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Ryland Wrenn Olive, Jr.

UNSTEADY FLOW IN A SMOOTH PIPE AFTER INSTANTANEOUS OPENING OF A DOWNSTREAM VALVE

PART IV. MATHEMATICAL ANALYSIS

A THESIS

Presented to

the Faculty of the Graduate Division

Georgia Institute of Technology

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

by

Ryland Wrenn Olive, Jr.

June, 1956

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Approved:

Date Approved by Chairman: May 14, 1956

ACKNOWLEDGMENTS

The author wishes to express gratitude to Dr. M. B. Sledd and Dr. J. A. Nohel for their patient guidance of the preparation of this thesis and to thank Dr. M. R. Carstens for his assistance and for extending the opportunity to work with this project. The author appreciates the work of Mrs. D. J. Rhoades, Mrs. E. D. McCalla, and Mrs. R. W. Olive, Jr. who typed the manuscript.

FOREWORD

This thesis, which is primarily mathematical in nature, is the fourth in a series of theses reporting the results of a project sponsored by the National Science Foundation and supervised by Dr. M. R. Carstens of the School of Civil Engineering. Its predecessors, which are based primarily on experimental investigations, are

Unsteady Flow in a Smooth Pipe After Instantaneous Opening of a Downstream Valve

Part I. "Mean Flow Characteristics - Velocity," by B. G. Christopher;

Part II. "Transition from Laminar to Turbulent Flow," by J. B. Trimble;

Part III. "Mean Flow Characteristics - Pressure and Boundary Shear," by J. E. Roller.

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LIST OF SYMBOLS

English Letters

A	a constant
a	the pipe radius, a constant
B	a constant
C	a constant
c	a constant
D	a constant
E	a constant
e	the base of natural logarithms
F	a constant
$f(t)$	the function representing pressure gradient, $\frac{\partial p^*}{\partial z}$
g	gravitational acceleration, a constant
$g(t)$	an arbitrary function
$H(r, t)$	an unspecified function used in illustrating the superposition principle
h	vertical distance in the force field of gravity
h_o	the vertical distance, h, from the pipe centerline at the inlet plane to the reservoir surface
J_o, J_1	Bessel functions of the first kind, orders zero and one respectively
k	the separation constant used in separation of variables

LIST OF SYMBOLS (Continued)

English Letters

k_n	specific values of k determined by the equation $J_o(k_n a) = 0$
L	the length of the pipe, a constant
$M(t)$, $N(r)$	functions used in separation of variables
m	an index integer; $m = 1, 2, \ldots,$ etc.
n	an index integer; $n = 1, 2, \ldots,$ etc.
p	instantaneous pressure at a point
p^*	instantaneous piezometric pressure at a point, $p^* = p + \rho gh$
p^*_1	piezometric pressure at pipe inlet
p^*_2	piezometric pressure at pipe inlet
p^*_o	piezometric pressure within the reservoir
p^*_Ω	piezometric pressure at pipe inlet plane at onset of turbulence
$q(t)$	an unspecified function used in illustrating the superposition principle
r	radial dimension in cylindrical coordinate system
T, T_m	particular values of t
t	time variable
t_Ω	time at onset of turbulence
u	variable in homogeneous equation corresponding to v in non-homogeneous equation, also solution for equation in u

LIST OF SYMBOLS (Continued)

English Letters

v or v_z, v_r, v_θ	components of instantaneous velocity at a point in cylindrical coordinate system
$\bar{v} = v / \frac{p_o^* a^2}{4L\mu}$	dimensionless instantaneous velocity at a point, where $\frac{p_o^* a^2}{4L\mu}$ denotes the Hagen-Poiseuille velocity
$V \quad \int_o^1 \bar{v}\,\beta\, d\beta$	instantaneous mean velocity
$w(t)$	a step-function approximation to $q(t)$
$\underline{\bar{X}}_r$, $\underline{\bar{X}}_\theta$, $\underline{\bar{X}}_z$	components of body forces (excluding gravity) in cylindrical coordinates
z	length dimension in cylindrical coordinates

Greek Letters

$\alpha = \frac{\mu t}{\rho a^2}$	dimensionless time parameter
$\beta = \frac{r}{a}$	dimensionless radial distance parameter
γ_n	roots of the equation $J_o(\gamma_n) = 0$
θ	angular displacement in cylindrical coordinates
$\wedge$	an index integer; $\wedge = 1, 2, \ldots,$ etc.
μ	coefficient of dynamic viscosity
$\nu = \frac{\mu}{\rho}$	coefficient of kinematic viscosity
ξ_m	a value of t such that $T_{m-1} < \xi_m < T_m$

LIST OF SYMBOLS (Continued)

Greek Letters

ρ	mass density
$\Pi_o = V/\sqrt{2gh_o}$	dimensionless velocity parameter
$\Pi_1 = L/2a$	dimensionless distance parameter
$\Pi_2 = \frac{\sqrt{gh_o}}{L^2}\,t$	dimensionless time parameter
$\Pi_3 = \frac{gh_o a^3}{8L\nu^2}$	dimensionless pressure head parameter
τ	dummy variable of integration

Other Symbols

d, ∂, $'$	denote the differentiation operation according to common usage.

LIST OF ILLUSTRATIONS AND TABLES

SUMMARY

Considered in this thesis are certain aspects of starting flow in a pipe having at its inlet a large reservoir and at its outlet a valve abruptly opened to the atmosphere. The fluid is considered viscous and incompressible. The analysis is restricted to the period of transient, laminar flow.

By applying the assumption that the radial component of velocity be zero (and other more plausible assumptions), the Navier-Stokes and continuity equations are simplified to a single linear partial differential equation. This equation was deduced and solved by Szymanski for the case of a constant pressure gradient. For the present thesis, in both the equation and the solution, the pressure gradient is an arbitrary function of time. By approximating the unknown pressure gradient variation by a linear function of time and substituting this linear function for the arbitrary function of time in the solution to the simplified equation, a solution which represents an approximation of the case of reservoir-pipe flow is obtained. This last solution is then shown to be a formal and a rigorous solution of the simplified equation with its boundary and initial conditions, and the uniqueness of this solution is established.

The validity of the solution obtained here is investigated from the standpoint of agreement with physical facts by comparison with experimental data from two sources. Data from the thesis by Christopher

(Part 1 of this investigation) is employed for comparison of the mean flow velocity. Data from investigations by Crausse is used for comparison of the velocity profile shape. The mean flow velocity and velocity profile shape computed from Szymanski's solution are also shown.

The velocity profile shapes for this solution and Szymanski's solution do not show boundary layer development commensurate with Crausse's data. The mean flow velocity computed from this solution is a poorer approximation to Christopher's data at the beginning of flow, but a better approximation at the onset of turbulence than the velocity computed from Szymanski's solution.

INTRODUCTION

This thesis deals with some mathematical aspects of the initiation of flow in a smooth pipe which has at its inlet a large reservoir and at its outlet a valve abruptly opened to the atmosphere.

Assumptions and Approximations.--Strict applicability of the analysis to the physical problem is limited by the following assumptions and approximations.

(1) The viscosity and density of the fluid are constant.

(2) The flow is laminar. Thus, in particular, the analysis does not apply after the onset of turbulence.

(3) The pipe is straight, its cross-section is circular, and the interior surface is smooth.

(4) The flow is axially symmetric and the angular component of velocity is zero.

(5) The effect of opening the valve is assumed to be that of abruptly applying a pressure gradient throughout the fluid in the pipe rather than abruptly applying a pressure difference to the fluid at the outlet plane.

(6) The radial component of the velocity is zero.

From the standpoint of agreement with the physical facts, the sixth assumption is the least justifiable, particularly in the region just downstream from the pipe inlet. It is made solely for the purpose of simplifying the mathematical analysis.

Related Literature.--Szymanski [1] and later Gerbes [2] investigated unsteady flow in a smooth pipe by simplifying the Navier-Stokes and continuity equations. This system of quasi-linear partial differential equations was reduced to a single second-order partial differential equation linearized by assuming, as is done in this thesis, that the radial component of velocity is zero. In the simplified equation the pressure gradient appears as an arbitrary function of time. By particularizing this arbitrary function of time, both Szymanski and Gerbes gave solutions[1] of the simplified equation for two different pressure gradients. First, the pressure gradient was assumed constant; second, the pressure gradient was assumed a harmonically varying function of time.

Purpose of the Research.--In this thesis, a solution of the simplified equation is deduced without particularizing the function which represents the pressure gradient; that is, in the solution itself the pressure gradient appears as an arbitrary function of time. A linear approximation of this arbitrary function is then introduced in order to obtain for the reservoir-fed pipe a more realistic solution than the one given by Szymanski for a constant pressure gradient.

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[1]Gerbes obtained the Laplace transform of a solution in which the pressure gradient is an arbitrary function of time. Completion of Gerbes' solution then required substitution of the Laplace transform of a particular pressure gradient into the Laplace transform of the solution. The inverse Laplace transform then became the desired solution.

CHAPTER I

THE PROBLEM AND SOLUTION FROM A PRIMARILY PHYSICAL VIEWPOINT

This chapter is intended to be an uninterrupted presentation, from the physical viewpoint, of the work accomplished. Mathematical details not facilitating continuity of presentation of the overall concept are deferred until Chapter II.

<u>Notation and Nomenclature</u>.--A straight pipe, circular in cross-section, extends from a large reservoir. Cylindrical coordinates (z, r, θ) are oriented to the pipe as follows (see Figure 1).

(1) z is the dimension of axial length. The pipe inlet plane is at $z = 0$ and the outlet plane at $z = L$.

(2) r is the dimension of radial distance. The pipe centerline is at $r = 0$ and the inside surface of the pipe at $r = a$.

(3) θ is the angular dimension of rotation about the z axis. The position of $\theta = 0$ and the direction of increasing values of θ do not enter into the analysis.

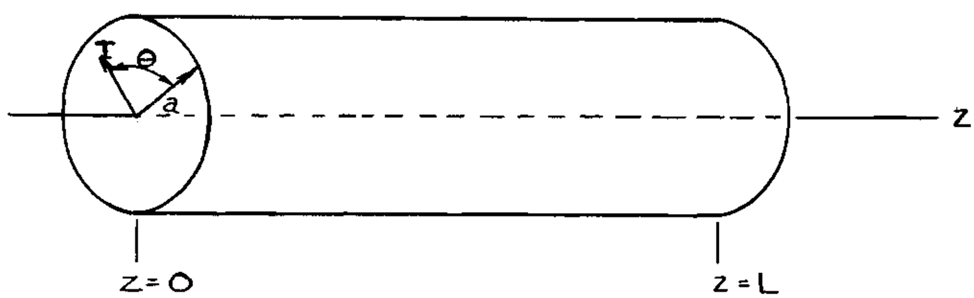

Fig. 1. Orientation of Cylindrical Coordinates to the Pipe

The symbols pertaining to the fluid and the fluid motion are as follows:

(1) ρ denotes mass density, μ denotes dynamic viscosity, and ν (or $\frac{\mu}{\rho}$) denotes kinematic viscosity. These quantities are assumed constant.

(2) v_z, v_r, and v_θ denote components of instantaneous velocity in the z, r, θ space.

(3) $\bar{X}_z$, $\bar{X}_r$, and $\bar{X}_\theta$ denote components of body forces, other than gravity, in the z, r, θ space.

(4) t denotes the time.

(5) p^* denotes the instantaneous piezometric pressure at a point, and is defined by the equation $p^* = p + \rho gh$, in which p is the pressure potential and ρgh is the potential due to gravity. p_1^* and p_2^* refer to piezometric pressures in the pipe inlet and outlet planes respectively. p_o^* denotes the piezometric pressure exerted by a column (of height h_o) of the fluid being considered.

(6) h_o is the vertical distance from the pipe axis at the inlet plane to the surface of the reservoir.

<u>Solution of the Navier-Stokes Equations for an Arbitrary Variation of Pressure Gradient</u>.--For axially symmetric incompressible flow with $v = 0$, the Navier-Stokes equations in cylindrical coordinates [3] are:

$$\bar{X}_r - \frac{1}{\rho}\frac{\partial p^*}{\partial r} + \frac{\mu}{\rho}\nabla^2 v_r - \frac{\mu}{\rho}\frac{v_r}{r} = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\,, \qquad (1)$$

$$\bar{X}_z - \frac{1}{\rho}\frac{\partial p^*}{\partial z} + \frac{\mu}{\rho}\nabla^2 v_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\,. \qquad (2)$$

The equation of continuity is

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(rv_z) = 0. \tag{3}$$

∇^2 denotes the Laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad .$$

The following assumptions are made:

(1) The body forces $\overline{X}_r$, $\overline{X}_z$, and $\overline{X}_\theta$ are zero.

(2) The velocity and pressure are independent of θ.

(3) The radial component of velocity v_r is zero (this assumption in conjunction with assumption (2) is linearizing).

(4) v_z and its partial derivatives appearing in equations (1), (2), and (3) are continuous.

With these assumptions, the equations (1), (2), and (3) simplify to the equation considered by Szymanski [4], namely

$$\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) - \frac{\partial v}{\partial t} = \frac{1}{\rho}\frac{\partial p^*}{\partial z}, \tag{4}$$

where v_z has been denoted by v.

The boundary and initial conditions to accompany equation (4) are determined as follows:

(1) From the theory of flow of a viscous fluid, the tangential component of velocity at a stationary surface has the value zero. Thus

$$r = a, \ v = 0 \text{ for } t \geqq 0. \tag{5}$$

(2) v being independent of θ, and the continuity of $\frac{\partial v}{\partial r}$ require that $\frac{\partial v}{\partial r}$ be zero at the pipe centerline. Thus

$$r = 0, \quad \frac{\partial v}{\partial r} = 0 \quad \text{for} \quad t \geqq 0. \tag{6}$$

(3) The fluid is initially at rest. Thus

$$t = 0, \quad v = 0 \quad \text{for} \quad 0 \leqq r \leqq a. \tag{7}$$

Thus the problem has been reduced to finding the unique solution of equation (4) satisfying the boundary conditions (5) and (6) and the initial condition (7).

Some significant properties of the pressure gradient may be noted during the manipulations yielding equation (4) (see Chapter II, pages 16 and 17). These properties are:

(1) The pressure is a function of z and t only.

(2) The pressure gradient, $\frac{\partial p^*}{\partial z}$, is a function of t only. Thus $\frac{\partial p^*}{\partial z} = f(t)$, in which f(t) is an arbitrary function of time.

(3) The pressure at any instant is a linear function of z.

Since pressure gradient (not pressure level) influences the fluid motion, p_2^* is assigned the value zero and the pressure gradient can be written $\frac{p_1^*}{L}$.

By the change of variable

$$v = u - \frac{1}{\rho} \int_0^t f(\tau)\, d\tau, \tag{8}$$

equation (4) with boundary and initial conditions (5), (6), and (7) is transformed from a non-homogeneous linear partial differential equation with homogeneous boundary and initial conditions to a homogeneous linear partial differential equation with non-homogeneous boundary conditions and a homogeneous initial condition. This new system is the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\rho}{\mu}\frac{\partial u}{\partial t}=0 \tag{9}$$

with the boundary conditions

$$r=a, \quad u=\frac{1}{\rho}\int_0^t f(\tau)\,d\tau \quad \text{for} \quad t \geqq 0, \tag{10}$$

$$r=0, \quad \frac{\partial u}{\partial r}=0 \quad \text{for} \quad t \geqq 0, \tag{11}$$

and the initial condition

$$t=0, \quad u=0 \quad \text{for} \quad 0 \leqq r \leqq a. \tag{12}$$

This system for u is solved by employing the principle of superposition embodied in Duhamel's theorem [5]. First, the boundary condition in equation (10) is replaced by the condition

$$r=a, \quad u=1 \quad \text{for} \quad t \geqq 0. \tag{13}$$

Separation of variables yields a solution of equation (9) satisfying the boundary conditions (13) and (11) and the initial condition (12) of the form

$$u=1-\frac{2}{a}\sum_{n=1}^{\infty} e^{-k_n^2 \mu/\rho\, t}\,\frac{J_o(k_n r)}{k_n J_1(k_n a)}\,. \tag{14}$$

Here J_o and J_1 denote Bessel functions of the first kind of orders zero and one respectively; the positive numbers k_n are determined from roots of the equation $J_o(ka) = 0$. Equation (14) can be shown to be the Szymanski solution for the case where $\frac{\partial p^*}{\partial z}$ has the value unity. Now using superposition, a solution of equation (9) satisfying the boundary conditions (10) and (11) and the initial condition (12) is obtained. Reverting to the variable v by means of equation (8) gives the solution of equation (4) satisfying the boundary conditions (5) and (6) and the initial condition (7) in the form

$$v = -\frac{2}{a} \sum_{n=1}^{\infty} e^{-k_n^2 \frac{\mu}{\rho} t} \frac{J_o(k_n r)}{k_n J_1(k_n a)} \frac{1}{\rho} \int_o^t e^{k_n^2 \frac{\mu}{\rho} \tau} f(\tau)\, d\tau, \tag{15}$$

in which $f(t) = \frac{\partial p^*}{\partial z}$.

<u>Solution Incorporating an Approximation to the Pressure Gradient Change for Reservoir-Pipe Flow</u>.--The time dependence of the pressure gradient associated with the reservoir-pipe flow is needed to complete the solution. In lieu of a better approximation, a linear pressure gradient variation,

$$f(t) = \frac{p_o^*}{L} - \frac{p_o^* - p_{\Omega}^*}{L}\left(\frac{t}{t_{\Omega}}\right), \tag{16}$$

is assumed. Here p_{Ω}^* and t_{Ω} denote conditions at either the onset of turbulence or the cessation of unsteady flow.

Substitution of equation (16) into equation (15) then gives a solution of the simplified Navier-Stokes equation for a linear temporal variation of pressure gradient. This solution is

$$v = \left(\frac{p_o^*}{L}\right)\left(\frac{2}{a\mu}\right)\sum_{n=1}^{\infty} \frac{J_o(k_n r)}{k_n^3 J_1(k_n a)}\left[1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right]$$

$$-\left(\frac{2}{a}\right)\left(\frac{1}{t_n}\right)\left(\frac{p_o^* - p_n^*}{L}\right)\sum_{n=1}^{\infty} \frac{J_o(k_n r)}{k_n^3 J_1(k_n a)}\left[t - \left(\frac{\rho}{\mu}\right)\left(\frac{1}{k_n}\right)^2\left(1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right)\right]. \quad (17)$$

In Chapter II, equation (17) is shown to be a rigorous solution of equation (4) satisfying the boundary conditions (5) and (6) and the initial condition (7).

Numerical Work and Results.--The velocities of equation (17) and Szymanski's solution (corresponding to a particular physical situation which was investigated experimentally by Christopher) have been computed. The constants in equation (17) and Szymanski's solution are evaluated using Christopher's data.[1] To facilitate checking these computations (using the results presented by Szymanski [6] as a reference) the solutions are converted to dimensionless variables defined by

$$\alpha = \frac{\mu}{\rho}\frac{t}{a^2}, \quad (18)$$

$$\beta = \frac{r}{a}, \quad (19)$$

and

$$\bar{v} = v\left[\frac{4L}{p_o^* a^2}\right] \quad (20)$$

(note that $\frac{p_o^* a^2}{4L}$ is the Hagen-Poiseuille velocity at the pipe centerline

[1] This data corresponds to Figure 21, page 55 of Christopher's thesis.

for a pipe of length L and radius a, and with the inlet and outlet pressures p_o^* and zero respectively).

In the Appendix, Figure 2, velocity profiles for this solution, the Szymanski solution, and Crausse's measured data [7] are shown. The profile shape (not velocity level) is to be considered; therefore the velocities shown have been divided by the flow weighted mean of the velocity over the pipe cross-section ($\int_0^1 v\ 2\beta\, d\beta$). Figure 3 is then a comparison of the relative distributions of velocity over the pipe cross-section at a particular instant.

In the Appendix, Figure 3, mean flow velocities as functions of time are shown for three cases: the present solution, the Szymanski solution, and Christopher's measured data. The velocities are presented in terms of the dimensionless parameters of Christopher's thesis. These parameters are

$$\Pi_0 = V/\sqrt{2gh_o}\ , \tag{21}$$

$$\Pi_1 = L/2a\ , \tag{22}$$

$$\Pi_2 = \frac{\sqrt{gh_o}}{L^2}\, t\ , \tag{23}$$

and

$$\Pi_3 = \frac{gh_o a^3}{8L\nu^2}\ , \tag{24}$$

in which $V = \int_0^1 v\ 2\beta\, d\beta$.

Table 1 of the Appendix contains terms involved in the computation of the first nine elements of the series

$$\sum_{n=1}^{\infty} \frac{J_o(k_n \beta)}{k_n^3 J_1(k_n)} \qquad (25)$$

which are in both the Szymanski solution and the dimensionless form of equation (17).

CHAPTER II

THE MATHEMATICAL DETAILS OF THE SOLUTION

In this chapter certain mathematical details of the problem are examined. First, the simplification of the Navier-Stokes and continuity equations is explained. Second, the simplified equation is solved without rigorous justification of the processes involved. Third, this solution is shown to be a formal solution of the simplified equation and to satisfy the appropriate boundary and initial conditions. Fourth, this solution is shown to be an actual solution. Finally, the uniqueness of the solution is established.

Simplification of the Navier-Stokes and Continuity Equations.-- The Navier-Stokes and continuity equations for axially symmetric, incompressible flow with $v_\Theta = 0$ are

$$\bar{X}_r - \frac{1}{\rho}\frac{\partial p^*}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \Theta^2} + \frac{\partial^2 v_r}{\partial z^2}\right) - \frac{\mu}{\rho}\frac{v_r}{r}$$

$$= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z}\,, \tag{1}$$

$$\bar{X}_z - \frac{1}{\rho}\frac{\partial {}^*p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \Theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right.$$

$$= \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\,, \tag{2}$$

and

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(rv_z) = 0. \tag{3}$$

The following assumptions are employed to simplify the above equations.

(1) $\bar{X}_r$, $\bar{X}_z$, and v_r are zero.

(2) v_r, v_z, and p^* are independent of θ.

(3) v_r, v_z, and their derivatives appearing in equations (1), (2), and (3) are continuous.

(4) The terms involving $\frac{1}{r}$ and $\frac{1}{r^2}$ are continuous at $r = 0$.

(5) ρ and μ are constants.

Furthermore, the equations (1), (2), and (3) are considered only in the region $0 \leq r \leq a$, $0 \leq z \leq L$, and $t \geq 0$.

For equation (1), $v_r = 0$ (assumption 1) implies that $\frac{\partial v_r}{\partial r} = 0$, $\frac{\partial^2 v_r}{\partial r^2} = 0$, and $\frac{\partial^2 v_r}{\partial z^2} = 0$. Then $\frac{1}{r}\frac{\partial v_r}{\partial r} = 0$ for $r \neq 0$ and, with this term continuous (assumption 4), $\frac{1}{r}\frac{\partial v_r}{\partial r} = 0$ for all r. Since v_r is independent of θ (assumption 2), $\frac{\partial^2 v_r}{\partial \theta^2} = 0$. Equation (1) then reduces to

$$\frac{\partial p^*}{\partial r} = 0, \tag{26}$$

implying that p^* is independent of r as well as of θ. Thus

$$p^* = p^*(z, t). \tag{27}$$

Equation (3) reduces to $r\frac{\partial v_z}{\partial z} = 0$ (under assumption 1), and therefore $\frac{\partial v_z}{\partial z} = 0$ for $r \neq 0$. But $\frac{\partial v_z}{\partial z}$ is continuous (assumption 3) so that

$\frac{\partial v}{\partial z} = 0$ at $r = 0$. Thus equation (3) becomes

$$\frac{\partial v_z}{\partial z} = 0, \tag{28}$$

and v_z is independent of z as well as of Θ. Thus

$$v_z = v_z(r, t). \tag{29}$$

For equation (2), since v_z is independent of Θ, $\frac{\partial^2 v_z}{\partial \Theta^2} = \frac{\partial v_z}{\partial \Theta} = 0$. Then, with assumption 4, $\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \Theta^2} = 0$ everywhere; and equation (28) demands that $\frac{\partial^2 v_z}{\partial z^2} = 0$. Thus equation (2), reduced and rearranged, becomes

$$\frac{\mu}{\rho}\left(\frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) - \frac{\partial v_z}{\partial t} = \frac{1}{\rho}\frac{\partial p^*}{\partial z}\,. \tag{30}$$

Now according to equations (27) and (29), the left member of equation (30) is a function of r and t, and the right member is a function of z and t. The equality is to be valid for all r, z, and t; therefore both members of equation (30) must be functions of t only. Thus $\frac{\partial p^*}{\partial z} = f(t)$, where $f(t)$ is an arbitrary function of time, and equation (30) may be written

$$\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) - \frac{\partial v}{\partial t} = \frac{1}{\rho} f(t), \tag{31}$$

where v denotes v_z.

Integration of $\frac{\partial p^*}{\partial z} = f(t)$ gives

$$p^* = z\, f(t) + g(t) \tag{32}$$

($g(t)$ is another arbitrary function of time), and for a given t, p^* must vary linearly with z. Furthermore, since only values of the partial derivative $\frac{\partial p^*}{\partial z}$ (not values of p^*) directly influence the equations (26) and (31), p^* may be assigned the value zero at $z = L$ and

$$p^* = f(t)\ (z - L). \tag{33}$$

<u>Formal Solution of the Simplified Equation</u>.--The formal solution of equation (31) (equivalent to equation (4) of Chapter I), satisfying the boundary conditions (5) and (6) and the initial condition (7), will be obtained by using the equations in the variable u, where u is defined by equation (8) of Chapter I. In the variable u, the differential equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\rho}{\mu}\frac{\partial u}{\partial t} = 0; \tag{9}$$

the boundary conditions become

$$r = a, \quad u = \frac{1}{\rho}\int_0^t f(\tau)\, d\tau \quad \text{for} \quad t \geq 0; \tag{10}$$

$$r = 0, \quad \frac{\partial u}{\partial r} = 0 \quad \text{for} \quad t \geq 0; \tag{11}$$

and the initial condition becomes

$$t = 0, \quad u = 0 \quad \text{for} \quad 0 \leq r \leq a. \tag{12}$$

The separation of variables technique is applied by assuming the existence of a solution of equation (9) in the form

$$u = \Big[M(t)\Big]\Big[N(r)\Big]. \tag{34}$$

Here M (differentiable) and N (twice differentiable) are functions to be determined. With the derivatives of u obtained from equation (34), equation (9) can be written in the form

$$\frac{N''}{N} + \frac{1}{r}\frac{N'}{N} = \frac{\rho}{\mu}\frac{M'}{M} . \tag{35}$$

The left member of equation (35) is a function of r only, and the right member is a function of t only; therefore, the equality can hold for all r and t under consideration if and only if both members are equal to a constant.

If this separation constant were positive, solutions for M(t) would arise which become infinite as t increases and require that u and v become infinite as t increases. The steady flow solution for v is known to be finite; therefore, the separation constant cannot be positive. If the separation constant were complex, M(t) would have complex solutions and u would have complex solutions. Only real solutions are of interest here. If the separation constant were zero, M(t) would have solutions of the form $M(t) = A$, where A is an arbitrary constant, and N(r) would have solutions of the form $N = F + G \log r$ where F and G are arbitrary constants. If the separation constant were negative, solutions for M(t) would have the form

$$M(t) = B\, e^{-k^2 \frac{\mu}{\rho} t} ; \tag{36}$$

and solutions for N(r) would have the form

$$N(r) = D\, J_o(kr) + E\, Y_o(kr), \tag{37}$$

where J_o and Y_o denote Bessel functions of order zero of the first and second kinds, respectively, and D and E denote arbitrary constants. Since log r and $Y_o(kr)$ do not remain finite as $r \to 0$, they cannot be included as terms of solutions for u if u is to be continuous on $0 \leq r \leq a$, and therefore the constants G and E are chosen zero. Thus the real, finite separable solutions of equation (9) are of the form

$$u = b + c\ J_o(kr)\ e^{-k^2 \frac{\mu}{\rho} t}, \tag{38}$$

where the constants b and c and the separation constant k are to be determined in such a way that the boundary conditions (10) and (11) and the initial condition (12) are satisfied.

To determine the constants b, c, and k, the boundary condition (10) is first replaced by the condition

$$r = a, \quad u(r,\ t) = 1 \quad \text{for} \quad t \geq 0 \tag{39}$$

(the remaining conditions, (11) and (12), are unchanged). Boundary condition (11) is satisfied by any solution of the form in equation (38), for since $\frac{\partial u(0,\ t)}{\partial r} = J_1(0)\ e^{-k^2 \frac{\mu}{\rho} t}$ and since $J_1(0) = 0$, $\frac{\partial u}{\partial r}(0,\ t) = 0$ for $t \geq 0$. Boundary condition (58) requires that

$$1 = b + c\ J_o(ka)\ e^{-k^2 \frac{\mu}{\rho} t}. \tag{40}$$

This condition will be satisfied if

$$b = 1 \tag{41}$$

and

$$J_o(ka) = 0. \tag{42}$$

It is well known that the function $J_o(x)$ has a countable infinity of zeros. These zeros may be designated by γ_n, $n = 1, 2, \ldots$, etc. Since $J_o(-x) = J_o(x)$ and $J_o(0) = 1 > 0$, only positive values of γ will be considered. Then the admissible values of the constant k are chosen according to the relation

$$k_n = \frac{\gamma_n}{a}, \tag{43}$$

where $n = 1, 2, \ldots$, etc. Thus

$$u_n(r, t) = 1 + C_n\, J_o(k_n r)\, e^{-k_n^2 \frac{\mu}{\rho} t},$$

$n = 1, 2, \ldots$, etc., are formal solutions of equation (7) satisfying boundary conditions (11) and (39). Since equation (9) is linear,

$$u(r, t) = 1 + \sum_{n=1}^{\infty} C_n\, J_o(k_n r)\, e^{-k_n^2 \frac{\mu}{\rho} t} \tag{44}$$

is also a formal solution of equation (9) satisfying conditions (11) and (39).

The initial condition, equation (12) requires that

$$\sum_{n=1}^{\infty} C_n\, J_o(k_n r) = -1. \tag{45}$$

This determines the coefficients C_n as the Fourier-Bessel expansion coefficients of the function -1 [8]. These coefficients are, formally,

$$C_n = -\frac{2}{(k_n a)\, J_1(k_n a)} \tag{46}$$

where $J_1(k_n a) \neq 0$. Thus the solution (44) becomes

$$u(r, t) = 1 - \frac{2}{a} \sum_{n=1}^{\infty} e^{-k_n^2 \frac{\mu}{\rho} t} \frac{J_0(k_n r)}{k_n J_1(k_n a)} . \tag{47}$$

To solve the homogeneous equation with its original boundary and initial conditions, superposition will be used. If $u = H(r, t)$ is a solution of equation (9) with the boundary condition of equation (39), then by the linearity of equation (9), $u = [Q][H(r, t)]$ is a solution of equation (9) for the condition

$$r = a, \quad u = Q \quad \text{for} \quad t \geq 0,$$

where Q is a constant. Let $H(r, t)$ be defined to be zero for $t < 0$. At $r = a$, let a function $q(t)$ (arbitrary for the present) be approximated on the interval $0 \leq t \leq T$ by the function $\omega(t)$ in the following manner:

$$0 = T_0 \leq t \leq T_1, \quad \omega(t) = q(T_0) = q(0);$$

$$T_1 \leq t \leq T_2, \quad \omega(t) = q(T_1);$$

$$\ldots \text{ etc. } \ldots$$

$$T_{\wedge -1} \leq t \leq T_{\wedge} = T, \quad \omega(t) = q(T_{\wedge -1});$$

where $0 = T_0 < T_1 < T_2 \ldots < T_m \ldots < T_{\wedge -1} < T_{\wedge} = T$. Since equation (9) is linear, with condition (10) replaced by

$$r = a, \quad u = \omega(t) \quad \text{for} \quad t \geq 0, \tag{48}$$

a solution of equation (9) at the time T will be

$$u(r, T) \simeq q(0) \Big[H(r, T)\Big] \quad \sum_{m=1}^{\Lambda} \Big[q(T_m) - q(T_{m-1})\Big]\Big[H(r, T - T_m)\Big]. \qquad (49)$$

Now, if $q(t)$ is continuous on $0 \leq t \leq T$ and differentiable on $0 < t < T$, then by the mean value theorem of the differential calculus

$$\Big[q(T_m) - q(T_{m-1})\Big] = q'(\xi_m)\Big[(T_m - T_{m-1})\Big], \qquad (50)$$

where $T_m < \xi_m < T_{m-1}$. Substituting equation (50) into equation (49) gives

$$u(r,T) = q(0)\Big[H(r,T)\Big] \quad \sum_{m=1}^{\Lambda} q'(\xi_m) \Big[(T_m - T_{m-1})\Big]\Big[H(r,T - T_m)\Big] \qquad (51)$$

where $T_{m-1} < \xi_m < T_m$. By Bliss's theorem [9],

$$\lim_{\|T_m - T_{m-1}\| \to 0} \quad \sum_{m=1}^{\Lambda} q'(\xi_m) \Big[(T_m - T_{m-1})\Big]\Big[H(r, T - T_m)\Big]$$

$$= \int_0^T q'(\tau) \; H(r, T - \tau) \; d\tau, \qquad (52)$$

for $q(t)$ and $H(r, T - t)$ continuous on $0 \leq t \leq T$. Here $\|T_m - T_{m-1}\|$ denotes the value of the maximum of the numbers $(T_1 - T_o)$, $(T_2 - T_1)$, ..., $(T_\Lambda - T_{\Lambda-1})$; and as $\|T_m - T_{m-1}\| \to 0$, $\Lambda \to \infty$. Replacing T by the arbitrary value t, and taking the limit of (49) as $\|T_m - T_{m-1}\| \to 0$, gives the formal solution

$$u(r, t) = q(0) \Big[H(r, t)\Big] \int_0^t q'(\tau) \Big[H(r, t - \tau)\Big] d\tau. \qquad (53)$$

Relating the preceding concept to the present problem, $H(r, t)$ is identified with the solution (47) and $q(t)$ with $-\frac{1}{\rho}\int_0^t f(\tau)\, d\tau$. Thus the formal solution of equation (9) satisfying the boundary conditions (10) and (11) and the initial condition (12) is

$$u(r, t) = \frac{1}{\rho}\int_0^t f(\tau)\, d\tau$$

$$- \frac{2}{a}\sum_{n=1}^{\infty} \frac{J_0(k_n r)}{k_n\, J_1(k_n a)} \left(e^{-k_n^2 \frac{\mu}{\rho} t}\right) \frac{1}{\rho}\int_0^t e^{k_n^2 \frac{\mu}{\rho}\tau} f(\tau)\, d\tau. \qquad (54)$$

Note that in passing from (53) to (54) summation and integration are interchanged. Reverting to the variable v and rearranging gives a solution that should satisfy equation (4) and the conditions (5), (6), and (7). This solution is

$$v = - \frac{2}{a}\sum_{n=1}^{\infty} e^{-k_n^2 \frac{\mu}{\rho} t}\, \frac{J_0(k_n r)}{k_n\, J_1(k_n a)}\, \frac{1}{\rho}\int_0^t e^{k_n^2 \frac{\mu}{\rho}\tau} f(\tau)\, d\tau \qquad (15)$$

(this equation is identical with the one shown by Carslaw and Jaeger [10] for heat flow in an infinite rod).

In obtaining equation (15), all questions of vigor were ignored. Thus (15) must yet be shown to be an actual solution of equation (4) satisfying the conditions (5), (6), and (7); and the uniqueness of the solution must also be established.

<u>Formal Verification of the Solution for a Linear Variation of Pressure Gradient</u>.--The assumed linear pressure gradient change and the solution involving an arbitrary pressure gradient change are

$$f(t) = -\left(\frac{p_o^*}{L}\right) + \left(\frac{p_o^* - p_\Omega^*}{L}\right)\left(\frac{t}{t_\Omega}\right) \qquad (16)$$

and

$$v = -\frac{2}{a}\sum_{n=1}^{\infty} e^{-k_n^2 \frac{\mu}{\rho} t} \frac{J_o(k_n r)}{k_n\ J_1(k_n a)} \frac{1}{\rho}\int_o^t e^{k_n^2 \frac{\mu}{\rho}\tau} f(\tau)\ d\tau. \tag{15}$$

Substitution of equation (16) into equation (15) gives, upon integration and algebraic modification,

$$v = \left\{\left(\frac{p_o^*}{L}\right)\left(\frac{2}{a\mu}\right)\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n^3\ J_1(k_n a)}\left[1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right]\right\}$$

$$-\left\{\left(\frac{2}{a\mu}\right)\left(\frac{1}{t_\Omega}\right)\left(\frac{p_o^* - p_\Omega^*}{L}\right)\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n^3 J_1(k_n a)}\left[t - \left(\frac{\rho}{\mu}\right)\left(\frac{1}{k_n}\right)^2\left(1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right)\right]\right\}. \tag{17}$$

It will now be shown that equation (17) formally satisfies equation (4) with the conditions (5), (6), and (7).

The derivative relations

$$\frac{d}{dr} J_o(k_n r) = -k_n\ J_1(k_n r)$$

and

$$\frac{d}{dr} J_1(k_n r) = k_n\ J_o(k_n r) - \frac{1}{r} J_1(k_n r) \tag{55}$$

are known properties of the Bessel functions. Using these derivatives to differentiate equation (26) gives formally,

$$\frac{\partial v}{\partial r} = \left\{\left(\frac{p_o^*}{L}\right)\left(\frac{2}{a\mu}\right)\sum_{n=1}^{\infty}\frac{J_1(k_n r)}{k_n^2\ J_1(k_n a)}\left[e^{-k_n^2 \frac{\mu}{\rho} t} - 1\right]\right\}$$

$$+\left\{\left(\frac{2}{a\mu}\right)\left(\frac{1}{t_\Omega}\right)\left(\frac{p_o^* - p_\Omega^*}{L}\right)\sum_{n=1}^{\infty}\frac{J_1(k_n r)}{k_n^2 J_1(k_n a)}\left[t - \left(\frac{\rho}{\mu}\right)\left(\frac{1}{k_n}\right)^2\left(1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right)\right]\right\}, \tag{56}$$

and

$$\frac{\partial^2 v}{\partial r^2} = \left\{\left(\frac{p_o^*}{L}\right)\left(\frac{2}{a\mu}\right)\sum_{n=1}^{\infty}\frac{1}{k_n^2 J_1(k_n a)}\left[k_n J_o(k_n r) - \frac{1}{r} J_1(k_n r)\right]\left[e^{-k_n^2 \frac{\mu}{\rho} t} - 1\right]\right\}$$

$$+\left\{\left(\frac{2}{a\mu}\right)\left(\frac{1}{t_\Omega}\right)\left(\frac{p_o^* - p_\Omega^*}{L}\right)\sum_{n=1}^{\infty}\frac{1}{k_n^2 J_1(k_n a)}\left[k_n J_o(k_n r) - \frac{1}{r} J_1(k_n r)\right]\right.$$

$$\left.\cdot\left[t - \left(\frac{\rho}{\mu}\right)\left(\frac{1}{k_n}\right)^2\left(1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right)\right]\right\}, \tag{57}$$

and

$$\frac{\partial v}{\partial t} = \left\{\left(\frac{p_o^*}{L}\right)\left(\frac{2}{a\mu}\right)\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n^3 J_1(k_n a)}\left[\left(k_n^2\right)\left(\frac{\mu}{\rho}\right)\left(e^{-k_n^2 \frac{\mu}{\rho} t}\right)\right]\right\}$$

$$-\left\{\left(\frac{2}{a\mu}\right)\left(\frac{1}{t_\Omega}\right)\left(\frac{p_o^* - p_\Omega^*}{L}\right)\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n^3 J_1(k_n a)}\left[1 - e^{-k_n^2 \frac{\mu}{\rho} t}\right]\right\}. \tag{58}$$

Substitution of equations (56), (57), and (58) into equation (4) gives, upon algebraic reduction,

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n J_1(k_n a)}\left(\frac{1}{\rho}\right)\left[-\left(\frac{p_o^*}{L}\right)+\left(\frac{p_o^* - p_\Omega^*}{L}\right)\left(\frac{t}{t_\Omega}\right)\right]$$

$$= \left(\frac{1}{\rho}\right)\left[-\left(\frac{p_o^*}{L}\right)+\left(\frac{p_o^* - p_\Omega^*}{L}\right)\left(\frac{t}{t_\Omega}\right)\right]. \tag{59}$$

With the assumption that $f(t)$ of equation (16) is never zero, equation (59) becomes

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n\ J_1(k_n a)} = 1. \tag{60}$$

The boundary and initial conditions (equations (5) and (6) and equation (7)) are easily shown to be satisfied by equation (15), and thus equation (15) formally satisfies equation (4) with conditions (5), (6), and (7) provided equation (60) is satisfied. The formal procedure used above will be verified in the next section.

Rigorous Verification of the Formal Solution.--For the solution, equation (15), to be an actual solution on a region in r and t,

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_o(k_n r)}{k_n\ J_1(k_n a)} = 1$$

must hold on that region and the differentiation involved in obtaining $\frac{\partial v}{\partial r}$, $\frac{\partial^2 v}{\partial r^2}$, and $\frac{\partial v}{\partial t}$ must be meaningful on that region. Most of the issues encountered in verifying that equation (15) is a rigorous solution have been discussed by Szymanski in the verification of his solutions. Szymanski's results will be used wherever possible.

To establish the convergence properties of certain series, Szymanski cited the following theorems:

Theorem A. Every function, $\phi(r)$, with its first two derivatives continuous in the interval (0, 1) and vanishing for r = 1 can be expanded in a uniformly and absolutely convergent series of the form

$$\sum_{n=1}^{\infty} C_n \, J_o(k_n r)$$

where

$$C_n = \frac{1}{\int_o^1 r \, J(k_n r) \, dr} \int_o^1 r \, \phi(r) \, J(k_n r) \, dr.$$

Theorem B. If the function $\sqrt{r}\,\phi(r)$ is piecewise continuous in the interval (0, 1), the series of Theorem A converges to the value

$$\frac{1}{2}\left[\phi(r-0) + \phi(r+0)\right]$$

for every point r in the interval (0, 1) in the vicinity of which $\phi(r)$ is of bounded variation. If, moreover, the function $\phi(r)$ is continuous in an interval (a, b) which in turn is contained in another interval (α, β) where the function is of bounded variation, the series converges uniformly to $\phi(r)$ in the interval (a, b).

Corollary to Theorem B. If for all points in the vicinity of the point r = 0, the conditions of Theorem B are fulfilled, and if the function $\phi(r)$ is continuous at the point $r = 0$ and the series of Theorem A is uniformly convergent near $r = 0$, this series converges to the value $\phi(r)$ at the point $r = 0$.

Using the preceding theorems, Szymanski noted that the series

$$\sum_{n=1}^{\infty} \frac{J_o(k_n r)}{a \, k_n \, J_1(k_n a)}$$

is the expansion of $\phi(r) = 1/2$ in a series of the form in Theorem A, and that this series converges to the value 1/2 for $0 \leq r \leq a$. Thus

$$2 \sum_{n=1}^{\infty} \frac{J_o(k_n r)}{a\ k_n\ J_1(k_n a)} = 1$$

except at $r = a$. At $r = a$, since each term of the series is zero, the series converges to the value zero.

By a well known theorem [11] let a series, whose terms are functions of a single variable having a continuous derivative on some interval, converge on this interval. Also let a new series, formed by differentiating the original series term by term, converge uniformly on this interval. Then the derivative of the sum of the original series is equal to the sum of the derived series. In the preceding section, the derivatives $\frac{\partial v}{\partial r}$, $\frac{\partial^2 v}{\partial r^2}$, and $\frac{\partial v}{\partial t}$ were obtained formally. If for these series the variable not involved in the differentiation is considered constant, the above theorem may be applied. Then the theorem may be applied for all values of the variable not involved in the differentiation.

Thus it is necessary to consider the uniform convergence of

$$\sum_{n=1}^{\infty} \frac{J_o(k_n r)}{(a\ k_n)^3\ J_1(k_n a)}\, e^{-k_n^2 \frac{\mu}{\rho} t} \tag{61}$$

and

$$\sum_{n=1}^{\infty} \frac{J_o(k_n r)}{(a\ k_n)\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t} \tag{62}$$

with respect to r and with respect to t, and also to consider the uniform convergence of

$$\sum_{n=1}^{\infty} \frac{J_1(k_n r)}{(a\ k_n)^2\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t}\ , \tag{63}$$

$$\sum_{n=1}^{\infty} \frac{J_1(k_n r)}{(a\ k_n)^3\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t}\ , \tag{64}$$

and

$$\sum_{n=1}^{\infty} \frac{J_1(k_n r)}{(a\ k_n)^4\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t} \tag{65}$$

with respect to r. Szymanski has shown that series (61) converges uniformly and absolutely for $0 \leq r \leq a$ and for $0 \leq t \leq T$ where T is arbitrary, and that series (62) converges uniformly over the same closed region except for the point $r = a$, $t = 0$.

Szymanski has also shown that for sufficiently large n,

$$\left| \frac{J_1(k_n r)}{(a\ k_n)^3\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t} \right| \leq 2\sqrt{\pi}\ \frac{1}{(k_n^{5/2})}\ ; \tag{66}$$

and since $\sum_{n=1}^{\infty} \frac{1}{(k_n^{5/2})}$ converges, the series (64) converges uniformly and absolutely with respect to r and t. It is a simple matter to extend this reasoning to the series (63) and (65). For multiplying equation (66) by ($a\ k_n$) gives, for sufficiently large n,

$$\left| \frac{J_1(k_n r)}{(a\ k_n)^2\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t} \right| \leq 2\sqrt{\pi}\ \frac{1}{(a\ k_n)^{3/2}}\ ; \tag{67}$$

and multiplying equation (66) by $(\frac{1}{a\ k_n})$ gives, for sufficiently large n,

$$\left| \frac{J_1(k_n r)}{(a\ k_n)^4\ J_1(k_n a)}\ e^{-k_n^2 \frac{\mu}{\rho} t} \right| \leq 2\sqrt{\pi}\ \frac{1}{(a\ k_n)^{7/2}}\ . \tag{68}$$

As Szymanski shows,

$$(a\ k_n) = \left[n\pi - \frac{\pi}{4} + \frac{1}{8\pi n} + O(\frac{1}{n^2}) \right]$$

(where $O(\frac{1}{n^2})$ represents a function which vanishes for increasing n to the same order as $(\frac{1}{n^2})$). Also, for positive n,

$$\left[n\pi - \frac{\pi}{4} + \frac{1}{8\pi n} + O(\frac{1}{n^2}) \right] > \left[3n - 1 \right] > \left[3n - n \right] > \left[n \right]\ ;$$

then $a\ k_n > n$ and $\frac{1}{a\ k_n} < \frac{1}{n}$. Thus $\frac{1}{(a\ k_n)^{7/2}} < \frac{1}{(a\ k_n)^{3/2}} < \frac{1}{n^{3/2}}$, and the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ implies uniform and absolute convergence of series (63) and (65).

Now define equation (17) to be a solution at $r = 0$ if $v(0) = \lim_{r \to 0} v$. Equation (17) consists of terms involving series (62) and (63) which are both uniformly convergent on $0 \leq r \leq b < a$. Then v defined by equation (17) is continuous at $r = 0$ and, according to the above definition, is a solution at $r = 0$.

Thus for $t \geq 0$ and $0 \leq r \leq a$, equation (17) has been shown to be an actual solution of the equation (4) with the boundary conditions (5) and (6) and the initial condition (7).

Uniqueness of the Solution.--To establish the uniqueness of a solution of equation (4) satisfying boundary conditions (5) and (6) and initial condition (7), it clearly suffices to consider uniqueness of the solution to the problem represented by equation (9) and conditions (10), (11), and (12).

Let u_1 and u_2 be two solutions of equation (9) satisfying conditions (10), (11), and (12). Then because equation (9) is linear, the difference $u = u_2 - u_1$ is also a continuous solution. The solution u has the following properties: at $r = a$, $u = u_2(a, t) - u_1(a, t) = 0$. At $r = 0$, $\frac{\partial u}{\partial r} = \frac{\partial}{\partial r} u_2(0, t) - \frac{\partial}{\partial r} u_1(0, t) = 0$. At $t = 0$, $u = u_2(r, 0) - u_1(r, 0) = 0$.

Multiplying equation (9) by r and u gives

$$ru \frac{\partial^2 u}{\partial r^2} + u \frac{\partial u}{\partial r} - \frac{\rho}{\mu} ru \frac{\partial u}{\partial t} = 0 \;, \tag{69}$$

and integration of this equation over the region $0 \leq r \leq a$, $0 \leq t \leq T$ (here T denotes an arbitrary value of t) gives

$$\int_0^T \int_0^a \left[ru \frac{\partial^2 u}{\partial r^2} + u \frac{\partial u}{\partial r} - \frac{\rho}{\mu} ru \frac{\partial u}{\partial t} \right] dr\ dt = 0 \ . \tag{70}$$

Now

$$\frac{\partial}{\partial r} \left[u \frac{\partial u}{\partial r} \right] = u \frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial u}{\partial r}\right)^2 \tag{71}$$

and

$$u \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (u^2) \ . \tag{72}$$

Substituting for $u \frac{\partial^2 u}{\partial r^2}$ and $u \frac{\partial u}{\partial t}$ in equation (70) their values from equations (71) and (72) gives

$$\int_0^T \int_0^a r \frac{\partial}{\partial r} \left(u \frac{\partial u}{\partial t}\right) dr\ dt + \int_0^T \int_0^a u \frac{\partial u}{\partial r}\ dr\ dt$$

$$= \int_0^T \int_0^a r \left(\frac{\partial u}{\partial r}\right)^2 dr\ dt + \frac{\rho}{2\mu} \int_0^T \int_0^a r \frac{\partial}{\partial t} (u^2)\ dr\ dt \ . \tag{73}$$

Since $\frac{\partial u}{\partial t}$ is continuous, the order of integration may be interchanged and

$$\frac{\rho}{2\mu}\int_0^T\int_0^a r\,\frac{\partial}{\partial t}(u^2)\,dr\,dt = \frac{\rho}{2\mu}\int_0^a r\left[u(r,\ T)\right]^2 dr.$$

Also, integration by parts with respect to r in the first integral of equation (73) gives

$$\int_0^T\int_0^a \left[r\,\frac{\partial}{\partial r}\left(u\,\frac{\partial u}{\partial r}\right)\right] dr\,dt$$

$$= \int_0^T \left\{\left(ru\,\frac{\partial u}{\partial r}\right)\Big|_0^a\right\} dt - \int_0^T\int_0^a u\,\frac{\partial u}{\partial r}\,dr\,dt.$$

But at $r = a$, $u = 0$ and at $r = 0$, $\frac{\partial u}{\partial r} = 0$; so that $\int_0^T \left\{\left(ru\,\frac{\partial u}{\partial r}\right)\Big|_0^a\right\} dt = 0.$

These results permit reducing equation (73) to

$$\int_0^T\int_0^a r\left(\frac{\partial u}{\partial r}\right)^2 dr\,dt + \frac{\rho}{2\mu}\int_0^a r\left[u(r,\ T)\right]^2 dr = 0\ . \tag{74}$$

Both integrands of equation (74) are non-negative and continuous and $\frac{\rho}{2\mu}$ is non-negative. Therefore, both integrands must be identically zero for the equality to hold. Furthermore, since r is not identically zero on the interval of integration and $\frac{\partial u}{\partial r}$ is continuous, $r\left(\frac{\partial u}{\partial r}\right)^2 = 0$ implies that $\frac{\partial u}{\partial r} = 0$ and hence $u(r,\ t)$ is constant for $0 \leq r \leq a$, $0 \leq t \leq T$. The second integrand being zero and $u(r,\ t)$ being continuous imply that $u(r,\ T) \equiv 0$ for $0 \leq r \leq a$, $0 \leq t \leq T$. Since T is an arbitrary value of t, $u(r, t) \equiv 0$, and hence $u_2 \equiv u_1$. Violation of the condition that $u_2 \equiv u_1$ would imply a contradiction of equation (74), and uniqueness is therefore established.

DISCUSSION OF RESULTS

The assumption that the radial component of velocity is zero appears to have drastic effects on the range of applicability of the solution. In the first place this assumption immediately prohibits accounting for inlet effects. This assumption, with the other assumptions, also requires that the instantaneous velocity profile at any two cross sections of the pipe be the same. Thus, at any instant, the boundary layer thickness must be constant and boundary layer growth with distance alone is not possible.

For the comparison of velocity profile shapes (Figure 2), four curves are shown. Two of these were obtained from experimental data by Crausse (who photographed aluminum particles suspended in water flowing through a transparent tube section). The curves from Crausse's data represent conditions at two different times, one time being larger and one smaller than the time corresponding to the two remaining curves. The two remaining curves represent the Szymanski solution for a constant pressure gradient and the present solution for a linear temporal change of pressure gradient (the constants for the Szymanski solution and the constants - including conditions at the onset of turbulence - for the present solution were obtained from run number six of Christopher's data.

The shape of the velocity profile obtained from the present solution is almost identical to the shape of the velocity profile from Szymanski's solution, at least for the particular case shown. Since for this particular physical situation the inlet pressure at the onset of turbulence is about six tenths of the starting inlet pressure, it is unlikely that the velocity profiles differ significantly for any of the physical situations investigated by Christopher.

Neither the present velocity profile nor the Szymanski velocity profile shows the extent of boundary layer development indicated by Crausse's data.

For the comparison of mean flow accelerations (Figure 3), three curves are shown. Christopher's data for run number six is used for one curve. The constants associated with this run are used to compute the values for the other two curves, namely the curve for the Szymanski solution using a constant pressure gradient and the curve for the present solution using a linear temporal variation of pressure gradient. It appears that the validity of the present solution could be significantly improved by making a non-linear approximation to the pressure gradient.

The Table of the Appendix contains computations for the first nine terms of the series

$$\sum_{n=1}^{\infty} \frac{J_o\,(k_n \beta)}{(k_n)^3\; J_1\,(k_n)}$$

at the stations β = 0.1, 0.2, 0.3, ..., 0.9. With constants and exponential terms for a specific physical situation evaluated, these numbers may then be used to calculate values of the present solution and the Szymanski solution.

Errors may occur in the last figure of each column. With $(k_n\beta)$ given to five decimal places, $J_o\ (k_n\beta)$ and $J_1\ (k_n\beta)$ may have errors of $\pm$ .00003 and the succeeding columns will be subject to error from this source.

CONCLUSIONS

1. Subject to assumptions, a solution is obtained for viscous, incompressible flow in a straight pipe within which the pressure gradient varies linearly with time.

2. This solution cannot account for inlet effects and cannot account for a boundary layer growth with distance along the pipe.

3. The velocity profile shape implied by this solution is not consistent with experimental data (at least for the comparison presented).

4. This solution, using a linear temporal variation of pressure gradient, gives more realistic mean velocity near the onset of turbulence and less realistic mean velocity at the beginning of flow than the Szymanski solution using a constant pressure gradient (at least for the comparison presented).

APPENDIX

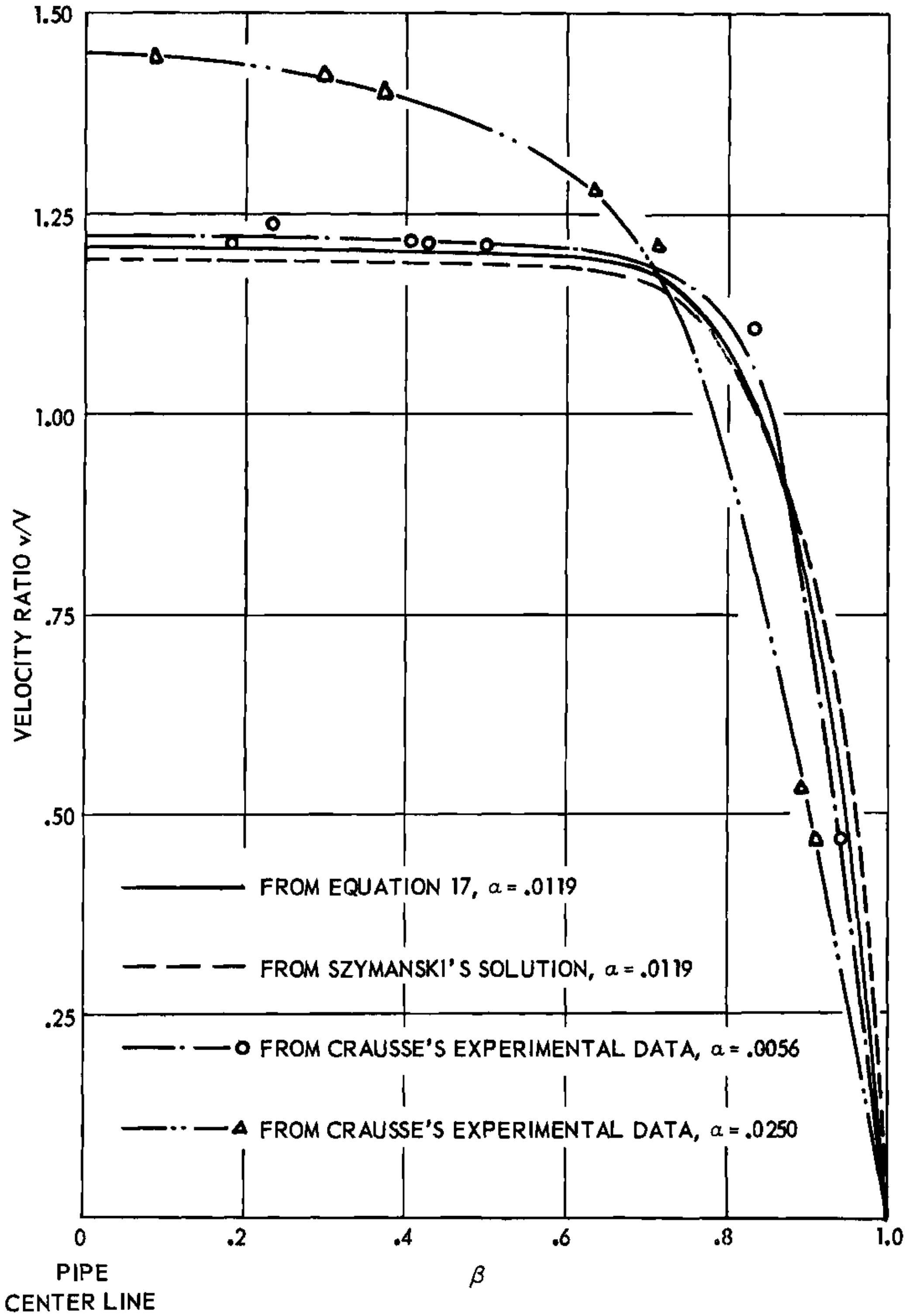

Figure 2. Comparison of Velocity Profile Shapes.

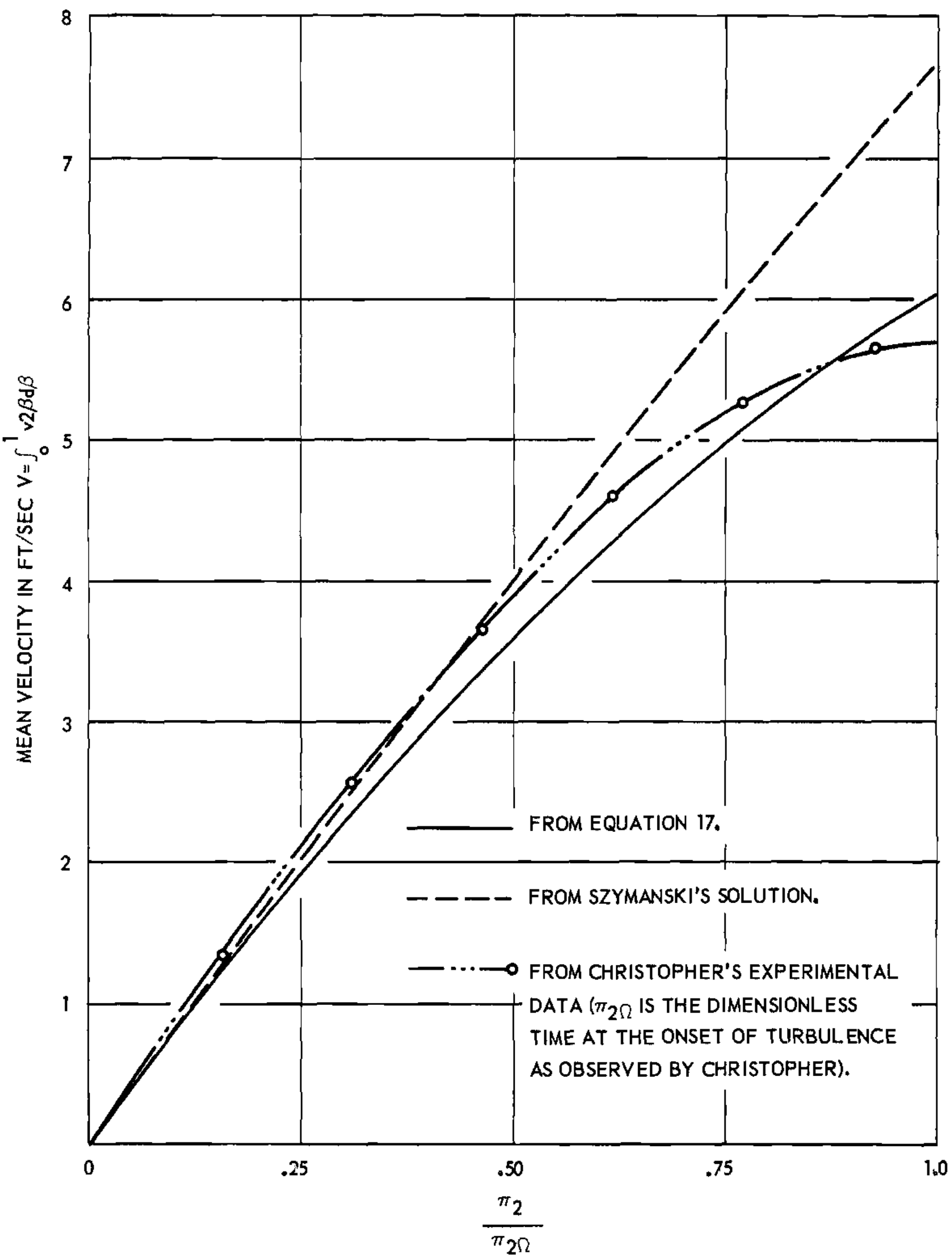

Figure 3. Comparison of Mean Flow Velocity versus Time.

TABLE

Computation of Nine Terms of the Series $\sum_{n=1}^{\infty} \frac{J_o(k_n\beta)}{(k_n)^3 J_1(k_n)}$

n	k_n	$(k_n)^2$	$(k_n)^3$	$J_1(k_n)$	$\frac{J_o(0)}{(k_n)^3 J_1(k_n)}$
1	+ 2.4048	+ 5.7831	+ 13.907	+ .51914	+ .13851
2	+ 5.5201	+ 30.471	+ 168.21	- .34025	- .017473
3	+ 8.6537	+ 74.886	+ 648.05	+ .27145	+ .0056847
4	+ 11.7915	+139.04	+ 1639.5	- .23246	- .0026239
5	+ 14.9309	+222.93	+ 3328.6	+ .20654	+ .0014546
6	+ 18.0711	+326.56	+ 5901.4	- .18772	- .00090268
7	+ 21.2116	+449.93	+ 9543.8	+ .17326	+ .00060476
8	+ 24.2535	+593.04	+14442.	- .16170	- .00042822
9	+ 27.4935	+755.89	+20782.	+ .15218	+ .00031619

n	$J_o(.1\ k_n)$	$J_o(.2\ k_n)$	$J_o(.3\ k_n)$	$J_o(.4\ k_n)$	$J_o(.5\ k_n)$
1	+ .98559	+ .94300	+ .87405	+ .78171	+ .66993
2	+ .92525	+ .71774	+ .42334	+ .10591	- .16839
3	+ .82136	+ .38021	- .09496	- .37448	- .35628
4	+ .68147	+ .02439	- .38497	- .26604	+ .12078
5	+ .51567	- .25533	- .32520	- .14296	+ .27087
6	+ .33585	- .39309	- .03386	+ .29340	- .09893
7	+ .15461	- .37038	+ .23649	+ .04613	- .22704
8	- .01569	- .21895	+ .28774	- .22638	+ .08584
9	- .16383	- .00722	+ .10972	- .17164	+ .19930

n	$J_o(.6\ k_n)$	$J_o(.7\ k_n)$	$J_o(.8\ k_n)$	$J_o(.9\ k_n)$
1	+ .54344	+ .40759	+ .26796	+ .13028
2	- .34692	- .40254	- .33896	- .18799
3	- .11262	+ .16624	+ .29882	+ .21775
4	+ .29959	+ .10815	- .18261	- .22637
5	- .08002	- .24019	+ .03524	+ .21694
6	- .19696	+ .16979	+ .09581	- .19227
7	+ .18004	- .01713	+ .17277	+ .15555
8	+ .06556	- .16510	+ .17844	- .11053
9	- .19640	+ .16777	- .12004	+ .06122

TABLE (Continued)

n	$J_o(.1k_n)/(k_n)^3\ J_1(k_n)$	$J_o(.2k_n)/(k_n)^3\ J_1(k_n)$	$J_o(.3k_n)/(k_n)^3\ J_1(k_n)$
1	+ .13651	+ .13061	+ .12106
2	- .016167	- .012541	- .0073969
3	+ .0046700	+ .0021617	- .0005399
4	- .0017881	- .00006400	+ .0010101
5	+ .00075008	- .00037140	- .00047303
6	- .00030317	+ .00035483	+ .00030564
7	+ .000093501	- .00022399	+ .00014302
8	+ .000006719	+ .000094658	- .000046984
9	- .000051802	- .00000228	+ .000034693

n	$J_o(.4k_n)/k_n)^3\ J_1(k_n)$	$J_o(.5k_n)/(k_n)^3\ J_1(k_n)$	$J_o(.6k_n)/(k_n)^3\ J_1(k_n)$
1	+ .10827	+ .092791	+ .075271
2	- .0018505	+ .0029422	+ .0060616
3	- .0021292	- .0020257	- .00064032
4	+ .00069806	- .00031691	- .00078607
5	+ .00020795	+ .00039400	- .0001164
6	- .00026485	+ .00008930	+ .00017779
7	+ .00002790	- .00013730	+ .00010888
8	+ .000096939	- .00003676	- .00002807
9	- .000054271	+ .000063017	- .000062101

n	$J_o(.7k_n)/(k_n)^3\ J_1(k_n)$	$J_o(.8k_n)/(k_n)^3\ J_1(k_n)$	$J_o(.9k_n)/(k_n)^3\ J_1(k_n)$
1	+ .056455	+ .037115	+ .018120
2	+ .0070335	+ .0059225	+ .0032847
3	+ .00094519	+ .0016990	+ .0012381
4	- .00028377	+ .00047915	+ .00059397
5	- .00034938	+ .00005126	+ .00031556
6	- .00015327	- .00008649	+ .00017356
7	+ .00001036	- .00010448	+ .000094070
8	+ .000070698	- .000076411	+ .000047331
9	+ .000053048	- .000037956	+ .00001936

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