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THE DESIGN OF A PRESSURE CONTROL

SYSTEM FOR A COMPRESSIBLE

FLOW TEST APPARATUS

A THESIS

Presented to

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THE DESIGN OF A PRESSURE CONTROL

SYSTEM FOR A COMPRESSIBLE

FLOW TEST APPARATUS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

This thesis is concerned with the design of a test apparatus for the study of rarified flows. The function of the apparatus is to provide regulated air pressures upstream and downstream of a test section. The test section is assumed to behave pneumatically as a variable area nozzle. Only the analysis portion of the design is considered here.

The apparatus consist of two control valves, two tanks, a test section, a vacuum pump, and a two-loop feedback control system. Atmospheric air is used as the supply since the pressure range of interest is below atmospheric.

The analysis of the system is divided into two parts. The steady-flow analysis portion develops the relations necessary to determine the valve and vacuum pump sizes. The second portion of the analysis deals with controlling the pressures automatically with a feedback control system.

Although much of the analysis is specifically for the application at hand, some generality concerned with the controlling of pressures in a series of volumes and restrictions is achieved.

NOMENCLATURE

A,a	area, square inches; dimensionless area ratio
A_e	nozzle exit area, square inches
A*	nozzle throat area, square inches
A_m	valve orifice area, square inches
C_1, C_2, C_3	constants used to calculate mass flow rate
C_d	coefficient of discharge
c	speed of sound, ft/sec
E	steady-state error
e	constant, 2.71828 ...
$F(P_o/P_1)$	function of pressure ratio used to calculate mass flow rate
f	area change factor
G	dimensionless feedback gain
S	maximum angular speed, RPM
$H(\xi)$	function of ξ used in calculations
K	feedback gain
K_n	Knudsen Number, dimensionless
k	specific heat ratio, dimensionless
M	Mach Number, dimensionless
m	mass, pounds-mass
N	number of valve stem turns to shutoff, revolutions
P,p	pressure, mmHg; dimensionless pressure ratio
Q	volume flow rate, ft^3/min
R	gas constant, 19.16 mmHg-ft^3/lbm-^oR

S	maximum angular speed, RPM
T	temperature, degrees Rankine
T_c	time constant, seconds
t	time, seconds
U, u	reference pressure input, mmHg
V	volume, cubic feet
v	velocity, ft/sec
w	mass flow rate, lbm/sec
x	characteristic dimension of flow channel, inches
Z	dimensionless volume ratio
α	intermediate calculation constant
β	intermediate calculation constant
γ	area coefficient
λ	mean free path, inches
ρ	density, lbm/ft^3
τ	dimensionless time ratio
ω_n	natural frequency, radians/sec
ζ	damping coefficient, dimensionless

CHAPTER I

INTRODUCTION

This thesis was motivated by a project to design and build a test apparatus to study the flow of air through models of fluidic devices at high Mach Number and low Reynolds Number. In order to achieve a reasonable size for the models, low pressures are required. This necessitates a vacuum pump in the system.

The need to provide a means of controlling the pressure in the tanks upstream and downstream of the test section gives rise to the design utilizing the upstream and downstream control valves in Figure I. The problem was then generalized to designing a test apparatus with the configuration of Figure 1 but, instead of restricting the design to fluidic models in the test section, any restriction characterized by a cross section area and Mach Number is considered.

An early consideration is the determination of the size of the upstream valve, the downstream valve, and vacuum pump that is necessary to produce the desired steady-flow conditions. In Chapter III, this analysis is made first for the general case and then for fluidic models.

A second consideration is that of designing an automatic feedback control system for properly manipulating the valves. It has been found that maintaining constant pressure in the tank by manual regulation of the valve is difficult. Ordinary differential pressure regulators which depend on the fluid itself for energy are

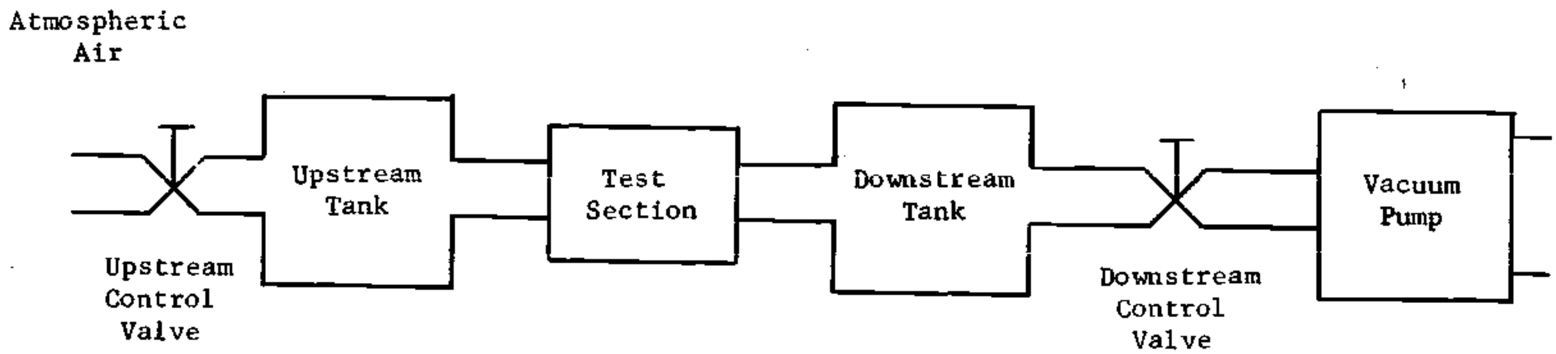

Figure 1. Diagram of Test Apparatus

not feasible because of the low pressures involved. Therefore some type of feedback control with power amplification is required for providing the regulation. Two such systems were analyzed: one using proportional control with saturation and the other using an integral control with saturation.

The transient response was analyzed to determine the choice of parameters that gives the best results. In this respect the upstream and downstream tanks are considered part of the control system since their volumes affect the response of the system. The automatic control system analysis, like the steady-flow analysis, is performed first in general and then for the fluidic models. This analysis is in Chapter IV.

Chapter II develops the equations that are used to describe the flow through valves, nozzles, and volumes. The equations are based on the one-dimensional, isentropic flow of a perfect gas. The constants are evaluated for air at standard temperatures and pressures.

CHAPTER II

COMPRESSIBLE FLOW THROUGH VOLUMES AND RESTRICTIONS

This chapter presents the equations used to describe the flow of air through the various components of the apparatus. The flow through nozzles, orifices, and valves is discussed as well as the flow of a compressible fluid through volumes.

Units and Constants

A consistent set of units is utilized in numerical examples. Millimeters of mercury are used as the units for pressure since these units are common for vacuum systems. The other units are common in general engineering usage. A summary of these units is given in Table 1.

Table 1. Units and Abbreviations

Quantity	Units	Abbreviation
mass	pounds mass	lbm
density	pounds mass per cubic foot	lbm/ft^3
pressure	millimeters of mercury	mmHg
area	square inches	in^2
mass flow rate	pounds mass per second	lbm/sec
volume flow rate	cubic feet per minute	CFM

temperature	degrees, Rankine Scale (^oF + 460)	^oR
velocity	feet per second	ft/sec
angular velocity	revolutions per minute	RPM

Flow Through Nozzles

Perfect Gas Law

Air is assumed to obey the perfect gas law

$$p = \rho R T \tag{2.1}$$

where R=19.16 $\frac{\text{mmHg ft}^3}{\text{lbm }^oR}$.

Isentropic Process

The flow of air through nozzles is assumed to be isentropic.

Hence

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^k \tag{2.2}$$

and

$$\frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}} \tag{2.3}$$

where k is the usual ratio of specific heats and the subscript, o, indicates the value at any reference state at the same entropy.

Mach Number

The Mach Number, M, is defined as

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Area Ratio

With one-dimensional flow, the continuity equation yields

$$w = \rho v A \tag{2.9}$$

where

w = mass flow rate
ρ = density
v = velocity
A = area

Since the mass flow rate must be equal at any two stations, equation (2.9) can be used to obtain

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \tag{2.10}$$

Solving for the area ratio yields

$$\frac{A_1}{A_2} = \frac{\rho_2 v_2}{\rho_1 v_1} \tag{2.11}$$

Using equation (2.4), this can be written

$$\frac{A_1}{A_2} = \frac{\rho_2 M_2 \sqrt{T_2}}{\rho_1 M_1 \sqrt{T_1}} \tag{2.12}$$

and with equation (2.1) and equation (2.3), the result is

$$\frac{A_1}{A_2} = \frac{M_2}{M_1}\left(\frac{T_2}{T_1}\right)^{\frac{k+1}{2(k-1)}} \tag{2.13}$$

Finally, using equation (2.7), this becomes

$$\frac{A_1}{A_2} = \frac{M_2}{M_1}\left[\frac{2+(k-1)M_1^2}{2+(k-1)M_2^2}\right]^{\frac{k+1}{2(k-1)}} \tag{2.14}$$

Putting in the throat condition at station 2, equation (2.14) becomes

$$\frac{A_1}{A^*} = \frac{1}{M_1}\left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M_1^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{2.15}$$

This last form is the most useful since it allows the necessary cross section of an isentropic nozzle to be calculated as a function of the throat area and the Mach Number.

Critical Pressure Ratio

The critical pressure ratio, p_{cr}/p_o, can be found from equation (2.8) by letting M=1. This is

$$\frac{p_{cr}}{p_o} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{2.16}$$

With K=1.4, equation (2.16) becomes

$$\frac{p_{cr}}{p_o} = .5283 \tag{2.17}$$

Mass Flow Rate

As given earlier, the mass flow rate for one-dimensional flow can be written

$$w = \rho v A \tag{2.18}$$

If equation (2.1), equation (2.4), and equation (2.7) are combined with equation (2.18), the result is

$$w = \rho A \sqrt{\frac{k}{RT_o}}\; M \sqrt{1 + \frac{k-1}{2} M^2} \tag{2.19}$$

Letting $C_1 = \sqrt{k/RT_o}$, equation (2.19) becomes

$$w = C_1 \rho A M \sqrt{1 + .2M^2} \tag{2.20}$$

For air at 537°R

$$C_1 = 7.672 \times 10^{-4} \frac{\text{lbm}}{\text{sec mmHg IN}^2} \tag{2.21}$$

Using equation (2.8), equation (2.20) can be written

$$w = p_o A \sqrt{\frac{k}{RT_o}} \frac{M}{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k+1}{2(k-1)}}} \tag{2.22}$$

Equations (2.19) through (2.22) have given the mass flow rate in terms of the Mach Number. The mass flow rate can also be written in terms of the pressure ratio. This equation is

$$w = p_o A \sqrt{\frac{2k}{(k-1)RT_o}} \left(\frac{p}{p_o}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}}} \tag{2.23}$$

Equation (2.23) is valid for both subsonic and supersonic flow. Letting $C_2 = \sqrt{\frac{2k}{(k-1)RT_o}}$, equation (2.23) becomes

$$w = C_2 p_o A \left(\frac{p}{p_o}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}}} \tag{2.24}$$

For air at 537°R,

$$C_2 = 1.716 \times 10^{-3} \frac{lbm}{sec \; mmHg \; in^2} \tag{2.25}$$

All of the mass flow equations developed thus far are, under the conditions stated, valid for all positive Mach Numbers and all pressure ratios between zero and one. It is necessary, however, for the nozzles to have certain geometries for these conditions to be reached isentropically. In a practical situation, the geometry is fixed and the flow conditions vary. It would then be beneficial to discuss the mass flow rates in relation to fixed geometries.

Consider first the strictly convergent, or subsonic nozzle. The Mach Number can never be greater than unity, and therefore equation (2.19), equation (2.20), and equation (2.22) are only valid for $0<M\leq 1$. Equation (2.23) and equation (2.24) are only valid for $p_{cr}/p_o \leq p/p_o \leq 1$.

When the ratio of the exit pressure and the stagnation pressure is less than or equal to the critical pressure ratio, the nozzle is choked and the mass flow rate is at a maximum. This maximum is

$$W = p_o A_e \sqrt{\frac{k}{RT_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \tag{2.26}$$

The area, A_e, is the exit area of the nozzle.

Letting

$$C_3 = \sqrt{\frac{k}{RT_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \tag{2.27}$$

equation (2.20) can be written

$$w = C_3 p_o A_e \tag{2.28}$$

For air at 537°R

$$C_3 = 4.441 \times 10^{-4} \frac{\text{lbm}}{\text{sec mmHg in}^2} \tag{2.29}$$

The other nozzle that will be considered is one that initially converges and then diverges. This is called a convergent-divergent or supersonic nozzle. The minimum area in the nozzle is the throat area and will be designated by A*. For the flow in the supersonic nozzle to be isentropic, two conditions must be met simultaneously. The area ratio as given by equation (2.15) and the pressure ratio as given by equation (2.8) must both be satisfied at the nozzle exit. When this happens for supersonic nozzles, all of the mass flow equations will apply without restriction.

What happens when these conditions are not met simultaneously will be discussed in the next section. However, for any convergent-divergent nozzle, the mass flow can be written in terms of the throat area, A*, with only the restriction that the ratio of the exit pressure and the stagnation pressure be less than the critical pressure ratio. This mass flow rate is the same as given in equation (2.26) with A* in the place of A_e. Thus

$$w = p_o A^* \sqrt{\frac{k}{R T_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \tag{2.30}$$

With

$$C_3 = \sqrt{\frac{k}{RT_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \qquad (2.31)$$

equation (2.30) becomes

$$w = C_3 p_o A^* \qquad (2.32)$$

and the value of C_3 given in equation (2.29) can be used.

Convergent-Divergent Nozzles with Subsonic Flow

The mass flow rate relations that have already been developed for supersonic nozzles have been for the case where the nozzles were at least choked. This section discusses the relationship between the mass flow and the pressure ratio when the nozzle is not choked.

Figure 2(a) shows a convergent-divergent nozzle. Figure 2(b) shows the relationship between the pressure ratio along the nozzle axis for four different back pressure ratios and Figure 2(c) shows the relationship between the mass flow rate at the full range of back pressure ratios. Also indicated on Figure 2(c) are the four cases of Figure 2(b).

Consider the flow represented by line (1). In this situation the pressure decreases in the converging section and the Mach Number increases. The Mach Number, however, never reaches unity nor does the pressure ever reach the critical pressure. As the flow passes the throat, the pressure begins to increase and the Mach Number begins to decrease. Assuming that the flow is isentropic throughout, the mass flow rate can be found using equation (2.24). Using the exit area, A_e, and the back pressure, p_B, this becomes

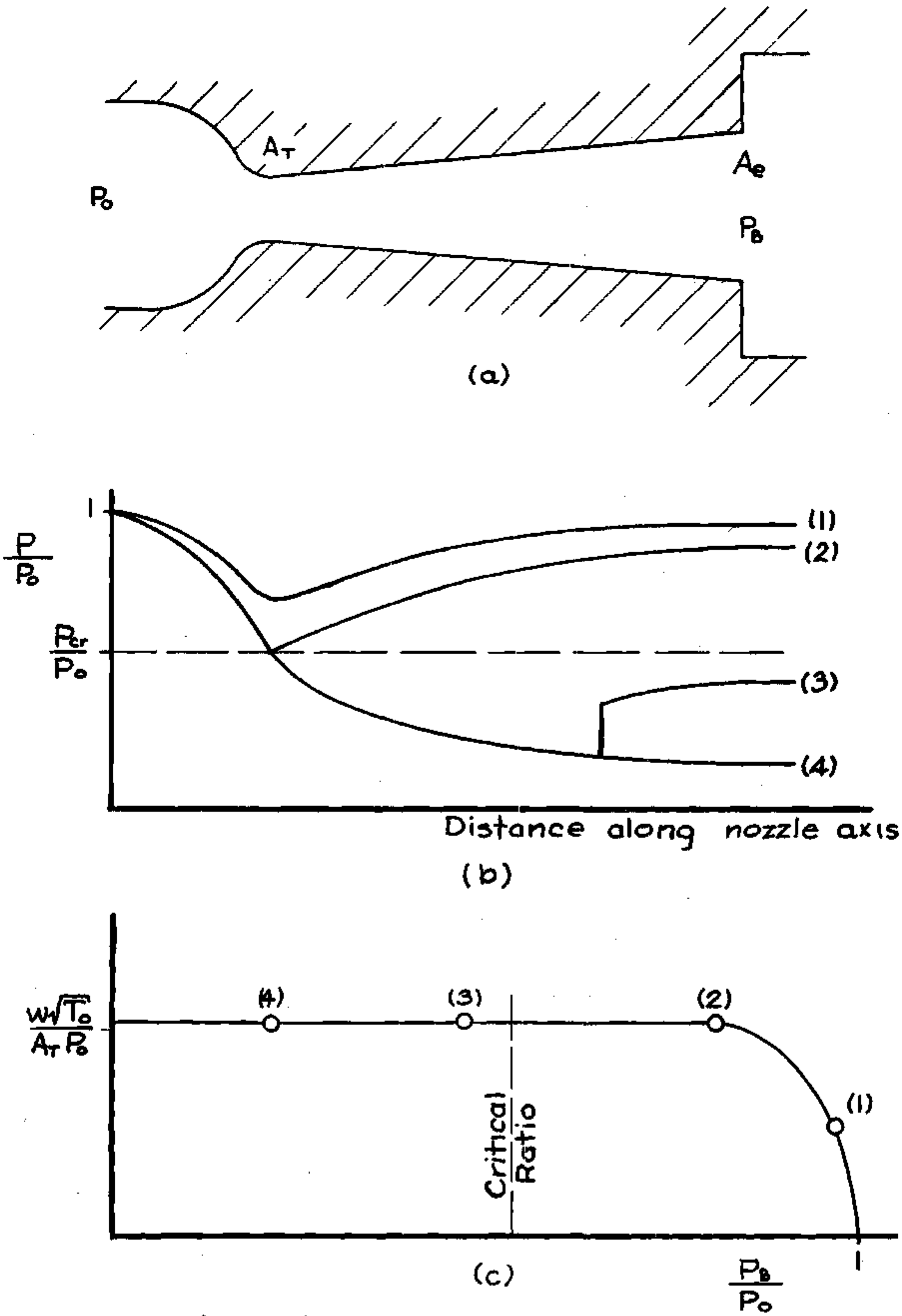

Figure 2. Supersonic Nozzle Flow

$$w = C_2 p_o A_e \left(\frac{p_B}{p_o}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_B}{p_o}\right)^{\frac{k-1}{k}}} \tag{2.33}$$

As the pressure is lowered more, the situation depicted by line (2) results. The Mach Number at the throat reaches 1.0 and any further decrease in the back pressure, as shown by lines (3) and (4), will not affect the mass flow rate. This is the range referred to earlier where equation (2.30) applies. Notice that the pressure ratio at which the nozzle chokes is greater than the critical pressure ratio.

It would be of interest to find the pressure ratio at which the nozzle chokes. Suppose the nozzle is designed for a Mach Number of M_d. The ratio of the exit area, A_e, and the throat area, A^*, is

$$\frac{A_e}{A^*} = \frac{1}{M_d}\left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M_d^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{2.34}$$

If the flow depicted by line (4) is assumed (i.e. isentropic and supersonic), the mass flow rate can be found by equation (2.22). This is

$$w = p_o A_e \sqrt{\frac{k}{RT_o}} \; \frac{M_d}{\left(1 + \frac{k-1}{2} M_d^2\right)^{\frac{k+1}{2(k-1)}}} \tag{2.35}$$

The mass flow rate found from equation (2.33) will be equal to the mass flow rate given by equation (2.35) when the nozzle becomes choked. Thus the solution of the equation

$$\sqrt{\frac{2}{k-1}}\left(\frac{p_B}{p_o}\right)^{\frac{1}{k}}\sqrt{1-\left(\frac{p_B}{p_o}\right)^{\frac{k-1}{k}}} = M_d\left(1+\frac{k+1}{2}M_d\right)^{-\frac{k+1}{2(k-1)}} \qquad (2.36)$$

will give the ratio at which the nozzle chokes.

For any particular M_d there will be two values of p_B/p_o that satisfy equation (2.36). One pressure ratio will be below the critical pressure ratio and will be the same as that given by equation (2.8). The other solution is greater than the critical pressure ratio and is the solution sought here.

Equation (2.36) cannot be solved explicitly for p_B/p_o; however Table 2 gives a few sample values.

Table 2. Pressure Ratios for Choking in Supersonic Nozzles

M_d	A_e/A^*	Pressure Ratio for Choking
1.0	1.00	.528
1.5	1.18	.778
2.0	1.69	.907
2.5	2.67	.967
3.0	4.23	.987
4.0	10.72	.998

Flow Through Valves and Orifices

All the equations that have been developed for the mass flow rates through subsonic nozzles can also be applied to valves and orifices if a coefficient of discharge, C_d, is used. The coefficient of discharge is necessary since the mass flow rate through a valve or orifice will not be as large as the mass flow rate through a subsonic nozzle of the same area. For example, the flow through a choked subsonic nozzle

$$w = C_3 p_o A_e \tag{2.28}$$

For a valve or orifice of the same area, the flow is

$$w = C_d C_3 p_o A_e \tag{2.37}$$

However , if the product of C_d and A_e is thought of as an effective area, then equation (2.28) can be used for the flow through valves and orifices with the understanding that the area is the effective area instead of the actual area. Throughout the remainder of this thesis the areas associated with any valve or orifice will be assumed to be the effective area instead of the actual area unless otherwise stated. Discharge coefficients for many valves and orifices can be found in reference 1.

For convenience let a function, $F\left(\frac{p_2}{p_1}\right)$, be defined:

$$F\left(\frac{p_2}{p_1}\right) = \begin{cases} \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}} \ , & \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \le \left(\frac{p_2}{p_1}\right) < 1 \\ \left(\frac{k-1}{2}\right)^{\frac{1}{2}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \ , & \left(\frac{p_2}{p_1}\right) \le \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \end{cases} \tag{2.38}$$

The mass flow rate through a valve or orifice can be written

$$W_{12} = C_2 \, p_1 \, A_{12} \, F\left(\frac{p_2}{p_1}\right) \tag{2.39}$$

where C_2 for air is given by equation (2.25).

With the constants evaluated for air, equation (2.38) becomes

$$F\left(\frac{P_2}{P_1}\right) = \begin{cases} \left(\frac{P_2}{P_1}\right)^{.7183} \sqrt{1 - \left(\frac{P_2}{P_1}\right)^{.2857}} \\ .2588 \end{cases} \tag{2.40}$$

It is noted in equation (2.39), that when the valve is choked,

$$C_2 F\left(\frac{P_2}{P_1}\right) = C_3 \quad , \quad \frac{P_2}{P_1} \leq .5283 \tag{2.41}$$

Flow Through Volumes

Consider the system shown in Figure 3.

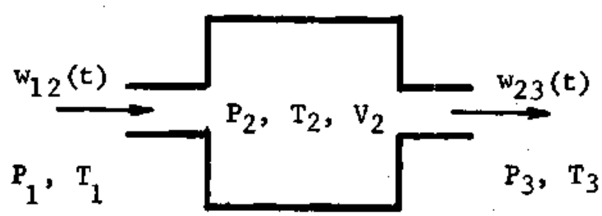

Figure 3. Flow Through a Volume

From the conservation of mass principle, the total mass in the volume at any time, t, is

$$m_2(t) = \int_0^t \left[W_{12}(\tau) - W_{23}(\tau) \right] d\tau + m_0 \tag{2.42}$$

The mass, m_o, is the initial mass in the volume.

If it is assumed that the fluid properties are homogeneous throughout the volume, and the fluid is a perfect gas, then the mass, m_2, can also be written

$$m_2 = \frac{p_2 V_2}{R T_2} \tag{2.43}$$

It is known that the temperature and the pressure in the inlet jet will be different from the temperature and pressure in the remainder of the volume. Yet it is assumed that the jet will be small compared to the volume and that the error caused by assuming homogeneous properties is small.

The initial temperature of the fluid in Figure 3 is T_1. If it is assumed that there is no heat transfer to or from the fluid as it enters the volume, and that the fluid in the volume is at rest, then $T_2=T_1$. This equality is a result of the conservation of energy equation for a perfect gas.

With $T_2=T_1$ and equation (2.43), equation (2.41) becomes

$$\frac{V_2 p_2}{R T_1} = \int_0^t \left[W_{12}(\tau) - W_{23}(\tau) \right] d\tau + m_o \tag{2.44}$$

When equation (2.44) is differentiated with respect to time, the result is

$$\frac{dp_2}{dt} = \frac{R T_1}{V_2} \left[W_{12}(t) - W_{23}(t) \right] \tag{2.45}$$

This result will be used later to describe the transient behavior of air through tanks.

CHAPTER III

ANALYSIS OF THE TEST APPARATUS UNDER STEADY-FLOW CONDITIONS

This chapter uses the relations developed in Chapter II to determine the required size of the upstream and downstream valves, and the required vacuum pump capacity.

The first part of this chapter considers the somewhat general case of designing the system where the test section is characterized only by a throat area and a Mach Number. An analysis is presented which provides a method of finding the upstream and downstream valve effective areas as a function of this throat area and Mach Number. Also a method is presented for estimating the vacuum pump size.

The last part of this chapter is devoted to designing the system for a particular type of test model. These are to be models of fluidic devices. The flow conditions at which the models are to operate and the dimensions of the models are given. Therefore the mass flow rates are known and the specific size of the upstream and downstream valves can be found. The vacuum pump size is estimated and a method is presented that will allow any specific vacuum pump with its particular pumping characteristics to be evaluated.

Figure 4 shows the notation that will be used in this chapter. The supply pressure, p_o, is the standard atmospheric pressure of 760 mmHg and the temperature, T_o, is 537^oR. The two control valves are characterized by their effective areas of A_1 and A_2. The pressures in the upstream and downstream tanks are p_1 and p_2 respectively.

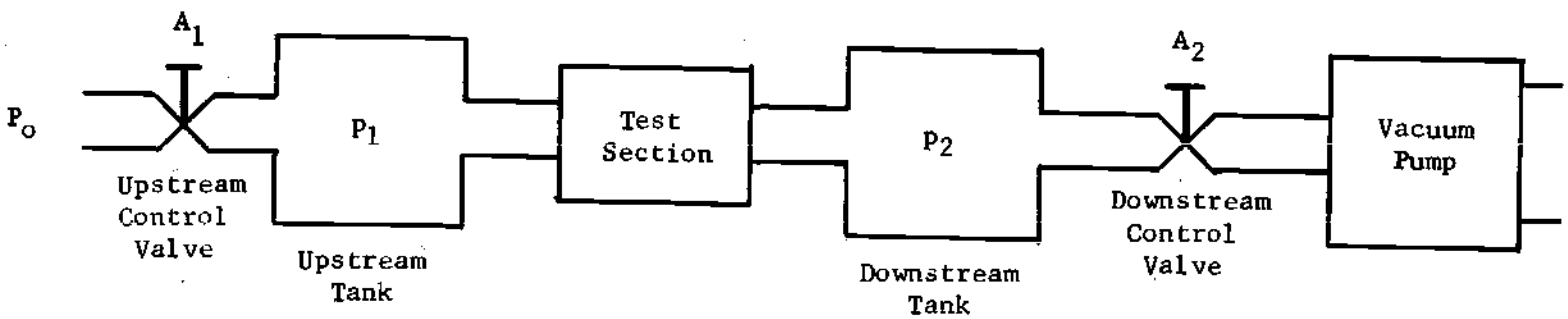

Figure 4. Diagram of Apparatus

The pressure p_3 is the exit pressure of the downstream valve as well as the vacuum pump inlet pressure.

Analysis for General Case

Upstream Valve: General

The function of the upstream valve is to control the pressure in the upstream tank. The analysis in this section is to help determine the valve size necessary to provide this control.

The mass flow rate through any valve can be found from equation (2.38). For the upstream valve this is

$$w = C_2 p_o A_1 F\left(\frac{p_1}{p_o}\right) \tag{3.1}$$

For steady flow, the mass flow rates through the valves, as well as the nozzle, must be equal. Solving equation (3.1) for A_1 yields

$$A_1 = \frac{w}{C_2 p_o F\left(\frac{p_1}{p_o}\right)} \tag{3.2}$$

It can be seen that the upstream valve effective area, A_1, is a function of the system mass flow rate, w; the atmospheric pressure, p_o; the upstream tank pressure, p_1; and the constant, C_2. With the atmospheric pressure considered to be constant, the valve area becomes a function of the system mass flow rate and the upstream tank pressure. It might also be mentioned that if the upstream tank pressure is below the critical pressure, then the valve area is only a function of the system mass flow rate.

It would be desirable to relate the system mass flow rate to the nozzle conditions. There are several ways to write the mass flow rate but here, as mentioned previously, the exit Mach Number and the throat area will be used. This gives rise to two cases: subsonic flow and supersonic flow. The mass flow rate is thus written (see equation (2.19) and equation (2.28))

$$w = \begin{cases} C_1 p_1 A^* \dfrac{M}{(1 + .2M^2)^3} & , \quad M < 1 \\ C_3 p_1 A^* & , \quad M \geq 1 \end{cases} \tag{3.3}$$

The constants C_1 and C_3 are given by equation (2.21) and equation (2.29) respectively. The area, A*, is defined in both the subsonic and supersonic case as the smallest cross section area of the nozzle. For the subsonic nozzles this is the exit area and for the supersonic nozzles this is the throat area.

By substituting the mass flow rate of equation (3.3) into equation (3.2), it can be written

$$\frac{A_1}{A^*} = \begin{cases} \dfrac{C_1}{C_2} \dfrac{\left(\frac{p_1}{p_0}\right)}{F\left(\frac{p_1}{p_0}\right)} \dfrac{M}{(1 + .2M^2)^3} & , M < 1 \\ \dfrac{C_3}{C_2} \dfrac{\left(\frac{p_1}{p_0}\right)}{F\left(\frac{p_1}{p_0}\right)} & , M \geq 1 \end{cases} \tag{3.4}$$

Figure 5 is a plot of equation (3.4) and can be used to find the necessary effective area for the upstream valve as a function of A*, p_1, and M. The Mach Number, M, is the Mach Number at the exit of nozzle in the test section. This also assumes the upstream or ambient

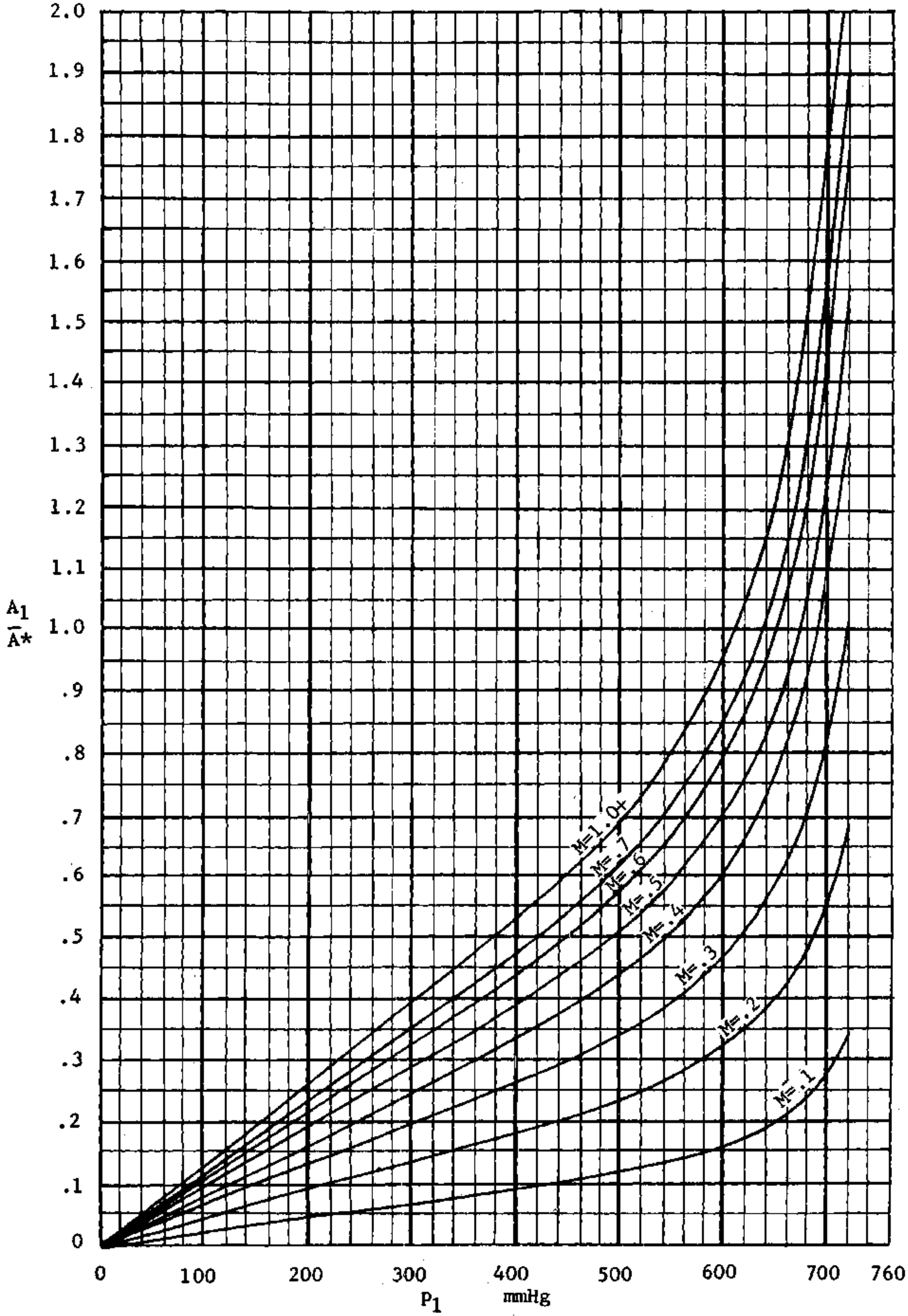

Figure 5. Design Chart for Upstream Tank

conditions to be those of the atmosphere.

Downstream Valve: General

The function of the downstream valve is to control the pressure in the downstream tank and this analysis is to help determine the size of the valve. There are two important differences between this analysis for the downstream valve and the previous analysis for the upstream valve. The first difference is that the upstream valve controls the pressure by regulating the inlet flow, while the downstream valve controls the pressure by regulating the exit flow. The second difference is that the relation between the pressure and the mass flow rate at the exit of the downstream valve are not derived from isentropic flow theory since the relations are influenced by the characteristics of the vacuum pump. Therefore, under some flow conditions there are no analytical relations to describe the flow through the downstream valve.

This second difference complicates the downstream valve analysis to the extent that the analysis is not easily performed in general. However, as will be shown, some useful results can be achieved for the general case.

The mass flow rate through the downstream valve can be found from equation (2.39) and is

$$w = C_2 p_2 A_2 F\left(\frac{p_3}{p_2}\right) \tag{3.5}$$

Solving for A_2 yields

$$A_2 = \frac{w}{C_2 p_2 F\left(\frac{p_3}{p_2}\right)} \tag{3.6}$$

The pressure, p_3, is the pressure at the inlet of the vacuum pump as well as at the exit of the downstream valve. Usually the inlet pressure a vacuum pump can produce is a function of the mass flow rate and varies for each pump. Therefore A_2 in equation (3.5) cannot be evaluated unless a specific pump curve is used and then only with an iterative solution. For the present consider the downstream valve to be choked and therefore knowledge of p_3 will not be necessary.

With the downstream valve choked, equation (3.6) becomes

$$A_2 = \frac{w}{C_3 p_2} \tag{3.7}$$

For the subsonic nozzle, the mass flow rate can be written

$$w = C_1 p_2 A^* M \sqrt{1 + .2M^2} \quad , \quad M \leq 1 \tag{3.8}$$

Putting this into equation (3.7) gives

$$\frac{A_2}{A^*} = \frac{C_1}{C_3} M \sqrt{1 + .2M^2} \quad , \quad M \leq 1 \tag{3.9}$$

which is only a function of the Mach Number.

For the supersonic nozzle the mass flow rate can be written

$$w = C_3 p_1 A^* \quad , \quad M > 1 \tag{3.10}$$

Putting this into equation (3.7) gives

$$\frac{A_2}{A^*} = \frac{p_1}{p_2} \quad , \quad M > 1 \tag{3.11}$$

By using equation (2.8), equation (3.11) becomes

$$\frac{A_2}{A^*} = (1+.2M^2)^{3.5} \quad , \quad M>1 \tag{3.12}$$

and again this is only a function of the Mach Number. Therefore if the downstream valve is choked, the area ration, A_2/A^*, is only a function of the Nozzle Mach Number. Table 3 gives this area ratio for several Mach Numbers.

Table 3. Downstream Valve Area Ratio and Nozzle Mach Number

M	A_2/A^*
.1	.1730
.2	.3470
.4	.7022
.6	1.074
.8	1.468
1.0	1.893
1.5	3.671
2.0	7.825
2.5	17.08
3.0	36.74
3.5	76.28
4.0	152.0

Two other implications can be made from equation (3.9) and equation (3.12). Consider the case where a nozzle of some specified A* is in the test section. If the size of the downstream valve is only a function of the nozzle Mach Number (when the downstream valve is choked), then the Nozzle Mach Number must only be a function of the downstream valve area. Consequently, once a Mach Number is set by the downstream valve, the upstream valve will raise and lower both pressures but will not change the pressure ratio or Mach Number.

This could be used advantageously when performing the experiment.

For the second implication, again consider A* fixed. Any particular valve used for the downstream valve will have some maximum effective area. This maximum effective area according to equation (3.8) and equation (3.11) will determine the maximum Mach Number that can be achieved. Table 3 can then be interpreted as giving the minimum necessary area ratio to achieve the corresponding Mach Number. If the downstream valve is choked, then the area ratio given will be adequate, but, if the downstream valve is not choked, an even larger area ratio may be required.

Vacuum Pump: General

A vacuum pump is a constant displacement device . For this reason the capacity of vacuum pumps is usually given as a volume flow rate. However, because of clearance volumes and leakages, the volume flow rate does change with inlet pressure and even reduced completely to zero for very low pressures. Therefore the capacity is usually given as volume flow rate versus inlet pressure. It must be determined then what volume flow rate is necessary to accommodate the flow conditions in the test section.

When determining the pump capacity, the same complication associated with the subsonic downstream valve is present. However for the general case, this problem will be circumvented by only considering a method of estimating the needed capacity. Later when determining the pump capacity needed for the fluidic models, the subsonic downstream valve will be considered.

There are two assumptions that lead to simple direct computations of the pump capacity. One assumption will lead to a maximum capacity and the other leads to a minimum capacity. Thus the two methods that are developed next can be used to bracket the necessary pump capacity and this is what is meant by estimating the capacity.

From the perfect gas law and the one dimensional continuity equation, the relation between the mass flow rate and the volume flow rate, Q, is

$$Q = \frac{wRT}{p} \tag{3.13}$$

The volume flow rate, Q_2, at the downstream tank conditions can be written

$$Q_2 = \frac{wRT_o}{p_2} \tag{3.14}$$

And similarly for the volume flow rate at the pump inlet conditions

$$Q_3 = \frac{wRT_o}{p_3} \tag{3.15}$$

In equation (3.15) it was assumed that there is a sufficiently large volume at the pump inlet for the air to come to rest after leaving the downstream valve and consequently for the temperature to become T_o.

Equations (3.14) and (3.15) can be solved for w and equated. The result is

$$Q_2 p_2 = Q_3 p_3 \tag{3.16}$$

Thus the volume flow rate, Q_3, can be written

$$Q_3 = \frac{Q_2}{\left(\frac{p_3}{p_2}\right)} \tag{3.17}$$

The problem now is to find p_3/p_2 and Q_2.

Using equation (2.19), the mass flow rate for a subsonic nozzle can be written

$$w = p_2 A^* \sqrt{\frac{k}{RT_o}}\; M \sqrt{1 + \frac{k-1}{2} M^2} \quad , \quad M \leq 1 \tag{3.18}$$

Putting this into equation (3.14) yields

$$\frac{Q_2}{A^*} = \sqrt{kRT_o}\; M \sqrt{1 + \frac{k-1}{2} M^2} \quad , \quad M \leq 1 \tag{3.19}$$

Thus the ratio Q_2/A^* is only a function of the Mach Number for subsonic flow.

With a supersonic nozzle the mass flow is

$$w = p_1 A^* \sqrt{\frac{k}{RT_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \quad , \quad M \geq 1 \tag{3.20}$$

Equation (3.20) can be written in terms of p_2 and the nozzle Mach Number by using equation (2.8). The result is

$$w = p_2 A^* \sqrt{\frac{k}{RT_o}} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \quad , \quad M \geq 1 \tag{3.21}$$

Putting equation (3.21) into equation (3.14) yields

$$\frac{Q_2}{A^*} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \sqrt{kRT_0} \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} , \quad M \geq 1 \qquad (3.22)$$

From equation (3.17) and (3.18) it can be seen that Q_2 is only a function of the Mach Number and A*.

Going back to equation (3.17) it can be seen that Q_3 can be found in terms of the Mach Number and A* if the pressure ratio across the downstream valve is known. However for the present consider the maximum and minimum values the pressure ratio can have. The largest p_3/p_2 can become is unity which means that there is no pressure drop across the downstream valve. This is not very realistic but it does provide a minimum vacuum pump capacity. There is no limit to how small the ratio p_3/p_2 can become and therefore it would seem as though there were no maximum Q_3. Once the downstream valve becomes choked, no further decrease in the pressure ratio will affect the flow through the downstream valve. Thus supplying enough pump capacity to choke the downstream valve will be the maximum capacity needed to accommodate that particular Q_2.

The pressure ratio at choking in

$$\frac{p_3}{p_2} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \qquad (3.23)$$

And the maximum pump capacity required for any particular Q_2 is

$$Q_3 = \frac{Q_3}{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}} \tag{3.24}$$

Using equation (3.24), equation (3.19), and equation (3.22), the volume flow rate necessary to choke the downstream valve can be written

$$\frac{Q_3}{A^*} = \begin{cases} \dfrac{\sqrt{kRT_0}}{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}} M\sqrt{1+\frac{k-1}{2}M^2} \quad , & M \leq 1 \\ \sqrt{\dfrac{kRT_0(k+1)}{2}} \left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} , & M > 1 \end{cases} \tag{3.25}$$

With the constants in equation (3.25) evaluated, the result is

$$\frac{Q_3}{A^*} = \begin{cases} 896.5\, M(1+.2M^2)^{.5} & , M \leq 1 \\ 518.8\,(1+.2M^2)^{3.5} & , M > 1 \end{cases} \tag{3.26}$$

with Q_3 in CFM, and A^* in in^2.

Table 4 gives several values of Q_2/A^*, Q_3/A^* with the downstream valve just choked, and the corresponding Mach Number.

If the Mach Number and the throat area are known, then the corresponding volume flow rate can be found from Table 4. There is still no way to determine the required vacuum pump capacity if the downstream valve is not choked. Assuming no pressure drop across the downstream valve, Column (1) can be used to give the volume flow rate required at the downstream tank pressure. The volume flow rate re-

Table 4. Required Pump Capacity

M	(1) Q_2/A^*	(2) Q_3/A^*	(3) A_2/A^*
.1	52.16	98.74	.1736
.2	95.11	180.0	.3470
.4	192.5	364.3	.7022
.6	294.3	557.0	1.074
.8	402.4	761.8	1.468
1.0	518.8	982.1	1.893
1.5	1006	1904	3.671
2.0	2145	4060	7.825
2.5	4683	8865	17.08
3.0	1.007×10^4	1.906×10^4	36.74
3.5	2.091×10^4	3.958×10^4	76.28
4.0	4.166×10^4	7.885×10^4	152.0

quired will always be larger than this since there will always be some pressure drop across the downstream valve. Assume that the downstream valve has the area obtained from Column (3). (Column (3) is obtained from Table 3). If the vacuum pump meets the volume flow rate requirements obtained from Column (2) at a pressure of .5283 times the downstream tank pressure, then no additional capacity will be required. If the downstream valve can be made larger than that found in Column (3), then some capacity that is between the capacities given in Column (1) and Column (2) will be required.

Analysis for Fluidic Models

The remainder of this chapter considers the design of the test apparatus for a specific test model. These test models are of fluidic devices.

Model Description

The fluidic models are similar to compressible flow nozzles and, as far as the analysis of the flow is concerned, they are considered to be compressible flow nozzles. Both the subsonic and the supersonic nozzles are present.

The nozzle walls are cut from a one inch thick aluminum plate, and sandwiched between two cover plates. Before the walls are clamped between the plates, they can be adjusted to give any desired throat and exit area.

For this particular study, all of the nozzles are to have an exit area of 0.2 in^2. This means that with the nozzle construction described, the walls at the exit will always be spaced 0.2 inches apart at the exit, and the throat area is changed to vary the Mach Number.

Some of the models will also have a control jet as exemplified in Figure 6. The interaction of this control jet with the main jet will also be an objective of the fluidic model study.

Range of Operation

It is necessary to determine the range of operation before the size of the system components can be determined. For the fluidic models, it was desired to have a capability of producing Mach Numbers of 0.5 to 3.5 and Knudsen Number of 10^{-1} to 10^{-4}.

The Mach and Knudsen Number range can be used to find the corresponding pressure range. In order to produce a Mach Number of M at the model exit, the upstream and downstream tank pressures must be related by

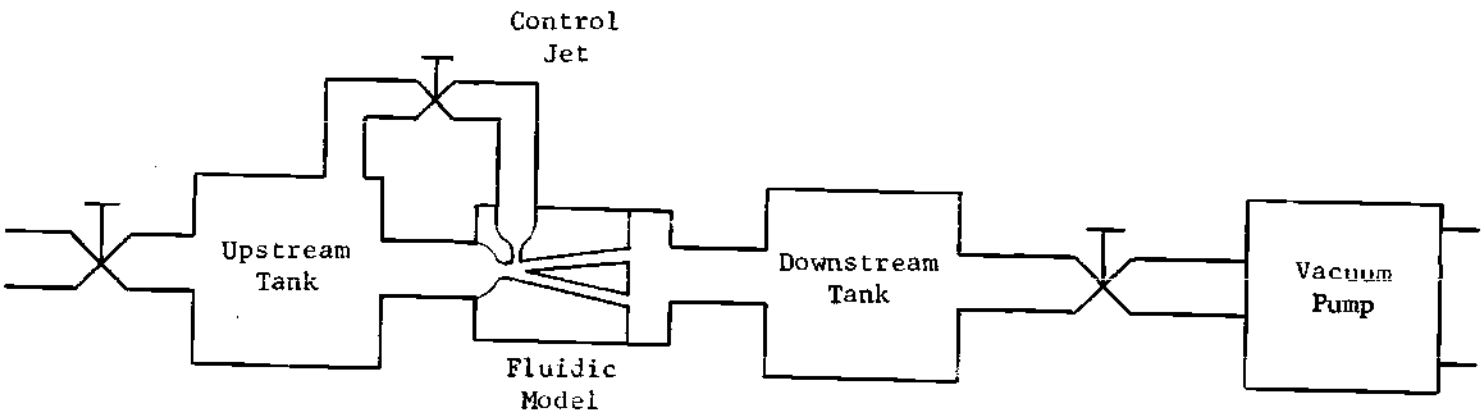

Figure 6. Diagram of Test Apparatus with Fluidic Model

$$\frac{p_1}{p_2} = (1+.2M^2)^{3.5} \tag{3.27}$$

Taking the logarithm of both sides yields

$$\log_{10} p_1 = \log_{10} p_3 + 3.5\log_{10}(1+.2M^2) \tag{3.28}$$

Equation (3.28) is a linear relation between $\log_{10}p_2$ and $\log_{10}p_3$ for a constant Mach Number. Therefore equation (3.28) is a straight line on log-log graph paper.

The Knudsen Number can also be found in terms of p_1 and p_2. The definition of the Knudsen Number, Kn, is

$$K_N = \frac{\lambda}{x} \tag{3.29}$$

where λ is the mean free path of the gas molecules and x is a characteristic dimension of the flow area.

To determine the mean free path of the air molecules, it was assumed that the product of the density and the mean free path is constant. Thus

$$\rho\lambda = K \tag{3.30}$$

where ρ is in lbm/ft^3, λ is in inches, and K=1.966x10^{-7} $\frac{\text{lbm in.}}{\text{ft}^3}$

Using the equation of state for a perfect gas along with equation (3.29) and equation (3.30), the downstream tank pressure can be written

$$p_2 = \frac{KRT_2}{xK_N} \tag{3.31}$$

The temperature, T_2, used in the above calculation is the temperature of the fluid jet as it passes through the nozzle exit which is different from the temperature of the fluid when it comes to rest in the downstream tank.

Using equation (2.3), the temperature ratio can be written

$$\frac{T_2}{T_0} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} \tag{3.32}$$

Equation (3.31) thus becomes

$$p_2 = \frac{KRT_0}{x K_N} \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} \tag{3.33}$$

The characteristic dimension, x, used to calculate the Knudsen Number was chosen to be the distance between the nozzle walls at the exit. This dimension is 0.2 inches.

With the constants evaluated, equation (3.33) becomes

$$p_2 = \frac{14.57}{K_N} \left(\frac{p_2}{p_1}\right)^{.2857} \tag{3.34}$$

Taking the logarithm gives

$$\log_{10} p_1 = -16.952 \log_{10} K_N - 2.5 \log_{10} p_2 \tag{3.35}$$

For a constant Knudsen Number equation (3.35) is a linear relation between $\log_{10} p_1$ and $\log_{10} p_2$.

Figure 7 displays conveniently the pressures that must be produced in the upstream and downstream tank to obtain the required

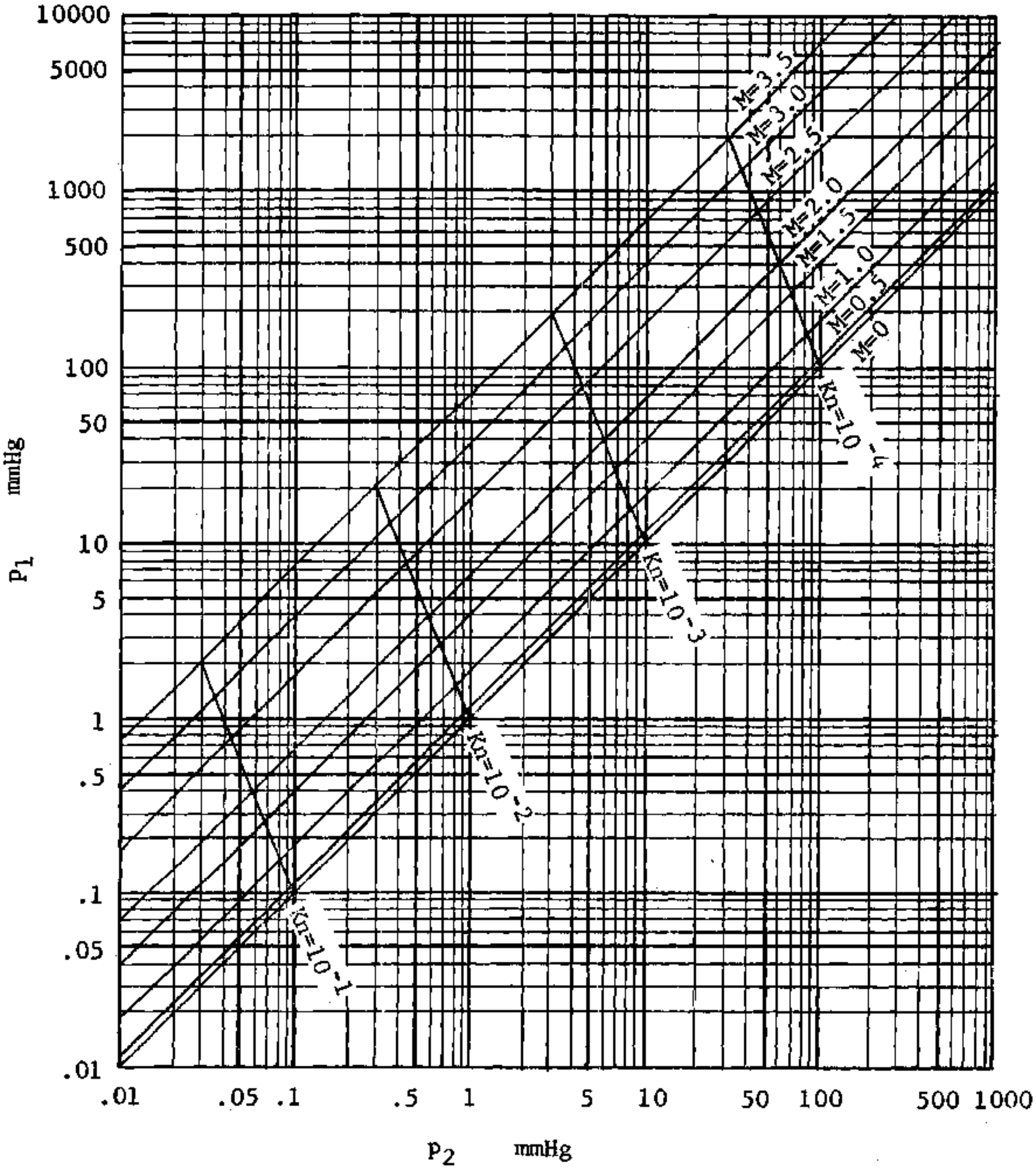

Figure 7. Range of Operation

Mach and Knudsen Numbers. The region of interest lies between the line for M=3.5 and M=0.5, and between the lines for $Kn=10^{-4}$ and $Kn=10^{-1}$.

Upstream Valve: Fluidic Models

With the pressure range shown in Figure 7 and a means of calculating the mass flow rate, the required size of the upstream valve can be determined.

By using equation (2.39), the mass flow rate through the upstream valve is

$$w = C_2 p_0 A_1 F\left(\frac{p_1}{p_0}\right) \tag{3.36}$$

Solving equation (3.34) for A_1 gives

$$A_1 = \frac{w}{C_2 p_0 F\left(\frac{p_1}{p_0}\right)} \tag{3.37}$$

The mass flow rate through the nozzle can be found by using equation (2.22). The result is

$$w = C_1 p_1 A_e \frac{M}{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k+1}{2(k-1)}}} \tag{3.38}$$

By equating the mass flow rates, equation (3.35) becomes

$$A_1 = \frac{C_1 p_1 A_e M}{C_2 p_0 F\left(\frac{p_1}{p_0}\right)\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k+1}{2(k-1)}}} \tag{3.39}$$

Equation (3.39) now allows the effective area of the upstream valve to be calculated for any known Mach Number and any upstream tank pressure, p_1.

The smallest valve of A_1 that will be required can be found by putting in the conditions at a Mach Number of 0.5 and a Knudsen Number of 10^{-1}. At that point the upstream tank pressure, p_1, is 0.12 mmHg. With this upstream tank pressure, the upstream valve is choked and equation (3.39) becomes

$$A_1 = \frac{C_1 p_1 A_e M}{C_2 p_0 \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k+1}{2(k-1)}}} \tag{3.40}$$

With p_0=760 mmHg, p_1=0.12 mmHg, Ae=0.2in.^{2}, and M=0.5, then A_1 is

$$A_1 = 2.24 \times 10^{-5} \quad \text{in}^2 \tag{3.41}$$

It is assumed that a valve can provide good regulation at an area of 10% of its maximum flow area, then the area of $2.24\text{x}10^{-5}$ in^2 could be achieved by a valve with maximum effective area of $2.24\text{x}10^{-4}$ in^2. With a discharge coefficient of 0.75, the actual area would be $2.987\text{x}10^{-4}$ in^2 which corresponds to a valve orifice .0195 inches in diameter. Thus a very small needle valve will be required for good regulation at low upstream tank pressures.

Since valves are limited to the range of effective area at which they can provide good control and since the desired range of operation is so large, it will be necessary to provide several graduated valves in parallel to provide good control. Graduations by a factor of 10 would probably provide good control or graduations by a larger factor could be used if economy dictated. The analysis thus far has only provided a method of determining the smallest valve. Finding the largest upstream valve area that is required presents a different problem. As seen in Figure 7, the highest upstream tank pressure that is required is above atmospheric pressure. Since atmospheric air is being used as the supply air, the upstream tank cannot have a pressure greater than 760 mmHg. How close the upstream tank pressure comes to being atmospheric depends on how large the upstream valve is. It would be of interest to find the maximum pressures that can be obtained for some given maximum valve opening.

First consider the case where the upstream valve is choked. The mass flow rate through the valve is

$$w = C_3 p_o A_1 \quad , \quad p_1 \leq 402 \text{ mm Hg} \tag{3.42}$$

The mass flow rate through the nozzle is given by equation (3.38). Equating the mass flow rates gives

$$p_1 = \frac{C_3 p_o A_1}{C_1 A_e} \; \frac{(1 + .2M^2)^{3.5}}{M} \tag{3.43}$$

Let

$$\beta = \frac{C_3 A_1 \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k+1}{2(k-1)}}}{C_1 A_e M} \tag{3.44}$$

Equation (3.43) becomes

$$p_1 = p_o \beta \quad , \quad \beta < .5283 \tag{3.45}$$

If β>.5283, then the upstream valve is not choked. The mass flow rate from equation (2.39) is

$$W = C_2 p_0 A_1 \left(\frac{p_1}{p_0}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_1}{p_0}\right)^{\frac{k-1}{k}}} \ , \ \frac{p_1}{p_0} > .5283 \qquad (3.46)$$

Equating the mass flow rates and substituting in the definition of β yields

$$\left(\frac{p_1}{p_0}\right)^{-1} \left(\frac{p_1}{p_0}\right)^{\frac{1}{k}} \sqrt{1 + \left(\frac{p_1}{p_0}\right)^{\frac{k-1}{k}}} = \frac{C_3}{C_2 \beta} \ , \ \beta > .5283 \qquad (3.47)$$

Squaring both sides of equation (3.47) yields

$$\left(\frac{p_1}{p_0}\right)^{2\left(\frac{1-k}{k}\right)} - \left(\frac{p_1}{p_0}\right)^{\left(\frac{1-k}{k}\right)} - \left(\frac{C_3}{C_2 \beta}\right)^2 = 0, \ \beta > .5283 \qquad (3.48)$$

which is a quadratic equation in $\left(\frac{P_1}{P_0}\right)^{\frac{1-k}{k}}$ and can be solved to give

$$p_1 = p_0 \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{C_3}{C_2 \beta}\right)^2} \right]^{\frac{k}{1-k}} , \ \beta > .5283 \qquad (3.49)$$

With the constants evaluated, equation (3.49) is

$$p_1 = p_0 \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \left(\frac{.5176}{\beta}\right)^2} \right]^{-3.5} , \ \beta > .5283 \qquad (3.50)$$

Thus the largest pressure, p_1, that can be reached for any particular A_1 can be found from equation (3.45) and equation (3.50). An example using these equations is given in a later section.

Downstream Valve: Fluidic Models

As mentioned previously in connection with the more general discussion of the downstream valve, determining the proper valve size is complicated by the fact that vacuum pump characteristics are usually given graphically. Assuming the downstream valve to be choked, which is usually the case anyhow, allows the valve analysis to be conducted independent of the characteristic of the pump. Such will be the case here.

The mass flow rate through the downstream valve is

$$w = C_3 A_2 p_2 \tag{3.51}$$

and the mass flow rate through the nozzle is (equation (2.20))

$$w = C_1 p_2 A_e M \sqrt{1 + .2M^2} \tag{3.52}$$

Equating these gives the following equation for A_2:

$$A_2 = \frac{C_1}{C_3} A_e \sqrt{1 + .2M^2} \tag{3.53}$$

Equation (3.53) shows that the necessary downstream valve area is only a function of the desired Mach Number. Several values are shown in Table 5.

Table 5. Downstream Valve Effective Area and Nozzle Mach Number

M	A_2 (in^2)
.5	.177
1.0	.349
1.5	.624
2.0	.927
2.5	1.269
3.0	1.735
3.5	2.192

As can be seen from Table 5, one butterfly valve will probably provide the necessary control. Characteristics of available valves would have to be examined to determine this.

Vacuum Pump: Fluidic Models

The only method given thus far for determining the vacuum pump capacity is to determine the maximum and minimum capacity required. This will be the method used in this section. The next section, however, offers an alternative yet it is still desirable to have an estimate of the required capacity.

The volume flow rate at the downstream tank condition is

$$Q_2 = \frac{wRT_o}{p_2} \tag{3.54}$$

The mass flow rate for the fluidic models is (equation (2.19))

$$w = p_2 A_e \sqrt{\frac{k}{RT_o}}\, M \sqrt{1 + \frac{k-1}{2} M^2}\ , \quad M > 0 \tag{3.55}$$

Thus

$$Q_2 = A_e\sqrt{kRT_0}\, M\sqrt{1+\frac{k-1}{2}M^2}\ ,\ M>0 \tag{3.56}$$

With the constants evaluated,

$$Q_2 = 94.7\, M\sqrt{1+.2M^2}\ ,\ M>0 \tag{3.57}$$

Equation (3.57) is the volume flow rate at the downstream tank conditions and if it is taken to be the volume flow rate that the pump must handle it will be a minimum capacity since it assumes no pressure drop across the downstream valve. As discussed in the section which determined the pump capacity for the general case, the maximum capacity required can be found by dividing Q_2 in equation (3.57) by .5283.

Table 6 gives a summary of these capacities. The minimum capacity is at the downstream tank pressure but the maximum capacity must be at a pressure of 0.5283 times the downstream tank pressure.

Table 6. Maximum and Minimum Volume Flow Rates

M	Maximum (CFM)	Minimum (CFM)
.5	91.8	48.5
1.0	197	104
2.0	481	254
3.0	901	476
3.5	1166	616

Fixed Downstream Valve Area

The last section assumed no pressure drop across the downstream valve. This section takes the pressure drop into account.

Assume that a graph of volume flow rate versus the inlet pressure for a vacuum pump is available. Solving equation (3.15) for the mass flow rate gives w as a function of A_3 and p_3 which are available from the assumed graph. This relation is

$$w = \frac{Q_3 p_3}{RT_0} \qquad (3.58)$$

Consider first the case where the valve is choked. The mass flow rate is

$$w = C_3 p_2 A_2 \qquad (3.59)$$

Equating the mass flow rates gives

$$\frac{p_3}{p_2} = \frac{C_3 A_2 R T_0}{Q_3} \qquad (3.60)$$

Letting

$$\alpha = \frac{60\, C_3 A_2 R T_0}{Q_3} \qquad (3.61)$$

Equation (3.60) can be written

$$p_2 = \frac{p_3}{\alpha}\ , \quad \alpha \leq .5283 \qquad (3.62)$$

If $\alpha > .5283$ then the downstream valve is not choked. The mass flow rate from equation (2.39) is

$$w = C_2 p_2 A_2 \left(\frac{p_3}{p_2}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_3}{p_2}\right)^{\frac{k-1}{k}}} \quad , \left(\frac{p_3}{p_2}\right) < .5283 \tag{3.63}$$

Equating this to equation (3.58) and using the definition of α, the result is

$$\left(\frac{p_3}{p_2}\right)^{-1} \left(\frac{p_3}{p_2}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_3}{p_2}\right)^{\frac{k-1}{k}}} = \frac{C_3}{C_2 \alpha} \quad , \quad \alpha > .5283 \tag{3.64}$$

Similar to equation (3.47), this can be solved to yield

$$p_2 = p_3 \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + \left(\frac{.5176}{\alpha}\right)^2} \right]^{3.5} \quad , \quad \alpha > .5283 \tag{3.65}$$

From this equation, p_2 can be found if the pump capacity in CFM, the pump inlet pressure in mmHg, and the downstream valve area in in^2. is known.

Now if p_1 can be found, a line can be drawn on Figure 7 showing the limiting conditions that can be reached with any particular vacuum pump and downstream valve.

From equation (2.24), the mass flow rate through the nozzle can be written as

$$w = C_2 p_1 A_e \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}} \tag{3.66}$$

This can be equated to the mass flow rate in equation (3.58) yielding

$$C_2 p_1 A_e \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}} = \frac{Q_3 p_3}{R T_o} \tag{3.67}$$

Dividing by p_2 gives

$$\left(\frac{p_1}{p_2}\right)\left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}} = \frac{Q_3 p_3}{60 R T_o C_2 A_e p_2} \tag{3.68}$$

By letting

$$\beta = \frac{2 Q_3 p_3}{60 R T_o C_2 A_e p_2} \tag{3.69}$$

equation (3.68) can be solved to give

$$p_1 = p_2 \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \beta^2}\right]^{3.5} \tag{3.70}$$

These equations will be used in the next section to plot a limiting curve for a particular vacuum pump and downstream valve.

An Example of Equipment Capabilities

In previous sections, equations have been developed for determining the capabilities of the equipment. This section works an example using these equations for the fluidic devices.

First consider an upstream valve with an orifice diameter of 0.5 in. The orifice area is 0.1963 in^2 and, with a coefficient of discharge of 0.75, the effective area is 0.1425. Using equation (3.50) and equation (3.44), the maximum upstream tank pressure can be found for each nozzle Mach Number. Then by using equation (2.8) the downstream pressure can be found by using the same Mach Number. This line is shown in Figure 9 designated by A_1.

With the pump capacity estimated by methods given in the last section, a vacuum pump of the approximate capacity can be selected. For this example, the pump curve given in Figure 8 is used.

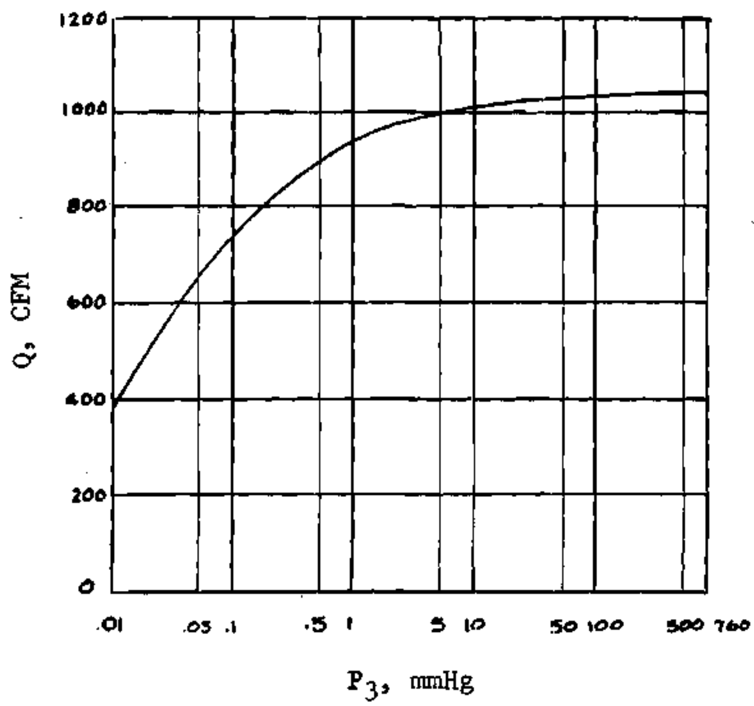

Figure 8. Sample Vacuum Pump Characteristics

Let the downstream valve have a 2 in. diameter orifice with a coefficient of discharge of .75. The effective area then is 2.356 in^2. The object now is to find the largest Knudsen Number and Mach Number that can be reached with this pump and this downstream valve. Equation

(3.61), (3.62), (3.65), (3.67), and (3.70) can be used for this purpose. The resulting line is shown on Figure 9 designated by A_2. No conditions above this line can be reached.

Since this line indicates that the region around $K=10^{-1}$ and M=3.5 cannot be reached, it would be of interest to know if this is a vacuum pump limitation or a downstream valve limitation. If no pressure drop across the downstream valve is assumed and a line is drawn from the vacuum pump curves, then any region excluded would definitely be the result of inadequate vacuum pump capacity. The region between the two lines can possibly be recovered with a larger downstream valve. The line showing the limitations caused by the vacuum pump can be found by using equation (3.69) and equation (3.70) with p_2 equal to p_3. This line is also shown on Figure 9.

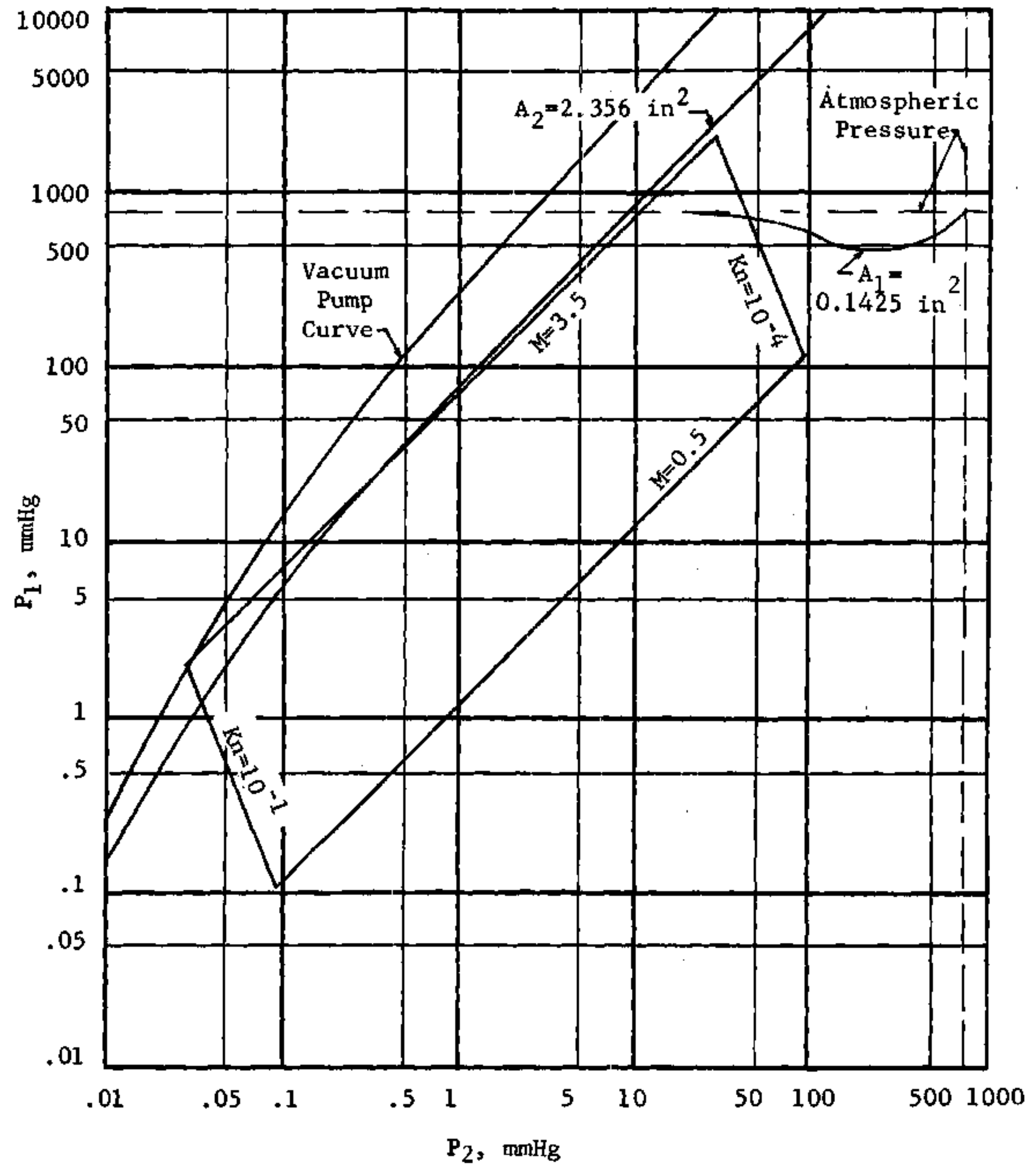

Figure 9. Range of Operation and Equipment Capabilities

CHAPTER IV

AUTOMATIC CONTROL SYSTEM ANALYSIS

This chapter analyzes two control systems for automatically regulating the pressures in the upstream and downstream tanks.

The first system is a representation of a system that has d.c. motors operating the upstream and downstream control valves. The error signal, which is the difference between the reference input signal and the signal from the pressure transducer, is amplified to supply the motor input voltage. The motors act as integrators, integrating their input signal. For this reason the first system is referred to in this thesis as the integral control system.

A second system was analyzed for the possibility of improving the performance. The second system assumed a position controller to maintain the control valve area proportioned to the error signal. The strategy is to use a very large feedback gain so that the error signal is forced to be small. This system is referred to in this thesis as the proportional control system.

Integral Control System

The integral control system was chosen because it would be comparitively easy to implement. A diagram of the system is shown in Figure 10.

The voltage to the valve motors is supplied by the power amplifier. The output of the power amplifier is assumed to be pro-

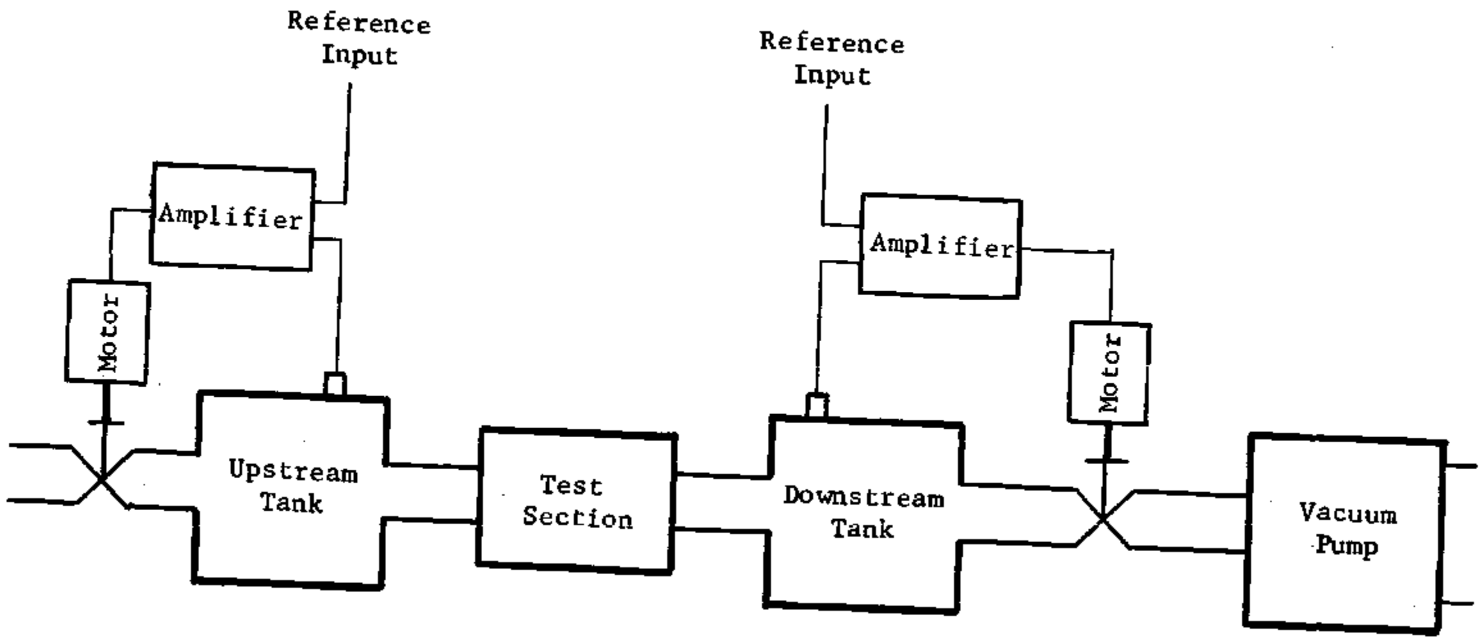

Figure 10. Apparatus with Integral Controller

portional to the difference between the reference input and the pressure transducer signal. It is assumed that the power amplifier as well as the pressure transducer act as a straight gain. Also it is assumed that the speed of the motor is proportional to its input voltage and there is no associated inertia. The flow area of the control valves are considered to be linearly related to the angular position of the valve stem. Thus the flow area of the control valves will be proportional to the position of the motor shaft.

Using equation (2.44) to describe the transient behavior of the pressures in the volume, the system equation can be written (Figure 11)

$$\frac{da_1(t)}{dt} = K_1\left[u_1(t) - p_1(t)\right] \tag{4.1a}$$

$$\frac{dp_1(t)}{dt} = \frac{RT_o}{V_1}\left[w_{01}(t) - w_{12}(t)\right] \tag{4.1b}$$

$$\frac{da_2(t)}{dt} = K_2\left[p_2(t) - u_2(t)\right] \tag{4.1c}$$

$$\frac{dp_2(t)}{dt} = \frac{RT_o}{V_2}\left[w_{12}(t) - w_{23}(t)\right] \tag{4.1d}$$

The reference inputs are $u_1(t)$ and $u_2(t)$. The gains associated with the pressure transducer, the power amplifier, and the valve are contained in the constants K_1 and K_2. The mass flow rates are $w_{01}(t)$, $w_{12}(t)$, and $w_{23}(t)$. The volume V_1 is the upstream tank volume and V_2 is the downstream tank volume. It is assumed (Chapter I) that the temperature in the tanks is T_o, and R is the gas constant for air.

In general the mass flow rates through the restrictions will be non-linear functions of the associated pressures. Considering the

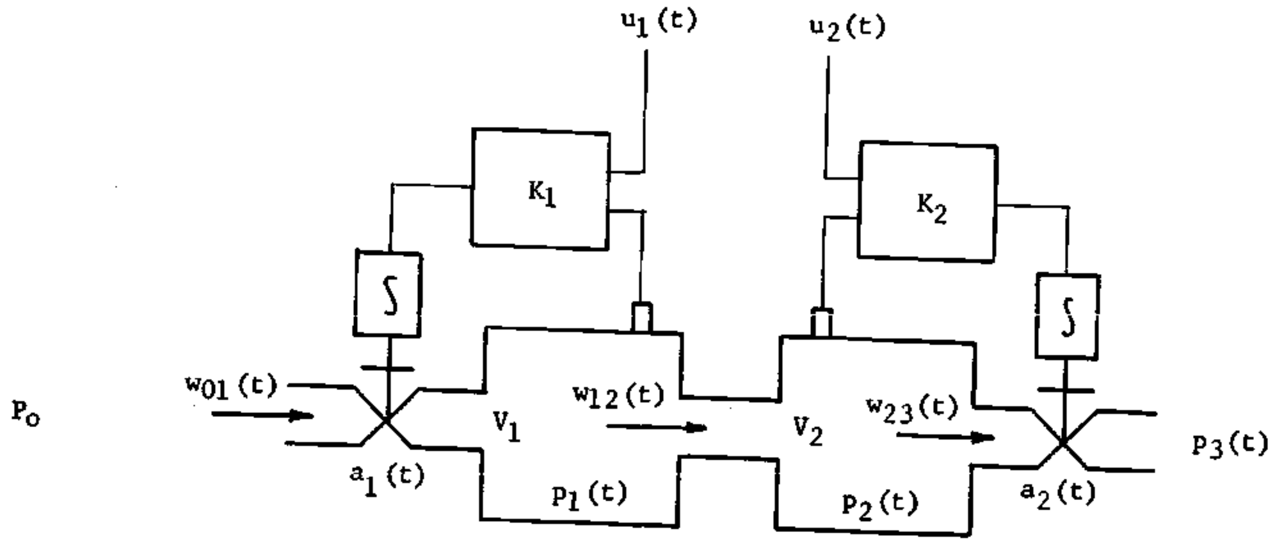

Figure 11. Diagram of Apparatus with Integral Controller

mass flow rates to be only functions of the time varying parameters, equations (4.1) can be written

$$\frac{da_1(t)}{dt} = K_1\left[u_1(t) - p_1(t)\right] \tag{4.2a}$$

$$\frac{dp_1(t)}{dt} = \frac{RT_0}{V_1}\left[w_{01}(a_1(t), p_1(t)) - w_{12}(p_1(t), p_2(t))\right] \tag{4.2b}$$

$$\frac{da_2(t)}{dt} = K_2\left[p_2(t) - u_2(t)\right] \tag{4.2c}$$

$$\frac{dp_2(t)}{dt} = \frac{RT_0}{V_2}\left[w_{12}(p_1(t), p_2(t)) - w_{23}(a_2(t), p_2(t), p_3(t))\right] \tag{4.2d}$$

The mass flow rate, w_{01}, can be obtained from equation (2.39) as a function of the pressure ratio and the valve area. The mass flow rate, w_{12}, which is the mass flow rate through the nozzle, can be found from equations (2.33) until the flow becomes choked and by equation (2.32) after it becomes choked. The pressure ratio at which the flow becomes choked must be found from equation (2.36). The mass flow rate, w_{23}, is a little more difficult. This is the mass flow rate through the downstream valve. When the downstream valve is subsonic, the mass flow rate through it is a function of p_2 and p_3. The pressure p_3 is the pressure at the inlet to the vacuum pump and it is, in turn, a function of the mass flow rate. In general the relation between the mass flow rate and p_3 is given graphically and not analytically. This leads to difficulties when trying to solve the system equations analytically. The difficulties could possibly be overcome by obtaining an analytical curve fit of the vacuum pump characteristics, but even so it is not likely the mass flow rate could be found as an explicit function of p_2 and p_3.

Also it is possible to consider the inlet to the vacuum pump to be a small volume resulting in the system in Figure 12. The system equations become 5th degree instead of 4th degree, but the advantage is that all the required functions are known explicity since $w_p(t)$ as a function of $p_3(t)$ could be found. If V_3 is small compared to V_1 is small compared to V_1 and V_2, the response of the system in Figure 12 should be very similar to the response of the system in Figure 11.

However for the present case the downstream control valve will be considered to be choked and the mass flow rate through the downstream valve will be independent of the pressure at the vacuum pump inlet. Thus the mass flow rate can be found from equation (2.39).

Considerable simplifications occur when the flow through a nozzle or valve becomes choked. These simplified cases will be used as a starting point to analyze the response of the system.

When the nozzle becomes choked, the flow becomes independent of the pressure in the downstream tank. Therefore the upstream and downstream systems can be analyzed independently. The following two sections do just this.

Upstream Tank System with Integral Controller

This section considers the transient behavior of the upstream tank with an integral control system.

With the nozzle choked, the upstream tank system is independent of the pressure in the downstream tank. This system appears in Figure 13.

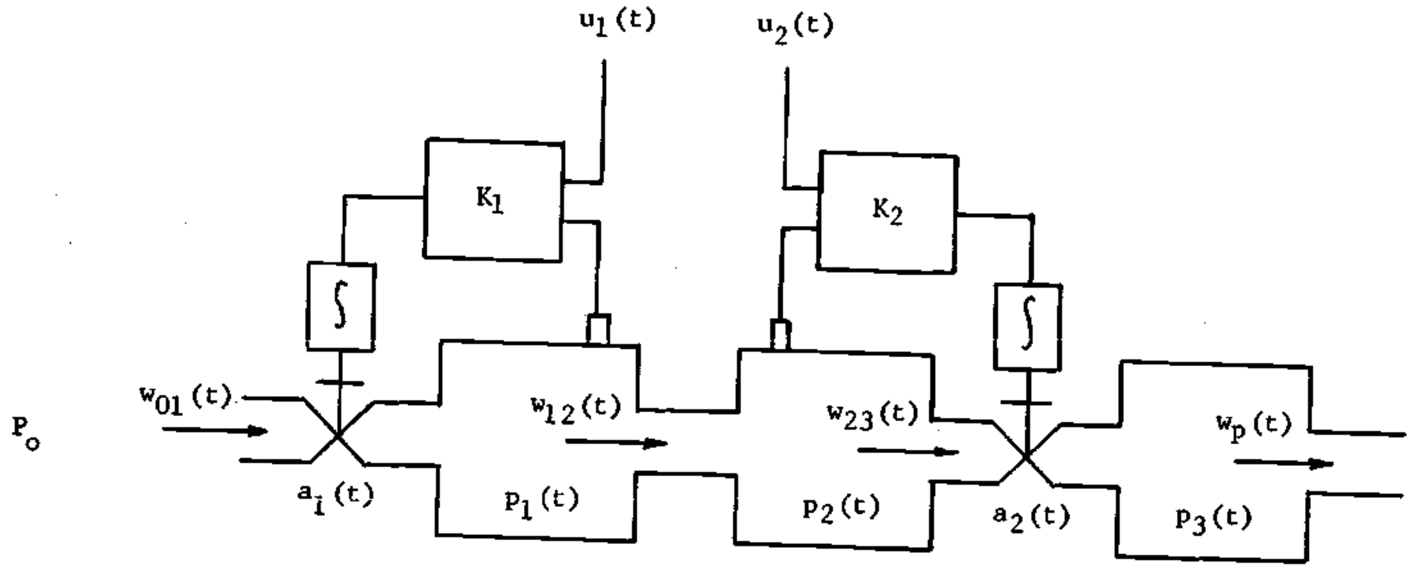

Figure 12. Modified Apparatus with Integral Controller

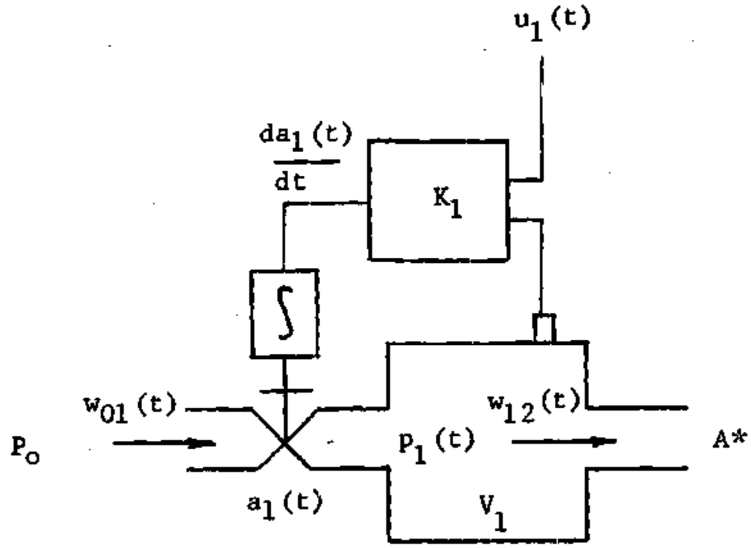

Figure 13. Upstream Tank System
with Integral Controller

The system equations can be written

$$\frac{da_1(t)}{dt} = K_1\left[u_1(t) - p_1(t)\right] \tag{4.3a}$$

$$\frac{dp_1(t)}{dt} = \frac{RT_0}{V_1}\left[w_{01}(a_1(t), p_1(t)) - w_{12}(p_1(t))\right] \tag{4.3b}$$

Since it is assumed that the nozzle is choked, the mass flow rate, w_{12}, is given by equation (2.23). Thus

$$w_{12}(t) = C_3 A^* p_1(t) \tag{4.4}$$

If the assumption is made that the upstream control valve is also choked, then the system equations become linear. Equations (4.3) become

$$\frac{da_1(t)}{dt} = K_1\left[u_1(t) - p_1(t)\right] \tag{4.5a}$$

$$\frac{dp_2(t)}{dt} \simeq \frac{C_3 R T_0}{V_1}\left[P_0 a_1(t) - A^* p_1(t)\right] \tag{4.5b}$$

Equations (4.5) can be reduced to a convenient form if $u_1(t)$ is assumed to be constant. There are two cases of interest where $u_1(t)$ would be constant. If the response to a step input is desired, then u_1 can be assumed to be a constant with the initial conditions supplying the step input. If the system is to be considered a regulator, then u_1 would be constant, and the response of the system to a disturbance would be analyzed.

Let $u_1(t)=U_1$, and define the following dimensionless variables as

$$p(t) = \frac{p_1(t)}{U_1} \tag{4.6}$$

$$a(t) = \frac{P_1}{U_1 A^*}\, a_1(t) \tag{4.7}$$

Equations (4.5) can be written in terms of the variables defined in equation (4.6) and equation (4.7) to give

$$\frac{da(t)}{dt} = \frac{P_0 K_1}{A^*}\left[1 - p(t)\right] \tag{4.8a}$$

$$\frac{dp(t)}{dt} = \frac{C_3 R T_0 A^*}{V_1}\left[a(t) - p(t)\right] \tag{4.8b}$$

Assuming equations (4.8) are stable (as will be determined later), equating the left-hand side to zero gives the steady-state values of a(t) and p(t).

Denoting the steady-state points by p_{ss} and a_{ss}, the equations are

$$\frac{p_o K_1}{A^*}(1 - p_{ss}) = 0 \tag{4.9a}$$

$$\frac{C_3 R T_o A^*}{V_1}(a_{ss} - p_{ss}) = 0 \tag{4.9b}$$

The solutions to equations (4.9) are

$$p_{ss} = 1 \tag{4.10a}$$

$$a_{ss} = 1 \tag{4.10b}$$

Let two new variables be defined

$$\Delta p(t) = p(t) - 1 \tag{4.11a}$$

$$\Delta a(t) = a(t) - 1 \tag{4.11b}$$

The variables $\Delta p(t)$ and $\Delta a(t)$ have a physical significance. Using the definitions of p(t) and a(t) along with equations (4.10), the steady-state values of the variables $p_1(t)$ and $a_1(t)$ are

$$p_{1ss} = U_1 \tag{4.12a}$$

$$a_{1ss} = \frac{U_1 A^*}{p_o} \tag{4.12b}$$

where p_{1ss} and a_{1ss} are the steady-state values.

The variable $\Delta p(t)$ can be written

$$\begin{aligned} \Delta p(t) &= p(t) - 1 \\ &= \frac{p_1(t)}{U_1} - 1 \\ &= \frac{p_1(t) - U_1}{U_1} \\ &= \frac{p_1(t) - p_{1ss}}{p_{1ss}} \end{aligned} \tag{4.13}$$

Thus $\Delta p(t)$ is the fractional deviation of the upstream tank pressure from its steady-state value.

Similarily, $\Delta a(t)$ can be written

$$\Delta a(t) = \frac{a_1(t) - a_{1ss}}{a_{1ss}} \tag{4.14}$$

The variable $\Delta a(t)$ is the fractional deviation of $a_1(t)$ from its steady-state value.

The system equations (4.8) can be written in terms of the new variables as

$$\frac{d\Delta a(t)}{dt} = -\frac{P_o K_1}{A^*} \Delta p(t) \tag{4.15a}$$

$$\frac{d\,\Delta p(t)}{dt} = \frac{C_3 R T_o A^*}{V_1}\left[\Delta a(t) - \Delta p(t)\right] \tag{4.15b}$$

Differentiating equation (4.15b) and combining it with equation (4.15a) yields

$$\frac{d^2\Delta p(t)}{dt^2} + \frac{C_3 R T_o A^*}{V_1}\frac{d\Delta p(t)}{dt} + \frac{P_o K_1 C_3 R T_o}{V_1}\Delta p(t) = 0 \tag{4.16}$$

Equation (4.16) can be written in the form,

$$\frac{d^2\Delta p(t)}{dt^2} + 2\zeta\omega_N \frac{d\Delta p(t)}{dt} + \omega_N^2 \Delta p(t) = 0 \tag{4.17}$$

where

$$\omega_N = \left[\frac{P_o K_1 C_3 R T_o}{V_1}\right]^{\frac{1}{2}} \tag{4.18}$$

$$\zeta = \frac{A^*}{2}\left[\frac{C_3 R T_o}{V_1 K_1 P_o}\right]^{\frac{1}{2}} \tag{4.19}$$

Equation (4.17) is the well known second order differential equation. The response is governed by the initial condition, the damping ratio, ζ, and the natural frequency, ω_n. For any given set of initial conditions, the amplitude of the response is determined by the damping ratio. The natural frequency is a time scale and has no effect on the amplitude.

Before it can be determined what control system parameters should be selected to give the best results, a performance criteria needs to be established. One obvious procedure would be to assume a typical reference input and then evaluate the systems response to that input. However this control system was intended primarily to be a regulator and the response to a change in the reference input would not be particularly important so long as the response was reasonably well behaved. And since the system was intended to be a regulator, the control systems ability to recover from a disturbance would be of interest. One likely source of a disturbance would be the application of the control jet during operation. It was decided then that the control systems response to a step application of the control jet would be the subject of analysis to determine the best control system.

Consider the fluidic model shown in Figure 6. If the main jet, with area A_m, is choked then the mass flow rate will be

$$w_m = C_3 p_1 A_m \tag{4.20}$$

Let A_J be the control jet area. When the control jet is opened an additional mass flow rate of

$$w_J = C_3 p_1 A_J \tag{4.21}$$

results.

The total mass flow rate between the tanks is

$$W_{12} = w_m + w_J = C_3 p_1 (A_m + A_J) \tag{4.22}$$

Since the mass flow is proportional to the sum of the areas, the application can then be reflected by a step change in the area A* between the two tanks. Therefore the control systems will be evaluated according to their response to a step change in A*.

It is necessary now to find what initial conditions correspond to the sudden application of the control jet.

Assume that the system was at its steady-state values before the application of the disturbance. Then for time before the disturbance

$$\Delta p(t) = 0 \quad , t < 0 \tag{4.23}$$

Since $\Delta p(t) = \frac{p_1(t)}{u} - 1$, $\Delta p(t)$ does not change in value for a step change in A*. Thus

$$\Delta p(t) = 0 \quad , t = 0 \tag{4.24}$$

Let the value of the exit area of the upstream tank, A*, before the application of the control jet be

$$A^* = A_0^* \tag{4.25}$$

Let the area of the control jet be defined as a fraction of the nozzle area such that after the application of the control jet, the area is

$$A^* = f A_o^* \quad , t \geq 0 \tag{4.26}$$

The natural frequency is

$$\omega_N = \left[\frac{p_o K_1 C_3 R T_o}{V_1}\right]^{\frac{1}{2}} \tag{4.27}$$

which does not depend on A* and therefore is not altered by a change in A*.

The damping ratio is

$$\zeta = \frac{A^*}{2}\left[\frac{C_3 R T_o}{V_1 K_1 p_o}\right]^{\frac{1}{2}} \quad , t<0 \tag{4.28}$$

which is proportioned to A*. After the application of the jet, the damping ratio becomes

$$\zeta = \frac{f A_o^*}{2}\left[\frac{C_3 R T_o}{V_1 K_1 P_o}\right]^{\frac{1}{2}} , t \geq 0 \tag{4.29}$$

The quantity $\Delta a(t)$ is

$$\Delta a(t) = \frac{P_o\, a_1(t)}{U_1 \; A_o^*} - 1 = 0 \quad , \quad t<0 \tag{4.30}$$

From equation (4.30), it can be obtained

$$a_1(t) = \frac{U_1 A_o^*}{P_o} \quad , \quad t<0 \tag{4.31}$$

The quantity, $\Delta a(t)$, for $t \geq 0$, is

$$\Delta a(t) = \frac{P_o\, a_1(t)}{U_1 f\, A_o^*} - 1 \quad , \; t \geq 0 \tag{4.32}$$

Using equation (4.31) and (4.32) to find $\Delta a(t)$ at $t=0$ gives

$$\Delta a(0) = \frac{1}{f} - 1$$

or

$$\Delta a(0) = \frac{1-f}{f} \tag{4.33}$$

Equations (4.15) can be written in terms of ζ and ωn to give

$$\frac{d\Delta a(t)}{dt} = -\frac{\omega_N}{\zeta}\, \Delta p(t) \tag{4.34a}$$

$$\frac{d\Delta p(t)}{dt} = 2\zeta\,\omega_N\left[\Delta a(t) - \Delta p(t)\right] \tag{4.35b}$$

Putting the value of $\Delta a(0)$ from equation (4.33) and the value of $\Delta p(0)$ from equation (4.24) into equation (4.32b) results in

$$\frac{d\Delta p(0)}{dt} = 2\zeta\omega_N\left[\Delta a(0) - \Delta p(0)\right]$$

$$= \frac{2\zeta\,\omega_N\,(1-f)}{f} \tag{4.36}$$

In summary, the equation describing the transient pressure changes in the upstream tank is

$$\frac{d^2\Delta p(t)}{dt^2} + 2\zeta\omega_N \frac{d\Delta p(t)}{dt} + \omega_N^2\, \Delta p(t) = 0 \tag{4.18}$$

with the initial conditions

$$\Delta p(0) = 0 \tag{4.24}$$

$$\frac{d\Delta p(t)}{dt} = \frac{2\zeta\omega_N(1-f)}{f} \tag{4.36}$$

Figures 14 and 15 show the solution of these equations for various ζ and f. The natural frequency, ωn, the damping ratio, ζ, and the area increase factor, f, determine the response of the differential equation in equation (4.18). Consider that some f is chosen as typical and it is desired to find the best response possible with the proper choice of ζ and ωn. The quantities ζ and ωn are functions of many factors. The temperature, T_0, the supply pressure, P_0, and the constants, C_3, and R, are considered to be fixed. Also it will be assumed that the area, A*, is fixed. The only two variables that can be manipulated to achieve the best ζ and ωn are the tank volume, V_1, and the feedback gain, K_1. The tank volume V_1 is a variable since this is one of the quantities to be specified by this design. The gain, K_1, represents the gain of several elements in the feedback circuit. However, since a power amplifier is included in the feedback circuit, it will be assumed that the amplifier gain can be varied enough to allow any overall gain to be achieved.

The strategy for selecting the proper K_1, and V_1 will be to assume that no matter what V_1 is chosen, a K_1 can be selected that will allow any ζ to be obtained. An equation to give the value of K_1 necessary to produce any particular ζ can be obtained by solving equation (4.20) for K_1. This is

$$K_1 = \frac{C_3 R T_0}{V_1 P_0}\left(\frac{A^*}{2\zeta}\right)^2 \tag{4.37}$$

Using equation (4.37) to eliminate K_1 from equation (4.20) yields

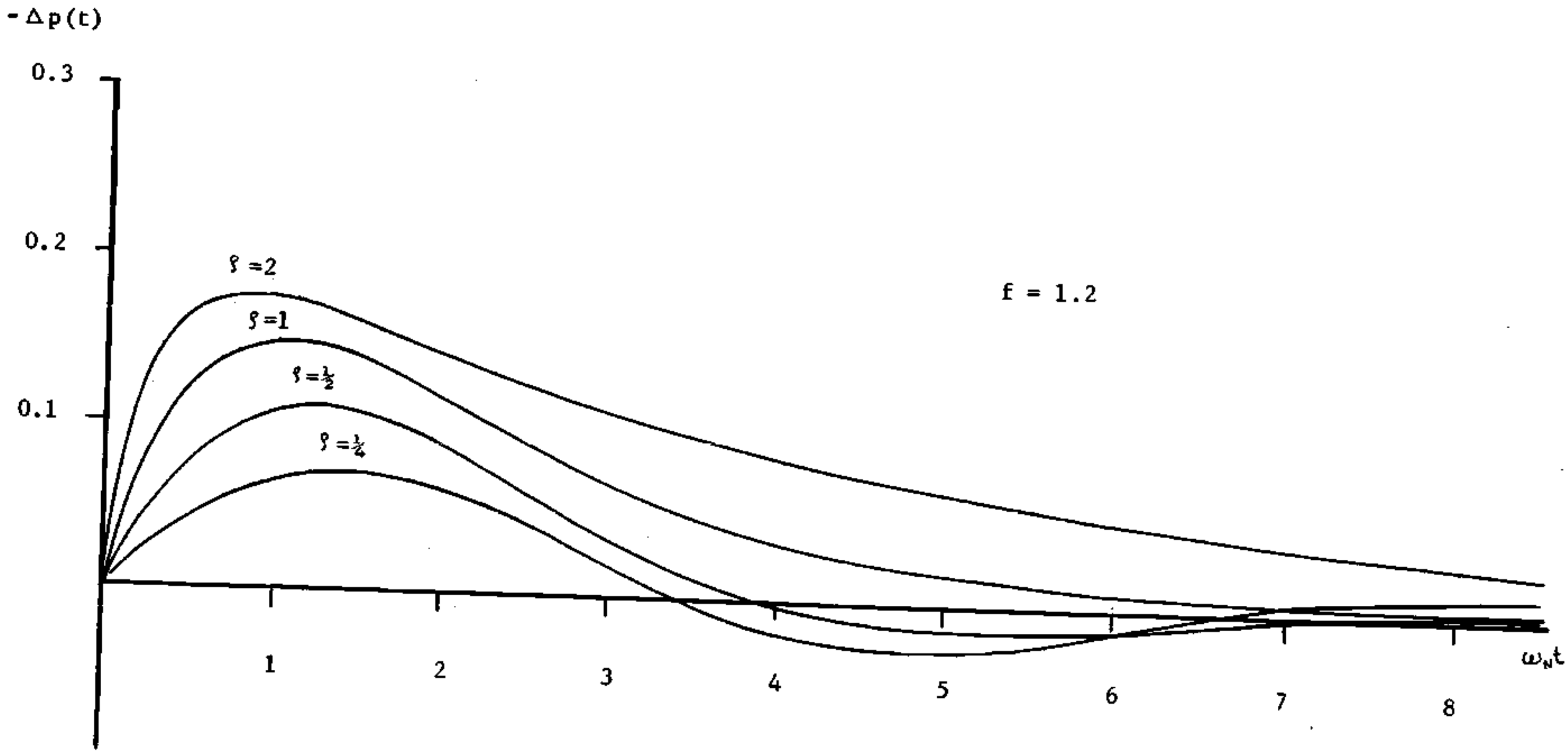

Figure 14. Upstream Pressure Response with Integral Controller, ξ Variable

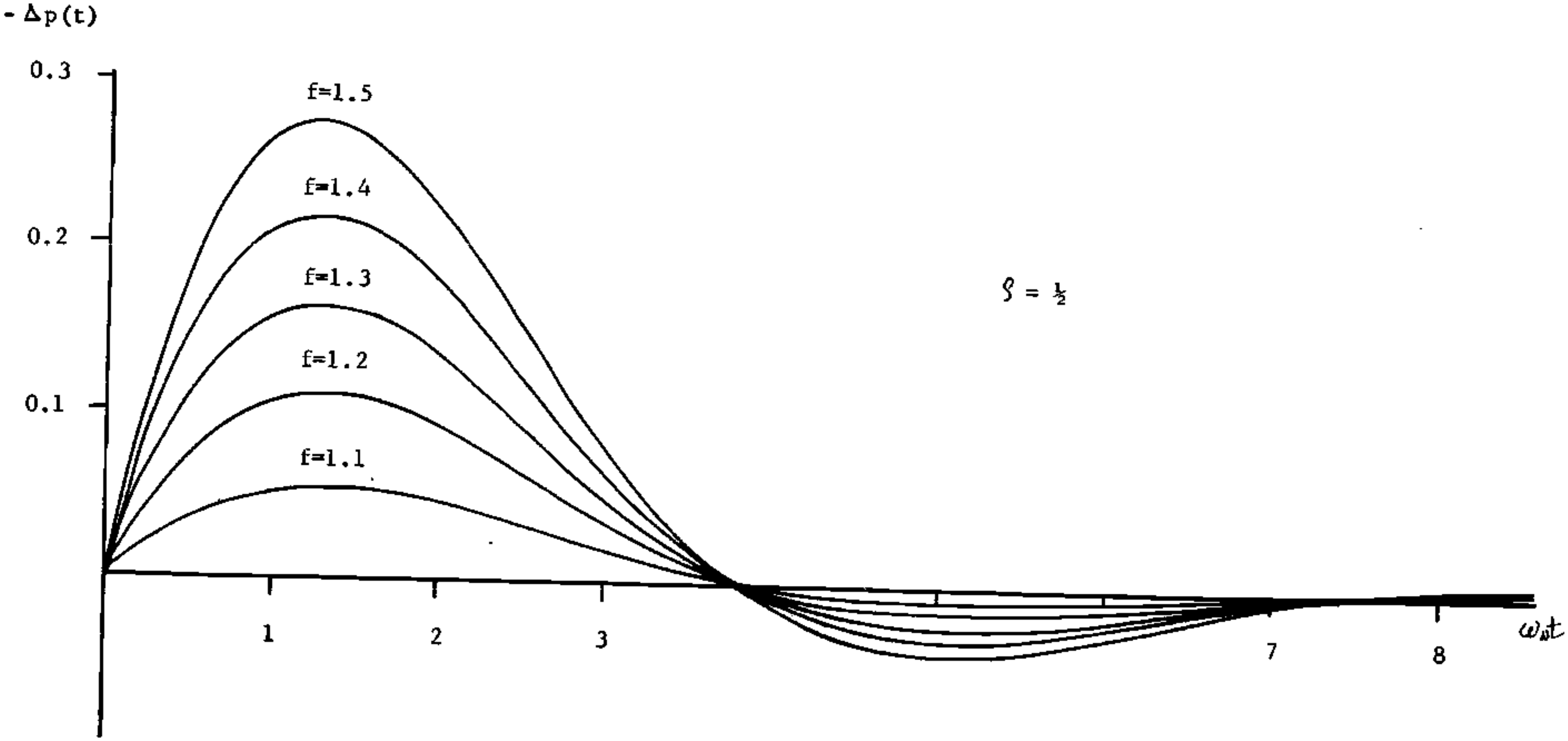

Figure 15. Upstream Pressure Response with Integral Controller, f Variable

$$\omega_N = \frac{C_3 R T_o A^*}{2 \zeta V_1} \tag{4.38}$$

with ωn in radians/sec.

Following the strategy of holding ζ constant (by appropriate choice of K_1) while making ωn as large as possible calls for, according to equation (4.38), making V_1 as small as possible. And as seen from equation (4.37), the appropriate choice of K_1 becomes increasingly large as V_1 becomes small. It is obvious that this strategy cannot be carried to the ultimate conclusion of making $V_1 = 0$ and $K_1 = \infty$. What then is the best design with the available information? If it were possible to find a maximum gain, K_{max}, then the best design would be to calculate the volume from equation (4.37). This would be

$$V_1 = \frac{C_3 R T_o}{P_o K_{max}} \left(\frac{A^*}{2 \zeta} \right)^2 \tag{4.39}$$

because this would result in the largest possible ωn while still exercising control over ζ. However at this point it would be difficult to obtain a reliable value for the maximum feedback gain, K_1.

There are other possible equipment limitations that might be easier to estimate and indeed may be more of a limitation than that of a maximum feedback gain. Saturation of the feedback elements seems a likely candidate and will be the one used here. Whether it be the amplifier, valve motor, or some other element in the feedback circuit that saturates, it will be assumed to be manifested in a upper limit on the speed at which the valve can be driven.

Let the maximum rate at which the valve stem can be turned be designed by S. Let the Maximum effective valve area be A_m and let the valve have N terms from full open to full closed. Assume that the effective valve area is proportional to the number of turns the stem is from shutoff. If r represents the number of turns the stem is from shutoff, then the effective area, A, can be expressed

$$A = \frac{r A_m}{N} \tag{4.40}$$

Differentiating equation (4.40) yields

$$\frac{dA}{dt} = \frac{A_m}{N}\frac{dr}{dt} \tag{4.41}$$

Using equation (4.41), it can be written

$$\left(\frac{dA}{dt}\right)_{sat} = \frac{A_m S}{N} \tag{4.42}$$

where $(dA/dt)_{sat}$ is the valve area rate at saturation.

Equation (4.38) becomes

$$\left(\frac{dA}{dt}\right)_{sat} = \frac{A_m S}{60 N} \tag{4.43}$$

where $(dA/dt)_{sat}$ is in in^2/sec, A_m is in in.^{2}, S in RPM, and N is in revolutions. This then will allow the area rate to be calculated from a knowledge of the valve orifice area and the maximum motor speed.

Since the system equations are most conviently utilized in terms of the dimensionless parameter, it would be desirable to relate the actual maximum area rate to a maximum allowable value of some dimensionless parameter

The definition of $\Delta a(t)$ is

$$\Delta a(t) = \frac{p_o}{U_1 A^*} a_1(t) - 1 \tag{4.44}$$

Differentiating (4.44) gives

$$\frac{d\Delta a(t)}{dt} = \frac{p_o}{U_1 A^*} \frac{da_1(t)}{dt} \tag{4.45}$$

Thus we obtain

$$\left(\frac{d\Delta a(t)}{dt}\right)_{sat} = \frac{p_o}{U_1 A^*} \left(\frac{da_1(t)}{dt}\right)_{sat} \tag{4.46}$$

Equation (4.46) gives the maximum $d\Delta a(t)/dt$ that will be allowed.

When the control jet is applied, it is desirable to have the absolute value of $\frac{d\Delta a(t)}{dt}$ at its maximum value equal to the saturation value in equation (4.46). Thus it is necessary to find where the maximum $\frac{d\Delta a(t)}{dt}$ occurs.

From equation (4.34a) it is found that

$$\left|\frac{d\Delta a(t)}{dt}\right|_{max} = \frac{\omega_N}{\delta}\left|\Delta p(t)\right|_{max} \tag{4.47}$$

Thus the maximum $\frac{d\Delta a(t)}{dt}$ can be found from knowing the maximum $\Delta p(t)$.

To find the maximum $\Delta p(t)$ consider the system equation,

$$\frac{d^2\Delta p(t)}{dt^2} + 2\zeta\omega_N \frac{d\,\Delta p(t)}{dt} + \omega_N^2\,\Delta p(t) = 0 \tag{4.48}$$

with the initial conditions

$$\Delta p(0) = 0 \tag{4.49}$$

$$\frac{d\,\Delta p(0)}{dt} = \frac{2\zeta\omega_N(1-f)}{f} \tag{4.50}$$

If the damping ratio is less than 1, then the solution is

$$\Delta p(t) = \frac{2(1-f)}{f\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_N t} \sin \omega_N \sqrt{1-\zeta^2}\, t \tag{4.51}$$

To find where the maximum $\Delta p(t)$ occurs, differentiate equation (4.51) and set the derivative equate to zero.

$$\frac{d\Delta p(t)}{dt} = \frac{2\zeta\omega_N(1-f)}{f\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_N t} \sin\left(\omega_N\sqrt{1-\zeta^2} + \phi\right) \tag{4.52a}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} \tag{4.52b}$$

The derivative is zero when

$$\omega_N\sqrt{1-\zeta^2}\, t + \phi = N\pi \ , \ N = 0, 1, 2 \cdots \tag{4.53a}$$

or

$$t = \frac{N\pi - \phi}{\omega_N\sqrt{1-\zeta^2}} \ , \ N = 0, 1, 2 \cdots \tag{4.53b}$$

As can be seen from Figure 14, the largest peak is the first. Then the maximum occurs at n=0; and this value of t is

$$t = \frac{-\phi}{\omega_N \sqrt{1-\zeta^2}} \qquad (4.54)$$

Putting this value of t into equation (4.51) yields

$$\Delta p_{max} = \frac{2\zeta(1-f)}{\sqrt{1-\zeta^2}\, f}\, e^{\frac{\zeta\phi}{\sqrt{1-\zeta^2}}} \sin(-\phi) \qquad (4.55)$$

Now let

$$H(\zeta) = \frac{2\zeta}{\sqrt{1-\zeta^2}}\, e^{\frac{\zeta\phi}{\sqrt{1-\zeta^2}}} \sin(-\phi) \qquad (4.56)$$

Equation (4.55) becomes

$$\Delta p_{max} = \left(\frac{1-f}{f}\right) H(\zeta) \qquad (4.57)$$

and Table 7 can be used to find $H(\zeta)$ for several values of ζ.

Table 7. $H(\zeta)$ Functions for Several Values of ζ

ζ	$H(\zeta)$
0	0
.1	.1725
.2	.3024
.3	.4030
.4	.4823
.5	.5463
.6	.5986
.7	.6421
.8	.6784
.9	.7074

All the equations are now available to find the desired volume of the upstream tank.

From equation (3.46), the value of $\left(\frac{d\Delta a(t)}{dt}\right)_{sat}$ is

$$\left(\frac{d\Delta a(t)}{dt}\right)_{sat} = \frac{P_o}{U_1 A_o^*}\left(\frac{da_1(t)}{dt}\right)_{sat} \tag{4.58}$$

and with equation (4.43) this can be written

$$\left(\frac{d\Delta a(t)}{dt}\right)_{sat} = \frac{P_o \; A_m S}{U_1 A_o^* \; 60N} \tag{4.59}$$

which gives the $\frac{d\Delta a(t)}{dt}$ at saturation in terms of more easily estimated parameters.

Now $\left(\frac{d\Delta a(t)}{dt}\right)_{max}$, which is the maximum value of $\frac{d\Delta a(t)}{dt}$ that occurs after application of control jet, needs to be found.

Equation (3.47) gives

$$\left|\frac{d\Delta a(t)}{dt}\right|_{max} = \frac{\omega_N}{\zeta}\left|\Delta p(t)\right|_{max} \tag{4.60}$$

and $\left|\Delta p(t)\right|_{max}$ can be found from equation (4.57) to give

$$\left|\frac{d\Delta a(t)}{dt}\right|_{max} = \frac{\omega_N}{\zeta}\left|\left(\frac{1-\zeta}{\zeta}\right)H(\zeta)\right| \tag{4.61}$$

As stated in deriving equation (4.38) it is assumed that no matter what V_1 is chosen, a K_1 can be found that will permit any ζ to be obtained. With this assumption, ω_N could then be written in terms of ζ and V_1 which is equation (4.38). Putting this into equation (4.62) gives

$$\left|\frac{d\,\Delta a(t)}{dt}\right|_{max} = \frac{f C_3 R T_o A_o^*}{2\rho^2 V_1}\left|\left(\frac{1-f}{f}\right) H(\rho)\right| \tag{4.62}$$

To prevent saturation, it must be true that

$$\left|\frac{d\Delta a(t)}{dt}\right|_{max} \leq \left(\frac{d\Delta a(t)}{dt}\right)_{sat} \tag{4.63}$$

The equality corresponds to the largest ω_N and smallest V_1; therefore equating equation (4.59) and equation (4.62) gives

$$\frac{f C_3 R T_o A_o^*}{2\rho^2 V_1}\left|\left(\frac{1-f}{f}\right) H(\rho)\right| = \frac{P_o A_m S}{U_1 A_o^* 60 N} \tag{4.64}$$

Solving for V_1 yields

$$V_1 = \frac{C_3 R T_o A_o^{*2}\left|(1-f)\,H(\rho)\right| U_1 60 N}{2\rho^2 P_o A_m S} \tag{4.65}$$

This equation can be used to calculate the value of V_1. Notice that equation (4.65) contains U_1 which would indicate that the value of V_1 is dependent on the pressure at which the system is operating. For the fluidic models the pressure varies over a range of several orders of magnitude. Even though it is true that V_1 does depend on U_1, it is not necessarily true that this will cause V_1 to vary over several orders of magnitude. Suppose that a_{1ss} is the steady-state position corresponding to U_1. Now let A_m be some constant, γ, times this equilibrium value. Thus

$$A_m = \gamma a_{1ss}$$

This implies that the valve is selected with at least some consideration as to the range of operation. Putting this into equation (4.65) gives

$$V_1 = \frac{C_3 R T_o A_o^{*2} \left|(1-f)\, H(\zeta)\right| U_1\, 60N}{2\zeta^2 P_o \gamma\, a_{1ss} S} \tag{4.66}$$

At equilibrium it is known that (equation (4.12b)

$$P_o a_{1ss} = U_1 A_o^* \tag{4.67}$$

and equation (4.66) becomes

$$V_1 = \frac{C_3 R T_o A_o^* \left|(1-f)\, H(\zeta)\right| 60N}{2\zeta^2 \gamma S} \tag{4.68}$$

As an example consider calculating the volume for a typical system. Since most of the upstream pressures call for a small orifice, let a needle valve be used. For a needle valve, a value of N=10 would be typical. Let the damping ratio be 0.7. Since needle valves stems for vacuum application are packed very tightly, a high torque motor with gear reduction will probably be required. Therefore let the maximum speed be 60 RPM. If parallel valves are used with each successive valve having an orifice area of 10 times the previous, the value of γ would be between 1 and 10 depending on the operating point. However, considering the worst case (the case giving the largest volume), γ=1 will be used. Therefore the constants will be

$$\zeta = .7$$
$$A_o^* = .2 \text{ in}^2$$
$$\gamma = 1$$
$$S = 60 \text{ RPM}$$
$$N = 10$$
$$f = 1.2$$

This gives

$$V_1 = .239 \text{ ft}^3 \tag{4.69}$$

The natural frequency can be found from equation (4.38). In cycles/sec this is

$$\omega_N = \frac{.7274 \, f A_o^*}{2 \zeta V_1} \tag{4.70}$$

with A_o^* in in^2 and V_1 in ft^3. For this example, it is

$$\omega_N = .5217 \text{ cycles/sec} \tag{4.71}$$

The value of Δp_{max} was fixed by the damping ratio. Its value is

$$\Delta p_{max} = -.107 \tag{4.72}$$

or

$$\Delta p_{max} = -10.7\% \tag{4.73}$$

Downstream Tank System with Integral Controller

The downstream tank is similar to the upstream tank.

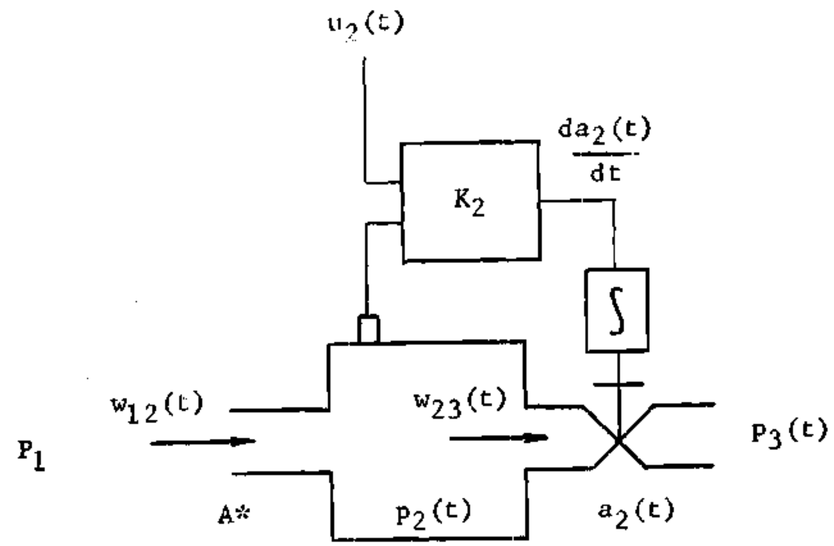

Figure 16. Downstream Tank System with Integral Controller

The system equations can be written (Figure 16)

$$\frac{da_2(t)}{dt} = K_2[u_2(t) - p_2(t)] \tag{4.74a}$$

$$\frac{dp_2(t)}{dt} = \frac{RT_0}{V_2}[w_{12}(p_1, p_2) - w_{23}(a_2, p_3)] \tag{4.74b}$$

Assume that the nozzle and the downstream control valve are choked, and assume that the upstream tank pressure, $p_1(t)$, is equal to a constant, P_1.

The mass flow rates for choked flow can be obtained from equation (2.28). The resulting equations are

$$\frac{da_2(t)}{dt} = K_2[u_2(t) - p_2(t)] \tag{4.75a}$$

$$\frac{dp_2(t)}{dt} = \frac{C_3 R T_0}{V_2}\left[P_1 A^* - p_2(t)\, a_2(t)\right] \tag{4.75b}$$

The principle difference between the equation for the upstream tank and the equations above for the downstream tank is that in equation (4.75b) there is a product of two state variables. Thus, unlike the upstream tank analysis of the last section, the equations for the downstream tank are not linear.

An analytical solution to these nonlinear equations is difficult to find. An analog computer was therefore used to find the solution.

In order to reduce the number of variables, dimensionless quantities were formed. Letting the reference input, $u_2(t)$, be a constant, U_2, the equations (4.75) can be written

$$\frac{U_2}{P_1 A^*}\frac{da_2(t)}{dt} = \frac{K_2 U_2^2}{P_1 A^*}\left[\frac{p_2(t)}{U_2} - 1\right] \tag{4.76a}$$

$$\frac{1}{U_2}\frac{dp_2(t)}{dt} = \frac{C_3 R T_0 P_1 A^*}{V_2 U_2}\left[1 - \frac{p_2(t)}{U_2}\ \frac{U_2}{P_1 A^*}\, a_2(t)\right] \tag{4.76b}$$

Let the dimensionless quantities be defined

$$p(t) = \frac{p_2(t)}{U_2} \tag{4.77}$$

$$a(t) = \frac{U_2}{P_1 A^*}\, a_2(t) \tag{4.78}$$

Using these dimensionless quantities, equation (4.76) can be written

$$\frac{da(t)}{dt} = \frac{K_2 U_2^2}{P_1 A^*}\left[p(t) - 1\right] \tag{4.79a}$$

$$\frac{d\,p(t)}{dt} = \frac{C_3 R T_o P_1 A^*}{V_2 U_2}\left[1 - p(t)a(t)\right] \tag{4.79b}$$

If equations (4.79) are linearlized, the resulting second order system of equations will have a natural frequency and damping ratio given by

$$\omega_N = \left[\frac{C_3\, R T_o K_2 U_2}{V_2}\right]^{\frac{1}{2}} \tag{4.80}$$

$$\zeta = \frac{P_1 A^*}{2 U_2}\left[\frac{C_3\, R T_o}{V_2 K_2 U_2}\right]^{\frac{1}{2}} \tag{4.81}$$

With the constants of equations (4.79) written in terms of these parameters, the result is

$$\frac{da(t)}{dt} = \frac{\omega_N}{2\zeta}\left[p(t) - 1\right] \tag{4.82a}$$

$$\frac{dp(t)}{dt} = 2\zeta\omega_N\left[1 - p(\tau)a(\tau)\right] \tag{4.82b}$$

If equations (4.70) are time scaled by the relation

$$\tau = \omega_N t \tag{4.83}$$

Then equations (4.70) can be written

$$\frac{da(\tau)}{d\tau} = \frac{1}{2\zeta}\left[p(\tau) - 1\right] \tag{4.84a}$$

$$\frac{d\,p(\tau)}{d\tau} = 2\zeta\left[1 - p(\tau)a(\tau)\right] \tag{4.84b}$$

The variables $\Delta a(\tau)$ and $\Delta p(\tau)$ can also be introduced. Let these be defined as

$$\Delta a(\tau) = a(\tau) - 1 \qquad (4.85)$$

$$\Delta p(\tau) = p(\tau) - 1 \qquad (4.86)$$

These variables have the same interpretation as did their counterparts in the upstream tank. The quantity $\Delta p(\tau)$ represents the fractional deviation of the pressure in the downstream tank from the reference input, and the quantity $\Delta a(\tau)$ is the fractional deviation the downstream control valve area is from its steady-state position.

Equations (4.84) can be written in terms of these new variables

$$\frac{d\Delta a(\tau)}{d\tau} = \frac{1}{2\zeta}\,\Delta p(\tau) \qquad (4.87a)$$

$$\frac{d\Delta p(\tau)}{d\tau} = 2\zeta\left[\Delta p(\tau)\,\Delta a(\tau) + \Delta a(\tau) + \Delta p(\tau)\right] \qquad (4.87b)$$

It is desired now to find the initial conditions that correspond to a sudden application of the control jet. Let A_o^* be the inlet area before the jet is applied and let fA_o^* be the area after the control jet is applied.

The natural frequency is

$$\omega_N = \left[\frac{C_3\, R T_o K_2 U_2}{V_2}\right]^{\frac{1}{2}} \qquad (4.88)$$

which is unchanged for an increase in A*.

The damping ratio, ζ , is

$$\zeta = \frac{P_1 A^*}{2 U_2} \left[\frac{C_3 R T_0}{V_2 K_2 U_2} \right]^{\frac{1}{2}} \tag{4.89}$$

After the application of the control jet, the damping ratio becomes

$$\zeta = \frac{f P_1 A_0^*}{2 U_2} \left[\frac{C_3 R T_0}{V_2 K_2 U_2} \right]^{\frac{1}{2}} \tag{4.90}$$

The quantity $\Delta p(\tau)$ is

$$\Delta p(\tau) = \frac{p_2(\tau)}{U_2} - 1 \tag{4.91}$$

Since it is assumed that the system is at equilibrium for $t<0$, then

$$\Delta p(\tau) = 0 \quad , \quad \tau < 0 \tag{4.92}$$

because $p_2(\tau)=U_2$ at equilibrium. And $\Delta p(\tau)$ is unchanged as the jet is applied because $\Delta p(\tau)$ does not depend on A^*.

The quantity $\Delta a(\tau)$ is

$$\Delta a(\tau) = \frac{U_2}{P_1 A^*} a_2(\tau) - 1 \tag{4.93}$$

at equilibrium and therefore for $t<0$,

$$a_2(\tau) = \frac{P_1 A_0^*}{U_2} \quad , \quad \tau < 0 \tag{4.94}$$

For $\tau \geq 0$, $\Delta a(\tau)$ is

$$\Delta a(\tau) = \frac{U_2 a_2(\tau)}{P_1 f A_o^*} - 1 , \quad \tau \geq 0 \tag{4.95}$$

Using equation (4.95) and (4.94), $\Delta a(\tau)$ at t=0 is

$$\Delta a(0) = \frac{1-f}{f} \tag{4.96}$$

Therefore, the system equations are

$$\frac{d\Delta a(\tau)}{d\tau} = \frac{1}{2\zeta} \Delta p(\tau) \tag{4.87a}$$

$$\frac{d\Delta p(\tau)}{d\tau} = -2\zeta\left[\Delta p(\tau)\,\Delta a(\tau) + \Delta a(\tau) + \Delta p(\tau)\right] \tag{4.87b}$$

with the initial conditions

$$\Delta a(0) = \frac{1-f}{f} \tag{4.96}$$

$$\Delta p(0) = 0 \tag{4.97}$$

Notice that these equations only depend on the damping ratio, ζ, and the fractional area increase, f.

Figure 17 and 18 give the response of the downstream system for several values of ζ and f.

The same procedure will be followed to find the size of the downstream tank as was followed to find the volume of the upstream tank. The size of the tank will be made large enough so that the downstream valve motor will not saturate under a sudden application of the control jet.

It is assumed that the feedback gain, K_2, can be made as large or as small as necessary to give any desired damping ratio, ζ. The

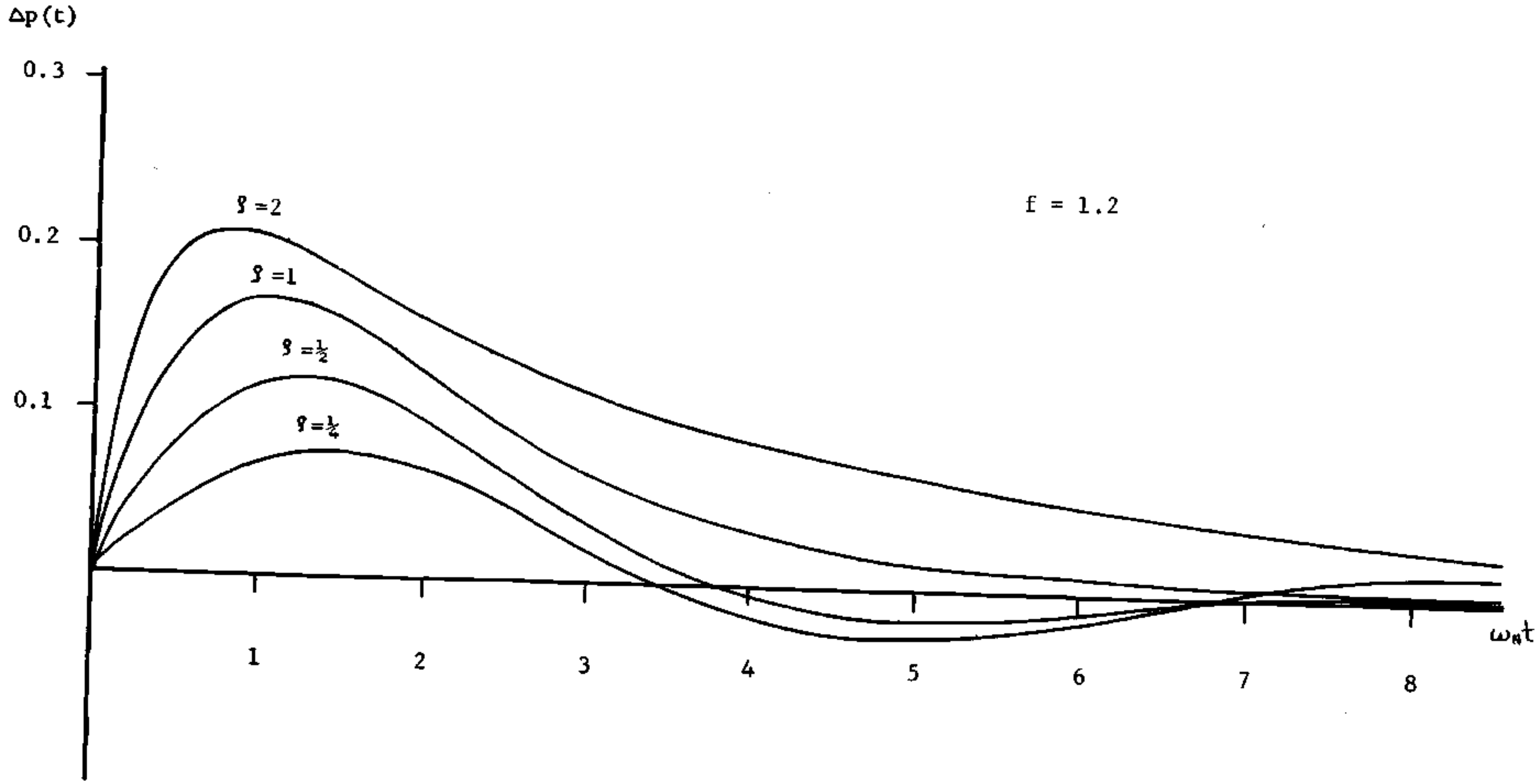

Figure 17. Downstream Pressure Response with Integral Controller, ς Variable

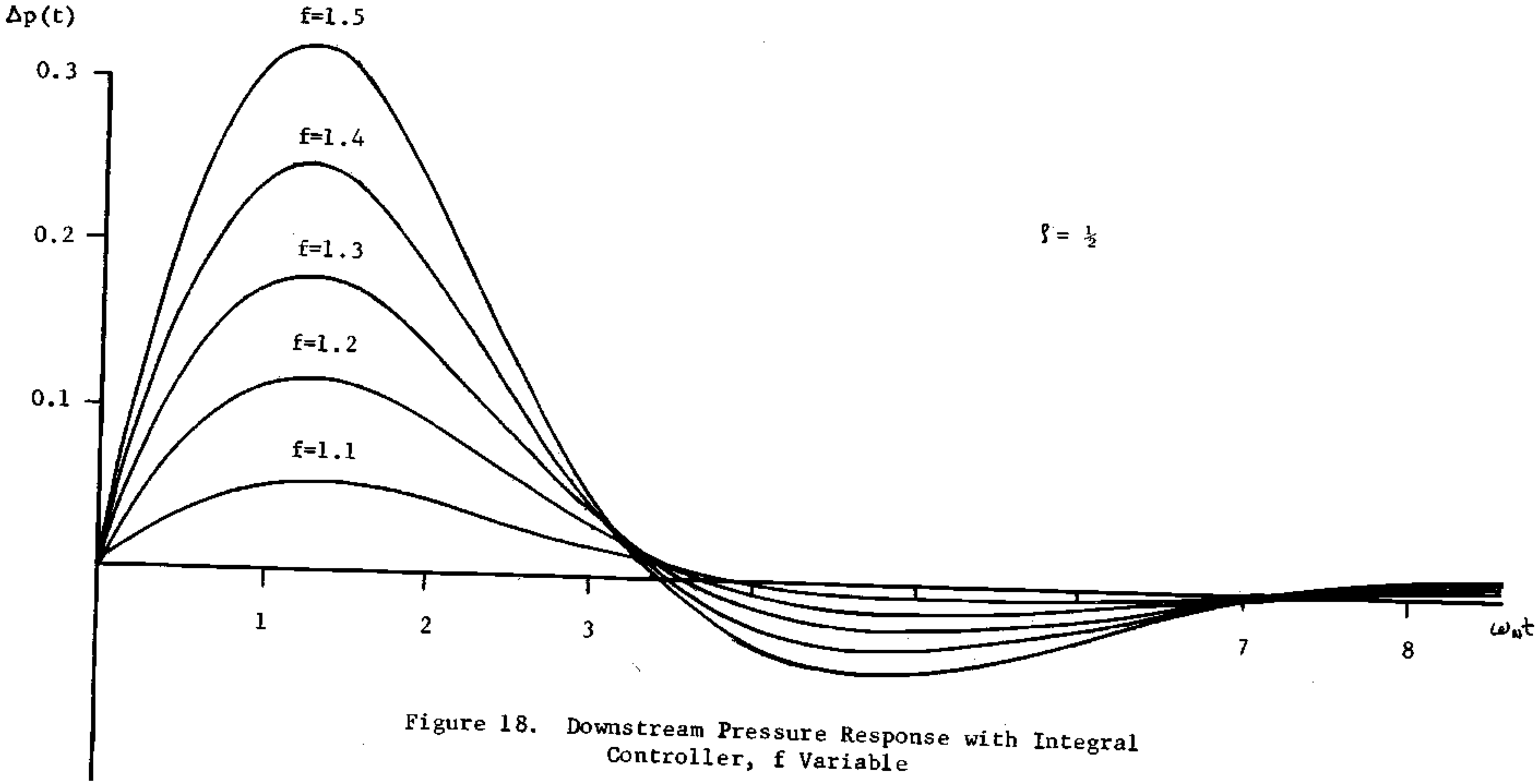

Figure 18. Downstream Pressure Response with Integral Controller, f Variable

equation giving the damping ratio in terms of the downstream system components can be solved for the feedback gain to give

$$K_2 = \frac{C_3 R T_0}{V_2 U_2} \left[\frac{P_1 A^*}{2 U_2 \zeta} \right]^2 \tag{4.98}$$

If equation (4.98) is used to eliminate K_2 from the natural frequency, the result is

$$\omega_N = \frac{C_3 R T_0 P_1 A^*}{V_2 2 U_2 \zeta} \tag{4.99}$$

Now from equation (4.99), the downstream tank volume can be obtained in terms of ωn, ζ, and the other system parameters as

$$V_2 = \frac{C_3 R T_0 P_1 A^*}{\omega_N 2 U_2 \zeta} \tag{4.100}$$

The natural frequency at which the downstream valve motor saturates needs to be found.

Equation (4.43) gives the maximum area rate in terms of valve and motor characteristics. Thus

$$\left(\frac{da_2}{dt}\right)_{sat} = \frac{A_m S}{60 N} \tag{4.101}$$

And then from the definition of $\Delta a(t)$

$$\left(\frac{d\Delta a(t)}{dt}\right)_{sat} = \frac{U_2}{P_1 A^*} \left(\frac{da_2(t)}{dt}\right)_{sat} \tag{4.102}$$

Combining these equations,

$$\left(\frac{d\Delta a(t)}{dt}\right)_{sat} = \frac{U_2 A m S}{P_1 A^* 60 N} \tag{4.103}$$

Equation (4.82a) can be used to give

$$\left(\frac{d\Delta a(t)}{dt}\right)_{max} = \left(\frac{da(t)}{dt}\right)_{max} = \frac{\omega_N}{2\zeta} \Delta p(t)_{max} \tag{4.104}$$

Equation (4.104) gives the natural frequency in terms of $\left(\frac{d\Delta a(t)}{dt}\right)_{max}$ and $p(t)_{max}$.

$$\omega_N = \frac{2\zeta\left(\frac{d\Delta a(t)}{dt}\right)_{max}}{\Delta p(t)_{max}} \tag{4.105}$$

By equating $\left(\frac{d\Delta a(t)}{dt}\right)_{max}$ to $\left(\frac{d\Delta a(t)}{dt}\right)_{sat}$, the result of combining equation (4.105), equation (4.103) and equation (4.102) is

$$V_2 = 60\left(\frac{P_1 A^*}{U_2}\right)^2 \frac{C_3 R T_0 \Delta p(t)_{max} N}{4\zeta^2 A m S} \tag{4.106}$$

The quantity $\left(\frac{P_1 A^*}{U_2}\right)^2$ can be found as a function of the Mach Number. Let this be written

$$\left(\frac{P_1 A^*}{U_2}\right)^2 = \frac{A_e^2}{\left[\frac{U_2 A_e}{P_1 A^*}\right]^2} \tag{4.107}$$

The denominator on the right hand side of the equality can be calculated in terms of the mach Number from the equations of Chapter II. A few of these values are given in Table 8.

Table 8. Table of $\left(\frac{U_2 A_e}{P_1 A^*}\right)$ for
Several Mach Numbers

M	$\left(\frac{U_2 A_e}{P_1 A^*}\right)$
1.0	.5283
1.5	.3204
2.0	.2157
2.5	.1543
3.0	.1153
3.5	.0892

As mentioned before, the control required of the downstream valve could probably be supplied by one butterfly valve. Choosing the constants that might be considered typical, let the following constants be

$$\begin{aligned} \rho &= .7 \\ M &= 1.0 \\ f &= 1.2 \\ A_e &= .2 in^2 \\ A_m &= 12 in^2 \\ S &= 10 RPM \\ N &= \tfrac{1}{4} \end{aligned}$$

The Mach Number, M, given as 1 is used in conjunction with Table 8.

From equation (4.106), the tank volume is

$$V_2 = .325 \ ft^3 \tag{4.108}$$

The natural frequency from Equation (4.99) is

$$\omega_N = \frac{C_3 R T_0 P_1 f A_0^*}{V_2 2 U_2 \rho} \tag{4.99}$$

The ratio P_1/U_2 can be found by knowing the Mach Number and in this example, M=1. The natural frequency is

$$\omega_N = 4.56 \text{ rad/sec}$$

$$= .726 \text{ cycles/sec} \tag{4.109}$$

The maximum overshoot is

$$\Delta p(t)_{max} = 0.16 \tag{4.110}$$

or

$$\Delta p(t)_{max} = 16\% \tag{4.111}$$

Summary of Design Proceedure for Integral Control System

This section is intended to bring together the more important design considerations given throughout the integral control system analysis and to describe a proceedure for designing such a system.

The integral control system consist of a pressure transducer, a power amplifier, and a motor driven valve, for the upstream as well as the downstream tank. By assuming the Mach Number in the nozzle to be one or greater the two tanks can be analyzed independently. The primary difference between the two analyses is that the upstream system controls the pressure in the upstream tank by controlling the inlet flow and the downstream system controls the pressure in the downstream tank by controlling the outlet flow. For the upstream tank system this led to a linear second order system, and for the downstream tank system the result was a nonlinear second order system that in many ways resembles the linear upstream system response.

The systems would be functioning primarily as a regulator and therefore the response of the system to a disturbance would be the criteria for determining the system parameters. A step application of the control jet of a fluidic amplifier was selected as the disturbance.

The equations for the upstream tank system response, with the initial conditions that correspond to step increase in the nozzle throat area (the nozzle throat area increase is equivalent to the application of the control jet) are given by equation (4.18), equation (4.24), and equation (4.36). The solution to these equations is given graphically in Figure 14 and Figure 15. Correspondingly, the equations for the downstream tank system are given by equations (4.87), equation (4.96), and equation (4.97). The solution of the downstream system equations as determined by an analog computer is given in Figure 17 and Figure 18. The variable $\Delta p(t)$ plotted in these figures is the normalized pressure error.

As mentioned before the upstream tank is a linear second order system and the downstream tank is a nonlinear second order system, but for the range of variables involved, the downstream tank response is similar to the upstream tank response. The variable f is a measure of the size of the control jet in relation to the nozzle throat area. If f is considered to be constant then the overshoot or magnitude of the pressure error for a sudden application of the control jet is solely a function of the parameter ζ . For the upstream tank system, ζ is the conventional damping ratio and is defined in equation (4.19). For the downstream tank, ζ should perhaps not be called the damping ratio since the system is nonlinear, but it excersizes the same influence on the downstream system performance as does the damping ratio in that the magnitude of the normalized pressure error is solely a function of ζ for a constant f. This same transitional quality from linear to nonlinear is also true of ω_N. In the upstream tank system, ω_N (defined by equation (4.18)) is the natural frequency and therefore determines the time scale of the response. In the downstream tank system, the quantity ω_N (defined by equation (4.80)) cannot be called a natural frequency in the strictest sense but influences the response as would the natural frequency.

From the definitions of ζ and ω_N, it can be seen that they may be made any desired value by the appropriate values of the feedback gains, K_1 and K_2 (referred to collectively as K), and the tank volumes, V_1 and V_2 (referred to collectively as V). When attempting to obtain the best system response it becomes obvious that some limitation must be placed on the range of system parameters or else the conclusion is reached that the natural frequency can be made arbitrarily large by making the tank volume small. Saturation of the speed of the valve motor was the limitation imposed here.

By thinking of ζ as an independent variable and the feedback gain as a dependent variable, i.e. ζ can be chosen freely and the feedback gain must be chosen to accomodate this choice, the strategy for choosing the tank volume becomes that of making ω_N as large as possible (by making V small) without causing saturation of the valve motor. It is necessary then to determine the relation between the motor speed and the other parameters. The relation between the maximum area rate and motor speed is dependent on the size and type of valve used. Therefore the parameters A_m and N were introduced to reflect the valve type and size. The area, A_m, is the size of the maximum valve orifice area and N is the number of valve stems turns from full opened to full closed.

Thus far, the comments made on the operation on the integral system have applied equally to the upstream as well as the downstream tank systems. However the following discussion points out a difference in the two systems.

The relation between ω_N and V_1 for the upstream tank is still dependent on the pressure at which the upstream system was operating. But it is very likely that the size of the upstream valve will be dependent on the pressures at which the upstream tank is operating. The variable γ was used to reflect this dependence and thus eliminates the dependence on the upstream tank pressure.

With all these considerations, equation (4.68) gives the upstream tank volume in terms of these various parameters. The variable S in this equation is the maximum valve motor speed in RPM. The function $H(\zeta)$ is given in Table 7. Then with V_1 chosen, equation (4.70) can be used to determine if the resulting ω_N is acceptable.

For the downstream tank the relation between ω_N and V_2 is not dependent on the absolute value of the tank pressure as with the upstream tank, but dependent on the Mach Number at which the nozzle is operating. The worst case (largest V_2) for the choked nozzle is M=1. Also since the downstream tank system equations could not be solved analytically, $\Delta p(t)_{max}$ (the maximum value taken on by $\Delta p(t)$)

must be estimated from Figures 17 and 18. Equation (4.106) can therefore be used to find V_2. The ratio $\left(\frac{P_1 A^*}{U_2}\right)^2$ can be found from equation (4.107) and Table 8. Equation (4.99) can be used to check the resulting .

This completes the design of the integral control system since the minimum V_1 and V_2 can be calculated for some given maximum area rate and some chosen ζ. The resulting ω_N can be calculated to determine if it is suitable large. If ω_N is not large enough, equipment must be selected that allows a larger maximum area rate.

Proportional Control System

This section considers the analysis of the test apparatus with the proportional control system. With this control system the upstream and downstream valve areas are proportional to the error signal whereas the valve area rate was proportional to the error signal with the integral control system. The proportional control system is analyzed in an effort to find a control system with better transient characteristics than the integral control system. Even if the control characteristics of proportional control system are more desirable than the characteristics of the integral control system, the disadvantage of the proportional control system is that it would be more difficult to implement. Since the valve area must be proportional to the error signal, there must be some element to sense the valve area. This is an additional element that would not be required with the integral control system.

Figure 19 shows the complete proportional control system. As

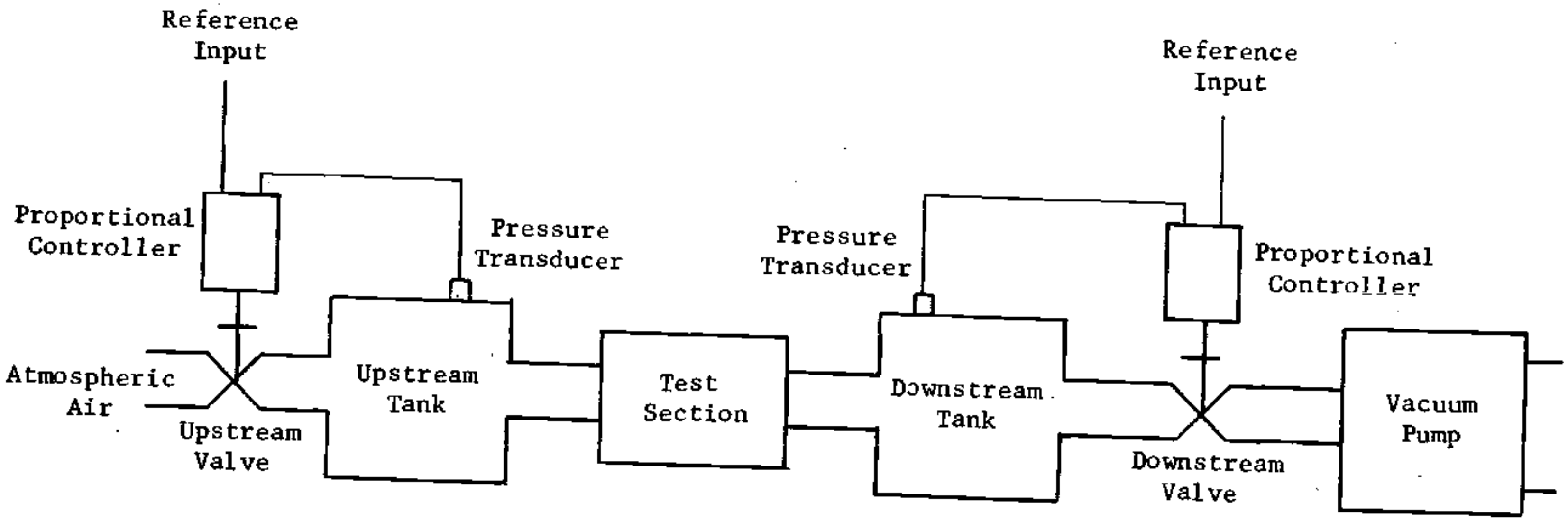

Figure 19. Apparatus with Proportional Controller

before, the upstream and downstream tanks can be analyzed separately if the test section is assumed to be choked. Thus the following two sections determine the size of, first, the upstream tank and, second, the downstream tank, both with the proportional control system. Also the transient response of the proportional control system will be analyzed and compared to the response of the integral control system.

The Upstream Tank System With Proportional Control

When the test section is choked, the equations for the upstream tank (Figure 20) can be written

$$\frac{dp_1(t)}{dt} = \frac{C_3 R T_0}{V_1}\left[P_0 a_1(t) - A^* p_2(t)\right] \tag{4.112a}$$

$$a_1(t) = K_1\left[u_1(t) - p_1(t)\right] \tag{4.112b}$$

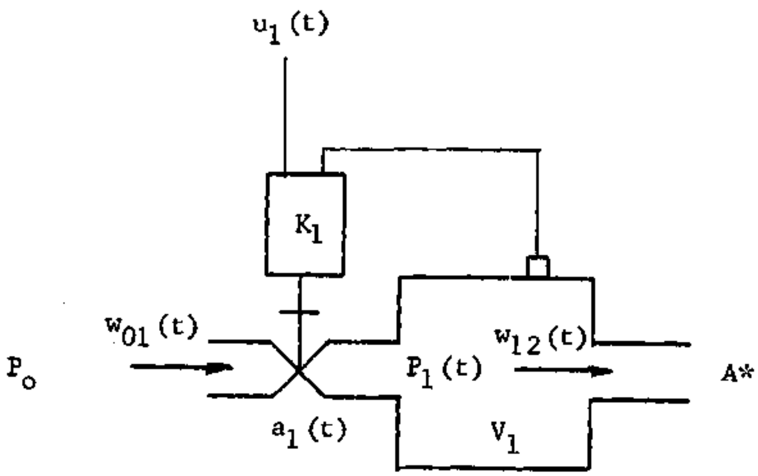

Figure 20. Upstream Tank System with Proportional Controller

The test section is characterized by A*.

Equation (4.98a) and equation (4.98b) can be combined into one equation yielding

$$\frac{dp_1(t)}{dt} = \frac{C_3 R T_0}{V_1}\left[P_0 K_1 u_1(t) - (P_0 K_1 + A^*)\, p_1(t)\right] \tag{4.113}$$

If it is assumed that

$$u_1(t) = U_1 \tag{4.114}$$

$$p(t) = \frac{p_1(t)}{U_1} \tag{4.115}$$

$$\Delta p(t) = p(t) - 1 \tag{4.116}$$

then equation (4.113) becomes

$$\frac{d\Delta p(t)}{dt} = -\frac{C_3 R T_0}{V_1}\left[A^* - (P_0 K_1 + A^*)\,\Delta p(t)\right] \tag{4.117}$$

In addition, let the two parameters, G and Z, be defined

$$G = \frac{P_0 K_1}{A^*} \tag{4.118}$$

$$Z = \frac{V_1}{C_3 R T_0 A^*} \tag{4.119}$$

The purpose for defining G and Z by the above equations is that G and Z are proportional to the two design variables, K_1 and V_1. Therefore, for example, the effect of making V_1 small can be seen by examining the effects of making Z small.

Equation (4.117) is now written

$$\frac{d\Delta p(t)}{dt} = -\frac{1}{Z}\left[1 + (1+G)\Delta p(t)\right] \tag{4.120}$$

Equation (4.120) is a first order, linear differential equation and the solution is

$$\Delta p(t) = \frac{\left[1 + (1+G)\Delta p(0)\right]e^{-\frac{1+G}{Z}t} - 1}{1+G} \tag{4.122}$$

where $\Delta p(0)$ is the initial condition of $\Delta p(t)$.

The time constant, T_c, is

$$T_c = \frac{Z}{1+G} \tag{4.121}$$

It would be desirable to adjust the design parameters so that the time constant can be minimized. This can be accomplished (equation (4.121)) by either making Z small or by making G large or both.

Another factor to consider is the steady-state error. This is, of course, unlike the integral control system which had no steady-state error. The error, Δp_{ss}, can be found by equating equations (4.120) to zero. The result is

$$\Delta p_{ss} = -\frac{1}{1+G} \tag{4.123}$$

Since $\Delta p(t)$ can be written

$$\Delta p(t) = \frac{p_1(t) - U_1}{U_1} \tag{4.124}$$

then the error as defined in equation (4.123) is the fractional deviation of $p_1(t)$ from the desired value of U_1. Making Δp_{ss} small is another consideration when choosing the design parameter. And Δp_{ss} can be made small by making G large. Therefore, making G large by making K_1 large will reduce the time constant and the steady-state error. As with the integral control system, determining the practical limit on how large to make K_1 is difficult; other factors may become more important before K_1 reaches its upper limit. Again it will be considered that saturation in the feedback is the limiting factor. The time constant and the steady-state error will be made as small as possible without allowing the valve motor to saturate.

If the variable a(t) is defined to be

$$a(t) = \frac{P_0 a_1(t)}{U_1 A^*} \tag{4.125}$$

then equation (4.112b) can be written

$$a(t) = -G\,\Delta p(t) \tag{4.126}$$

Differentiating this equation gives

$$\frac{da(t)}{dt} = -G\,\frac{d\Delta p(t)}{dt} \tag{4.127}$$

Equation (4.127) provides a means to calculate the maximum area rate, $\frac{da(t)}{dt}$, from a knowledge of the maximum $\frac{d\Delta p(t)}{dt}$.

To find the maximum $\frac{d\Delta p(t)}{dt}$, consider the response of equation (4.120) to a step input. A plot of $\frac{d\Delta p(t)}{dt}$ versus $\Delta p(t)$ under such an input is shown in Figure 21. The arrows show the movements of the states as time progresses. The movement is always to decrease $\frac{d\Delta p(t)}{dt}$; therefore, the largest $\frac{d\Delta p(t)}{dt}$ must occur at the application of the step. As before the following analysis will be for the application of the control jet.

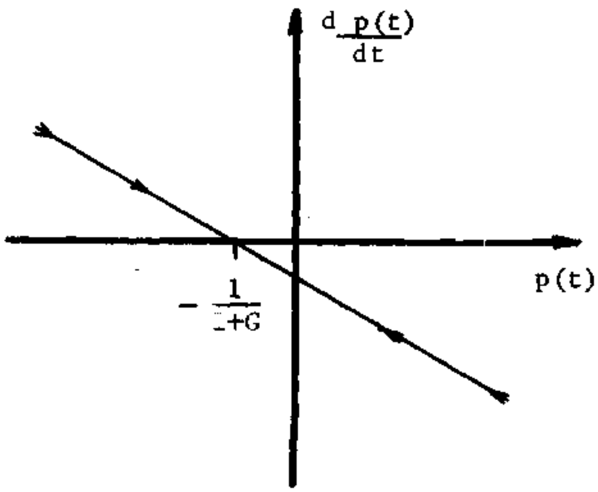

Figure 21. Plot of Equation (4.120)

By putting equation (4.120) into equation (4.127), the result is

$$\frac{da(t)}{dt} = \frac{G}{Z}\left[1 + (1+G)\Delta p(t)\right] \tag{4.128}$$

The largest $\frac{da(t)}{dt}$ occurs at t=0. Therefore

$$\left(\frac{da(t)}{dt}\right)_{max} = \frac{G}{Z}\left[1 + (1+G)\Delta p(0)\right] \tag{4.129}$$

By assuming that the system is at equilibrium before the application of the control jet, $\Delta p(0)$ can be written

$$\Delta p(o) = -\frac{1}{1+G_o} \tag{4.130}$$

where

$$G_o = \frac{P_o K_1}{A_o^*} \tag{4.131}$$

As before, the area after the application of the control jet is

$$A^* = f A_o^* \quad , t \geq o \tag{4.132}$$

Putting equation (4.130) into equation (4.129) gives

$$\left(\frac{da(t)}{dt}\right)_{max} = \frac{f-1}{f} \; \frac{1}{Z_o} \; \frac{G_o^2}{1+G_o} \tag{4.133}$$

where

$$Z_o = \frac{V_1}{C_3 R T_o A_o^*} \tag{4.134}$$

The time constant, T_c, can be written

$$T_c = \frac{Z_o}{f+G_o} \tag{4.135}$$

From equation (4.119), it can be seen that adjusting the parameter to make the time constant and steady-state error small, i.e. making Z_o small and G_o large, has the effect of making $\left(\frac{da(t)}{dt}\right)_{max}$ large. And since $\left(\frac{da(t)}{dt}\right)_{max}$ must be kept below some maximum value, it would be beneficial to examine these equations for tradeoffs.

Using equation (4.120) to eliminate Z_o from equation (4.133) gives

$$\left(\frac{da(t)}{dt}\right)_{max} = \frac{f-1}{f}\,\frac{1}{T_c}\,\frac{G_o^2}{G_o^2 + (1+f)\,G_o + f} \tag{4.136}$$

For large G_o, equation (4.136) is seen to be primarily a function of the time constant T_c.

As stated before it is desired to have $\left(\frac{da(t)}{dt}\right)_{max}$ as large as possible without exceeding some limit. Let this limit be designated by $\left(\frac{da}{dt}\right)_{sat}$. In equation (4.136) it was found that the maximum area rate was primarily a function of T_c. By letting the maximum area rate, $\left(\frac{da(t)}{dt}\right)_{max}$, equal the maximum possible, i.e. $\left(\frac{da}{dt}\right)_{sat}$, then it follows

$$T_c \approx \frac{f-1}{f}\,\frac{1}{\left(\frac{da}{dt}\right)_{sat}} \tag{4.137}$$

Let the steady-state error, after the application of the control jet, be (equation (4.124))

$$E = \frac{P_1(t) - U_1}{U_1} \approx -\frac{f}{f + G_o} \tag{4.138}$$

where E is the error.

Considering f to be fixed, the error is only a function of G_o. It would be necessary then to make G_o large so that the steady-state error will be small. Also, since E is only a function of G_o, specifying a permissable E fixes G_o.

Combining equation (4.136) and equation (4.139) to eliminate G_o results in

$$Z_o = -\frac{f T_c}{E} \tag{4.139}$$

and by using the definition of Z_o, it can be written

$$V_1 = \frac{-f T_c C_3 R T_o A_o^*}{E} \tag{4.140}$$

Equation (4.140) gives the upstream tank volume, V_1, as a function of the time constant, T_c, and the steady-state error, E. The time constant is primarily a function of the maximum area rate permissible for large G_o and this makes choosing V_1 primarily a tradeoff between keeping V_1 small and keeping E small.

The area rate at saturation can be estimated using equation (4.43). This is

$$\left(\frac{da_1}{dt}\right)_{sat} = \frac{A_m S}{60 N} \tag{4.141}$$

Then from equation (4.125), it is obtained that

$$\left(\frac{da}{dt}\right)_{sat} = \frac{P_o A_m S}{f U_1 A_o^* 60 N} \tag{4.142}$$

Define two additional variables, γ and A_1, to be

$$\gamma = \frac{A_m}{A_1} \tag{4.143}$$

$$A_1 = \frac{U_1 A_o^*}{P_o} \tag{4.144}$$

The area A_1 will not be the value of $a_1(t)$ at equilibrium since there is a steady-state error with the proportional control system. However if the steady-state error is small, then the difference between $a_1(t)$ at equilibrium and A_1 will be small. Then γ is essentially the constant that relates the maximum valve area to the equilibrium position of the valve. Now equation (4.142) can be written

$$\left(\frac{da}{dt}\right)_{sat} = \frac{\gamma S}{f60N} \tag{4.145}$$

Using constants similar to the numerical example of the integral controller, let the following constants be

$\gamma = 1$
$S = 60RPM$
$f = 1.2$
$N = 10$

then

$$\left(\frac{da}{dt}\right)_{max} = 0.0833 \tag{4.146}$$

Putting this value of $\left(\frac{da}{dt}\right)_{sat}$ into equation (4.139) gives

$$T \approx 2 \text{ sec} \tag{4.147}$$

Equation (4.140) then gives

$$V_1 = \frac{-2.62}{E} \tag{4.148}$$

Downstream Tank System with Proportional Control

When the test section is choked, the equations for the downstream tank system (Figure 22) can be written

$$\frac{dp_2(t)}{dt} = \frac{C_3 R T_0}{V_2}\left[A^* P_1 - p_2(t)\, a_2(t)\right] \tag{4.149a}$$

$$a_2(t) = K_2\left[p_2(t) - u_2(t)\right] \tag{4.149b}$$

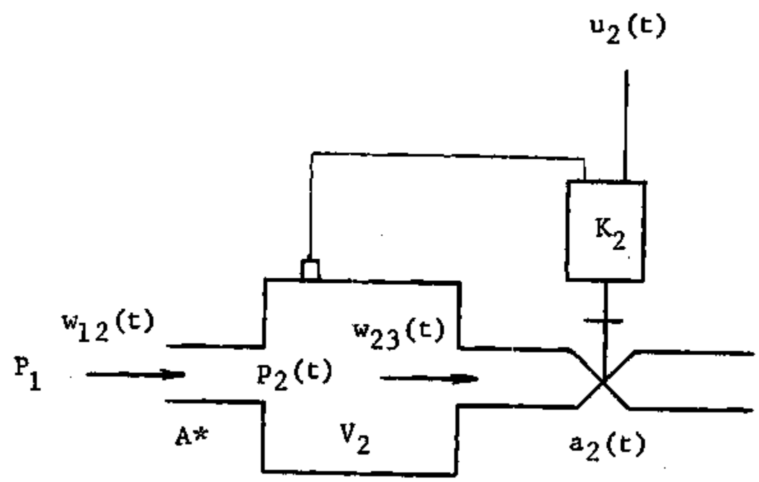

Figure 22. Downstream Tank System with Proportional Controller

Let the following variables be defined as:

$$u_2(t) = U_2 \tag{4.150a}$$

$$p(t) = \frac{p_2(t)}{U_2} \tag{4.150b}$$

$$G = \frac{K_2 U_2^2}{A^* P_1} \tag{4.150c}$$

$$Z = \frac{U_2 V_2}{C_3 R T_0 A^* P_1} \tag{4.150d}$$

$$a(t) = \frac{U_2}{P_1 A^*}\, a_2(t) \tag{4.150e}$$

With these definitions, equations (4.149) can be written

$$\frac{d\Delta p(t)}{dt} = -\frac{1}{Z}\left[1 - a(t)\left(\Delta p(t) - 1\right)\right] \quad (4.151a)$$

$$a(t) = G\,\Delta p(t) \quad (4.151b)$$

Equations (4.151) can be combined to yield

$$\frac{d\,\Delta p(t)}{dt} = -\frac{1}{Z}\left[G\Delta p^2(t) + G\,\Delta p(t) - 1\right] \quad (4.152)$$

Equation (4.152) is a first order equation but it is not linear as was the upstream tank system equation. The solution to equation (4.152) is

$$\Delta p(t) = \frac{(D-1) + (1+D)Be^{-\frac{GD}{Z}t}}{2\left[1 - B\,e^{-\frac{GD}{Z}t}\right]} \quad (4.153a)$$

where

$$D = \left(1 + \frac{4}{G}\right)^{\frac{1}{2}} \quad (4.153b)$$

and

$$B = \frac{2\Delta p(0) + 1 - D}{2\Delta p(0) + 1 + D} \quad (4.153c)$$

It is interesting to compare the non-linear downstream tank system response to the linear upstream tank system response.

Define two new variables, q(t) and τ, in terms of the upstream tank variable to be

$$q(t) = -(1+G)\,\Delta p(t) \tag{4.154a}$$

$$\tau = \frac{1+G}{Z}\,t \tag{4.154b}$$

With the new variables, all the upstream tank system responses can be represented by one plot. If we let the new variable have the same definition for the downstream tank system except for the minus sign in equation (4.154a), then it too can be plotted on the same coordinates but the response will still be a function of G. Figure 23 shows a plot of the upstream tank system response as well as for the downstream tank system response with G=1 and G=10. The initial condition is q(0)=.25.

From the figure it can be seen that when G becomes large, the two responses become very similar.

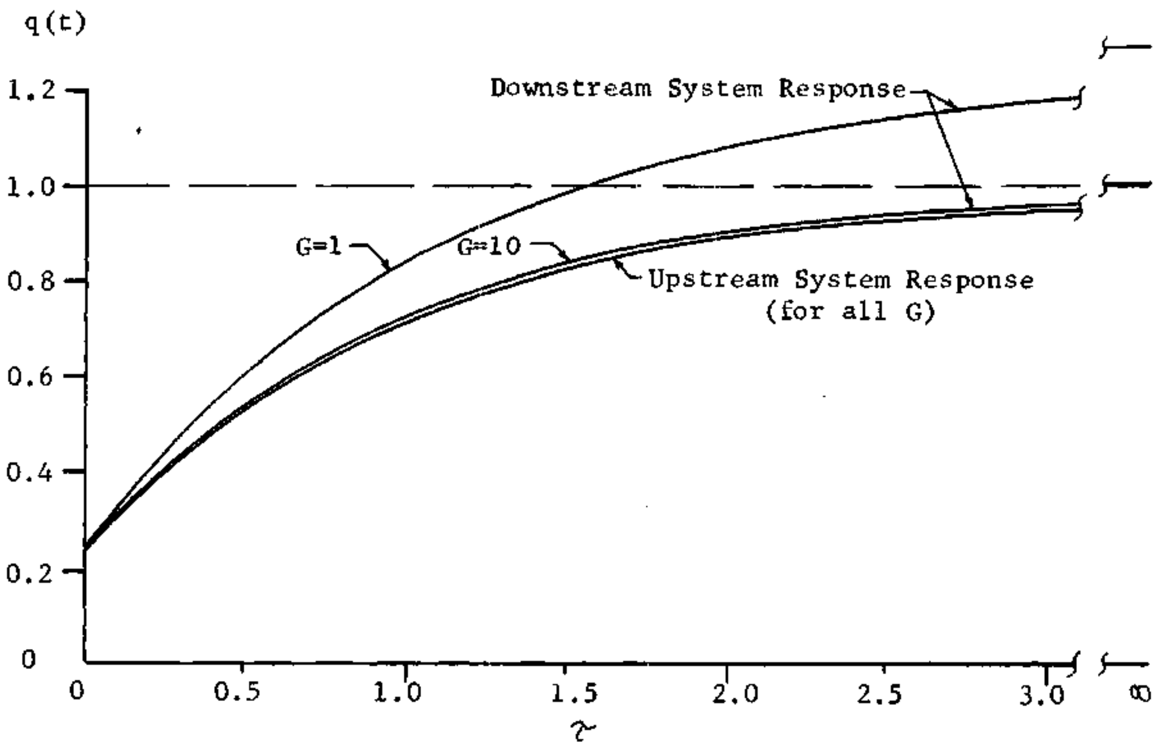

Figure 23. Upstream and Downstream System Response with Proportional Controller

Therefore the same design procedure would be valid for determining the size of the downstream tank as was used for the upstream tank. The equivalent of the time constant for the downstream tank is

$$T_c = \frac{Z}{G\left(1+\frac{4}{G}\right)^{\frac{1}{2}}} \tag{4.155}$$

The steady-state error is

$$\Delta p_{ss} = -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{G}} \tag{4.156}$$

Table 9 gives a few values of the steady-state error as a function of G.

Table 9. Steady-State Error for Downstream Tank System

G	p_{ss} (%)
1	61.8
2	36.6
5	17.1
10	9.2
25	3.9
50	2.0
100	1.0

Again the maximum area rate (saturation) will be the factor that will determine how large to make G and how small to make Z.

By differentiating equation (4.151b) and by combining it with equation (4.152), an equation giving the area rate as a function of $\Delta p(t)$ can be written as

$$\frac{da(t)}{dt} = -\frac{G}{Z}\left[G\Delta p^2(t) + G\Delta p(t) - 1\right] \qquad (4.157)$$

Figure 24 is a plot of this equation.

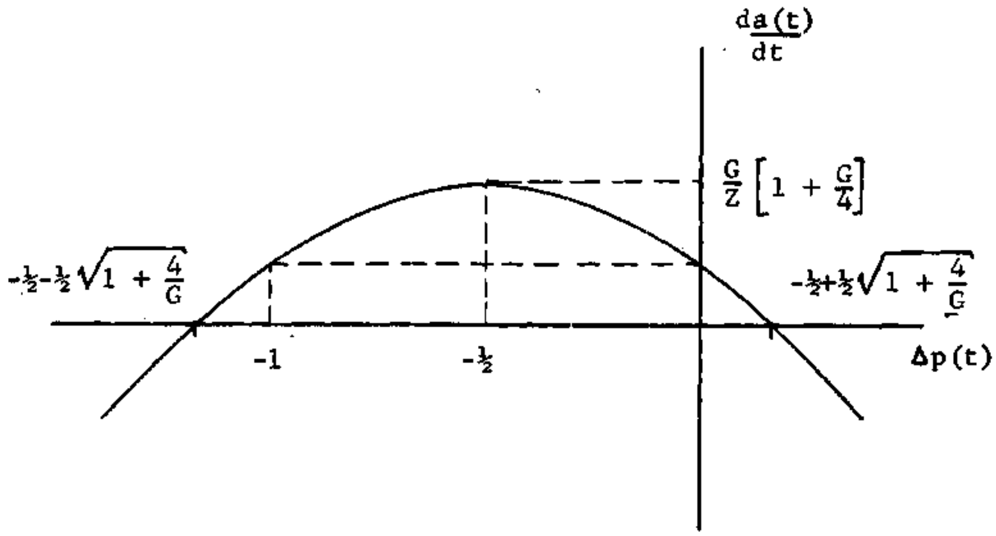

Figure 24. Plot of Equation (4.157)

The arrows in Figure 24 indicate the movement of the state along the line for some initial displacement from the equilibrium position.

From the figure it might appear as if the system could become unstable if the initial conditions were to put the system on the left-most portion of the curve. However since the point at which the curve crosses the $\Delta p(t)$ axis on the left is to the left of the -1 point, and since

$$\Delta p(t) = \frac{p_2(t) - U_2}{U_2} \qquad (4.158)$$

then it would be necessary for either $p_2(t)$ or U_2 to be negative in order to reach this portion of the curve. The pressure $p_2(t)$ will not be negative and it would be simple to constrain the reference input, U_2, to be always positive. Thus the system should not be unstable.

Now the maximum area rate needs to be found for the step application of the control jet. Let the size of the test section area, A*, be designated by A_o^* before the application of the control jet. Also let the following variables be defined as:

$$a_o(t) = \frac{a_2(t)\,U_2}{A_o^*\,P_1} \tag{4.159}$$

$$G_o = \frac{K_2\,U_2^2}{A_o^*\,P_1} \tag{4.160}$$

$$Z_o = \frac{U_2\,V_2}{C_3 R T_o A_o^* P_1} \tag{4.161}$$

$$\Delta p_{oss} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{G_o}} \tag{4.162}$$

$$E = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4f}{G_o}} \tag{4.163}$$

Since the dimensionless parameter $a(t)$ will change in value with a change in A* even if $a_2(t)$ does not change, then so will $\frac{da(t)}{dt}$. The subscript will designate this difference.

Figure 25 shows two trajectories: one for the control jet closed and one for the control jet open.

For the closed jet, the equation is

$$\frac{da_o(t)}{dt} = -\frac{G_o}{Z_o}\left[G_o \Delta p^2(t) + G_o \Delta p(t) - 1\right] \tag{4.164}$$

and for the open control jet the equation is

$$\frac{da_o(t)}{dt} = -\frac{G_o}{Z_o}\left[G_o \Delta p^2(t) + G_o \Delta p(t) - f\right] \qquad (4.165)$$

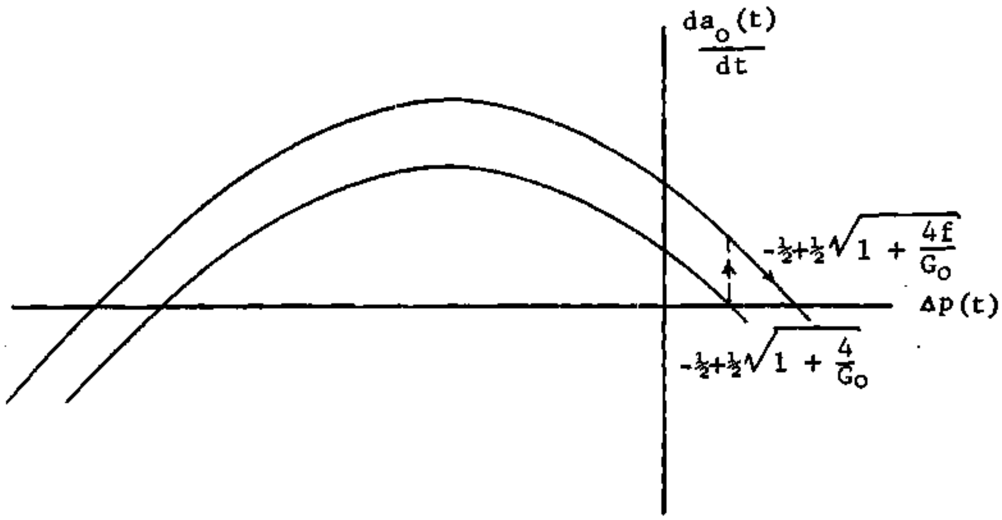

Figure 25. Response of Downstream Pressure to Control Jet Disturbance

It is assumed that $\Delta p(t)$ is at the equilibrium point with the jet closed and since $\Delta p(t)$ does not change value instantaneously with the opening of the jet (does not depend on A*), then the initial condition for the response corresponds to projecting the equilibrium point straight up until it intersects the upper line. This is obviously the largest $\frac{da_o(t)}{dt}$ that will occur for a step jet application.

By substituting equation (4.164) into equation (4.166) the maximum area rate is found to be

$$\left(\frac{da_o(t)}{dt}\right)_{max} = \frac{G_o}{Z_o}\left[f-1\right] \qquad (4.166)$$

Using (4.155) to eliminate T_c gives

$$\left(\frac{da_o(t)}{dt}\right)_{max} = \frac{(f-1)}{T_c\left(1+\frac{f4}{G_o}\right)^{\frac{1}{2}}} \tag{4.167}$$

which shows that for large G_o,

$$T_c \approx \frac{(f-1)}{\left(\frac{da_o(t)}{dt}\right)_{max}} \tag{4.168}$$

This is similar to the upstream tank in the respect that the time constant, T_c, is primarily a function of the maximum area rate allowable. By using equation (4.43) and equation (4.154), the saturation limit can be obtained as

$$\left(\frac{da_o(t)}{dt}\right)_{sat} = \frac{U_2 A_m S}{f P_1 A_o^* 60 N} \tag{4.169}$$

The ratio $\frac{U2}{P_1 A_o^*}$ can be obtained from Table 8.
c

If the constants are

$M = 1.0$
$A_e = .2 in^2$
$A_m = 12 in^2$
$S = 10 RPM$
$N = \frac{1}{4}$

then

$$\left(\frac{da_o(t)}{dt}\right)_{sat} = 17.6 \tag{4.170}$$

Putting this into equation (4.169) gives

$$T \approx 0.0113 \text{ sec} \tag{4.171}$$

The desired volume can be found from equation (4.161) and equation (4.155) as

$$V_2 = C_3 R T_o T_c G_o \left(\frac{A_o^* P_1}{U_2}\right)\left(1 + \frac{4f}{G_o}\right)^{\frac{1}{2}} \tag{4.172}$$

and the corresponding steady-state error is

$$E = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4f}{G_o}} \tag{4.173}$$

Several values of V_2 and E are given in Table 10.

Table 10. Values of V_2 and E for T=0.067

G_o	$V_2(ft^3)$	E(%)
1	.0471	70.4
2	.0721	42.2
5	.137	20.0
10	.238	10.8
25	.534	4.6
50	1.02	2.3
100	2.00	1.2

Summary of Design Proceedure for Proportional Control System

The proportional control system differs from the integral control system in that the valve area itself is proportional to the error signal with the proportional control system instead of having the valve area rate proportional to the error signal as with the integral control system. This difference causes the proportional control systems to be first order instead of second order like the integral control system. With the proportional control system the object is to make the feedback gain large so that the error will be small.

The upstream tank and the downstream tank system were analyzed separately. The upstream tank system was found to be a linear first order equation while the downstream tank system was a non-linear first order equation.

With the integral control system, the response characteristics used to determine the control parameter valves were the overshoot and settling time. These were characterized conveniently by ζ and ω_n and controlled by the feedback gain and tank volume. For the proportional system the factors which determine the performance are rise time (so called in first order system) and steady-state error. Recall that there was no steady-state error with the integral control system.

These are characterized conveniently by T_c and E. The parameter T_c is the time constant when applied to the upstream tank and similar to a time constant when applied to the downstream tank system. The parameter E is the normalized steady-state error in both the upstream and downstream tank systems. And in turn T_c and E can be found in terms of two more parameters G and Z.

The parameter G is defined by equation (4.118) for the upstream tank and by equation (4.150c) for the downstream tank. In both cases, it is proportional to the feedback gain. The parameter Z is defined by equation (4.119) for the upstream tank and by equation (4.150d) for the downstream tank. The parameter Z was thus defined for convenience and because it is proportional to the tank volumes.

Again the control jet application was used as the disturbance, and saturation of the valve motor was set to reflect equipment limitations.

Several simplifications can be made if we assume that the feedback gain is sufficiently large. In order to justify this, notice in equation (4.124) and equation (4.156) that the steady-state error is only a function of G. Thus specifying some G fixes the steady-state error. For errors of 10% or smaller, G must be 10 or larger. If G is 10 or larger then it is sufficiently large to make the desired simplifications.

The first simplification is made with regard to Figure 23. Even though the analytical solutions look quite different, the upstream and downstream tank system responses are essentially the same for G equal to or greater than 10. Therefore unless stated to the contrary, the following discussion is pertinent to both the upstream and downstream systems.

For large G and f fixed, the minimum time constant, T_c, is found to be only a function of the maximum allowable normalized area rate. As the normalized area rate increases, the time constant decreases. With γ, S, and N (as defined for the integral control system) used, the maximum normalized area rate can be found from equation (4.145). The upstream system time constant is thus approximated by equation (1.37) and the downstream system (pseudo) time constant is given by equation (4.168).

Since the minimum time constant is primarily a function of the maximum normalized area rate and not the gain, G, then equation (4.140) shows that the upstream tank size is only a function of the steady-state error. Therefore, to make E small, V_1 must be made large. This tradeoff between E and tank volume is also true for the downstream system; however, G cannot be eliminated to give a direct relation between V_2 and E as in equation (4.140). Instead equation (4.172) and equation (4.173) must be used with G_0 as the independent variable.

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