SPLINES AND THEIR APPLICATION TO THE APPROXIMATION OF LINEAR FUNCTIONALS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Jorge Jesus Moré

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

May, 1968

SPLINES AND THEIR APPLICATION TO THE

APPROXIMATION OF LINEAR FUNCTIONALS

Approved:

Chairman

Date approved by Chairman: May 14, 1968

To Jeannette

"The best thing for being sad," replied Merlyn, beginning to puff and blow, "is to learn something. That is the only thing that never fails. You may grow old and trembling in your anatomies, you may lie awake at night listening to the disorder of your veins, you may miss your only love, you may see the world about you devastated by evil lunatics, or know your honour trampled in the sewers of baser minds. There is only one thing for it then--to learn."

—T. H. White
The Once and Future King

ACKNOWLEDGMENTS

I would like to express my gratitude to those who through their understanding and consideration helped me during the writing of the thesis. In particular I would like to mention my thesis advisor, Dr. William J. Kammerer, for his interest and help in the development of this work, and Dr. George C. Caldwell for trying to make the following pages more readable. I would also like to thank Dr. David L. Finn for his reading of the thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

INTRODUCTION

The theory of splines and its application to technological problems has rapidly grown in recent years, and yet it was only last summer that the first book on the subject appeared. See [3]. The purpose of this thesis is to provide an elementary development of spline theory in such a way that the theorems and methods of proof can be easily generalized to the more general versions of splines. This program is carried out in the second chapter.

We carry out this program by the introduction of splines with multiple knots. This approach has the advantage that the splines used are general enough to cover most of the numerical applications, and simple enough so that the reader will not be lost in the mass of details that enters in the study of "generalized splines." The approximation properties of splines are stated and proved, as well as the first and second integral relations for splines. Convergence results for low order derivatives are presented and proved, but convergence results for high order derivatives are only indicated since their proof is cumbersome and does not add to the understanding of the general theory. However, all the machinery necessary for their proof is developed, and the interested reader is given appropriate references.

What then is the purpose of Chapter I? The first chapter gives the most primitive version of splines with simple knots, and starts to develop the theory through the use of B-splines. Although this approach is somewhat lengthy in comparison to that of the second chapter, it has the advantage of introducing the B-splines, which are important in the study of variation diminishing approximation methods (see [16] and [23]). The subsequent introduction of "natural splines" eases the way for the definition of the splines with multiple knots so that it will not appear to be "pulled out of the air."

In the third chapter we indicate some of the directions in which splines have been generalized: monosplines, G-splines, and generalized splines. The discussion of monosplines is particularly brief, since we only give definitions, state the main results, and make some appropriate comments. The sections on G-splines and generalized splines are more lengthy, and we indicate how these are the natural extensions of splines with multiple knots. The section on G-splines is complete in the sense that the theorems and proofs stated are analogous to those found in the second chapter, and the reader should have no problems in providing the details. In all cases we give references to where further results can be found.

In the last chapter we derive the relationship between the approximation of linear functionals in the sense of Sard [18] and spline theory. The equations that define a cubic spline are then derived and used to improve upon the convergence results of the second chapter. We close the thesis with some numerical examples to show that the theoretical results are substantiated by computational experience.

CHAPTER I

SPLINES WITH SIMPLE KNOTS

A. Motivation and Definition

Polynomial interpolation to a function is, in many cases, not desirable. For one thing, as the number of interpolation points increases without bounds, the resulting polynomials do not necessarily converge to the function. Worse yet, the derivatives of the interpolating polynomial are, in general, unrelated to the derivatives of the function. Splines were introduced as an effort to improve upon these results.

In this study, splines are used to generalize and improve increasingly complex interpolation problems. The first problem that we consider will be called the Lagrange interpolation problem since the main formula is attributed to Lagrange. A description of the problem follows.

Suppose f(x) is defined at the n+1 distinct points $x_0, x_1, \cdots, x_n$, where

$$x_0 < x_1 < \cdots < x_n.$$

The Lagrange interpolation problem then consists of finding a polynomial of lowest degree that interpolates f at these points. The solution to this problem is unique, and consists of a polynomial of degree at most n.

If we try to interpolate f with a polynomial of degree less than n, we find that this, in general, is impossible. (See Theorem A.1 in the Appendix.) We might, therefore, want to consider the following problem: Given an integer k, $1 \leq k \leq n + 1$, find a function s(x) such that

$$\text{a)} \qquad s(x_i) = f(x_i), \qquad 0 \leq i \leq n$$

and

$$\text{b)} \qquad \int_{x_0}^{x_n} [s^{(k)}(x)]^2 dx \qquad \text{is a minimum.}$$

Loosely speaking, we are trying to find a function that a) interpolates f at the desired points, and b) resembles a polynomial of degree k-1. Let us then introduce the solution to this problem.

Let $\{x_j\}_{-\infty}^{\infty}$ be a sequence of points such that

$$\cdots < x_{-2} < x_{-1} < x_0 < x_1 < \cdots$$

and with no limit points.

<u>Definition 1.1</u> Let k be a given non-negative integer. A real function f(x) is called a spline of degree $k > 0$, and denoted by $s_k(x)$ if, and only if,

a) f(x) is a polynomial of degree at most k on each subinterval (x_j, x_{j+1}), and

b) $f(x)$ has k-1 continuous derivatives for all x, i.e. $f \in C^{k-1}(-\infty,\infty)$.

A real function $f(x)$ is called a spline of degree $k = 0$ if, and only if, $f(x)$ is a step function with possible discontinuities at the points x_j.

In subsequent work, a spline of degree k will be referred to as a k-spline, while the points x_j will be called the "knots" or "joints" of the k-spline. If we are trying to interpolate a function and one or more of its derivatives at a particular knot, this knot is termed multiple; if we are only trying to interpolate the function, the knot will be said to be simple.

The k-splines exhibit a remarkable number of properties. Among them we have

$$s_k^{(j)}(x) = s_{k-j}(x), \qquad 0 \le j \le k,$$

and

$$\int s_k(x)dx = s_{k+1}(x).$$

If $s_k \in C^k(-\infty,\infty)$, then $s_k(x)$ is a polynomial of degree at most k since in this case $s_k^{(k)}(x) \equiv$ constant.

The k-splines also form an infinite dimensional linear space. To prove this, let us exhibit a basis. We need the following definitions.

Definition 1.2 Let k be a non-negative integer. The truncated power function x_+^k is defined as

$$x_+^k = \begin{cases} x^k & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Note that x_+^k is a k-spline with a knot at the origin. We are thus justified in making the following definition.

Definition 1.3 A B-spline with knots at $x_j, \cdots, x_{j+k+1}$ is a k-spline defined by

$$B_{kj}(x) = \sum_{i=j}^{j+k+1} \frac{(x_i - x)_+^k}{w'(x_i)},$$

where $w(x) = (x-x_j) \cdots (x-x_{j+k+1})$.

Note that this is just the (k+1)-st divided difference of the function $g(y) = (y-x)_+^k$ based on the points $x_j, \cdots, x_{j+k+1}$. Since $g(y) = (y-x)^k$ for $y > x$, and $g(y) = 0$ for $y < x$, we have that $B_{kj}(x) = 0$ for $x < x_j$ and $B_{kj}(x) = 0$ for $x > x_{j+k+1}$. Other interesting properties of the B-splines can be found in [7]. We will just limit ourselves to sketching a few B-splines, and to proving that they form a base for the k-splines.

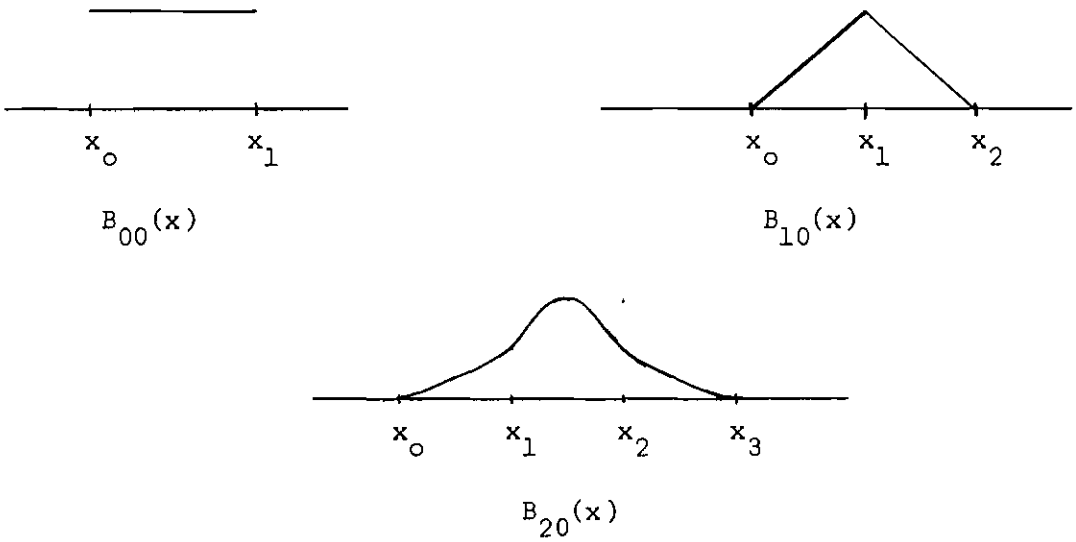

Figure 1. The First Three B-Splines

We will need several lemmas to prove that the B-splines form a base. The first lemma establishes the minimum "span" of a k-spline, where span means the number of consecutive subintervals on which a spline is not identically zero.

Lemma 1.1 Let ℓ be an integer, $0 < \ell \le k$. If $s_k(x) = 0$ everywhere outside of (x_o, x_ℓ), then $s_k(x) = 0$ for all x.

Proof: Since $s_k^{(k)}(x)$ is a step function, we can write

$$s_k^{(k)}(x) = \sum_{j=0}^{\ell} a_j (x_j - x)_+^0 \tag{1.1}$$

so that

$$s_k(x) = \sum_{j=0}^{\ell} \frac{a_j}{k!} (x_j - x)_+^k + p_{k-1}(x)$$

where $p_{k-1}(x)$ is a polynomial of degree k-1. But $s_k(x) \equiv 0$ for $x > x_\ell$ so that $p_{k-1}(x) \equiv 0$; and since $s_k(x) \equiv 0$ for $x < x_o$,

$$\sum_{j=0}^{\ell} a_j (x_j - x)^k = 0$$

for all x.

We want to show that $a_j = 0$, $j = 0, \cdots, \ell$. Assume that $\ell = k$, then repeated differentiation of (1.1) and setting $x = 0$, we obtain the (k+1) linear equations in the unknowns a_j

$$\sum_{j=0}^{k} a_j x_j^i = 0 \qquad 0 \leq i \leq k,$$

whose coefficient matrix has a Vandermonde determinant, and hence non-singular. Thus, $a_j = 0$, $j = 0, \cdots, \ell$ if $\ell = k$. But a glance at (1.1) reveals that this will still be the case if $0 < \ell < k$. This concludes the proof since we have already shown that $p_{k-1}(x) \equiv 0$.

<u>Lemma 1.2</u> If $s_k(x)$ is a spline that vanishes for $x < x_0$, then it can be uniquely represented as

$$s_k(x) = \sum_{j=0}^{\infty} c_j \, B_{kj}(x).$$

<u>Proof</u>: Since $s_k^{(k)}(x)$ is a step function, we can set

$$a_i = s_k^{(k)}(x), \qquad x_{i-1} < x < x_i.$$

Similarly, we can set

$$b_{ij} = B_{kj}^{(k)}(x), \qquad x_{i-1} < x < x_i,$$

and we want to show that the system

$$a_i = \sum_{j=0}^{\infty} b_{ij}c_j \qquad (i \geq 1)$$

can be solved uniquely for the c_i. To see this, note that $b_{ij} = 0$ if $j \geq i$ so that

$$a_i = \sum_{j=0}^{i-1} b_{ij}c_j \quad (i \geq 1)$$

can be solved recursively for the c_i since $b_{i,i-1} \neq 0$. With the c_i so determined, consider

$$g(x) = s_k(x) - \sum_{j=0}^{\infty} c_j B_{kj}(x).$$

Then $g^{(k)}(x) = 0$ for all $x \neq x_i$, and thus $g^{(k-1)}(x)$ is a step function with possible jumps at the knots. However, $g \in C^{k-1}(-\infty,\infty)$ so $g^{(k-1)}(x) \equiv$ constant and hence g is a polynomial of degree at most $k-1$. But $g(x) = 0$ for $x < x_0$ so $g(x) \equiv 0$ as desired.

<u>Lemma 1.3</u> The B-splines

$$B_{k,-k},\ B_{k,-k+1},\ \cdots,\ B_{k,0}$$

are linearly independent on (x_0,x_1), and therefore form a base for the polynomials in (x_0,x_1).

<u>Proof</u>: Assume that $s_k(x) \equiv 0$ in (x_0,x_1), where

$$s_k(x) = \sum_{j=-k}^{0} c_j\, B_{kj}(x),$$

and consider the function

$$g(x) = \begin{cases} s_k(x) & x < x_0, \\ 0 & x \geq x_0. \end{cases}$$

Then $g(x)$ is a k-spline which by Lemma 1.1 vanishes everywhere. In particular, $s_k(x) = 0$ for $x < x_0$ so that by the uniqueness in Lemma 1.2, $c_j = 0$, $j = -k, \cdots, 0$, as desired.

We can now prove

<u>Theorem 1.1</u> Every k-spline with knots $\{x_j\}_{-\infty}^{\infty}$ can be uniquely represented as

$$s_k(x) = \sum_{-\infty}^{\infty} c_j \; B_{kj}(x), \tag{1.2}$$

where the c_j are appropriate constants. Conversely, (1.2) is a k-spline.

<u>Proof</u>: That (1.2) is a k-spline is clear. Suppose now that s_k is an arbitrary k-spline. Then $s_k(x)$ is a polynomial in (x_0,x_1), so that by Lemma 1.3 it can be uniquely represented in (x_0,x_1) by

$$s_k(x) = \sum_{j=-k}^{0} c_j \; B_{kj}(x), \qquad x_0 < x < x_1.$$

Thus,

$$s_k^*(x) = s_k(x) - \sum_{j=-k}^{0} c_j \; B_{kj}(x) \tag{1.3}$$

is a k-spline that vanishes in (x_0,x_1), so we can write

$$s_k^*(x) = s_{k0}(x) + s_{k1}(x) \tag{1.4}$$

where $s_{k0}(x)$ and $s_{k1}(x)$ are k-splines vanishing for $x < x_0$ and $x > x_1$, respectively. By Lemma 1.2, they can be uniquely represented by

$$s_{k0}(x) = \sum_{j=1}^{\infty} c_j \; B_{kj}(x)$$

and

$$s_{k1}(x) = \sum_{j=-\infty}^{-k-1} c_j \, B_{kj}(x).$$

Placing these last results together with (1.3) and (1.4), we obtain (1.2).

B. Natural Splines

We are now in a position to begin solving the problem stated at the beginning of this chapter by the introduction of "natural splines." The term "natural" refers to splines that reduce to a polynomial outside a compact interval.

Let $x_0,\cdots,x_n$ be n + 1 distinct points in an interval [a,b] such that

$$a \le x_0 < x_1 < \cdots < x_n \le b.$$

Now choose arbitrary but fixed reals x_j for $j > n$, and $j < 0$ so as to obtain a sequence of knots.

Definition 1.4 Let k be a positive integer. A natural spline of degree 2k-1 is a 2k-1 spline with $\{x_j\}$ for knots, which reduces to a polynomial of degree at most k-1 outside of $[x_0,x_n]$. The space of all natural splines of degree 2k-1 will be denoted by S_{2k-1}, and its elements by s(x); the subscript being understood.

If we are given a function f defined at $x_0,\cdots,x_n$, then as already noted there is a unique interpolating polynomial $p_n(x)$ of degree at most n. Note that S_{2k-1} will contain $p_n(x)$ if $k = n + 1$, and if $k > n + 1$ S_{2k-1} contains an infinite variety of interpolating polynomials. Let us then assume that

$$1 \le k \le n + 1. \tag{1.5}$$

<u>Theorem 1.2</u> Let $f \in C^k[a,b]$. Then there is a unique natural spline in S_{2k-1}, $s(x)$, such that

$$s(x_j) = f(x_j), \qquad 0 \le j \le n. \tag{1.6}$$

Moreover, $s(x)$ is also characterized up to an additive polynomial $p_{k-1}(x)$ by the condition that

$$\int_a^b [s^{(k)}(x) - f^{(k)}(x)]^2 dx \tag{1.7}$$

is a minimum.

<u>Proof</u>: Let us try to find a (k-1)-spline $s_{k-1}(x)$ with x_j, $0 \le j \le n$, for knots, vanishing outside of $[x_0,x_n]$, such that

$$\int_a^b [s_{k-1}(x) - f^{(k)}(x)]^2 dx \tag{1.8}$$

is a minimum.

By Theorem 1.1 we can write

$$s_{k-1}(x) = \sum_{j=0}^{n-k} c_j\, B_{k-1,j}(x);$$

using this representation and taking partial derivatives with respect to the c_j, we obtain

$$\int_a^b \Big[\sum_{j=0}^{n-k} c_j\, B_{k-1,j}(x) - f^{(k)}(x)\Big]B_{k-1,i}(x)dx = 0 \tag{1.9}$$

for $i = 0,\cdots,n-k$.

The determinant of this system of linear equations in the unknowns c_j is the Gramian of the functions $B_{k-1,j}(x)$. Since they are linearly independent on [a,b], the Gramian does not vanish, and the above system will have a unique solution $s^*_{k-1}(x)$. If we integrate this solution k times, we obtain a natural spline s(x) which contains an arbitrary polynomial $p_{k-1}(x)$ of degree k-1. We will now show that s(x) is uniquely determined and satisfies (1.6) and (1.7).

That s(x) satisfies (1.7) is trivial since $s^{(k)}(x) = s^*_{k-1}(x)$ is the solution of the problem (1.8).

To show that it satisfies (1.6), define

$$g(x) = s(x) - f(x). \tag{1.10}$$

Then

$$g^{(k)}(x) = s^*_{k-1}(x) - f^{(k)}(x)$$

$$\int_a^b B_{k-1,i}(x)\, g^{(k)}(x)\, dx = 0, \qquad 0 \le i \le n-k .$$

Corollary A.1 in the Appendix now guarantees that

$$g[x_i,\cdots,x_{i+k}] = 0 \qquad 0 \le i \le n-k$$

so that Theorem A.1 in the Appendix applies, and we conclude that g can be interpolated at $x_0,\cdots,x_n$ by a unique polynomial of degree k-1. A glance at (1.10) then reveals that the arbitrary polynomial in s(x) is actually uniquely determined, and that

$$s(x_i) = f(x_i), \qquad 0 \le i \le n,$$

as desired.

We are now ready to present the solution to our problem.

<u>Theorem 1.3</u> Given n + 1 points in the plane, (x_j,y_j), $0 \le j \le n$, with

$$x_0 < x_1 < \cdots < x_n$$

and an integer k, $1 \le k \le n+1$, choose an interval [a,b] containing all x_j and suppose that s(x) is the unique natural spline in S_{2k-1} such that

$$s(x_i) = y_i \qquad 0 \le i \le n$$

If $f \in C^k[a,b]$ is such that

$$f(x_i) = y_i \qquad 0 \le i \le n$$

then

$$\int_a^b [f^{(k)}(x)]^2 dx \ge \int_a^b [s^{(k)}(x)]^2 dx \tag{1.11}$$

with equality if, and only if, $f(x) \equiv s(x)$ on $[a,b]$.

Proof: (1.11) follows if we prove that

$$\int_a^b [f^{(k)}(x)]^2 dx = \int_a^b [f^{(k)}(x) - s^{(k)}(x)]^2 dx + \int_a^b [s^{(k)}(x)]^2 dx. \tag{1.12}$$

To prove (1.12) note that the difference between the left-hand side and the right-hand side is

$$2 \int_a^b s^{(k)}(x)[f^{(k)}(x) - s^{(k)}(x)]dx.$$

Repeated integration by parts transforms the above integral into

$$\sum_{j=1}^{k-1} (-1)^{j+1} s^{(k+j-1)}(x)[f^{(k-j)}(x) - s^{(k-j)}(x)]\Big|_a^b + \tag{1.13}$$

$$(-1)^{k-1} \int_a^b s^{(2k-1)}(x)[f^{(1)}(x) - s^{(1)}(x)]dx.$$

The sum vanishes since $s^{(k+j-1)}(x) \equiv 0$ on $[a,x_0]$ and $[x_n,b]$ for $j = 1,\cdots,k-1$. The integral vanishes since $s^{(2k-1)}(x)$ vanishes on $[a,x_0)$ and $(x_n,b]$, is constant on (x_{i-1},x_i) $i = 1,\cdots,n$, and $s(x_i) = f(x_i)$ for $i = 0,1,\cdots,n$. Thus, (1.12) holds. From (1.12) we see that equality holds in (1.11) if, and only if, $f(x) = s(x) + p_{k-1}(x)$. But $s(x_j) = f(x_j)$, $0 \leq j \leq n$, so $p_{k-1}(x)$ is a polynomial of degree at most $k - 1 \leq n$ with $n + 1$ zeros. Hence, $p_{k-1}(x) \equiv 0$, and the proof is complete.

At this moment we have enough machinery to prove a number of important results. However, they would only be special cases of theorems proved in the next chapter by more powerful methods. We thus refer the reader to the next chapter.

CHAPTER II

SPLINES WITH MULTIPLE KNOTS

A. Motivation and Definition

In the first chapter we considered a generalization of the Lagrange interpolation problem, and we found that the solution to this problem was a natural spline with simple knots.

Let us now consider the problem of Hermite interpolation. To this end, suppose we are given n + 1 distinct points, $x_0, \cdots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b \qquad (2.1)$$

and suppose $f \in C^k[a,b]$. The Hermite interpolation problem then consists of finding a polynomial p(x) of lowest degree such that

$$p^{(j)}(x_i) = f^{(j)}(x_i), \qquad j = 0,\cdots,m_i-1$$

for $i = 0,1,\cdots,n$ and for $1 \le m_i \le k$.

This problem is known [8] to have a unique solution in a polynomial of degree less than

$$\sum_{i=0}^{n} m_i .$$

What we now want to do is to generalize natural splines so that Theorem 1.3 holds, and have these splines interpolate $f(x)$. By interpolate we mean

<u>Definition 2.1</u> Let $f \in C^k[a,b]$. A function $g(x)$ is said to interpolate $f(x)$ if

$$g^{(j)}(x_i) = f^{(j)}(x_i), \quad j = 0,1,\cdots,m_i-1,$$

for each i, $0 \leq i \leq n$, where $1 \leq m_i \leq k$.

Note that if $m_i = 1$ for all i, then $g(x)$ would interpolate $f(x)$ in the sense of the Lagrange interpolation problem.

To see how to generalize natural splines, note that if Theorem 1.3 is to hold, then (1.13) must vanish, so that Theorem 1.3 would still hold if natural splines had been defined as real valued functions $s(x)$ such that

a) $s(x)$ is a polynomial of degree at most $2k-1$ in each interval (x_j,x_{j+1}),

b) $s \in C^{2k-2}(-\infty,\infty)$, and

c) $s^{(k+j-1)}(a) = s^{(k+j-1)}(b) = 0$ for $j = 1,2,\cdots,k-1$.

The correct generalization of natural splines to take into account the Hermite problem is then

<u>Definition 2.2</u> Let m_i and k be given integers with $1 \leq m_i \leq k$ for $i = 0,\cdots,n$ and suppose we have $n + 1$ distinct points such that (2.1) holds. A real function $s(x)$ is called a spline of degree $2k-1$ with

multiplicity m_i at x_i $(1 \le i \le n-1)$ if

a) $s(x)$ is a polynomial of degree at most 2k-1 in each interval (x_i, x_{i+1}),

b) $s \in C^{k-1}[a,b]$,

c) $s^{(k+j-1)}(x_i) = 0$ for $j = 1, \cdots, k-m_i$ and $i = o, n$,

d) $s \in C^{(2k-m_i-1)}$ at x_i for $i = 1, \cdots, n-1$.

Several remarks are in order. First, note that natural splines satisfy the definition when $m_i = 1$ for all i, but since the splines of this definition are not defined outside of [a,b], they are not natural splines. Second, condition b) is just for ease of reference since it is implied by the other three. Finally, if either $m_o = k$ or $m_n = k$, then condition c) is to be regarded as vacuous.

B. Existence, Uniqueness, and Minimal Properties

Let us assume for the moment the existence of interpolating splines and show that it implies their uniqueness. This requires several lemmas; the first one is usually called the first integral relation for splines.

Lemma 2.1 Suppose $s(x)$ interpolates $f(x)$. Then

$$\int_a^b [f^{(k)}(x)]^2 dx = \int_a^b [f^{(k)}(x) - s^{(k)}(x)]^2 dx + \int_a^b [s^{(k)}(x)]^2 dx \quad (2.2)$$

Proof: The difference between the right and left-hand side of (2.2) is

$$2 \int_a^b s^{(k)}(x)[f^{(k)}(x) - s^{(k)}(x)]dx \tag{2.3}$$

which can be written as

$$2 \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} s^{(k)}(x)[f^{(k)}(x) - s^{(k)}(x)]dx \tag{2.4}$$

We evaluate each of the integrals by repeated integration by parts; we obtain

$$\sum_{j=1}^{k} (-1)^{j+1} s^{(k+j-1)}(x)[f^{(k-j)}(x) - s^{(k-j)}(x)]\Big|_{x_{i-1}}^{x_i}$$

We now show that (2.4) vanishes. Let $m = \min(m_i)$. Then

$$\sum_{j=k-m+1}^{k} (-1)^{j+1} s^{(k+j-1)}(x)[f^{(k-j)}(x) - s^{(k-j)}(x)]\Big|_{x_{i-1}}^{x_i} \tag{2.5}$$

vanishes since $f^{(k-j)}(x_i) = s^{(k-j)}(x_i)$ for $j = k-m+1,\cdots,k$ and $i = 0,\cdots,n$. If $m = k$ we are through; otherwise we can write

$$\sum_{i=1}^{n} \sum_{j=1}^{k-m} (-1)^{j+1} s^{(k+j-1)}(x)[f^{(k-j)}(x) - s^{(k-j)}(x)]\Big|_{x_{i-1}}^{x_i}$$

as

$$\sum_{j=1}^{k-m} (-1)^{j+1} s^{k+j-1}(x)[f^{(k-j)}(x) - s^{(k-j)}(x)]\Big|_{a}^{b} \tag{2.6}$$

since, if $s^{(k+j-1)}(x)$ is not continuous at x_i, then $f^{(k-j)}(x_i) = s^{(k-j)}(x_i)$. But (2.6) vanishes since $s^{(k+j-1)}(a) = s^{(k+j-1)}(b) = 0$ for $j \geq 1$. Retracing our steps, we see that every integral in (2.4) vanishes and thus (2.3) also vanishes. This concludes the proof.

Lemma 2.2 If $s_1(x)$ and $s_2(x)$ are two splines that interpolate $f(x)$ then

$$s_2(x) - s_1(x) = p_{k-1}(x) \text{ on } [a,b]$$

Proof: Note that $s_2(x) - s_1(x)$ interpolates $s_2(x) - f(x)$ so that Lemma 2.1 yields

$$\int_a^b [s_2^{(k)}(x) - f^{(k)}(x)]^2 dx = \int_a^b [s_1^{(k)}(x) - f^{(k)}(x)]^2 dx$$

$$+ \int_a^b [s_2^{(k)}(x) - s_1^{(k)}(x)]^2 dx.$$

Moreover, the above equation is also valid if we interchange the subscripts; but this means that

$$\int_a^b [s_2^{(k)}(x) - s_1^{(k)}(x)]^2 dx = 0.$$

Therefore, $s_2^{(k)}(x) - s_1^{(k)}(x) = 0$ except at the knots. However, $s_i \in C^{k-1}[a,b]$ for $i = 1,2$ so $s_2^{(k)}(x) - s_1^{(k)}(x) \equiv 0$ and the theorem follows.

Theorem 2.1 (Uniqueness) There is at most one spline that interpolates $f(x)$ provided $k \leq n + 1$.

Proof: If $s_1(x)$, $s_2(x)$ are two splines that interpolate $f(x)$ then by Lemma 2.2

$$s_2(x) - s_1(x) = p_{k-1}(x) \text{ on } [a,b]$$

But $s_2(x)$, $s_1(x)$ interpolate $f(x)$ so $p_{k-1}(x)$ is a polynomial of degree at most $k - 1 \leq n$ with the $n + 1$ zeroes $x_o,\cdots,x_n$. Hence, $p_{k-1}(x) \equiv 0$ as desired.

Theorem 2.2 (Existence) There is exactly one spline that interpolates $f(x)$ provided $k \leq n + 1$.

Proof: Since every spline $s(x)$ is uniquely defined by n polynomials of degree at most $2k-1$, the problem of specifying $s(x)$ is linear, and has $2nk$ unknowns. Let us count the number of equations: condition c) of Definition 2.2 gives $2k - m_o - m_n$ equations, while condition d) provides at each interior knot $2k - m_i$ equations for a total of $\sum_{i=1}^{n-1} 2k - m_i$; the requirement that $s(x)$ interpolate $f(x)$ provides an additional $\sum_{i=0}^{n} m_i$ equations, so there are $2nk$ equations. Since the number of equations equals the number of unknowns, it suffices to show that $s(x)$ interpolates $f(x) \equiv 0$ if, and only if, $s(x) \equiv 0$. But this is clear from Theorem 2.1 since the zero function is a spline. The proof is therefore complete.

Now that we have proved the existence and uniqueness of splines that interpolate $f(x)$, the minimal properties follow immediately. The first minimum property is the analogue of Theorem 1.3 for the Hermite problem, and is usually referred to as the minimum norm property for splines.

<u>Theorem 2.3</u> Let $f \in C^k[a,b]$, and suppose we are given $n + 1$ points x_j, $0 \le j \le n$, with

$$a = x_0 < x_1 < \cdots < x_n = b$$

and an integer k, $1 \le k \le n + 1$. Let $s(x)$ be the unique spline that interpolates $f(x)$. If $g \in C^k[a,b]$ also interpolates $f(x)$, then

$$\int_a^b [g^{(k)}(x)]^2 dx \ge \int_a^b [s^{(k)}(x)]^2 dx \tag{2.7}$$

with equality if, and only if, $g(x) \equiv s(x)$ on $[a,b]$.

<u>Proof:</u> Since $s(x)$ interpolates $g(x)$, Lemma 2.1 applies and yields

$$\int_a^b [g^{(k)}(x)]^2 dx = \int_a^b [g^{(k)}(x) - s^{(k)}(x)]^2 dx + \int_a^b [s^{(k)}(x)]^2 dx$$

and (2.7) follows. If equality holds in (2.7) then from the above relation, $g^{(k)}(x) = s^{(k)}(x)$ except possibly at the knots. But $g, s \varepsilon C^{(k-1)}[a,b]$ so $g^{(k)}(x) \equiv s^{(k)}(x)$ and then

$$g(x) = s(x) + p_{k-1}(x) \text{ on } [a,b].$$

However, s(x) interpolates g(x), so $p_{k-1}(x)$ is a polynomial of degree at most $k - 1 \leq n$ with the $n + 1$ zeros $x_0, \cdots, x_n$. Hence, $p_{k-1}(x) \equiv 0$ as desired. If $g(x) \equiv s(x)$ on [a,b], (2.7) follows trivially.

The next minimum property is sometimes called the best approximation property of splines.

<u>Theorem 2.4</u> Suppose that $s_o(x)$ and s(x) are splines and that s(x) is the unique spline that interpolates f(x). Then

$$\int_a^b [s_o^{(k)}(x) - f^{(k)}(x)]^2 dx \geq \int_a^b [s^{(k)}(x) - f^{(k)}(x)]^2 dx \qquad (2.8)$$

with equality if, and only if, $s_o(x) = s(x) + p_{k-1}(x)$ on [a,b].

<u>Proof</u>: Since $s_o(x) - s(x)$ interpolates $s_o(x) - f(x)$, Lemma 2.1 applies and yields

$$\int_a^b [s_o^{(k)}(x) - f^{(k)}(x)]^2 dx = \int_a^b [s^{(k)}(x) - f^{(k)}(x)]^2 dx$$

$$+ \int_a^b [s_o^{(k)}(x) - s^{(k)}(x)]^2 dx$$

and (2.8) follows. The conditions of equality are verified as in the previous proof.

At this stage it is apparent that splines as described by Definition 2.2 are too restrictive. The key result is Lemma 2.1; from it followed Lemma 2.2, and if $k \leq n + 1$, so does Theorem 2.1 and both minimal properties. We cannot expect to relax the continuity requirements of part d) of Definition 2.2 and still have a well-defined problem; but we can certainly modify the boundary conditions of part c) so that Lemma 2.1 holds, and the linear problem of Theorem 2.1 is still uniquely solvable.

Observe that Lemma 2.1 is still valid if (2.5) and (2.6) vanish. Moreover, (2.6) vanishes if $f^{(k-j)}(x_i) = s^{(k-j)}(x_i)$ for $i = o,n$ and $j = 1,\cdots,k-m$ so that if $s(x)$ interpolates $f(x)$ then $f^{(k-j)}(x_i) = s^{(k-j)}(x_i)$ for $0 \leq i \leq n$ and $j = k - m + 1,\cdots,k$ since $m \leq k$ and, therefore, (2.5) also vanishes. Since $m \geq 1$ also, we make the following

<u>Definition 2.3</u> Let $f \varepsilon C^k[a,b]$. A function $g(x)$ is said to interpolate $f(x)$ in a Type A problem if

$$g^{(j)}(x_i) = f^{(j)}(x_i) \qquad j = 0,1,\cdots,m_i-1$$

for each i, $1 \leq i \leq n - 1$, and

$$g^{(j)}(x_i) = f^{(j)}(x_i) \qquad j = 0,1,\cdots,k-1$$

for $i = 0,n$.

We would now like to modify Condition c) of Definition 2.2 so as to obtain splines that solve problems of Type A. Note that both Lemmas 2.1 and 2.2 hold if s(x) interpolates f(x) in a Type A problem. Moreover, Theorem 2.1 holds without the restriction that $k \leq n + 1$ since now $p_{k-1}^{(j)}(x_i) = 0$ for $j = 0,1,\cdots,k-1$ and $i = 0,n$ so that $p_{k-1}(x) \equiv 0$.

To obtain the analogue of Theorem 2.2 we must then specify k-1 boundary conditions at each end point in such a way that s(x) interpolates $f(x) \equiv 0$ in a Type A problem if, and only if, $s(x) \equiv 0$. The following definition is therefore appropriate.

Definition 2.4 A spline of degree 2k-1 with multiplicity m_i at x_i ($1 \leq i \leq n - 1$) is said to be of Type A if instead of requiring that

$$s^{(k+j-1)}(x_i) = 0 \quad \text{for } j = 1,\cdots,k-m_i \quad \text{and } i = 0,n,$$

we require that s(x) interpolate some function in a Type A problem.

Our previous discussion then provides the proof for the following

Theorem 2.5 There is exactly one spline of Type A that interpolates f(x) in a Type A problem.

Moreover, it is trivial to verify that the minimum norm and best approximation properties also holds for splines of Type A.

The way is now open for introducing more types of splines which will solve their corresponding interpolation problems. See for example [3], where they consider the case $m_i = q$ for all i, and the splines are said to be of deficiency q.

C. Convergence Properties

The convergence properties of splines are truly remarkable, and very easy to obtain. They fare a lot better than say, polynomial approximation. (To compare with other kinds of interpolating processes, see [8]).

In stating these properties, it is convenient to introduce the following

Definition 2.5 If g(x) is square-integrable on [a,b], set

$$\|g\| = \left(\int_a^b [g(x)]^2 dx\right)^{1/2} .$$

Let us now consider a sequence of partitions P_i of [a,b]

$$P_i: \quad a = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} = b$$

and let $N(P_i) = \max\{x_{i,j} - x_{i,j-1} : 1 \le j \le n_i\}$ denote the norm of the partition.

<u>Theorem 2.6</u> Let $f \in C^k[a,b]$, and let $\{P_i\}$ be a sequence of partitions of $[a,b]$ such that $N(P_i) \to 0$ as $i \to \infty$. If $s_i(x)$ interpolates $f(x)$ on P_i, then

$\{s_i^{(j)}(x)\}$ converges uniformly to $f^{(j)}(x)$ on $[a,b]$ for $j = 0,1, \cdots,k-1$.

<u>Proof</u>: We first prove the theorem for $j = k-1$. Chose $i \geq k$ so that P_i contains at least $k + 1$ points. Then $s_i(x)$ exists, and is uniquely determined. Moreover, for any x in $[a,b]$, there is an interval I in $[a,b]$ which contains x and k points of P_i. Thus, $f(t) - s_i(t)$ has k zeros in I, and by repeated application of Rolle's theorem we conclude that $f^{(k-1)}(t) - s_i^{(k-1)}(t)$ has at least one zero in I; say $\bar{x}$. Then for x in I

$$|x-\bar{x}| \leq k\ N(P_i)$$

and

$$|f^{(k-1)}(x) - s_i^{(k-1)}(x)| = \left|\int_{\bar{x}}^{x} [f^{(k)}(t) - s_i^{(k)}(t)]dt\right|.$$

An application of the Cauchy-Schwartz inequality yields

$$|f^{(k-1)}(x) - s_i^{(k-1)}(x)| \leq |x - \bar{x}|^{1/2} \left|\int_{\bar{x}}^{x} [f^{(k)}(t) - s_i^{(k)}(t)]^2 dt\right|^{1/2} \quad (2.9)$$

and, since the first integral relation, Lemma 2.1, holds,

$$|f^{(k-1)}(x) - s_i^{(k-1)}(x)| \le k^{1/2} \; N(P_i)^{1/2} \; \|f^{(k)}\| \tag{2.10}$$

for all x in [a,b]. The result of the theorem follows for $j = k - 1$.

If $0 \le j \le k - 1$, repeated use of (2.9) shows that there is a constant c such that

$$|f^{(j)}(x) - s_i^{(j)}(x)| \le c \; N(P_i)^{(k-j-1/2)} \; \|f^{(k)}\| .$$

Since this holds for all x in [a,b], we are through.

Theorem 2.6 yields two inequalities that deserve special attention. We single them out as

Corollary 2.1 Under the hypothesis of Theorem 2.6 there is a constant c_1, dependent on j and k, but independent of i, such that for $i \ge k$,

$$|f^{(j)}(x) - s_i^{(j)}(x)| \le c_1 \; N(P_i)^{k-j-1/2} \; \|f^{(k)} - s_i^{(k)}\|$$

$$\le c_1 \; N(P_i)^{k-j-1/2} \; \|f^{(k)}\|$$

for $j = 0,1,\cdots,k-1$.

Proof: The first inequality follows by observing that we can replace $\|f^k\|$ with $\|f^{(k)} - s_i^{(k)}\|$ in (2.10) due to (2.9). The second inequality follows from the first integral relation.

Since the properties of splines used in the proof of Theorem 2.6 also hold for splines of Type A, we have proved

Corollary 2.2 Theorem 2.6 holds for splines of Type A. In addition, the inequalities of Corollary 2.1 hold for all $i \geq 1$.

If we are interested in a norm squared error bound, the results of Corollaries (2.1) and (2.2) can be improved. To do so, we need the following

Lemma 2.3 If $g'(x)$ is integrable on $[a,b]$, and $g(a) \cdot g(b) = 0$, then

$$\int_a^b [g(t)]^2 dt \leq (b-a)^2 \int_a^b [g'(t)]^2 dt$$

Proof: Suppose $g(a) = 0$. Then for each x in $[a,b]$

$$g(x) = \int_a^x g'(t)dt.$$

An application of the Cauchy-Schwarz inequality yields

$$[g(x)]^2 \leq (x-a) \int_a^x [g'(t)]^2 dt \leq (b-a) \int_a^b [g'(t)]^2 dt$$

and our result follows upon integration. If $g(b) = 0$, the proof is analogous and is therefore omitted.

<u>Corollary 2.3</u> Under the hypothesis of Theorem 2.6 there is a constant c_2, dependent on j and k, but independent of i, such that for $i \geq k$,

$$\|f^{(j)} - s_i^{(j)}\| \leq c_2\, N(P_i)^{k-j}\, \|f^{(k)} - s_i^{(k)}\|$$

$$\leq c_2\, N(P_i)^{k-j}\, \|f^{(k)}\|$$

for $j = 0,1,\cdots,k$.

<u>Proof:</u> We only prove the first inequality, since the second inequality follows from Lemma 2.1. If $j = k$, there is nothing to prove; so assume $j = k-1$. Since $i \geq k$, we can divide $[a,b]$ into m intervals, $I_1,\cdots,I_m$, such that each interval, with the possible exception of I_m, contains k consecutive points of P_i. Then $f(x) - s_i(x)$ has k zeros on each I_r $(1 \leq r \leq m - 1)$, and by Rolle's theorem, $f^{(k-1)}(x) - s_i^{(k-1)}(x)$ has at least one zero on I_r, say $\bar{x}_r$. If we now define $\bar{x}_o = a$, $\bar{x}_m = b$, we have that

$$|\bar{x}_r - \bar{x}_{r-1}| \leq 2k\, N(P_i),\ 1 \leq r \leq m.$$

By Lemma 2.3

$$\int_{\bar{x}_{r-1}}^{\bar{x}_r} [f^{(k-1)}(x) - s_i^{(k-1)}(x)]^2 dx \leq 4k^2 N(P_i)^2 \int_{\bar{x}_{r-1}}^{\bar{x}_r} [f^{(k)}(x) - s_i^{(k)}(x)]^2 dx,$$

and our result follows by summing the above inequalities. The proof for $0 \leq j < k - 1$ is analogous and is omitted.

Corollary 2.4 The results of Corollary 2.3 hold for splines of Type A and for all $i \geq 1$.

The results of Theorem 2.6 can be improved if we assume that $f \in C^{2k}[a,b]$ and consider interpolation in a Type A problem. The results that we will obtain can be extended to other types of interpolation, but not to the type of Definition 2.1, since our proofs rest heavily on the following second integral relation for splines.

Theorem 2.7 Let $f \in C^{2k}[a,b]$, and suppose $s(x)$ interpolates $f(x)$ in a Type A problem. Then

$$\int_a^b [f^{(k)}(x) - s^{(k)}(x)]^2 dx = (-1)^k \int_a^b [f(x) - s(x)] f^{(2k)}(x) dx \qquad (2.10)$$

Proof: The left-hand side of (2.10) can be written as

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f^{(k)}(x) - s^{(k)}(x)]^2 dx \qquad (2.11)$$

We evaluate each of the integrals by integrating by parts k times; we obtain

$$\sum_{j=1}^{k} (-1)^{j+1}[f^{(k-j)}(x) - s^{(k-j)}(x)][f^{(k+j-1)}(x) - s^{(k+j-1)}(x)]\Big|_{x_{i-1}}^{x_i} \quad (2.12)$$

$$+ (-1)^k \int_{x_{i-1}}^{x_i} [f(x) - s(x)][f^{(2k)}(x) - s^{(2k)}(x)]dx.$$

Since $s^{(2k)}(x) \equiv 0$ in (x_{i-1},x_i), (2.11) will reduce to the right-hand side of (2.10) if we show that

$$\sum_{i=1}^{n} \Delta(x_i,x_{i-1}) \quad (2.13)$$

vanishes, where $\Delta(x_i,x_{i-1})$ is the first term appearing in (2.12).

Let $m = \min(m_i)$; then for $j \geq k - m + 1$, the addends of $\Delta(x_i,x_{i-1})$ vanish due to interpolation in the first factor of the addends. If $m = k$, we are through. Otherwise, both factors are continuous, and (2.13) reduces to

$$\sum_{j=1}^{k-m} (-1)^{j+1} [f^{(k-j)}(x) - s^{(k-j)}(x)][f^{(k+j-1)}(x) - s^{(k+j-1)}(x)]\Big|_{b}^{a} ,$$

which again vanishes due to interpolation at the end-points. Since (2.13) vanishes, we are through.

We can now use Theorem 2.7 to raise the order of convergence of Corollary 2.2.

Theorem 2.8 Let $f \in C^{2k}[a,b]$, and suppose $s(x)$ interpolates $f(x)$ in a Type A problem on some partition P. Then there is a constant c_3 dependent on j and k, such that

$$|f^{(j)}(x) - s^{(j)}(x)| \leq c_3 \; N(P)^{2k-j-1/2} \|f^{(2k)}\|$$

for $j = 0,1,\cdots,k-1$.

Proof: By Corollary 2.2, the first inequality of Corollary 2.1 holds on any partition P. Hence,

$$|f^{(j)}(x) - s^{(j)}(x)| \leq c_1 \; N(P)^{(k-j-1/2)} \|f^{(k)} - s^{(k)}\| \; . \qquad (2.14)$$

By Theorem 2.7,

$$\|f^{(k)} - s^{(k)}\|^2 = (-1)^k \int_a^b [f(x) - s(x)] \; f^{(2k)}(x)dx,$$

and, applying the Cauchy-Schwarz inequality,

$$\|f^{(k)} - s^{(k)}\|^4 \leq \|f^{(2k)}\|^2 \int_a^b [f(x) - s(x)]^2 dx.$$

If we now use the first inequality mentioned in Corollary 2.4 with $j = 0$,

$$\|f^{(k)} - s^{(k)}\|^4 \leq c_2^2 \; N(P)^{2k} \|f^{(k)} - s^{(k)}\|^2 \|f^{(2k)}\|^2 \; ,$$

and if we solve for $\|f^{(k)} - s^{(k)}\|$ in this last inequality, and use it in (2.14), we obtain the desired result.

The result of Theorem 2.8 indicates that we might expect uniform convergence of derivatives of order up to 2k-1 for $f \in C^{2k}[a,b]$. This is indeed the case, if we restrict our partitions in such a way that the quantities

$$\frac{N(P_i)}{n(P_i)} \tag{2.15}$$

remain bounded. Here,

$$n(P_i) = \min \{x_{i,j} - x_{i,j-1}: \ 1 \le j \le n_i\} .$$

The proofs can be found in [25].

CHAPTER III

GENERALIZATIONS

A. Monosplines

Splines handle the problem of approximation in the least square sense with great ease; to attack the problem of approximation in the uniform sense, the concept of monosplines was introduced.

Let $x_1, \cdots, x_n$ be n given reals such that

$$x_1 < x_2 < \cdots < x_n. \tag{3.1}$$

Definition 3.1 A monospline of class (k,n) with knots (3.1) is a function M of the form

$$M(x) = x^k + s_{k-1}(x), \tag{3.2}$$

where $s_{k-1}(x)$ is a (k-1) spline with (3.1) for knots. See Definition 1.1.

It is tacitly understood that $k \geq 1$ and $n \geq 0$. If $n = 0$, then we have no knots, and the monospline (3.2) reduces to a polynomial of degree k with leading coefficient unity.

The main theorem concerning approximation by monosplines was proved by R. S. Johnson [11].

Theorem 3.1 For each (k,n) there exists a unique monospline $M^*_{k,n}$ of class (k,n) which deviates least from zero on [-1,1]. For $k \geq 2$, $M^*_{k,n}$ achieves its maximum absolute deviation, with alternating signs, at precisely k + 2n + 1 points of [-1,1], including both end-points, and this condition determines $M^*_{k,n}$ uniquely.

Thus, $M^*_{k,o}$ is in fact the Tchebycheff polynomial which is known to satisfy the stated characterization property.

Monosplines have also been used by I. J. Schoenberg to construct best quadrature formulas. See [19] and [22]. For details on the numerical procedures used to construct the monosplines, see [4].

B. G-Splines

If we wanted to generalize splines in the same way that we did in Chapters I and II, we would want to consider an interpolation problem that generalizes the Lagrange and Hermite interpolation problems. This problem was first considered by G. D. Birkhoff in 1906, and is known as the Hermite-Birkhoff interpolation problem. It is described below.

Suppose we are given n + 1 distinct points $x_0,\cdots,x_n$ such that

$$x_0 < x_1 < \cdots < x_n$$

and a (n+1) x k matrix $E = (e_{ij})$, $i = 0,1,\cdots,n$, $j = 0,1,\cdots,k-1$, such that each row of E has at least one nonzero element. Define the set e by

$$e = \{(i,j): \ e_{ij} = 1\},$$

and suppose we are given reals $y_i^{(j)}$, (i,j) in e. Then, the Hermite-Birkhoff interpolation problem consists of finding the polynomial of least degree such that

$$p^{(j)}(x_i) = y_i^{(j)}, \qquad (i,j) \text{ in e.} \tag{3.3}$$

If we now try to obtain a function s(x) which satisfies (3.3) and such that

$$\int_a^b [s^{(k)}(x)]^2 dx$$

is a minimum, we obtain the natural G-splines which are described as

Definition 3.2 A function s(x) is called a natural G-spline for the knots $x_0,\cdots,x_n$, the matrix E, and *order* k, if

a) s(x) is a polynomial of degree at most 2k-1 on (x_i,x_{i+1})

b) $s \in C^{k-1}(-\infty,\infty)$

c) s(x) is a polynomial of degree at most k - 1 in $(-\infty,x_0]$ and $[x_n,\infty)$

d) $s \in C^{(2k-j-1)}$ at x_i if $e_{ij} = 0$.

Note that if E is such that $e_{io} = 1$ for $0 \le i \le n$, and zero otherwise, we obtain the natural splines of degree 2k-1 of Definition 1.4, while if $e_{ij} = 1$ for all j, $0 \le j < m_i \le k$, and $i = 1,\cdots,n-1$ with $e_{0,0} = e_{n,0} = 1$ and all other entries zero, the splines with multiple knots of Definition 2.2 emerge out of the above definition.

The proofs of existence, uniqueness, and minimal properties for these splines are completely analogous to those found in Chapter II. The requirement that $k \leq n + 1$, which guaranteed that the polynomial appearing in Lemma 2.2 was identically zero, must now be replaced by the following: For each integer i, $0 \leq i \leq n$, if $e_{i,0} \in e$, let k_i be the greatest positive integer such that $e_{i,0}, \cdots, e_{i,k_i-1}$ are all in e. If $e_{i0} = 0$, define k_i to be zero. We then have the following.

<u>Theorem 3.2</u> If $\sum_{i=0}^{n} k_i \geq k$, then the natural G-spline for the knots $x_0, \cdots, x_n$, the matrix E, and order k, exists and is unique.

The natural G-splines will then satisfy the first integral relation, the minimum norm and best approximation properties, and the convergence results of the splines of Definition 2.2. Once again, these splines do not satisfy the second integral relation due to requirement c) of Definition 3.2. The replacement of c) by some other interpolation condition gives rise to splines of different types. For example, to obtain a generalization of splines of Type A we would replace requirement c) of Definition 3.2 by

$$c')\qquad e_{o,j} = e_{n,j} = 1, \qquad 0 \leq j \leq k - 1.$$

The splines obtained in this manner satisfy all the properties of natural G-splines, and in addition, the second integral relation with the corresponding convergence results. This procedure is followed by Schultz and Varga [25]. For more results on natural G-splines see [24]; more details on the Hermite-Birkhoff interpolation problem can be found in [21].

C. Generalized Splines

One of the most important generalizations of spline functions occurred when it was noticed that the requirement that s(x) be a polynomial of degree at most 2k-1 on each subinterval could be replaced by

a') s(x) is a solution of $D^{2k}(s) = 0$

on each interval (x_i, x_{i+1}), where $D = \frac{d}{dx}$.

So why not replace the operator D with the kth order linear differential operator defined by

$$L(u) = \sum_{j=0}^{k} a_j(x)\, D^j(u)$$

for any $u \in C^k[a,b]$? This is indeed possible, and with the formal adjoint of L, L^*, being defined by

$$L^*(u) = \sum_{j=0}^{k} (-1)^j\, D^j(a_j u),$$

We can replace condition a') by

a") s(x) is a solution of $L^*L(s)$ in each interval (x_i, x_{i+1}).

In doing this, it was noticed that the proofs of the theorems in the first two chapters would not be altered if we replaced the class $C^k[a,b]$ by the more restrictive class $K^k[a,b]$ of all functions f, such that $f \varepsilon C^{(k-1)}[a,b]$, $f^{(k-1)}$ is absolutely continuous, and $f^{(k)} \varepsilon$ $L^2[a,b]$.

The generalization in this direction was started by I. J. Schoenberg in 1964 [20] with his "trigonometric splines" by considering a linear differential operator with constant coefficients. The generalization to non-constant coefficients was made by T.N.E. Grevelle [10] who used the method of Lagrange's multipliers to obtain the extremal properties. Later, Ahlberg, Nilson, and Walsh [1], considered the relationship between the linear differential operator and its adjoint to obtain the extremal properties, and at the same time introduced the concept of types of splines. In a later paper [2], they proved some convergence results for generalized splines: uniform convergence for derivatives of order up to k-1, and convergence in the mean for the kth derivative; and for partitions restricted as in (2.15), uniform convergence of derivatives of order up to 2k-2. In the same paper, they noticed that the validity of the first integral relation was not altered for splines with knots of equal multiplicity, and thus obtained convergence of lower order derivatives for the so-called *splines of deficiency*.

The case of multiple nodes when the operator L has Polya's property W was done by Karlin and Ziegler [12]; the resulting splines being termed *Tchebycheffian splines*. Other results in this direction can be found in [9], [13], [26], and [27]. In 1967, Ahlberg, Nilson , and Walsh [3] treated the case of multiple nodes for a general linear differential operator, obtaining the convergence results of both lower and higher order derivatives.

The complete generalization of the above results was done by Schultz and Varga [25]. The L-splines which they introduced have for special cases all of the splines discussed in this section, and they have been able to improve previous convergence results; in particular, uniform convergence of the derivative of order 2k-1 for partitions restricted as in (2.15). Moreover, with this same restriction on the partitions, and $f \in K^k[a,b]$, they were able to show that the result of Theorem 2.6 is sharp.

Since in the case where $L = D^k$, L-splines are a special case of G-splines, the way is open to generalize G-splines to an arbitrary linear differential operators.

CHAPTER IV

THE APPROXIMATION OF LINEAR FUNCTIONALS

A. Theoretical Results

In this chapter we discuss the relationship between spline theory and the approximation of linear functionals of the form

$$Lf = \sum_{j=0}^{k} \int_a^b a_j(x)\, f^{(j)}(x)dx + \sum_{j=0}^{k} \sum_{i=1}^{r} b_{ij}\, f^{(j)}(x_{ij}), \tag{4.0}$$

where $f \in C^k[a,b]$, the functions $a_j(x)$ are piecewise continuous on [a,b], and the points x_{ij} lie in [a,b]. Special cases of (4.0) are

$$\int_a^b f(x)dx, \quad f'(x), \cdots, \quad f^{(k)}(x),$$

where x is fixed in [a,b], so that it includes most of the functionals studied in numerical analysis.

If we want to approximate (4.0), it is natural to do so by a functional of the form

$$\sum_{i=0}^{n} \sum_{j=0}^{m_i-1} c_{ij}\, f^{(j)}(x_i), \tag{4.1}$$

where the constants c_{ij} are picked so as to make

$$Rf = Lf - \sum_{i=0}^{n} \sum_{j=0}^{m_i - 1} c_{ij} \, f^{(j)}(x_i) \qquad (4.2)$$

small in some sense. Here $1 \leq m_i \leq k$ for $i = 0,1,\cdots,n$ and

$$a \leq x_0 < x_1 < \cdots < x_n \leq b,$$

so the approximating functional (4.1) can either be of the open type ($a < x_0$, $x_n < b$), or of the closed type ($a = x_0$, $x_n = b$). In subsequent work, the approximating functional (4.1) will be abbreviated as

$$\sum_{i,j} c_{ij} \, f^{(j)}(x_i).$$

We will now consider two schemes for making (4.2) small: the classical procedure and Sard's procedure [18].

The classical procedure of numerical analysis determines the $N = \sum_{i=0}^{n} m_i$ constants c_{ij} by requiring that $Rf = 0$ whenever f is a polynomial of degree at most $N - 1$. By linearity, this is equivalent to requiring $Rf = 0$ for $f(x) = x^{(N-1-r)}/(N-1-r)!$ for $r = 0,1,\cdots,N-1$, i.e.,

$$L\left[\frac{x^{(N-1-r)}}{(N-1-r)!}\right] = \sum_{i,j} c_{ij} \frac{x_i^{(N-1-r-j)}}{(N-1-r-j)!} \qquad (4.3)$$

for $r = 0,1,\cdots,N-1$, and where $\frac{x^r}{r!} = 0$ if $r < 0$ and $\frac{x^r}{r!} = 1$ if $r = 0$. However, (4.3) is equivalent to requiring that

$$p^{(r)}(x_i) = L\left[\frac{x^{(N-1-r)}}{(N-1-r)!}\right]$$

for $r = 0,\cdots,m_i-1$, and $i = 0,1,\cdots,n$, where

$$p(x) = \sum_{i,j} c_{ij} \frac{x^{(N-1-j)}}{(N-1-j)!} \tag{4.4}$$

Since $p(x)$ is a polynomial of degree $N-1$, we know that these conditions uniquely determine $p(x)$ and, therefore, the constants c_{ij}.

Sard's procedure is a generalization of the classical procedure and requires that $Rf = 0$ whenever f is a polynomial of degree at most $k-1$, where $k < N$. This, however, still leaves $N-k$ free parameters c_{ij}, so that the remaining parameters are determined as follows.

Peano's theorem, which is stated in the Appendix, allows us to write

$$Rf = \int_a^b f^{(k)}(t)\ K(t)\ dt, \tag{4.5}$$

and

$$K(t) = \frac{1}{(k-1)!} R_x[(x-t)_+^{k-1}],$$

where R_x denotes the functional R applied to $(x-t)_+^{k-1}$ as a function of x. The remaining parameters are then determined by requiring that

$$\int_a^b [K(t)]^2 dt \tag{4.6}$$

be a minimum. The resulting formula (4.1) is then called the best approximation to Lf on [a,b] for the set of knots $x_0, \cdots, x_n$. The reason for this name follows from the fact that if we apply Schwartz's inequality to (4.5) we obtain

$$(Rf)^2 \le \|f^{(k)}\|^2 \int_a^b [K(t)]^2 dt,$$

so that if we restrict ourselves to a class of functions for which $\|f^{(k)}(x)\| \le M$, then the minimization of (4.6) gives rise to a formula (4.1) that minimizes Rf within this class.

Let us now relate the results of the classical procedure to spline theory and see how it anticipates the solution to Sard's procedure.

Theorem 2.2 guarantees the existence and uniqueness of 2k-1 splines s_{ij} with multiplicity m_i at x_i such that

$$s_{ij}^{(q)}(x_r) = \begin{cases} 1 & \text{if } (i,j) = (r,q) \\ \\ 0 & \text{if } (i,j) \ne (r,q) \end{cases}$$

for $q = 0, \cdots, m_r - 1$ and $r = 0, \cdots, n$.

We can then write (4.4) as

$$p(x) = \sum_{i,j} p^{(j)}(x_i)\, s_{ij}(x)$$

so that

$$Lp = \sum_{i,j} p^{(j)}(x_i)\ L\ s_{ij}. \tag{4.7}$$

But $Rp = 0$, and in the classical procedure the c_{ij} are uniquely determined; comparing (4.2) and (4.7), we conclude that

$$c_{ij} = L\ s_{ij}$$

if $Rf = 0$ whenever f is a polynomial of degree $N-1$ or less. The same result holds in Sard's procedure.

Theorem 4.1 The best approximation (4.1) to a linear functional L of the form (4.0), in the sense of Sard, is obtained by operating with L on the unique $2k-1$ spline

$$s(x) = \sum_{i,j} f^{(j)}(x_i)\ s_{ij}(x)$$

which interpolates $f(x)$ in the sense of Definition 2.1.

Proof: We want to show that if the constants c_{ij} are chosen so that $Rf = 0$ whenever f is a polynomial of degree at most $k-1$, then we must choose $c_{ij} = L\ s_{ij}$ in order to minimize (4.6). In order to do this, consider

$$Rf = Lf - \sum_{i,j} c_{ij}\, f^{(j)}(x_i), \tag{4.8}$$

where $c_{ij} = L\, s_{ij}$, and

$$\hat{R}f = Lf - \sum_{i,j} \hat{c}_{ij}\, f^{(j)}(x_i), \tag{4.9}$$

where the $\hat{c}_{ij}$ are chosen so that

$$\hat{R}f = 0$$

whenever f is a polynomial of degree at most k-1. When $c_{ij} = L\, s_{ij}$

$$Rf = 0$$

whenever f is a polynomial of degree at most k-1; therefore, Peano's theorem in the Appendix can be used to deduce that

$$Rf = \int_a^b K(t)\, f^{(k)}(t)dt \quad \text{and} \quad \hat{R}f = \int_a^b \hat{K}(t)\, f^{(k)}(t)dt,$$

where

$$K(t) = R_x\left[\frac{(x-t)_+^{k-1}}{(k-1)!}\right] \quad \text{and} \quad \hat{K}(t) = \hat{R}_x\left[\frac{(x-t)_+^{k-1}}{(k-1)!}\right]. \tag{4.10}$$

Consider

$$g(x) = \hat{K}(x) - K(x)$$

which by (4.8), (4.9) and (4.10) can be expressed as

$$g(x) = \sum_{i,j} d_{ij} \frac{(x-x_i)_+^{k-1-j}}{(k-1-j)!}, \qquad (4.11)$$

where

$$d_{ij} = c_{ij} - \hat{c}_{ij}. \qquad (4.12)$$

Then, by (4.11), s(x) is a spline with multiplicity m_i at x_i if

$$s^{(k)}(x) = g(x);$$

so we can write

$$s(x) = \sum_{i,j} s^{(j)}(x_i)\, s_{ij}(x);$$

and since $c_{ij} = L\, s_{ij}$ in (4.8), Rs = 0. Therefore,

$$\int_a^b K(t)\, s^{(k)}(t)dt = \int_a^b K(t)\, g(t)dt = 0,$$

and, hence,

$$\int_a^b [\hat{K}(t)]^2 dt = \int_a^b [K(t)]^2 dt + \int_a^b [g(t)]^2 dt,$$

which shows, by (4.11) and (4.12), that the choice of constants c_{ij} that minimizes (4.6) is

$$c_{ij} = Ls_{ij}.$$

Before we proceed to apply this theorem to certain specific functionals, note that the restriction $k \leq N$ is just the one required for the existence of natural G-splines so that this theorem can be easily extended to cover these splines.

B. Numerical Results for Cubic Splines

In the previous section we saw that once we obtain the unique 2k-1 spline which interpolates $f(x)$ at the points x_j, we will be able to obtain the best approximation to $L(f)$ on $[a,b]$. In this section we propose to show how to obtain this spline when $k = 2$, i.e. for cubic splines.

The problem is as follows. We are given $f \in C^2[a,b]$ and $n + 1$ points in the plane,

$$(x_0,y_o),\cdots,(x_n,y_n)$$

such that $y_i = f(x_i)$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We want to determine a cubic spline for which $s(x_i) = y_i$, $0 \leq i \leq n$. We have already seen that we need two end conditions to specify the spline completely. If we want the theorems of the second chapter to hold, we want $s''(x_i) = 0$ for $i = 0,n$, or for Type A interpolation, $s'(x_i) = f'(x_i)$ for $i = 0,n$. Another condition that arises naturally is to require $s''(x_i) = f''(x_i)$ for $i = 0,n$. This type of interpolation is described by:

<u>Definition 4.1</u> Let $f \in C^k[a,b]$. A function $g(x)$ is said to interpolate $f(x)$ in a Type B problem if

$$g^{(j)}(x_i) = f^{(j)}(x_i) \qquad j = 0, 1, \cdots, m_i - 1$$

for each i, $0 \leq i \leq n$, and

$$g^{(j)}(x_i) = f^{(j)}(x_i) \qquad j = k, \cdots, 2k - m_i - 1$$

for $i = 0,n$.

We will therefore also consider splines of the following type.

<u>Definition 4.2</u> A spline of degree $2k-1$ with multiplicity m_i at x_i $(1 \leq i \leq n-1)$ is said to be of Type B if instead of requiring that

$$s^{(k+j-1)}(x_i) = 0 \quad \text{for} \quad j = 1,\cdots,k-m_i \quad \text{and} \quad i = 0,n,$$

we require that s(x) interpolate some function in a Type B problem.

These splines are the generalization of the splines of Definition 2.2 and although they do not satisfy the first integral relation of Lemma 2.1, we can prove:

Theorem 4.2 There is exactly one spline of Type B that interpolates f(x) in a Type B problem provided $k \leq n + 1$.

Proof: If $s_1(x)$ and $s_2(x)$ are two splines that interpolate f(x) in a Type B problem, then $s(x) = s_1(x) - s_2(x)$ is a spline of multiplicity m_i at x_i that interpolates the zero function in the sense of Definition 2.1. By Theorem 2.1, $s(x) \equiv 0$. Uniqueness is therefore established, and existence follows by the argument of Theorem 2.2.

We also note that the second integral relation is valid for Type B splines. Moreover, Theorem 2.8 holds since the first inequality of Corollary 2.1 is independent of the type of spline.

Let us now derive the equations that define a cubic spline with simple knots, i.e. $m_i = 1$ for all i, and in so doing, illustrate and improve the general results of the second chapter.

Since s"(x) is continuous on [a,b] we can write

$$s''(x) = m_{j-1} \frac{x_j - x}{h_j} + m_j \frac{x - x_{j-1}}{h_j}, \tag{4.13}$$

where $h_j = x_j - x_{j-1}$, and $m_j = s''(x_j)$ for $j = 1,\cdots,n$. Then

$$s'(x) = -m_{j-1} \frac{(x_j-x)^2}{2\,h_j} + m_j \frac{(x-x_{j-1})^2}{2\,h_j} + c_1,$$

and

$$s(x) = m_{j-1} \frac{(x_j-x)^3}{6\,h_j} + m_j \frac{(x-x_{j-1})^3}{6\,h_j} + c_1 x + c_2.$$

The constants c_1 and c_2 can be evaluated by noting that $s(x_j) = y_j$, $s(x_{j-1}) = y_{j-1}$;

$$c_1 = \frac{y_j - y_{j-1}}{h_j} - (m_j - m_{j-1}) \frac{h_j}{6}$$

$$c_2 = \frac{y_{j-1}\,x_j - y_j\,x_{j-1}}{h_j} - (m_{j-1}\,x_j - m_j\,x_{j-1}) \frac{h_j}{6}.$$

The resulting expressions for $s(x)$ and $s'(x)$ are:

$$s(x) = m_{j-1} \frac{(x_j-x)^3}{6\,h_j} + m_j \frac{(x-x_{j-1})^3}{6\,h_j} + \qquad (4.14)$$

$$\left[y_j - \frac{m_j\,h_j^2}{6}\right] \frac{x - x_{j-1}}{h_j} + \left[y_{j-1} - \frac{m_{j-1}\,h_j^2}{6}\right] \frac{x_j - x}{h_j}\,,$$

and

$$s'(x) = -m_{j-1} \frac{(x_j - x)^2}{2\,h_j} + m_j \frac{(x - x_{j-1})^2}{2\,h_j} + \frac{y_j - y_{j-1}}{h_j} - \frac{m_j - m_{j-1}}{6} h_j . \tag{4.15}$$

We must still require the continuity of $s'(x)$ at $x = x_j$, $1 \le j \le n - 1$. From (4.15) we obtain

$$\frac{h_j}{6} m_{j-1} + \frac{h_j + h_{j+1}}{3} m_j + \frac{h_{j+1}}{6} m_{j+1} = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} . \tag{4.16}$$

If we define

$$a_j = \frac{h_j}{h_j + h_{j+1}} \qquad c_j = \frac{h_{j+1}}{h_j + h_{j+1}}$$

$$d_j = 6 \frac{\left[\frac{y_{j+1} - y_j}{h_{j+1}}\right] - \left[\frac{y_j - y_{j-1}}{h_j}\right]}{h_j + h_{j+1}}$$

we can write (4.16) as

$$a_j\, m_{j-1} + 2\, m_j + c_j\, m_{j+1} = d_j \tag{4.17}$$

for $1 \leq j \leq n - 1$. The end conditions that we are considering can be written in the form

$$2\, m_0 + c_0\, m_1 = d_0$$
$$a_n\, m_{n-1} + 2\, m_n = d_n, \tag{4.18}$$

where $|a_n| \leq 2$, $|c_0| \leq 2$. Equations (4.17) and (4.18) then define a tridiagonal system of n linear equations in n unknowns which is irreducibly diagonally dominant if $a_n \cdot c_0 \neq 0$, and strictly diagonally dominant if $|a_n| < 2$, $|c_0| < 2$. In either case the system is nonsingular. See Varga [28] for details.

Before we indicate how to solve equations (4.17) and (4.18), let us show how these equations can be used to improve the results of the second chapter.

Theorem 4.3 Let $f \in C^2[a,b]$, and let $s(x)$ be the unique cubic spline of Type B that interpolates $f(x)$. Then

$$|f^{(j)}(x) - s^{(j)}(x)| \leq 5h^{2-j} \cdot \max_{|y-z| \leq h} |f^{(j)}(y) - f^{(j)}(z)|$$

for $j = 0,1,2$, where

$$h = \max \{x_i - x_{i-1}:\ 1 \leq i \leq n\}.$$

<u>Proof</u>: Let $w_j = \max\limits_{|y-z| \le h} |f^{(j)}(y) - f^{(j)}(z)|$, $b_j = s''(x_j) - f''(x_j)$, and note that

$$d_j = 6\,\frac{s[x_{j+1},x_j] - s[x_j,x_{j-1}]}{h_j + h_{j+1}} = 6s[x_{j-1},x_j,x_{j+1}]$$

so that by (A.3) of the Appendix,

$$d_j = 3f''(\bar{x}_j), \quad x_{j-1} < \bar{x}_j < x_{j+1}.$$

We can then write (4.17) as

$$h_j\, b_{j-1} + 2(h_{j+1} + h_j)b_j + h_{j+1}\, b_{j+1} =$$

$$h_j[f''(\bar{x}_j) - f''(x_{j-1})] + h_{j+1}[f''(\bar{x}_j) - f''(x_{j-1})]$$

$$+ 2(h_j + h_{j+1})\,[f''(\bar{x}_j) - f''(x_j)].$$

Now let $b = \max\{|b_j|: 0 \le j \le n\}$; solve for b_j, and use the triangle inequality to obtain

$$2(h_j+h_{j+1})\,|b_j| \le 2\,h_j\,w_2 + 2\,h_{j+1}\,w_2 +$$

$$2(h_j+h_{j+1})w_2 + h_j\,b + h_{j+1}\,b.$$

Hence, $2\,|b_j| \le 4\,w_2 + b$ for $1 \le j \le n-1$, and since $b_0 = b_n = 0$, $|b| \le 4\,w_2$, so that $|b_j| \le 4\,w_2$ for all j.

Since $s''(x)$ is linear between knots, it follows that

$$|s''(x) - f''(x)| \le |s''(x) - f''(x_i)| + |f''(x_i) - f''(x)| \le 5\,w_2$$

for all x in [a,b] as desired. The proofs for j = 0,1 are similar to those of Theorem 2.6 and are omitted.

<u>Corollary 4.1</u> Let $f \in C^2[a,b]$, and let $\{P_i\}$ be a sequence of partitions of [a,b] such that $N(P_i) \to 0$ as $i \to \infty$. If $s_i(x)$ denotes the unique cubic spline interpolating f(x) on P_i in a Type B problem, then $\{s_i^{(j)}(x)\}$ converges uniformly to $f^{(j)}(x)$ on [a,b] for j = 0,1,2.

<u>Proof</u>: Just note that $\lim_{h\to 0} \max_{|y-z| \le h} |f^{(j)}(y) - f^{(j)}(z)| = 0$ for j = 0,1,2.

This theorem is still valid for Type A interpolation problems since from (4.15) we have

$$2\,m_0 + m_1 = \frac{6}{h_1}\left[\frac{y_1 - y_0}{h_1} - f'(x_0)\right]$$

$$m_{n-1} + 2\,m_n = \frac{6}{h_n}\left[f'(x_n) - \frac{y_n - y_{n-1}}{h_n}\right],$$

and using Taylor's theorem,

$$2\ m_0 + m_1 = 3\ f''(\bar{x}), \qquad x_0 < \bar{x}_1 < x_1$$

$$m_{n-1} + 2\ m_n = 3\ f''(\bar{x}_n), \qquad x_{n-1} < \bar{x}_n < x_n.$$

Consequently,

$$2\ |b_i| \leq 4\ w_2 + b$$

for $0 \leq i \leq n$, so that the theorem with its corollary follows. Ahlberg, Nilson, and Walsh [3] have derived similar results with different assumptions on the smoothness of f.

To solve equations (4.17) and (4.18), we resort to the following direct algorithm which is stable with respect to growth of rounding errors. See [28].

Define

$$u_o = \frac{c_o}{2} \qquad u_i = \frac{c_i}{2 - a_i\ u_{i-1}}, \quad 1 \leq i \leq n - 1$$

$$g_o = \frac{d_o}{2} \qquad g_i = \frac{d_i - a_i\ g_{i-1}}{2 - a_i\ u_{i-1}}\ , \ 1 \leq i \leq n.$$

Then

$$s''(x_n) = g_n, \; s''(x_i) = g_i - u_i \, s''(x_{i+1}), \quad 0 \le 1 \le n - 1 \; .$$

In order to illustrate the convergence and approximating properties of cubic splines we performed several numerical experiments. To discuss the results of these experiments we need the following

Definition 4.3 Let y be an approximation to x. Then the kth decimal place of y is significant if $|x-y| \le 0.5 \; 10^{-k}$.

All the experiments were done on the interval [-5,5] so that $a = -5$, $b = 5$. The subintervals were of equal length so that $a_i = c_i = \frac{1}{2}$ for $i = 1,\cdots,n-1$.

The result of using (4.14) to interpolate $f(x) = \dfrac{1}{1 + x^2}$ can be described as follows: With 50 subintervals the interpolating spline gave results with four significant digits in all three types of interpolation problems. In Type A and B problems, there was a gain of one significant digit around the endpoints. Table 1 below shows some of the results.

Table 1. Approximation of $f(x) = \dfrac{1}{1 + x^2}$

x	f(x)	$m_o = m_n = 0$	Type A	Type B
0.10	0.990099	0.989988	0.989988	0.989988
1.30	0.371747	0.371748	0.371748	0.371748
2.50	0.137931	0.137930	0.137930	0.137930
3.70	0.068074	0.068073	0.068073	0.068073
4.90	0.039984	0.039999	0.039984	0.039984

With 100 subintervals there were five significant digits around the origin. The number of significant digits increased gradually as we went out towards the endpoints reaching a maximum of eight significant digits around ± 1.

For $f(x) = |x|$, there was one significant digit around the origin. We rapidly acquired significant digits as we approached the endpoints; the approximation being exact after ± 1 for 100 subintervals, and ± 2 for 50 subintervals. For $f(x) = \frac{|x| \; x^2}{6}$, which is a spline, the results obtained were exact.

One of the most useful properties of splines is the excellent approximation to the derivative of the approximated function. Table 2 shows some typical results of using (4.15) to approximate the derivative of $f(x) = \frac{1}{1 + x^2}$. Fifty subintervals were used.

Table 2. Approximation of the Derivative of $f(x) = \frac{1}{1 + x^2}$

x	f'(x)	$m_o = m_n = 0$	Type A	Type B
0.10	-0.19605	-0.19687	-0.19627	-0.19627
1.30	-0.35931	-0.35929	-0.35929	-0.35929
2.50	-0.09512	-0.09513	-0.09512	-0.09512
3.70	-0.03429	-0.03429	-0.03429	-0.03429
4.90	-0.01566	-0.01578	-0.01566	-0.01566

For Type A and B interpolation problems, the above results show that there were three significant digits around the origin, five around ± 2, and seven near the end points. In the other type of interpolation there were two significant digits near the end points. For 100 subintervals there was a gain of two significant digits throughout the interval for all three types of interpolation.

If we use (4.15) on $f(x) = \frac{x|x|}{2}$, interpolation becomes exact around ± 0.95 for 100 subintervals, and around ± 1.7 for 50 subintervals. In both cases there was only one significant digit at the origin.

In spite of the results of Corollary 4.1, (4.13) is rarely used to approximate the second derivative of a function. Instead, (4.15) is used twice; once on the function, and then on the resulting spline. The results of using both methods in a Type A problem are shown below. One hundred subintervals were used.

Table 3. Approximation of the Second Derivative of $f(x) = 1/1+x^2$ in a Type A Problem

x	f"(x)	s"(x)	Spline on Spline
0.05	-1.970187	-1.960204	-1.969939
0.95	0.495926	0.494363	0.495941
1.95	0.187921	0.188059	0.187921
2.95	0.054977	0.055011	0.054977
3.95	0.020019	0.020028	0.020019
4.95	0.008743	0.087456	0.008744

Although the improvement is remarkable, the spline on spline method requires $f \in C^3[a,b]$. The above remark was verified by experiments on the spline $f(x) = \frac{|x|x^2}{6}$ where (4.13) gave exact results throughout the interval, but the spline on spline method had only one significant digit around the origin where the spline fails to have a continuous third derivative.

The preceding numerical experiments are limited, but indicate the usefulness of splines in approximation problems. Splines have also been used to solve ordinary differential equations with limited success. See [14] and [15]. The use of splines in boundary value problems is treated in [5], [6] and [29], while smoothing by spline functions is discussed in [17]. Finally, [3] is an excellent overall reference to numerical applications of splines.

APPENDIX

APPENDIX

Suppose f(x) is defined at the distinct points $x_0, x_1, \cdots, x_n, \cdots$ where we assume that

$$x_0 < x_1 < \cdots < x_n < \cdots.$$

The first divided difference of f is defined by

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

The nth divided difference of f is defined by

$$f[x_0, \cdots, x_n] = \frac{f[x_0, \cdots, x_{n-1}] - f[x_1, \cdots, x_n].}{x_0 - x_n}$$

An induction argument then shows that we can write

$$f[x_0, \cdots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{w'(x_j)}, \tag{A.1}$$

where $w(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$.

From (A.1) it follows that

a) divided differences are symmetric with respect to their arguments, and

b) the operation of taking divided differences is linear.

Another induction argument yields

$$f(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + \cdots +$$

$$(x-x_0) \cdots (x-x_{n-1})\, f[x_0,\cdots,x_n] +$$

$$(x-x_0) \cdots (x-x_n)\, f[x_0,\cdots,x_n,x].$$

By making use of (A.1), it is easy to show that

$$\lim_{x \to x_i} (x-x_0) \cdots (x-x_n)\, f[x_0,\cdots,x_n,x] = 0, \qquad 0 \le i \le n$$

so that

$$P_n(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + \cdots + \tag{A.2}$$

$$+ (x-x_0) \cdots (x-x_{n-1})\, f[x_0,\cdots,x_n]$$

is the unique interpolating polynomial for f at $x_0,\cdots,x_n$.

Suppose now that $f \in C^n[x_0,x_n]$, and consider the function

$$\phi(x) = f(x) - P_n(x).$$

Then $\phi(x) = 0$ at the $n + 1$ points $x_0,\cdots,x_n$, so that by Rolle's theorem its first derivative $\phi^{(1)}$ must vanish at n points. Repeated use of this argument yields the result that $\phi^{(n)}$ must vanish at least once in (x_0,x_n), say at $\bar{x}$. Then,

$$\phi^{(n)}(\bar{x}) = f^{(n)}(\bar{x}) - n!\, f[x_0,\cdots,x_n] = 0,$$

or

$$f[x_0,\cdots,x_n] = \frac{f^{(n)}(\bar{x})}{n!}, \quad x_0 < \bar{x} < x_n. \tag{A.3}$$

The above equation has for consequence that, if f is a polynomial of degree n with leading coefficient a_n, then

$$f[x_0,\cdots,x_n] = a_n.$$

In particular, the nth divided difference of a polynomial of degree at most n - 1 vanishes. We have proved the necessity of the condition in

<u>Theorem A.1</u> Let f(x) be defined at the n + 1 distinct points $x_0,\cdots,x_n$. A necessary and sufficient condition to guarantee the existence and uniqueness of a polynomial of degree k - 1, $k \le n$, that interpolates f at $x_0,\cdots,x_n$, is that

$$f[x_i,\cdots,x_{i+k}] = 0$$

for all i, $0 \le i \le n - k$.

Proof: To prove the sufficiency of the condition we show that the unique interpolating polynomial as given by (A.2) is actually of degree $k - 1$. We do this by proving that

$$f[x_0,\cdots,x_{k+j}] = 0$$

for $0 \leq j \leq n - k$. That this is actually the case follows from the definition of a $(k+j)$th divided difference which shows that it is a linear combination of kth divided differences. Since our condition guarantees that all kth divided differences vanish, we are through.

Suppose R is a linear functional of the form

$$Rf = \sum_{j=0}^{k} \int_a^b a_j(x)\, f^{(j)}(x)dx + \sum_{j=0}^{k} \sum_{i=1}^{\ell} b_{ij}\, f^{(j)}(x_{ij}),$$

where $f \in C^k[a,b]$, the functions $a_j(x)$ are piecewise continuous on $[a,b]$, and the points x_{ij} lie in $[a,b]$. We then have the following

Theorem A.2 (Peano) If $Rf = 0$ whenever f is a polynomial of degree at most $k - 1$, then for all $f \in C^k[a,b]$

$$Rf = \int_a^b f^{(k)}(t)\, K(t)dt,$$

where

$$K(t) = \frac{1}{(k-1)!} \; R_x[(x-t)_+^{k-1}].$$

The notation R_x means that the functional R is applied to $(x-t)_+^{k-1}$ as a function of x.

Proof: By Taylor's theorem, for $x \in [a,b]$

$$f(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(a)}{j!} (x-a)^j + \frac{1}{(k-1)!} \int_a^x f^{(k)}(t) \; (x-t)^{k-1} dt,$$

and, since Rf = 0 whenever f is a polynomial of degree at most k - 1,

$$Rf = \frac{1}{(k-1)!} R_x \int_a^x f^{(k)}(t) \; (x-t)^{k-1} dt.$$

The theorem follows from the observation that this last integral can be written as

$$\int_a^b f^{(k)}(t) \; (x-t)_+^{k-1} dt$$

and that the form of our functional R allows an interchange of R with the integral sign.

Corollary A.1 If $g \in C^k[x_0,x_k]$, then

$$g[x_0,\cdots,x_k] = \frac{1}{(k-1)!} \int_{x_0}^{x_k} B_{k-1,0}(x) \; g^{(k)}(x) dx.$$

Proof: Apply Peano's theorem to

$$Rg = g[x_0,\cdots,x_k] = \sum_{j=0}^{k} \frac{g(x_j)}{w'(x_j)} .$$

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Ahlberg, J. H., E. N. Nilson, and J. L. Walsh: Fundamental properties of generalized splines. Proc. Nat. Acad. Sci. U.S.A. 52, 1412-1419 (1964).

2. _______, Convergence properties of generalized splines. Proc. Nat. Acad. Sci. U.S.A. 54, 344-350 (1965).

3. _______, The Theory of Splines and their Applications. New York: Academic Press, 1967.

4. Barrondale, I., and A. Young: A note on numerical procedures for approximation by spline functions. Comput. J, 9.318-320 (1966).

5. Birkhoff, G., C. deBoor, B. Swartz, and B. Wendroff: Rayleigh-Ritz approximation by piecewise cubic polynomials. SIAM J. Numer. Anal. 3, 188-203 (1966).

6. Ciarlet, P. G., M. H. Schultz, and R. S. Varga: Numerical methods of high-order accuracy for nonlinear boundary value problems. I. One dimensional problem. Numer. Math. 9, 394-430 (1967).

7. Curry, H. B. and I. J. Schoenberg: On Pólya frequency functions IV. J. Analyse Math. 17, 71-107 (1966). (MRC Tech. Summary Report 567 (1965))

8. Davis, P. J.: Interpolation and Approximation, New York: Blaisdell Publishing Company, 1963.

9. Fitzgerald, C. H., and L. Schumaker: A differential equation approach to interpolation at extremal points. (20 pp.) MRC Tech. Summary Report 731 (1967).

10. Greville, T. N. E.: Numerical procedures for interpolation by spline functions. SIAM J. Numer. Anal. 1, 53-68 (1965). (MRC Tech. Summary Report 450 (1964))

11. Johnson, R. S.: On monosplines of least deviation. Trans. Amer. Math. Soc. 96, 458-477 (1960).

12. Karlin, S., and Z. Ziegler: Tchebycheffian spline functions. SIAM J. Numer. Anal. 3, 514-543 (1966).

13. _______, and L. Schumaker: The fundamental theorem of algebra for Tchebycheffian monosplines. J. Analyse Math. 20, 233-270 (1967).

14. Loscalzo, F. R., and I. J. Schoenberg: On the use of spline functions for the approximation of solutions of ordinary differential equations (8 pp.) MRC Tech Summary Report 723 (1967).

15. _______, and T. D. Talbot: Spline function approximations for solutions of ordinary differential equations. SIAM J. Numer. Anal. 4, 433-445 (1967).

16. Marsden, M., and I. J. Schoenberg: On variation diminishing spline approximation methods (27 pp.) MRC Tech. Summary Report 694 (1966).

17. Reinsch, C. H.: Smoothing by spline functions. Numer. Math. 10, 177-183 (1967).

18. Sard, A.: Linear Approximation, Math Surveys No. 9. Providence, R. I.: Amer. Math. Soc. 1963.

19. Schoenberg, I. J.: On monosplines of least deviation and best quadrature formulae. SIAM J. Numer. Anal. 2, 144-170 (1965).

20. _______, On trigonometric spline interpolation. J. Math. Mech. 13, 795-825 (1964).

21. _______, On Hermite-Birkhoff interpolation. J. Math. Anal. Appl. 16, 538-543 (1966). (MRC Tech. Summary Report 659 (1966).)

22. _______, On monosplines of least square deviation and best quadrature formulae II. SIAM J. Numer. Anal. 3, 321-328 (1966).

23. _______, On spline functions (52 pp.). MRC Tech. Summary Report 625 (1966).

24. _______, On the Ahlberg-Nilson extension of spline interpolation: the g-splines and their optimal properties (35 pp.). MRC Tech. Summary Report 716 (1966).

25. Schultz, M. H., and R. S. Varga: L-Splines. Numer. Math. 10, 345-369 (1967).

26 Schumaker, L. L.: Uniform approximation by Tchebycheffian spline functions. (28 pp.) MRC Tech. Summary Report 768 (1967).

27. _______, Uniform approximation by Tchebycheffian spline functions. II. Free knots. (15 pp.) MRC Tech. Summary Report 810 (1967).

28. Varga, R. S.: Matrix Iterative Analysis, New Jersey: Prentice-Hall, Inc. 1962.

29. Wendroff, B.: Bounds for eigenvalues of some differential operators by the Rayleigh-Ritz method. Math. Comp. 19, 218-224 (1965).

OTHER REFERENCES

Ahlberg, J. H., and E. N. Nilson: Convergence properties of the spline fit. SIAM J. Appl. Math. 11, 95-104 (1963).

_______, Orthogonality properties of spline functions. J. Math. Anal. Appl. 11, 321-337 (1965).

_______, The approximation of linear functionals. SIAM J. Numer. Anal. 3, 173-182 (1966).

_______, and J. L. Walsh: Extremal, orthogonality, and convergence properties of multidimensional splines. J. Math. Anal. Appl. 12, 27-48 (1965).

_______, Best approximation and convergence properties of higher-order spline approximations. J. Math. Mech. 14, 231-244 (1965).

Birkhoff, G.: Error bounds for spline interpolation. J. Math. Mech. 13, 827-835 (1964).

_______, Local spline approximation by moments. J. Math. Mech. 16, 987-990 (1967).

_______, and C. de Boor: Piecewise polynomial interpolation and approximation. Approximation of Functions, H. L. Garabedian (ed.) (pp. 164-190). Amsterdam: Elsevier Publishing Company, 1965.

_______, and H. L. Garabedian: Smooth surface interpolation. J. Math. and Phys. 39, 258-268 (1960).

de Boor, C.: Bicubic spline interpolation. J. Math. and Phys. 41, 212-218 (1962).

_______, Best approximation properties of spline functions of the odd degree. J. Math. Mech. 12, 747-749 (1963).

_______, and R. E. Lynch: On splines and their minimum properties. J. Math. Mech. 15, 953-969 (1966).

Greville, T. N. E.: Interpolation by generalized spline functions. (44 pp.) MRC Tech. Summary Report 476 (1964).

_______, Spline functions, interpolation, and numerical quadrature. Mathematical Methods for Digital Computers, Volume 2, A. Ralston and H. S. Wilf (eds.) (pp. 156-168). New York: John Wiley and Sons, 1967.

Rice, J. R.: Characterization of Chebyshev approximations by splines. SIAM J. Numer. Anal. 4, 557-565 (1967).

Schoenberg, I. J.: Contributions to the problem of approximation of equidistant data by analytic functions. Parts A and B. Quart. Appl. Math. 4, 45-99, 112-141 (1946).

_______, Spline functions, convex curves and mechanical quadrature. Bull. Amer. Math. Soc. 64, 352-357 (1958).

_______, On best approximation of linear operators. Nederl. Akad. Wetensch. 67, 155-163 (1964).

_______, On interpolation by spline functions and its minimal properties. Proc. Conf. on Approximation Theory in Oberwolfach, Germany, 1963 (pp. 109-129). Basel: Birhäuser, 1964.

_______, Spline functions and the problem of graduation. Proc. Nat. Acad. Sci. U.S.A. 52, 947-950 (1964).

_______, Spline interpolation and the higher derivatives. Proc. Nat. Acad. Sci. U.S.A. 51, 24-28 (1964).

_______, and A. Whitney: On Pólya frequency functions III. Trans. Amer. Math. Soc. 74, 246-259 (1953).

Secrest, D.: Error bounds for interpolation and differentiation by the use of spline functions. SIAM J. Numer. Anal. 2, 440-447 (1965).

Sharma, A., and A. Meir: Degree of approximation of spline interpolation. J. Math. Mech. 15, 759-767 (1966).

Walsh, J. L., J. H. Ahlberg, and E. N. Nilson: Best approximation properties of the spline fit. J. Math. Mech. 11, 224-234 (1962).