PERIODIC-REVIEW POLICIES FOR A SYSTEM WITH EMERGENCY ORDERS

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
H. Milton Stewart School of Industrial and Systems Engineering

Georgia Institute of Technology
December 2008

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I dedicate this work to those who are the essence of my life: Soledad, Joaquin, and Maximiliano.

ACKNOWLEDGEMENTS

This doctoral dissertation is the result of almost four years of hard work, with invaluable help from my advisor Doctor Christos Alexopoulos, who patiently went over many revisions, and my co-advisor Doctor Mark Ferguson, who provided valuable insights, motivation, and reality checks.

I would also like to extend my gratitude to Doctors Hayriye Ayhan, David Goldsman, and Paul M. Griffin. I appreciate their time and effort in serving on my Ph.D. committee.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

Most major modern manufacturers use some combination of transportation modes to source parts from overseas facilities. Often, they use ocean freight as a regular mode to meet their more predictable requirements and air freight as an emergency mode to meet unexpected imbalances between supply and demand. The vast majority of publications in the literature assume that both supply modes are available in every period or that the times between regular order placement opportunities are equal to the regular order lead-time. The restriction that regular orders can be placed at a lower frequency than emergency orders results in a periodic Markov decision process that is significantly more complex to optimize than when both modes are available at every period. The inclusion of a setup cost for the emergency mode further increases the difficulty as it is necessary to optimize functions that are not convex.

This thesis achieves two goals. The first goal is to close the aforementioned gap in the literature by studying an inventory system with two potential supply modes having different frequencies for order placement opportunities and a setup cost for emergency orders. For a regular order lead-time equal to two periods, we derive an optimal policy that minimizes the expected total discounted cost, and provide a value iteration algorithm for computing the parameters of the optimal policy. Computational experience indicates that this policy remains optimal for lead-times exceeding two periods. Since the algorithm for computing the optimal policy requires significant computational effort, we also develop and evaluate two heuristic policies whose operational parameters can be computed with relatively small computational effort.

The second goal is the development of a multi-echelon inventory system simulator with the flexibility to model and evaluate various inventory related decisions such as inventory allocation policies or reorder policies of the type depicted in this thesis. We achieve this goal with the Inventory Simulator Workbench, ISW. This simulator includes a graphical user

interface to draw inventory networks and specify inventory policies and parameters. Since the simulator is developed in Java, we further achieve the goal of providing a multiplatform simulator that can be expanded following the rules of Object-oriented Programming.

CHAPTER I

INTRODUCTION

Manufacturers and assemblers in the U.S.A. and Europe are increasingly sourcing parts from overseas facilities, as the lower purchase costs more than compensate for the increased transportation costs of changing from a local sourced part to a part sourced from overseas. Transportation of overseas parts often takes two forms: ocean freight which incurs a lower transportation cost, but is slower and more restrictive, or air freight which incurs a higher transportation cost, but is faster and more flexible in terms of delivery times. Most major modern manufacturers use some combination of both modes, shipping parts via ocean freight to meet their more predictable requirements and using air freight to meet unexpected imbalances between supply and demand.

Ocean freight places strong restrictions on a firm in terms of when an order can be placed and expected at the destination. Contracts with ocean freight carriers often require the cargo to be at the dock and ready to load by a certain day of the week, or month, when the vessel is scheduled to embark. After a long intercontinental trip, ship arrivals are arranged: this forces vessels to spend a minimum amount of time at ports, adding up to a very restrictive schedule for goods shipped in this mode. For example, Ormeci [43] reports that 90% of the fastest 30% of ocean freight services from Hamburg to Charleston, SC and 80% of the fastest 30% of services between Hong Kong and Long Beach, CA are scheduled to arrive between Friday and Sunday. With regard to travel times, a freight trip between Hong Kong and Long Beach takes about 11 to 15 days, while a trip from Hamburg to Savannah, GA takes about 11 days. Air freight, on the other hand, often has the advantage of offering one or more flights within a day and travel times less than 24 hours between destinations mentioned above.

Ocean freight costs are commonly computed as a function of consolidated volume units (container size). Since deliveries occur on a regular cycle and shipment quantities are usually very large, a distribution system typically exists to transport the parts from the port to the factory(ies). Hence, the transportation cost for ocean freight is often viewed as variable (per unit) in nature. In the absence of emergency shipment options (air freight is not considered), a base-stock policy would be optimal. Shipping parts by air, on the other hand, not only incurs variable costs computed according to the weight of the pallets, but often also includes additional fixed costs such as the cost of sending dedicated trucks to the airport and the cost of expediting the shipment through customs (these costs usually do not depend on the actual quantity of parts that was ordered; see Ormeci [43]). The total costs associated with these two transportation modes differ significantly, with the variable per pound cost for air freight typically being about five times the variable per pound cost of ocean freight (Beyer and Ward [7]).

Because of the aforementioned restrictions, many firms face an ordering problem with two delivery modes: a low cost mode with long lead-times and even longer time intervals between orders, and a high cost mode with short lead-times, short ordering intervals, and a fixed plus variable order cost. In the automobile industry, for example, Chiang [13] reports that Hotai Motor Co. Ltd., the distributor of Toyota Motor Co. products in Taiwan, replenishes the inventory of auto parts by ocean freight as well as by air. In the former case, Hotai places orders for thousands of auto parts once per week (there is an order-up-to level for each part) from Toyota Motor Co. in Japan. In the latter case, if the inventory level of a part falls below a "warning" point, Hotai has the opportunity to place an emergency order to be shipped via air freight every day. In the computer industry, Hewlett-Packard Corp. manufactures a major subassembly for network servers in Singapore and ships it to four distribution centers worldwide, where the assembly is completed based on customer specifications. Again, HP uses two modes of transportation (air and ocean) between the factory and the distribution centers (Beyer and Ward [7]).

This research derives an optimal inventory policy for a system that can only place regular orders at a fixed frequency but can always resort to an emergency mode with variable (per unit) cost and a setup cost.

Example 1 To understand the cost benefits of such an optimal policy, consider a system

Table 1: Parameters for Case Study in Example 1

Demand Distribution	Poisson(2)
Regular Order Lead-time	2
Regular Review Cycle	Regular Order Lead-time + 3
Emergency Order Variable Cost	5
Emergency Order Fixed Cost	2, 5, 50
Regular Order Variable Cost	1
Backorder Penalty Cost	10
Holding Cost	1
Discount Factor	0.99

with the parameters listed in Table 1. Specifically, we compare the expected total discounted cost under the optimal policy described in Section 2.4 against the expected cost associated with two different heuristics. The first heuristic combines two policies, one for the regular mode and another for the emergency mode. In particular, we use an order-up-to-R policy to control orders under the regular mode with parameters obtained under the assumption that the emergency mode is not available. The policy of choice for the emergency mode is an (s, S) policy with parameters computed assuming that the regular mode is not available. The parameters for this first heuristic can be obtained from the literature or there exist easy to compute methods to estimate them. The second heuristic uses the order-up-to-R policies presented in Chiang and Gutierrez [13], and would be optimal if orders placed using the emergency mode had no fixed costs.

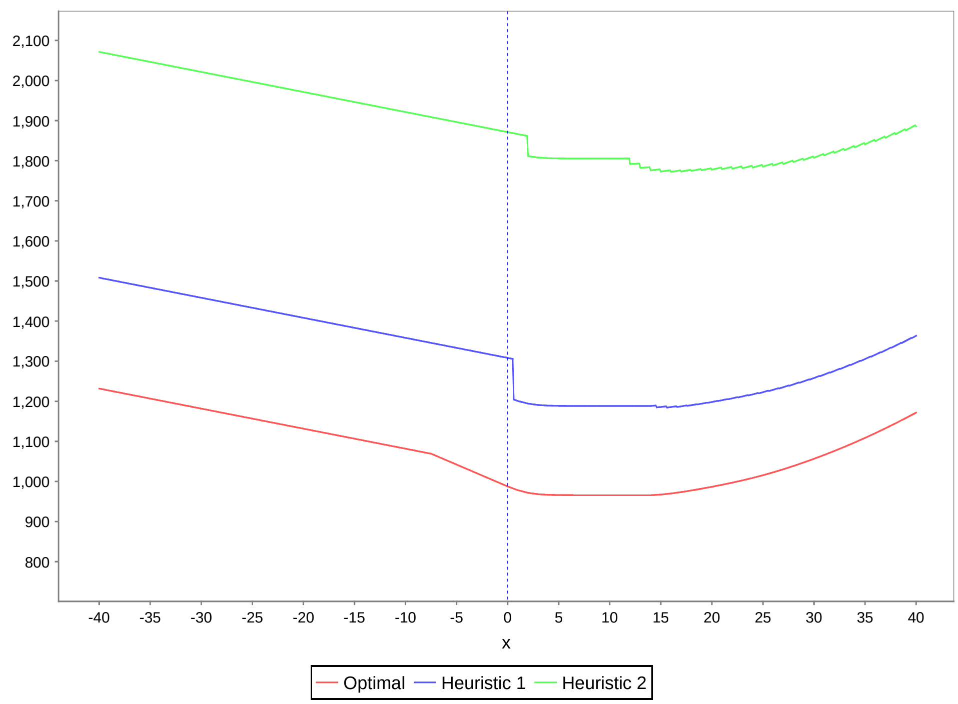
Using stochastic dynamic programming, we computed the expected total cost functions for the optimal policy and the two heuristics using the set of parameters shown in Table 1. Table 2 displays the maximum percentage difference between the optimal cost and the optimal costs obtained by the heuristics when the initial inventory takes values from the interval $[-40, 40]$. Figure 1 displays the three cost functions in terms of the initial inventory x for a fixed cost of $K = 50$.

The economic benefits displayed in this example, and further shown later in this thesis, could be surpassed by the computational burden of implementing the algorithms required to obtain the optimal policy parameters. This motivates the development of two heuristic

Table 2: Maximum Differences for the Optimal Costs for the Designs in Table 1

	Maximum Difference Optimal versus First Heuristic	Maximum Difference Optimal versus Second Heuristic
$K = 2$	13.6%	1.1%
$K = 5$	12.7%	5.3%
$K = 50$	32.8%	91.5%

Figure 1: Comparison of Cost Functions in Terms of the Initial Inventory x



policies (that are more effective than the heuristics of Example 1) along with a study for comparing them versus the optimal policy.

The need for evaluation of the proposed optimal policy and the heuristics via simulation and the lack of public-domain, user-friendly simulation suites tailored for multi-echelon inventory systems motivated the development of the Inventory Simulator Workbench (ISW). The ISW is a flexible tool because it allows the user to draw networks of supplier and demand nodes, while its graphical user interface (GUI) and editing tools facilitate the implementation of complicated inventory policies.

This thesis proceeds as follows: Chapter II provides an optimal policy for the system under study and Chapter III develops and analyzes two heuristics. Chapter IV describes the development the ISW in Java. Chapter V ends with conclusions and future research directions.

CHAPTER II

OPTIMAL INVENTORY POLICY

The goal of this chapter is to obtain an optimal policy for the system described in Chapter I. We proceed as follows: In Section 2.1 we review the related literature, in Section 2.2 we provide the key assumptions used in this research, and in Section 2.3 we define the necessary notation. In Section 2.4 we state (without proof) the optimal inventory policy for the system under study for a regular-order lead-time equal to two periods and present numerical evidence for the extension of the result to longer lead-times. In Section 2.5 we define the set of optimality equations for the dynamic program based on the underlying Markov decision process. In Sections 2.6 and 2.7 we establish the optimality of the policy described in Section 2.4 for a regular-order lead-time of two periods. In Section 2.8 we provide characteristics of the optimal policy. In Section 2.9 we demonstrate a stopping criterion, we state an algorithm to compute the optimal parameters, and we define an experimental design. In Section 2.10 we draw conclusions for this chapter. The appendices include auxiliary results from the literature.

2.1 Literature Review

The problem of using multiple supply modes efficiently is complex and has been studied by several authors; see Minner [38] for a recent review. We classify the relevant literature in two groups. The first group includes algorithms for computing the parameters of heuristic inventory control policies. This group starts with the publication of Moinzadeh and Nahmias [39], who develop an approximate model of an inventory control system with two options for resupply on every period, one of them with a shorter lead-time. They study the application of two (r, Q) simultaneous ordering policies based on the current inventory position. Whenever the reorder point r_1 for the regular mode is reached, an order of size Q_1 is placed. If the emergency reorder point r_2 is reached within the replenishment lead-time of the regular order, an order of size Q_2 is placed, but only if this order will arrive before the delivery of the outstanding regular order. Moinzadeh and Schmidt [40] study a continuous-review inventory control policy for a system with two supply modes available at any time. Orders are placed whenever the inventory position is below a target value of R. This study covers a lost sales case and a backlogging case, modeling systems with an ordering process that allows up to R outstanding orders in the first case and an unlimited number in the second case. The proposed approximate policy is based not only on the inventory position, but also considers the arrival time of an order to determine the amount and type of order to place.

Johansen and Thortenson [33] study a continuous-review inventory system with an emergency supply mode to hedge against demand uncertainty during a regular mode replenishment lead-time. The demand is modelled as a stationary Poisson process and a standard (r, Q) policy is used for controlling the regular replenishment orders, which are assumed to have a relatively long and constant lead-time. On the other hand, emergency orders also have a constant but shorter lead-time. They assume that only one regular order may be outstanding at any time and that, during that time period, emergency orders are issued according to reorder points and order-up-to levels depending on the time remaining until the regular order is delivered. Two algorithms are provided, the first to minimize the expected total inventory cost rate with state-dependent emergency orders and the second to find the best state-independent emergency order policy.

Tagaras and Vlachos [59] consider a periodic-review inventory system with two replenishment modes. Regular orders are placed periodically following a base-stock policy based on the inventory position. The system also has the option of placing emergency orders, characterized by a shorter lead-time but higher acquisition cost. During a regular replenishment cycle, the necessity and size of an emergency order is determined according to a base-stock policy based on the net inventory. The timing of an emergency order is such that this order arrives and can be used to satisfy the demand in the time period just before the arrival of a regular order. Axsäter [1] models a continuous-review system which at any time has the possibility to place regular or emergency orders, the latest with an additional cost but shorter lead-times. He also proposes a heuristic to determine the timing and size of an emergency order while keeping an (r, Q) policy for regular orders that is optimal in the absence of the emergency mode.

The second literature group includes publications that derive optimal inventory policies. This body of knowledge starts with Barankin [2], who studies a single-period inventory model with two potential supply modes, with lead-times equal to one and zero periods and linear ordering costs. Daniel [16] treats an extension of Barankin's model to multiple planning periods and derives the form of an optimal policy assuming that the size of an emergency order is bounded from above by a given constant. Fukuda [24] considers an inventory model with two or three supply modes available in every period and both variable and fixed plus variable acquisition cost structures. The modes have delivery lead-times that differ by one period. Chiang and Gutierrez [13] consider two supply modes, namely a regular and an emergency mode. The emergency mode can be used during any period, but regular-mode orders can be placed only at a frequency that is lower than the emergency mode frequency. The authors state that order-up-to-R policies are optimal at both emergency and regular review periods, with the size of a regular order depending on the size of the emergency order. They also derive a stopping rule for a value iteration algorithm to compute the optimal parameters. Chiang [11] further restricts the last model assuming that regular and emergency mode lead-times differ by one period and devises a simple algorithm to compute the optimal policy parameters.

Sethi, Yan and Zhang [53] study a periodic-review inventory system with fast and slow delivery modes, setup cost, and regular demand forecast updates. At the start of each period, on-hand inventory and demand information are updated. At the same time, decisions on how much to order using fast and slow delivery modes are made. Those orders are delivered at the end of the current period and at the end of the next period, respectively. A forecast-update-dependent (s, S) type policy is shown to be optimal.

Bylka [9] presents a periodic-review capacitated lot sizing model with limited backlogging and a possibility of emergency orders at every review period with no time lag. He models this system as a discrete-time Markov decision process, and describes a simple and efficient value iteration algorithm for finding an optimal policy.

Most of the aforementioned papers assume that both supply modes, emergency and regular, are available in every period and that the times between regular order placement opportunities are equal to the regular order lead-time. The exceptions include Chiang and Gutierrez [12] and Chiang [11] who also assume that the regular supply mode is not always available but they consider only a variable cost for both modes. The restriction that regular orders can be placed at a lower frequency than emergency orders results in a periodic Markov decision process that is more complex to optimize than when both modes are available at every period. The inclusion of a setup cost for the emergency mode further increases the difficulty as it is necessary to optimize functions that are not convex. In this chapter we obtain an optimal policy for this inventory system when the regular mode lead-time equals two periods and provide a value iteration algorithm to compute its optimal parameters.

2.2 System Description and Key Model Assumptions

Our major assumptions are as follows.

Assumption 1 *The lead-time of regular orders is larger than the lead-time of emergency orders.*

The case where the lead-times of the regular and emergency orders are equal corresponds to a system where the decision is between two modes that differ only in their cost structure. We do not consider this possibility because it does not conform with our motivation. On the other hand, if the lead-time of a regular order is shorter than that of an emergency order, there would be no reason to use a slower and more expensive emergency mode.

Assumption 2 *On any period, at most one emergency order can be outstanding. Similarly, there can be only one regular order outstanding during a regular review cycle.*

Specifically, regular orders have a lead-time of τ periods and can be placed every m periods ($m > \tau$). The respective time periods are named *regular review* periods while the elapsed time between successive regular review periods is called a *regular review cycle.* Emergency orders also have a lead-time that is shorter than the time interval between the respective placement opportunity epochs.

This assumption may represent a decision maker who will observe the result of previous similar decisions before making a new purchase commitment. Furthermore, the case where more than one order of the same kind can be outstanding during a period or regular order cycle requires a substantially different model than the one under consideration and is left for future research.

Assumption 3 *Regular and emergency order opportunities exist at a fixed frequency. Further, the length of the regular review cycle and the lead-time for regular orders are multiples of the emergency lead-time.*

Under these assumptions, we define emergency lead-times to be one period and, therefore, a regular review occurs every m emergency review periods. We consider an infinite planning horizon and set the first period as a regular review period.

As mentioned in Porteus [47], this type of periodic-review is characteristic of systems that can only place orders to suppliers at a fixed frequency (e.g., once a day or once a week) or the supplier's transportation system has a fixed schedule. In either case, even systems that have online recording of transactions, which could motivate a continuous-review argument, must be modelled with periodic-review intervals.

Note that air freight transportation for emergency orders can be achieved within one or two days, and inventory review cycles as well as regular order lead-times are usually measured in days or weeks. Thus, restricting regular review cycle lengths and regular order lead-times to be multiples of the emergency lead-time simplifies notation and clarifies the model without a loss of generality.

Assumption 4 *Unsatisfied demand is fully backlogged.*

We assume that demand that cannot be satisfied with inventory on hand is fulfilled later with a penalty or backorder cost. In the case of overseas suppliers, the term "demand" refers to distributors or assembly plants that, in the absence of parts provided by their supply chain, would wait for a future arrival (usually at the cost of altering production plans or not having inventory to satisfy customers) and would not seek alternative providers.

Assumption 5 *The cost to place a regular order has only a variable component while emergency orders have a fixed (setup) and a variable (unit cost)component.*

This assumption is dictated by our motivational setting in Chapter I and differentiates this work from Chiang and Gutierrez [12]. We do not assume any relationship between the magnitude of the regular and emergency variable costs.

2.3 Notation

We adopt the following notation.

- Periods: (i, j) where $i \in \{1, 2, \ldots\}$ denotes the regular review cycle and $j \in \{0, \ldots, m-1\}$ is the number of periods elapsed after the last regular review period. Using the modulus function "mod", we define

  $$j^+ := (j+1) \bmod m$$

  $$j^- := (j-1) \bmod m$$

  so that period 0 follows period $m-1$.

- Inventory state variables: The state of the system is described with the following variables:
  - x: Inventory on hand (or net inventory) at the beginning of a period.
  - z: Emergency inventory-position after an emergency order is placed. This variable does not include outstanding regular orders.
  - r: Size of a regular order in transit at the start of a period.
  - $\mathbf{x} := (x, r, j)$: The vector corresponding to the inventory state (net inventory and amount of regular order on transit) augmented with the period j within the regular review cycle.
  - $\mathbf{X}$: The state space of the system.

- Decision variables:

- y^e: Size of the emergency order placed during an emergency review opportunity. Note that $z = x + y^e$.
- y: Size of the regular order placed during a regular order placement opportunity. We have $y = 0$ for $j \neq 0$.
- $\mathbf{d}$: The vector containing the decision variables. We have $\mathbf{d} := (z, y) = (x+y^e, y)$.

- $\mathcal{D}(\mathbf{x})$: The decision space as a function of the state of the system.

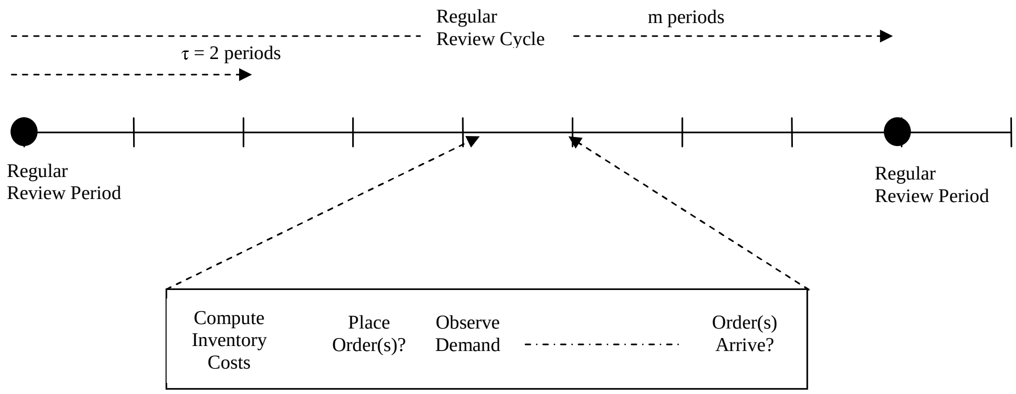
$$\mathcal{D}(\mathbf{x}) = \begin{cases} [x, \infty) \times [0, \infty) & \text{for } j = 0 \\ [x, \infty) \times \{0\} & \text{for } j \neq 0 \end{cases}$$

- Indicator function: $\delta(u) = 1$ if $u > 0$ and 0 otherwise.

The following costs are involved:

- Inventory cost $L(x)$: Holding and shortage cost per period when the inventory on hand at the beginning of the period is x.
- Acquisition costs: The acquisition cost for y^e units in emergency mode is $K\delta(y^e)+c^e y^e$. The acquisition cost for y units in regular mode is $c^r y$.
- Single-period discount factor: $\alpha \in (0, 1)$.

Figure 2: Timeline of Events within a Regular Review Cycle



We also make the following common assumptions.

Assumption 6 .

(a) The expected inventory cost during a single period, $\mathrm{E}_\xi L(x-\xi)$*, is a convex function.*

(b) The following events happen sequentially in every period (see Figure 2):

- *Inventory costs for the period are computed according to the amount of inventory on hand at the beginning of the period.*
- *The inventory policy is applied (i.e., the inventory position is observed and orders are placed), and acquisition costs are computed.*
- *Demand is observed and fulfilled with available inventory.*
- *At the end of the period, outstanding orders (emergency orders placed on the same period and, if applicable, regular orders placed* $\tau - 1$ *periods ago) arrive.*

(c) The demands observed in successive periods form an independent and identically distributed (i.i.d.) sequence of nonnegative random variables with a discrete distribution having a finite first moment. Let $\xi_{(i,j)}$ *be the demand in period* (i,j) *with* $\mathrm{E}[\xi_{(i,j)}] := \mu_\xi < \infty$*.*

Assumption 7 *We assume the following limiting properties:*

$$c^e x + \alpha \mathrm{E}_\xi L(x-\xi) \to \infty \qquad \textit{as } x \to -\infty \tag{1}$$

and

$$\mathrm{E}_\xi L(x-\xi) \to \infty \qquad \textit{as } |x| \to \infty \tag{2}$$

This is a common assumption required to prove the existence of the minimum on the cost functions that we define later (see for example Heyman and Sobel [29, p. 311]). On any period, the inventory cost function $L(x)$, as a function of the initial inventory x, includes both holding and backorder costs. Equation (1) requires that the rate of backorder cost be larger than that of the emergency order variable cost. This should be the case in a real-world system; otherwise, there would be no monetary justification for the emergency mode. Equation (2) requires that holding and backorder costs increase without bound as the inventory or backorder levels increase. In our setting, we expect large amounts of items shipped by slow freight, hence we can assume an unbounded storage capacity and, consequently, unbounded holding costs. Similarly, since we have not assumed bounded demands, we could observe large amounts of backorders.

The goal is to find an inventory policy that minimizes the infinite horizon expected total discounted cost.

2.4 Optimal Inventory Policy

The main result is the structure of the optimal inventory policy stated by the following theorem for $\tau = 2$. The proof of this theorem is given in Section 2.7.1. Its validity for the general case $\tau > 2$ is discussed in Section 2.7.2.

Theorem 1 *For $\tau = 2$, there exist constants (s_j, S_j) for $j \in \{0, \dots, m-1\}$ and a function $Q(z)$ such that the following inventory policy is optimal in state (x, r, j).*

(a) For $j = 0$: if $x < s_0$, place an emergency order for $y^e = S_0 - x$ units; otherwise, do not order ($y^e = 0$). Further, place a regular order of size $Q(y^e + x)$.

(b) For $j = 1$: if $x + r < s_1$, place an emergency order for $S_1 - (x + r)$ units; otherwise, do not order.

(c) For $j \in \{2, \ldots, m-1\}$: if $x < s_j$ place an emergency order for $S_j - x$ units; otherwise, do not order.

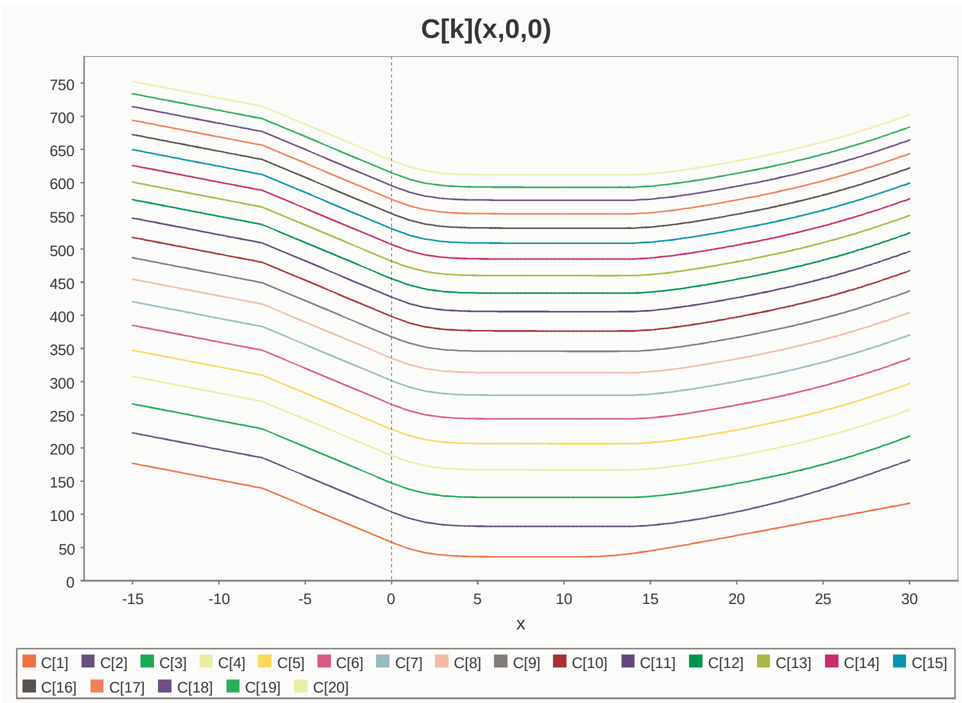
Example 1 (continued) Table 3 shows the parameters for the optimal policy and the two heuristics described in Example 1. Recall that under the first heuristic, the policy for regular orders at regular review periods is an order-up-to-R policy with parameters obtained under the assumption that the emergency mode is not available. Similarly, the emergency mode uses an (s, S) policy with parameters computed assuming that the regular mode is not available. The second heuristic uses the order-up-to-R policies presented in Chiang and Gutierrez [13]. As expected, the parameters of the optimal policy and those of the second heuristic differ for every period within a regular review cycle. This is illustrated in Table 3 where we use the auxiliary variable w_1 to specify the regular order size function $Q(z) = \max\{w_1 - z, 0\}$.

Table 3: Policy Parameters for the Designs in Table 1

Optimal Policy						
	w_1	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	12	(0.8, 2.0)	(2.6, 5.0)	(2.6, 5.0)	(2.6, 4.0)	(2.3, 4.0)
$K = 5$	12	(0.2, 2.0)	(2.0, 6.0)	(2.0, 6.0)	(2.1, 5.0)	(1.8, 4.0)
$K = 50$	13	(-7.5, 2.0)	(0.9, 9.0)	(1.0, 8.0)	(0.5, 6.0)	(-1.2, 4.0)
First Heuristic						
	R	(s, S)				
$K = 2$	14	(2.6, 5.0)				
$K = 5$	14	(2.0, 6.0)				
$K = 50$	14	(0.6, 14.0)				
Second Heuristic						
	w_1	R_0	R_1	R_2	R_3	R_4
$K = 2, 5, 50$	11	2	4	4	4	3

To compute the parameters of the optimal policy for this example we used the algorithm described later in Section 2.9.2. Figure 3 shows the sequence of optimal cost-to-go functions $C_k(x, 0, 0)$ (to be defined formally in Section 2.5.1) for Example 1 with an emergency order fixed cost $K = 50$. Recall that the cost savings are illustrated in Table 2 and Figure 1.

Figure 3: Convergence of the Functions $C_k(x, 0, 0)$ for Example 1



2.5 *Optimality Equations and Dynamic Programming Model*

The (random) total discounted cost for a finite horizon of N regular review cycles, say $T_N(x_{(1,0)})$, when the net inventory at the beginning of the first cycle is $x_{(1,0)}$, can be expressed as

$$\begin{aligned} T_N(x_{(1,0)}) \quad := \quad & \sum_{i=1}^{N} \Bigg[\sum_{j=0}^{m-1} \alpha^{(i-1)m+j} K\delta(z_{(i,j)} - x_{(i,j)}) + c^e(z_{(i,j)} - x_{(i,j)}) \\ & + \alpha^{(i-1)m} c^r y_{(i,0)} \\ & + \sum_{j=0, j \neq \tau-1}^{m-1} \alpha^{(i-1)m+j+1} L(z_{(i,j)} - \xi_{(i,j)}) \\ & + \alpha^{(i-1)m+\tau} L(z_{(i,\tau-1)} + r_{(i,\tau-1)} - \xi_{(i,\tau-1)}) \Bigg] \qquad (3) \end{aligned}$$

where $z_{(i,j)} \geq x_{(i,j)}$ and $y_{(i,0)} \geq 0$. The expression for $T_N(x_{(1,0)})$ does not include the first period's inventory cost $L(x_{(1,0)})$ since it is only a function of the initial inventory, thus a constant for this decision problem. The objective is to minimize the expected total discounted cost $\mathrm{E}[\lim_{N\to\infty} T_N(x_{(1,0)})]$.

Note that our system can be formulated as a periodic Markov reward process, where the undiscounted single-period cost function, denoted by $c(\mathbf{x}, \mathbf{d}, \xi)$, depends not only on the state of the inventory, the decision variables and the demand ξ, but also on the period within the regular cycle. This motivates the inclusion of the variable j in the state $\mathbf{x}$. From the terms of equation (3) we have:

$$\begin{aligned}
c[(x,r,0),(z,y),\xi] &:= K\delta(z-x) + c^e(z-x) + c^r y + \alpha L(z-\xi) \\
c[(x,r,j),(z,y),\xi] &:= K\delta(z-x) + c^e(z-x) + \alpha L(z-\xi) \\
&\qquad \text{for } j \in \{1,\ldots,m-1\} \setminus \{\tau-1\} \\
c[(x,r,\tau-1),(z,y),\xi] &:= K\delta(z-x) + c^e(z-x) + \alpha L(z+r-\xi)
\end{aligned}$$

2.5.1 Optimality Equations

Let $C(x, r, j)$ be the optimal expected total discounted cost-to-go (current single-stage cost plus the discounted total cost from next period and onwards) when we start at an arbitrary period $j \in \{0, \ldots, m-1\}$ with x units on hand and r outstanding units from a regular order. Clearly, $C(x, r, j) = C(x, 0, j)$ for $j \in \{\tau, \ldots, m-1\}$.

In the definitions of the optimality equations as well as in the definitions of the cost to go functions of the dynamic program we use "min" instead of "inf" because we prove that for all $\mathbf{x}$ there exists a decision $\mathbf{d}$ in $\mathcal{D}(\mathbf{x})$ that achieves the infimum.

Based on the single-stage costs $c(\mathbf{x}, \mathbf{d}, \xi)$ and the transition function $f(\mathbf{x}, \mathbf{d}, \xi)$, which describes the evolution of the system, the optimality equations (see Bertsekas [5, Proposition 3.6.1]) are

$$C(\mathbf{x}) = \min_{\mathbf{d}\in\mathcal{D}(\mathbf{x})} \mathrm{E}_\xi[c(\mathbf{x}, \mathbf{d}, \xi) + \alpha C(f(\mathbf{x}, \mathbf{d}, \xi))]$$

where the expectation is taken with respect to the random demand ξ. These functions are the same as to those presented in Section 2.5.2, but without the subindex k.

2.5.2 Dynamic Programming Model

To establish the structural properties of the optimal policy, we use the value iteration approach. Here $C_k(x, r, j)$ is the optimal expected discounted cost-to-go for a finite horizon problem with k stages, when we start at an arbitrary period $j \in \{0, \ldots, m-1\}$ with x units on hand and r outstanding units of regular order. In Section 2.7 we will prove that $C_k(\mathbf{x}) \to C(\mathbf{x})$ as $k \to \infty$ for all $\mathbf{x} = (x, r, j)$.

We use the following indexed auxiliary functions (the last two are indexed in k):

$$G(z) := c^e z + \alpha \mathrm{E}_\xi L(z - \xi) \tag{4}$$

$$V_k(z, r, j) := G(z) + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, r, j^+) \tag{5}$$

$$H_k(z) := \min_{y \geq 0} \{c^r y + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, y, 1)\} \tag{6}$$

We define $C_0(x, z, j) := 0$ for all j. For $k \geq 1$, the cost functions $C_k(\mathbf{x})$ are computed recursively as follows. For $j = 0$:

$$\begin{aligned} C_k(x, r, 0) &:= \min_{y \geq 0, z \geq x} \{K\delta(z - x) + c^e(z - x) + c^r y + \alpha \mathrm{E}_\xi L(z - \xi) + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, y, 1)\} \\ &= \min_{z \geq x} \{K\delta(z - x) + G(z) + H_k(z)\} - c^e x \end{aligned}$$

For $j \in \{1, \ldots, \tau - 2\}$:

$$\begin{aligned} C_k(x, r, j) &:= \min_{z \geq x} \{K\delta(z - x) + c^e(z - x) + \alpha \mathrm{E}_\xi L(z - \xi) + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, r, j^+)\} \\ &= \min_{z \geq x} \{K\delta(z - x) + V_k(z, r, j)\} - c^e x \end{aligned}$$

For $j = \tau - 1$:

$$\begin{aligned} C_k(x, r, \tau - 1) &:= \min_{z \geq x} \{K\delta(z - x) + c^e(z - x) + \alpha \mathrm{E}_\xi L(z + r - \xi) + \alpha \mathrm{E}_\xi C_{k-1}(z + r - \xi, 0, \tau)\} \\ &= \min_{z \geq x} \{K\delta(z - x) + V_k(z + r, 0, \tau - 1)\} - c^e(x + r) \end{aligned}$$

For $j \in \{\tau, \ldots, m - 1\}$:

$$\begin{aligned} C_k(x, r, j) &:= \min_{z \geq x} \{K\delta(z - x) + c^e(z - x) + \alpha \mathrm{E}_\xi L(z - \xi) + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, 0, j^+)\} \\ &= \min_{z \geq x} \{K\delta(z - x) + V_k(z, 0, j)\} - c^e x \end{aligned}$$

2.6 K-Convexity of $C_k(x, z, j)$

The goal of this section is to show that $C_k(x, z, j)$ is K-convex with respect to x. To achieve this, we first consider some auxiliary lemmas. Lemmas 5–7 will be proved under the additional assumption $\tau = 2$. The proofs refer to various definitions and lemmas from the literature delegated to Appendix A.

Lemma 2 *For $k > l$, $(j + l) \in \{1, \ldots, \tau - 1\}$, and a function $\rho : \mathbb{R}_+^2 \to \mathbb{R}_+$, we have:*

(a) If for all x and $l \geq 1$

$$\mathrm{E}_\xi C_{k-l}(x - \xi, r_1, j + l) - \mathrm{E}_\xi C_{k-l}(x - \xi, r_2, j + l) \leq \rho(r_1, r_2)$$

then

$$C_k(x, r_1, j) - C_k(x, r_2, j) \leq \alpha^l \rho(r_1, r_2)$$

and

$$\mathrm{E}_\xi C_k(x - \xi, r_1, j) - \mathrm{E}_\xi C_k(x - \xi, r_2, j) \leq \alpha^l \rho(r_1, r_2)$$

(b) If for all x and $l \geq 1$

$$\mathrm{E}_\xi C_{k-l}(x - \xi, r_1, j + l) - \mathrm{E}_\xi C_{k-l}(x - \xi, r_2, j + l) \geq \rho(r_1, r_2)$$

then

$$C_k(x, r_1, j) - C_k(x, r_2, j) \geq \alpha^l \rho(r_1, r_2)$$

and

$$\mathrm{E}_\xi C_k(x - \xi, r_1, j) - \mathrm{E}_\xi C_k(x - \xi, r_2, j) \geq \alpha^l \rho(r_1, r_2)$$

Proof

Part (a). The proof uses induction on the variable $n \in \{1, \ldots, l\}$. For $n = 1$, stage $k - l + 1$, and period $j + l - 1$, we have that for all x

$$\begin{aligned} & V_{k-l+1}(x, r_1, j + l - 1) - V_{k-l+1}(x, r_2, j + l - 1) \\ &= \alpha \mathrm{E}_\xi [C_{k-l}(x - \xi, r_1, j + l) - C_{k-l}(x - \xi, r_2, j + l)] \\ &\leq \alpha \rho(r_1, r_2) \end{aligned}$$

and

$$\begin{aligned}
C_{k-l+1}(x, r_1, j+l-1) &= \min_{z \ge x}\{K\delta(z-x) + V_{k-l+1}(x, r_1, j+l-1)\} - c^e x \\
&\le \min_{z \ge x}\{K\delta(z-x) + V_{k-l+1}(x, r_2, j+l-1) + \alpha\rho(r_1, r_2)\} - c^e x \\
&= \alpha\rho(r_1, r_2) + C_{k-l+1}(x, r_2, j+l-1)
\end{aligned}$$

Hence

$$C_{k-l+1}(x, r_1, j+l-1) - C_{k-l+1}(x, r_2, j+l-1) \le \alpha\rho(r_1, r_2)$$

and

$$\mathrm{E}_\xi C_{k-l+1}(x-\xi, r_1, j+l-1) - \mathrm{E}_\xi C_{k-l+1}(x-\xi, r_2, j+l-1) \le \alpha\rho(r_1, r_2)$$

Assume that

$$\mathrm{E}_\xi C_{k-l+n-1}(x-\xi, r_1, j+l-n+1) - \mathrm{E}_\xi C_{k-l+n-1}(x-\xi, r_2, j+l-n+1) \le \alpha^{n-1}\rho(r_1, r_2)$$

Then

$$\begin{aligned}
&V_{k-l+n}(x, r_1, j+l-n) - V_{k-l+n}(x, r_2, j+l-n) \\
&\quad = \alpha\mathrm{E}_\xi[C_{k-l+n-1}(x-\xi, r_1, j+l-n+1) - C_{k-l+n-1}(x-\xi, r_2, j+l-n+1)] \\
&\quad \le \alpha^n\rho(r_1, r_2)
\end{aligned}$$

and

$$\begin{aligned}
C_{k-l+n}(x, r_1, j+l-n) &= \min_{z \ge x}\{K\delta(z-x) + V_{k-l+n}(x, r_1, j+l-n)\} - c^e x \\
&\le \min_{z \ge x}\{K\delta(z-x) + V_{k-l+n}(x, r_2, j+l-n) + \alpha^n\rho(r_1, r_2)\} - c^e x \\
&= \alpha^n\rho(r_1, r_2) + C_{k-l+n}(x, r_2, j-l)
\end{aligned}$$

This completes the induction argument.

Part (b). Again, we use induction on n. For $n = 1$, stage $k-l+1$, and period $j+l-1$, we have that for all x

$$\begin{aligned}
&V_{k-l+1}(x, r_1, j+l-1) - V_{k-l+1}(x, r_2, j+l-1) \\
&\quad = \alpha\mathrm{E}_\xi[C_{k-l}(x-\xi, r_1, j+l) - C_{k-j}(x-\xi, r_2, j+l)] \\
&\quad \ge \alpha\rho(r_1, r_2)
\end{aligned}$$

and

$$
\begin{aligned}
C_{k-l+1}(x, r_1, j+l-1) &= \min_{z \geq x}\{K\delta(z-x) + V_{k-l+1}(x, r_1, j+l-1)\} - c^e x \\
&\geq \min_{z \geq x}\{K\delta(z-x) + V_{k-l+1}(x, r_2, j+l-1) + \alpha\rho(r_1, r_2)\} - c^e x \\
&= \alpha\rho(r_1, r_2) + C_{k-l+1}(x, r_2, j+l-1)
\end{aligned}
$$

Hence

$$
C_{k-l+1}(x, r_1, j+l-1) - C_{k-l+1}(x, r_2, j+l-1) \geq \alpha\rho(r_1, r_2)
$$

and

$$
\mathrm{E}_\xi C_{k-l+1}(x-\xi, r_1, j+l-1) - \mathrm{E}_\xi C_{k-l+1}(x-\xi, r_2, j+l-1) \geq \alpha\rho(r_1, r_2)
$$

Assume that

$$
\mathrm{E}_\xi C_{k-l+n-1}(x-\xi, r_1, j+l-n+1) - \mathrm{E}_\xi C_{k-l+n-1}(x-\xi, r_2, j+l-n+1) \geq \alpha^{n-1}\rho(r_1, r_2)
$$

Then we have

$$
\begin{aligned}
& V_{k-l+n}(x, r_1, j+l-n) - V_{k-l+n}(x, r_2, j+l-n) \\
&= \alpha \mathrm{E}_\xi[C_{k-l+n}(x-\xi, r_1, j+l-n+1) - C_{k-l+n}(x-\xi, r_2, j+l-n+1)] \\
&\geq \alpha^n \rho(r_1, r_2)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{k-l+n}(x, r_1, j+l-n) &= \min_{z \geq x}\{K\delta(z-x) + V_{k-l+n}(x, r_1, j+l-n)\} - c^e x \\
&\geq \min_{z \geq x}\{K\delta(z-x) + V_{k-l+n}(x, r_2, j+l-n) + \alpha^n\rho(r_1, r_2)\} - c^e x \\
&= \alpha^n\rho(r_1, r_2) + C_{k-l+n}(x, r_2, j+l-n)
\end{aligned}
$$

This completes the proof. ∎

Lemma 3 *The cost functions $C_k(\cdot)$ satisfy the following properties:*

(a) For all $\gamma \geq 0$ and $j \in \{1, \ldots, m-1\}$:

(a.1) $C_k(x, r, j) \leq C_k(x+\gamma, r, j) + K + c^e\gamma$.

(a.2) $\mathrm{E}_\xi C_k(x-\xi,r,j) \le \mathrm{E}_\xi C_k(x+\gamma-\xi,r,j) + K + c^e\gamma$.

(b) If $C_k(x,r,j) < \infty$ for all x, r, and j, then the following limits and expectations can be exchanged:

$$\lim_{r\to\infty} \mathrm{E}_\xi C_k(x-\xi,r,j) = \mathrm{E}_\xi \lim_{r\to\infty} C_k(x-\xi,r,j)$$

and

$$\lim_{|x|\to\infty} \mathrm{E}_\xi C_k(x-\xi,r,j) = \mathrm{E}_\xi \lim_{|x|\to\infty} C_k(x-\xi,r,j)$$

(c) $C_k(x,r,\tau-1) = C_k(x+r,0,\tau-1)$.

(d) For all $\gamma,\gamma_1,\gamma_2 \ge 0$ and $j \in \{1,\ldots,m-1\}$:

(d.1) $C_k(x,r,j) \le C_k(x,r+\gamma,j) + K + c^e\gamma$.

(d.2) $\mathrm{E}_\xi C_k(x-\xi,r,j) \le \mathrm{E}_\xi C_k(x-\xi,r+\gamma,j) + K + c^e$.

(d.3) $C_k(x,r,j) \le C_k(x+\gamma_1,r+\gamma_2,j) + K + c^e(\gamma_1+\gamma_2)$.

(d.4) $\mathrm{E}_\xi C_k(x-\xi,r,j) \le \mathrm{E}_\xi C_k(x+\gamma_1-\xi,r+\gamma_2,j) + K + c^e(\gamma_1+\gamma_2)$.

Proof

Part (a.1). By the definition of $C_k(x,r,j)$, for $j \in \{1,\ldots,m-1\} \setminus \{\tau-1\}$, we have

$$\begin{aligned}
C_k(x,r,j) &= \min_{z\ge x}\{K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_{k-1}(z-\xi,r,j+1)\} - c^e x \\
&\le \min_{z\ge x+\gamma}\{K\delta(z-x-\gamma) + G(z) + \alpha \mathrm{E}_\xi C_{k-1}(z-\xi,r,j+1)\} + K - c^e x \\
&\qquad \text{[by Lemma A.3 in Appendix A]} \\
&= C_k(x+\gamma,r,j) + K + c^e\gamma
\end{aligned}$$

For $j = \tau - 1$, we have

$$\begin{aligned}
C_k(x,r,\tau-1) &= \min_{z\ge x}\{K\delta(z-x) + G(z+r) + \alpha \mathrm{E}_\xi C_{k-1}(z+r-\xi,0,\tau)\} - c^e(x+r) \\
&\le \min_{z\ge x+\gamma}\{K\delta(z-x-\gamma) + G(z+r) + \alpha \mathrm{E}_\xi C_{k-1}(z+r-\xi,0,\tau)\} \\
&\qquad +K - c^e(x+r) \quad \text{[by Lemma A.3 in Appendix A]} \\
&= C_k(x+\gamma,r,\tau-1) + K + c^e\gamma
\end{aligned}$$

Part (a.2). The proof follows directly from part (a.1).

Part (b). Since the random variable ξ is nonnegative, part (a.1) implies that, w.p.1,

$$C_k(x-\xi,r,j) \le C_k(x,r,j) + K + c^e\xi$$

Taking expectations yields

$$\mathrm{E}_\xi[C_k(x,r,j) + K + c^e\xi] = C_k(x,r,j) + K + c^e\mu_\xi < \infty$$

for all x, r, and j. Then the Dominated Convergence Theorem (Billingsley [8, Theorem 16.4]) implies

$$\lim_{r\to\infty} \mathrm{E}_\xi C_k(x-\xi,r,j) = \mathrm{E}_\xi \lim_{r\to\infty} C_k(x-\xi,r,j)$$

and

$$\lim_{|x|\to\infty} \mathrm{E}_\xi C_k(x-\xi,r,j) = \mathrm{E}_\xi \lim_{|x|\to\infty} C_k(z-\xi,r,j)$$

Part (c). We have

$$\begin{aligned}
C_k(x,r,\tau-1) &= \min_{z\ge x}\{K\delta(z-x) + G(z+r) + \alpha\mathrm{E}_\xi C_k(z+r-\xi,0,\tau)\} - c^e(x+r) \\
&= \min\{G(x+r) + \alpha\mathrm{E}_\xi C_k(x+r-\xi,0,\tau), \\
&\qquad K + \min_{z>x} G(z+r) + \alpha\mathrm{E}_\xi C_k(z+r-\xi,0,\tau)\} - c^e(x+r) \\
&= \min\{G(x+r) + \alpha\mathrm{E}_\xi C_k(x+r-\xi,0,\tau), \\
&\qquad K + \min_{w>x+r} G(w) + \alpha\mathrm{E}_\xi C_k(w-\xi,0,\tau)\} - c^e(x+r) \\
&= C_k(x+r,0,\tau-1)
\end{aligned}$$

Part (d.1). For $j=\tau-1$ we have

$$\begin{aligned}
C_{k-\tau+1}(x,r,\tau-1) &= C_{k-\tau+1}(x+r,0,\tau-1) \quad \text{[by part (c)]} \\
&\le C_{k-\tau+1}(x+r+\gamma,0,\tau-1) + K + c^e\gamma \quad \text{[by part (a.1)]} \\
&= C_{k-\tau+1}(x,r+\gamma,\tau-1) + K + c^e\gamma \quad \text{[by part (c)]}
\end{aligned}$$

Hence, the hypothesis is true for all x and $j=\tau-1$. By Lemma 2 with $j\in\{1,\ldots,\tau-2\}$ and $l=\tau-1-j$ we have

$$\begin{aligned}
C_k(x,r,j) &\le C_k(x,r+\gamma,j) + \alpha^{\tau-1-j}(K+c^e\gamma) \\
&\le C_k(x,r+\gamma,j) + K + c^e\gamma
\end{aligned}$$

Therefore, the hypothesis is true for $1 < j < \tau - 1$. Since $C_k(x, r, j)$ is constant in r when $j \in \{\tau, \ldots, m-1\}$, the result holds as an equality.

Part (d.2). The proof follows directly from part (d.1).

Part (d.3). For $j = \tau - 1$ and $\gamma_1, \gamma_2 \geq 0$ we have

$$\begin{aligned} C_{k-\tau+1}(x, r, \tau-1) &= C_{k-\tau+1}(x+r, 0, \tau-1) \quad \text{[by part (c)]} \\ &\leq C_{k-\tau+1}(x+r+\gamma_1+\gamma_2, 0, \tau-1) + K + c^e(\gamma_1+\gamma_2) \\ &\qquad \text{[by part (a.1)]} \\ &= C_{k-\tau+1}(x+\gamma_1, r+\gamma_2, \tau-1) + K + c^e(\gamma_1+\gamma_2) \end{aligned}$$

Hence, the hypothesis is true for all x and $j = \tau - 1$. By Lemma 2 we have for $j \in \{1, \ldots, \tau-2\}$ and $l = \tau - 1 - j$

$$\begin{aligned} C_k(x, r, j) &\leq C_k(x+\gamma_1, r+\gamma_2, j) + \alpha^{\tau-1-j}[K + c^e(\gamma_1+\gamma_2)] \\ &\leq C_{k-\tau+1+l}(x, r+\gamma, l) + K + c^e(\gamma_1+\gamma_2) \end{aligned}$$

Therefore, the hypothesis is true for $1 < j < \tau - 1$. Since $C_k(x, r, j)$ is constant in r for $j \in \{\tau, \ldots, m-1\}$, it follows that

$$\begin{aligned} C_{k-\tau+1+j}(x, r, j) &= C_{k-\tau+1+j}(x, r+\gamma_2, j) \\ &\leq C_{k-\tau+1+j}(x+\gamma_1, r+\gamma_2, j) + K + c^e\gamma_1 \\ &\qquad \text{[by part (a.1)]} \\ &\leq C_{k-\tau+1+j}(x, r+\gamma, j) + K + c^e(\gamma_1+\gamma_2) \end{aligned}$$

Part (d.4). The proof follows directly from part (d.3). ∎

Lemma 4 establishes additional properties for the functions $C_k(\cdot)$ under the assumption that the functions $C_{k-1}(\cdot)$ are finite, nonnegative, and K-convex in x. The functions $\sigma_{k,j}(r)$ and $\Sigma_{k,j}(r)$ in part (b) will be used in Section 2.8 and are related to the functions $\mathrm{E}_\xi C_k(x - \xi, r, j)$ in the same manner as the functions $S(r)$ and $s(r)$ are related to the cost-to-go functions $C_k(x, r, j)$.

Lemma 4 *For fixed r, the cost functions $C_k(\cdot)$ satisfy the following properties:*

(a) If $C_{k-1}(x,r,2)$ is K-convex in x and $C_{k-1}(x,r,2) < \infty$, then there exists a function $s_{k,1}(r)$ such that for all $x \leq z \leq s_{k,1}(r)$:

(a.1) $C_k(x,r,1) = C_k(z,r,1) + c^e(z-x)$.

(a.2) $\mathrm{E}_\xi C_k(x-\xi,r,1) = \mathrm{E}_\xi C_k(z-\xi,r,1) + c^e(z-x)$.

(b) If $C_{k-1}(x,r,j^+)$ is K-convex in x and $C_{k-1}(x,r,j^+) < \infty$, then there exist functions $\sigma_{k,j}(r)$ and $\Sigma_{k,j}(r)$ such that:

(b.1) $\mathrm{E}_\xi C_k(\sigma_{k,j}(r)-\xi,r,j) = \mathrm{E}_\xi C_k(\Sigma_{k,j}(r)-\xi,r,j) + K$.

(b.2) $\mathrm{E}_\xi C_k(\Sigma_{k,j}(r)-\xi,r,j) \leq \mathrm{E}_\xi C_k(x-\xi,r,j)$ *for all* x.

(b.3) $\mathrm{E}_\xi C_k(x-\xi,r,j) \leq \mathrm{E}_\xi C_k(z-\xi,r,j) + K$ *for* $z \geq x \geq \sigma_{k,j}(r)$.

(b.4) $\mathrm{E}_\xi C_k(x-\xi,r,j) \geq \mathrm{E}_\xi C_k(z-\xi,r,j)$ *for* $x \leq z \leq \sigma_{k,j}(r)$.

(b.5) $\mathrm{E}_\xi C_k(\Sigma_{k,j}(r)-\xi,r,j) \geq C_k(S_{k,j}(r),r,j)$.

(c) If $C_{k-1}(x,r,\tau-1)$ is K-convex in x, then $C_k(x,r,j)$ is continuous in r for all j.

Proof

Part (a.1). Since the random variable ξ is nonnegative, then by Lemma 3(a.1) we have $C_{k-1}(x-\xi,r,2) \leq C_{k-1}(x,r,2) + K + c^e\xi$, w.p.1. It follows that

$$\mathrm{E}_\xi C_{k-1}(x-\xi,r,2) \leq C_{k-1}(x,r,2) + K + c^e\mu_\xi < \infty$$

For fixed r, by parts (a), (c) and (g) of Lemma A.1 in Appendix A, we have $V_k(z,r,1) = G(z)+\alpha\mathrm{E}_\xi C_{k-1}(z-\xi,r,2)$ is K-convex in z. Since $G(z) \to \infty$ as $|z| \to \infty$ and $C_{k-1}(x,r,2) \geq 0$, then by part (h) of Lemma A.1 in Appendix A, there exist functions of r, $(s_{k,1}(r), S_{k,1}(r))$ such that

$$C_k(x,r,1) = \begin{cases} V_k(x,r,1) - c^e x & \text{if } x \geq s_{k,1}(r) \\ K + V_k(S_{k,1}(r),r,1) - c^e x & \text{if } x < s_{k,1}(r) \end{cases}$$

We conclude that for $x \leq z \leq s_{k,1}(r)$

$$C_k(x,r,1) = C_k(z,r,1) + c^e(z-x)$$

Part (a.2). Since $\xi \geq 0$, then $x \leq z \leq s_{k,1}(r)$ implies $x - \xi \leq z - \xi < s_{k,1}(r)$ and therefore $C_k(x-\xi, r, 1) = C_k(z-\xi, r, 1) + c^e(z-x)$, w.p.1. Taking expectations completes the proof.

Part (b). By the same argument used in the proof of part (a), we have

$$C_k(x,r,j) = \begin{cases} V_k(x,r,j) - c^e x & \text{if } x \geq s_{k,j}(r) \\ K + V_k(S_{k,j}(r), r, j) - c^e x & \text{if } x < s_{k,j}(r) \end{cases}$$

Further, $C_k(x,r,j)$ is K-convex in x by Lemma A.2 in Appendix A. $\mathrm{E}_\xi C_k(x-\xi, r, j)$ is also K-convex in x by part (g) of Lemma A.1 in Appendix A. Since for sufficiently large positive x, we have $C_k(x,r,j) = \mathrm{E}_\xi L(x-\xi) + \mathrm{E}_\xi C_{k-1}(x-\xi, r, j+1)$, equation (2) and $C_{k-1}(x-\xi, r, j+1) \geq 0$ imply

$$\lim_{x \to +\infty} C_k(x,r,j) \geq \lim_{x \to +\infty} \mathrm{E}_\xi L(x-\xi) = \infty$$

On the other hand, for sufficiently small $x < 0$ we have

$$C_k(x,r,j) = K + G(S_{k,j}) + \mathrm{E}_\xi C_{k-1}(S_{k,j} - \xi, r, j+1) - c^e x$$

Hence, $\lim_{x \to -\infty} C_k(x,r,j) = \infty$ and Lemma 3(b) implies

$$\lim_{|x| \to \infty} \mathrm{E}_\xi C_k(x-\xi, r, j) = \mathrm{E}_\xi \lim_{|x| \to \infty} C_k(x-\xi, r, j) = \infty$$

By part (h) of Lemma A.1 in Appendix A, there exist functions $\sigma_{k,j}(r)$ and $\Sigma_{k,j}(r)$ such that (b.1)–(b.4) hold.

Part (b.5). We have

$$\begin{aligned} \mathrm{E}_\xi C_k(\Sigma_{k,j}(r) - \xi, r, j) &= \min_x \mathrm{E}_\xi C_k(x-\xi, r, j) \\ &\geq \min_x \mathrm{E}_\xi \min_x C_k(x-\xi, r, j) \\ &= \min_x \mathrm{E}_\xi C_k(S_{k,j}(r), r, j) \\ &= C_k(S_{k,j}(r), r, j) \end{aligned}$$

Part (c). We first observe that $C_k(x,r,j)$ is a function of $C_{k-1}(x,r,j^+)$ and therefore, a function of $C_{k-l}(x, r, (j+l) \bmod m)$, for all $l \leq k$.

For $k < \tau - 1 - j$, $C_k(x,r,j)$ depends on functions that are constant in r; hence it is trivially continuous in r. Therefore, we consider the case $k \geq \tau - 1 - j$.

Since the functions $C_k(x, r, j)$ are constant in r for $j \in \{0\} \cup \{\tau, \ldots, m-1\}$, they are trivially continuous in r.

For $j \in \{1, \ldots, \tau - 1\}$ we write the cost functions in terms of the variable l as $C_{k-(\tau-1-j)+l}(x, r, \tau - 1 - l)$ and establish continuity by induction on $l \in \{0, \ldots, \tau - 1 - j\}$.

We start the induction argument at $l = 0$, where we have $C_{k-\tau+1+j}(x, r, \tau - 1) = C_{k-\tau+1+j}(x + r, 0, \tau - 1)$ by Lemma 3(c). Since $C_{k-\tau+1+j}(x, 0, \tau - 1)$ is K-convex in x by assumption, we conclude that $C_{k-\tau+1+j}(x + r, 0, \tau - 1)$ is K-convex in r. Hence, by part (e) of Lemma A.1 in Appendix A, $C_{k-\tau+1+j}(x, r, \tau - 1)$ is continuous in r. By part (b) of Lemma 3, $\mathrm{E}_\xi C_{k-\tau+1+j}(x - \xi, r, \tau - 1)$ is also continuous in r.

The continuity of $\mathrm{E}_\xi C_{k-\tau+1+j}(x - \xi, r, \tau - 1)$ at a point $r = r_0$ implies that for all $\epsilon > 0$ there exists a $\delta(\epsilon; x, r_0) > 0$ such that for all $r \geq 0$ with $|r - r_0| < \delta(\epsilon; x, r_0)$ we have

$$|\mathrm{E}_\xi C_{k-\tau+1+j}(x - \xi, r_0, \tau - 1) - \mathrm{E}_\xi C_{k-\tau+1+j}(x - \xi, r, \tau - 1)| < \epsilon$$

By Lemma 2 we have that for $l = \tau - 1 - j$,

$$|r - r_0| < \delta(\epsilon; x, r_0) \Rightarrow |C_k(x, r_0, j) - C_k(x, r, j)| < \alpha^{\tau-1-j}\epsilon < \epsilon$$

Therefore $C_k(x, r, j)$ is continuous in r for $j \in \{1, \ldots, \tau - 1\}$. This completes the proof. ∎

Lemmas 5 and 6 below establish K-convexity of $H_k(z)$ when $\tau = 2$. In this case we rewrite $H_k(z)$, defined in (6), as follows:

$$\begin{aligned} H_k(z) &= \min_{y \geq 0} \{c^r y + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, y, 1)\} \\ &= \min_{y \geq 0} \{c^r y + \alpha \mathrm{E}_\xi C_{k-1}(z + y - \xi, 0, 1)\} \\ &\qquad \text{[by Lemma 3(c)]} \\ &= \min_{w \geq z} \{(w - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)\} \qquad (7) \end{aligned}$$

For given z, let $w_k(z)$ be the argument that attains the minimum in (7). That is,

$$w_k(z) := \mathrm{argmin}_{w \geq z} \{(w - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)\} \qquad (8)$$

$w_k(z)$ is related to the optimal regular order function $Q_k(z)$ defined in Equation (10) by $Q_k(z) = w_k(z) - z$. Note that $w_k(z)$ corresponds to an inventory position that includes outstanding emergency and regular order quantities.

Lemma 5 *If $\tau = 2$, then $w_k(a) = w_k(z)$ for all $a \in [z, w_k(z)]$.*

Proof By the definition of $w_k(z)$, for all $w \geq z$ we have

$$[w_k(z) - z]c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(z) - \xi, 0, 1) \leq [w - z]c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)$$

Adding $(z - a)c^r$ to both sides yields

$$[w_k(z) - a]c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(z) - \xi, 0, 1) \leq (w - a)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)$$

Since $z \leq a \leq w_k(z)$ and $w \geq a$, it follows that $w_k(z) = w_k(a)$. ∎

Lemma 6 *Suppose that $\tau = 2$. For all k, if $\mathrm{E}_\xi C_{k-1}(x - \xi, 0, 1)$ is K-convex in x, then the function $H_k(x)$ defined in equation (7) is also K-convex.*

Proof The proof is based on Definition A.1(b) in Appendix A. Let $x_1 \leq x_2$, $\lambda \in [0, 1]$, $\bar{\lambda} = 1 - \lambda$ and $x_\lambda = \lambda x_1 + \bar{\lambda} x_2$.

By Lemma 5, if $x_2 \leq w_k(x_1)$, we have $w_k(x_2) = w_k(x_1)$. Alternatively, if $x_2 > w_k(x_1)$, we have $w_k(x_2) \geq x_2 > w_k(x_1)$. Therefore, $x_1 \leq x_2$ implies $w_k(x_1) \leq w_k(x_2)$. Since $x_2 \geq x_\lambda$, we analyze the following two cases.

Case 1: $x_\lambda \leq w_k(x_1)$. We have

$$\begin{aligned}
\lambda H_k(x_1) + \bar{\lambda}[H_k(x_2) + K] &= \lambda[(w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&\qquad + \bar{\lambda}[(w_k(x_2) - x_2)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&= (\lambda w_k(x_1) + \bar{\lambda} w_k(x_2) - x_\lambda)c^r + \lambda \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1) \\
&\qquad + \bar{\lambda}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&\geq (w_\lambda - x_\lambda)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_\lambda - \xi, 0, 1) \\
&\qquad [\text{for } w_\lambda = \lambda w_k(x_1) + \bar{\lambda} w_k(x_2) \\
&\qquad\quad \text{by the } K\text{-convexity of } \alpha \mathrm{E}_\xi C_{k-1}(x - \xi, 0, 1)] \\
&\geq \min_{w \geq x_\lambda} \{(w - x_\lambda)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)\} \\
&\qquad [\text{since } x_\lambda \leq w_k(x_\lambda)] \\
&= H_k(x_\lambda)
\end{aligned}$$

Case 2: $w_k(x_1) < x_\lambda$. Since

$$\begin{aligned}
H_k(x_1) &= (w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1) \\
&\leq (w_k(x_2) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) \\
&= (w_k(x_2) - w_k(x_1))c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + (w_k(x_1) - x_1)c^r
\end{aligned}$$

we have

$$\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1) \leq (w_k(x_2) - w_k(x_1))c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) \qquad (9)$$

We will show that, in this region, the line that joins the points $(x_1, H_k(x_1))$ and $(x_2, H_k(x_2) + K)$, denoted as $g_1(x)$, lies above the line that joins the points $(w_k(x_1), H_k(w_k(x_1)))$ and $(w_k(x_2), H_k(w_k(x_2)) + K)$, denoted as $g_2(x)$, which in turn lies above the point $(x, H_k(x))$. By Definition A.1(b) in Appendix A, this will establish K-convexity for $H_k(x)$. Figure 4 illustrates this line of thinking for $K = 5$.

We have

$$\begin{aligned}
g_1(x) \quad := \quad & \frac{x_2 - x}{x_2 - x_1}[(w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
& + \frac{x - x_1}{x_2 - x_1}[(w_k(x_2) - x_2)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K]
\end{aligned}$$

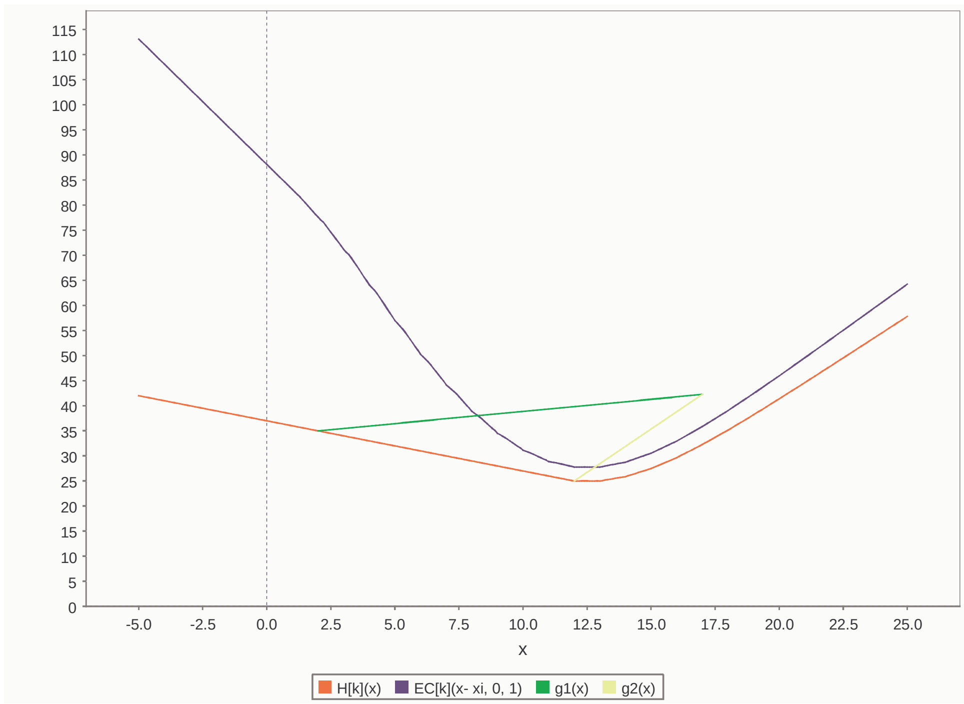
Note that for x_λ as defined above and $\lambda = (x_2 - x_\lambda)/(x_2 - x_1)$ we have

$$g_1(x_\lambda) = \lambda H_k(x_1) + \bar{\lambda}[H_k(x_2) + K]$$

On the other hand,

$$\begin{aligned}
g_2(x) \quad := \quad & \frac{w_k(x_2) - x}{w_k(x_2) - w_k(x_1)}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
& + \frac{x - w_k(x_1)}{w_k(x_2) - w_k(x_1)}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K]
\end{aligned}$$

Figure 4: Descriptive Graph for Case 2. Displayed are $H_k(x)$, $g_1(x)$, $g_2(x)$, and $\mathrm{E}_\xi C_k(x - \xi, 0, 1)$ for $x_1 = 2$, $x_2 = 17$. Note that $w_k(x_1) = 11$ and $w_k(x_2) = 17$.



At $x = w_k(x_1)$ we have

$$
\begin{aligned}
g_1(w_k(x_1)) &= \frac{x_2 - w_k(x_1)}{x_2 - x_1}[(w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&\quad + \frac{w_k(x_1) - x_1}{x_2 - x_1}[(w_k(x_2) - x_2)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&= \frac{x_2 - w_k(x_1)}{x_2 - x_1}[(w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&\quad + \frac{w_k(x_1) - x_1}{x_2 - x_1}[(w_k(x_2) - w_k(x_1))c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) \\
&\quad + (w_k(x_1) - x_2)c^r + K] \\
&\geq \frac{x_2 - w_k(x_1)}{x_2 - x_1}[(w_k(x_1) - x_1)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&\quad + \frac{w_k(x_1) - x_1}{x_2 - x_1}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] + (w_k(x_1) - x_2)c^r + K] \\
&\qquad \text{[by equation (9)]} \\
&= \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&= g_2(w_k(x_1))
\end{aligned}
$$

while at $x = x_2$ we have

$$\begin{aligned}
g_2(x_2) &= \frac{w_k(x_2) - x_2}{w_k(x_2) - w_k(x_1)}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1)] \\
&\quad + \frac{x_2 - w_k(x_1)}{w_k(x_2) - w_k(x_1)}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&\leq \frac{w_k(x_2) - x_2}{w_k(x_2) - w_k(x_1)}[(w_k(x_2) - w_k(x_1))c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1)] \\
&\quad + \frac{x_2 - w_k(x_1)}{w_k(x_2) - w_k(x_1)}[\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&\qquad \text{[by equation (9)]} \\
&= (w_k(x_2) - x_2)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + \frac{x_2 - w_k(x_1)}{w_k(x_2) - w_k(x_1)}K \\
&\leq (w_k(x_2) - x_2)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K \\
&= g_1(x_2)
\end{aligned}$$

Since both $g_2(x)$ and $g_1(x)$ are linear, we conclude that $g_1(x) \geq g_2(x)$ for $x \in [w_k(x_1), x_2]$.

Recall that by definition (8), $x \leq w_k(x)$. Now, at $x_\lambda = \lambda x_1 + \bar{\lambda} x_2$ such that $w_k(x_1) \leq x_\lambda \leq x_2 \leq w_k(x_2)$, we have

$$\begin{aligned}
\lambda H_k(x_1) + \bar{\lambda}[H_k(x_2) + K] &= g_1(x_\lambda) \\
&\geq g_2(x_\lambda) \\
&= \frac{w_k(x_2) - x_\lambda}{w_k(x_2) - w_k(x_1)} \quad \alpha \mathrm{E}_\xi C_{k-1}(w_k(x_1) - \xi, 0, 1) \\
&\quad + \frac{x_\lambda - w_k(x_1)}{w_k(x_2) - w_k(x_1)} \quad [\alpha \mathrm{E}_\xi C_{k-1}(w_k(x_2) - \xi, 0, 1) + K] \\
&\geq \alpha \mathrm{E}_\xi C_{k-1}(x_\lambda - \xi, 0, 1) \\
&\qquad \text{[by the K-convexity of $\mathrm{E}_\xi C_{k-1}(x - \xi, 0, 1)$]} \\
&\geq \min_{w \geq x_\lambda} (w - x_\lambda)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1) \\
&= H_k(x_\lambda)
\end{aligned}$$

This completes the proof. ▮

The next lemma states the structure of the optimal decisions for the functions $C_k(x, r, j)$.

Lemma 7 *Suppose that $\tau = 2$. Then for $k = 0, 1, \ldots$, $x \in \mathbb{R}$, and $r \in \mathbb{R}_+$:*

1. *$C_k(x, r, j)$ is K-convex in x.*

2. *For all $(x, r, j) \in \mathbf{X}$, $C_k(x, r, j) < \infty$.*

3. *There exist constants $(s_{k,j}, S_{k,j})$ for $j \in \{0, \ldots, m-1\}$, and a function $Q_k(x)$ such that the following inventory control policy is optimal at iteration k for an inventory state (x, r, j).*

   (a) *For $j = 0$: if $x < s_{k,0}$, place an emergency order for $y^{\mathrm{e}} = S_{k,0} - x$ units; otherwise, do not order ($y^{\mathrm{e}} = 0$). Further, place a regular order of size $Q_k(y^{\mathrm{e}}+x)$.*

   (b) *For $j = 1$: if $x + r < s_{k,1}$, place an emergency order for $S_{k,1} - (x + r)$ units; otherwise, do not order.*

   (c) *For $j \in \{2, \ldots, m-1\}$: if $x < s_{k,j}$ place an emergency order for $S_{k,j} - x$ units; otherwise, do not order.*

Proof We first observe that since $\mathrm{E}_\xi L(z - \xi)$ is convex and $\mathrm{E}_\xi L(z - \xi) \to \infty$ as $|z| \to \infty$ by Assumption 7, then $\mathrm{E}_\xi L(z - \xi) < \infty$ and $G(z) = c^e z + \mathrm{E}_\xi L(z - \xi)$ is convex and finite for $z \in \mathbb{R}$. Hence, $G(z) \to \infty$ as $|z| \to \infty$.

For $k \geq 1$ we define

$$Q_k(z) := \mathrm{argmin}_{y \geq 0}\{c^r y + \alpha \mathrm{E}_\xi C_{k-1}(z - \xi, y, 1)\} \tag{10}$$

Note that for any emergency inventory position z, the function $Q_k(z)$ returns the size of the optimal regular order.

The proof uses induction on k. Since $C_0(x, r, j) = 0$ for all (x, r, j), we start with $k = 1$. We have

$$H_1(z) = \min_{y \geq 0}\{c^r y + \alpha \mathrm{E}_\xi C_0(z - \xi, y, 1)\} = \min_{y \geq 0} c^r y$$

Since $c^r > 0$, then $Q_1(z) = 0$ and $H_1(z) = 0$. Hence, $H_1(z)$ is K-convex in z.

Now we consider

$$\begin{aligned} C_1(x, r, 0) &= \min_{z \geq x}\{K\delta(z - x) + G(z) + H_1(z)\} - c^e x \\ &= \min_{z \geq x}\{K\delta(z - x) + G(z)\} - c^e x \end{aligned}$$

Since $G(z)$ is convex by part (a) of Lemma A.1 in Appendix A and $G(z) \to \infty$ as $|z| \to \infty$ by Assumption 7, then by parts (a) and (h) of Lemma A.1 in Appendix A, there exists a pair of constants $(s_{1,0}, S_{1,0})$ such that for all r,

$$C_1(x,r,0) = \begin{cases} G(x) - c^e x & \text{if } x \geq s_{1,0} \\ K + G(S_{1,0}) - c^e x & \text{if } x < s_{1,0} \end{cases}$$

Further, $C_1(x,r,0)$ is finite and K-convex in x by Lemma A.2 in Appendix A.

Note that the optimal inventory policy that yields such a cost function can be stated as: if the net inventory at the beginning of the period, x, is less than $s_{1,0}$, place an emergency order of size $S_{1,0} - x$; otherwise, do not order. This type of cost functions will appear several times during this proof and hence will prove part (a) of this lemma.

For $j = 1$ we have

$$\begin{aligned} C_1(x,r,1) &= \min_{z \geq x} \{K\delta(z-x) + G(z+r) + \alpha \mathrm{E}_\xi C_0(z+r-\xi,0,2)\} - c^e(x+r) \\ &= \min_{z \geq x} \{K\delta(z-x) + G(z+r)\} - c^e(x+r) \end{aligned}$$

Again, since $G(z)$ is convex and $G(z) \to \infty$ as $|z| \to \infty$, then for a given r and by parts (a) and (h) of Lemma A.1 in Appendix A, there exists a pair of constants $(s_{1,1}, S_{1,1})$ such that

$$C_1(x,r,1) = \begin{cases} G(x+r) - c^e(x+r) & \text{if } x+r \geq s_{1,1} \\ K + G(S_{1,1}) - c^e(x+r) & \text{if } x+r < s_{1,1} \end{cases}$$

Hence $C_1(x,r,1)$ is finite and K-convex in x by Lemma A.2 in Appendix A.

For $j = 2$ we have

$$\begin{aligned} C_1(x,r,2) &= \min_{z \geq x} \{K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_0(z-\xi,0,3)\} - c^e x \\ &= \min_{z \geq x} \{K\delta(z-x) + G(z)\} - c^e x \end{aligned}$$

Since $G(z)$ is convex and $G(z) \to \infty$ as $|z| \to \infty$, then by parts (a) and (h) of Lemma A.1 in Appendix A, there exists a pair of constants $(s_{1,2}, S_{1,2})$ such that

$$C_1(x,r,2) = \begin{cases} G(x) - c^e x & \text{if } x \geq s_{1,2} \\ K + G(S_{1,2}) - c^e x & \text{if } x < s_{1,2} \end{cases}$$

Further, $C_1(x, r, 2)$ is finite and K-convex in x by Lemma A.2 in Appendix A.

Using similar arguments, we can establish that for any r and $j \in \{3, \dots, m-1\}$, $C_1(x, r, j)$ is finite and K-convex in x.

Now we assume that for fixed r this lemma holds for a given k. For $j \in \{2, \dots, m-1\}$, we have

$$C_{k+1}(x, r, j) = \min_{z \geq x}\{K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_k(z-\xi, 0, j^+)\} - c^e x$$

Recall that the random variable ξ is nonnegative and by Lemma 3(a.1) we have that, w.p.1, $C_k(x-\xi, r, j) \leq C_k(x, r, j) + K + c^e\xi$. Then

$$\mathrm{E}_\xi C_k(x-\xi, r, j) \leq C_k(x, r, j) + K + c^e \mu_\xi < \infty$$

where the finiteness follows from the induction hypothesis and $\mu_\xi < \infty$. Therefore, by part (g) of Lemma A.1 in Appendix A, $\mathrm{E}_\xi C_k(x-\xi, 0, j)$ is K-convex in x. By parts (a) and (c) of Lemma A.1 in Appendix A, $V_{k+1}(z, r, j) = G(z) + \mathrm{E}_\xi C_k(z-\xi, r, j)$ is, in turn, K-convex in z. Since $G(z) \to \infty$ as $|z| \to \infty$, by part (h) of Lemma A.1 in Appendix A there exists a pair of constants $(s_{k+1,j}, S_{k+1,j})$ such that

$$C_{k+1}(x, r, j) = \begin{cases} V_{k+1}(x, 0, j) - c^e x & \text{if } x \geq s_{k+1,j} \\ K + V_{k+1}(S_{k+1,j}, 0, j) - c^e x & \text{if } x < s_{k+1,j} \end{cases}$$

Further, $C_{k+1}(x, r, j)$ is finite and, for fixed r, is K-convex in x by Lemma A.2 in Appendix A.

For $j = 1$ we have

$$\begin{aligned} C_{k+1}(x, r, 1) &= \min_{z \geq x}\{K\delta(z-x) + G(z+r) + \alpha \mathrm{E}_\xi C_k(z+r-\xi, 0, 2)\} - c^e(x+r) \\ &= \min_{z \geq x}\{K\delta(z-x) + V_{k+1}(x+r, 0, 1)\} - c^e(x+r) \end{aligned}$$

By the same argument used for $j \in \{2, \dots, m-1\}$ we can prove that $\mathrm{E}_\xi C_k(x-\xi, r, j) < \infty$. Then by parts (a)–(c) and (g) of Lemma A.1 in Appendix A, and for fixed r, $V_{k+1}(z + r, 0, 1) = G(z+r) + \alpha \mathrm{E}_\xi C_k(z+r-\xi, 0, 2)$ is K-convex in $z+r$. Since $G(z) \to \infty$ as $|z| \to \infty$, by part (h) of Lemma A.1 in Appendix A there exists a pair of constants $(s_{k+1,1}, S_{k+1,1})$

such that

$$C_{k+1}(x,r,1) = \begin{cases} V_{k+1}(x+r,0,1) - c^e(x+r) & \text{if } x+r \geq s_{k+1,1} \\ K + V_{k+1}(S_{k+1,1},0,1) - c^e(x+r) & \text{if } x+r < s_{k+1,1} \end{cases}$$

Also $C_{k+1}(x,r,1)$ is finite and, for fixed r, is K-convex in x by Lemma A.2 in Appendix A.

For $j=0$ we have

$$H_{k+1}(z) = \min_{y\geq 0}\{c^r y + \alpha \mathrm{E}_\xi C_k(z-\xi,y,1)\}$$

Since $\mathrm{E}_\xi C_k(z-\xi,y,1) \geq 0$, it follows that

$$\lim_{y\to\infty}\{c^r y + \alpha \mathrm{E}_\xi C_k(z-\xi,y,1)\} \geq \lim_{y\to+\infty} c^r y = +\infty$$

By Lemma 3(b) and Lemma 4(c), $\mathrm{E}_\xi[C_k(z-\xi,y,1)]$ is continuous in y. Hence we conclude that $H_{k+1}(z)$ achieves its infimum in $[0,\infty)$ and therefore $Q_{k+1}(z)$ defined in equation (10) exists for all z.

Now we look at

$$\begin{aligned} C_{k+1}(x,0,0) &= \min_{y\geq 0, z\geq x}\{c^r y + K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_k(z-\xi,y,0)\} - c^e x \\ &= \min_{z\geq x}\{K\delta(z-x) + G(z) + H_{k+1}(z)\} - c^e x \end{aligned}$$

Since $\alpha \mathrm{E}_\xi C_k(z-\xi,y,0)$ is K-convex by the induction hypothesis, then, by Lemma 6, $H_{k+1}(z)$ is K-convex. In turn, parts (a) and (c) of Lemma A.1 in Appendix A imply that $G(z)+H_{k+1}(z)$ is K-convex. Since $G(z)\to\infty$ as $|z|\to\infty$ and $H_{k+1}(z)\geq 0$, we have $G(z)+H_{k+1}(z)\to\infty$ as $|z|\to\infty$. Therefore, by Lemma A.1(h), there exists a pair of constants $(s_{k+1,0}, S_{k+1,0})$ such that

$$C_{k+1}(x,r,0) = \begin{cases} G(x) + H_{k+1}(x) - c^e x & \text{if } x \geq s_{k+1,0} \\ K + G(S_{k+1,0}) + H_{k+1}(S_{k+1,0}) - c^e x & \text{if } x < s_{k+1,0} \end{cases}$$

and $C_{k+1}(x,r,0)$ is K-convex by Theorem A.2 in Appendix A.

We have shown that, if the net inventory at the beginning of a regular-review period, x, is less than $s_{k+1,0}$, then an optimal inventory policy places places an emergency order of size $S_{k+1,0}-x$; otherwise, it does not place an order using the emergency mode. Note that the function $C_{k+1}(x,0,0)$, and hence the operational parameters $s_{k+1,0}$ and $S_{k+1,0}$, include the cost of an optimal regular order decision, $H_{k+1}(\cdot)$. From the definitions of $Q_k(z)$ in equation (10) and $C_{k+1}(x,0,0)$, at regular review epochs an optimal inventory policy also places an order for $Q_{k+1}(z)$ items using the regular mode, where $z = y^e + x$ is the inventory position after an emergency order is placed. This completes the proof. ∎

2.7 *Existence and Structure of* $C_\infty(x, z, j)$

We now prove that our dynamic programming cost functions $C_k(x,r,j)$ converge to the optimality functions $C(x,r,j)$ as $k \to \infty$.

Lemma 8 *For all initial states* $\mathbf{x} = (x,r,j)$, $\lim_{k\to\infty} C_k(\mathbf{x}) = C(\mathbf{x})$.

Proof Note that the single-stage cost functions $c(\mathbf{x},\mathbf{d},\xi)$ are nonnegative for all (x,r,j). Hence, we have a *Negative Dynamic Program* (Bertsekas [5, p. 124]). Define the level sets of $C_k(x,r,j)$ by

$$
\begin{aligned}
U_k(x,r,m-1,\lambda) &:= \{z \geq x \mid K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_k(z-\xi,0,0) - c^e x \leq \lambda\} \\
&\;\;\vdots \\
U_k(x,r,\tau-1,\lambda) &:= \{z \geq x \mid K\delta(z-x) + G(z+r) + \alpha \mathrm{E}_\xi C_k(z+r-\xi,\tau) - c^e(x+r) \leq \lambda\} \\
U_k(x,r,\tau-2,\lambda) &:= \{z \geq x \mid K\delta(z-x) + G(z) + \alpha \mathrm{E}_\xi C_k(z-\xi,r,\tau-1) - c^e x \leq \lambda\} \\
&\;\;\vdots \\
U_k(x,r,0,\lambda) &:= \{z \geq x, y \geq 0 \mid K\delta(z-x) + G(z) + c^r y + \alpha \mathrm{E}_\xi C_k(z-\xi,y,1) - c^e x \leq \lambda\}
\end{aligned}
$$

These sets are bounded since the functions inside the curly braces tend to ∞ as either $|z| \to \infty$ or $r \to \infty$. The sets are also closed since $C_k(x,r,j)$ is continuous in x and r by Lemma 4. Hence, the sets $U_k(x,r,j,\lambda)$ are compact subsets of $\mathbf{X}$ for all λ. The proof of the Lemma follows from Lemma A.4 in Appendix A. ∎

2.7.1 Structure of the Optimal Inventory Policy

The following lemma establishes the K-convexity of $C(x,r,j)$.

Lemma 9 *If $C_k(x, r, j)$ is K-convex in x for all $k = 0, 1, \ldots$, then $C(x, r, j)$ is K-convex in x.*

Proof Again, we use Definition A.1(b) in Appendix A. Let $x_1 \leq x_2$, $\lambda \in [0, 1]$, $\bar{\lambda} = 1 - \lambda$ and $x_\lambda = \lambda x_1 + \bar{\lambda} x_2$. Then

$$\begin{aligned} C(x_\lambda, r, j) &= \lim_{k\to\infty} C_k(x_\lambda, r, j) \quad \text{[by Lemma 8]} \\ &\leq \lim_{k\to\infty} \{\lambda C_k(x_1, r, j) + \bar{\lambda}[C_k(x_2, r, j) + K]\} \\ &\qquad \text{[by the } K\text{-convexity of } C_k(\mathbf{x})] \\ &= \lambda C(x_1, r, j) + \bar{\lambda}[C(x_2, r, j) + K] \\ &\qquad \text{[by Lemma 8]} \end{aligned}$$

This completes the proof. ∎

The proof of Theorem 1 is now obvious.

Proof of Theorem 1 For $\tau = 2$, we have by Lemma 7 that the functions $C_k(x, r, j)$ are K-convex in x; hence, by Lemma 9 the limiting functions $C(x, r, j)$ are K-convex in x. The same arguments used in Lemma 7 establish the optimal policy described in this theorem. ∎

2.7.2 Extension of the Optimality Proof

We conjecture that Lemma 9 can be extended the the general case $\tau > 2$.

Conjecture 10 *For $k = 0, 1, \ldots$, $x \in \mathbb{R}$, and $r \in \mathbb{R}_+$:*

1. *$C_k(x, r, j)$ is K-convex in x for fixed r.*
2. *For all $(x, r, j) \in \mathbf{X}$, $C_k(x, r, j) < \infty$.*
3. *There exist constants $(s_{k,j}, S_{k,j})$ for $j \in \{0\} \cup \{\tau - 1, \ldots, m - 1\}$, functions $(s_{k,j}(r), S_{k,j}(r))$ for $j \in \{1, \ldots, \tau - 2\}$, and a function $Q_k(x)$ such that the following inventory control policy is optimal at iteration k for an inventory state (x, r, j).*
   (a) *For $j = 0$: if $x < s_{k,0}$, place an emergency order for $y^e = S_{k,0} - x$ units; otherwise, do not order ($y^e = 0$). Further, place a regular order of size $Q_k(y^e + x)$.*

(b) *For* $j \in \{1, \ldots, \tau - 2\}$*: if* $x < s_{k,j}(r)$*, place an emergency order for* $S_{k,j}(r) - x$ *units; otherwise, do not order.*

(c) *For* $j = \tau - 1$*: if* $x + r < s_{k,\tau-1}$*, place an emergency order for* $S_{k,\tau-1} - (x + r)$ *units; otherwise, do not order.*

(d) *For* $j \in \{\tau, \ldots, m - 1\}$*: if* $x < s_{k,j}$ *place an emergency order for* $S_{k,j} - x$ *units; otherwise, do not order.*

The following discussion lists properties that we have proved and the gap we attempt to close with a numerical argument.

By the arguments in the proof of Lemma 7, for stage $k = 1$ and $j = 1$ we have

$$\begin{aligned} C_1(x, r, 1) &= \min_{z \geq x}\{K\delta(z - x) + G(z) + \alpha \mathrm{E}_\xi C_0(z - \xi, r, 2)\} - c^e x \\ &= \min_{z \geq x}\{K\delta(z - x) + G(z)\} - c^e x \end{aligned}$$

Since $G(z)$ is convex and $G(z) \to \infty$ as $|z| \to \infty$, then by parts (a) and (h) of Lemma A.1 in Appendix A, there exists a pair of constants $(s_{1,1}, S_{1,1})$ such that for all r

$$C_1(x, r, 1) = \begin{cases} G(x) - c^e x & \text{if } x \geq s_{1,1} \\ K + G(S_{1,1}) - c^e x & \text{if } x < s_{1,1} \end{cases}$$

Also, $C_1(x, r, 1)$ is finite and K-convex in x by Lemma A.2 in Appendix A.

Using similar arguments as for $C_1(x, r, 1)$, we can establish that, for fixed r and $j \in \{1, \ldots, \tau - 2\}$, the functions $C_1(x, r, j)$ are finite and K-convex with respect to x.

Similarly, if we assume that items (a) through (d) of Conjecture 10 are valid for stage k, then we can argue that for $j \in \{1, \ldots, \tau - 2\}$, we have

$$\begin{aligned} C_{k+1}(x, r, j) &= \min_{z \geq x}\{K\delta(z - x) + G(z) + \alpha \mathrm{E}_\xi C_k(z - \xi, r, j + 1)\} - c^e x \\ &= \min_{z \geq x}\{K\delta(z - x) + V_{k+1}(x, r, j)\} - c^e x \end{aligned}$$

Since $G(z) \to \infty$ as $|z| \to \infty$, then by part (h) of Lemma A.1 in Appendix A there exist functions, $(s_{k+1,j}(r), S_{k+1,j}(r))$ such that

$$C_{k+1}(x, r, j) = \begin{cases} V_{k+1}(x, r, j) - c^e x & \text{if } x \geq s_{k+1,j}(r) \\ K + V_{k+1}(S_{k+1,j}(r), r, j) - c^e x & \text{if } x < s_{k+1,j}(r) \end{cases}$$

Further, for fixed r, $C_{k+1}(x,r,j)$ is K-convex in x by Lemma A.2 in Appendix A, and finite.

By the same argument of Lemma 7 the functions $C_{k+1}(x,r,j)$ are K-convex for $j \in \{1,\ldots,m-1\}$. To complete the same induction argument used in Lemma 7 though, $H_k(x)$ must be shown to be K-convex. Although we have not been able to prove this property for $\tau > 2$, we have computed the dynamic program limiting functions $C(x,r,j)$ for the experimental design of Table 4 with $\tau \in \{3,4,5\}$. At each stage k, we verified numerically that the functions $H_k(x)$ are indeed K-convex. Hence, by the argument stated in Section 2.7.1, if the conjecture is true, then the optimal policy for the system described in Chapter I has a structure of the form stated in this conjecture.

2.8 Characteristics of the Optimal Regular Orders

In a periodic-review inventory system with a single supply mode available on every period, an alternative supplier or transportation mode available only on certain time periods represents an opportunity to improve the expected total cost. Recall the dynamic program functions for periods $j = 0$ and $j = \tau - 1$:

$$C_k(x,r,0) = \min_{y\geq 0, z\geq x}\{K\delta(z-x) + c^e(z-x) + c^r y + \alpha \mathrm{E}_\xi L(z-\xi) + \alpha \mathrm{E}_\xi C_{k-1}(z-\xi, y, 1)\}$$

$$C_k(x,r,\tau-1) = \min_{z\geq x}\{K\delta(z-x) + c^e(z-x) + \alpha \mathrm{E}_\xi L(z+r-\xi) + \alpha \mathrm{E}_\xi C_{k-1}(z+r-\xi, 0, \tau)\}$$

Note that the addition of the regular supply mode may reduce the optimal cost. Further, emergency supply decisions made between the time of the regular order placement and its arrival must consider the outstanding regular order, as our optimal policy does.

The following lemmas are defined for the dynamic program cost functions $C_k(\cdot)$ but the results can be extended for the optimal cost functions $C(\cdot)$. They state bounds for the regular order size that provide insights into the optimal solution and are used in the algorithm of Section 2.9.2.

Recall from equation (10) that given an emergency inventory position z, the optimal regular order size is

$$Q_k(z) = \mathrm{argmin}_{y\geq 0}\{c^r y + \alpha \mathrm{E}_\xi C_{k-1}(z-\xi, y, 1)\}$$

The following lemma obtains an upper bound for the size of a regular order when the unit cost for regular order is larger than the discounted unit cost for emergency orders (over $\tau - 1$ periods). The last condition is typically invalid for small τ or large α.

Lemma 11 *Suppose that $c^r \geq \alpha^{\tau-1} c^e$. Then for any iteration step k, emergency inventory position z, and $y \geq K/(\alpha^{1-\tau} c^r - c^e)$ we have*

$$\alpha \mathrm{E}_\xi C_k(z - \xi, 0, 1) \leq c^r y + \alpha \mathrm{E}_\xi C_k(z - \xi, y, 1)$$

Proof Let

$$\bar{y} := \frac{K}{\alpha^{1-\tau} c^r - c^e}$$

Since $y(\alpha^{1-\tau} c^r - c^e) \geq K$ for $y \geq \bar{y}$, then for $j = \tau - 2$ we have

$$\begin{aligned}
&\mathrm{E}_\xi[C_{k-\tau+1}(z - \xi, 0, \tau - 1) - \mathrm{E}_\xi C_{k-\tau+1}(z - \xi, y, \tau - 1)] \\
&\quad = \mathrm{E}_\xi[C_{k-\tau+1}(z - \xi, 0, \tau - 1) - C_{k-\tau+1}(z - \xi + y, 0, \tau - 1)] \\
&\qquad\quad \text{[by Lemma 3(c)]} \\
&\quad \leq (K + y c^e) \quad \text{[by Lemma 3(b)]} \\
&\quad \leq y(\alpha^{1-\tau} c^r - c^e + c^e) \\
&\quad = y \alpha^{1-\tau} c^r
\end{aligned}$$

The remainder of the proof follows from Lemma 2 with $j = 1$ and $l = \tau - 2$. ∎

The next lemma implies that the economic benefits that can be obtained using regular orders are bounded by an exponentially decreasing function of the regular order lead-time; hence, no regular orders should be placed if this lead-time is too large.

Lemma 12 *For any $k \geq 1$, $\gamma \geq 0$, and for periods $j \in \{1, \ldots, \tau - 1\}$,*

$$C_k(x, r, j) - C_k(x, r + \gamma, j) \leq \alpha^{(\tau-1-j)}(K + \gamma c^e)$$

In particular, for $r > 0$ and $j = 1$,

$$\mathrm{E}_\xi C_k(x - \xi, 0, 1) - \mathrm{E}_\xi C_k(x - \xi, r, 1) \leq \alpha^{\tau-2}(K + r c^e)$$

Proof For $j = \tau - 1$, by Lemma 3(a.2) we have

$$\begin{aligned} & \mathrm{E}_\xi[C_{k-\tau+1}(x-\xi, r, \tau-1)] - \mathrm{E}_\xi[C_{k-\tau+1}(x-\xi, r+\gamma, \tau-1)] \\ = \;& \mathrm{E}_\xi[C_{k-\tau+1}(x+r-\xi, 0, \tau-1) - C_{k-\tau+1}(x+r+\gamma-\xi, 0, \tau-1)] \\ \leq \;& (K + \gamma c^e) \end{aligned}$$

Now, by Lemma 2 with $j \in \{1, \ldots, \tau - 1\}$ and $l = \tau - 1 - j$, we conclude that

$$C_k(x, r, j) - C_k(x, r+\gamma, j) \leq \alpha^{\tau-1-j}(K + \gamma c^e)$$

In particular, for $j = 1$ we have

$$\mathrm{E}_\xi C_k(x-\xi, 0, 1) - \mathrm{E}_\xi C_k(x-\xi, r, 1) \leq \alpha^{\tau-2}(K + rc^e)$$

∎

The next lemma applies to the case where $\tau = 2$ and $z \leq s_{k-1,1}$. Recall the definition of $w_k(z)$ from equation (8): $w_k(z) = \mathrm{argmin}_{w \geq z}\{(w-z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w-\xi, 0, 1)\}$.

Lemma 13 *Suppose that $\tau = 2$. Then for $s_{k-1,1}$ and $\Sigma_{k-1,1}$ defined in Lemma 4:*

(a) If $c^r \leq \alpha c^e$, then for $z \leq s_{k-1,1}$, $w_k(z) = w_k(s_{k-1,1})$.

(b) If $c^r > \alpha c^e$, then $w_k(z) = z$ for $z < \bar{z} < s_{k-1,1}$ where

$$\bar{z} = s_{k-1,1} - \alpha \frac{\mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1) - \mathrm{E}_\xi[C_{k-1}(\Sigma_{k-1,1} - \xi, 0, 1)]}{c^r - \alpha c^e} \tag{11}$$

Further, $\bar{z}$ satisfies

$$\bar{z} \geq s_{k-1,1} - \alpha \frac{K + c^e \mu_\xi}{c^r \alpha c^e} \tag{12}$$

Proof

Part (a). To prove this part of the lemma we compare the value of the objective function at $w_k(s_{k-1,1})$ with the value of the function for any $w \geq z$. We consider two cases, $w \geq s_{k-1,1}$ and $z \leq w < s_{k-1,1}$. For $z \leq s_{k-1,1} \leq w$, by the optimality of $w_k(s_{k-1,1})$ we have

$$[w_k(s_{k-1,1}) - s_{k-1,1}]c^r + \alpha \mathrm{E}_\xi C_{k-1}[w_k(s_{k-1,1}) - \xi, 0, 1] \leq (w - s_{k-1,1})c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)$$

Adding $(s_{k-1,1} - z)c^r$ to both sides yields

$$[w_k(s_{k-1,1}) - z]c^r + \alpha \mathrm{E}_\xi C_{k-1}[w_k(s_{k-1,1}) - \xi, 0, 1] \leq (w - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1)$$

For $z \leq w < s_{k-1,1}$, we have

$$\begin{aligned}
(w - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1) &= (w - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1) + \alpha c^e(s_{k-1,1} - w) \\
&\quad \text{[by Lemma 4(a.2)]} \\
&= (s_{k-1,1} - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1) \\
&\quad +(s_{k-1,1} - w)(\alpha c^e - c^r) \\
&\geq (s_{k-1,1} - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1) \\
&\quad \text{[since } w < s_{k-1,1} \text{ and } c^r \leq \alpha c^e\text{]} \\
&\geq (s_{k-1,1} - z)c^r + (w_k(s_{k-1,1}) - s_{k-1,1})c^r \\
&\quad + \alpha \mathrm{E}_\xi C_{k-1}(w_k(s_{k-1,1}) - \xi, 0, 1) \\
&\quad \text{[by the optimality of } w_k(s_{k-1,1})\text{]} \\
&= (w_k(s_{k-1,1}) - z)c^r + \alpha \mathrm{E}_\xi C_{k-1}(w_k(s_{k-1,1}) - \xi, 0, 1)
\end{aligned}$$

We conclude that $w_k(z) = w_k(s_{k-1,1})$.

Part (b). For $z \leq w \leq s_{k-1,1}$ and $c^r > \alpha c^e$, Lemma 4(a.2) implies

$$\begin{aligned}
\alpha \mathrm{E}_\xi C_{k-1}(z - \xi, 0, 1) &= \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1) + \alpha c^e(w - z) \\
&< \alpha \mathrm{E}_\xi C_{k-1}(w - \xi, 0, 1) + c^r(w - z)
\end{aligned}$$

Before we continue, we note that $\bar{z}$ is the horizontal coordinate of the point where the line

$$\psi_1(z) := \alpha \mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1} - \xi, 0, 1) + c^r(s_{k-1,1} - z)$$

intercepts the line

$$\psi_2(z) := \alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1) + \alpha c^e(s_{k-1,1} - z)$$

Since $\mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1} - \xi, 0, 1) < \mathrm{E}_\xi C_{k-1}(s_{k-1,1} - \xi, 0, 1)$ and $c^r > \alpha c^e$, then for $z < \bar{z}$ we have

$$\psi_1(z) \geq \psi_2(z) \tag{13}$$

For $z \leq \bar{z} < s_{k-1,1} < w$, we have

$$\begin{aligned}
\alpha \mathrm{E}_\xi C_{k-1}(z-\xi,0,1) &= \alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1}-\xi,0,1) + \alpha c^e (s_{k-1,1}-z) \\
&\quad \text{[by Lemma 4(a.2)]} \\
&\leq \alpha \mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1}-\xi,0,1) + c^r (s_{k-1,1}-z) \\
&\quad \text{[by inequality (13)]} \\
&\leq \alpha \mathrm{E}_\xi C_{k-1}(w-\xi,0,1) + c^r (s_{k-1,1}-z) \\
&\quad \text{[by Lemma 4(b.2)]} \\
&< \alpha \mathrm{E}_\xi C_{k-1}(w-\xi,0,1) + c^r (w-z) \\
&\quad \text{[since } s_{k-1,1} < w\text{]}
\end{aligned}$$

We conclude that for $z \leq \bar{z}$ and $z \leq w$

$$\alpha \mathrm{E}_\xi C_{k-1}(z-\xi,0,1) \leq \alpha \mathrm{E}_\xi C_{k-1}(w-\xi,0,1) + c^r (w-z)$$

To obtain the lower bound in (12), we observe that for a nonnegative random variable ξ, by Lemma 4(a.1) we have, w.p.1,

$$C_{k-1}(s_{k-1,1}-\xi,0,1) = C_{k-1}(s_{k-1,1},0,1) + c^e \xi$$

Therefore

$$\mathrm{E}_\xi C_{k-1}(s_{k-1,1}-\xi,0,1) = C_{k-1}(s_{k-1,1},0,1) + c^e \mathrm{E}[\xi]$$

and

$$\begin{aligned}
&\mathrm{E}_\xi C_{k-1}(s_{k-1,1}-\xi,0,1) - \mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1}-\xi,0,1) \\
&= C_{k-1}(s_{k-1,1},0,1) + c^e \mu_\xi - \mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1}-\xi,0,1) \\
&\leq C_{k-1}(s_{k-1,1},0,1) + c^e \mu_\xi - C_{k-1}(S_{k-1,1},0,1) \\
&\quad \text{[by Lemma 4(b.2)]} \\
&= K + c^e \mu_\xi
\end{aligned}$$

Finally

$$\begin{aligned}
\bar{z} &= s_{k-1,1} - \frac{\alpha \mathrm{E}_\xi C_{k-1}(s_{k-1,1}-\xi,0,1) - \alpha \mathrm{E}_\xi C_{k-1}(\Sigma_{k-1,1}-\xi,0,1)}{c^r - \alpha c^e} \\
&\geq s_{k-1,1} - \alpha \frac{K + c^e \mu_\xi}{c^r - \alpha c^e}
\end{aligned}$$

which completes the proof. ∎

2.9 Computing Parameter Values for the Optimal Policy

This section covers the numerical computation of the optimal policy parameters. We first develop a stopping criterion for the dynamic program defined in Section 2.5.2, and then present an algorithm to compute the optimal parameters of the inventory policy. Finally, we obtain the optimal parameters for an experimental design.

2.9.1 Stopping Criterion

We begin by providing bounds for the operational parameters based on the values of the dynamic program cost functions $C_k(\cdot)$. These bounds can be used as a stopping rule for the value iteration algorithm.

We use the notation and arguments of Bertsekas [5]. First, for any function $J : \mathbf{X} \to \mathbb{R}$ we define the mapping T in terms of the single-stage costs $c(\mathbf{x}, \mathbf{d}, \xi)$ and the transition function $f(\mathbf{x}, \mathbf{d}, \xi)$ by

$$(TJ)(\mathbf{x}) := \min_{\mathbf{d} \in \mathcal{D}(\mathbf{x})} \mathrm{E}_{\xi}\{c(\mathbf{x}, \mathbf{d}, \xi) + \alpha J(f(\mathbf{x}, \mathbf{d}, \xi))\}$$

and denote the composition of the mapping T with itself k times as $(T^k J)(\mathbf{x})$. In the dynamic program defined in Section 2.5.2, we have $C_k(x, r, j) = T^k C_0(x, r, j)$. Further, since we have defined a negative dynamic program, it follows that

$$C_0(x, r, j) \leq C_1(x, r, j) \leq \cdots \leq C_k(x, r, j) \leq \cdots \leq C(x, r, j)$$

The following lemma is from Bertsekas [5, Exercise 1.9].

Lemma 14 *Let $\mathbf{X}$ be a set and $B(\mathbf{X})$ be the set of all real valued bounded functions on $\mathbf{X}$. Let T be a mapping with the following two properties:*

(a) $TJ \leq TJ'$ for all J, $J' \in B(\mathbf{X})$ with $J \leq J'$

(b) For every scalar $b \neq 0$ and all $\mathbf{x} \in \mathbf{X}$

$$\alpha_1 \leq \frac{(T(J + b\mathbf{e}))(\mathbf{x}) - (TJ)(\mathbf{x})}{b} \leq \alpha_2$$

where α_1, α_2 are two scalars with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, and $\mathbf{e}$ is the vector of ones with the same dimensions as $\mathbf{X}$.

Then for all $J \in B(\mathbf{X})$ and $\mathbf{x} \in \mathbf{X}$

$$\begin{aligned}(T^k J)(\mathbf{x}) + \underline{c}_k &\leq (T^{k+1} J)(\mathbf{x}) + \underline{c}_{k+1} \\ &\leq J^*(\mathbf{x}) \\ &\leq (T^{k+1} J)(\mathbf{x}) + \bar{c}_{k+1} \\ &\leq (T^k J)(\mathbf{x}) + \bar{c}_k\end{aligned}$$

where

$$\underline{c}_k = \min\Big\{\frac{\alpha_1}{1-\alpha_1} \min_{\mathbf{x}\in\mathbf{X}} [(T^k J)(\mathbf{x}) - (T^{k-1} J)(\mathbf{x})], \frac{\alpha_2}{1-\alpha_2} \min_{\mathbf{x}\in\mathbf{X}} [(T^k J)(\mathbf{x}) - (T^{k-1} J)(\mathbf{x})\Big\}$$

$$\bar{c}_k = \max\Big\{\frac{\alpha_1}{1-\alpha_1} \max_{\mathbf{x}\in\mathbf{X}} [(T^k J)(\mathbf{x}) - (T^{k-1} J)(\mathbf{x})], \frac{\alpha_2}{1-\alpha_2} \max_{\mathbf{x}\in\mathbf{X}} [(T^k J)(\mathbf{x}) - (T^{k-1} J)(\mathbf{x})\Big\}$$

The following lemma is from Bertsekas [6, Proposition 5.12].

Lemma 15 *Consider the mapping*

$$H(\mathbf{x}, \mathbf{d}, J) = \mathrm{E}_\xi\{c(\mathbf{x}, \mathbf{d}, \xi) + \alpha J(f(\mathbf{x}, \mathbf{d}, \xi)) \mid (\mathbf{x}, \mathbf{d})\}$$

If $c(\mathbf{x}, \mathbf{d}, \xi) \geq 0$, w.p.1, for all $\mathbf{x} \in \mathbf{X}$ and $\mathbf{d} \in \mathcal{D}$, then for all scalars $b > 0$

$$H(\mathbf{x}, \mathbf{d}, J) \leq H(\mathbf{x}, \mathbf{d}, J + b) \leq H(\mathbf{x}, \mathbf{d}, J) + \alpha b \qquad \textit{for all } \mathbf{x} \in \mathbf{X} \textit{ and } \mathbf{d} \in \mathcal{D}$$

The next lemma can be used to define a stopping rule for the search of the parameters $(s_j(r), S_j(r))$.

Lemma 16 *For fixed r, $S_j(r) \in [\underline{S}_{k,j}(r), \overline{S}_{k,j}(r)]$ and $s_j(r) \in [\underline{s}_{k,j}(r), \bar{s}_{k,j}(r)]$ where*

$$\begin{aligned}\overline{S}_{k,j}(r) &:= \{x > S_{k,j}(r) \mid C_k(x, r, j) = C_k(S_{k,j}, r, j) + \bar{c}_k - \underline{c}_k\} \\ \underline{S}_{k,j}(r) &:= \{x < S_{k,j}(r) \mid C(x, r, j) = C(S_{k,j}, r, j) + \bar{c}_k - \underline{c}_k\} \\ \bar{s}_{k,j}(r) &:= \{x < S_j(r) \mid C_k(x, r, j) = C_k(s_{k,j}, r, j) + \underline{c}_k - \bar{c}_k\} \\ \underline{s}_{k,j}(r) &:= \{x < S_j(r) \mid C_k(x, r, j) = C(s_{k,j}, r, j) + \bar{c}_k - \underline{c}_k\}\end{aligned}$$

Proof First note that by Lemma 15 we can use the bounds for $C_k(\cdot)$ defined in Lemma 14. Also note that the definition of K-convexity does not require the minimum of this type of function to be unique. Nevertheless, the minimizing argument that defines the optimal policy and satisfies parts (i)–(iv) of Lemma A.1 corresponds to the smallest minimizer.

For any state (x, r, j), by Lemma 14 we have

$$C_k(x, r, j) + \underline{c}_k \leq C(x, r, j) \leq C_k(x, r, j) + \bar{c}_k \tag{14}$$

Therefore, for any x and fixed (r, j) we have

$$C_k(S_j(r), r, j) \leq C_k(x, r, j) \leq C_k(x, r, j) + \bar{c}_k$$

In particular, for $x = S_{k,j}(r)$ we have

$$C_k(S_j(r), r, j) \leq C_k(S_{k,j}(r), r, j) \leq C_k(S_{k,j}(r), r, j) + \bar{c}_k \tag{15}$$

Hence, for fixed (r, j)

$$S_j(r) \in \{x \mid C(x, r, j) \leq C_k(S_{k,j}(r), r, j) + \bar{c}_k\} \tag{16}$$

On the other hand, from equation (14) we have $C(x, r, j) \geq C_k(x, r, j) + \underline{c}_k$; hence we conclude that

$$S_j(r) \in \{x \mid C_k(x, r, j) + \underline{c}_k \leq C_k(S_{k,j}(r), r, j) + \bar{c}_k\} \tag{17}$$

Equations (16) and (17) imply $S_j(r) \in [\underline{S}_{k,j}(r), \bar{S}_{k,j}(r)]$.

Now recall that for fixed (r, j)

$$s_j(r) = \{x < S_j(r) \mid C(x, r, j) = C(S_j(r), r, j) + K\}$$

Equation (14), $S_j(r) = \text{argmin}_x C(x, r, j)$, and $S_{k,j}(r) = \text{argmin}_x C_k(x, r, j)$, imply that for fixed (r, j)

$$C_k(S_{k,j}(r), r, j) + \underline{c}_k \leq C(S_j(r), r, j) \leq C_k(S_{k,j}(r), r, j) + \bar{c}_k$$

Therefore

$$C_k(s_{k,j}(r), r, j) + \underline{c}_k \leq C(s_j(r), r, j) \leq C_k(s_{k,j}(r), r, j) + \bar{c}_k$$

and

$$s_j(r) \in \{x \mid C_k(s_{k,j}(r), r, j) + \underline{c}_k \leq C(x, r, j) \leq C_k(s_{k,j}(r), r, j) + \overline{c}_k\}$$

By the left inequality of (14), we have

$$s_j(r) \in \{x \mid C_k(x, r, j) + \underline{c}_k \leq C_k(s_{k,j}(r), r, j) + \overline{c}_k\}$$

and the condition $x < S_j(r)$ implies $s_j(r) \geq \underline{s}$. By the right inequality of (14), we also have

$$s_j(r) \in \{x \mid C_k(x, r, j) + \overline{c}_k \geq C_k(s_{k,j}(r), r, j) + \underline{c}_k\}$$

and the condition $x < S_j(r)$ implies $s_j(r) \leq \overline{s}$. This completes the proof. ▮

2.9.2 Algorithm

To compute the parameters of the optimal policy for $\tau = 2$ we solve the dynamic program described in Section 2.5.2 using the next algorithm. An implementation of this algorithm is available upon request.

Step 0. Set $j = m - 1$, $k = 1$, and $C_0(x, r, j) = 0$ for all (x, r, j).

Step 1. For $j \geq 1$, set

$$V_k(x, 0, j) = G(x) + \alpha \mathrm{E}_\xi C_{k-1}(x - \xi, 0, j)$$

and perform a grid search for the value $S_{k,j}$ that minimizes the function $V_k(x, 0, j)$ with respect to x. For this search, use the lower bound defined in Lemma 13 and the upper bound defined by part h(iv) of Lemma A.1. Next, find the value $s_{k,j}$ such that $s_{k,j} < S_{k,j}$ and $V_k(s_{k,j}, 0, j) = V_k(S_{k,j}, 0, j) + K$. These two values define the function $C_k(x, 0, j)$ by

$$C_k(x, 0, j) = \begin{cases} V_k(x, 0, j) - c^e x & \text{if } x \geq s_{k,j} \\ K + V_k(S_{k,j}, 0, j) - c^e x & \text{if } x < s_{k,j} \end{cases}$$

For $j = 0$ proceed as follows: for every value of z perform a grid search for the smallest $y \geq 0$ that minimizes $c^r y + \alpha \mathrm{E}_\xi C_k(z - \xi, y, 1)$ with respect to y. This yields

$$Q_k(z) = \mathrm{argmin}_{y \geq 0}\{c^r y + \alpha \mathrm{E}_\xi C_k(z - \xi, y, 1)\}$$

and

$$H_k(z) = c^r Q_k(z) + \alpha \mathrm{E}_\xi C_k(z, Q_k(z), 1)$$

Then, perform a grid search for the argument $S_{k,0}$ that minimizes $G(z) + H_k(z)$. Proceed with finding the value $s_{k,0}$ such that $s_{k,0} < S_{k,0}$ and

$$G(s_{k,0}) + H_k(s_{k,0}) = G(S_{k,0}) + H_k(S_{k,0}) + K$$

These three parameters define the function $C_k(x, r, 0)$ as

$$C_k(x, 0, 0) = \begin{cases} G(x) + H_k(x) - c^e x & \text{if } x \geq s_{k,0} \\ \\ K + G(S_{k,j}) + H_k(S_{k,j}) - c^e x & \text{if } x < s_{k,0} \end{cases}$$

Step 2. If $C_k(x, r, j)$ does not satisfy the stopping criterion defined in Section 2.9.1, set $k = k + 1$, $j = j - 1$ and go to Step 1. Otherwise, deliver $S_j = S_{k,j}$, $s_j = s_{k,j}$, and $Q(z) = Q_k(z)$.

Example 1 (continued) Computing the parameters of the optimal policy for Example 1 with this algorithm we obtain the following values for the first review cycle (stages k from 1 to 5 and periods j from 4 to 0): $(s_{1,4}, S_{1,4}) = (-9.5, 2)$, $(s_{2,3}, S_{2,3}) = (-1.6, 4)$, $(s_{3,2}, S_{3,2}) = (0.4, 6)$, $(s_{4,1}, S_{4,1}) = (1, 7)$, $(s_{5,0}, S_{5,0}) = (-7.5, 2)$, and $Q_5(z) = \max(11 - z, 0)$.

2.9.3 Numerical Computations

In this section we compute the optimal policy for a grid of 3888 cases specified in Table 4. This design and the optimal cost functions will be used in Section 3.3 to evaluate the proposed heuristics.

We consider an inventory cost function of the form $L(x) = h\delta(x) + p\delta(-x)$, where x is the net inventory at the end of the current period, h is the unit holding cost, and p is the unit backorder penalty cost. We consider two types of demand distributions, namely Poisson and negative binomial with 9 combinations of means and coefficients of variation to account for central tendency and deviation about the mean. The negative binomial distribution with parameters $r > 0$ and $p \in (0, 1)$ has probability mass function

$$\Pr(X = k) = \frac{\Gamma(k + r)}{k!\Gamma(r)} p^r (1 - p)^k$$

Table 4: Experimental Design

Factor	Levels	Number of Levels
Demand Distribution	Poisson($\sigma^2/\mu = 1$) Negative Binomial($\sigma^2/\mu = 3$) Negative Binomial($\sigma^2/\mu = 9$)	3
Mean Demand (μ)	2, 4, 8	3
Regular Order Lead-time	2, 3, 4	3
Review Cycle	Regular Order Lead-time + 1, 4	2
Emergency Order Variable Cost	3, 5, 7	3
Emergency Order Fixed Cost	5, 50	2
Regular Order Variable Cost	1, 2	2
Backorder Penalty Cost	8, 15	2
Holding Cost	1	1
Discount Factor	0.999, 0.99, 0.9	3
Total Number of Cases		3888

where $\Gamma(\cdot)$ is the gamma function. We also consider regular order lead-times equal to 2, 3 and 4 and two variable costs for regular orders. The variable costs for emergency orders that are approximately 5 times as large as the variable costs for regular orders. Since the single-stage cost functions are linear with respect to the emergency variable cost and the backorder penalty cost, the holding cost is set equal to 1. By the same reasoning, we consider the backorder penalty cost rate in three levels corresponding to approximately 100%, 200% and 300% of the emergency variable cost. The setup costs for emergency orders are 50%, 100%, and 1000% of the emergency variable cost. Since for real systems a single-period discount factor is not expected to be lower than 0.9, we consider the discount factors of 0.9, 0.99, and 0.999. We further note that both equations of Assumption 7 are satisfied.

A subset of the experimental results is tabulated in Section B.1.

2.10 Concluding Remarks

In this chapter we have proved the optimality of an (s, S) type policy when the regular lead-time is two periods, provided a dynamic program to estimate the parameters of the optimal policy, and argued the extension of the optimal policy to the case $\tau > 2$. We also obtained the parameters for these policies.

Event though our proof for the case $\tau = 2$ could restrict the applicability of this result to many real situations, our numerical results give confidence that the structure of the optimal policy can be applied in a fair number of scenarios.

In the next chapter, we will develop approximate policies that are less time consuming and yield near optimal parameters.

CHAPTER III

APPROXIMATE POLICIES

The policy described in Chapter II was proven to be optimal for the case of $\tau = 2$ and conjectured to remain optimal for $\tau > 2$. It does require, however, significant computational effort. The goal of this chapter is to obtain heuristic policies whose operational parameters can be computed with relatively small computational effort. The chapter proceeds as follows. In Section 3.1 we review the related literature related, in Section 3.2 we present two heuristic approaches, and in Section 3.3 we compare their performance. Finally, in Section 3.4 we present conclusions for this chapter.

3.1 Literature Review

Most heuristics for multi-period inventory policies originate from approximations to limiting results based on renewal theory, approximations to expectations and differential equations, or approximations to the value iteration algorithm. We will present the relevant literature following these three directions focusing primarily on policies that include a fixed cost.

The first direction starts with Roberts [49] who derives the asymptotic behavior of the discounted renewal function for (s, S) policies and obtains approximations for the difference $S - s$ when the values of the fixed cost and the backorder penalty cost are large. Hadley and Whitin [28, Ch. 4] assume that the expected number of backorders is negligible and that the supplier lead-time is a random variable to present two approximations for the estimation of the operational parameters of an (r, Q) policy. Wagner [63] compares the performance of several policies by estimating the optimal parameters using search heuristics, approximating the asymptotic quantities derived by Roberts [49], using continuous-review models instead of periodic-review models, or adopting a batch base-stock policy.

Ehrhardt [18] estimates the optimal parameters of an (s, S) policy using a regression model whose structure is obtained from the work of Roberts [49] and only requires knowledge of the first two moments of the demand distribution. The regression coefficients are estimated from an experimental design of 288 configurations with known optimal parameters. This is the original Power Approximation (PA) method. Ehrhardt et al. [20] study stocking rules for a warehouse facility whose demand is comprised of replenishment orders from other facilities that follow (s, S) policies. Using simulation they search for stationary (s, S) policies. The best performing approximation found is a rule that is an adjustment of the power approximation of Ehrhardt [18] to an (auto) correlated demand process. This policy is close to optimal when the demand's mean and variance are known exactly, and reasonably close when statistical estimates are used. Ehrhardt and Mosier [21] present a revision to the PA method incorporating modifications to ensure the homogeneity in the units chosen to measure demand and the proper limiting behavior of the quantity $S - s$ when the variance of the demand is small. Ehrhardt [19] studies policies for systems with random lead-times assuming that replenishment orders do not cross in time and that the supplier's random lead-time is independent of the size of the order. For the minimization of the expected discounted cost problem, he presents the optimality conditions for an (s, S) policy and shows how to modify the PA method to estimate the operational parameters of the optimal policy.

Sahin and Sinha [51] show simple conditions under which two policy approximations based on asymptotic renewal theory are accurate. The approximations sunder consideration are the Revised PA of Ehrhardt and Mosier [21] and a linear approximation of the cost rate in the renewal function.

Tijms and Groenevelt [60] evaluate approximations for (s, S) policies for periodic and continuous-review systems. They allow stochastic lead-times for replenishment orders provided that the probability of orders crossing in time is negligible. Their inventory policies take into account a constraint on the service level, defined as the fraction of demand met directly with inventory at hand. In particular, they use renewal theory to find the reorder level as a function of the amount $S - s$.

The second direction of research, based on expectations and approximation of differential equations, begins with Sivazlian [56] who uses computational methods to estimate Laplace transforms and obtain the solution of the differential equations involving the parameters of an (s, S) policy for a demand following a gamma distribution. Naddor [41] presents heuristics to estimate the operational parameters of order-up-to-R, (r, Q), and (s, S) policies when the acquisition costs do not include a fixed setup cost. The heuristics are motivaed from the optimal results for six inventory systems. The author also extends the heuristics to multi-item inventory systems.

Shore [55] employs approximations for the quantiles of a random variable loss function to derive explicit approximate solutions to the standard newsboy problem, the (r, Q) model, and a periodic order-up-to-R model.

Sivazlian and Wei [57] analyze a multicommodity inventory system which operates under a given (s, S) policy. They approximate an integral of the expected backlog level with a bivariate exponential function to obtain first a closed-form expression for the Laplace transform of the expected backlog level and then approximate operational parameters.

Kapalka, Katircioglu and Puterman [34] study optimal (s, S) policies for a large number of products and locations of a Western Canadian retailer. They evaluate the long-run average cost and service level for a fixed (s, S) policy and then use a search procedure to locate the optimal parameters. The search procedure is based on an updating scheme for the transition probability matrix of the underlying Markov chain, bounds on S, and monotonicity assumptions on the cost and service level functions.

Kleinau and Thonemann [35] present an alternative approach for solving inventory-control problems that is based on Genetic Programming. They apply their procedure to a single-echelon system with deterministic demand, a single-echelon system with Poisson process demand, and a serial two-echelon system with Poisson process demand under continuous review.

The work presented in this chapter does not follow the first two research directions because they are based on analytical results that are hard to obtain for the inventory policy described in Theorem 1. Since we have already shown the convergence of a dynamic program, we follow the third direction of research which begins with the work of Norman and White [42], who present approximate solutions for the policy iteration algorithms introduced by Howard [31] by replacing probability distributions with their expectations and using the value of the states in the corresponding deterministic system under its optimal policy to determine an approximate policy for the stochastic system through a single application of the policy improvement step.

Porteus [46] introduces an adjustment to the approach in [42]. In the policy evaluation step, his approach maintains the current period's probabilistic reward but approximates the random demand in the transition function by its expected value. Freeland and Porteus [23] evaluate the approximation presented in Porteus [46] and compare its performance against the approximation presented by Wagner et al. [63]. Freeland and Porteus [22] simplify the method presented in Porteus [46] by assuming that the shortage cost is relatively large and that the variance of the demand is relatively small. Porteus [47] introduces three new methods to compute the operational parameters of an (s, S) policy, two of them based on Freeland and Porteus [22] and a third one based on a continuous-review approximation. The development of the heuristics presented in this chapter is based on the same simplification principle presented by Porteus [46] albeit in a more complex system.

3.2 Inventory-Policy Heuristics

In this section we develop two inventory-policy heuristics based on a simplification of the value iteration algorithm. This simplification, which we call Deterministic Model, is an approximation of the underlying Markov reward process defined in Section 2.5.

3.2.1 Extension of Definitions

We extend some definitions of Section 2.5 to the corresponding limiting function $C(x, r, j)$. As in (5), we have

$$V(z, r, j) := G(z) + \alpha \mathrm{E}_\xi C(z - \xi, r, j^+) \tag{18}$$

Parallel to (7), we define

$$H(z) := \min_{w \geq z} \{(w - z)c^r + \alpha \mathrm{E}_\xi C(w - \xi, 0, 1)\} \tag{19}$$

For given z, we define $w(z)$ to be the argument that attains the minimum in (19). That is,

$$w(z) := \mathrm{argmin}_{w \geq z} \{(w - z)c^r + \alpha \mathrm{E}_\xi C(w - \xi, 0, 1)\} \tag{20}$$

3.2.2 Deterministic Model

We define the deterministic model as the reward process with state space $\mathbf{X}$, decision space $\mathcal{D}$, single-period cost functions $c(\mathbf{x}, \mathbf{d}, \xi)$ as defined in Sections 2.3 and 2.5, and transition function

$$f[(x,r,j),(z,y)] = \begin{cases} (z-\mu_\xi, r, j^+) & \text{if } j \notin \{0, \tau-1\} \\ (z+r-\mu_\xi, 0, 1) & \text{if } j = \tau-1 \\ (z-\mu_\xi, y, 1) & \text{if } j = 0 \end{cases}$$

3.2.3 Heuristic Policy 1 (HP1)

Since the proofs in Chapter II are valid for a deterministic demand distribution, the optimal policy for the deterministic model retains the structure presented in Theorem 1.

The first inventory-policy heuristic is the optimal policy for the deterministic model. As proved in Section 2.7, the optimal parameters for this system can be obtained from the limit functions of a dynamic programming model, hence the algorithm presented in Section 2.9.2 can be used to compute the operational parameters.

3.2.4 Heuristic Policy 2 (HP2)

This policy is applicable to the case $\tau = 2$ and is based on the user-defined parameter R and the operational parameters (s_j, S_j), $j \in \{0, \ldots, m-1\}$, that are computed with the algorithm presented later in Section 3.2.6.1. In state (x, r, j) the following actions are taken:

(a) For $j = 0$: if $x < s_0$, place an emergency order for $y^e = S_0 - x$ units; otherwise, do not order ($y^e = 0$). Further, place a regular order of size $\max(R - y^e - x, 0)$.

(b) For $j = 1$: if $x + r < s_1$, place an emergency order for $S_1 - (x + r)$ units; otherwise, do not order.

(c) For $j \in \{2, \ldots, m-1\}$: if $x < s_j$ place an emergency order for $S_j - x$ units; otherwise, do not order.

This policy is very similar to the optimal policy presented in Theorem 1, but the regular order size function is simplified. This change not only eases the application of the policy but also allows us to develop a simpler minimization model in order to obtain the operational parameters. To motivate this simplification, we note that in all the results presented in Section B.1, there exists a value $R > S_0$ such that $Q(z) = R - z$ for $z \leq R$. In most of the experiments, there is no benefit to place regular orders when $z > R$, so an order-up-to-R with $Q(z) = \max(R - z, 0)$ is optimal. Even though there are a few experiments where this is not the case, for the optimal policy stated in Theorem 1 the emergency inventory-position z at period $j = 0$ will always be in the interval $[s_0, \max_j S_j]$, hence the function $Q(z)$ for $z > R$ can be simplified to $Q(z) = 0$ and an order-up-to-R policy with $Q(z) = \max(R - z, 0)$ is a good approximation.

3.2.4.1 Characteristics of HP2

In this section we show that, under HP2, the emergency inventory-position of the deterministic model follows cycles of length m and, consequently, the expression for the total expected discounted cost stated in equation (3) can be simplified. This simpler form and the fact that $V(x, r, j)$, defined in equation (18), achieves a minimum at S_j are the basis to formulate a minimization model that returns the operational parameters for HP2.

First, we show that the emergency inventory-position of the deterministic model controlled with HP2, follows cycles with length m.

Lemma 17 *When the deterministic model is controlled with the HP2 policy, the path of the emergency inventory-position z follows cycles with length m. The cycles start at period $j = 2$ following the first regular-order placement opportunity.*

Proof We prove this lemma by showing that at every period $j = 2$ following the first regular-order placement opportunity, the emergency inventory-position is constant. Since the transition function is deterministic, the sample path will start following cycles proving the lemma.

We first look at the period following following the first regular order opportunity, i.e., period $(1, 1)$. Recall that if $x_{1,1} + r < s_1$, the policy will raise the emergency inventory-position to $z_{1,1} = S_1 - r$; otherwise, $z_{1,1} = x_{1,1}$. Also, we note that since $x_{1,1} = z_{1,0} - \mu_\xi$ and $r = y = R - z_{1,0}$, we have $x_{1,1} + r = R - \mu_\xi$. Hence, according to the inventory policy, if $x_{1,1} + r = R - \mu_\xi \geq s_1$, then $x_{1,2} = z_{1,1} + r - \mu_\xi = R - 2\mu_\xi$. On the other hand, $x_{1,1} + r = R - \mu_\xi < s_1$ implies $x_{1,2} = z_{1,1} + r - \mu_\xi = S_1 - \mu_\xi$. We observe that the decision level is constant and that the inventory at hand at the beginning of period $j = 2$, and consequently the emergency inventory-position, does not depend on previous inventory levels. Further,

$$x_{1,2} = \begin{cases} R - 2\mu_\xi & \text{if } R - \mu_\xi \geq s_1 \\ S_1 - \mu_\xi & \text{if } R - \mu_\xi < s_1 \end{cases}$$

This argument is valid for any of the following regular review cycles, hence we have $x_{1,2} = x_{2,2} = \cdots$.

Since the transition function and the reorder decisions are deterministic, then two regular-order cycles with same net inventory at the beginning of period $j = 2$, i.e., $x_{k,2} = x_{k+1,2}$, will have the same emergency inventory-position path, proving the lemma. ▮

As a consequence of the above lemma, we show that the total size of orders placed during a cycle is a constant.

Corollary 18 *Under the assumptions of Lemma 17, in every cycle of the emergency inventory-position sample path we have*

$$R - z_{i,0} + \sum_{j=2}^{m-1} y^e_{i,j} + \sum_{j=0}^{1} y^e_{i+1,j} = m\mu_\xi$$

Proof By Lemma 17, we have that for all regular review cycles i, $z_{i,0} = z_{i+1,0}$. Since for the deterministic model

$$z_{i+1,0} = z_{i,0} + R - z_{i,0} + \sum_{j=2}^{m-1} y^e_{i,j} + \sum_{j=0}^{1} y^e_{i+1,j} - m\mu_\xi$$

the result follows. ▮

3.2.5 Optimization Model for HP2

In this section we formulate a minimization model to obtain the optimal parameters for the deterministic model controlled with HP2.

3.2.5.1 *Notation*

The following variables will be used to define the optimization model. In the following definitions, we distinguish the path cycle-related elements by adding a bar on top of them.

- l: Specific initial period for cost accounting denoted by the number of periods after the last regular review epoch, $l \in \{0, 1, \ldots, m-1\}$. In other words, this is the period within a cycle where costs will start to be accrued.
- x_l: Inventory on hand at the beginning of period l.
- y_j^e: Size of emergency order placed in period j before starting the first cycle.
- $Q = R - z_0$: Size of regular order at period $j = 0$ before starting the first cycle.
- z_j: Emergency inventory-position at period j before starting the first cycle. This is an auxiliary variable such that

$$z_j = \begin{cases} z_{j^-} + y_j^e - \mu_\xi & \text{if } j \neq 2 \\ z_1 + y_2^e - \mu_\xi + Q & \text{if } j = 2 \end{cases} \tag{21}$$

- $\bar{y}_j^e$: Size of emergency order placed in period j during a cycle.
- $\bar{Q} = R - \bar{z}_0$: Size of regular order at period 0 during a cycle.
- $\bar{z}_j$: Emergency inventory-position at period j during a cycle. This is an auxiliary variable such that

$$\bar{z}_j = \begin{cases} \bar{z}_{j^-} + \bar{y}_j^e - \mu_\xi & \text{if } j \neq 2 \\ \bar{z}_1 + \bar{y}_2^e - \mu_\xi + \bar{Q} & \text{if } j = 2 \end{cases} \tag{22}$$

3.2.5.2 *Objective Function*

Since the path of the emergency inventory-position follows cycles, the total expected discounted cost for the deterministic model can be broken in two parts: the function that accounts for the costs before the start of the first cycle, which we name the pre-cycle cost function, and the function that represents the infinite sum of the remaining cycle costs, which we name the cycle cost function. Our goal is to form an objective function $\Psi_l(x_l, R, y^e_{j^+}, \ldots, y^e_1, \bar{z}_2, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1})$ with the same value, under the optimal policy, as the function $V(x, 0, l)$ defined in equation (18). By the definition of S_l provided in Lemma 7, the minimization of this objective function will provide the optimal parameters for HP2.

Since under the optimal policy no emergency orders are placed for an initial inventory of S_l units, we do not include emergency orders on the first period. We define the pre-cycle cost function for an initial period $l \in \{1, 2, \ldots, m-1\}$ as

$$\begin{aligned}
\varphi_l(x_l, R, y^e_{l^+}, \ldots, y^e_1) \quad := \quad & \alpha \mathrm{E}_\xi L(x_l - \xi) + \alpha[K\delta(y^e_{l^+}) + c^e(y^e_{l^+}) + \alpha \mathrm{E}_\xi L(z_{l^+} - \xi)] + \cdots \\
& + \alpha^{m-l}[c^r(R - z_0) + K\delta(y^e_0) + c^e(y^e_0) + \alpha \mathrm{E}_\xi L(z_0 - \xi)] \\
& + \alpha^{1+m-l}[K\delta(y^e_1) + c^e(y^e_1) + \alpha \mathrm{E}_\xi L(z_1 - \xi)]
\end{aligned}$$

and for $l = 0$ as

$$\varphi_0(x_0, R, y^e_1) \quad := \quad c^r(R - x_0) + \alpha \mathrm{E}_\xi L(x_0 - \xi) + \alpha[K\delta(y^e_1) + c^e(y^e_1) + \alpha \mathrm{E}_\xi L(z_1 - \xi)]$$

For $l \neq 0$, we replace the redundant variable z_0 to get

$$\begin{aligned}
\varphi_l(x_l, R, y^e_{l^+}, \ldots, y^e_1) \quad := \quad & \alpha \mathrm{E}_\xi L(x_l - \xi) + \alpha[K\delta(y^e_{l^+}) + c^e(y^e_{l^+}) + \alpha \mathrm{E}_\xi L(z_{l^+} - \xi)] + \cdots \\
& + \alpha^{m-l}[c^r[R - x_l + (m-1-l)\mu_\xi - y^e_{l^+} - \cdots - y^e_{m-1} - y^e_0] \\
& \quad + K\delta(y^e_0) + c^e(y^e_0) + \alpha \mathrm{E}_\xi L(z_0 - \xi)] \\
& + \alpha^{1+m-l}[K\delta(y^e_1) + c^e(y^e_1) + \alpha \mathrm{E}_\xi L(z_1 - \xi)]
\end{aligned}$$

We now define the cost function of a single cycle as

$$\begin{aligned}
\phi(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1}) \quad := \quad & K\delta(\bar{y}^e_2) + c^e \bar{y}^e_2 + \alpha \mathrm{E}_\xi L(\bar{z}_2 - \xi) + \cdots \\
& + \alpha^{m-3}[K\delta(\bar{y}^e_{m-1}) + c^e \bar{y}^e_{m-1} + \alpha \mathrm{E}_\xi L(\bar{z}_{m-1} - \xi)] \\
& + \alpha^{m-2}[K\delta(\bar{y}^e_0) + c^e \bar{y}^e_0 + c^r(R - \bar{z}_0) + \alpha \mathrm{E}_\xi L(\bar{z}_0 - \xi)] \\
& + \alpha^{m-1}[K\delta(\bar{y}^e_1) + c^e \bar{y}^e_1 + \alpha \mathrm{E}_\xi L(\bar{z}_1 + \bar{Q} - \xi)]
\end{aligned}$$

Next, we write $\bar{z}_0$ in terms of $\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots$, and $\bar{y}^e_{m-1}$

$$\begin{aligned}
\bar{z}_0 &= \bar{z}_{m-1} - \mu_\xi + \bar{y}^e_0 \\
&= \bar{z}_{m-2} - 2\mu_\xi + \bar{y}^e_0 + \bar{y}^e_{m-2} \\
&= \bar{z}_2 - (m-2)\mu_\xi + \bar{y}^e_0 + \sum_{j=3}^{m-1} \bar{y}^e_j
\end{aligned}$$

By Corollary 18, we also have

$$\begin{aligned}
\bar{y}^e_2 &= m\mu_\xi - R + \bar{z}_0 - \sum_{j=0, j\neq 2}^{m-1} \bar{y}^e_j \\
&= m\mu_\xi - R + \bar{z}_2 - (m-2)\mu_\xi + \bar{y}^e_0 + \sum_{j=3}^{m-1} \bar{y}^e_j - \sum_{j=0, j\neq 2}^{m-1} \bar{y}^e_j \\
&= \bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1}) &= K\delta(\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1) \\
&\quad + c^e(\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1) + \alpha \mathrm{E}_\xi L(\bar{z}_2 - \xi) + \cdots \\
&\quad + \alpha^{m-3}[K\delta(\bar{y}^e_{m-1}) + c^e \bar{y}^e_{m-1} + \alpha \mathrm{E}_\xi L(\bar{z}_{m-1} - \xi)] \\
&\quad + \alpha^{m-2}[K\delta(\bar{y}^e_0) + c^e \bar{y}^e_0 + c^r(R - \bar{z}_0) + \alpha \mathrm{E}_\xi L(\bar{z}_0 - \xi)] \\
&\quad + \alpha^{m-1}[K\delta(\bar{y}^e_1) + c^e \bar{y}^e_1 + \alpha \mathrm{E}_\xi L(\bar{z}_1 + \bar{Q} - \xi)]
\end{aligned}$$

We express the auxiliary variables $\bar{z}_j$ in terms of $\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots$ using equation (22) to obtain

$$\begin{aligned}
&\phi(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1}) = K\delta(\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1) \\
&\qquad + c^e(\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1) + \alpha \mathrm{E}_\xi L(\bar{z}_2 - \xi) + \cdots \\
&\quad + \alpha^{m-3}[K\delta(\bar{y}^e_{m-1}) + c^e \bar{y}^e_{m-1} + \alpha \mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_{m-1} - (m-3)\mu_\xi - \xi)] \\
&\quad + \alpha^{m-2}[K\delta(\bar{y}^e_0) + c^e \bar{y}^e_0 + c^r[R - (\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi)] \\
&\qquad + \alpha \mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi - \xi)] \\
&\quad + \alpha^{m-1}[K\delta(\bar{y}^e_1) + c^e \bar{y}^e_1 + \alpha \mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_1 - (m-1)\mu_\xi - \xi)]
\end{aligned}$$

To simplify the notation, we define the function $\kappa : \mathbb{R}^m \to \mathbb{R}$, as

$$\begin{aligned}\kappa(R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \dots, \bar{y}^e_{m-1}) \quad &:= \quad K[\delta(\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1) + \cdots \\ &\qquad + \alpha^{m-3}\delta(\bar{y}^e_{m-1}) + \alpha^{m-2}\delta(\bar{y}^e_0) + \alpha^{m-1}\delta(\bar{y}^e_1)]\end{aligned}$$

and rewrite

$$\begin{aligned}\phi(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \dots, \bar{y}^e_{m-1}) &= \kappa(R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \dots, \bar{y}^e_{m-1}) \\ &+ \alpha^{m-2}c^r[R - (\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi)] \\ &+ c^e[\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1 + \alpha\bar{y}^e_3 + \ldots + \alpha^{m-3}\bar{y}^e_{m-1} + \alpha^{m-2}\bar{y}^e_0 + \alpha^{m-1}\bar{y}^e_1] \\ &+ \alpha \mathrm{E}_\xi L[\bar{z}_2 - \mu_\xi - \xi] + \cdots + \alpha^{m-2}\mathrm{E}_\xi L[\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_{m-1} - (m-3)\mu_\xi - \xi] \\ &+ \alpha^{m-1}\mathrm{E}_\xi L[\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi - \xi] \\ &+ \alpha^m \mathrm{E}_\xi L[\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_1 - (m-1)\mu_\xi - \xi]\end{aligned}$$

To build the objective function, we start with the total expected discounted cost and correct it with the term term $c^e x_l$ to account for the relation $V(S_l, 0, l) = C(S_l, 0, l) + c^e S_l$, as established in Lemma 7. Therefore, we have

$$\begin{aligned}\Psi_l(x_l, R, y^e_{j+}, \dots, y^e_1, \bar{z}_2, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \dots, \bar{y}^e_{m-1}) \quad &:= \quad c^e x_l + \varphi_l(x_l, R, y^e_{j+}, \dots, y^e_1) \\ &\quad + \alpha^{2+m-j}\phi(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \cdots, \bar{y}^e_{m-1})/(1-\alpha^m)\end{aligned}$$

3.2.5.3 Constraints

Corollary 18 defined the first constraint for our model. The following lemma establishes bounds for $\bar{z}_2$.

Lemma 19 *Under the assumptions of Lemma 17, the emergency inventory-position $\bar{z}_2$ satisfies the following bounds*

$$R - 2\mu_\xi + \bar{y}^e_1 \le \bar{z}_2 \le R + (m-2)\mu_\xi - \bar{y}^e_0 - \sum_{j=3}^{m-1} \bar{y}^e_j$$

Proof Since $Q(z) \geq 0$ and $Q(\bar{z}_0) = R^* - \bar{z}_0$, we have

$$\begin{aligned} R &\geq \bar{z}_0 = \bar{z}_{m-1} - \mu_\xi + \bar{y}_0^e \\ &= \bar{z}_{m-2} - 2\mu_\xi + \bar{y}_0^e + \bar{y}_{m-2}^e \\ &= \bar{z}_2 - (m-2)\mu_\xi + \bar{y}_0^e + \sum_{j=3}^{m-1} \bar{y}_j^e \end{aligned}$$

This implies

$$\bar{z}_2 \leq R + (m-2)\mu_\xi - \bar{y}_0^e - \sum_{j=3}^{m-1} \bar{y}_j^e$$

With regard to the lower bound, Corollary 18 implies

$$\bar{y}_2^e = m\mu_\xi - (R - \bar{z}_0) - \sum_{j=0, j\neq 2}^{m-1} \bar{y}_j^e \geq 0$$

Hence

$$\begin{aligned} 0 &\leq m\mu_\xi - (R - \bar{z}_0) - \sum_{j=0, j\neq 2}^{m-1} \bar{y}_j^e \\ &= m\mu_\xi - (R - (\bar{z}_2 - (m-2)\mu_\xi + \sum_{j=3}^{m-1} \bar{y}_j^e + \bar{y}_0^e) - \sum_{j=0, j\neq 2}^{m-1} \bar{y}_j^e \\ &= \bar{z}_2 - R + 2\mu_\xi - \bar{y}_1^e \end{aligned}$$

This completes the proof. ∎

3.2.5.4 Model Formulation

Using the objective function and constraints we formulate the following minimization problem

$$\min \Psi_l(x_l, R, y^e_{j+}, \ldots, y^e_1, \bar{z}_2, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1}) \tag{23}$$

subject to

$$\begin{aligned}
& R - (\bar{z}_2 - 2\mu_\xi + \bar{y}^e_0 + \bar{y}^e_1) + \sum \bar{y}^e_j = m\mu_\xi \\
& \bar{z}_2 \geq (1-m)\mu_\xi + R + \sum_{j=1, j\neq 2}^{m-1} \bar{y}^e_j \\
& \bar{z}_2 \leq R + (m-2)\mu_\xi - \bar{y}^e_0 - \sum_{j=3}^{m-1} \bar{y}^e_j \\
& y^e_j, \bar{y}^e_j \geq 0 \quad \forall j
\end{aligned} \tag{24}$$

3.2.6 Relaxed Model Formulation

To avoid having a discontinuous objective function, we relax equation (23) by adding the auxiliary variables ρ_j and model the fixed emergency order cost in period j with $K\rho_j$ such that $y^e_j \leq M\rho_j$ and $0 \leq \rho_j \leq 1$, for some constant M. Using the new variables, we rewrite the cost function of a single cycle as

$$\begin{aligned}
& \phi'(\bar{z}_2, R, \bar{y}^e_0, \bar{y}^e_1, \bar{y}^e_3, \ldots, \bar{y}^e_{m-1}, \rho_0, \rho_1, \rho_3, \ldots, \rho_{m-1}) \\
:= \quad & K[\bar{\rho}_2 + \alpha\bar{\rho}_3 + \cdots + \alpha^{m-3}\bar{\rho}_{m-1} + \alpha^{m-2}\bar{\rho}_0 + \alpha^{m-2}\bar{\rho}_1] \\
& + \alpha^{m-2} c^r [R - (\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi)] \\
& + c^e[\bar{z}_2 - R + 2\mu_\xi - \bar{y}^e_1 + \alpha\bar{y}^e_3 + \cdots + \alpha^{m-3}\bar{y}^e_{m-1} + \alpha^{m-2}\bar{y}^e_0 + \alpha^{m-1}\bar{y}^e_1] \\
& + \alpha \mathrm{E}_\xi L(\bar{z}_2 - \mu_\xi - \xi) + \cdots + \alpha^{m-2}\mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_{m-1} - (m-3)\mu_\xi - \xi) \\
& + \alpha^{m-1}\mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_0 - (m-2)\mu_\xi - \xi)) \\
& + \alpha^m \mathrm{E}_\xi L(\bar{z}_2 + \bar{y}^e_3 + \cdots + \bar{y}^e_1 - (m-1)\mu_\xi - \xi)]
\end{aligned}$$

and the pre-cycle cost function as

$$
\begin{aligned}
&\varphi_l'(x_l, R, y_{l+}^e, \ldots, y_1^e, \rho_{l+}, \ldots, \rho_1) \\
&:= \quad c^e x_l + \alpha \mathrm{E}_\xi L(x_l - \xi) + \alpha[K\rho_{l+} + c^e(y_{l+}^e) + \alpha \mathrm{E}_\xi L(z_{l+} - \xi)] + \cdots \\
&\qquad + \alpha^{m-l}\{c^r[R - x_l + (m-1-l)\mu_\xi - y_{l+}^e - \cdots - y_{m-1}^e - y_0^e] \\
&\qquad\quad + K\rho_0 + c^e(y_0^e) + \alpha \mathrm{E}_\xi L(z_0 - \xi)\} \\
&\qquad + \alpha^{1+m-l}[K\rho_1 + c^e(y_1^e) + \alpha \mathrm{E}_\xi L(z_1 - \xi)]
\end{aligned}
$$

The resulting nonlinear minimization model is

$$
\begin{aligned}
&\min \Psi_l(x_l, R, y_{l+}^e, \ldots, y_1^e, \rho_{l+}, \ldots, \rho_1, \bar{z}_2, \bar{y}_0^e, \bar{y}_1^e, \bar{y}_3^e, \ldots, \bar{y}_{m-1}^e, \rho_0, \rho_1, \rho_3, \ldots, \rho_{m-1}) \\
&\text{subject to} \\
&\quad R - (\bar{z}_2 - 2\mu_\xi + \bar{y}_0^e + \bar{y}_1^e) + \sum \bar{y}_j^e = m\mu_\xi \\
&\quad \bar{z}_2 \geq (1-m)\mu_\xi + R^* + \sum_{j=1, j\neq 2}^{m-1} \bar{y}_j^e \\
&\quad \bar{z}_2 \leq R^* + (m-2)\mu_\xi - \bar{y}_0^e - \sum_{j=3}^{m-1} \bar{y}_j^e \\
&\quad y_j^e \leq M\rho_j \\
&\quad \bar{y}_j^e \leq M\bar{\rho}_j \\
&\quad y_j^e, \bar{y}_j^e \quad \forall j \\
&\quad 1 \geq \bar{\rho}_j \geq 0 \quad \forall j \\
&\quad 1 \geq \rho_j \geq 0 \quad \forall j
\end{aligned}
\tag{25}
$$

The value $M = 4$ was used for the experiments of Table 4.

3.2.6.1 Algorithm

The algorithm uses a simplified steepest ascent heuristic. On each step, the algorithm searches for the best improving direction verifying the effect of a single variable change (that is, changing either R, y_j^e, $\bar{y}_j^e$, ρ_j, $\bar{\rho}_j$ or x_l). Let $\mathrm{SSA}_l(\bar{z}_2)$ be the smallest value of the objective function found by the simplified steepest ascent method for the fixed values of $\bar{z}_2$, that is,

$$
\mathrm{SSA}_l(\bar{z}_2) = \Psi_l(x_l^*, R^*, y_{l+}^*, \ldots, y_1^*, \rho_{l+}^*, \ldots, \rho_1^*, \bar{z}_2, \bar{y}_0^*, \bar{y}_1^*, \bar{y}_3^*, \ldots, \bar{y}_{m-1}^*, \rho_0^*, \rho_1^*, \rho_3^*, \ldots, \rho_{m-1}^*)
$$

The search for the best value $\bar{z}_2$ proceeds in the following manner:

Step 0. Set $k = 0$, $R = m\mu_\xi$, $\bar{z}_2 = R - 2\mu_\xi$, $x_l = 0$, $y_j^e = 0$, $\bar{y}_j^e = 0$, searchStep = 0.1, and minVal = SSA($\bar{z}_2$).

If $\mathrm{SSA}_l(\bar{z}_2 - \mathrm{searchStep}) < \mathrm{SSA}_l(\bar{z}_2)$, then set searchStep = −0.1.

Step 1. Set $\bar{z}_2 = \bar{z}_2 + \mathrm{searchStep}$, $R = \bar{z}_2 + 2\mu_\xi$, $x_l = 0$, $y_j^e = 0$, and $\bar{y}_j^e = 0$.

Step 2. If $\mathrm{SSA}_l(\bar{z}_2) < \mathrm{minVal}$, then set $\mathrm{minVal} = \mathrm{SSA}_l(\bar{z}_2)$ and go to Step 1. Otherwise, set $S_l = x_l$ and $R^* = R$.

3.3 Comparisons of Heuristics

We implemented the algorithm of Section 2.9.2, HP1 and HP2 with Java in order to obtain the operational parameters and the expected cost-to-go function for each of the experiments of Section 2.9.3. Different implementations of the algorithm of Section 2.9.2 and HP1 are required for $\tau = 2$ and $\tau > 2$ because the bounds and properties presented in Section 2.8, which are used to speed up the algorithm of Section 2.9.2, are different for each case.

To compare each heuristic against the optimal solution, we use the following criteria: implementation difficulty, speed, and accuracy.

3.3.1 Implementation Difficulty

Since both the optimal solution and HP1 use the same Java code, there is no difference between them with regard to implementation difficulty. The implementation of HP2 algorithm requires less coding.

3.3.2 Speed

We executed all programs on a computer with two 2.4GHz Xeon processors and 2GB RAM running under the Linux operating system (Vanilla Linux kernel, version 2.4.20). Tables 5 and 6 display the minimum, maximum and average time it took to solve each of the experiments described in Table 4.

Table 5: Maximum, Minimum, and Average Time Required for the Experimental Design in Table 4 when $\tau = 2$

	Max time (min)	Min time (min)	Average Time (min)
Optimal Solution	3.51	0.40	1.53
HP1	0.26	0.03	0.10
HP2	0.44	0.001	0.06

Table 6: Maximum, Minimum, and Average Time Required for the Experimental Design in Table 4 when $\tau = 3, 4$

	Max time (min)	Min time (min)	Average Time (min)
Optimal Solution	3803.95	4.04	295.09
HP1	72.49	0.66	17.98

3.3.3 Accuracy

For each of the experiments of Section 2.9.3, we searched for the largest difference between the expected total costs produced by the heuristics and the optimal policy using an initial inventory in the range $[-40, 40]$, that is $\max_{x\in[-40,40]}\{C_{\mathrm{HP}}(x, 0, 0) - C(x, 0, 0)\}$, where $C_{\mathrm{HP}}(x, 0, 0)$ is the expected total cost function under a heuristic policy.

Tables 7 and 8 display the maximum, minimum, and average largest difference for these experiments. The tables also show an estimated percent difference histogram. For example, in 75.3% of the experiments, the largest difference between HP1 and the optimal solution was less than 2%. Although HP2 is, on average, 40 % faster than HP1 (Table 5), the latest heuristic has a smaller cost difference with the optimal expected total cost.

Table 7: Maximum, Minimum, Average and Histogram for the Largest Differences Between the Heuristics and the Optimal Policy when $\tau = 2$

	Max	Min	Average	2%	5%	10%	15%	20%
HP1	13.4%	0%	1.5%	75.3%	97.1%	99.8%	100%	100%
HP2	25.4%	0%	3.1%	41.9%	82.7%	96.4%	99.2%	99.8%

Table 8: Maximum, Minimum, Average and Histogram for the Largest Differences Between the Heuristics and the Optimal Policy when $\tau = 3, 4$

	Max	Min	Average	2%	5%	10%	15%	20%
HP1	11.0%	0.002%	3.3%	27.6%	79.3%	98.5%	100%	100%

3.4 Concluding Remarks

In this chapter we presented two heuristic inventory policies and compared them against the optimal policy described in Chapter II in terms of implementation difficulty, speed and accuracy for the experimental design of 3888 cases listed in Table 4. Based on this substantial experimentation, both heuristics provided a significant reduction in computational time without adding substantial errors in the total expected costs when the regular lead-time is two periods.

The next chapter presents an inventory simulator suite that will allow the estimation of performance measures which are hard to obtain by analytical means.

CHAPTER IV

INVENTORY SYSTEM SIMULATOR

Several simulators have been developed to solve inventory problems in a supply chain. Some simulators have been developed for academic research such as Pope's "Inventory Management Simulation" [44], Bernstein's "Inventory Simulator" [4], Wedel's "Otto's Inventory Simulation" [64], Jacobs' "Supply Chain Inventory System Design Exercise" [32], and Snyder's "BaseStockSim" [58]; see also Przasnyski [48] and Adi Ben-Israel [3]. Since these simulators were developed for educational purposes, they provide limited flexibility to model and test more complex systems involving non-trivial inventory allocation policies or reorder policies of the type depicted in this thesis. Such flexibility can be provided by Object-oriented simulations such as the one as presented in Rossetti, Miman, Varghese and Xiang [50] or simulation libraries such as DSOL [17]. Since these require coding (in both cases in Java), they have limited use by those who can effectively program those languages as opposed to graphical simulations.

Various commercial simulations have also been developed such as "The SIMPLE_1 programming language" [14], the "Financial and Inventory Simulator" [15], "VALOGIX" [61], and the "Supply Chain Guru" [36].

This chapter describes the implementation of a multi-echelon inventory system simulator developed in Java. It proceeds as follows: Section 4.1 contains the User's Guide, Section 4.2 presents test cases to validate various models created with the simulator, and Section 4.3 provides software documentation.

4.1 User's Guide

4.1.1 Introduction

The Inventory Simulation Workbench (ISW) is a Java-based simulation suite that allows a user to develop a network of inventory systems by means of nodes and supply arcs in a graphical environment, define experimental settings, and observe the results of a simulation.

4.1.2 Acknowledgements

ISW is based on:

- The open source Distributed Simulation Object Library (DSOL) developed at the Delft University of Technology, The Netherlands, and available at http://www.simulation.tudelft.nl/ (August 21, 2008) .
- The open source Java Graph Visualization and Layout library, JGraph. Available at http://www.jgraph.com/ (August 21, 2008).

4.1.3 Copyright

- DSOL: GNU Lesser General Public License available at http://www.gnu.org/copyleft/lesser.html (August 21, 2008).
- JGraph: Library General Public License (LGPL) version 2.1 and JGraph License version 1.1, available online at http://www.jgraph.com/license.html (August 21, 2008).
- ISW: GNU Lesser General Public License.

4.1.4 Installation and System Requirements

The distribution of ISW is through a compressed and Java executable jar file. No installation is required.

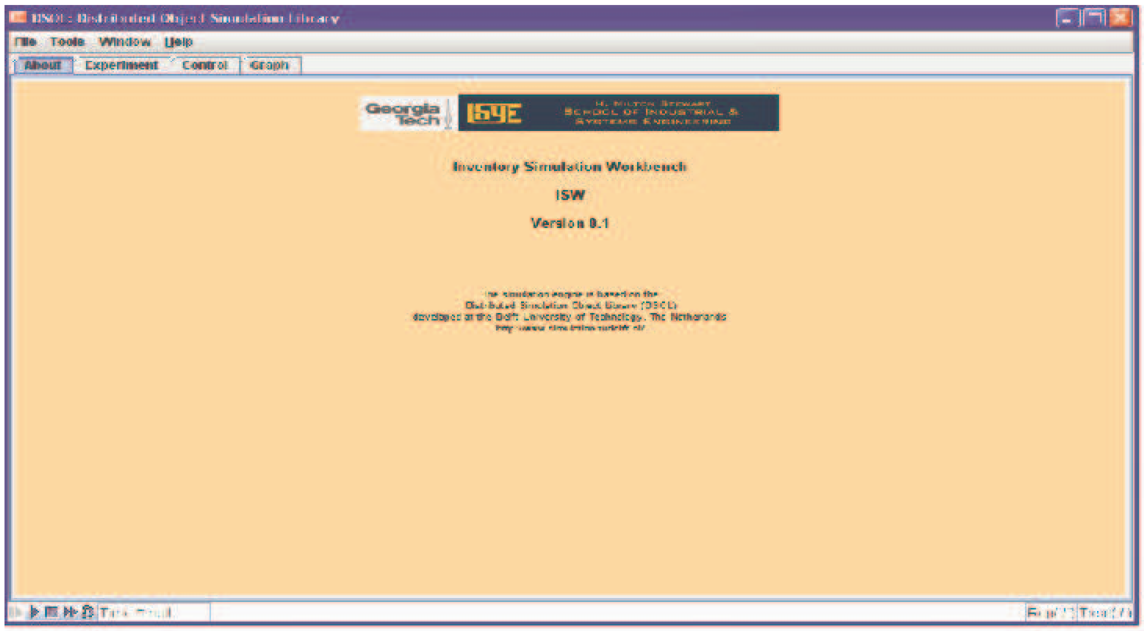
- Operating Systems: Windows, Linux, Unix, and Mac OS X.
- Hardware Requirements: 1GB RAM.
- Software Requirements: Java runtime environment J2SE version 1.5 or higher.

4.1.5 Quick Start

To illustrate the use of ISW, we will explain how to run the tutorial model `rQPolicySim`. Double-click the `inventory.jar` file or run the command `java -jar inventory.jar` in a command prompt window or shell. The main window should open. Open the (r, Q) inventory tutorial model using the menu `Help-Tutorial-rQPolicySim.xml`. As shown in

Figure 5, three tabs should have been created on the main window: `Experiment`, `Control`, and `Graph`.

Figure 5: ISW Main Window



The `Control` tab allows the user to setup simulation parameters such as the number of replications and warmup interval (more details available in Section 4.1.8). The `Graph` tab allows the user to review and modify the inventory network (more details can be found in Section 4.1.7).

The rQPolicySim model represents a single-item, single-echelon, continuous-review inventory system managed with an (r, Q) policy. To view the parameters of each of the nodes in the network, select the `Graph` tab and right-click on any selected node or arc. This will open a dialog allowing the user to view and edit any node parameters (more details about node parameters can be found in Section 4.1.7). To add new nodes (demand, supplier or node manager) or arcs, the user can utilize the buttons on the top of the panel.

When changes to the inventory network and the simulation parameters have been completed, the model may be saved using the `File-Save` menu (or the shortcut CTRL+S). Since the included tutorial files are distributed in a compressed jar file, the simulation model will not create the log nor record statistical results; hence, it is advised to save the model as a local file before running the model.

The user may run the simulation using the buttons at the bottom left corner of the window. The `Run` button will start the simulation, with the current simulation time being displayed at the bottom right corner of the window. Similarly, the simulation can be paused or stopped with the buttons at the bottom left corner of the window. To observe the inventory statistics or graphs, open the Statistics window (use the `Window-Statistics` menu or the key shortcut F2). Once the statistic window is opened, any chart or statistic may be "dragged and dropped" from the statistic panel on the left to the display panel on the right side of the window (more details can be found in Section 4.1.6).

To compare the results against a different model, close the current experiment (use the `File-Close` menu or the key shortcut CTRL+F4) and open or create a new inventory network. Run the simulation and open the Statistics window. Now, both experiment results are available in the left panel in folders named with the date and time that the experiment was run, so any combination of statistics can be 'dragged and dropped" to the display panel for comparison.

4.1.6 The Workbench Menus

The following menu items are available:

- File
  - New File: Creates a new model with no components.
  - Open File: Opens an existing model file (model files have extension .xml).
  - Save File: Saves the existing model. The model's parent directory becomes the experiment directory where the log and statistic files are saved.
  - Open URL: Opens an existing model file from a URL address.
  - Close: Closes the current model.
  - Open Recent: Provides quick access to the most recent successfully opened models.
  - Exit: Exits the ISW.

- Tools
  - Pause Simulator At: Provides a way to pause the simulation at a given time.
- Window
  - Context: Displays current experiment information such as experiment name, date, and time.
  - Statistics: Displays a two-panel window (see Figure 6). The left panel displays the experiments simulated during the current session, providing access to statistics and charts for each experiment (see Section 4.1.9 for the available measures of performance and charts). The right panel displays selected statistics or graphs. To display any performance measure statistic or graph, click and drag the object from the left panel to any of the cells in the right panel. The right panel has buttons to add or delete rows and columns of cells.

    The displayed plots can be zoomed in and out, saved or printed by right-clicking on them.

Figure 6: Statistics WIndow

- Event List: Displays a table with the simulation events currently scheduled (future event list).
- Logging: Displays different types of logs (inventory, simulation, etc). To review the inventory events, open the `gatech.isye.smtet.trace` log. This display and the simulation log are useful debugging tools.
- Memory: Displays the current system memory usage.

• Help

- Tutorial: Provides access to tutorial examples as explained in Section 4.1.12.

4.1.7 The Inventory Model

ISW involves two different models, the Inventory Model and the Simulation Model. The user graphically creates a network of inventory nodes and arcs connecting the nodes using a library of elements. This network is the Inventory Model. When the simulation is started, the Inventory Model is appropriately translated to a Simulation Model using the DSOL library. To expand the available library of inventory elements, the user can define new elements following the procedure described in Section 4.1.13.

The four basic elements used to build the Inventory Model are: `Demand`, `Supplier`, `Node Manager`, and `Supply Arc`. The basic structure of the inventory network is that of a directed tree, where the direction of the arcs represents flow of goods (as opposed to flow of orders). `Orders` are created by a `Demand Node` or by any `Node Manager` that requires replenishment of goods, and are sent to a `Supplier` or to another `Node Manager`, where they can be fulfilled with inventory or production. In order to ensure that all orders are satisfied, it is required that all roots are of type `Demand` and all leafs are of type `Supplier`.

The behavior of the inventory elements is defined by some smaller, and usually simpler, elements. For example, the behavior of a `Demand` node is mainly defined by the following three `Random Variable` elements: the starting time, the interarrival time, and the demand quantity.

To describe the elements of the Inventory Model, we follow a top-bottom approach, describing basic components first and then their subcomponents.

Demand: A `Demand` is a node that generates `Orders` for a single type of item according to a stochastic process. The following parameters define a `Demand`:

(a) Demand Class: This corresponds to the Java class that implements the Demand according to the DSOL paradigm. Default value: `Demand`.

(b) Demand Id: A unique identifier for this node. Default value: `dem`.

(c) Item Type: The type of product that this node will demand. Default value: `prod`.

(d) Quantity Distribution: A `Random Variable` that models the amount of items requested in the order. Default value: `DistributionDiscreteConstant(1)`.

(e) Interval Distribution: A `Random Variable` that models the time between orders. Default value: `DistributionConstant(1)`.

(f) Start-time Distribution: A `Random Variable` that defines the time of the first order. Default value: `DistributionConstant(0)`.

(g) Maximum Number Creations: A constant that defines the maximum number of orders generated by the node. Default value: `java.lang.Long.MAX_VALUE`.

Supplier: A `Supplier` is a node that provides a single type of item. These nodes have unlimited availability of resources, hence they do not require replenishment. The following parameters define a `Supplier`:

(a) Supplier Class: This corresponds to the Java class that implements the Supplier according to the DSOL paradigm. Default Value: `Supplier`.

(b) Supplier Id: A unique identifier for this node. Default value: `supp`.

(c) Item Type: The type of product that this node will supply. Default value: `prod`.

(d) Lead-time: This `Random Variable` parameter defines the lead-time of the `Supplier` due to transportation. If a shipment from this node follows a `Supply Arc` with a specified lead-time, the arc delay prevails over the node value. Default value: `DistributionConstant(0)`.

Node Manager: A `Node Manager` node models a production, assembly or inventory warehouse organization. The following parameters define a `Node Manager`:

(a) Node Class: This corresponds to the Java class that implements the `Node Manager` according to the DSOL paradigm. Two different classes are provided in the ISW database:

- NodeManager: This class models an organization that behaves independently of the rest of the network.
- CentralizedNodeManager: This class models an organization that is an element of a multi-echelon inventory management. Henceforth, its behavior (and its components behavior as well) may depend on other nodes.

Default value: `NodeManager`.

(b) Node Id: A unique identifier for this node. Default value: `man`.

(c) Transport Mode: This `Transport` parameter models transportation and its related costs. The `Transport Mode` element provides the flexibility to model different transport delays at different costs. If a shipment from this node follows a `Supply Arc` with a specified lead-time, the arc delay prevails over the `Transport` value. Please see `Transport` for information about available modes. Default value: `SingleModeTransport`.

(d) Inventories: A `Node Manager` node may have one or more `Inventories`. Each element of type `Inventory` models the storage, replenishment policy and accounting of a single item inventory. Please see `Inventory` for information about available inventories and their management. Default value: Single `PeriodicReviewInventory` element.

Supply Arc: A `Supply Arc` represents the flow of a single item. Different arcs should connect demands and suppliers for different item types. The following parameters define a `Supply Arc`:

- Demand Node Id: The `Node Id` of the destination node.

- Supply Node Id: The `Node Id` of the origin node.
- Item Type: The type of product that moves through this arc. Default value: `prod`.
- Lead-time: This `Random Variable` parameter models the transportation lead-time of the items moving through this arc. Default Value: 0.

Inventory: An `Inventory` models the storage, replenishment management, and accounting of a single item. We first describe the `Inventory` functionalities and then its parameters.

The following functionalities are implemented either by the inventory element or by its parameters:

- Inventory Reviewing: Timing for inventory-position review and resupply decision. This functionality is implemented by the `Inventory Class`.
- Inventory Production: This functionality defines how the inventory obtains the stored items and is implemented by the `Inventory Production/Warehousing` element.
- Reorder Decision: This functionality defines whether to place an `Order` to a supplier. The `Order` includes the order quantity and the type of order (see the `Order` element for more details). This functionality is implemented by the `Inventory Policy` element.
- Costing: This functionality keeps inventory level statistics and computes the inventory costs and revenues. It is implemented by the `Cost Function` element.
- Allocation: This functionality describes the allocation of inventory to pending orders in a `Distribution Warehouse` inventory. It is implemented by the `Allocation Policy` element and works only on the `Distribution Warehouse` inventory type.
- Event Prioritization: This functionality defines the sequence in which the

inventory-related events are executed on every period (highest numbered priority has precedence). The following events may be prioritized:

  - Replenishment.
  - Demand.
  - Review.
  - Costing.

For obvious reasons, this functionality is important mainly in `PeriodicReviewInventory` systems.

The following parameters define the inventory behavior:

(a) Inventory Class: This corresponds to the Java class that implements the inventory review functionality.

   The following classes are provided in the ISW database:

   - PeriodicReviewInventory: Models an inventory with periodic review.
   - ContinuousReviewInventory: Models an inventory with continuous review.
   - DistributionWarehouse: Models an inventory with periodic review and allocation policy defined by an `AllocationPolicy` element. The allocation event for this type of inventory has a low priority (priority 1, see `Event Execution Priority` for details).
   - NeverReviewInventory: Implements an inventory that is never reviewed.
   - UpstreamSynchronizedReviewInventory: Models an inventory whose review epochs happen after the upstream inventory has completed its review.

   Default value: `PeriodicReviewInventory`.

(b) Item Type: The type of product that this inventory will hold. Default value: `prod`.

(c) Backorder: This is a `Boolean` field that defines whether the node will backlog orders. Default value: `true`.

(d) Partial Shipment: This is a `Boolean` field that defines whether the inventory will deliver partially satisfied orders. Default value: `true`.

(e) Inventory Policy: Depending on the state of the inventory (some complex cases may include inventory levels, upstream inventory levels, and current time), this element defines whether to place an `Order` to a supplier. The `Order` includes order quantity and type (see the `Order` element for more details). The following policies are implemented in the database:

- BaseStock: Implements a base-stock inventory policy defined by a single parameter, the base-stock level. That is, when the inventory position is less than the base-stock level, it will reorder enough items to raise the inventory position up to the base-stock level. For details, see Hopp and Spearman [30, p. 69].
- BaseStock_Batch: Implements a batch ordering policy. This policy has a single parameter, the reorder point as defined in Cachon [10]. That is, when the inventory position is less than or equal to the reorder point, it will reorder enough quantity (in batches) to raise the inventory position above the reorder point.
- BaseStock_sSPolicy: In a periodic-review inventory with two supply modes (regular and emergency), this policy implements a base-stock policy for the regular mode and an (s, S) policy for the emergency mode. The emergency mode is available at all time periods while the regular mode is available at a fixed frequency. This policy place `TwoModesSingleItemOrder` orders.
- CentralizedSerialBaseStock: Implements a base-stock policy in a serial supply chain. For details, see Shang and Song [54].
- NeverOrderPolicy: Implements a policy that will never place an order.
- rQPolicy: Implements an (r, Q) policy. That is, when the inventory position is less than r, it will place an `Order` for Q items. For details, see Zheng [65].
- sSPolicy: Implements an (s, S) policy. That is, when the inventory position

is less than s, it will place an `Order` to raise the inventory position up to S. For details, see Porteus [45, p. 103].

Default value: `NeverOrderPolicy`.

(f) Inventory Production/Warehouse: This element models the production department or warehouse that supplies the items of the inventory. Two classes are available in the database:

- Warehouse: This is a warehouse with no production capability.
- BTOProductionDepartment: This is a model of a build-to-order (assembly) production facility.

(g) Cost Function: This element keeps inventory level statistics and computes the corresponding inventory costs. The following costs are accounted for: holding, backorder, on-transit, and purchase (includes acquisition and transport costs charged by the Transport element).

Two classes are available in the database:

- ContinuousTimeCostFunction: This object computes the average inventory level as a time-average statistic. Cost statistics are computed based on a linear holding rate, backorder rate, and on-transit rate.
- DiscreteTimeCostFunction: This object performs the same computations as the BasicInvCostFunction, but instead of time-averages, it uses the average of periodic samples to compute the inventory costs. This type of cost function should be used for all periodic-review inventories.

Default value: `ContinuousTimeCostFunction`.

(h) Allocation Policy: In a `DistributionWarehouse` inventory, this element implements the allocation of available inventory to all pending orders. The `Allocation Policy` will only work well in the `DistributionWarehouse` inventory type, since this type of inventory will delay the allocation decision until all orders have been received. Two classes are available in the database:

- FIFOAllocation: This class allocates the available inventory to pending orders following a First-In-First-Out (FIFO) order.
- RandomAllocation: This class allocates the available inventory to pending orders in a random fashion (with equal probabilities).
- RelativeNeedAllocation: If there is not enough inventory at-hand to satisfy all pending orders, this class will allocate sequentially each item available to the requesting `Node Manager` with the largest number of pending orders. If the inventory has enough inventory at-hand, this class will allocate the items following a FIFO order.

Default value: `FIFOAllocation`.

(i) Event Execution Priorities: This is a set of priorities (numbers) that defines the sequence in which some inventory-related events (demand, review, replenishment, and costing) are executed on every period. The event with highest priority is given preference. The range of priority values is $\{0, 1, \ldots, 10\}$ and the following sets are available in the database:

- Default: With this set of priorities all four events have the same priority (value 5). Hence, events will be executed in the order that they were scheduled by the model.
- Cachon: This set of priorities defines the sequence demand-review-replenishment-costing as defined in Cachon [10].
- Lystadt: This set of priorities defines the sequence replenishment-demand-review-costing as defined in Lystadt and Ferguson [37].
- Scarf: This set of priorities defines the sequence review-replenishment-demand-costing as defined in Scarf [52].
- Veinott: This set of priorities defines the sequence review-demand-replenishment-costing as described in Veinott and Wagner [62].

Random Variable: A Random Variable provides a stream of pseudo-random samples according to a specified distribution. The following real-valued random variables are included in the database:

- DistributionBeta(α_1, α_2): Implements the Beta random variable with expected value $\alpha_1/(\alpha_1 + \alpha_2)$.
- DistributionConstant(c): Implements a constant random variable with value c.
- DistributionErlang(k, β): Implements an Erlang random variable of order k and expected value $k\beta$.
- DistributionExponential(μ): Implements an exponential random variable with mean μ.
- DistributionGamma(α, β): Implements a gamma random variable with shape parameter $\alpha > 0$ and expected value $\alpha\beta$.
- DistributionLogNormal(μ, σ): Implements a lognormal random variable with mean $e^{\mu+\sigma^2/2}$ and variance $e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.
- DistributionNormal(μ, σ): Implements a normal random variable with mean μ and standard deviation σ.
- DistributionPearson5(α, β): Implements a Pearson type 5 distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$.
- DistributionPearson6($\alpha_1, \alpha_2, \beta$): Implements a Pearson type 6 distribution with shape parameters $\alpha_1 > 0$, $\alpha_2 > 0$ and scale parameter $\beta > 0$.
- DistributionTriangular(a, b, c): Implements a triangular random variable with minimum value a, mode b, and maximum value c.
- DistributionUniform(a, b): Implements a uniform random variable with minimum value a and maximum value b.
- DistributionWeibull(α, β): Implements a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$.

The following integer-valued random variables are included in the database:

- DistributionBernoulli(p): Implements a Bernoulli random variable with mean p.

- DistributionDiscreteConstant: Implements a discrete random variable with value c.
- DistributionGeometric(p): Implements a geometric random variable with parameter $p \in (0, 1)$ and probability mass function $\Pr(X = k) = p(1 - p)^k$ for $k \in \{0, 1, 2, \ldots\}$.
- DistributionBinomial(n, p): Implements a Binomial distribution with n trials an probability of success p.
- DistributionDiscreteUniform(a, b): Implements a random variable with equal probability for the integer numbers in the set $\{a, \ldots, b\}$.
- DistributionNegBinomial(r, p): Implements a negative binomial random variable with parameters $r > 0$ and $p \in (0, 1)$, and probability mass function

$$\Pr(X = k) = \frac{\Gamma(k + r)}{k!\Gamma(r)} p^r (1 - p)^k$$

- DistributionPoisson(λ): Implements a Poisson random variable with mean λ.

Also included are the DistReader. This object read samples from a user-defined file.

Transport: This element models the transportation of goods between nodes. Two transportation classes are provided in the database:

- SingleModeTransport: This class models the transportation of any order simulating its lead-time and its costs.
- Reg_EmergModeTransport: This class models the behavior of two transportation modes for the same node, a regular mode and an emergency mode. With this type of transportation, goods required by a `TwoModesSingleItemOrder` order will be transported by the regular mode or by the emergency mode, as specified in the order. On the other hand, goods required by a `SingleItemOrder` will be transported using the regular mode. For further details about orders, see below.

Order: The user does not have to specify any parameter for an `Order` but a short description is included for completeness. This element specifies a supply request including the type and amount of items required. Two classes are used in order to specify the urgency (by requesting different types of transport) of the `Order`:

- SingleItemOrder: This `Order` is the basic type of order and will be delivered using the regular mode of transportation.
- TwoModesSingleItemOrder: This `Order` requires the use of two modes of transportation, regular and emergency, to deliver the goods (see `Reg_EmergModeTransport` for further details), hence it specifies the amount to be delivered by regular means and the amount to be delivered by emergency means.

4.1.8 The Simulation Model

The user interacts with the `Simulation Model` by defining the following treatment parameters in the `Control` tab:

- Number of Replications.
- Warmup Time: Time to start the computation of statistics.
- Run Length: Length of a replication. `INF` sets this value to infinity.
- Time Units: Defines the units of time to be used in the simulation. The options are `MILLISECOND, SECOND, MINUTE, HOUR, DAY, WEEK, YEAR`, and `UNIT` (generic).
- Record Log: Defines whether to record a log of events. If set to `true`, the simulation will create a subdirectory named `log` in the same directory where the ISW file is saved and will write in this folder the log of every replication run. Default value: `true`.
- Record Sample Path: Defines whether to record the sample paths of various processes related to the measures of performance under consideration. If set to `true`, the simulation will create a subdirectory named `stat` in the same directory where the ISW file is saved, and will record in this folder the sample path of every statistic in a file with the same name as the statistic (see Section 4.1.9). Default value: `false`.

4.1.9 Measures of Performance

For each inventory in the network, the following statistics are maintained and can be displayed as explained in Section 4.1.6. Statistics are updated during the data collection period that starts at the `warmup time` and ends with the replication (`run length`).

- Inventory at Hand: Number of items in stock.
- Backorder Level: Number of backordered items.
- Inventory on Transit: Number of items in transit.
- Service Level: Fraction of fully satisfied orders.
- Time Between Replenishment(s): Time between order placements.
- Total Purchase Cost: Total cost incurred in orders to suppliers.
- Total Cost: At the end of each replication, the total cost is computed as the sum of the total purchase, total inventory holding, total backorder penalty, and total in-transit costs.
- Cost Per Period: At the end of each replication the cost per period is computed as the total cost divided by (total) number of periods.
- Total Revenue: Total income from orders received.

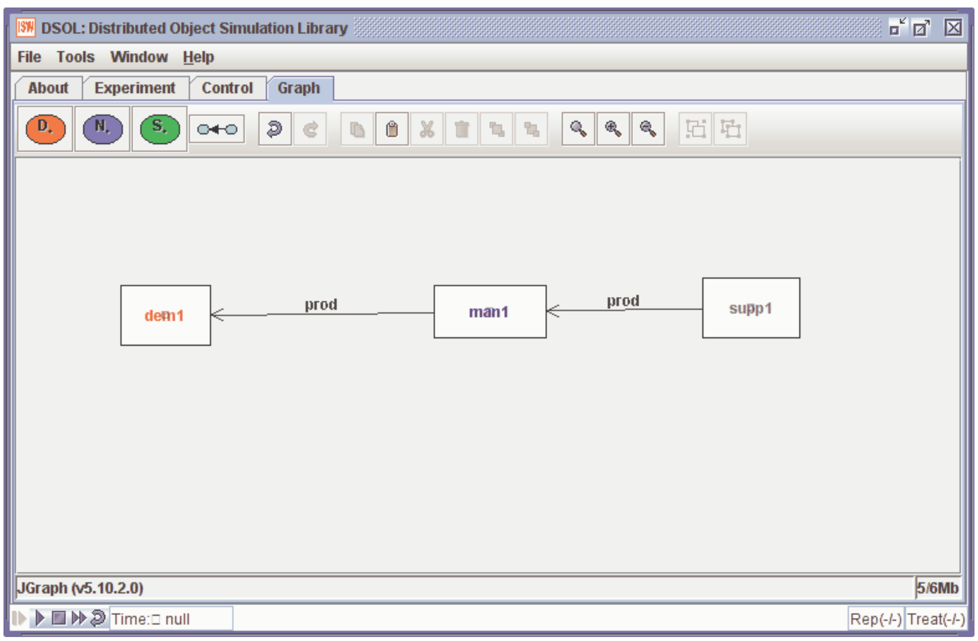
If the number of replications is 5 or less, ISW will create a time graph of the inventory at hand and the backorder level for each inventory in the network. When more than 5 replications are selected, no graphs will be created in order to conserve memory.

4.1.10 Building a Model

To build a model, two approaches can be followed: start from scratch (use the `File-New` menu or press CTRL+N) or modify an existing model (use the `File-Open` menu or press CTRL+O) and later save the model (use the `File-Save` menu or CTRL+S). In either case, a new `Graph` tab will be created where the user can add, remove or modify the nodes and arcs of the inventory network with the menu buttons available on the tab. Similarly, a new `Control` tab will be created where the user can change the simulation parameters.

To understand the details of building a simulation model in ISW, we describe the steps to build and setup the parameters for the `sSPolicySim.xml` tutorial available from the `Help` menu. This tutorial example models a single-item, single-echelon inventory network. The inventory is reviewed periodically and an (s, S) policy (`sSPolicy` element) is applied. Figure 7 displays the network. We explain how to build this network by stating the goals and the required commands.

Figure 7: (s, S) Inventory Network



Start ISW: Double-click the file inventory.jar or run the command `java -jar inventory.jar` in a command window or shell.

Create a new model: Type CTRL+N or use the menu `File-New`. Select `Graph` tab.

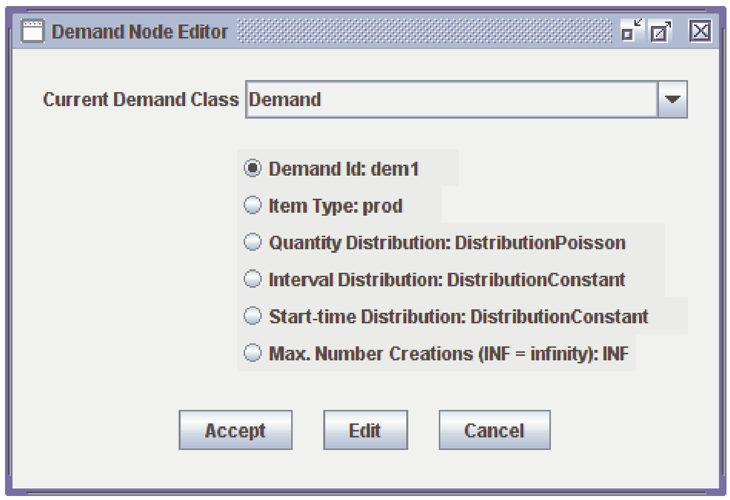
Create a Demand: Click on `Insert Demand Node` button on the toolbar. A demand node named `dem[0]` is created. Later we will change the node identification to `dem1`.

Relocate a Node: Using the arrow cursor on top of the `dem[0]` node, click and drag the node to the desired location.

Set Demand Parameters: Right-click on top of `dem[0]`. A `Demand Node Editor` dialog will open.

Change Demand Id: Select the `Demand Id` radio button and click the `Edit` button. The `Identification Editing` dialog will open. Type the new node name, `dem1`, and select `Accept`. If the `Cancel` button is selected, no changes are made. See Figure 8.

Figure 8: Demand Node Editor

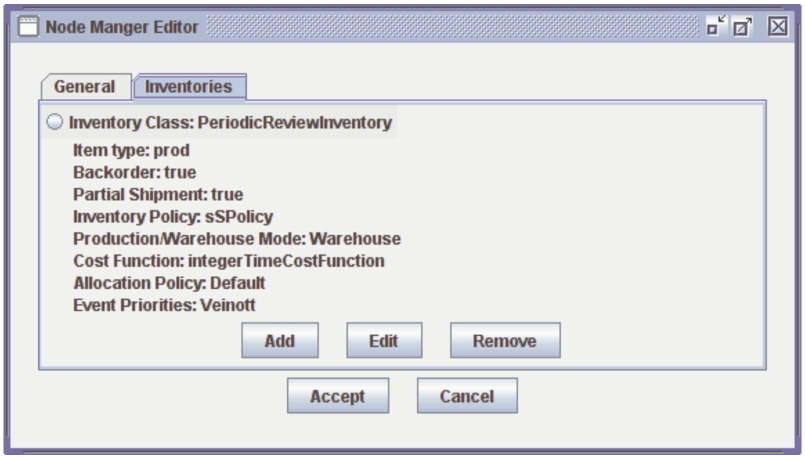


Change Quantity Distribution: Select the `Quantity Distribution` button and then click `Edit`. A `Random Variable` editing dialog will open. In the combo box search for `DistributionPoisson`. Two input boxes will be created, one for the λ parameter (mean), and one for the stream number. For the λ parameter enter 21 and `Accept` the changes to the `Random Variable` and the `Demand Node Editor` dialogs.

Create Node Manager: Click on `Insert Node Manager` button on the toolbar. A Node Manager named `man[0]` is created. Later we will change the node identification to `man1`.

Set Node Parameters: Right-click on top of `man[0]`. A `Node Manager Editor` dialog will open. See Figure 9. Change the `Node Id` to `man1` following the same procedure as explained for the `Demand` node.

Figure 9: Node Manager Editor



Modify Inventory Parameters: Select the `Inventories` tab in the `Node Manager Editor` dialog. Select the inventory by clicking on the radio button and then click the `Edit` button. An `Inventory Editing` dialog will open.

Change the Inventory Policy: Select the Inventory policy and click `Edit`. An `Inventory Reorder Policy Selection` dialog will open. To change the parameters of the current policy we must first change the policy. In the combo box select `sSPolicy`. Two input boxes will be added, one for the reorder point s and another one for the order-up-to value S. Set $s = 15$ and $S = 65$. `Accept` the changes in the `Inventory Reorder Policy Selection` dialog.

Change the Initial Inventory: Select the `Production/Warehouse Mode` button and click `Edit`. An `Inventory Production Mode Selection` dialog will be displayed. In the combo box select `Warehouse`. One input box will be created for the initial inventory. Type 65 for the initial inventory and accept the changes in the `Inventory Production Mode Selection` dialog.

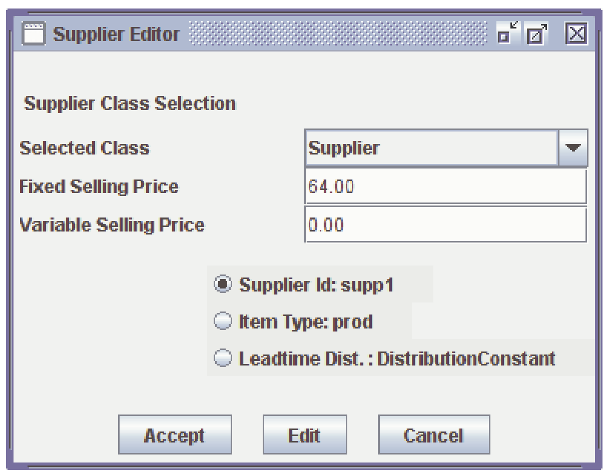
Define the Cost Function: Select the `Cost Function` button and click `Edit`. A `Cost Function Selection` dialog will be displayed. In the combo box select `DiscreteTimeCostFunction`. Type 1 for the holding rate and 9 for the bakorder rate. Accept the changes in the `Cost Function` selection dialog.

Modify Event Priorities: Select the `Event Priorities` button and click `Edit`. In the combo box select the set `Veinott` and `Accept` the changes. Also `Accept` the changes in the `Inventory Editing` dialog. `Accept` the changes in the `Node Manager Editor` dialog.

Create Supplier Node: Click on `Insert Supplier Node` button on the toolbar. A Supplier Node named `supp[0]` is created. Relocate the node as desired and rename it as `supp1`.

Set Supplier Parameters Right-click on top of `supp[0]`. A `Supplier Editor` dialog will open; see Figure 10. Rename the node to supp1 and type in the `Fixed Selling Price` box the value 64. `Accept` the changes.

Figure 10: Supplier Editor



Create Supply Arc: Move the cursor on top of the `man1` node until a hand cursor appears. Click and drag towards the dem1 node. Drop the button on top of the dem1 node. This will create a supply arc from `man1` to `dem1`. To edit the arc parameters, select the arc and right-click. In the same manner, create a supply arc from `supp1` to `man1`.

Change Simulation Parameters: Click on the `Control` tab and edit the simulation parameters.

4.1.11 Running the Model and Viewing Results

When the Inventory Network is ready, the user can run, pause or stop the simulation using the buttons available for this purpose in the bottom left corner of the ISW window. If `Record Log` was selected, a log (text) file will be created in the directory containing the ISW file.

4.1.12 Tutorial Examples

The ISW examples are accessed in the `Help` menu. The following examples are provided:

(a) EOQPolicySim.xml: This is the model of a single-echelon, continuous-review, deterministic inventory system that uses an `rQPolicy` policy.

(b) baseStockPolicySim.xml: This is the model of a single-echelon, continuous-review inventory system that uses a `Base_Stock` policy.

(c) rQPolicySim.xml: This is the model of a single-echelon, continuous-review inventory system that uses an `rQPolicy` policy.

(d) sSPolicySim.xml: This is the model of a single-echelon, periodic-review inventory system that uses a `sSPolicy` policy.

(e) cachonPolicySim.xml: This is the model of a two-echelon, periodic-review inventory system that uses a random allocation policy as explained in Cachon [10].

(f) serialSupplyChainSim.xml: This is the model of a four-echelon, periodic-review inventory system that uses a `CentralizedSerialBaseStock` policy.

4.1.13 Expanding the Workbench

The user may expand any of the elements defined in Section 4.1.7, Inventory Model. To achieve this, the user must code the new Java class implementing the following interfaces:

- Demand: To code a `Demand` element, implement the interface `gatech.isye.sim.isw.model.manager.DemandInterface`.

- Supplier: To code a `Supplier` element, implement the interface `gatech.isye.sim.isw.model.manager.SupplierInterface`.

- Node Manager: To code a `Node Manager` element, implement the interface `gatech.isye.sim.isw.model.manager.NodeManagerInterface`.

- Inventory: To code an `Inventory` element, implement the interface `gatech.isye.sim.isw.model.inventory.SingleItemInventoryInterface`. To reduce the amount of required work, it is convenient to extend the abstract class `gatech.isye.sim.isw.model.inventory.BasicSingleItemInventory`.

- Inventory Policy: To code an `Inventory` element, implement the interface `gatech.isye.sim.isw.model.policies.reorder.InventoryPolicyInterface`.

- Inventory Production/Warehouse Mode: To code a `Production/Warehouse` element, implement the interface `gatech.isye.sim.isw.model.production.ProductionInterface`.

- Cost Function: To code a `Cost Function` element, implement the interface `gatech.isye.sim.isw.model.cost.InventoryCostFunctionInterface`.

- Allocation Policy: To code an `Allocation Policy` element, implement the interface `gatech.isye.sim.isw.model.policies.allocation.AllocationPolicyInterface`.

- Event Execution Priorities: No coding is necessary to implement new priority rules. Just add the new set of priorities to the database eventprioritiesDB.xml.

- Random Variable: To code a `Random Variable` element, implement the interface `nl.tudelft.simulation.jstats.distributions.DistContinuous` or `nl.tudelft.simulation.jstats.distributions.DistIntegerValued` depending on the type of distribution.

- Transport: To code a `Transport` element, implement the interface `gatech.isye.sim.isw.model.transport.TransportInterface`

4.2 Validation

In this section we validate various models created with ISW by computing 95% confidence intervals (CIs) based on independent replications and comparing these intervals against the true mean.

4.2.1 Single Echelon Warehouse Policies

Table 9 shows the simulation parameters used to validate the simulator for a single-echelon, single-item inventory system controlled with an EOQ or with an (s, S) policy. Table 10 displays the reorder point, average inventory, and total average cost obtained by the simulation of the EOQ policy compared to those presented in Hopp and Spearman [30, p. 51]. Table 11 displays the confidence intervals (CIs) for the expected cost per period under the (s, S) policy obtained by the ISW simulation compared to the expected values presented in Veinott and Wagner [62].

Table 9: Parameters for the EOQ Inventory and (s, S) Policy Simulations

		EOQ Policy	(s, S) Policy
Demand	Quantity	2	Poisson(λ)
	Interval	1	1
Node	Inventory Class	Continuous Review	Periodic Review
	Initial Inventory	101	S
	Inventory Holding Cost	0.05	1
	Backorder Penalty	0	9
	Inventory Policy	rQPolicy(0, 100)	sSPolicy(s, S)
Supplier	Priorities Set	default	Veinott
	Variable Cost	1	0
	Setup Cost	50	64
	Lead-time	0	0
Simulation Parameters	Replication Length	200 days	1500 days
	Warm-up Period	0	0
	Replications	N/A	100

Table 12 shows the parameters used to validate the ISW simulator for a single-echelon, single-item, inventory system controlled with a base-stock policy or with an (r, Q) policy. Table 13 displays the CIs for the expected inventory at hand and backorder level obtained by the ISW simulation compared to the expected values presented in Hopp and Spearman

Table 10: EOQ Inventory Simulation Results

	Reference Results	Simulation Results
Reorder Period	50	50
Average Inventory	50	50
Total Average Cost	5.5	5.5

Table 11: (s, S) Policy Simulation Results

λ	(s, S)	Reference Results Expected Cost per Period	Simulation Results 95% CI for Expected Cost per Period
21	(16,65)	50.41	[50.35, 50.47]
59	(51,126)	76.68	[76.57, 76.76]

[30, p. 72]. Table 14 displays the confidence intervals for the expected cost per period for two values of setup cost, K, compared to the expected values presented in Zheng [65]. In each case, the CI contains the true value.

Table 12: Parameters for the Base-Stock and (r, Q) Policy Simulations

		Base Stock Policy	(r, Q) Policy
Demand	Quantity	1	1
	Interval	Exponential(0.1)	Exponential(0.02)
Node	Inventory Class	Continuous Review	Continuous Review
	Initial Inventory	R	r
	Inventory Holding Cost	15	10
	Backorder Penalty	25	25
	Inventory Policy	Base_Stock(R)	rQPolicy(r, Q)
Supplier	Priorities Set	Default	default
	Variable Cost	0	0
	Setup Cost	0	K
	Lead-time	1	0
Simulation Parameters	Replication Length	100 days	50 days
	Warm-up Period	10	5
	Replications	100	100

Table 13: Results for the Base-Stock Policy Simulation

R		Expected Inventory at Hand	Expected Backorder Level
5	Reference Results	0.043	5.043
	95% CI	[0.04, 0.05]	[4.99, 5.11]
15	Reference Results	5.1	0.103
	95% CI	[5.03, 5.15]	[0.09, 0.11]

Table 14: Results for the (r, Q) Policy Simulation

K	(r, Q)	Reference Results Expected Cost per Period	Simulation Results 95% C.I. for Expected Cost per Period
1	(50, 7)	95.46	[93.47, 96.36]
100	(38, 40)	289.37	[288.02, 289.95]

4.2.2 Multi-echelon Policies

4.2.2.1 Serial Supply Chain

Table 15 shows the parameters used to validate the simulator for a four-echelon, single-item inventory system controlled with an echelon base-stock policy. The inventory network, shown in Figure 11, consists of a demand node that is supplied by a chain of four retailers. The last node, retailer 4, is supplied by an infinite capacity supplier. Table 16 displays the 95 % CIs for the expected inventory at hand and the backorder level and the expected values computed with the formulas provided in Gallego and Zipkin [26, Section 2.4] and Graves [27]. Again, the narrow CIs contain the true expected values.

4.2.2.2 Distribution Policy

To validate such a policy, we use a three-echelon inventory system depicted in Figure 12. The first echelon has four independent demand nodes that place orders to a second echelon of four retailers, each using a base-stock batch policy, (R_r, Q_r). In the third echelon, a warehouse distributes items to the retailers. A single supplier, with infinite capacity, supplies the warehouse. The warehouse places orders following a base-stock batch policy with parameters R_w and Q_w. Table 17 shows the parameters used to validate this system. Table 18 displays the confidence intervals for the expected inventory at hand and backorder

Figure 11: Serial Supply Chain Network

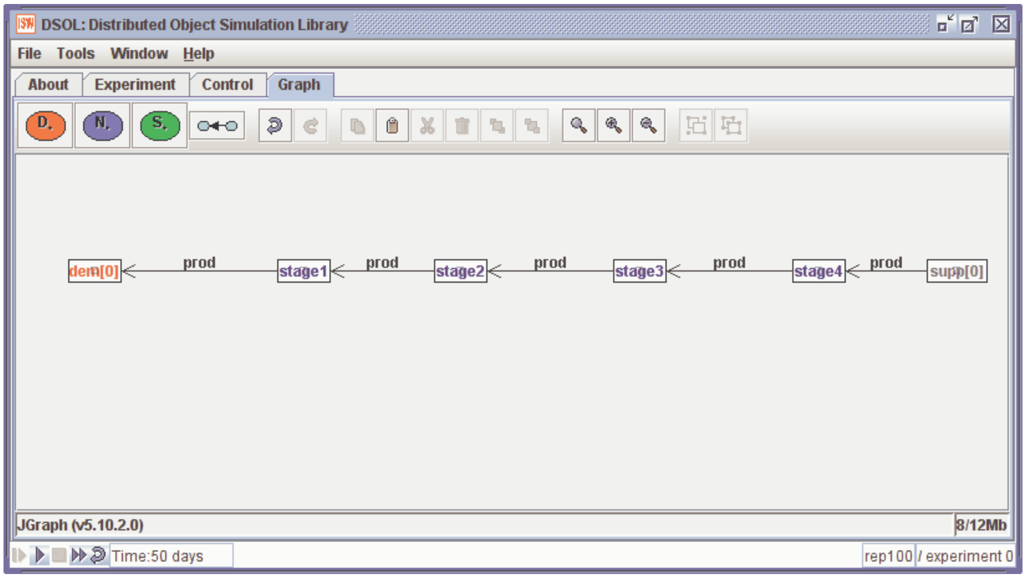


Table 15: Parameters for the Serial Supply Chain Simulation

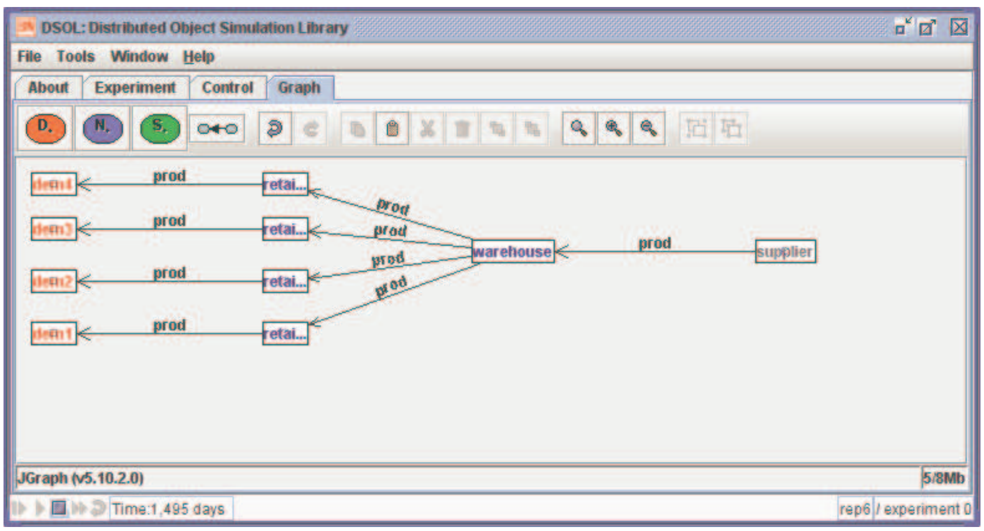
Demand	Quantity	1
	Interval	Exponential(0.0625)
Retailer 1	Inventory Class	Continuous Review
	Initial Inventory	8
	Inventory Holding Cost Rate	0.25
	Backorder Penalty	9
	Inventory Policy	CentralizedSerialBaseStock(8)
	Stage Lead-time	0
Retailer 2 Retailer 3 Retailer 4	Inventory Class	Synchronized Review
	Initial Inventory	5
	Inventory Holding Cost Rate	0.25
	Backorder Penalty	0
	Inventory Policy	CentralizedSerialBaseStock(13)
		CentralizedSerialBaseStock(18)
		CentralizedSerialBaseStock(22)
	Stage Lead-time	0.25
Supplier	Variable Cost	0
	Setup Cost	0
	Lead-time	0.25
Simulation Parameters	Replication Length	50 days
	Warm-up Period	10
	Number of Replications	100

Table 16: Results for the Serial Supply Chain Simulation

		Retailer 1	Retailer 2	Retailer 3	Retailer 4
Expected Inventory at Hand	Reference Results	3.309	1.066	1.059	0.781
	95% CI	[3.26, 3.35]	[1.04, 1.09]	[1.03, 1.08]	[0.77, 0.80]
Expected Backorder Level	Reference Results	0.215	0.906	0.840	0.781
	95% CI	[0.19, 0.24]	[0.87, 0.94]	[0.82, 0.87]	[0.77, 0.81]

level, and the expected values presented in Cachon [10].

Figure 12: Distribution Chain Network



4.2.3 Validation Conclusion

In all cases, the 95% CIs based on 100 independent replications contained the true mean. This creates a strong supporting argument for the validity of the simulation models built with ISW. Of course, the validity of models based on enhancements of the workbench will depend on the fidelity of the new classes and modules.

Table 17: Parameters for the Distribution Policy Simulation

Demand 1 Demand 2 Demand 3 Demand 4	Quantity	Poisson(0.1)
	Interval	1
Retailer 1 Retailer 2 Retailer 3 Retailer 4	Inventory Class	Periodic Review
	Initial Inventory	0
	Inventory Holding Cost Rate	1
	Backorder Penalty	20
	Inventory Policy	BaseStock_Batch(R_r, Q_r)
	Stage Lead-time	0
Warehouse	Inventory Class	Distribution Warehouse
	Initial Inventory	0
	Inventory Holding Cost Rate	1
	Backorder Penalty	0
	Inventory Policy	BaseStock_Batch(R_w, Q_w)
	Stage Lead-time	1
Supplier	Variable Cost	0
	Setup Cost	0
	Lead-time	1
Simulation Parameters	Replication Length	1500 days
	Warm-up Period	100
	Number of Replications	100

Table 18: Results for the Distribution Policy Simulation

(R_w, Q_w, R_r, Q_r)			Warehouse	Retailers
(0, 1, 0, 1)	Expected Inventory at Hand	Reference	0.45	3.09
		95% CI	[0.45, 0.46]	[3.01, 3.19]
	Expected Backorder Level	Reference	0.25	0.13
		95% CI	[0.25, 0.26]	[0.13, 0.13]
(0, 4, 0, 1)	Expected Inventory at Hand	Reference	1.78	3.21
		95% CI	[1.77, 1.79]	[3.20, 3.27]
	Expected Backorder Level	Reference	0.08	0.09
		95% CI	[0.08, 0.09]	[0.09, 0.10]
(-1, 1, 0, 4)	Expected Inventory at Hand	Reference	0.00	8.48
		95% CI	[0.00, 0.00]	[8.42, 8.50]
	Expected Backorder Level	Reference	0.80	0.08
		95% CI	[0.80, 0.82]	[0.08, 0.09]

4.3 Documentation

The documentation for ISW is provided in the form of a User Manual, HTML JAVA API documentation, and Class and Activity Diagrams using the Unified Modeling Language standard, UML 2.1. The User Manual and the the HTML files are provided in the distribution "inventory.jar" file.

4.3.1 Class Diagrams

The class diagram shows how the different entities relate to each other; in other words, it shows the static structures of the system. Since the code is written in Java packages grouping similar functionalities, we provide Class Diagrams for each of the relevant ISW packages.

- Order: The `order` package groups the objects that model the `Order` element of ISW. Figure 13 displays the class diagram.

- Production: The `production` package groups the objects that model the `Inventory Production` element of ISW. Figure 14 displays the class diagram.

- Cost: The `cost` package groups the objects that model the `Cost Function` element of the `Inventory`. Figure 15 displays the class diagram.

- Allocation: The `allocation` package groups the objects that model the `Allocation` element of ISW. Figure 16 displays the class diagram.

- Reorder: The `reorder` package groups the objects that model the `Inventory Policy` element of ISW. Figure 17 displays the class diagram.

- Inventory: The `inventory` package groups the objects that model the `Inventory` element of ISW. Figure 18 displays the class diagram without "getter" and "setter" methods.

- Transport: The `transport` package groups the objects that model the `Transport` element of ISW. Figure 19 displays the class diagram.

- Node Manager: The `manager` package groups the objects that model the `Node Manager` element of ISW. Figure 20 displays the class diagram.

Figure 13: Class Diagram for the Order Package

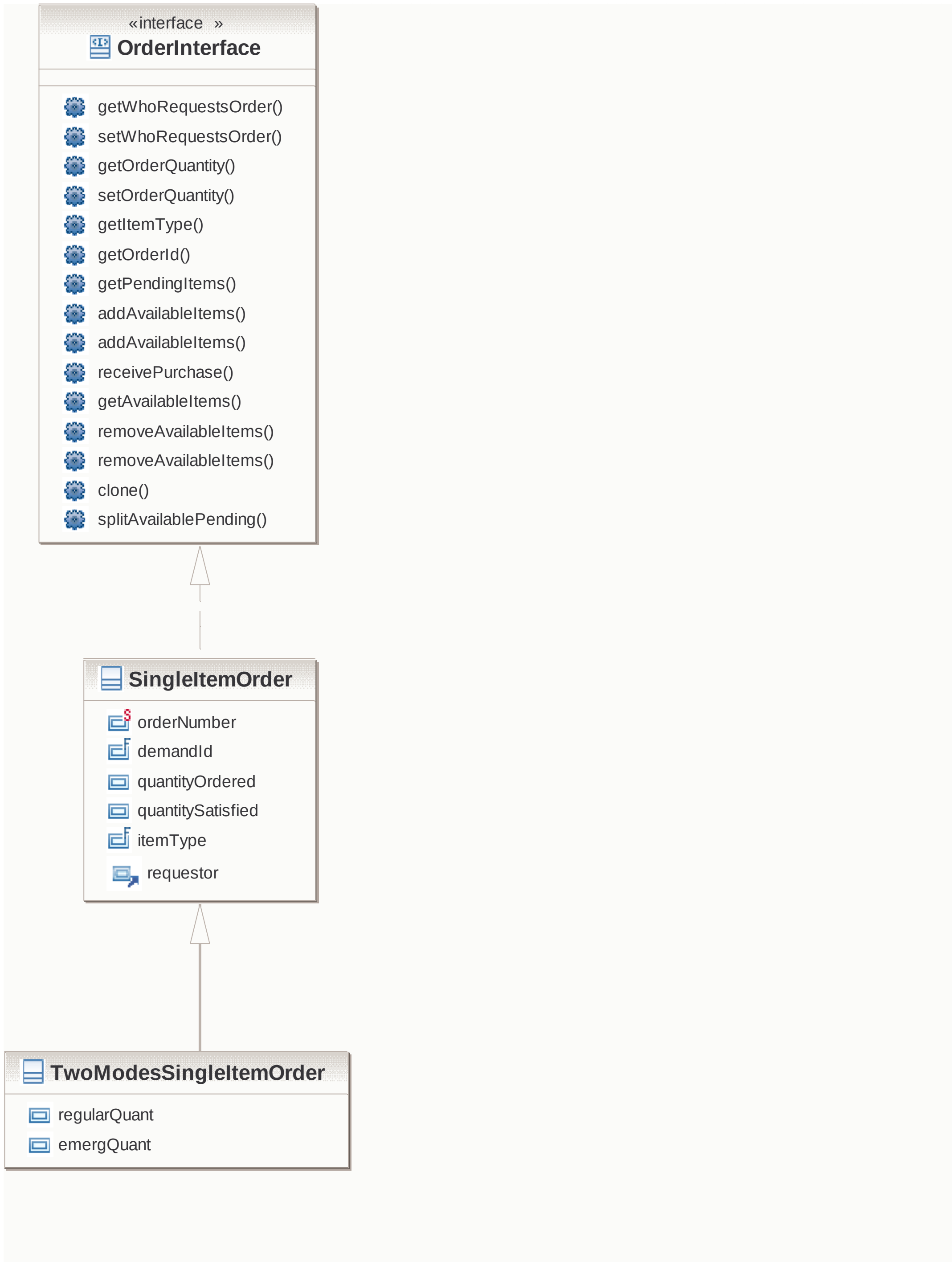


Figure 14: Class Diagram for the Production Package

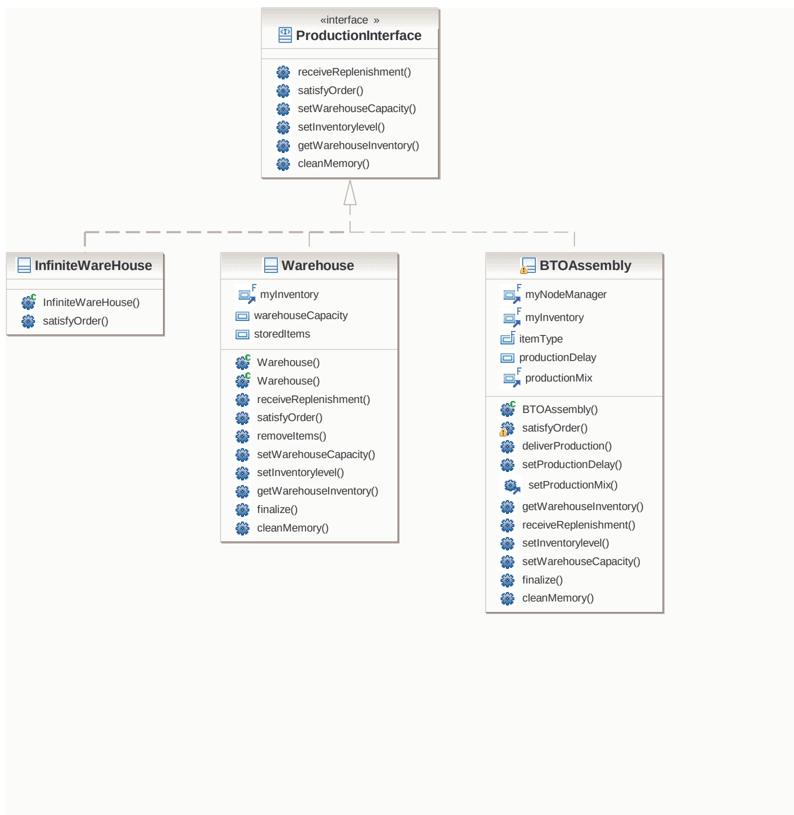


Figure 15: Class Diagram for the Cost Package

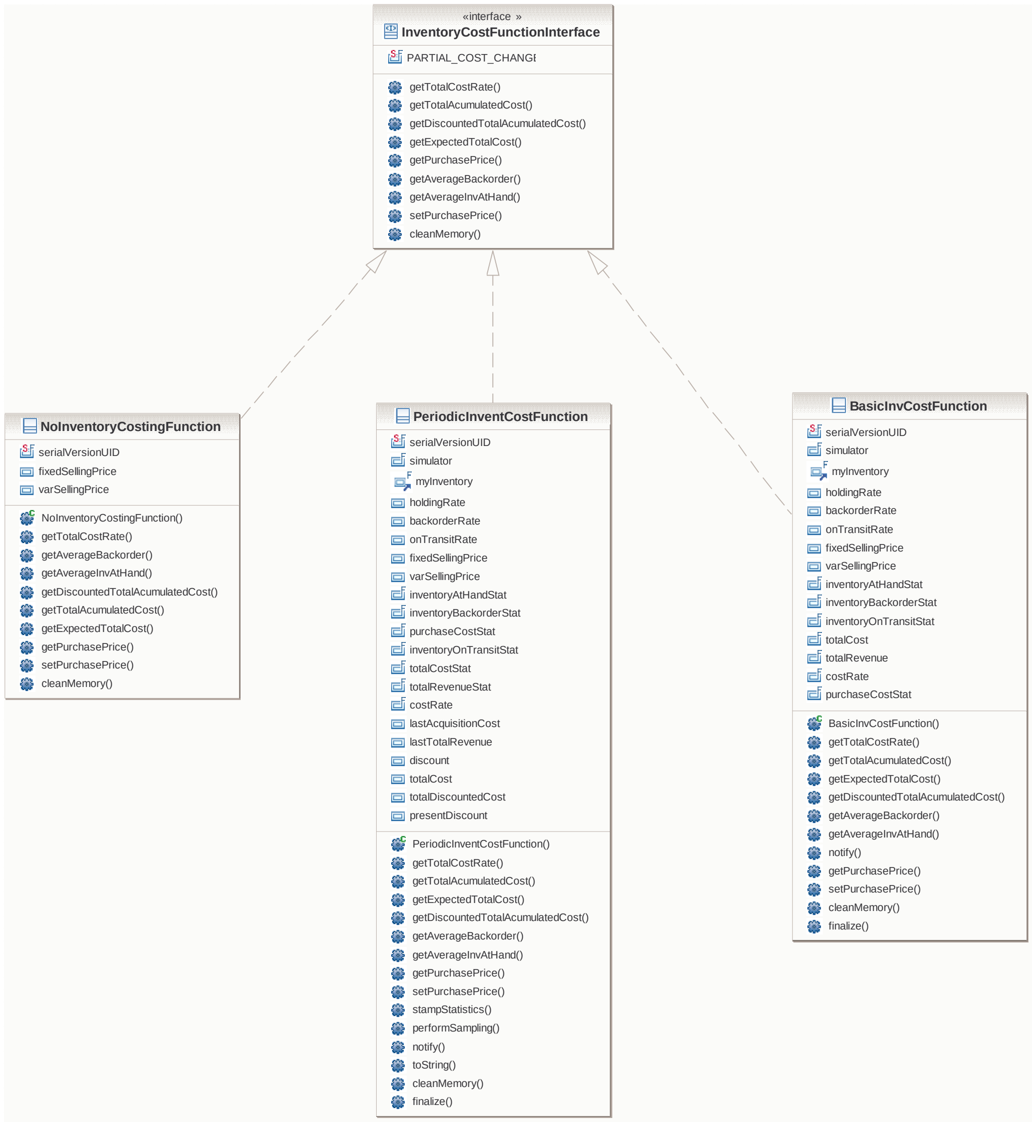


Figure 16: Class Diagram for the Allocation Package

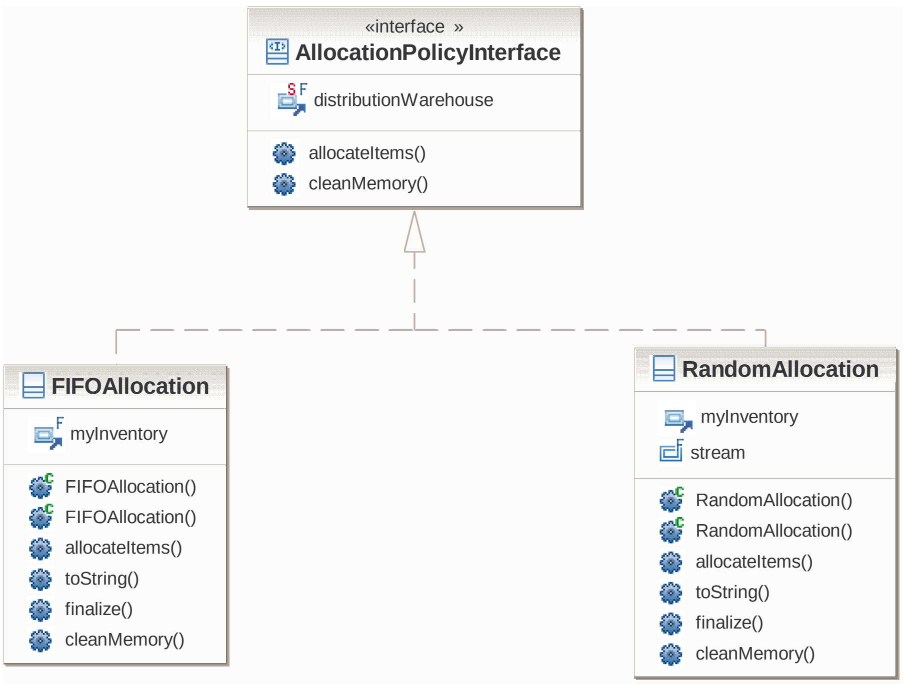


Figure 17: Class Diagram for the Reorder Package

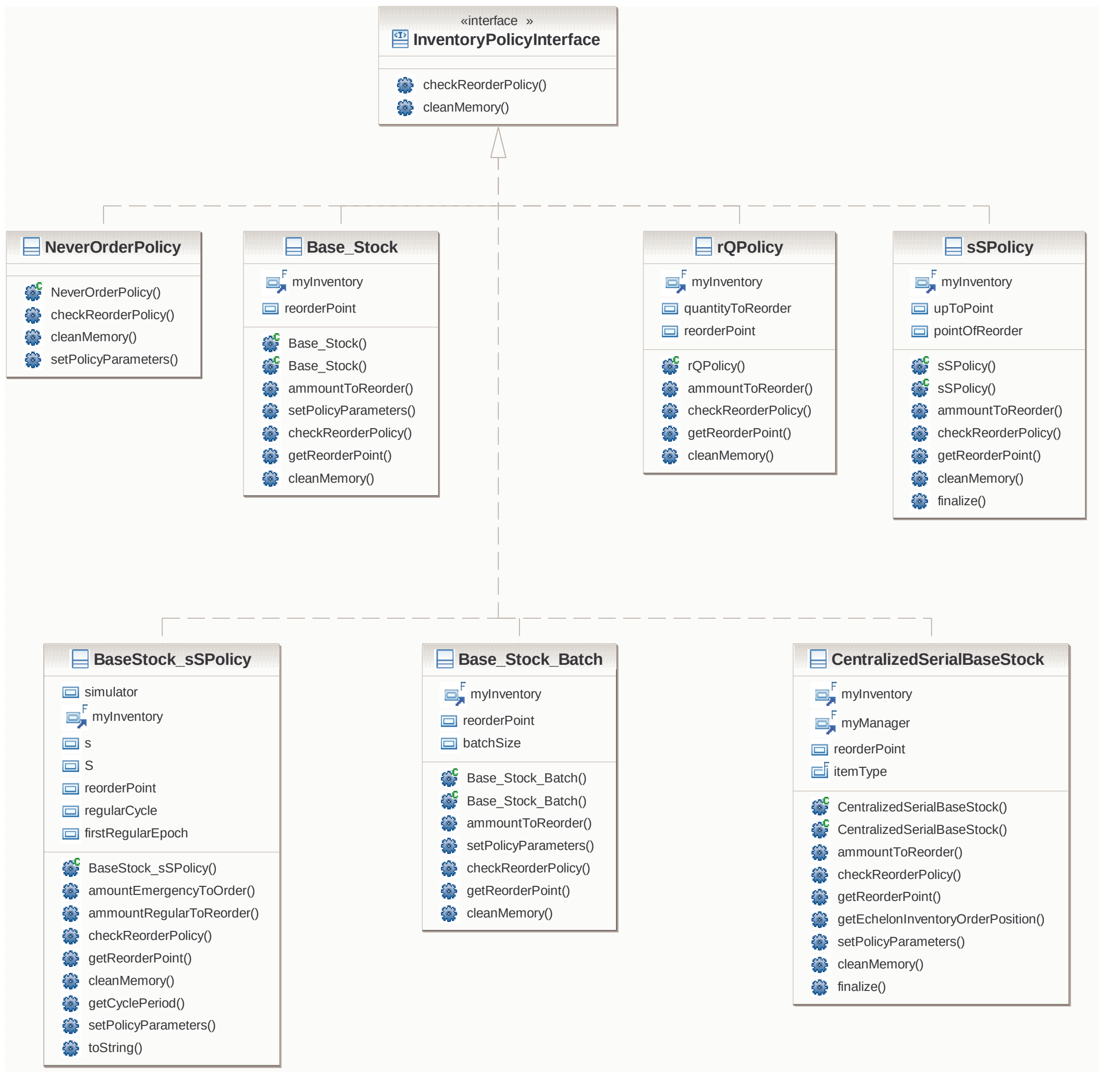


Figure 18: Class Diagram for the Inventory Package

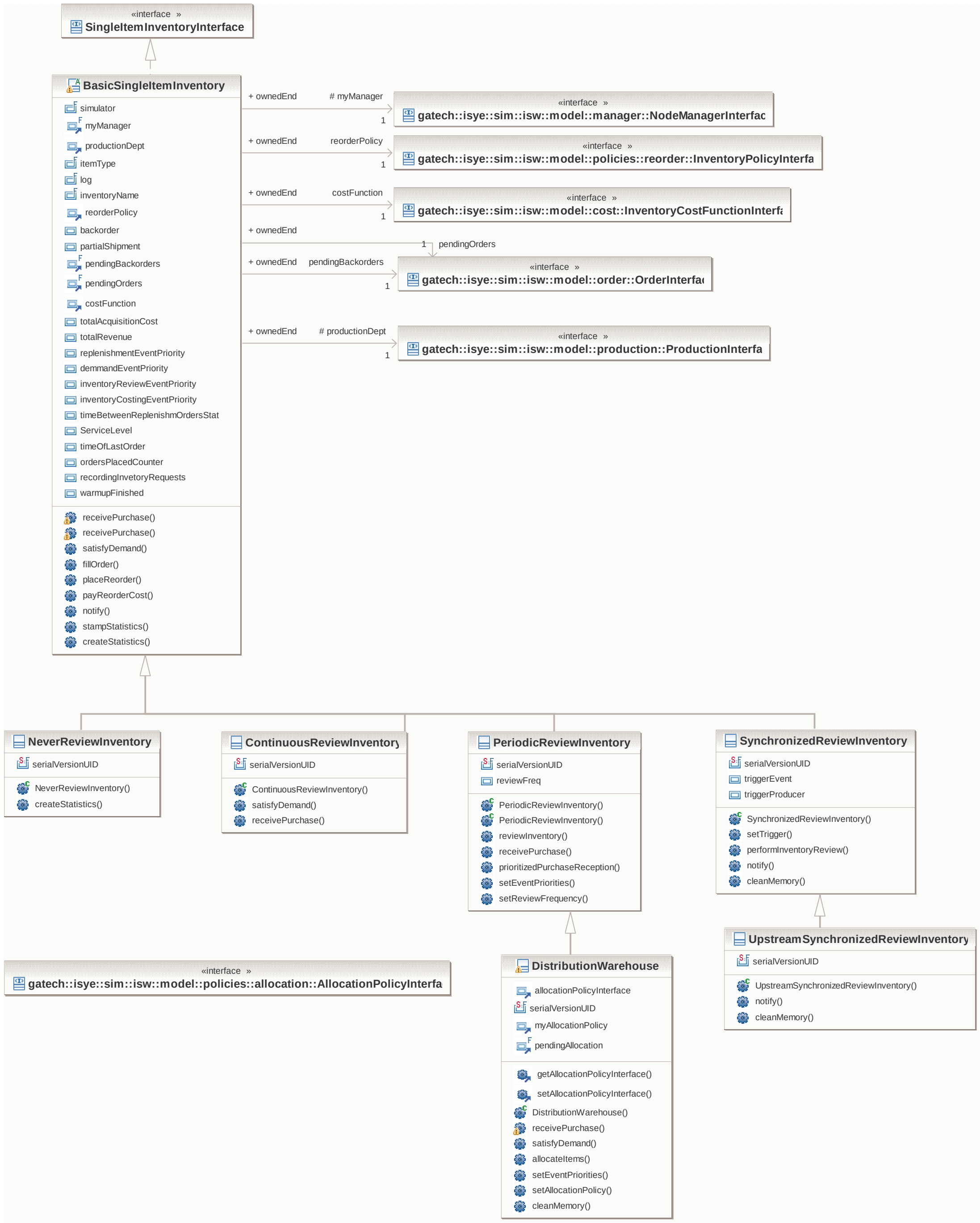


Figure 19: Class Diagram for the Transport Package

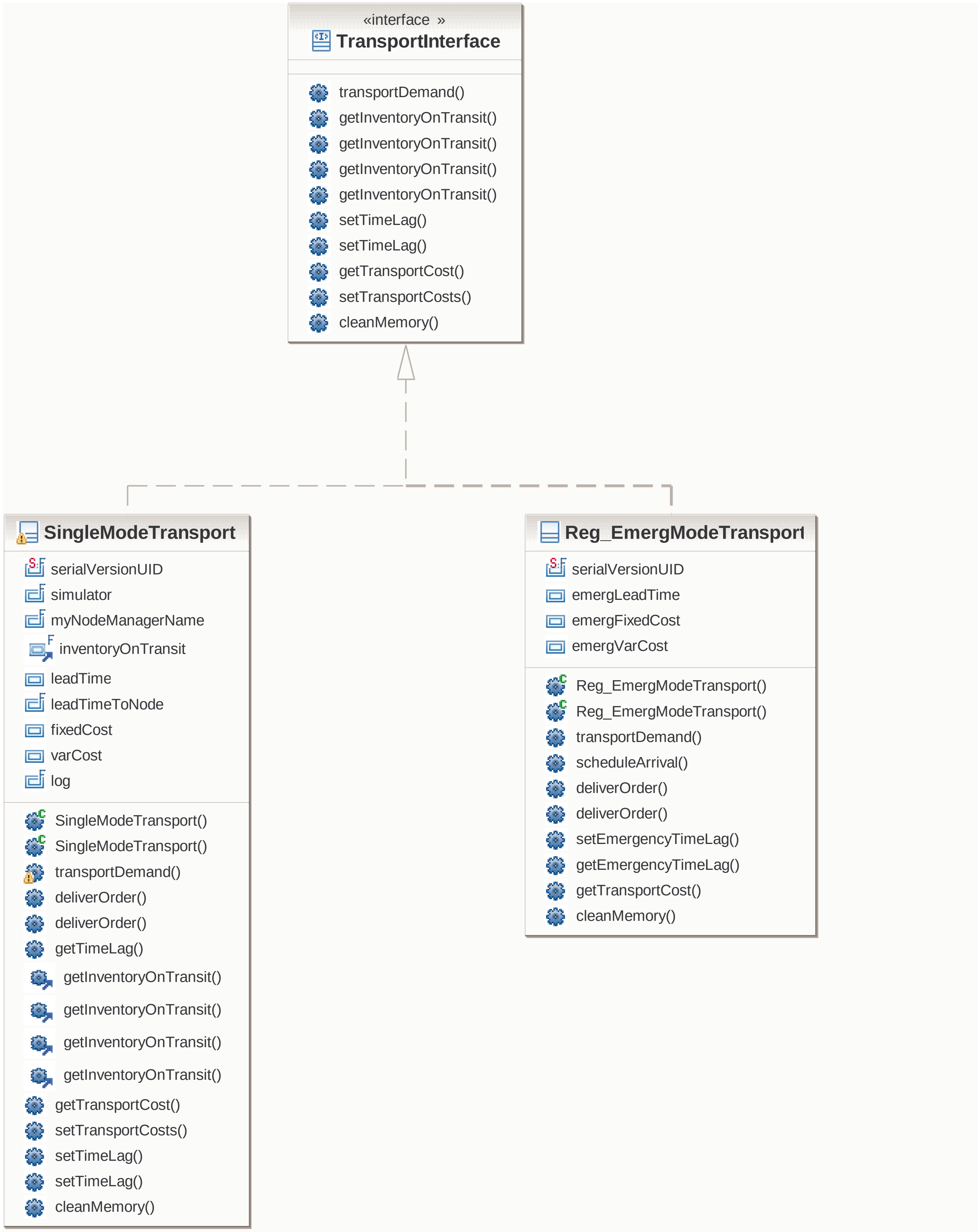
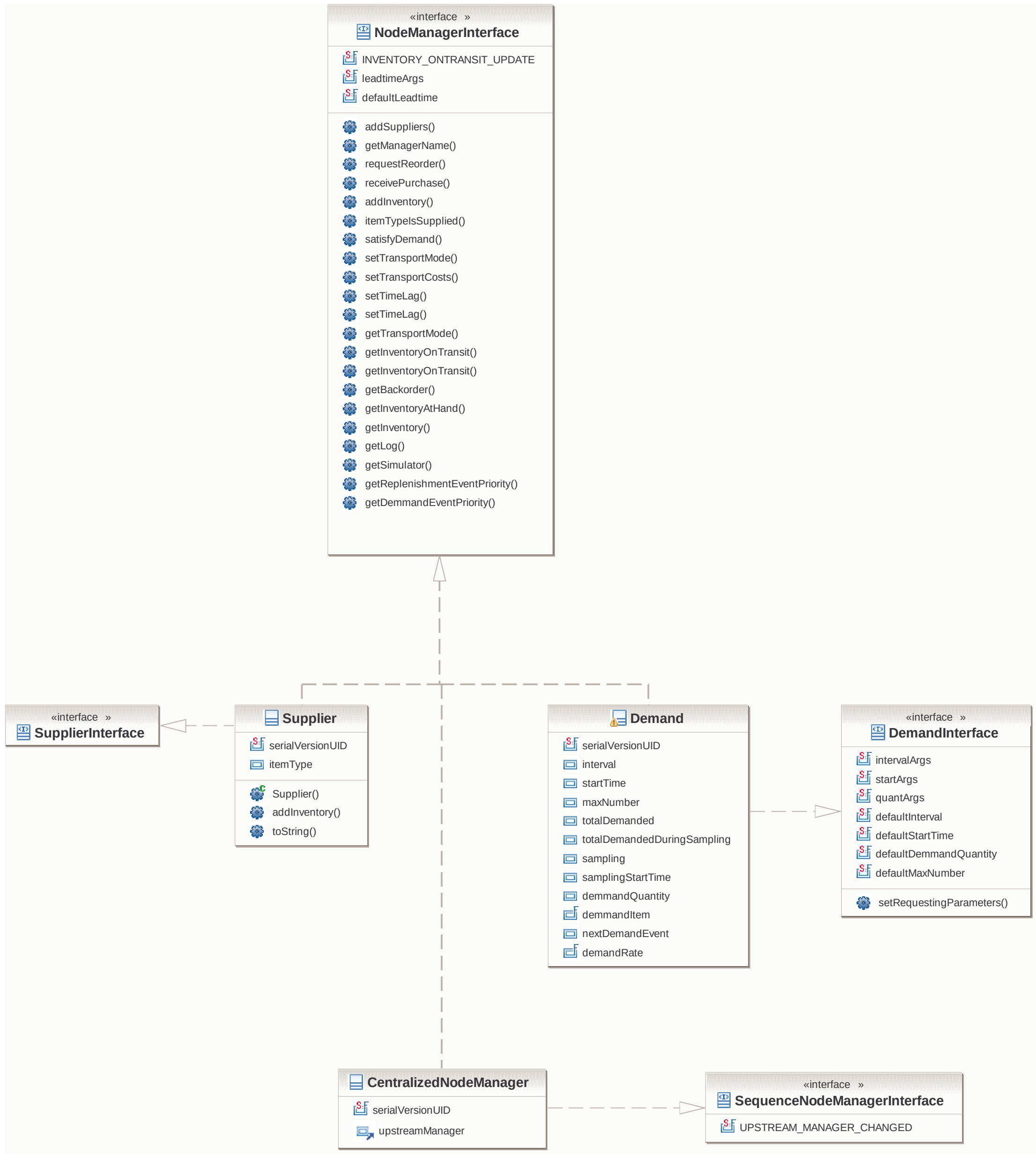


Figure 20: Class Diagram for the Manager Package



4.3.2 Activity Diagrams

Activity diagrams show the procedural flow of control between two or more class objects during the process of an activity. This section contains the diagrams for the following activities:

- Demand: The Demand activity in Figure 21 involves the flow of an order from its creation at a Demand node, fulfillment with production or stock at a Node Manager, and its transportation back to the requesting Demand node.
- Resupply: The Resupply activity in Figure 22 refers to the flow of orders required to replenish warehouse levels with inventory from an external Node Manager. This activity may be initiated at any `Node Manager`.
- Inventory Review: The Inventory Review in Figure 23 refers to the activities carried to verify inventory levels and place resupply orders in accordance with the inventory policy in use.

Figure 21: Demand Activity Diagram

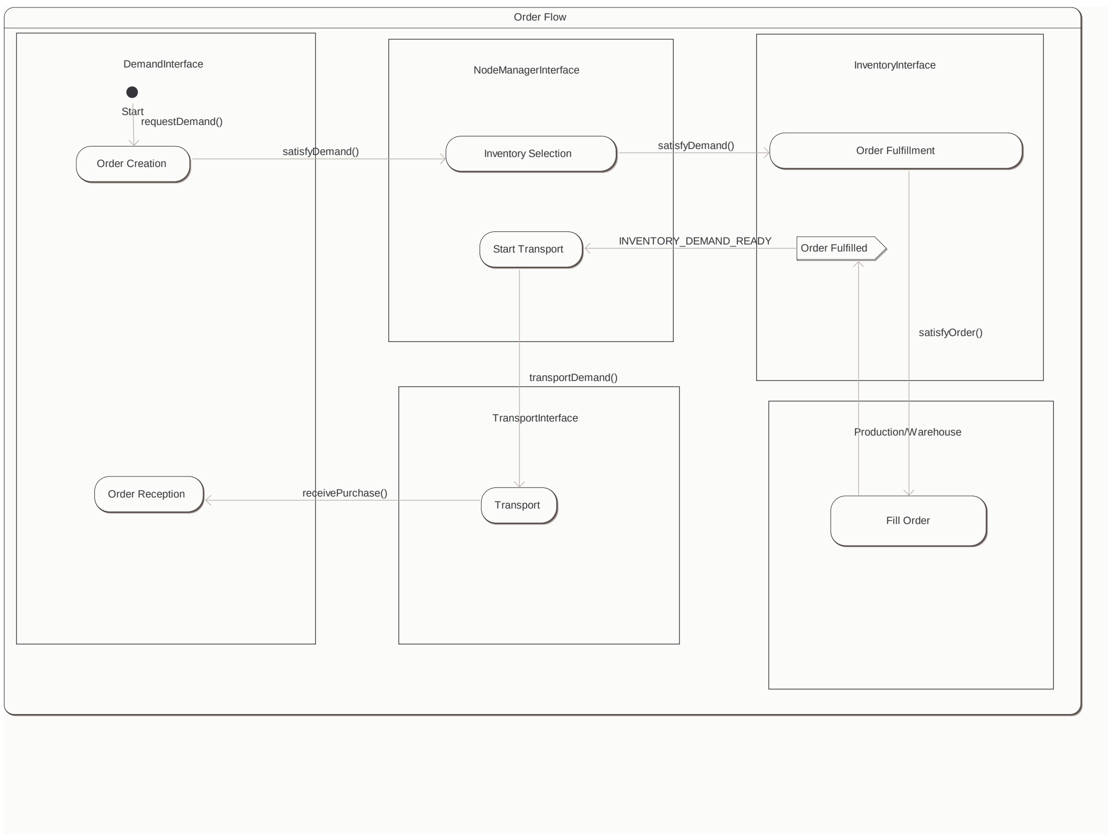


Figure 22: Resupply Activity Diagram

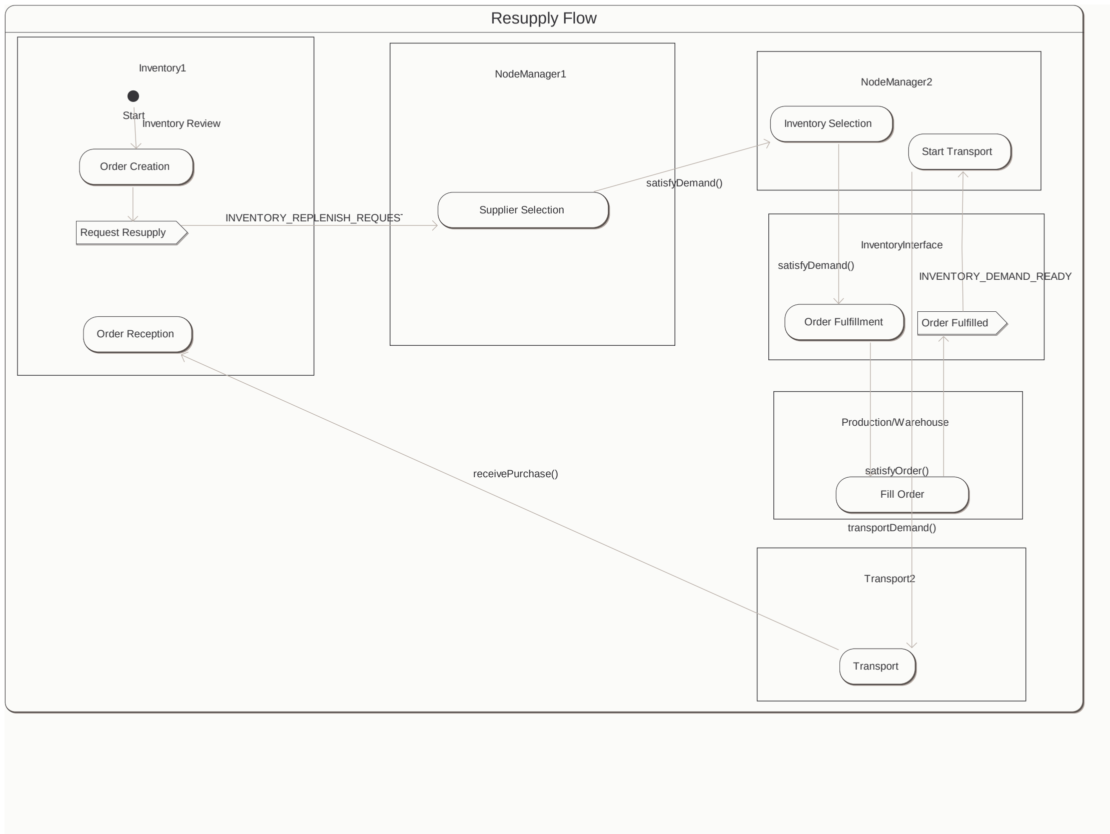
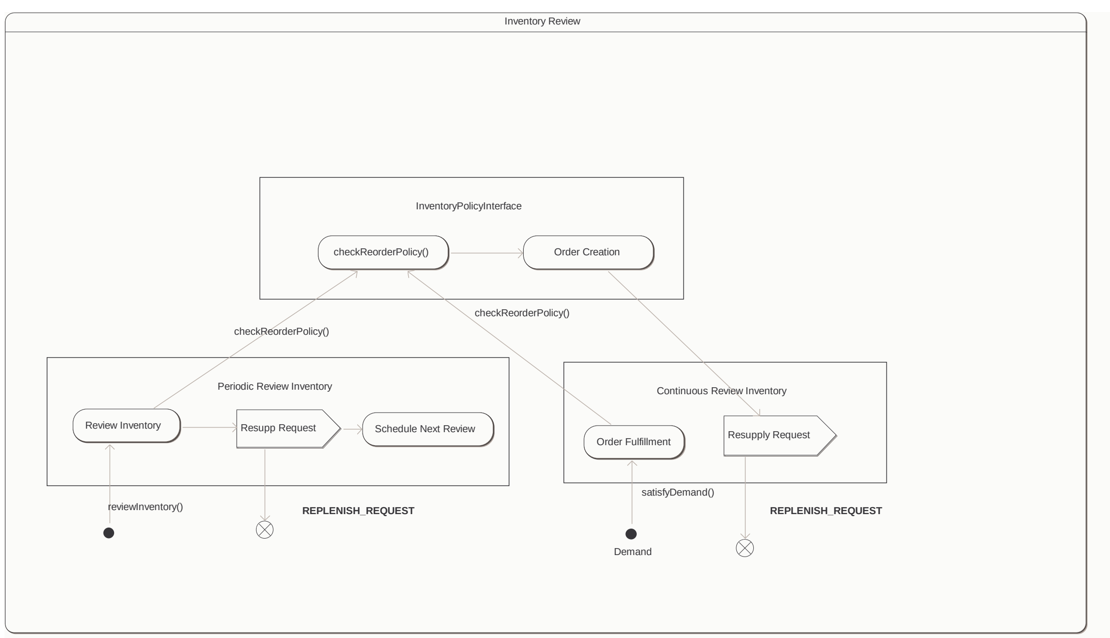


Figure 23: Inventory Review Activity Diagram



CHAPTER V

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

This thesis studied an inventory system with two potential supply modes: regular order placement opportunities happen at a fixed frequency, which defines regular review cycles, while emergency orders can be placed on every period within a regular review cycle. Regular and emergency orders incur a unit cost while the latest incurs an additional setup cost.

We made contributions on two fronts. First, for a regular order lead-time equal to two periods, we showed that the optimal policy with respect to the expected total discounted cost is of (s, S) type for emergency orders while the size of a regular order depends on the inventory position following a potential emergency order. Although we could not establish the optimality for regular order lead-times exceeding two periods, substantial experimental evidence supports the conjecture that the optimal policy retains the same structure. In addition, we developed a value iteration algorithm for computing the parameters of the optimal policy.

Since the optimal policy algorithm requires significant computational effort, we developed and evaluated two heuristic policies whose operational parameters can be computed with relatively small computational effort and compared them against the optimal policy in terms of implementation difficulty, speed and accuracy for the experimental design of 3888 cases listed in Table 4. The results indicated that both heuristics yield a significant reduction in computational time without adding substantial errors in the total expected costs.

The evaluation of the proposed optimal policy and the two heuristics required a simulation suite flexible enough to capture the specific problem dynamics. A search for such a tool exposed the lack of a public-domain, user-friendly, simulation package tailored for evaluating inventory systems. This motivated our second research front: the development of the Inventory Simulator Workbench (ISW). This inventory system simulator, written in

Java, provides the user with a graphical interface and the ability to model a large range of supply chain structures. Many classic supply chain structures are included with the package including the single-level base-stock, (r, Q), and (s, S) policies along with serial and distribution networks. In addition, other policies and network structures can be evaluated simply by modifying the network structure and choosing the appropriate policy for each location in the network. We envision that the ISW will fill a significant need in both academic research and in industry.

Our first goal for the future will be the expansion of Theorem 1 for regular order leadtimes larger than two periods. This development will contribute to the expansion of heuristic HP2 in Section 3.2.4. Further, we are planning to improve the search procedures of the algorithm in Section 2.9.2. Another goal is the enhancement and potential commercialization of the ISW.

APPENDIX A

PREVIOUS RESULTS

Definition A.1 *(K-convexity) A function $g : \mathbb{R} \to \mathbb{R}$ is K-convex, where $K \geq 0$ if either*

(a) Scarf [52]: For $a \geq 0$, $b > 0$,

$$K + g(a + y) \geq g(y) + a\left[\frac{g(y) - g(y - b)}{b}\right]$$

(b) Gallego [25]: For all $\lambda \in [0, 1]$, $\bar{\lambda} = 1 - \lambda$ and $y \geq x$:

$$g(\lambda x + \bar{\lambda} y) \leq \lambda g(x) + \bar{\lambda}[g(y) + K\delta(y - x)]$$

In other words, the line joining the points $(x, g(x))$ and $(y, g(y) + K)$ lies above the graph of $g(\cdot)$.

These definitions are equivalent.

The following lemmas are from Heyman and Sobel [29] and Bertsekas [5].

Lemma A.1 *For real valued functions $g(\cdot)$:*

(a) $g(\cdot)$ is 0-convex $\iff$ $g(\cdot)$ is convex on $\mathbb{R}$.

(b) $g(\cdot)$ is K-convex $\Rightarrow g(\cdot + u)$ is K-convex for all $u \in \mathbb{R}$.

(c) $g_i(\cdot)$ is K_i-convex, $i = 1, 2 \Rightarrow \alpha_1 g_1(\cdot) + \alpha_2 g_2(\cdot)$ is $\alpha_1 K_1 + \alpha_2 K_2$-convex, for all $\alpha_1 > 0$ and $\alpha_2 > 0$.

(d) $g(\cdot)$ is K-convex $\Rightarrow g(\cdot)$ is V-convex for all $V \geq K$.

(e) $g(\cdot)$ is K-convex $\Rightarrow g(\cdot)$ is continuous on $\mathbb{R}$.

(f) $g(\cdot)$ is K-convex $\Rightarrow g(\cdot)$ is differentiable on $\mathbb{R}$ except for at most countable many points.

(g) *If $g(y)$ is K-convex, then $E_w g(y-w)$ is also K-convex, provided that*

$$E_w|g(y-w)| < \infty \quad \text{for all } y$$

(h) *If $g(\cdot)$ is a K-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exist scalars s and S such that:*

(i) $g(S) \leq g(y) \quad \forall y \in \mathbb{R}$.

(ii) $g(S) + K = g(s) < g(y) \quad \forall y < s$.

(iii) *$g(y)$ is a decreasing function on $(-\infty, s)$.*

(iv) $g(y) \leq g(z) + K \quad \forall s \leq y \leq z$.

Lemma A.2 *Suppose $g(\cdot)$ is K-convex, attains its global minimum at S, and there is a value $s \leq S$ such that $g(s) \leq W + g(S)$, where $K \leq W$. Then*

$$f(x) = \inf_{z \geq x}\{W\delta(z-x) + g(z)\}$$

is V-convex.

Lemma A.3 *For a real-valued function $g(\cdot)$ on $\mathbb{R}$, $W \geq 0$ and $s \in \mathbb{R}$, let*

$$f(s) = \inf_{a \geq s}\{W\delta(a-s) + g(a)\}$$

Then for any $\gamma \geq 0$,

$$f(s) \leq f(s+\gamma) + W$$

The following lemma is from Bertsekas [5, Proposition 3.1.7] and is written in terms of the functions defined in this paper.

Lemma A.4 *If $c(\mathbf{x}, \mathbf{d}, \xi) \geq 0$ for all $(\mathbf{x}, \mathbf{d}, \xi)$, and the level sets $U_k(x, r, j, \lambda)$ defined by*

$$U_k(x, r, j, \lambda) = \{d \in \mathcal{D}(x, r, j) \mid \mathrm{E}_\xi[c(\mathbf{x}, \mathbf{d}, \xi) + \alpha C_k(f(\mathbf{x}, \mathbf{d}, \xi))] \leq \lambda\}$$

are compact subsets of a Euclidean space for every $x, r, \lambda \in \mathbb{R}$ and for all k greater than some integer $\bar{k}$, then $\lim_{k\to\infty} C_k(x, r, j) = C(x, r, j)$.

APPENDIX B

NUMERICAL RESULTS

B.1 Optimal Parameters for Regular Order Lead-time $\tau = 2$

The following tables display the optimal parameters for some cases of the experiment described in Table 4. In order to simplify the representation of the function $Q(z)$, we use the property stated in Lemma 13 and display the intervals where $w^*(z) \neq 0$. Hence if z is contained in some interval $[z_1, z_2]$, then $w(z) = z_2$ and $Q(z) = z_2 - z$; otherwise, $w(z) = z$ and $Q(z) = 0$.

Table 19: Results for the Poisson Distribution with $\lambda = 2$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 12.0]	(1.5, 3.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 5.0)	(2.8, 4.0)
$K = 5$	(-∞, 12.0]	(1.5, 3.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 5.0)	(2.8, 4.0)
$K = 50$	(-∞, 12.0]	(0.8, 2.0)	(2.2, 5.0)	(2.2, 5.0)	(2.2, 5.0)	(2.1, 4.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 12.0] [12.1, 13.1]	(0.9, 3.0)	(2.6, 6.0)	(2.6, 6.0)	(2.6, 5.0)	(2.4, 4.0)
$K = 5$	(-∞, 13.0]	(0.9, 3.0)	(2.6, 6.0)	(2.6, 6.0)	(2.6, 5.0)	(2.5, 4.0)
$K = 50$	(-∞, 14.0]	(-4.1, 2.0)	(0.7, 9.0)	(0.8, 8.0)	(0.8, 6.0)	(-0.3, 4.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 14.0]	(-3.4, 3.0)	(1.5, 10.0)	(1.6, 9.0)	(1.4, 7.0)	(0.2, 5.0)
$K = 5$	(-∞, 14.0]	(-3.4, 3.0)	(1.6, 10.0)	(1.7, 9.0)	(1.5, 7.0)	(0.3, 5.0)
$K = 50$	(-∞, 12.0]	(0.9, 2.0)	(2.6, 4.0)	(2.6, 4.0)	(2.6, 4.0)	(2.3, 3.0)

Table 20: Results for the Poisson Distribution with $\lambda = 2$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 11.0]	(1.8, 3.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 4.0)
$K = 5$	(-∞, 11.0]	(1.8, 3.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 5.0)	(2.9, 4.0)
$K = 50$	(-∞, 11.0]	(1.0, 3.0)	(2.2, 5.0)	(2.2, 5.0)	(2.2, 5.0)	(2.2, 4.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 12.0]	(1.2, 3.0)	(2.6, 6.0)	(2.6, 6.0)	(2.6, 6.0)	(2.5, 4.0)
$K = 5$	(-∞, 12.0]	(1.3, 3.0)	(2.6, 6.0)	(2.6, 6.0)	(2.7, 6.0)	(2.5, 4.0)
$K = 50$	(-∞, 13.0]	(-3.5, 3.0)	(0.6, 10.0)	(0.8, 8.0)	(0.8, 7.0)	(-0.1, 5.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 14.0]	(-2.9, 3.0)	(1.5, 11.0)	(1.6, 9.0)	(1.5, 7.0)	(0.4, 5.0)
$K = 5$	(-∞, 14.0]	(-2.8, 3.0)	(1.6, 11.0)	(1.7, 9.0)	(1.5, 7.0)	(0.4, 5.0)
$K = 50$	(-∞, 11.0]	(1.2, 2.0)	(2.6, 4.0)	(2.6, 4.0)	(2.6, 4.0)	(2.4, 4.0)

Table 21: Results for the Poisson Distribution with $\lambda = 4$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 23.0]	(3.7, 5.0)	(5.5, 8.0)	(5.5, 8.0)	(5.5, 7.0)	(5.6, 7.0)
$K = 5$	(-∞, 23.0]	(3.7, 5.0)	(5.5, 8.0)	(5.5, 8.0)	(5.5, 8.0)	(5.6, 7.0)
$K = 50$	(-∞, 22.0]	(2.7, 5.0)	(4.3, 8.0)	(4.4, 8.0)	(4.3, 8.0)	(4.6, 7.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 24.0]	(2.9, 5.0)	(4.8, 9.0)	(4.8, 9.0)	(4.7, 9.0)	(4.9, 8.0)
$K = 5$	(-∞, 24.0]	(2.9, 5.0)	(4.8, 10.0)	(4.8, 10.0)	(4.8, 9.0)	(4.9, 8.0)
$K = 50$	(-∞, 25.0]	(-2.5, 5.0)	(2.0, 17.0)	(2.3, 15.0)	(2.7, 12.0)	(2.2, 8.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 26.0]	(-1.7, 5.0)	(3.0, 19.0)	(3.4, 16.0)	(3.7, 13.0)	(2.8, 9.0)
$K = 5$	(-∞, 27.0]	(-1.7, 5.0)	(3.1, 19.0)	(3.5, 16.0)	(3.8, 13.0)	(2.8, 9.0)
$K = 50$	(-∞, 23.0]	(2.8, 4.0)	(4.9, 7.0)	(4.9, 7.0)	(4.9, 7.0)	(4.9, 6.0)

Table 22: Results for the Poisson Distribution with $\lambda = 4$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 21.0]	(3.9, 5.0)	(5.5, 8.0)	(5.5, 8.0)	(5.5, 7.0)	(5.5, 7.0)
$K = 5$	(-∞, 22.0]	(3.9, 5.0)	(5.5, 8.0)	(5.5, 8.0)	(5.5, 8.0)	(5.6, 7.0)
$K = 50$	(-∞, 20.0]	(3.1, 5.0)	(4.3, 8.0)	(4.4, 8.0)	(4.3, 8.0)	(4.5, 7.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 22.0]	(3.3, 5.0)	(4.8, 9.0)	(4.8, 9.0)	(4.7, 9.0)	(4.9, 8.0)
$K = 5$	(-∞, 23.0]	(3.3, 5.0)	(4.8, 10.0)	(4.8, 10.0)	(4.8, 10.0)	(5.0, 8.0)
$K = 50$	(-∞, 25.0]	(-1.7, 5.0)	(2.0, 18.0)	(2.3, 15.0)	(2.7, 12.0)	(2.3, 9.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 26.0]	(-1.2, 5.0)	(3.0, 20.0)	(3.3, 17.0)	(3.7, 13.0)	(3.0, 9.0)
$K = 5$	(-∞, 26.0]	(-1.1, 5.0)	(3.1, 20.0)	(3.4, 17.0)	(3.8, 13.0)	(3.0, 9.0)
$K = 50$	(-∞, 22.0]	(3.1, 4.0)	(4.9, 7.0)	(4.9, 7.0)	(4.9, 7.0)	(4.9, 7.0)

Table 23: Results for the Poisson Distribution with $\lambda = 8$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 40.0]	(7.8, 9.0)	(10.3, 12.0)	(10.3, 12.0)	(10.3, 12.0)	(10.3, 13.0)
$K = 5$	(-∞, 40.0]	(7.8, 9.0)	(10.3, 13.0)	(10.3, 13.0)	(10.3, 13.0)	(10.4, 13.0)
$K = 50$	(-∞, 40.0] [40.1, 40.2]	(6.7, 9.0)	(8.8, 12.0)	(8.8, 12.0)	(8.8, 12.0)	(8.9, 12.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 40.0] [40.1, 40.2]	(7.0, 9.0)	(9.3, 13.0)	(9.3, 13.0)	(9.3, 13.0)	(9.5, 13.0)
$K = 5$	(-∞, 40.0] [40.1, 40.2]	(7.0, 9.0)	(9.3, 13.0)	(9.3, 13.0)	(9.3, 13.0)	(9.5, 13.0)
$K = 50$	(-∞, 40.0] [40.1, 40.2]	(1.1, 9.0)	(5.3, 23.0)	(5.2, 28.0)	(6.0, 23.0)	(6.6, 16.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 40.0] [40.1, 40.2]	(1.9, 9.0)	(6.1, 36.0)	(6.6, 31.0)	(7.3, 24.0)	(7.7, 17.0)
$K = 5$	(-∞, 40.0] [40.1, 40.2]	(1.9, 9.0)	(6.2, 37.0)	(6.7, 31.0)	(7.4, 24.0)	(7.8, 17.0)
$K = 50$	(-∞, 40.0]	(6.6, 8.0)	(9.6, 12.0)	(9.6, 12.0)	(9.6, 12.0)	(9.6, 12.0)

Table 24: Results for the Poisson Distribution with $\lambda = 8$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 40.0]	(8.3, 10.0)	(10.3, 12.0)	(10.3, 12.0)	(10.3, 12.0)	(10.3, 13.0)
$K = 5$	(-∞, 40.0]	(8.4, 10.0)	(10.3, 13.0)	(10.3, 13.0)	(10.3, 13.0)	(10.4, 13.0)
$K = 50$	(-∞, 37.0]	(7.2, 10.0)	(8.8, 12.0)	(8.8, 12.0)	(8.8, 12.0)	(8.8, 12.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 40.0] [40.1, 40.2]	(7.5, 10.0)	(9.3, 13.0)	(9.3, 13.0)	(9.3, 13.0)	(9.4, 13.0)
$K = 5$	(-∞, 40.0] [40.1, 40.2]	(7.5, 10.0)	(9.3, 13.0)	(9.3, 13.0)	(9.3, 13.0)	(9.5, 13.0)
$K = 50$	(-∞, 40.0] [40.1, 40.2]	(1.9, 10.0)	(5.4, 23.0)	(5.2, 29.0)	(5.9, 24.0)	(6.7, 17.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 40.0] [40.1, 40.2]	(2.5, 10.0)	(6.1, 37.0)	(6.5, 31.0)	(7.2, 25.0)	(7.8, 18.0)
$K = 5$	(-∞, 40.0] [40.1, 40.2]	(2.6, 10.0)	(6.2, 37.0)	(6.7, 32.0)	(7.4, 25.0)	(7.9, 18.0)
$K = 50$	(-∞, 40.0]	(6.9, 8.0)	(9.6, 12.0)	(9.6, 12.0)	(9.6, 12.0)	(9.6, 12.0)

Table 25: Results for the Negative Binomial Distribution with $p = 1/3$ and $r = 1$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 13.0]	(1.0, 2.0)	(4.3, 7.0)	(4.2, 6.0)	(3.8, 6.0)	(2.9, 4.0)
$K = 5$	(-∞, 13.0]	(1.0, 2.0)	(4.5, 7.0)	(4.3, 6.0)	(3.8, 6.0)	(2.9, 4.0)
$K = 50$	(-∞, 13.0]	(0.2, 2.0)	(2.8, 6.0)	(2.8, 6.0)	(2.6, 5.0)	(1.9, 4.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 14.0]	(0.4, 2.0)	(3.7, 7.0)	(3.6, 7.0)	(3.3, 6.0)	(2.3, 5.0)
$K = 5$	(-∞, 14.0]	(0.4, 2.0)	(3.8, 7.0)	(3.7, 7.0)	(3.3, 6.0)	(2.4, 5.0)
$K = 50$	(-∞, 15.0]	(-4.7, 2.0)	(0.6, 9.0)	(0.5, 8.0)	(0.1, 7.0)	(-0.9, 5.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 16.0]	(-4.0, 2.0)	(1.7, 10.0)	(1.5, 9.0)	(0.8, 7.0)	(-0.4, 5.0)
$K = 5$	(-∞, 16.0]	(-3.9, 2.0)	(1.8, 10.0)	(1.6, 9.0)	(0.8, 7.0)	(-0.4, 5.0)
$K = 50$	(-∞, 14.0]	(0.2, 1.0)	(3.3, 5.0)	(3.2, 5.0)	(2.8, 4.0)	(2.0, 3.0)

Table 26: Results for the Negative Binomial Distribution with $p = 1/3$ and $r = 1$, $\tau = 2$, Backorder Cost = 15, Regular Review Cycle Length = 5, and Emergency Variable Cost = 5.

$\alpha = 0.9$	$w(z)$	(s_0, S_0)	(s_1, S_1)	(s_2, S_2)	(s_3, S_3)	(s_4, S_4)
$K = 2$	(-∞, 13.0]	(1.5, 3.0)	(4.4, 7.0)	(4.2, 6.0)	(3.9, 6.0)	(3.3, 5.0)
$K = 5$	(-∞, 13.0]	(1.5, 3.0)	(4.5, 7.0)	(4.3, 7.0)	(4.0, 6.0)	(3.3, 5.0)
$K = 50$	(-∞, 12.0]	(0.6, 3.0)	(2.8, 6.0)	(2.8, 6.0)	(2.7, 6.0)	(2.1, 5.0)
$\alpha = 0.99$						
$K = 2$	(-∞, 13.0]	(0.8, 3.0)	(3.7, 7.0)	(3.7, 7.0)	(3.4, 6.0)	(2.6, 5.0)
$K = 5$	(-∞, 13.0]	(0.8, 3.0)	(3.8, 8.0)	(3.7, 7.0)	(3.4, 6.0)	(2.6, 5.0)
$K = 50$	(-∞, 14.0]	(-4.0, 3.0)	(0.6, 9.0)	(0.6, 8.0)	(0.2, 7.0)	(-0.7, 5.0)
$\alpha = 0.999$						
$K = 2$	(-∞, 15.0]	(-3.4, 3.0)	(1.8, 11.0)	(1.5, 9.0)	(0.9, 8.0)	(-0.3, 6.0)
$K = 5$	(-∞, 16.0]	(-3.3, 3.0)	(1.9, 11.0)	(1.6, 9.0)	(1.0, 8.0)	(-0.2, 6.0)
$K = 50$	(-∞, 13.0]	(0.5, 2.0)	(3.3, 5.0)	(3.2, 5.0)	(2.9, 5.0)	(2.2, 4.0)

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