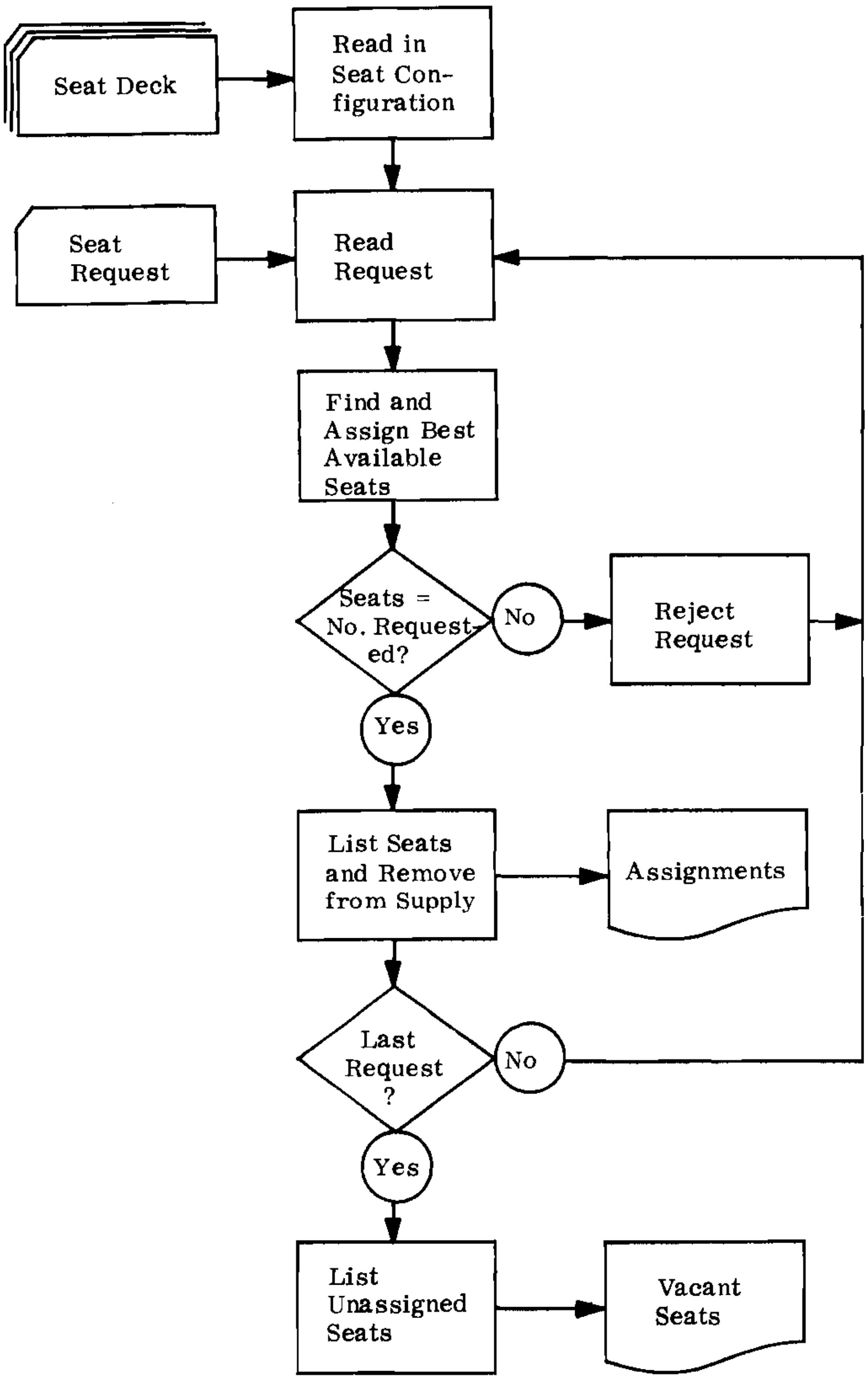


Figure 12. Model I Flow Chart

Section I

SEAT

ROW	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	13	14	15	16	25	26	27	28
2	9	10	11	12	17	18	19	20	29	30	31	32	45	46	47	48
3	21	22	23	24	33	34	35	36	49	50	51	52	69	70	71	72
4	37	38	39	40	53	54	55	56	73	74	75	76	97	98	99	100
5	57	58	59	60	77	78	79	80	101	102	103	104	129	130	131	132
6	81	82	83	84	105	106	107	108	133	134	135	136	161	162	163	164
7	109	110	111	112	137	138	139	140	165	166	167	168	193	194	195	196
8	141	142	143	144	169	170	171	172	197	198	199	200				
9	173	174	175	176	201	202	203	204								
10	205	206	207	208												

Seat Rank/

Figure 13. Large Seat Configuration

Section II

Seat Rank/

ROW \ SEAT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	41	42	43	44	61	62	63	64	85	86	87	88	113	114	115	116
2	65	66	67	68	89	90	91	92	117	118	119	120	145	146	147	148
3	93	94	95	96	121	122	123	124	149	150	151	152	177	178	179	180
4	125	126	127	128	153	154	155	156	181	182	183	184				
5	157	158	159	160	185	186	187	188								
6	189	190	191	192												

Figure 13 (concluded). Seat Configuration

Applicant Number	No. Requested
1	1
2	2
3	2
4	4
5	3
6	4
7	4
8	4
9	3
10	2
11	4
12	3
13	1
14	4
15	3
16	3
17	3
18	1
19	4
20	2
21	2
22	1
23	4
24	1
25	2
26	3
27	1
28	1

Applicant Number	No. Requested
29	6
30	4
*31	12
32	4
33	3
*34	16
35	9
36	2
37	4
38	2
39	10
40	6
*41	2
*42	2
*44	6
45	8
46	8
47	6
48	2
49	4
50	3
51	4
52	2
53	2
54	2
55	1
56	4

*Added to 2nd run

Figure 14. Model I Applicant List

list in the second application so that demand would exceed supply.

First Run. In the first application (supply > demand), 168 of 208 seats were assigned to 50 applicants, leaving 40 seats unassigned. Each applicant received the best available seats at the time of assignment in accordance with the number requested. Some good seats were left after the assignment, as shown in Figure 15, but an analysis of the results proved that a sufficient number of "adjacent" seats in a better location were not available when the assignment was made. It can also be seen that the number of seats requested can materially affect the results of an assignment. For example, Applicant 11 was assigned seats beginning with location 32 even though 26, 27, and 28 were available. Applicant 11 did not receive these seats because the number requested was four. Applicant 12 did receive the seats since the number requested was three -- exactly equal to the number available. It is therefore possible for an applicant to receive better seats than a predecessor if the number requested is smaller and the seats are available and adjacent.

Second Run. In the second application (supply < demand), 204 of 208 seats were assigned to 56 applicants, leaving four seats unassigned and a request of eight seats unfilled. Again, each applicant received the best available seats at the time of assignment in accordance with the number requested. Only four seats were left vacant after the assignment as shown in Figure 16, two single seats and one double grouping. Though more than eight seats were available in the stadium when Applicant 46 was processed (in priority order), eight "adjacent" seats were not available, and therefore, the request was not filled. If the priority of this request had been higher, in fact, one applicant higher, it would have been filled. It is notable that one request for seats equalled the length of a row, which is the maximum number that can be requested and avoid rejection. A request

Section I

SEAT

Seat Rank/Applicant Number

ROW	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1 / 1	2 / 2	3 / 2	4 / 3	5 / 3	6 / 4	7 / 4	8 / 4	13 / 4	14 / 7	15 / 7	16 / 7	25 / 7	26 / 12	27 / 12	28 / 12
2	9 / 5	10 / 5	11 / 5	12 / 6	17 / 6	18 / 6	19 / 6	20 / 8	29 / 8	30 / 8	31 / 8	32 / 11	45 / 11	46 / 11	47 / 11	48 / 22
3	21 / 9	22 / 9	23 / 9	24 / 10	33 / 10	34 / 13	35 / 14	36 / 14	49 / 14	50 / 14	51 / 19	52 / 19	69 / 19	70 / 19	71 / 36	72 / 36
4	37 / 15	38 / 15	39 / 15	40 / 16	53 / 16	54 / 16	55 / 20	56 / 20	73 / 30	74 / 30	75 / 30	76 / 30	97 / 48	98 / 48	99 / 52	100 / 52
5	57 / 21	58 / 21	59 / 23	60 / 23	77 / 23	78 / 23	79 / 32	80 / 32	101 / 32	102 / 32	103 / 40	104 / 40	129 / 40	130 / 40	131 / 40	132 / 40
6	81 / 33	82 / 33	83 / 33	84 / 35	105 / 35	106 / 35	107 / 35	108 / 35	133 / 35	134 / 35	135 / 35	136 / 35	161 / 54	162 / 54	163 / 55	164
7	109 / 45	110 / 45	111 / 45	112 / 45	137 / 45	138 / 45	139 / 45	140 / 45	165 / 56	166 / 56	167 / 56	168 / 56	193	194	195	196
8	141 / 49	142 / 49	143 / 49	144 / 49	169	170	171	172	197	198	199	200				
9	173	174	175	176	201	202	203	204								
10	205	206	207	208												

Figure 15. Model I Results (Supply > Demand)

Section II

ROW \ SEAT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	41/17	42/17	43/17	44/24	61/25	62/25	63/26	64/26	85/26	86/26	87/37	88/37	113/37	114/37	115/53	116/53
2	65/27	66/28	67/29	68/29	89/29	90/29	91/29	92/29	117/46	118/46	119/46	120/46	145/46	146/46	147/46	148/46
3	93/38	94/38	95/39	96/39	121/39	122/39	123/39	124/39	149/39	150/39	151/39	152/39	177/	178/	179/	180/
4	125/47	126/47	127/47	128/47	153/47	154/47	155/50	156/50	181/50	182/	183/	184/				
5	157/51	158/51	159/51	160/51	185/	186/	187/	188/								
6	189/	190/	191/	192/												

Seat Rank/Applicant Number

Figure 15 (concluded). Model I Results (Supply > Demand)

Section I

ROW \ SEAT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1/1	2/2	3/2	4/3	5/3	6/4	7/4	8/4	13/4	14/7	15/7	16/7	25/7	26/12	27/12	28/12
2	9/5	10/5	11/5	12/6	17/6	18/6	19/6	20/8	29/8	30/8	31/8	32/11	45/11	46/11	47/11	48/22
3	21/9	22/9	23/9	24/10	33/10	34/13	35/14	36/14	49/14	50/14	51/19	52/19	69/19	70/19	71/36	72/36
4	37/14	38/14	39/14	40/16	53/16	54/16	55/20	56/20	73/30	74/30	75/30	76/30	97/37	98/37	99/37	100/37
5	57/21	58/21	59/23	60/23	77/23	78/23	79/32	80/32	101/32	102/32	103/38	104/38	129/48	130/48	131/52	132/52
6	81/31	82/31	83/31	84/31	105/31	106/31	107/31	108/31	133/31	134/31	135/31	136/31	161/49	162/49	163/49	164/49
7	109/35	110/35	111/35	112/35	137/35	138/35	139/35	140/35	165/35	166/47	167/47	168/47	193/47	194/47	195/47	196/
8	141/42	142/42	143/42	144/42	169/42	170/42	171/42	172/42	197/51	198/51	199/51	200/51				
9	173/45	174/45	175/45	176/45	201/45	202/45	203/45	204/45								
10	205/56	206/56	207/56	208/56												

Seat Rank/Applicant Number

Figure 16. Model I Results (Supply < Demand)

Section I

SEAT

Seat Rank/Applicant Number

ROW \ SEAT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	41/17	42/17	43/17	44/17	61/24	62/25	63/25	64/26	85/26	86/26	87/33	88/33	113/33	114/41	115/41	116/55
2	65/27	66/28	67/29	68/29	89/29	90/29	91/29	92/29	117/40	118/40	119/40	120/40	145/40	146/40	147/53	148/53
3	93/34	94/34	95/34	96/34	121/34	122/34	123/34	124/34	149/34	150/34	151/34	152/34	177/34	178/34	179/34	180/34
4	125/39	126/39	127/39	128/39	153/39	154/39	155/39	156/39	181/39	182/39	183/54	184/54				
5	157/44	158/44	159/44	160/44	185/44	186/44	187/	188/								
6	189/50	190/50	191/50	192/												

Figure 16 (concluded). Model I Results (Supply < Demand) larger than the longest row available does not meet the condition that seats must be adjacent.

Model II

The "Reassignment" Model is applicable to stadium seating situations in which applicants have held seats previously and the location of these seats is taken into consideration in the assignment. If an applicant requests improvement in the location of his seats, better seats are assigned if available; if not, his original seats are reassigned. If an applicant that held seats from the previous assignment requests the same seats, they are reassigned. In both cases the model assures no worse seats than those previously held. To test Model II, results from Model I were used to provide a basis for reassignment.

Model Mechanics

The computer model developed to simulate the reassignment of applicants to seats is equipped with the same methods for checking the accuracy of inputs as Model I. Results of the model's operation include assignments, (reassignments and improvements), unfilled requests, and unassigned seats. Figure 17 illustrates the major inputs, functions, and outputs of Model II. The outputs are sufficiently detailed to permit manual adjustment of the results or, if necessary, revision of the inputs for subsequent reruns of the model.

The basic inputs to the model are applicant requests and seat records. The seat record is identical to the one described for Model I. Applicant requesters and seat records are sorted in priority order just as for Model I. Each request contains the priority (applicant number), applicant name, number of seats requested, improvement option indicator, and location of the first (best) seat held from the previous assignment. The improvement option indicator is

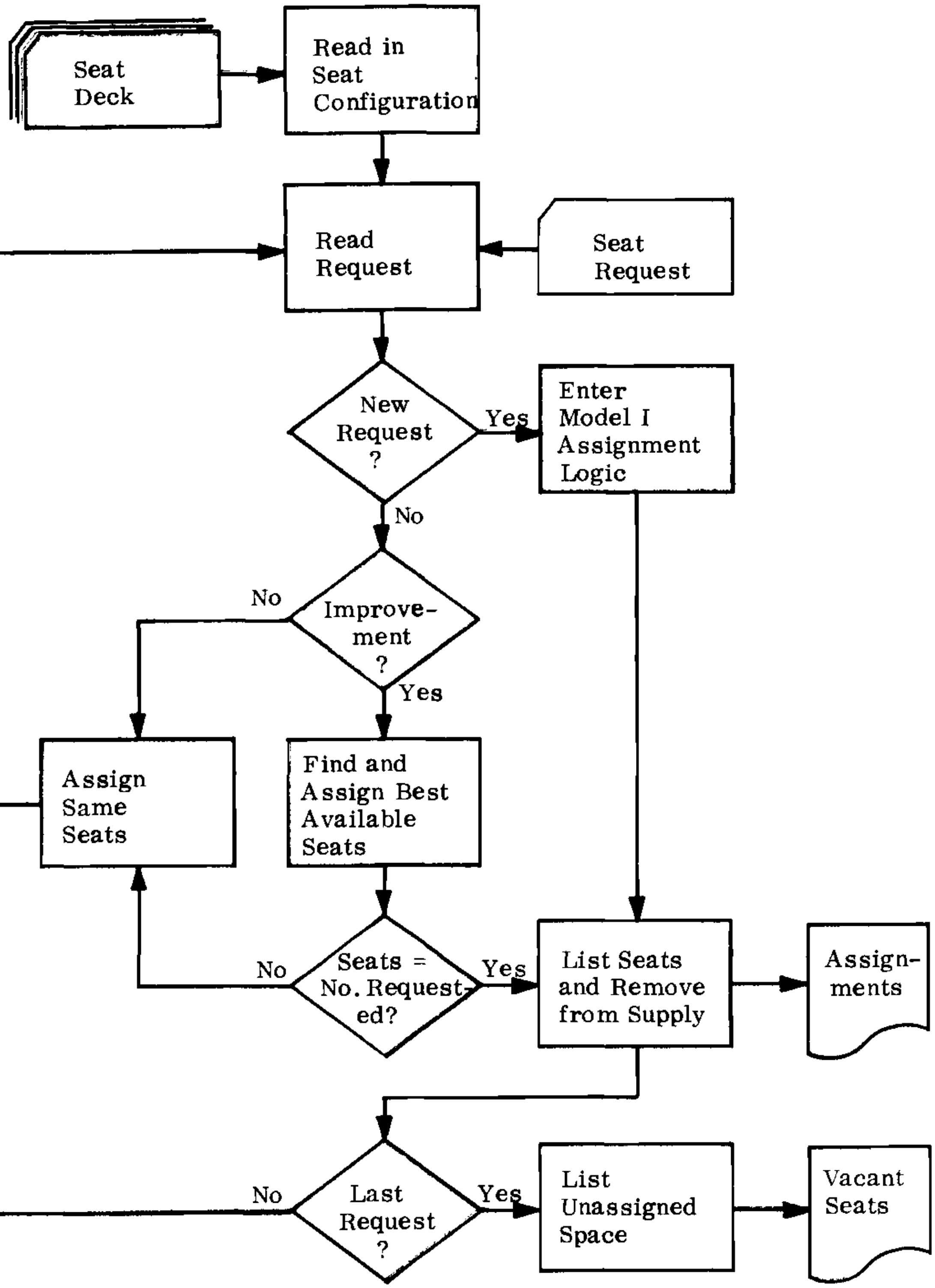


Figure 17. Model II Flow Chart

a numerical code by which the applicant requests improvement in or the return of previous seats. A "one" indicates improvement is desired while a "zero" indicates a request for the same seats. There is no option of improvement for new applicants since no basis for relocation to better seats exists. These applicants are processed in priority order and assigned the best seat available, as in Model I.

Reassignment is a two-pass process. In the first pass, applicants are tentatively assigned their previously held seats to avoid assigning them to other applicants inadvertantly. Applicants requesting the seats previously held are reassigned these locations permanently. In the second pass, the model scans the list of seats for each applicant requesting improvement to find the first unoccupied location. The location is compared with the first seat of the set held previously. If it is better and the adjacent seats are both available and sufficient in number to fill the request, the group of new seats is assigned, and the previously held seats are released for subsequent assignments. If it is not better, the applicant is assigned the seats previously held.

Model Results

The results discussed for Model II are based on the applicant list given in Figure 18. For each applicant the priority, number of seats requested, option of improvement choice, and first location (real seat) of previously held seats is given. It should be noted that some applicant numbers are missing from this list when compared to the list shown in Figure 14. Since the initial assignment produced by Model I was used as a basis for reassignment in Model II, certain applicant requests were deleted purposely to release seats for applicants requesting improvement. The seat configuration used in Model I was again used in Model II with the same ranking as shown in Figure 13.

Applicant Number	No. Requested	Improvement Option	Location Previous Seats
1	1	0	10101
2	2	0	10102
3	2	0	10104
4	4	0	10106
6	4	0	10204
7	4	1	10110
8	4	0	10208
9	3	1	10301
10	2	0	10304
12	3	1	10114
13	1	1	10306
14	4	0	10307
15	3	1	10401
17	2	0	20101
18	1	1	20104
19	4	0	10311
20	2	1	10407
21	2	0	10501
22	1	0	10216
23	4	0	10503
24	1	1	20105
25	2	1	20106

Applicant Number	No. Requested	Improvement Option	Location Previous Seats
28	1	0	20202
29	6	1	20203
30	4	1	10405
31	12	1	10601
32	4	1	10507
33	3	1	20111
34	16	1	20301
35	9	1	10701
36	2	1	10315
37	4	1	10413
38	2	1	10511
39	10	1	20401
40	6	1	20209
41	2	1	20114
42	8	1	10801
44	6	1	20105
45	8	1	10901
46	8	0	
47	6	1	10710
48	2	1	10513
49	4	1	10613
50	3	1	20601

Figure 18. Model II Applicant List

In this application supply was greater than demand and 183 of 208 seats were assigned to 44 applicants, leaving 25 seats unassigned. Applicant numbers 5, 11, 16, 26, 27, and 43 were omitted from the original list of 50 in the first application of Model I, which released 14 seats for assignment to the remaining applicants. As a result, 27 of the 30 applicants requesting improvement were assigned better seats than those previously held. Figure 19 shows the results of reassignment and improvement based on the option chosen by each applicant. Since a number of seats were released from the original assignment in Model I, it was possible to provide improvements in previous seat locations for a high percentage of the applicants. If no seats had been released, very little improvement would have been possible.

Model III

The "Selective Assignment" Model is applicable to stadium seating situations in which applicants are allowed to request specific seats. Previous seat locations are reassigned if the selected seat cannot be assigned and no better seats are available. As in Model II, if improvement is requested but cannot be provided, the applicant is reassigned his previous seats. To test Model III, results from Model I were again used to provide a basis for selective assignment and reassignment. Of course, new applicants could request specific seats but not improvement since no seats were held previously and there is no basis for improvement.

Model Mechanics

The computer model developed to simulate the selective assignment of applicants to seats is equipped with the same methods for checking the accuracy of inputs as Models I and II. Results of the model's operation include reassignments and specific assignments, unfilled requests, and unassigned seats. Figure

Section I

SEAT

ROW	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1/1	2/2	3/2	4/3	5/3	6/4	7/4	8/4	13/4	14/7	15/7	16/7	25/7	26/13	27/18	28/24
2	9/9	10/9	11/9	12/6	17/6	18/6	19/6	20/8	29/8	30/8	31/8	32/15	45/15	46/15	47/	48/22
3	21/12	22/12	23/12	24/10	33/10	34/	35/14	36/14	49/14	50/14	51/19	52/19	69/19	70/19	71/36	72/36
4	37/20	38/20	39/25	40/25	53/30	54/30	55/30	56/30	73/32	74/32	75/32	76/32	97/38	98/38	99/41	100/41
5	57/21	58/21	59/23	60/23	77/23	78/23	79/33	80/33	101/33	102/40	103/40	104/40	129/40	130/40	131/40	132/
6	81/34	82/34	83/34	84/34	105/34	106/34	107/34	108/34	133/34	134/34	135/34	136/34	161/34	162/34	163/34	164/34
7	109/39	110/39	111/39	112/39	137/39	138/39	139/39	140/39	165/39	166/39	167/49	168/49	193/49	194/49	195/	196/
8	141/44	142/44	143/44	144/44	169/44	170/44	171/50	172/50	197/50	198/	199/	200/				
9	173/46	174/46	175/46	176/46	201/46	202/46	203/46	204/46								
10	205/	206/	207/	208/												

Seat Rank/Applicant Number

Figure 19. Model II Results

Section II

ROW \ SEAT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	41/17	42/17	43/17	44/29	61/29	62/29	63/29	64/29	85/29	86/37	87/37	88/37	113/37	114/48	115/48	116/
2	65/	66/28	67/31	68/31	89/31	90/31	91/31	92/31	117/31	118/31	119/31	120/31	145/31	146/31	147/	148/
3	93/35	94/35	95/35	96/35	121/35	122/35	123/35	124/35	149/47	150/47	151/47	152/47	177/47	178/47	179/47	180/
4	125/42	126/42	127/42	128/42	153/42	154/42	155/42	156/42	181/	182/	183/	184/				
5	157/45	158/45	159/45	160/45	185/45	186/45	187/45	188/45								
6	189/	190/	191/	192/												

Seat Rank/Applicant Number

Figure 19 (concluded). Model II Results

20 illustrates the major inputs, functions, and outputs of Model III. The outputs are sufficiently detailed to permit manual adjustment of the results or, if necessary, revision of the inputs for subsequent reruns of the model.

The basic inputs to the model are applicant requests and seat records. The seat record is identical to the one described for Model I. Applicant requests and seat records are sorted in priority order just as for Models I and II. Each request contains the priority (applicant number), applicant name, number of seats requested, specific seat option indicator, location of first seat held from the previous assignment, and location of first of specific seats desired (if applicable). The specific seat option indicator is a numerical code (like the improvement option indicator in Model II) by which the applicant requests assignment of specific seats, best available seats, or the return of previous seats. A "one" indicates a request for specific seats (with the location of these seats included) while a "zero" indicates improvement is desired. New applicants can request specific seats by specifying the desired location, but since there is no basis for improvement this option cannot be taken. If a new applicant does not choose specific seats, the best available seats are assigned on the basis of priority, as in Models I and II.

Selective assignment is a two-pass process in which applicants that previously held seats are temporarily assigned their original seats. In the second pass, each applicant is examined in turn and assigned permanently specific seats, improved seats, or original seats. When applicants that held seats from a previous assignment are assigned to other seats, their original seats are released for subsequent applicants. Again, as in Model II, applicants are guaranteed no worse seats than those held previously, unless specific seats are requested and assigned - in which case the model does not consider the quality difference

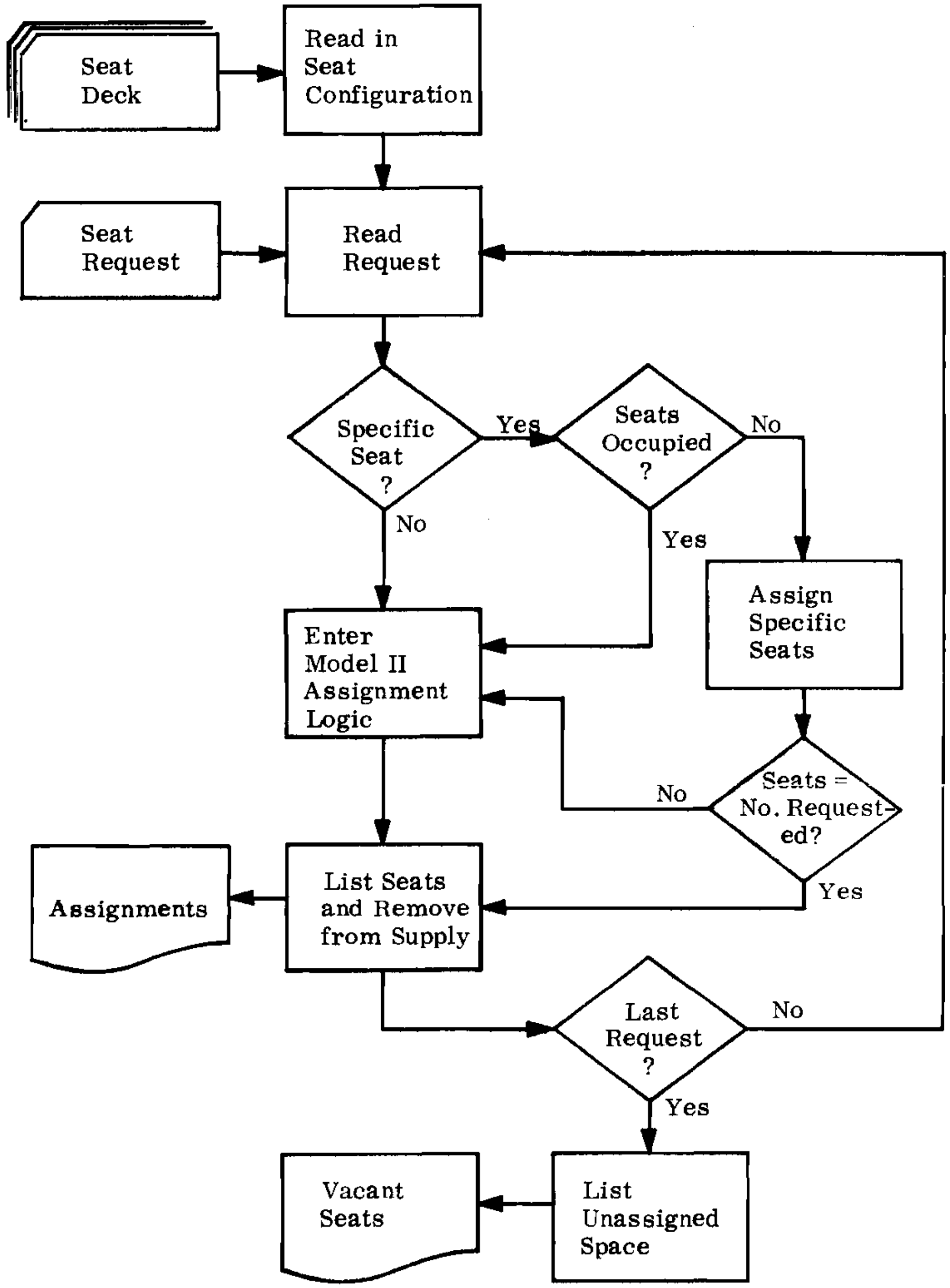


Figure 20. Model III Flow Chart

between the two sets of seats.

Model Results

The results discussed for Model II are based on the applicant list given in Figure 21. For each applicant the priority, number of seats requested, specific seat option indicator, first location (real seat) of previously held seats, and first location (real seat) of specific seats is given. It should be noted that the results obtained from the application of Model I were used as a basis for preparing the applicant list for this model run. Certain applicants were given the specific seat choice option while the remainder requested improvement. Again, some applicant numbers are missing from the list in order to release seats and test the specific seat assignment mechanism. The seat configuration used for Models I and II was used in the Model III application.

The demand for seats in this application was set exactly equal to supply, i.e., 54 requests for 208 seats equaled the 208 available. Seven requests for specific seats were included. The remaining applicant requests included choices of improvement, return of previous seats, and assignment of the best available seats (new requests). Requests for specific seats which were already occupied were included to test the occupancy check mechanism and the seat location mechanism. The model did not assign specific seats held previously by another applicant unless the seats were released as a result of a preceding assignment. Other applicants were assigned in essentially the same manner as in Model II. A high percentage of improvement was achieved among the applicants requesting that option. The results of selective assignments based on specific seat choices and improvement options taken are shown in Figure 22.

Model IV

The "Optimum Allocation" Model is applicable to situations in which the

Applicant Number	Number Re-quested	Specific Seat Option	Location Previous Seat	Specific Seat Desired
1	1		010101	
2	2		010102	
3	2		010104	
4	4		010106	010201
5	3			
6	4		010204	
7	4	1	010110	
8	4		010208	
9	3	1	010301	
10	2		010304	
12	3	1	010114	
13	1	1	010306	010401
14	4		010307	
15	3	1	010401	
16	3			
17	3		020101	
18	1	1	020104	
19	4		010311	
20	2	1	010407	
21	2		010501	
22	1		010206	
23	4		010503	
24	1	1	020105	
25	2	1	020106	
26	3			010502
27	1			
28	1		020202	

Applicant Number	Number Re-quested	Specific Seat Option	Location Previous Seat	Specific Seat Desired
29	6	1	020203	
30	4	1	010409	
31	12			
32	4			
33	3			
34	16			
35	9			
36	2			
37	4			020201
38	2			
39	10			
40	6			
41	2			
42	8			
44	6			
45	8			
46	8			
47	6			
48	2			
49	4			
50	3			
51	4			
52	2			010701
53	2			010804
54	2			
55	1			
56	4			010610

Figure 21. Model III Applicant List

Section I

SEAT

ROW	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1 / 1	2 / 2	3 / 2	4 / 3	5 / 3	6 / 4	7 / 4	8 / 4	13 / 4	14 / 7	15 / 7	16 / 7	25 / 7	26 / 12	27 / 12	28 / 12
2	9 / 5	10 / 5	11 / 5	12 / 6	17 / 6	18 / 6	19 / 6	20 / 8	29 / 8	30 / 8	31 / 8	32 / 13	45 / 20	46 / 20	47 / 55	48 / 22
3	21 / 9	22 / 9	23 / 9	24 / 10	33 / 10	34 / 18	35 / 14	36 / 14	49 / 14	50 / 14	51 / 19	52 / 19	69 / 19	70 / 19	71 / 36	72 / 36
4	37 / 15	38 / 15	39 / 15	40 / 16	53 / 16	54 / 16	55 / 25	56 / 25	73 / 32	74 / 32	75 / 32	76 / 32	97 / 37	98 / 37	99 / 37	100 / 37
5	57 / 21	58 / 21	59 / 23	60 / 23	77 / 23	78 / 23	79 / 33	80 / 33	101 / 33	102 / 38	103 / 38	104 / 41	129 / 41	130 / 50	131 / 50	132 / 50
6	81 / 31	82 / 31	83 / 31	84 / 31	105 / 31	106 / 31	107 / 31	108 / 31	133 / 31	134 / 31	135 / 31	136 / 31	161 / 49	162 / 49	163 / 49	164 / 49
7	109 / 39	110 / 39	111 / 39	112 / 39	137 / 39	138 / 39	139 / 39	140 / 39	165 / 39	166 / 39	167 / 51	168 / 51	193 / 51	194 / 51	195	196
8	141 / 44	142 / 44	143 / 44	144 / 44	169 / 44	170 / 44	171 / 52	172 / 52	197 / 56	198 / 56	199 / 56	200 / 56				
9	173 / 46	174 / 46	175 / 46	176 / 46	201 / 46	202 / 46	203 / 46	204 / 46								
10	205	206	207	208												

Seat Rank/Applicant Number

Figure 22. Model III Results

Section II

SEAT

ROW	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	41/17	42/17	43/17	44/24	61/26	62/26	63/26	64/29	85/29	86/29	87/29	88/29	113/29	114/48	115/48	116/
2	65/	66/28	67/30	68/30	89/30	90/30	91/35	92/35	117/35	118/35	119/35	120/35	145/35	146/35	147/35	148/
3	93/34	94/34	95/34	96/34	121/34	122/34	123/34	124/34	149/34	150/34	151/34	152/34	177/34	178/34	179/34	180/34
4	125/40	126/40	127/40	128/40	153/40	154/40	155/47	156/47	181/47	182/47	183/47	184/47				
5	157/45	158/45	159/45	160/45	185/45	186/45	187/45	188/45								
6	189/53	190/53	191/54	192/54												

Seat Rank/Applicant Number

Figure 22 (concluded). Model III Results effectiveness of assignment is of interest and the solution is to represent an optimum according to some measure. Each applicant is considered based on the cost of assignment to each section as compared with the cost of assignment of every other applicant to each section. The technique is based on graph theoretic methods applied to the analysis of networks using a minimum cost/ maximum flow criterion. Application of the model to a stadium situation proved that a minimum cost solution is produced when finite costs are associated with the potential assignment of each applicant to each section of seats.

Model Mechanics

The computer model used to allocate applicants to sections of seats is based on the logic of the "labeling algorithm" described in network flow theory. Essentially, the model receives the input of requests (which represent demands placed on the network) and sections of seats with specified capacities (which represent supply sources to draw form). The network is constructed from additional inputs which specify the relationship between each request and each section. Finite costs are included with each link connecting a request with a section. There can be no duplication of links since such ambiguity would produce an infeasible solution. All sections terminate at an artificial node called "source" and all requests terminate at an artificial node called "sink." The number of seats in each request represents a requirement for flow, while the number of seats in each section represents an upper-bound capacity. With this information the model progressively seeks a set of chain flows from source to sink which minimizes total cost.

The basic inputs to the model are requests, sections, and the connecting members - all of which are expressed as "links" in the network. For each link, a from node, to node, cost, and capacity must be specified. Outputs include the final state of the network and all the chain flows. Figure 23 illustrates the inputs, functions, and outputs of Model IV.

Model Results

The results discussed for Model IV are based on the network inputs, shown in Figure 24, including requests, sections, and connecting links (with costs). Flows in connecting links were made arbitrarily large to prevent restriction in allocation. To limit the number of seats assigned to any applicant in a given section, an upper-bound flow restriction must be placed on the connecting link. To force the allocation of seats in a given section to an applicant, a lower-bound flow must be specified for the connecting link. Such alternatives, may produce an infeasible solution or increase the minimum cost solution.

The seat configuration used in this application is represented in Figure 25 and corresponds to the "sink" links (j, 200) described with the inputs in Figure 24. There is no significance to the seat numbers in the sections since seats are grouped and handled as capacities. If the user wished to consider seats individually the relationship of each request to each seat would be necessary, but such an alternative would magnify the size of the network and input requirements substantially.

Based on the inputs described above, a network of links was constructed. The result is shown graphically in Figure 26. Note the complexity of the representation even when a small number of requests and sections is considered. Out of 45 possible choices of flow (applicants to sections), 13 chains were necessary to satisfy the requests and produce a minimum cost solution of four units. As the results in Figure 27 show, only one request was allocated entirely to the same section, i.e., Chain 5, Flow 10, Applicant Request 1 in Section 15. All the requests were filled and every seat was taken with a resultant flow of supply equal to demand.

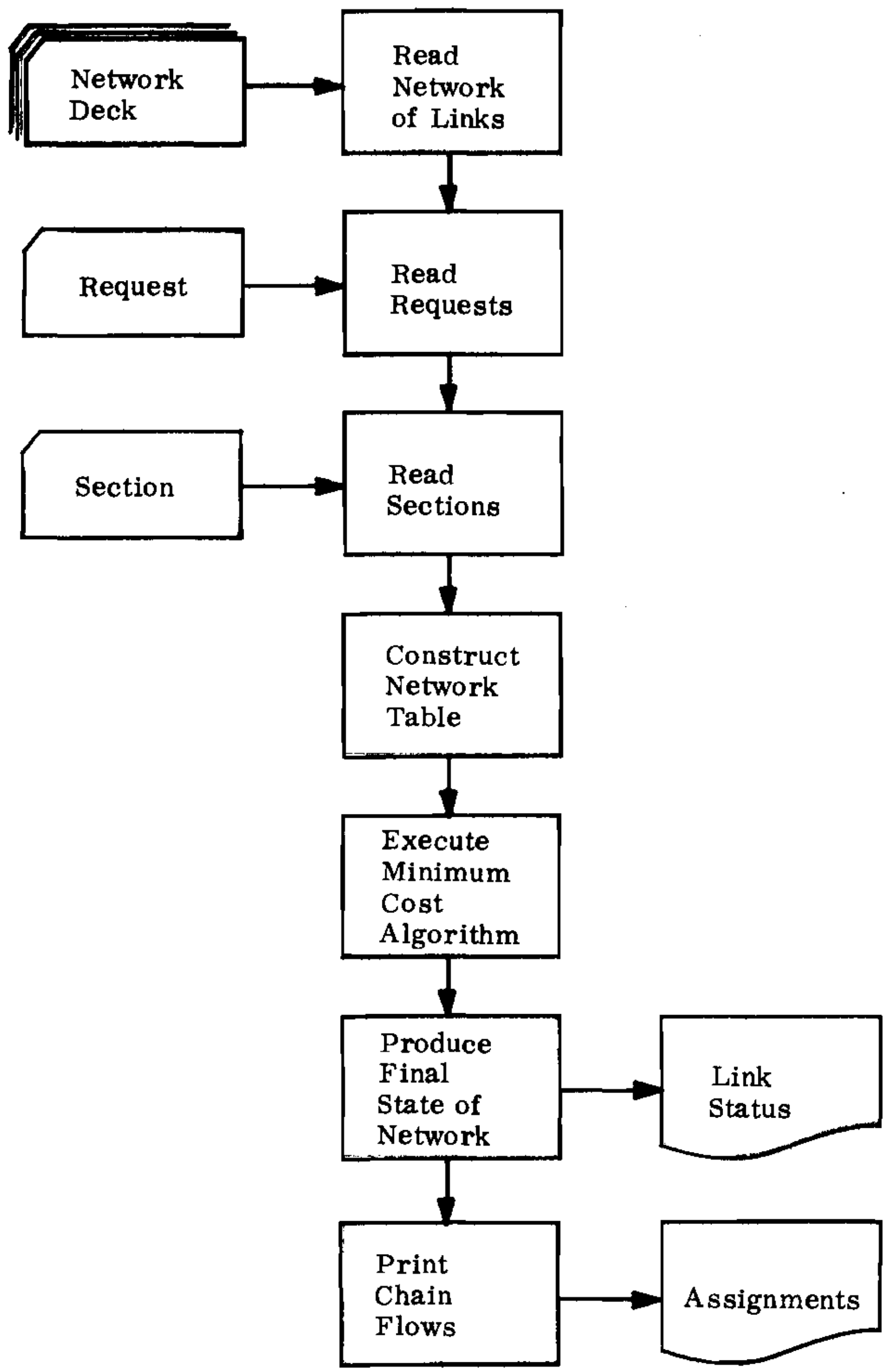


Figure 23. Model IV Flow Chart

From	To	Cost	Capacity
1	11	3	50000
1	12	4	50000
1	13	5	50000
1	14	2	50000
1	15	0	50000
1	16	2	50000
1	17	2	50000
1	18	2	50000
1	19	3	50000
2	11	3	50000
2	12	4	50000
2	13	5	50000
2	14	0	50000
2	15	1	50000
2	16	0	50000
2	17	2	50000
2	18	2	50000
2	19	3	50000
3	11	3	50000
3	12	4	50000
3	13	5	50000
3	14	1	50000
3	15	2	50000
3	16	0	50000
3	17	0	50000
3	18	0	50000
3	19	2	50000
4	11	0	50000
4	12	0	50000
4	13	1	50000
4	14	2	50000
4	15	3	50000
4	16	2	50000
4	17	1	50000
4	18	1	50000
4	19	0	50000
5	11	1	50000
5	12	0	50000
5	13	0	50000
5	14	2	50000
5	15	3	50000
5	16	2	50000
5	17	2	50000
5	18	2	50000
5	19	1	50000
1	100	0	10
2	100	0	15
3	100	0	20
4	100	0	15
5	100	0	10
200	11	0	6
200	12	0	8
200	13	0	6
200	14	0	9
200	15	0	12
200	16	0	9
200	17	0	6
200	18	0	8
200	19	0	6

Figure 24. Model IV Network Inputs

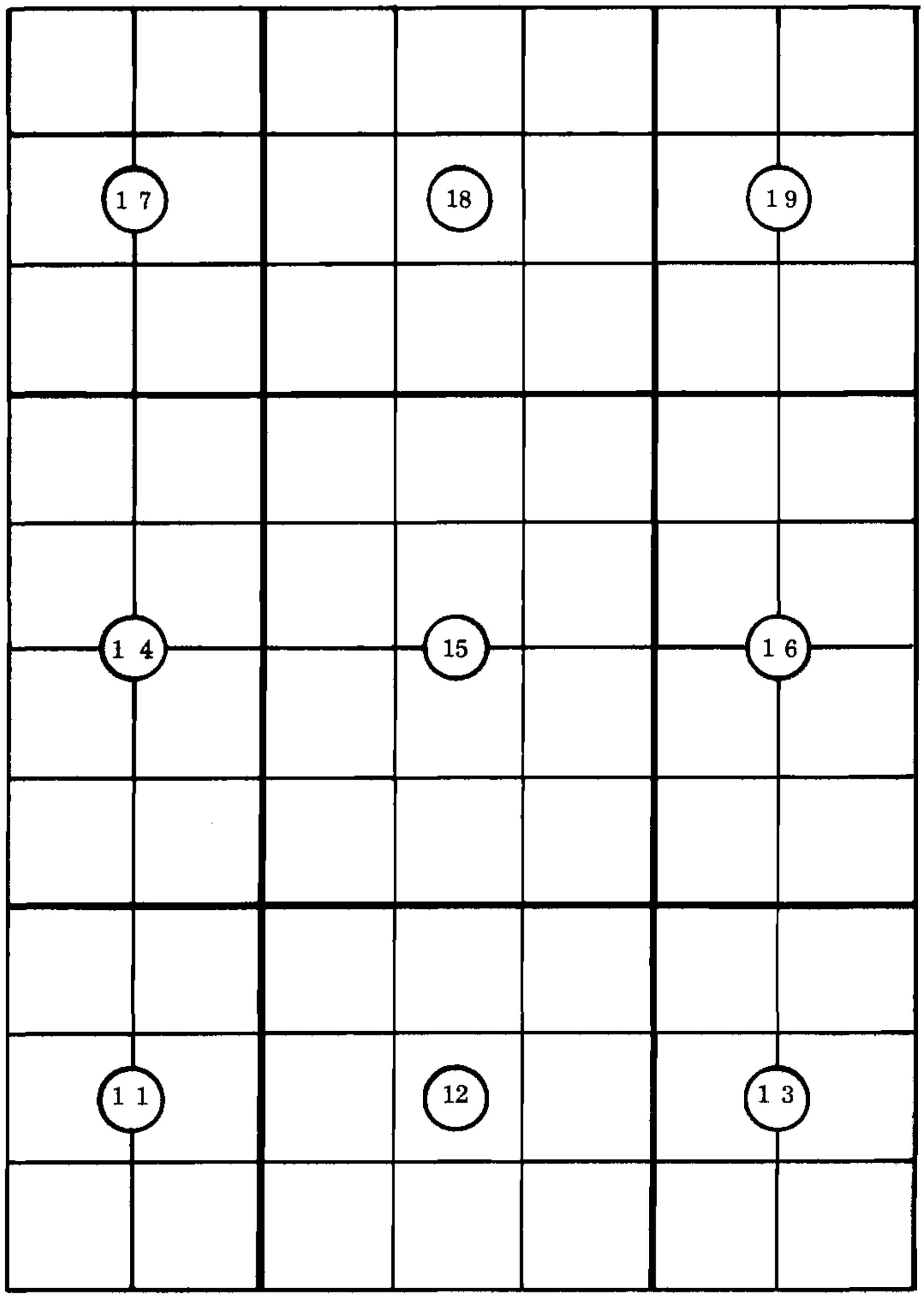


Figure 25. Model IV Section Configuration

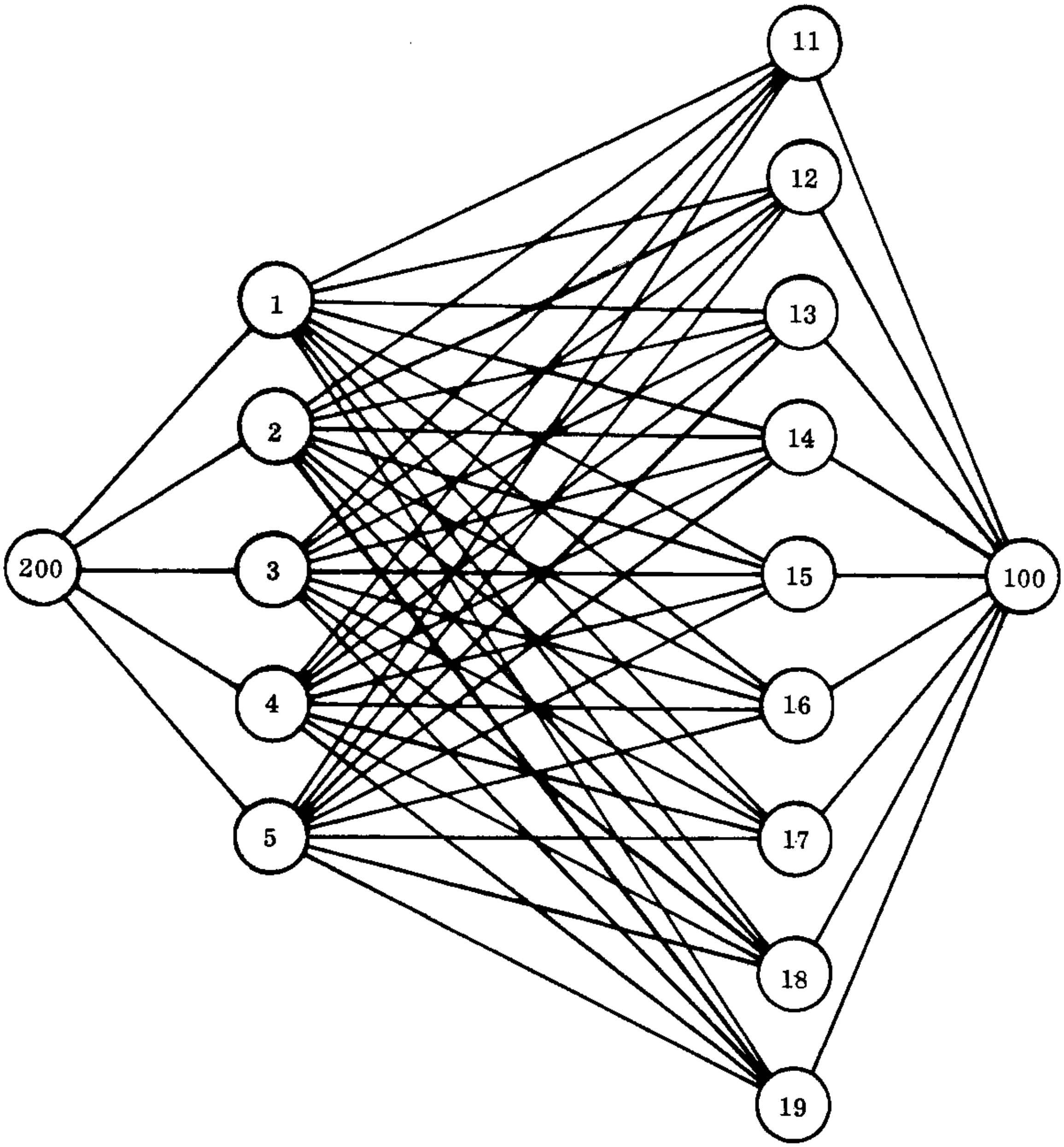


Figure 26. Network Construction

Chain	Flow (No. of Seats)	Section Number	Request Number	Cost of Assignment
1	6	11	4	0
2	8	12	4	0
3	6	13	5	0
4	9	14	2	0
5	10	15	1	0
6	6	16	2	0
7	3	16	3	0
8	6	17	3	0
9	8	18	3	0
10	1	19	4	0
11	4	12	5	0
12	2	16	3	1
	2	15	2	1
13	1	19	3	2
Total Cost of Assignment				4

Figure 27. Model IV Results

CHAPTER V

CONCLUSIONS

Stadium Seating

The dissimilarity of stadium seating situations make it necessary to develop specific models for specific situations rather than a general purpose model good for all applicants. Once the basic conditions concerning applicant status, seat quality, and assignment alternatives are established, the method of allocation can be developed. It is logical also to begin with the simple case and then progress to complex cases. The simplest case is the new stadium presented in Model I; a complex case is the historical reassignment case presented in Model II; a refinement of Model II is the selection of specific locations presented in Model III; and finally the effectiveness in the allocation of applicants to sections of seats on a least cost basis is presented in Model IV.

Utility of Models

Advantages

The development of the assignment technique and its refinement was the primary benefit of the research in this thesis. Once the basic assignment technique was developed, the alternatives presented in Models II and III were developed and programmed for the computer. In application the models can be used regardless of the stadium configuration. The assignment process described in Models I, II, and II selects the best available seats according to prescribed rules. If the effectiveness of assignment is important and applicants are restricted to groups of seats, Model IV provides an excellent vehicle for obtaining an optimum allocation at minimum cost. The assignment technique presented in the first three models is simple enough to permit refinements for more advanced applications.

Limitations

The technique designed for seat assignments in this thesis considers only the first seat of each set in determining the best location. It might be more desirable to consider the quality of an entire set in comparing one location to another. The models have no mechanism for choosing between two equally desirable seats, in fact, it was assumed that all seats were ranked uniquely. In a stadium assignment problem where it was necessary to examine seats on both sides of an arbitrary line simultaneously (say the 50-yard line), a revision of the assignment method would be necessary. This type of revision would probably not be difficult once certain basic rules were established. Similarly, the assignment of seats between the 40-yard lines could be incorporated in the existing assignment process. Such refinements can be made just as reassignment and selective assignment in Models II and III were made.

Need for Further Research

Investigation of the stadium seating problem beyond the developments discussed in this thesis would be desirable. More advanced models could be developed which consider assignment in general versus specific locations, groupings of seats rather than the first seat of the set, and optimum allocation to individual seats versus sections of seats. Both the establishment of priorities for applicants and the selection of desirability criteria for ranking seats could be studied. Solution of these problems would provide an excellent complement to the solution developed for seat assignment in this thesis.

BIBLIOGRAPHY

Literature Cited

1. Beckman, M.J., and T.C. Koopmans, "On Some Assignment Problems," Abstract, Operations Research, May 1953, Vol. 1, No. 3, pp. 148-149.

2. Beckwith, Richard E., and Ram Vaswani, "The Assignment Problem - A Special Case of Linear Programming," Journal of Industrial Engineering, Vol. VIII, No. 3, May-June, 1957, pp. 167-172.

3. Bindschelder, Andre E., and James M. Moore, "Optimal Location of New Machines in Existing Plant Layouts," Journal of Industrial Engineering, Vol. XII, No. 1, January-February, 1961, pp. 41-48.

4. Busacker, R.G., and P.J. Gowen, A Procedure for Determining a Family of Minimum-Cost Network Flow Patterns, Operations Research Office, ORO-TP-15, Nov., 1960.

5. Busacker, R.G., S.A. Coffin, and P.J. Gowen, Three General Network Flow Problems and Their Solutions, Research Analysis Corporation, RAC-SP-183, Nov., 1962.

6. Churchman, C. West, L. Ackoff Russell, and E. Leonard Arnoff, Introduction to Operations Research, John Wiley and Sons, New York, 1957, pp. 343-368.

7. Conway, R.W., and W.L. Maxwell, "A Note on the Assignment of Facility Location," Journal of Industrial Engineering, Vol. XII, No. 1, January-February, 1961, pp. 34-36.

8. Dantzig, George B., "Recent Advances in Linear Programming," Management Science, Vol. 2, No. 2, January, 1956, pp. 131-144.

9. Flagle, Charles D., William H. Huggins, and Robert H. Roy, Operations Research and Systems Engineering, The Johns Hopkins Press, Baltimore, 1960, pp. 391-399.

10. Ford, L.R.,Jr., and D.R. Fulkerson, A Simple Algorithm for Finding Maximal Network Flows and an Application to the Hitchcock Problem, The Rand Corporation, RM-1604, Dec., 1955.

11. Friedman, Lawrence, and Arthur J. Yaspan, "An Analysis of Stewardess Requirements and Scheduling for a Major Domestic Airline: Problem Technique," Naval Research Logistics Quarterly, Sept., 1957, 4:3:193-197.

Literature Cited (continued)

12. Gass, Saul I., "On the Distribution of Manhours to Meet Scheduled Requirements," Naval Research Logistics Quarterly, March, 1957, Vol. 4, No. 1, pp. 17-25.

13. Gass, Saul I., Linear Programming Methods and Applications, McGraw-Hill Co., New York, 1958, pp. 152-156, p. 185.

14. Jacobs, Walter, "Military Applications of Linear Programming," Second Symposium in Linear Programming, Vol. 1, H.A. Antosiewicz, Editor, Hq., U.S. Air Force, and National Bureau of Standards, Washington, D.C., 1955, pp. 1-27.

15. Koopmans, Tualling C., Activity Analysis of Production and Allocation, Proceedings of a Conference, John Wiley and Sons, Inc., New York, 1951.

16. Kuhn, H.W., "The Hungarian Method for the Assignment Problem," Naval Research Logistics Quarterly, Vol. 2, No. 1 and 2, March-June, 1955, pp. 83-98.

17. Kuhn, H.W., "Variants of the Hungarian Method for Assignment Problems," Naval Research Logistics Quarterly, Dec., 1956, 3:3:253-258.

18. McCloskey, Joseph F., and J.M. Coopinger, Operations Research for Management, Vol. II, The Johns Hopkins Press, Baltimore, 1956, pp. 340-344.

19. Miller, David W., and Martin K. Starr, Executive Decisions and Operations Research, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1960, pp. 280-290.

20. Moore, James M., "Optimal Locations for Multiple Machines," Journal of Industrial Engineering, Vol. XII, No. 5, September-October, 1961, pp. 307-313.

21. Orden, A. and D.F. Votaw, "Personnel Assignment Problem," Project SCOOP, Symposium on Linear Inequalities and Programming, Planning Research Division, Director of Management and Analysis Service, Comptroller, Hq., U.S. Air Force, and National Bureau of Standards, Washington, D.C., June 14-16, 1951, pp. 155-163.

22. Report of a NATO Conference, Operational Research in Practice, Pergamon Press, New York, 1958, pp. 94-103.

23. Sasieni, M., A. Yaspan, and L. Friedman, Operations Research: Methods and Problems, John Wiley and Sons, Inc., New York, 1959, pp. 183-194.

Literature-Cited (concluded)

24. Votaw, D. F., Jr., "Programming Under Conditions of Uncertainty," Second Symposium in Linear Programming, Vol. 1, H.A. Antosiewicz, Editor, Hq., U.S. Air Force, and National Bureau of Standards, Washington, D.C., 1955, pp. 1-27.

25. Votaw, D. F., "Solution of the Quota Problem by a Successive-Reduction Method," Operations Research, January-February 1958, 6:1:56-64.

Other References

Operations Research

1. Batchelor, J. M., Operations Research - An Annotated Bibliography, St. Louis University Press, St. Louis, Mo., Vol. 1 (1959), Vol. 2 (1962), and Vol. 3 (1963).

2. Case Institute of Technology, A Comprehensive Bibliography on Operations Research, John Wiley and Sons, New York, 1958.

Linear Programming

1. Rohde, F. V., "Bibliography on Linear Programming," Operations Research, Vol. 5, No. 1, Feb., 1957, pp. 45-62.

2. Wagner, H. M., "A supplemental Bibliography on Linear Programming," Operations Research, Vol. 5, No. 4, Aug., 1957, pp. 555-562.

3. Riley, Vera and S. I. Gass, Linear Programming and Associated Techniques, A Comprehensive Bibliography, The Johns Hopkins University, Operations Research Office, Chevy Chase, Md., May, 1958.