Perturbation theory of non-demolition measurements

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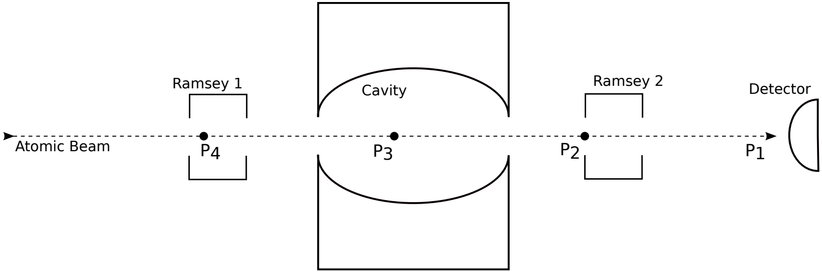
Based on Two Articles

- M Ballesteros, MF, J Fröhlich and B Schubnel: "Indirect acquisition of information in quantum mechanics." JSP 162 (2016)
- M Ballesteros, N Crawford, MF, J Fröhlich and B Schubnel: "Perturbation theory of non-demolition measurements." in preparation

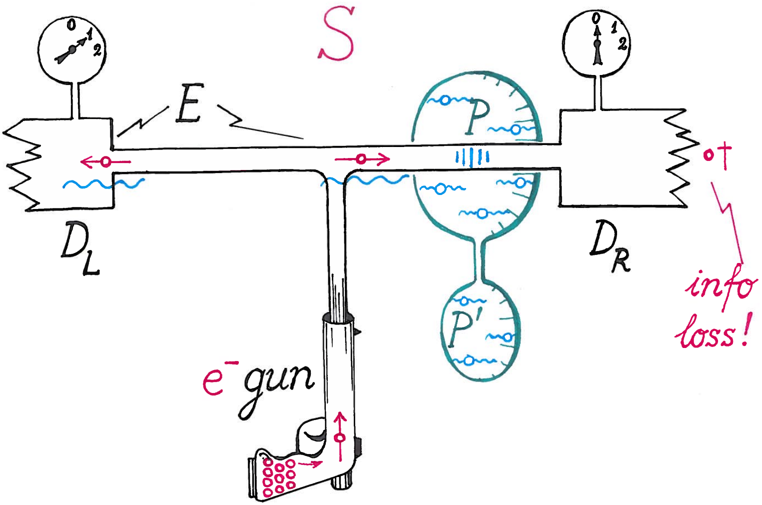
Haroche's experiment



Phenomena: After a long sequence of measurement results $\xi_1, \xi_2, \ldots$ the state of light in the cavity is close to a photon number state.

[Haroche et. al. Nature 2007, Siddiqi et. al. Nature 2013]

Electron gun



Mathematical Description

Jump operators V_ξ, $\xi \in \sigma$, with a normalization $\sum_{\xi\in\sigma} V_\xi^* V_\xi = 1$. Probability to measure ξ is $\mathrm{Tr}(V_\xi^* V_\xi \rho)$ and the state changes as

$$\rho \to \frac{V_\xi \rho V_\xi^*}{\mathrm{Tr}(V_\xi \rho V_\xi^*)}.$$

For an infinity history $\underline{\xi} = \xi_1, \xi_2, \ldots$ we put $V^{(n)}(\underline{\xi}) = V_{\xi_1} \ldots V_{\xi_n}$, the probability of a finite history $\xi_1, \ldots \xi_n$ is then

$$\mathbb{P}_\rho(\xi_1, \ldots, \xi_n) = \mathrm{Tr}((V^{(n)}(\underline{\xi}))^* V^{(n)}(\underline{\xi})\rho)$$

and the state evolves to

$$\rho_n(\underline{\xi}) = \frac{V^{(n)}(\underline{\xi})\rho(V^{(n)}(\underline{\xi}))^*}{\mathrm{Tr}(V^{(n)}(\underline{\xi})\rho(V^{(n)}(\underline{\xi}))^*)}.$$

Remarks

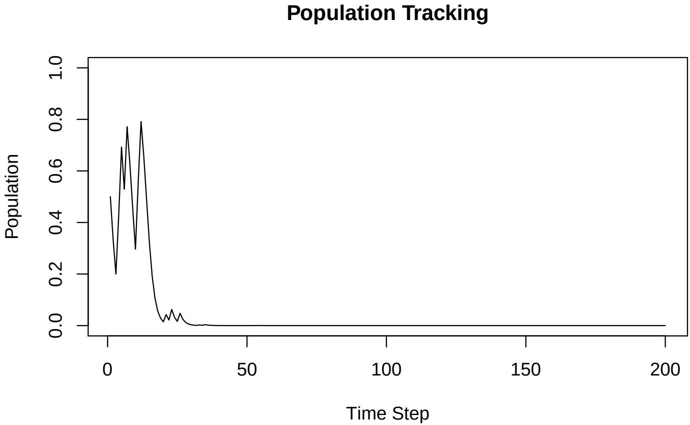
1. Products of i.i.d. random matrices studied by [Furstenberg, Kesten Annals of Stat. 1960] and many others with applications to 1D random Schrödinger e.g [Bougerol, Lacroix (1985)]
   Measurement in Quantum Mechanics also leads to study of product of matrices but with non i.i.d. measure.
2. The setting is an example of a finitely correlated state [Fannes, Nachtergaele, and Werner CMP 144].

Example $\varepsilon = 0$

For a two level system and $\sigma = \{e, g\}$ we put

$$V_e = e^{-i\varepsilon\sigma_1} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad V_g = e^{-i\varepsilon\sigma_1} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$$

For $\varepsilon = 0$ the population of $\mathcal{N} = \sigma_z$ approaches an eigenstate,

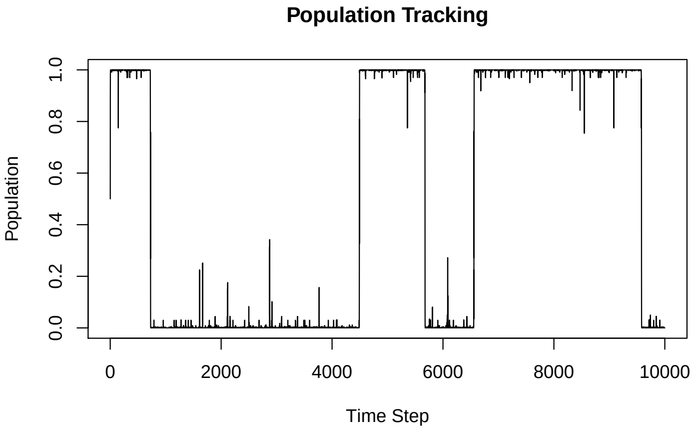


Example $\varepsilon \neq 0$

For a two level system and $\sigma = \{e, g\}$ we put

$$V_e = e^{-i\varepsilon\sigma_1} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad V_g = e^{-i\varepsilon\sigma_1} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$$

For $\varepsilon \neq 0$ the population of $\mathcal{N} = \sigma_z$ jumps between eigenstates,



Non-demolition case and its perturbations

For an observable $\mathcal{N}$ and a Hamiltonian H with $[H, \mathcal{N}] \neq 0$ we put

$$V_\xi^{(\varepsilon)} = e^{-i\varepsilon H} V_\xi(\mathcal{N})$$

for some complex functions $V_\xi(\cdot)$.
In the case $\varepsilon = 0$ we have $[V_\xi, V_{\xi'}] = 0$ and

$$\mathbb{P}_\rho(\xi_1, \ldots, \xi_n) = \int_{\sigma(\mathcal{N})} |V_{\xi_1}(\nu)|^2 \ldots |V_{\xi_n}(\nu)|^2 \mathrm{d}\lambda_\rho(\nu),$$

where λ_ρ is a spectral measure of $\mathcal{N}$. Interpreting ν as an unknown we define maximum likelihood estimate

$$\hat{\mathcal{N}}_k(\underline{\xi}) = \mathrm{argmax}_{\nu \in \sigma(\mathcal{N})} l_k(\nu | \underline{\xi}), \quad l_k(\nu|\underline{\xi}) := \frac{1}{k} \sum_{j=1}^{k} log |V_{\xi_j}(\nu)|^2.$$

Non-Demolition Case, $\varepsilon = 0$

For set $N \in \sigma(\mathcal{N})$ let let $\Pi(N)$ be the associated spectral projection and $S(\nu|N) = \inf_{\nu' \in N} \sum_{\xi \in \sigma} |V_\xi(\nu)|^2 [I(\nu|\xi) - I(\nu'|\xi)]$.

Theorem (Law of Large Numbers)

Suppose $\nu \to V_\xi(\nu)$ is injective and $V_\xi(\cdot)$ is continuos for all $\xi \in \sigma$, then the maximum likelihood estimator $\hat{\mathcal{N}}_k$ converges almost surely to a random variable $\hat{\mathcal{N}}_\infty$. For any Borel set $N \subset \sigma(\mathcal{N})$,

$$\mathbb{P}_\rho(\underline{\xi} : \lim_{k \to \infty} \hat{\mathcal{N}}_k \in N) = \mathrm{Tr}(\Pi(N)\rho).$$

Moreover if N is a closed subset of $\sigma(\mathcal{N})$ contained in the support of the measure λ_ρ then we have

$$-\lim_{k \to \infty} \frac{1}{k} \log \mathrm{Tr}(\Pi(N)\rho_k) = S(\hat{\mathcal{N}}_\infty | N), \quad \mathbb{P}_\rho - \textit{almost surely.}$$

Non-Demolition Case - references

When spectrum of $\mathcal{N}$ is discrete the Law of Large Numbers was proved in

- Maassen, Kümmerer 2006
- Bauer, Bernard PRA 2011
- Mabuchi et. al. IEEE 2004

The large deviation theory

- Bauer, Benoist, Bernard AHP 2013

Demolition Case

We make measurement times $t_1, t_2, \dots$ of $\xi_1, \xi_2, \dots$ random and distributed by Poisson distribution N_t, then the evolution is

$$\tau_\varepsilon^{(s)}(\underline{t}, \underline{\xi})\rho = e^{-i\varepsilon H(s-t_{N_s})} V_{\xi_{N_s}} \dots e^{-i\varepsilon H t_1} \rho e^{i\varepsilon H t_1} V_{\xi_1} \dots e^{i\varepsilon H(s-t_{N_s})}.$$

This is an unravelling of Lindblad evolution,

$$\mathbb{E}[\tau_\varepsilon^{(s)}] = exp(s\mathcal{L}_\varepsilon), \quad \mathcal{L}_\varepsilon \rho = -i\varepsilon[H, \rho] + \sum_{\xi \in \sigma} V_\xi \rho V_\xi^* - \rho.$$

For a sampling time $T > 0$ we define

$$\hat{\mathcal{N}}_s := \operatorname{argmax}_{\nu \in \sigma(\mathcal{N})} \frac{1}{N_{s+T} - N_s} \sum_{j=N_s}^{N_{s+T}} I(\nu|\xi_j).$$

To avoid dealing with overlapping data we set $\mathcal{M}_{jT} = \hat{\mathcal{N}}_{jT}$, for $j \in \mathbb{N}$ and extend the definition of $\mathcal{M}_t$ to all $t \geq 0$ by declaring it to be piecewise constant on the intervals $[jT, (j+1)T)$.

Demolition Case

Let $\mathcal{N} = \sum_\nu \nu P_\nu$ and define

$$\mathcal{P}\rho = \sum_{\nu \in \sigma(\mathcal{N})} P_\nu \rho P_\nu, \quad P_\nu = |\nu\rangle\langle\nu|.$$

We define an operator Q on the range of $\mathcal{P}$ by

$$\varepsilon^2 Q = -\mathcal{P}\mathcal{L}_\varepsilon \mathcal{P}_\perp \mathcal{L}_0^{-1} \mathcal{P}_\perp \mathcal{L}_\varepsilon \mathcal{P}.$$

The matrix Q defines a Markov Kernel on $\sigma(\mathcal{N})$ with elements

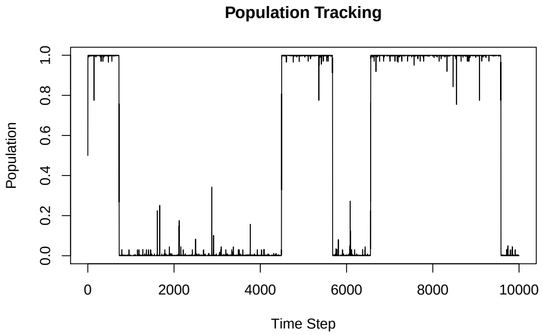
$$\mathrm{Tr}(P_{\nu'} Q P_\nu) = \begin{cases} \sum_{\beta \neq \nu} \frac{|\langle\beta|H|\nu\rangle|^2}{\sum_\xi V_\xi(\beta)\bar{V}_\xi(\nu) - 1} + c.c. & \text{for} \quad \nu = \nu' \\ -\frac{|\langle\nu'|H|\nu\rangle|^2}{\sum_\xi V_\xi(\nu')\bar{V}_\xi(\nu) - 1} + c.c. & \text{for} \quad \nu \neq \nu'. \end{cases}$$

Let Y_s be the continuous time Markov chain generated by Q started from an initial probability distribution $\pi_\rho(\nu) = \mathrm{Tr}(P_\nu \rho)$.

Demolition Case

Theorem (Distribution of Jumps)

Suppose $\nu \to V_\xi(\nu)$ is injective, and pick a positive I strictly smaller then $\min_{\nu,\nu'} S(\nu,\nu')$. Let $T = -\beta \log \varepsilon$, for some $\beta > \max\{2(1-e^{-I})^{-1}, (1-e^{-\frac{I}{2}})^{-1}\}$. Then under $\mathbb{P}_\rho^{(\varepsilon)}$, $\mathcal{M}_{\varepsilon^{-2}s}$ converges in law to Y_s, and the posterior density matrix $\rho_{\varepsilon^{-2}s}$converges in law to P_{Y_s}.



Thank you for your attention!