

Pricing of Game Options
in a market with stochastic interest rates

A Dissertation
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

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Georgia Institute of Technology
22 February 2005

Pricing of Game Options
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To God almighty.

ACKNOWLEDGEMENTS

First of all, I want to express my gratitude to my advisor, Dr. Robert P. Kertz whom has patiently and diligently guided me and reviewed this work.

I would like to thank the members of my defense committee who had to read through my work on such short notice.

I would like to thank the School of Mathematics at Georgia Tech that, throughout all my years of study has provided me with financial support, help and an amazing learning experience. In particular I must express my gratitude to: Dr. Shui-Nee Chow whom I accidentally encountered many years ago in the School of Matematics of the Universidad de Costa Rica and who invited me to come to Georgia Tech, to Dr. Alfred D. Andrew, who not only delivered a Georgia Tech application to me but also recommended me and offered me the hospitality of his home when I came to Georgia Tech, to Dr. Theodore P. Hill who visited the School of Mathematics at the Universidad de Costa Rica and recommended me, to Dr. William L. Green who was the graduate coordinator when I came to Tech and helped me through the process of becoming a Georgia Tech graduate student, to Dr. Evans M. Harrell, current graduate coordinator, and Dr. Luca Dieci and finally to all my professors here at Tech, which I am not mentioning individually for fear of forgetting to acknowledge someone. I particularly wish to express my gratitude to Ms. Cathy Jacobson; I will not forget her work, care and dedication.

I must also acknowledge the OAICEUCR (Office of International Affairs and External Cooperation of the Universidad de Costa Rica) and the Universidad de Costa Rica, which provided me with the additional financial aid required to take care of my family during all the years I have been a student at Georgia Tech.

My thanks go to all the friends I made while studying at Georgia Tech, and in particular, to Antonio Acosta (who offered me his friendship and hospitality from the first moment we met), as well as Luis Gonzalez, Teodoro Lara, Domingo Salazar, Maria do

Carmo Carbinatto, Tzvetan Stoyanov, Shobhana Murali, Russell Martin, Gianluigi del Magno, Hamed Maroofi, Jorge Rebaza Vasquez, Shuli Fu, Christophe Conseil, Kristi L. Chavez, Marcial Agueh, Jian Chen, Kasso Okoudjou, Djivede Armel Kelome, Ken Oberg, Ray Popovic, Victor Morales, Jose Miguel Renom, Jose Luis Sanchez, Becky Upchurch, Andrés Sercovich, Paul Wollan, Cinzia Elia, Sam Greenberg, Jorge Arroyo and Jose Enrique Figueroa Lopez; all of you encouraged me to continue. In addition I must offer an apology to all of those whom I should be acknowledging here but may have inadvertently left out.

I will always be in debt to my friend of many, many years, Pedro Diaz Navarro. Throughout all my years at Georgia Tech, he has been my eyes, my ears and my voice in Costa Rica.

I would like to express my deepest gratitude to my family for their encouragement and support. Gracias a mi Madre, Carmen Ureña. I thank my daughters: Daniela and Arantza, and my sons: Antonio and Gabriel. They are the light of my eyes; they gave me happiness when I was sad; they gave me love and made me feel strong when I was weak. Finally I would like to thank my wife, Marlene Blanco Chaves. She came to my life to make me whole; through the years she has been my stone pillar, my anchor and my lighthouse. I would have never succeeded in finishing this thesis without her patience, support and understanding.

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LIST OF SYMBOLS

$a^i(\cdot, \cdot)$	Denotes the appreciation rate of a zcb, same as $a(\cdot, \cdot)$, 50
$\alpha(\cdot, \cdot)$	Represents the diffusion matrix associated to our market model, ... 193
$\alpha(\cdot)$	Mean reversion parameter in the Hull White model of interest rates, 108
\mathcal{A}	Represents the infinitesimal generator of a given diffusion; we will use \mathcal{A}_Z instead when we want to make additional emphasis in the diffusion, Z in this case., 195
\mathcal{B}	The bank account (also denoted by \mathcal{S}_0) in \mathcal{M} , 42
$B^i(\cdot, \cdot)$	Denotes the price of a zcb, same as $B(\cdot, \cdot)$, 50
$\mathcal{B}([0, t])$	Denotes the Borel σ -algebra in $[0, t]$, $t \in]0, \infty[$, 47
B	The price process of security \mathcal{B} , 44
$b^i(\cdot, \cdot)$	Denotes the volatility of a zcb, same as $b(\cdot, \cdot)$, 50
$\beta(\cdot)$	Speed of mean reversion parameter in the Hull White model of interest rates, 108
$B(\cdot, \cdot)$	Denotes the price of a Zero Coupon Bond (zcb). That is, $B(t, T)$ is the time t price of a zcb of maturity T , 14
$\mathbf{Cpl}(\cdot)$	Denotes the price of a Caplet, 150
$\mathcal{CB}(\dots)$	Denotes the price of a Coupon Bond, $\mathcal{CB}(t, \dots)$ will be the time t price of a Coupon Bond (many more parameters may be used to uniquely identify this value, please see the text for details), 15
Ch	$Ch_X(\cdot)$ is the characteristic function of the random variable X , 113
\mathcal{C}_0	Continuation region., 190

C	Denotes a Coupon payment or a <i>regular</i> cash flow, 15
\cdot	Denotes the dot product (standard or canonical inner product) of two vectors, 43
$\text{diag}(\cdot)$	Denotes the $n \times n$ real diagonal matrix whose diagonal elements are given by the coordinates of its vector argument, 43
δ	The vector dividend rate process, 45
δ^i	The dividend rate process of security \mathcal{S}_i , 45
\dagger	Denotes matrix transposition, 43
$\mathcal{E}(\cdot)$	The Doléans exponential of a stochastic process, 58
$\mathcal{E}x$	Exercise region, 190
$E_{\mathcal{P}}(\cdot)$	Denotes the expectation with respect to probability measure \mathcal{P} , same as $E(\cdot)$, 48
$\eta(\cdot)$	The exponential of the integral of the speed of mean reversion parameter in the Hull White model of interest rates, 109
\mathcal{F}	The \mathcal{P} —augmentation of the natural filtration of W , 43
\mathcal{F}^X	The minimal σ —algebra generated by the random variables X_s , $0 \leq s \leq t$, 44
frp	Denotes a fixed rate payment (please see text for additional details), 25
$F_S(\cdot, \cdot)$	Denotes the forward price of security S , with respect to the zcb price process $B(\cdot, \cdot)$. If T is the maturity of the zcb, $F_S(t, T)$ denotes the time t value of security S in units of zcb value with maturity T , 197
Fr	Denotes a forward rate, 26

$\mathcal{FRA}(\dots)$	$\mathcal{FRA}(t, \dots)$ denotes the time t price of a forward rate agreement (many more parameters may be used to uniquely identify this value, please see the text for details), 26
$\Phi(\cdot)$	This symbol is used to denote the standard Normal distribution, ... 150
$\varphi_c(\cdot, \cdot)$	$\varphi_c(t, s)$ is the time fraction between dates t and s measured according to some preestablished conventions (here denoted as c), 7
gFr	Denotes a generalized forward rate, 26
$g\mathcal{FRA}(\dots)$	$g\mathcal{FRA}(t, \dots)$ denotes the time t price of a generalized forward rate agreement (many more parameters may be used to uniquely identify this value, please see the text for details), 26
$\gamma(\cdot)$	Volatility parameter in the Hull White model of interest rates, 108
i	Denotes the imaginary unit, that is the complex number $i = \sqrt{-1}$, . 113
$\mathcal{I}(\cdot, \cdot)$	Denotes the “intercept” part of the yield to maturity of a zcb under the Hull–White model of interest rates, 114
I	Denotes an interest, 12
$J(\cdot, \cdot)$	$J(\mathfrak{s}, \mathfrak{t})$ is the expected payoff of a game corresponding to the players strategies \mathfrak{s} and \mathfrak{t} , 70
$\mathcal{K}a$	Cancellation region., 190
κ	Cancellation time (game option), 67
\mathcal{K}	Denotes an interest rate (possibly constant), 15
κ	Denotes a strike price or a strike rate, 149
\mathcal{K}	Denotes a strike price or a strike rate, 147

\mathcal{K}	The forward strike price of an American Game Call,.....	208
\mathcal{L}	Represents the differential generator of a given diffusion; we will use \mathcal{L}_Z instead when we want to make additional emphasis in the underlying diffusion, Z .,	195
\mathcal{L}	The class of \mathcal{P} -a.s. integrable progressively measurable processes defined on $(\Omega, \mathcal{F}_T, \mathcal{P})$,	49
\mathcal{L}^2	The class of all square integrable progressively measurable processes defined on $(\Omega, \mathcal{F}_T, \mathcal{P})$,.....	48
L	The class of \mathcal{P} -integrable \mathcal{G} -measurable random variables,	49
L^2	The class of square \mathcal{P} -integrable \mathcal{G} -measurable random variables, ...	49
m	Denotes the expectation of the interest rate in the Hull White model of interest rates,.....	110
\mathfrak{V}	Denotes the variance of the interest rate in the Hull White model of interest rates,.....	110
\mathcal{M}	A market model,.....	42
$\max(\cdot, \cdot)$	Denotes the maximum between two or more quantities,	11
$\min(\cdot)$	Represents the minimum of its arguments,	68
$\min(\cdot, \cdot)$	Denotes the minimum between two or more quantities,.....	11
\vee	Alternative notation for the maximum between two quantities, that is $a \vee b = \max(a, b)$,.....	11
\wedge	Alternative notation for the minimum between two quantities, that is $a \wedge b = \min(a, b)$,	11
$(\cdot)^-$	Denotes the non-positive part and is defined as $(a)^- = -(a \wedge 0)$,....	11

μ	The vector process of appreciation rates, 45
μ^i	The appreciation rate process of security \mathcal{S}_i , 45
ν	Denotes the integral of the rate under the Hull–White model of interest rates, 111
\mathbb{N}	Denotes the set of natural numbers, 16
\mathbb{N}_n^*	Denotes the set of non–negative integers up to n , that is $\mathbb{N}_n^* = \{0\} \cup \mathbb{N}_n = \{i \in \mathbb{Z} : 0 \leq i \leq n\}$, 16
\mathbb{N}_n	Denotes the set of the first n natural numbers, that is $\mathbb{N}_n = \{i \in \mathbb{N} : i \leq n\}$, 16
\mathcal{N}	Denotes the Nominal (also known as Notional or Principal) amount paid by a contract, usually a bond., 14
$\ \cdot\ $	Denotes the Euclidean norm of an vector n –dimensional vector, 46
$\vec{\mathbf{1}}_n$	Represents the n –dimensional real vector whose entries are all equal to 1, 51
Ω	A sample space, 43
$\mathbb{1}_A$	Represents the indicator function of set $A \subset \Omega$, 59
$'$	Alternate notation for matrix transposition, 43
$(\cdot)^+$	Denotes the non–negative part and is defined as $(a)^+ = a \vee 0$, 11
πP	The active money vector process, 52
Φ	Used to represent the Buyer’s payoff in Chapter 5., 193
\mathcal{P}^F	Denotes the forward measure corresponding to the zcb price process $B(\cdot, \cdot)$. If T is the maturity of the zcb, \mathcal{P}_T^F , or simply \mathcal{P}^F , denotes the

maturity T forward measure defined by the maturity T zcb price process $B(\cdot, T)$,	200
\mathcal{P}^ε	The equivalent martingale measure from Girsanov's theorem, 58
\mathfrak{p}	Denotes a penalization, typically a positive amount —possibly time dependant, possibly random—to be payed as compensation for a determined action. We may also use the symbol p , 206
\mathcal{P}	A probability measure, 43
π	A portfolio, 51
π^0	Portfolio coefficient corresponding to the Bank account process, 51
π^i	Portfolio coefficient corresponding to security \mathcal{S}_i , 51
Π	A portfolio strategy, 51
Ψ	Used to represent the Seller's payoff in Chapter 5., 193
P	The vector of price process, 44
P^i	The price processes of security \mathcal{S}_i , 44
$\langle \cdot \rangle$	Denotes the quadratic variation of a stochastic process, 49
$\tilde{\mathcal{R}}$	Alternate Game Option payoff process (see (129)), same as $\tilde{\mathcal{R}}^{X,Y}$, . . . 74
r	The interest rate in the Hull White model, 108
\mathcal{R}	Game Option payoff process (Definition 3.4.1, and Definition 3.4.2), same as $\mathcal{R}^{X,Y} = \{\mathcal{R}^{X,Y}(s, t)\}_{(s,t) \in [0,T] \times [0,T]}$, 67
r	The interest rate process, 45
ρ	The risk premium process of market \mathcal{M} , 56

ϱ	Denotes the volatility of a given stock, 196
$\mathcal{S}(\cdot, \cdot)$	Denotes the “slope” part of the yield to maturity of a zcb under the Hull–White model of interest rates, 114
\mathcal{I}	Represents the set of strategies in an Dynkin game., 72
$\sigma_{r,p}$	Denotes Caplet “spot” volatility valid for the period $[T_r, T_p]$, 153
ς	Denotes the appreciation rate of a given stock, 196
\mathcal{S}_i	Any of the $1 + n$ securities in market \mathcal{M} , 42
$\sigma(\cdot)$	$\sigma(A)$ denotes the minimal σ –algebra generated by A , A being a collection of r.v., a collection of sets, a collection of σ –algebras or a mix of all of the above, 109
\mathfrak{s}	With a few exceptions, stopping times are represented by lower case letters in eufrat typography like $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}, \mathfrak{a}$, etc., 86
$\mathfrak{S}_{\mathcal{T}}$	Set of stopping times (with respect to the filtration \mathcal{F}) with values in $[0, \mathcal{T}]$, 66
$\mathfrak{S}_{\mathfrak{s}, \mathfrak{t}}$	Set of stopping times $\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$ such that $\mathfrak{s} \leq \mathfrak{w} \leq \mathfrak{t}$, 67
$\mathfrak{S}_{t,u}$	Set of stopping times (with respect to the filtration \mathcal{F}) with values in $[t, u] \subset [0, \mathcal{T}]$, 67
S^i	Denotes the price of the i th stock. Similarly, S will be used to represent the price of a given stock, 50
K	Denotes the strike price of an option, 206
sr	Denotes a Swap rate, 29
$\mathbb{S}(\cdot)$	$\mathbb{S}(t, \dots)$ denotes the time t price of a Swap (many more parameters may be used to uniquely identify this value, please see the text for details), 28

σ	The volatility matrix process, 45
σ^i	The volatility vector process of security \mathcal{S}_i , 45
σ^{ij}	The volatility process of security \mathcal{S}_i with respect to security \mathcal{S}_j , 45
$*$	Denotes discounting (when used as a superscript), 54
ξ	Exercise time (game option), 67
θ	The market price of risk process, 57
\mathcal{T}	Market \mathcal{M} 's time horizon, 43
\mathcal{U}	A σ -algebra, 43
\bigvee	Denotes the minimal σ -algebra generated by a union of σ -algebras. That is $\mathcal{F}_1 \bigvee \mathcal{F}_2 = \sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$, 109
\mathcal{V}	Price process of a gcc. As usual \mathcal{V}^* denotes the discounted price process, $\mathcal{V}_t^* = \mathcal{V}_t/B_t$, $t \in [0, \mathcal{T}]$, 103
\mathfrak{V}	The fair price of a gcc, 85
\underline{V}	Represents the lower value of a given game., 71
\overline{V}	Represents the upper value of a given game., 71
$Vrp.$	Denotes a variable rate payment, 25
$Vr(\cdot, \cdot)$	$Vr(t, s)$, $t \leq s$, denotes a variable interest rate known at time t and used to compute interest rate payments at time s . If nothing else is mentioned, it is safe to assume such a rate in simple compounded, .. 22
W	A d -dimensional Brownian Motion, 43
\wedge	$A \wedge B$ represents the minimum between A and B , 68

\mathcal{W}^{Π, w_0}	The wealth process corresponding to portfolio strategy Π with initial wealth w_0 , same as \mathcal{W}^{Π} and \mathcal{W} ,52
\mathfrak{X}	Exercise payoff process (game option), 67
$\mathcal{Y}(\cdot, \cdot)$	$\mathcal{Y}(t, T)$ denotes the yield to maturity of a zero coupon bond $B(t, T)$, 113
\mathfrak{Y}	Cancellation payoff process (game option), 67
$y(\cdot, \cdot)$	Denotes yield curve data, please see Definition 2.1.12, 20
Y	The vector yield process, 45
Y^i	The yield process of security \mathcal{S}_i , 45

SUMMARY

An in depth study of the pricing of Game contingent claims under a general diffusion market model, in which interest rate is non constant, is presented.

With the idea of providing a few numerical examples of the valuation of such claims, we present a detailed description of a Bootstrapping procedure to obtain interest rate information from Swaps rates. We also present a Stripping procedure that can be used to obtain initial spot (caplet) volatility from Market quotes on Caps/FLoors. These methods are of general application and could be used in the calibration of diffusion models of interest rate.

Then we show several examples of calibration of the Hull–White model of interest rates. Our calibration examples are later used in the numerical approximation of the value of a particular form of Game option.

CHAPTER I

INTRODUCTION

1.1 Opening thoughts

Particularly after the 1970's, Mathematics has found a fertile field of application in the Economical Sciences, and in particular in Finance.

With financial institutions eager for solutions to their most complex problems, new programs have been opened in several universities to satisfy the high demand for scientists and mathematicians trained in the solution of these new kinds of problems.

In Finance, an always present consideration is that of the need for an understanding of the sources of risk. Such studies may include the behaviour of interest rates, whose fluctuation is a source of market and credit risk, or the understanding of default and financial distress, main sources of default risk. For example, in the study of derivative pricing, mathematical market models are introduced to carefully account for all sources of systematic risk. Under various sets of additional conditions, such models are used to study the behavior of agents involved in a particular contract, the strategies available to those agents, and their ability to hedge their positions.

Although no clear definition of the concept of risk exists, the apparent random behavior of the many manifestations of the concepts we call risk make them suitable subjects for the general theory of stochastic processes and probability.

In a paper by Yuri Kifer published in *Finance and Stochastics*¹, the author proposes the study of a new kind of financial option called the *Game Option*.

A game option is a contract between two parties (a Holder —also known as the Investor, or the Buyer, and a Writer —also known as the Seller) based on a zero sum game of stopping, that allows the seller to terminate the contract and the holder to exercise his rights in the

¹[101] Kifer, Yuri. *Game options Finance and Stochastics*; Springer—Verlag, 4 443–463 (2000).

contract, at any time throughout the life of the contract. If the seller cancels or if the holder exercises, the contract ends with certain predetermined actions taken. This game option contains two payoff processes \mathfrak{X} (the cancellation payoff) and \mathfrak{Y} (the exercise payoff), with $0 \leq \mathfrak{Y}_t \leq \mathfrak{X}_t$, and a given maturity $T > 0$. If the seller decides to cancel the contract at a given time t , he will pay to the holder the amount \mathfrak{X}_t ; if on the other hand the holder decides to exercise his rights at time t , then the seller will pay \mathfrak{Y}_t to the holder. In any case, the contract ends at the moment of cancellation/exercise. If both, seller and holder, decide to act (that is to cancel or to exercise) at the same time t , the holder is given priority and is paid \mathfrak{Y}_t . If neither decides to act prior to maturity, the holder is paid \mathfrak{Y}_T at maturity.

Naturally, the goal of the holder is to maximize his expected gain, while the seller's goal (and this is the novelty of this contract) is not only to be able to hedge his position at any time but also to minimize the payment he will have to give to the holder.

Different from European options (where the seller's position is static after the settlement of the contract) and from American options (where the seller's activity is reduced to hedge his position while waiting for the holder to act), in a game option the two parties behave like adversaries in a zero-sum game. The goal of the buyer is to maximize his gain as much as possible; on the other hand, the goal of the writer is now more complex (when compared with the role of a writer in an American or European option), since he not only has to hedge his position (to ensure he will be able to cover his obligation in the contract) but also to minimize the payment he will have to make to the holder. From the pure mathematical point of view, such contracts offer an interesting source of problems of different levels of complexity. Since both parts of the contract can be active, at least two random times are required to model their decisions (a difference from American style options in which only one part is active and one stopping time can be used to signal the time of execution).

From the point of view of the Mathematician, the problem of finding a fair price for a Game Option is a very interesting one, it involves the theories of Stochastic Games, Game Theory, Optimal Stopping theory and the general theory of Stochastic processes and a fair amount of financial theory. If the numerical approximation of prices is also to be considered, the problem will also involve Partial differential equations, the theory of Diffusion processes,

numerical analysis, plus the knowledge of some market conventions required to combine real market data with the theoretical models.

Although we have no knowledge of derivatives or any other form of contracts of this exact kind being traded in a market, it is not difficult to us to see the applicability of this idea in the form of cheap insurance (it is not difficult to argue that the price of a game contingent claim should always be lower than the price of its corresponding American option counterpart), and in the form of embedded options (that is, options that are part of other contracts, for example a bond containing in its indenture a conversion feature in the name of the holder and a re-purchase/cancelation feature in the name of the writer).

In this thesis we extend the work of Kifer [101] on game options to a full fledged standard market model with non-constant stochastic interest rates. We show that under mild conditions a price for such contracts exists and a characterization of the investors strategies is given.

We also study the numerical pricing of a particular case of game option that does not only depend on the price of an underlying asset but that is also sensitive to changes in an interest rate.

1.2 Thesis outline

We have divided this Thesis into four main chapters. Chapter 2 offers an intuitive introduction to interest rates and to the basic Financial instruments known as Swaps, Forwards and Bonds. The chapter ends with a detailed description of a bootstrapping procedure, based on Market conventions, that allows us to obtain initial interest rates (initial yield curve) information from Swaps data. The input data in this bootstrapping procedure is a table of swap rates as it is typically quoted in the Swaps market. This data can be obtained from different sources (Bloomberg[©] for example).

Chapter 3 contains the theoretical part of our work, in this chapter we describe the market model used and its main properties. We show that the price of a Game contingent claim exists.

Chapter 4 is a detailed study of the Hull-White interest rate model and its calibration

to both initial yield and initial spot volatility. A detailed stripping procedure, based on market conventions, is given in this chapter that allows us to obtain initial spot volatility data from flat (Caps/Floors) volatility.

Finally, in Chapter 5 we offer the numerical valuation of an American Game Call on a Forward Price. This kind of contract gives us a nice simple example of a Game option that is also sensitive to changes in interest rates (in this case a zero coupon bond with a given maturity).

CHAPTER II

A LITTLE BIT OF FINANCE

2.1 Brief introduction to Financial Securities

In this chapter we will introduce several financially related terms and basic financial instruments either found in real markets or of theoretical importance due to their use in the study and actual pricing of other, more complex financial instruments.

Instruments as Zero coupon bonds (which constitute the central core around which many more financial instruments are constructed), forward rate agreements, interest rate swaps and caps will be introduced in the following sections; we will also consider their arbitrage pricing according to market practice. The material reviewed in the initial sections will be used as the background in the construction of Section §2.2.

The last section (§2.2) of the present chapter will deal with the bootstrapping of interest rate information from Swaps data taken from the market (in this particular case, swap quotes retrieved from Bloomberg[©]). Bootstrapping procedures are not only used to obtain graphical depictions of the behavior of interest rates but are also of key importance in any interest rate model calibration. Results from this chapter and in particular from section §2.2 will be used both in Chapter 4 where we present the calibration of Hull-White model of interest rates, and in Chapter 5, where some numerical examples will be presented.

Still, our exposition will be far from complete and will be driven mainly by our utilitarian needs, which are: a) to provide the reader with the minimal lexicon and tools required to read through this thesis, and b) to introduce the topics we need to give a detailed description of the calibration of Hull–White’s model of interest rates coming in a following chapter (see Chapter 4).

More detailed and complete descriptions of common financial instruments and lexicon can be found elsewhere. In particular, the textbooks by Bodie, Kane and Marcus ([17]) and Ross, Westerfield and Jaffe ([152]) should provide the reader with the general

lexicon and basic ideas related to simple financial instruments and their market pricing. Useful glossaries containing definitions of usual financial terms can be found in the Internet, for example Axone’s multilingual glossary (<http://tradition.axone.ch/>, financial terms in English, German, French and Italian are provided here, free of charge for non-commercial purposes), “Reuters Fixed Income Services Financial Glossary” (<http://www.ejv.com/bp/html/glossary3.html>), etc.. Musiela and Rutkowski ([133], in particular the second part of the book) and Brigo and Mercurio ([19]) provide descriptions of some of the instruments mentioned here. Additional comments regarding time fractions and day counting conventions are included in our Maple[©] worksheet ([78]) dealing with the bootstrapping of a yield curve, such a document can be obtained from the author. Additional sources are mentioned in the bibliography section of this work.

2.1.1 Interests and time fractions

The first two meanings of “interest”, according to the Merriam–Webster dictionary are:

- (1): *right, title, or legal share in something; (2): participation in advantage and responsibility*
- (a): *a charge for borrowed money generally a percentage of the amount borrowed; (b): the profit in goods or money that is made on invested capital; (c): an excess above what is due or expected*

It is thus clear that an *interest* is perceived as that amount that is received as the result of an investment or paid on behalf of money borrowed from someone. Similarly the corresponding *rate of interest* will be the fraction (or percentage) defined as interest over total amount invested or borrowed, per period of time.

Still, such a simple concept may be represented in many different ways related to the interpretation one gives to such a number or to the local practices regarding its uses. For example, interests are usually paid or received as a lump sum at the end of a given (predetermined) period of time (at or after the end of the contract), or in installments at predetermined dates after every certain period of time throughout the life of the contract.

As an example, you may have a savings account in a bank that pays a given *annual* interest rate on the money you keep in your account (maybe not for all of it, but for the average amount in a given, predetermined, period of time) and such interest is paid to your account in installments every month, the last Friday of the month. Or maybe, such interest is paid four times a year, etc.

As we can see, although the concept is a simple one, in practice its uses and definitions may turn a little complex.

A usual concept used in practice is that of the time for which an interest rate is valid, for example a year, a month, a semester, etc. Also important are the number of installments in which interest is paid (*the frequency per time period*), the times between installments, and if such amounts will be included into the computation for the future installments (*compounded interest*) or not (*simple interest*).

Also of importance are the conventions used in the computation (or measurement) of time periods (“time” usually stands for elapsed or remaining time, after or before a particular date, etc.). Which units of time are to be used? Which is the effective or base unit of time? One may compute time in seconds, or in days, etc. and such computation may be discrete or continuous. Although time may be measured in days, the effective unit of time used could be years and interests and interests rates computed and reported in years.

In practice, time is computed discretely in days, weeks, months, years, etc. But even if such convention is clearly stated some problems may come from the conversion from one basic period of time to another. For example not all months are of equal length (in days) and not all years are of equal length. Also of importance are the practices regarding business days, observance of holidays, etc.

Depending on the situation, time could be measured as *remaining time* (that is, how long until the end of the contract, or until a particular date) or as *elapsed time* (how long since the beginning of the contract, or since a particular date). For example, time to maturity (sometimes referred as **TtM**) is measured as remaining time, while most time fractions used in interest computations are measured as a fraction of elapsed time.

We call symbol $\varphi(t, s)$ the *time fraction* between dates t and s . Time fractions are

defined as the “number” of “small” time periods or subunits of time (designated using a given conventional unit of time, days usually, but they could be defined in terms of weeks, months, hours, etc.) per “base” time period or effective unit of time (also designated using a given conventional unit of time, usually years—in days—, —here years can be “standard” years of 365 days, or actual years, etc. see below—, but months, weeks, quarters –measured in days– could be used if days are used as small periods, etc.). In practice $\varphi(t, s)$ depends on “day counting conventions” (how to count how many days are between date t and date s), “business days conventions” (what to do if dates t and/or s are not business days) , “holiday conventions” (what to do if t and/or s happen to fall on a holiday), etc.

Notation 2.1.1. Common day counting conventions assume that years are measured in days (days of 24 hours), but months, weeks, quarters –measured in days– examples of day counting conventions are:

- Actual/Actual: assumes a year consists of 365 or 366 days (in case the year is a leap year)¹ and that days between dates t and s , t prior to s , are counted as the actual number of calendar days between those two dates, including the first but not the second. If we identify t and s with their corresponding Julian day number², the following formulas can be given to compute the corresponding time fraction. First, if both t and s are dates belonging to the same year and n is the actual number of days in that year

$$\varphi_{a/a}(t, s) = \frac{s - t}{n} \quad (1)$$

is the Actual/Actual time fraction in years. If t and s are dates belonging to two different years, let J_i be the first of January of the second year, J_f be the first of

¹This adds another problem, since different conventions could be used regarding the insertion, and in particular the place of insertion, of a leap day in a leap year. Additional information in this regard can be found in our Maple[©] worksheet [78] regarding interest rate bootstrapping (this document can be obtained from the author).

² For a definition of this concept please visit the Calendar FAQ, by Claus Tøndering at <http://www.tondering.dk/claus/cal/calendar26.html>. Similar information can be found in the first section of our Maple[©] worksheet [78] regarding interest rate bootstrapping (this document can be obtained from the author).

January of the final year, n_i the number of days in the first year, n_f the number of days in the final year, and y the number of years between first and final years (none of them included),

$$\varphi_{a/a}(t, s) = \frac{J_i - t}{n_i} + y + \frac{s - J_f}{n_f} \quad (2)$$

For example, the Actual/Actual time fraction (of days per year) between $t = 9/9/2003$ and $s = 3/4/2004$ is

$$\begin{aligned} \varphi_{a/a}(9/9/2003, 3/4/2004) &= \frac{22 + 31 + 30 + 31}{365} + 0 + \frac{31 + 29 + 3}{366} \\ &= \frac{21573}{44530} \sim 0.4844599 \end{aligned} \quad (3)$$

$\varphi_{a/a}(9/9/2003, 3/4/2004)$ is the fraction of a year between September 9th, 2003 and March 4th, 2004. Thus $\varphi_{a/a}(t, s)$ is also called the Actual/Actual year fraction between t and s .

- Actual/365: assumes a year consists of 365 days (that is, there is no distinction between leap and non-leap years). As in the Actual/Actual convention, days between dates t and s , t prior to s , are counted as the actual number of calendar days between those two dates, including the first but not the second. The corresponding time fraction can be computed as

$$\varphi_{a/365}(t, s) = \frac{s - t}{365} \quad (4)$$

Using the same sample dates $t = 9/9/2003$ and $s = 3/4/2004$ we obtain the Actual/365 time fraction (or year fraction) between September 9th, 2003 and March 4th, 2004 as

$$\begin{aligned} \varphi_{a/365}(9/9/2003, 3/4/2004) &= \frac{22 + 31 + 30 + 31 + 31 + 29 + 3}{365} \\ &= \frac{177}{365} \sim 0.4849315 \end{aligned} \quad (5)$$

- Actual/360: assumes that a year consists of 360 days and that in that regard there is no distinction between leap and non-leap years. As before, days between dates t and s , t prior to s , are counted as the actual number of calendar days between those two

dates, including the first but not the second. The corresponding time fraction can be computed as

$$\varphi_{a/360}(t, s) = \frac{s - t}{360} \quad (6)$$

Thus, the Actual/360 time fraction (or year fraction) between September 9th, 2003 and March 4th, 2004 is

$$\begin{aligned} \varphi_{a/360}(9/9/2003, 3/4/2004) &= \frac{22 + 31 + 30 + 31 + 31 + 29 + 3}{360} \\ &= \frac{177}{360} \sim 0.4916667 \end{aligned} \quad (7)$$

- 30/360: assumes that a year consists of 360 days and that all months are 30 day months, with no distinction between leap and non-leap years. Assuming dates t and s are Gregorian dates³, t prior to s , $t = m_i/d_i/y_i$ and $s = m_f/d_f/y_f$ we have

$$\varphi_{30/360}(t, s) = \frac{360(y_f - y_i) + 30(m_f - m_i - 1) + d_f \wedge 30 + (30 - d_i) \vee 0}{360} \quad (8)$$

From formula (8) we see that the 30/360 year fraction between September 9th, 2003 and March 4th, 2004 is

$$\begin{aligned} \varphi_{30/360}(9/9/2003, 3/4/2004) &= \frac{360 + 30(3 - 9 - 1) + \min(4, 30) + (30 - 9)^+}{360} \\ &= \frac{175}{360} = \frac{35}{72} \sim 0.4861111 \end{aligned} \quad (9)$$

Day counting conventions are also known as *day counting bases*. Once a day counting convention is selected, time fractions are to be computed using that day counting convention. If $\varphi(t, s)$ is the time fraction between dates t and s we may use the symbol φ to represent the day counting convention used. Similarly if we call φ the day counting convention selected, we will use the symbol $\varphi(t, s)$ to represent the corresponding time fraction between dates t and s .

If date t precedes date s , the symbol $\varphi(s, t)$ is to be interpreted in the sense of remaining time. That is, we still compute $\varphi(t, s)$ following the rules implied by the day counting basis φ , but the resulting number is used to measure the time remaining until date s .

³For calendar definitions and conventions please refer to Calendar FAQ, by Claus Tøndering at <http://www.tondering.dk/claus/cal/calendar26.html>, Footnote 2.

Notation 2.1.2. We will denote the maximum between two numbers (variables, functions, etc.) a and b as $\max(a, b) = a \vee b$. If one of the two numbers is zero, we use $(a)^+ = \max(a, 0) = a \vee 0$. Similarly, we will use $\min(a, b) = a \wedge b$ to represent the minimum between two quantities a and b , and if one of the numbers considered is zero, $-(a)^- = \min(a, 0) = a \wedge 0$. When needed, the notation $\max(\dots)$ and $\min(\dots)$ will be extended to represent the maximum and minimum of more than two quantities.

The previous list of day counting conventions and corresponding year fractions is by no means complete and has been given only as an example. Many more day counting conventions are used in practice.

In formula (8), taken from [19], as well as in our previous examples, dates are represented in standard American (USA) notation $m/d/y$, where m is the month number, d the day of the month, and y the year (year is AD –Anno Domini–, remember we assume Gregorian dates are used).

The reader should be aware that the formulas provided above must be corrected to include any possible “business day convention” or “holiday convention” in use.

Common “business day conventions” are, for example:

- **Prior day**, if date t falls on a Saturday, Sunday or any other non business day, such date should be corrected to the prior business date.
- **Next day**, if date t falls on a Saturday, Sunday or any other non business day, such date should be corrected to the next business date.

“Holiday conventions” correspond to the selection of observed holidays. If a date t falls on a holiday, the accepted “Holiday conventions” will dictate if such day is considered a business day or not. The following definitions assume that a particular day counting basis, a particular basic unit of time and a particular basic period of time, T , (expressed in terms of the selected unit of time, for example a time period of a year expressed as 360 days in accordance with a day counting convention of 30/360) have been selected and that two calendar dates t (the “start” or “initial” date) and s (the “terminal” or “final” date) are given. As we have done before, we will abuse notation and assume also that t and s can

be identified with their corresponding Julian day number (see Footnote 2) so that they can be compared using common order operators like “ $<$ ” and so that formulas from Notation 2.1.1 can be used.

Definition 2.1.1 (Simple compounded interest rate). *Denote the following: \mathcal{P} is the principal (that is the amount invested or borrowed), T is the basic period of time (or effective unit of time; years, for example), and $\varphi(t, s)$ the time fraction between times t and s (that is the “number” of time periods between times t and s), $t < s$. We define simple (compounded) interest rate per period of time T as such rate r_i that satisfies*

$$I = \mathcal{P}\varphi(t, s)r_i \quad (10)$$

where I is the total interest payed.

When interests are simple, the frequency of installments is taken as one.

If on the other hand, the frequency of installments (per basic period of time) is bigger than one and interests are accrued, then we have the following.

Definition 2.1.2 (Several times compounded interest rate). *Denote the following: \mathcal{P} is the principal, T the basic period of time, f is the frequency of installments per time period of length T , and $\varphi(t, s)$ the number of time periods between times t and s , $t < s$. We define the compounded interest rate per time period T of frequency f as the rate r_c such that*

$$I = \mathcal{P} \left(\left(1 + \frac{r_c}{f} \right)^{f\varphi(t, s)} - 1 \right) \quad (11)$$

where I is the total interest payed.

Continuous interest rates are also used, but are less common in practice⁴.

⁴I still remember the first savings account I ever had. That was when I was a kid back in Costa Rica, several years ago; it payed 6% annual rate continuously compounded over the average balance of the account (up to a million colones) in the three months prior to the payment of interests. Interest was computed four times a year, the last day of the third, sixth, ninth and twelfth month of the year and payed to the account the next (business) day.

Definition 2.1.3 (Continuously compounded interest rate). Denote the following: \mathcal{P} is the principal, T the basic period of time, and $\varphi(t, s)$ the number of time periods between times t and s , $t < s$. We define the continuously compounded interest rate per time period T as the rate r such that

$$I = \mathcal{P} \left(e^{r\varphi(t,s)} - 1 \right) \quad (12)$$

where I is the total interest paid.

From (11) and (12) it is clear that continuously compounded interest rates are the limiting case of several times compounded interest rates when the frequency of installments approaches infinity. On the other hand, if the time fraction $\varphi(t, s)$ decreases approaching zero, all three kinds of interest rates mentioned here are equivalent.

2.1.2 Bonds and Bond Pricing

In our work, a concept that shows up around every corner is that of a *bond*. To us

Definition 2.1.4. A **bond** is a contract according to which, two parties agree to exchange cash flows following a predetermined schedule.

Clearly, our definition is a bit too general, and thus encloses many other contracts not usually seen or regarded as bonds. Examples of bonds are: *Treasury Notes*, *Treasury Bills*, and of course *Treasury Bonds*. But bonds also include *Mortgages*, *loans*, *Interest rate swaps*, or a *Bank Account*. Several different kinds of embedded options and provisions can be attached to a bond in order to create more complex instruments, for example: *Callable Bonds*, *Convertible Bonds*, *lottery tickets*, etc. Here we are assuming that the cash flows are to be given in the same currency. If one allows for different currency cash flows, then our definition is general enough to cover almost all existing financial contracts (but of course this is not the issue, but a simple by-product of our simple definition).

Bonds, in practice, go from the very complex contracts to the very simple, the simplest of them being a Zero Coupon Bond.

Definition 2.1.5. A **Zero Coupon Bond (zcb)** is a bond consisting of two times t and T (the maturity of the bond), with $t < T$, and two cash flows: one from party A (the buyer),

$B(t, T) \leq 1$ given at time t , which we call the price of the zcb, and a second cash flow of size 1 from party B (the seller), given at time T . The zcb's are also known as discount bonds or simply zeroes.

In actual markets, bonds promise the payment of an amount N , known as the *nominal* or *principal* of the bond, and that quantity is rarely close to 1 unit of currency. The convenience of our definition of a zcb as a bond paying a nominal of 1 at maturity will be apparent in the coming pages.

Notation 2.1.3. In what follows we will use the notation

$$B(t, s) \tag{13}$$

to denote the time t price of a zcb whose maturity is s , $t \leq s$, and whose nominal or principal is one unit of currency.

Zero coupon bonds are of fundamental importance, both practical and theoretical. And as we will see in the following pages, they can be used to describe and, or, define more complex financial instruments.

In what follows, unless otherwise specified, we will assume that we are dealing with a “perfect world” in which there is no default risk or frictions and in which securities are perfectly divisible⁵. Even if uncertainty is allowed, we will assume that there is no arbitrage and that (quoted) interest rates are non-negative.

Under these conditions, the time T price of a T maturity zcb should be $B(T, T) = 1$.

Definition 2.1.6. The time t and maturity T , $t < T$, **spot rate** is defined as the theoretical yield (interest rate) of a (t, T) -zcb⁶.

⁵That is, investors are allowed to acquire or sell any fractional amount of a security. Although this assumption (when compared to real market practices where investors must trade in integer multiples of securities) may be considered extreme, it is in perfect accordance with our assumption of no frictions. Non divisibility forces investors to buy or sell more than what they should, which in turn translate into a form of friction. On the other hand, wealthy investors can achieve near zero friction and almost perfect divisibility in real markets. Thus this kind of assumption is not considered too far off.

⁶In practice, due to the relationship between $(0, T)$ -zcb spot rates and yield curves (see Definition 2.1.12) some tend to regard as spot rates only the theoretical yields of $(0, T)$ -zcb's.

Please refer to §2.1.3 for a detailed description of interest rates in relation to zcb's.

Zero coupon bonds have been in the market since very long ago, offered by governments as well as by companies, responding to the different needs and interests of lenders and investors. They may be pure discount bonds or synthetic. For example STRIPS⁷ are examples of synthetic zero coupon bonds. In the USA the STRIPS market has grown with the years. Now, data from the STRIPS market can be used to reconstruct the yield curve⁸ (see Definition 2.1.12) without the need to worry about carrying out a bootstrapping using all other Treasury securities. Although such simplification reduces the effort required, many other problems remain. For example, liquidity, seniority, or if a STRIP is coming from coupons or from principal, to mention a few. Please see [168] for a description of STRIPS.

Notation 2.1.4. In what follows we will use **family** to refer (in a loose way) to a set of contracts with a common set of characteristics, but differing in maturity. After some clarification, this use may be extended to situations where some other characteristic is allowed to change.

Remark 2.1.1. Due both to the size and efficiency of the market, we can assume that the bond market is arbitrage free. In our applications we will assume that there is a special family of zcb's and that such a family conforms to a family of arbitrage free zcb's.

Definition 2.1.7. A **Coupon Bond** starting at time T_0 is a contract in which one party (the issuer), for a price \mathcal{CB} paid up front, ensures to another party (the holder) the future payment, at times $\{T_i\}_{i \in \mathbb{N}_m}$, $T_0 < T_1 < T_2 < \dots < T_m$, of a certain number, m , of cash flows, $\{C_i\}_{i \in \mathbb{N}_m}$,

$$C_i = \varphi(T_{i-1}, T_i) \mathcal{K} \mathcal{N} \quad (14)$$

based on a rate⁹ \mathcal{K} relative to the day counting basis φ , and a nominal, \mathcal{N} , also called the

⁷The acronym STRIPS stands for Separate Trading of Registered Interest and Principal of Securities; basically, STRIPS are obtained when coupons and principal are stripped from a Treasury security.

⁸See [153] for a detailed analysis of this issue.

⁹We assume a particular day counting basis (as 30/360, or Actual/Actual for example) is used to compute the time fraction φ when interest payments related to rate \mathcal{K} are computed.

principal.

Cash flows (interest payments) $\{C_i\}_{i \in \mathbb{N}_m}$ are called coupons¹⁰. The times $\{T_i\}_{i \in \mathbb{N}_m}$ are called the coupon dates, and the last of such dates $T_m = T$ is also called the maturity of the bond. Dates $\{T_i\}_{i \in \mathbb{N}_{m-1}^*}$ are also known as reset dates. At maturity, the issuer of the Coupon Bond will pay the nominal \mathcal{N} (plus the last coupon) to the holder.

Notation 2.1.5. In what follows, as well as in the preceding definition, we will use the notation \mathbb{N}_m to represent the subset of \mathbb{N} containing the first (non null) m integers. That is, for $m \in \mathbb{N}$ we define $\mathbb{N}_m = \{x \in \mathbb{N} : x \leq m\}$. Similarly, for $m \in \mathbb{N}$ we define $\mathbb{N}_m^* = \{0\} \cup \mathbb{N}_m$.

Looking at the structure of cash flows in a Coupon Bond,

$$\begin{cases} C_i = \varphi(T_{i-1}, T_i) \mathcal{K} \mathcal{N} & 0 \leq i < m \\ \mathcal{N} + C_m = \mathcal{N} (1 + \varphi(T_{m-1}, T_m) \mathcal{K}) \end{cases} \quad (15)$$

it is easy to see that we can decompose a Coupon Bond into a portfolio of zcb's maturing at the coupon dates $\{T_i\}_{i \in \mathbb{N}_m}$. Since we can see $B(t, T_i)$ as the time t price of a dollar at time T_i , the total value of cash flows, at time t , $t \leq T_0$, of a Coupon Bond will be

$$\mathcal{N} B(t, T) + \sum_{i=1}^m C_i B(t, T_i) = \mathcal{N} \left(B(t, T) + \sum_{i=1}^m \varphi(T_{i-1}, T_i) \mathcal{K} B(t, T_i) \right) \quad (16)$$

Thus, for $t \leq T_0$, the time t fair price of a Coupon Bond of rate \mathcal{K} starting at time T_0 and with maturity $T = T_m$, paying coupons at times $\{T_i\}_{i \in \mathbb{N}_m}$ is

$$\mathcal{CB}(t, T_0, T_1, \dots, T_m, \varphi, \mathcal{K}, \mathcal{N}) = \mathcal{N} \left(B(t, T) + \sum_{i=1}^m \varphi(T_{i-1}, T_i) \mathcal{K} B(t, T_i) \right) \quad (17)$$

A contract very similar to a Coupon Bond is a generalized forward rate agreement (see Definition 2.1.14).

2.1.3 Zero coupon bonds and interest rates

In this section we will briefly describe some accepted conventions regarding interest rates and their relation with zero coupon bonds. In particular we will introduce here the concept of

¹⁰Such a name is due to the old practice of printing coupons in the contract that were stripped off or cut out by the holder at the coupon dates and carried to an office to exchange them for cash.

Yield curve. Together with *Swap curves*, *Cap curves*, etc., Yield curves are a very important piece of information from which much insight about the market can be obtained¹¹. In particular, Yield curves are one of the key ingredients in any interest rate model of the risk free rate. The topics presented in this section will be used in §2.2 and in Chapter 4, where we will present a calibration procedure for the Hull–White model (255) of interest rates.

In this section we will assume that dates are given relative to a known, fixed, conveniently chosen date. That is, we will measure time as elapsed time from that particular date. In that way the expression “time t ” will mean a date t days, months, etc. after our special initial date which conveniently corresponds to time $t = 0$.

Definition 2.1.8. *If T is the maturity of a given contract, and t represents a time prior to T , $t \leq T$, we define the **time to maturity** or **TtM** as $T - t$, the actual time count in the selected units of time (days, years, minutes, etc.).*

Interest rates can be defined based on the prices of zcb’s and day counting conventions as a measure of the return (interest) in an investment on bonds. For example, if $B(t, T)$ is the time t price of a (risk-less) zcb with maturity T of less than one year, and $\varphi(t, T)$ is the year fraction, with respect to day counting convention φ , one may see that the (yearly) interest in such an investment is given as the quotient of $1 - B(t, T)$ and $\varphi(t, T)B(t, T)$, thus we obtain

$$r_{\varphi}(t, T) = \frac{1 - B(t, T)}{\varphi(t, T)B(t, T)}$$

the simply compounded, annual rate¹² r reset at time t and with maturity T ¹³. If time t is well known or clearly implied from context we may reduce the notation writing r_T . Notice that the applied day counting convention influences the value of r , and thus the perception we have of the return in the investment. Similarly, if $\varphi(t, T)$ represents not the year fraction but the monthly fraction, etc. one may correspondingly change the interpretation of rate

¹¹See for example Bodie, Kane and Marcus [17] chapter 15 for an introduction to the analysis of Yield curves.

¹²If the day counting convention can be easily implied from the context, we will omit the subindex φ from the symbol representing the interest rate.

¹³Essentially if t is not zero, this is a “forward” rate, see Definition 2.1.13 and (36).

r . Thus, in general we may define

Definition 2.1.9. Simple compounded rate. *Let $B(t, T)$ be the time t price of a zcb of maturity T , and let φ represent a day counting convention, with $\varphi(t, T)$ being the corresponding time fraction between t and T (with respect to the convention). We define the simply compounded φ -rate $r_\varphi^s(t, T)$, with reset t and maturity T as*

$$r_\varphi^s(t, T) = \frac{1 - B(t, T)}{\varphi(t, T)B(t, T)} . \quad (18)$$

We use notation r_T^s for $r_\varphi^s(t, T)$ when both t and φ are known.

Clearly, Definition 2.1.9, implies also a pricing procedure for a simple compounded zcb defined with respect to a rate $r_\varphi^s(t, T)$ valid for the time period $[t, T[$, since

$$B(t, T, \varphi) = \frac{1}{1 + \varphi(t, T)r_\varphi^s(t, T)} \quad (19)$$

should be the time t price of a zcb yielding an interest of $r_\varphi^s(t, T)$ in a time fraction of $\varphi(t, T)$ maturing at time T . Thus we can also regard r as the simple compounded yield of the bond.

Examples of simple compounded rates are the interbank rates, or LIBOR rates. Simple compounded rates are also used when quoting the price of short maturity¹⁴ bonds.

Definition 2.1.9 does not describe the only possible type of interest rate formulation. In fact, Definition 2.1.9 may not be appropriate in some situations.

Consider the case in which an investment is placed on a long maturity bond,¹⁵ a 30 year maturity bond for example. Such a bond could be paying at maturity 4.3 times the initial investment. But such a payment, representing a return of 3.3 (or 330%) in 30 years,¹⁶ is not easy to compare to other short lived investments.

Therefore, we need to change our approach.

We assume that bonds are perfectly divisible, and that there are no transaction costs, nor liquidity or reinvestment risk. Consider the scenario where a sum (\$1 as usual) is

¹⁴less than one year

¹⁵In this context “long maturity” is use to refer to a maturity longer than one time period —year, month, etc.— while “short maturity” is mostly used in cases where the maturity is shorter than a time period.

¹⁶or $3.3/30 = 0.11 = 11\%$ yearly interest simple compounded

invested in one year maturity zcb's, and then, at maturity, all proceeds reinvested in a similar one year maturity zcb. Now say we carry over this scheme for thirty years, and that over that period of time, the prices of one year maturity zcb's do not change. Let r be the simple compounded (yearly) rate of such bonds. At the end of the thirty years the proceeds of our reinvestment scheme will be

$$(1 + r)^{30}$$

What should be the (constant) value of r that ensures $(1 + r)^{30} = 4.3$?

That will be the value of r that will allow us to replicate, through reinvestment on one year maturity zcb's, the earnings of a thirty year bond.

Solving we obtain $r \simeq 0.049821869 \simeq 4.98\%$, per year.

Thus r represents the yearly compounded rate of our 30 year maturity bond, with the advantage that we can now compare such a rate with short maturity investments in an easier way.

Definition 2.1.10. Annually compounded rate. *Let $B(t, T)$ be the time t price of a zcb of maturity T , and let φ represent a day counting convention ($\varphi(t, T)$ being the corresponding year fraction between t and T with respect to such a convention). We define the φ -annually compounded rate, $r_\varphi^a(t, T)$, with reset t and maturity T , as*

$$r_\varphi^a(t, T) = \left(\frac{1}{B(t, T)} \right)^{\frac{1}{\varphi(t, T)}} - 1 . \quad (20)$$

Similarly, if $r_\varphi^a(t, T)$ is the φ -annually compounded rate of our bond,

$$B(t, T) = \frac{1}{(1 + r_\varphi^a(t, T))^{\varphi(t, T)}} \quad (21)$$

as before, we may suppress t , T , a or φ when those are known or clearly implied from the context.

Similarly, if $\varphi(t, T)$ does not represent a year fraction, but any other time fraction (with respect to a given base time period), we can use Definition 2.1.10 to define rates compounded multiple times with respect to a different base time period.

Definition 2.1.11. Continuously compounded rate. As before, let $B(t, T)$ be the time t price of a zcb of maturity T , and let φ represent a day counting convention, being $\varphi(t, T)$ the corresponding time fraction between t and T . The continuously compounded interest rate $r_\varphi^c(t, T)$ is defined as

$$B(t, T) = \exp(-\varphi(t, T)r_\varphi^c(t, T)) \quad (22)$$

or equivalently as

$$r_\varphi^c(t, T) = -\frac{\ln B(t, T)}{\varphi(t, T)} \quad (23)$$

Many more definitions are used in the market, but we will not go into further details in this topic. The interested reader is referred to the bibliography.

Simple compounded and annually compounded rates are used to define the *Yield Curve*.

Note 2.1.1. It is also market use to quote short maturity zcb's using their corresponding simply compounded yield and to quote mid to long maturities using their annually compounded yield. In this way, yield curves can be defined for families of zcb's.

Definition 2.1.12. Yield Curve. Let $r^s(t_0, t)$ be the φ -simple compounded interest rate, $t - t_0 \leq 1$ year, quoted at time t_0 , and let $r^a(t_0, t)$ be the φ -annually compounded interest rate, $t - t_0 > 1$ year, as quoted at time t_0 . We define the *Yield Curve* as the graph of the maturity t yields (theoretical spot rates) $y(t_0, t)$, for $t_0 < t$, where

$$y(t_0, t) = \begin{cases} r^s(t_0, t) & t - t_0 \leq 1 \\ r^a(t_0, t) & t - t_0 > 1 \end{cases} \quad (24)$$

and time is measured in years following the day counting convention implied by φ .

As mentioned before, yield curves are one of the main ingredients in the calibration of an interest rate model. Thus, some attention must be paid to the construction of yield curves (sometimes referred as bootstrapping of interest rates).

In practice (that is in real markets) not all maturities are offered or quoted, thus instead of a curve, only a finite set of rates are available. This forces the introduction of conventions and uses proper to the market. For example, it is common practice to linearly interpolate

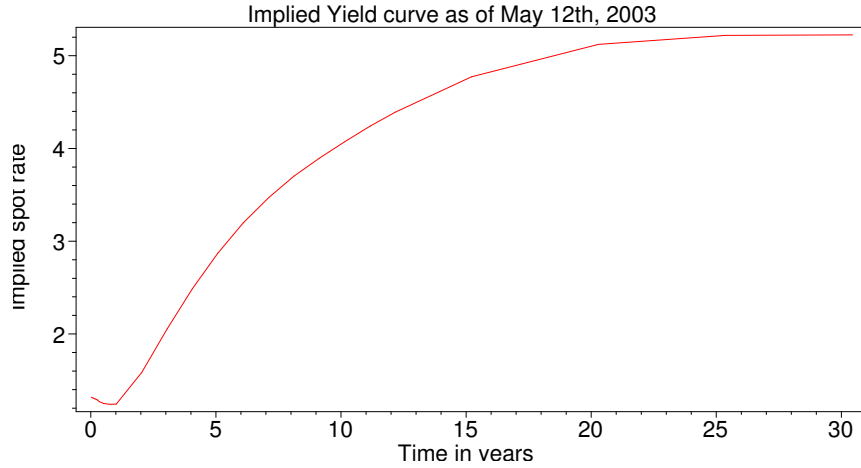


Figure 1: This plot shows an Implied Yield Curve extracted from Swap rate data as on May 12 2003. May 12 2003 was used as the reference date, and time is counted from that date on (thus $t = 1$ corresponds to Wed. May 12 2004, etc.).

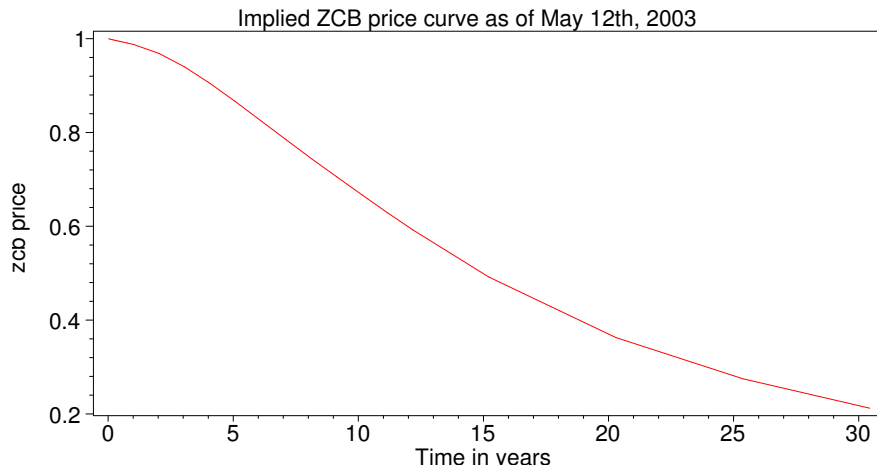


Figure 2: Implied zcb price curve extracted from Swap rate data as on May 12 2003

rates not available using the two closest maturities available. Both, Figure 1 and Figure 2, are using linear interpolation. This off course is not the only problem, bonds introduced into the market at some date will “age”; a two year maturity zcb introduced last year will be “like” a one year maturity zcb today. Thus we have the problem of *seniority*. Similarly, many other issues come in play, like *liquidity* (that is the ease to turn such instruments into cash), the discontinuance of a particular bond maturity, etc.

The number of available points in a Yield Curve also changes from market to market,

and from country to country. That number also changes depending on the methodology adopted to estimate those points. We will come back to this topic in §2.2.

Since only a finite number of maturities are available in a real market, it makes sense to talk about a *term structure*, to make reference to the set of available maturities of a given financial instrument; for example the term structure of zcb's. Similarly one may talk about the *term structure of interest rates* (for a given financial instrument), to make reference to the fact that only a finite number of maturities, and therefore of corresponding interest rates, is available.

In order to avoid many of the problems mentioned here, we will introduce in section §2.2, a bootstrapping procedure that uses information from the Swaps market. The following two subsections will introduce the background required.

2.1.4 General Forward Rate Agreements

Definition 2.1.13. *A Forward Rate Agreement or FRA is, simply put, a contract starting at time S in which two parties agree at time $t \leq S$ to exchange cash flows at a given date $T > S$ in the future; both cash payments are quoted as interest rate payments with respect to a nominal amount N which is not exchanged, and two interest rates K and $Vr(S, T)$. At time t rate K is known and constant throughout the life of the contract. On the other hand, rate $Vr(S, T)$ is only known at time S , $t \leq S < T$. The dates t , S , and T are known as the settlement, the reset or start, and the maturity dates, respectively.*

It is assumed that the rates involved are simply compounded and that each of them may carry a different day counting basis, which means that up to two different time fractions will be involved in the computation of the payments. If $\varphi_K(S, T)$ and $\varphi_{Vr}(S, T)$ are the time fractions between dates T and S corresponding to rates K and $Vr(S, T)$, respectively; interests paid by the two parties involved in an FRA are:

$$\begin{aligned} N\varphi_K(S, T)K \\ N\varphi_{Vr}(S, T)Vr(S, T) \end{aligned} \tag{25}$$

Forward rate agreements are usually priced to make their initial value equal to zero and are quoted in the market using the particular fixed rate $K = K_f(t, S, T)$, the forward

rate, that makes the contract valueless at settlement. Therefore, there are no payments at settlement or reset dates.

Consideration of this contract is given in several books and other bibliographical resources¹⁷. In our particular case, we are interested in the following simple generalization.

Definition 2.1.14. A **Generalized Forward Rate Agreement** or *gFRA* is a contract in which two parties agree at time t to exchange a series of variable interest rate cash flows, at times $\{T_i\}_{i \in \mathbb{N}_n}$, where $t \leq T_0 < T_1 < \dots < T_{n-1} < T_n = T$, with one fixed rate payment at time $T_n = T$. As in the case of an FRA, cash flows are quoted as interest rate



Figure 3: gFRA cash flow time schedule

payments with respect to a nominal amount \mathcal{N} which is not exchanged, and interest rates $\{Vr(T_{i-1}, T_i)\}_{i \in \mathbb{N}_n}$ and \mathcal{K} . The starting date of the contract is T_0 . At time t , $t \leq T_0$, rate \mathcal{K}

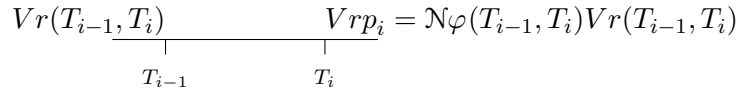


Figure 4: gFRA's i^{th} cash flow

is known and constant throughout the life of the contract. On the other hand, for $i \in \mathbb{N}_n$, the rate $Vr(T_{i-1}, T_i)$ is known at time T_{i-1} . The dates t , $\{T_i\}_{i \in \mathbb{N}_{n-1}^*}$, $\{T_i\}_{i \in \mathbb{N}_n}$, T_0 and T are known as the settlement, the resets, the installments, the starting and the maturity dates, respectively.

As discussed earlier, associated with each rate there is a day counting basis and corresponding time fractions $\varphi(T_{i-1}, T_i) = \varphi_{Vr}(T_{i-1}, T_i)$, $i \in \mathbb{N}_n$ (in the case of the variable interest rate Vr); and $\varphi_{\mathcal{K}}(T_0, T_n) = \varphi_{\mathcal{K}}(T_0, T)$ (in the case of the constant rate \mathcal{K}) used to determine the compounding of interests. If the variable rates in our contract follow the

¹⁷See for example [133], [55], [97] and [19], between many others.

Treasury yield curve, the natural choice for the day counting basis associated with such variable rates should be that used with the Treasury securities¹⁸.

Definition 2.1.15. *Both FRA's and gFRA's are classified into **Payer FRA's** and **Payer gFRA's** respectively or **Receiver FRA's** and **Receiver gFRA's** depending on the directions of the cash flows. A Payer FRA (resp. Payer gFRA) is one in which a fixed rate payment is given against the corresponding variable rate payment(s). Similarly, in a Receiver FRA (Receiver gFRA respectively) one fixed rate payment is received while a (n, in the case of a Receiver gFRA) variable rate payment(s) is (are) given.*

Clearly more generalizations are possible, for example one could consider the case in which reset and installment dates do not coincide at all, one could also consider the intermediary's fees, and other transaction costs, or contingencies such as default risk, etc. but those generalizations escape the scope of this work.

Later we will see that a gFRA could be seen as a portfolio of zeroes.

Assuming that the market contains enough zcb's (that is zcb's maturing at all reset and settlement dates considered in the gFRA) and that the market is free of arbitrage, we can price a gFRA as follows. First we will call φ the time fraction corresponding to the standard day counting basis used with the zcb's, which we assume is also the day counting basis quoted with the variable rate Vr ¹⁹. Similarly, we will call φ_K the time fraction corresponding to the day counting basis quoted with the fixed rate K . We will assume also that there is a standardized procedure to choose the value of Vr at the i^{th} reset date. For example, and also our implied assumption, this rate is made equal to the spot rate at time T_i valid for the period $[T_i, T_{i+1}]$ —see Definition 2.1.6—, that is the interest rate of a zcb starting at time T_i and maturing at time T_{i+1} . In order to simplify both notation and computations, we will assume that all periods $[T_i, T_{i+1}]$, $i \in \mathbb{N}$, are “short”

¹⁸Although someone could quote that rate with respect to a different day counting basis and perform the conversion internally, such a “generalization” seems of doubtful importance and will contribute to make formulas a lot more complex.

¹⁹As we mentioned before, one could assume those two day counting bases to be different, but such generalization will only complicate notation.

(that is periods of time shorter than a year). Our previous assumptions are equivalent to $B(T_i, T_{i+1}) (1 + \varphi(T_i, T_{i+1}) Vr(T_i, T_{i+1})) = 1$.

A gFRA involves $n+2$ times (please see Figure 3): t , the settlement date; T_0, T_1, \dots, T_{n-1} , the reset dates; and $T_1, T_2, \dots, T = T_n$, the n installment dates (note that T_n , the n^{th} installment date is also the maturity T of the gFRA).

For each $i \in \mathbb{N}_n$, at time T_{i-1} a variable rate Vr is quoted for the time interval $]T_{i-1}, T_i]$, “fixing” the floating payment to be given at time T_i . It is clear that the variable rate payment given by the first party at the end of the i^{th} time interval will be:

$$Vr p_i = \mathcal{N} \varphi(T_{i-1}, T_i) Vr(T_{i-1}, T_i) = \mathcal{N} \left(\frac{1}{B(T_{i-1}, T_i)} - 1 \right) \quad (26)$$

At time $T = T_n$ the second party will make its fixed rate payment to the first party

$$frp = \mathcal{N} \varphi_{\mathcal{K}}(T_0, T_n) \mathcal{K} \quad (27)$$

In terms of the prices of zcb’s, each payment can be “discounted” back to the settlement date t to obtain the time t total amount of cash flows. Since a zcb of maturity T_n will pay a unit of currency at time T_n , it is clear the time t “discounted” value of frp is

$$\mathcal{N} \varphi_{\mathcal{K}}(T_0, T_n) \mathcal{K} B(t, T_n) \quad (28)$$

Under our assumption that there are enough zcb’s in the market, and the assumption of no arbitrage, it is not hard to see that

$$B(t, T_i) = B(t, T_{i-1}) B(T_{i-1}, T_i) \quad (29)$$

Therefore, the time t value of a variable rate payment, made at time T_i , will be

$$\begin{aligned} \mathcal{N} \left(\frac{1}{B(T_{i-1}, T_i)} - 1 \right) B(t, T_i) &= \mathcal{N} \left(\frac{1}{B(T_{i-1}, T_i)} - 1 \right) B(t, T_{i-1}) B(T_{i-1}, T_i) \\ &= \mathcal{N} (B(t, T_{i-1}) - B(t, T_i)) \end{aligned} \quad (30)$$

Thus, the sum of all time t variable rate payments, made at times T_i , $i \in \mathbb{N}_n$, is

$$\mathcal{N} \sum_{i=1}^n (B(t, T_{i-1}) - B(t, T_i)) = \mathcal{N} (B(t, T_0) - B(t, T_n)) \quad (31)$$

Adding the payments to the first party (receiving fixed rate payments and giving variable rate payments), equations (28) to (31)²⁰ we obtain:

$$\begin{aligned} & \mathcal{N}\varphi_{\mathcal{K}}(T_0, T_n)\mathcal{K}B(t, T_n) - \mathcal{N}(B(t, T_0) - B(t, T_n)) \\ &= \mathcal{N}((\varphi_{\mathcal{K}}(T_0, T_n)\mathcal{K} + 1)B(t, T_n) - B(t, T_0)) \end{aligned} \quad (32)$$

Therefore, the time t value of a gFRA (receiving fixed rate, paying variable rate) is

$$g\mathcal{FRA}(t, T_0, T_1, \dots, T_n, \varphi, Vr, \varphi_{\mathcal{K}}, \mathcal{K}, \mathcal{N}) = \mathcal{N}((\varphi_{\mathcal{K}}(T_0, T_n)\mathcal{K} + 1)B(t, T_n) - B(t, T_0)) \quad (33)$$

We can see that there is a value $\mathcal{K} = gFr$, the generalized forward rate, that makes this contract valueless at time t . Such a fixed rate is:

$$gFr(t, T_0, T_n, \varphi_{\mathcal{K}}) = \frac{1}{\varphi_{\mathcal{K}}(T_0, T_n)} \left\{ \frac{B(t, T_0)}{B(t, T_n)} - 1 \right\} \quad (34)$$

Notice that if $n = 1$ we are in the case of an FRA. Thus if t , S and T are the settlement, the start, and the maturity dates; \mathcal{N} is the nominal, \mathcal{K} is the fixed rate, Vr the variable rate, and φ and $\varphi_{\mathcal{K}}$ are the corresponding day counting bases of the FRA, then

$$\mathcal{FRA}(t, S, T, \varphi, Vr, \varphi_{\mathcal{K}}, \mathcal{K}, \mathcal{N}) = \mathcal{N}((\varphi_{\mathcal{K}}(S, T)\mathcal{K} + 1)B(t, T) - B(t, S)) \quad (35)$$

is the time t value of an FRA, while

$$Fr(t, S, T, \varphi_{\mathcal{K}}) = \frac{1}{\varphi_{\mathcal{K}}(S, T)} \left\{ \frac{B(t, S)}{B(t, T)} - 1 \right\}, \quad (36)$$

the forward rate quoted at time t for the period $]S, T]$, is the fixed rate that renders the FRA contract valueless at inception.

Even more interesting, we notice that the time t value of a gFRA does not depend at all on the number of intermediate payments, but only on data associated with the settlement date, the first reset date (the start date) and the maturity date. It is this last property which will help us to easily price a more complex contract: a Swap.

2.1.5 Simplified version of Swaps

Instead of exchanging one fixed payment by a finite stream of floating rate payments, one may be interested in exchanging two streams of payments. When both streams are

²⁰Note that the cash flows are not given in the same direction, thus one must consider a change of sign.

computed as interest rate payments based on the same nominal and two given interest rates, the resulting contract is called a *Swap*. The next definition covers the particular case in which one of the rates is fixed.

Definition 2.1.16. *An **Interest Rate Swap** or simply a **Swap** is a contract in which two parties agree to exchange two cash flows consisting of interest rate payments based on the same nominal N , and two interest rates, K which is constant throughout the life of the contract and known at settlement, and Vr which is a variable rate, known only at reset dates $\{T_i\}_{i \in \mathbb{N}_{m-1}^*}$ and valid through the time period $]T_{n-1}, T_n]$, when reset at T_{n-1} , for $n \in \mathbb{N}_m$.*

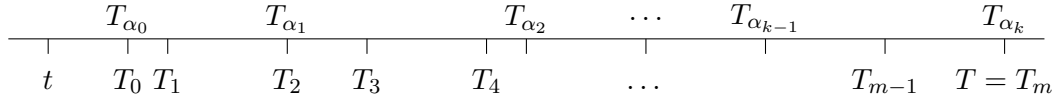


Figure 5: Swap's cash flow time schedule

In the case of a **Payer interest rate swap** or **Payer Swap**, k fixed rate payments ($k \leq m$) are given at times $\{T_{\alpha_j}\}_{j \in \mathbb{N}_k} \subset \{T_i\}_{i \in \mathbb{N}_m}$, while floating rate payments are received at times $\{T_i\}_{i \in \mathbb{N}_m}$. In the case of **Receiver interest rate swap** (or simply a **Receiver Swap**) floating rate payments are given while fixed rate payments are received. As in the case of a *gFRA*, nominals are not exchanged.

As before we will call t the settlement date, $T_0 = T_{\alpha_0}$ the start date, $\{T_i\}_{i \in \mathbb{N}_{m-1}^*}$ and $\{T_{\alpha_n}\}_{n \in \mathbb{N}_{k-1}^*}$ the reset dates, $\{T_i\}_{i \in \mathbb{N}_m}$ and $\{T_{\alpha_n}\}_{n \in \mathbb{N}_k}$ the installment dates, and $T = T_m = T_{\alpha_k}$ the maturity date.

In our definition of a swap we have assumed that the installment and reset dates of the fixed rate *leg* are (respectively) subsets of the installment and reset dates of the floating rate leg, we have assumed also that one of the rates is fixed. Those are qualities that could be relaxed giving rise to a more general version of a swap contract which escapes the scope of this document.

We observe that payments are exchanged at different times $\{T_i\}_{i \in \mathbb{N}_m}$ and $\{T_{\alpha_n}\}_{n \in \mathbb{N}_k}$

(this last set a subset of the former), but that the first reset date and the last installment date are the same for both cash streams.

Also we assume two different day counting bases φ and $\varphi_{\mathcal{K}}$ are used in the computation of interest rate payments with respect to rates Vr and \mathcal{K} respectively. In our applications, we will assume that the day counting basis used with rate Vr is the same as that used with the Treasuries; in fact we assume Vr is the zcb's rate.

Thus $\varphi(T_n, T_{n+1})$ is the time fraction between T_n and T_{n+1} measured according to the day counting basis corresponding to variable rate Vr , and $\varphi_{\mathcal{K}}(T_{\alpha_n}, T_{\alpha_{n+1}})$ is the time fraction between T_{α_n} and $T_{\alpha_{n+1}}$ measured according to the day counting basis corresponding to fixed rate \mathcal{K} .

In order to value this contract we need to account for all cash flows, considering their time t (the date at which the contract is quoted) value in terms of zcb prices. To avoid some notational complexity in the situation, we have assumed that all dates involved (maybe with the sole exception of the settlement date t which is assumed to satisfy $t \leq T_0 = T_{\alpha_0}$) are dates listed in the finite sequence $\{T_i\}_{i \in \mathbb{N}_m}$. Based on our discussion of the generalized forward rate agreement, we notice that our swaps can be decomposed into k generalized forward rate agreements with maturities $T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, \dots, T_{\alpha_k}$, settlement date t , floating rate payments (maybe not the same number in each period) at dates

$$T_{\alpha_{i-1}+1}, T_{\alpha_{i-1}+2}, \dots, T_{\alpha_{i-1}+j_i-1}, T_{\alpha_{i-1}+j_i} = T_{\alpha_i} \quad (37)$$

and reset dates

$$T_{\alpha_{i-1}}, T_{\alpha_{i-1}+1}, T_{\alpha_{i-1}+2}, \dots, T_{\alpha_{i-1}+j_i-1} = T_{\alpha_{i-1}} \quad (38)$$

where we have defined $j_i = \Delta\alpha_i = \alpha_i - \alpha_{i-1}$.

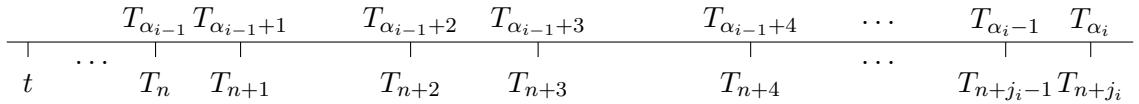


Figure 6: Swap's n^{th} gFRA cash flow time schedule

Thus, the Receiver swap's price at time t (receiving fixed, paying variable) will be

$$\begin{aligned}
& \mathcal{S}(t, \{T_i\}_{i \in \mathbb{N}_m}, \{T_{\alpha_j}\}_{j \in \mathbb{N}_k}, \varphi, Vr, \varphi_{\mathcal{K}}, \mathcal{K}, \mathcal{N}) \\
&= \sum_{i=1}^k g\mathcal{FRA}(t, \{T_{\alpha_{i-1}+j}\}_{j \in \mathbb{N}_{j_i}^*}, \varphi, Vr, \varphi_{\mathcal{K}}, \mathcal{K}, \mathcal{N}) \\
&= \mathcal{N} \sum_{i=1}^k \{(\varphi_{\mathcal{K}}(T_{\alpha_{i-1}}, T_{\alpha_i})\mathcal{K} + 1) B(t, T_{\alpha_i}) - B(t, T_{\alpha_{i-1}})\} \\
&= \mathcal{N} \left\{ B(t, T_{\alpha_k}) - B(t, T_{\alpha_0}) + \mathcal{K} \sum_{i=1}^k \varphi_{\mathcal{K}}(T_{\alpha_{i-1}}, T_{\alpha_i}) B(t, T_{\alpha_i}) \right\} \quad (39)
\end{aligned}$$

As in the case of gFRA's and FRA's, one can easily notice that there is a fixed rate $\mathcal{K} = Sr$ that makes the contract valueless at time t . Such a rate is commonly called the *swap rate*. Solving for \mathcal{K} in (39), we obtain:

$$Sr(t, \{T_{\alpha_i}\}_{i \in \mathbb{N}_k}, \varphi_{\mathcal{K}}) = \frac{B(t, T_{\alpha_0}) - B(t, T_{\alpha_k})}{\sum_{i=1}^k \varphi_{\mathcal{K}}(T_{\alpha_{i-1}}, T_{\alpha_i}) B(t, T_{\alpha_i})} \quad (40)$$

Remark 2.1.2. It is important to observe that as gFRA's reduce to FRA's when the number of floating rate payments collapse to one; Swaps will reduce to gFRA's when the number of fixed rate payments collapse to one, and furthermore, to FRA's when both legs of the Swap contain only one payment. In this sense all FRA's and all gFRA's are also Swaps.

Figure 7 shows 'bid' and 'ask' US dollar swap rates as reported in Bloomberg's page SWDF on May 12 2003.

As we mentioned at the end of §2.1.3 not all (theoretically possible) maturities of a given instrument are available in a real market. Even in a big market as the Swap market, only a reduced number of maturities are available.

For example, in the case of US markets, up to 24 swap rates are quoted, corresponding to 24 fixed maturities. Table 1 list the commonly available maturities in US markets.

This issue of availability is not only restricted to maturities, but as the reader may expect, to all other features of the Swap. For example, common contracts are available with annual (one payment per calendar year), semiannual (that is, two payments per year, spaced every six months) or quarter-quarter (four payments per year, spaced three calendar months) legs, being a common combination a semiannual/quarter-quarter. This issue also imposes an additional restriction to the available maturities, for example if a Swap's legs

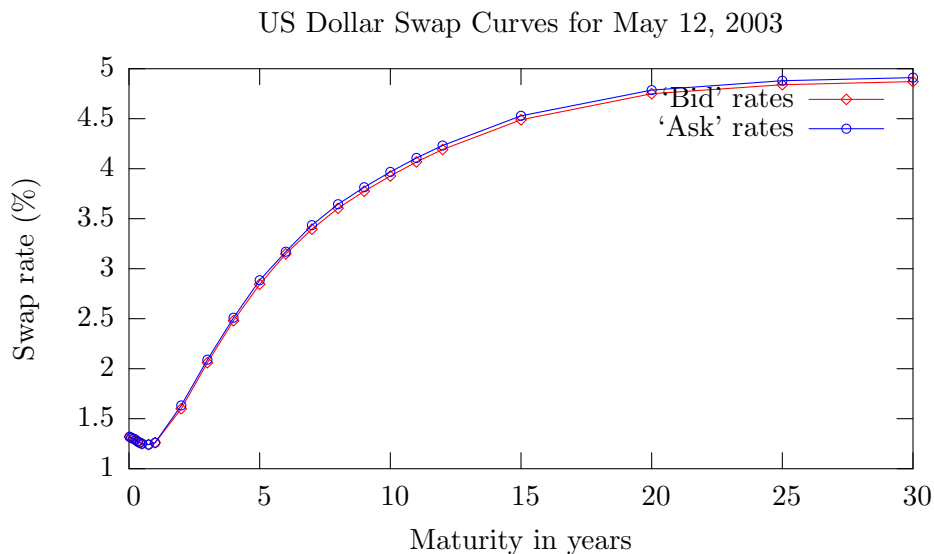


Figure 7: Swap rate curves as on May 12 2003. Shown are the swap rates (bid and ask quotes) corresponding to the 24 different maturities available in the US market.

Table 1: Usual swap rate maturities available in the US Market

Available Maturities				
Short		Long		
1 week	5 months	2 years	7 years	12 years
1 month	6 months	3 years	8 years	15 years
2 months	9 months	4 years	9 years	20 years
3 months	1 year	5 years	10 years	25 years
4 months		6 years	11 years	30 years

are semiannual and quarter-quarter, the shortest maturity that fits into such description is of six months (which will correspond to a gFRA with two quarterly floating rate payments and one fixed rate payment at maturity), with longer maturities being “multiples” of this short maturity.

To compensate for this kind of situation several market uses and conventions have been introduced. For example, in a family of semiannual/quarter-quarter Swaps, a nine (or seven and eight) month maturity Swap will have three “quarterly” payments (in the case of seven and eight months the last “quarter” is “short”) in one leg and one payment at the sixth month and another at maturity in its other leg. With the intention to obtain a Swap (rate)

curve, gFRA's and FRA's with maturities shorter than six months will also be included according to accepted conventions. The contracts usually included are four short maturity FRA's and two short maturity gFRA's. Namely, one week, one, two and three months maturity FRA's with same reference rates and same day counting conventions; and gFRA's with four and five months maturities, with payments at the third month and at maturity, and with same reference rates and same day counting conventions.

This explain the very short maturities in Table 1.

Definition 2.1.17. *A Swap (rate) curve is defined for a given family of Swap contracts as the graph of the Swap rates (at a given settlement date t_0) versus maturity. In practice, rates corresponding to unquoted maturities are linearly interpolated using the two closest maturities above and below.*

An example of Swap curve is provided in Figure 7.

2.2 *Bootstrapping of Yield information from Swaps data*

How did we generate the plot in Figure 1? In this section we describe the bootstrapping procedure used to obtain an Implied Yield Curve²¹ from Swap Rate data.

Interestingly, although the basic financial instruments are the zcb's and Coupon Bonds, it is the Swap market which exhibits higher activity, and volume. It is a fact that the Swap market is more liquid than the Bond market²². This being the case, since swaps can be seen as portfolios of zcb's, when looking for data on the yield curve it makes sense to use Swap

²¹By *Implied Yield Curve* we mean “the” zcb interest rates that should be in effect to make the swap rates computed through (40) to be those found (quoted) in the market.

²²According to the ISDA (International Swap and Derivatives Association, Inc.), mid year market survey —released on September 23rd, 2003—, the vanilla swap market has surpassed \$120 trillion NPOV (notational principal outstanding volume). Both, vanilla swaps and credit derivatives (credit default swaps, baskets and portfolio transactions) have exhibited a growth of 25% in the first semester of 2003. NPOV for equity derivatives (equity swaps, options and forwards) grew 14% to \$2.78 trillion, while credit derivatives experienced a growth of approximately 25% in the first six months of 2003; credit derivatives NOPV was reported as \$2.69 trillion. For more information visit the ISDA web site: <http://www.isda.org>.

The OCC (Office for the Comptroller of the Currency), on its “OCC Bank Derivatives Report, Second quarter 2003” reports that the USA's NPOV of all derivatives is on the order of \$65.8 trillion, 96% of which

data to obtain the perceived (by the “market”) yield curve. This indirect procedure is less complex than that of the bootstrapping of the yield curve from Bond market data²³ in the sense that it does not require from the practitioner a set of subjective decisions regarding the liquidity,²⁴ longevity and impact of some bond issues; complex details associated with series of bond prices pertaining to different emissions (that is, bonds that were created and placed in the market at different dates); statistical analysis of the data, etc.

Instead, one may look for current quotes on the swap market for swaps with respect to the underlying rate one wants to study, then using such data to obtain an “implied” rate. Obviously this procedure is tied to market structure and practices.

In this section we will briefly describe the bootstrapping of yields and the formation of the zcb price curve from swap data. We refer the reader to our Maple[©] worksheet, [78],

is accounted for by seven commercial banks, out of a total of 530 banks and trust companies, 99% of USA’s NPOV is associated with the top 25 banks and tc’s (trust companies). The US Swap market’s NPOV is on the order of \$38 trillion (99% held by the top 25 banks and tc’s). For more information visit the web site of the OCC at: <http://www.occ.treas.gov>; as of fall 2003, Quarterly Derivatives Fact Sheets are published under <http://www.occ.treas.gov/deriv/deriv.htm>.

More surveys, reports and information on this topic can be found at the BIS (Bank for International Settlements) web site: <http://www.bis.org>.

For comparison, according to the Bond Market Association (BMA), the total outstanding marketable US Treasury debt is on the order of \$3.38 trillion as of June 30th, 2003. Reports on the US Bond market can be found at the BMA’s web site: <http://www.bondmarkets.com/research>.

US treasury bulletin is also accessible on the web at: <http://www.fms.treas.gov/bulletin/index.html>.

Please see [168], [172] for some details on the Swap market.

²³For a good account on bootstrapping procedures used around the world, please see BIS’ Technical report on zero coupon yield curves [1]. Fisher, Nychka and Zervos ([59]) developed the method of predilection in the USA, Nelson and Siegel [136] describe the method still preferred in most countries. Both [59] and [136] describe direct procedures to bootstrap the yield curve from bond data. See also [153] for an analysis of the use of STRIP data.

Common to all these procedures is the detailed and careful examination of bond data and the rejection of some data based on several criteria.

²⁴For example, US Treasury department decided in October 2001 to suspend issuance of the 30 year bond. This will create an increasing problem when bootstrapping on long term maturities until a moment in which Treasury bond data will not be available for that particular maturity.

for examples and detailed procedures (such a document can be obtained from the author).

In the US, it is market practice to quote, not the price of a swap contract, but its corresponding swap rate (see (40) and Figure 7), with all other details on the swap contract implied by market conventions. For example, in the case of “US Dollar swap” contracts one finds swap rates quoted for a total of 24 non equally spaced maturities (see Table 1), namely: one week, one month, then monthly until six months, nine months, one year and annually until twelve years maturity, then fifteen years, and every five years afterward until thirty years maturity (in what follows we will use t_0 to represent both the first reset and the settlement date, t_1 for a one week maturity and similarly t_2 to t_{24} to represent maturities from one month to thirty years; we will also use φ to represent a day counting convention of *actual/360* and φ_K to represent *30/360*).

Table 2: Naming convention for available maturities

Available Maturities				
Short		Long		
$t_1 \equiv 1$ week	$t_6 \equiv 5$ months	$t_{10} \equiv 2$ years	$t_{15} \equiv 7$ years	$t_{20} \equiv 12$ years
$t_2 \equiv 1$ month	$t_7 \equiv 6$ months	$t_{11} \equiv 3$ years	$t_{16} \equiv 8$ years	$t_{21} \equiv 15$ years
$t_3 \equiv 2$ months	$t_8 \equiv 9$ months	$t_{12} \equiv 4$ years	$t_{17} \equiv 9$ years	$t_{22} \equiv 20$ years
$t_4 \equiv 3$ months	$t_9 \equiv 1$ year	$t_{13} \equiv 5$ years	$t_{18} \equiv 10$ years	$t_{23} \equiv 25$ years
$t_5 \equiv 4$ months		$t_{14} \equiv 6$ years	$t_{19} \equiv 11$ years	$t_{24} \equiv 30$ years

Those contracts assume a Receiver Semiannual/Quarter–Quarter²⁵ payment structure on two rates, one (constant throughout the maturity of the contract) using a 30/360 day counting convention and another (floating, that is reset every quarter) based on an *actual/360* day counting convention. Rates quoted are those (constant rates) that render the contract with zero initial value. As is clear from (40), such rates are tightly related to the corresponding zcb prices as perceived at the settlement of the swap, their number and the times between payments (these last two pieces of information are easily found based on the nature of the contract and the uses of the contracts in the market).

²⁵In this case, Semiannual/Quarter–Quarter refers to semiannual fixed rate payments against quarterly floating rate payments.

As mentioned before, for maturities less than a year, the structure of two semiannual fixed rate payments against four quarterly floating rate payments can not be satisfied.

Thus the convention in the market is to assume that, in the case of maturities shorter than a semester (six calendar months), one fixed rate payment is to be exchanged at maturity against one “variable” rate payment (turning the swaps into simple forward rate agreements). Both payments assume simple compounding; both use the same day counting convention (in the case of the US market, that of the underlying floating rate, which is actual/360). This choice forces the quoted swap rate to match the underlying floating rate (and changes the day counting basis at which such a rate is quoted for that segment of maturities).

In the cases of maturities of six months to nine months, market practice changes a little. For the six months case, one fixed rate payment at maturity is to be exchanged by two quarterly floating rate payments. Still actual/360 day counting is assumed for both rates; both assume simple compounding. Since only one fixed rate payment is exchanged such a swap contract reduces to a generalized forward contract. Therefore it is not hard to show (see (34)) that the implied zcb rate for a maturity of six months should match that of the quoted swap rate. For maturities from seven to nine months a similar convention is used, thus reducing such swaps to generalized forward rate agreements. Again it is not hard to see that the implied zcb rate should match the quoted swap rate. From (34) and (36) (assuming settlement and first reset date are the same) we have:

$$B(t_0, T) = \frac{1}{1 + \varphi(t_0, T)Sr(t_0, T)} \quad T = t_1, t_2, \dots, t_8 \quad (41)$$

which describes the zcb prices for maturities up to nine months.

If we denote the Swap curve data as $Sr(t_0, t)$ and the floating rate as $y(t_0, t)$, we may write:

$$y(t_0, t) = Sr(t_0, t) \quad \text{if} \quad t - t_0 < 1\text{year} \quad (42)$$

At maturities of one or more years, swaps ‘recover’ their quoted nature, so floating rate payments every quarter are to be exchanged by fixed rate payments every semester. Fixed rates are computed on a 30/360 day counting convention while the floating rate is to be the

equivalent actual/360 zcb rate. According to this convention, floating rates not explicitly known are to be linearly interpolated.

The advantage is that, at this stage, rates for zcb's with maturities of three months, six months and nine months are known. If one looks at formula (40), it is clear that the only unknown is that of the price of the longest maturity zcb involved, $B(t_0, t_9)$, the zcb of one year maturity. Therefore one can solve for it from (40) using the implied lower maturity rates and the market quoted swap rate (Table 2 shows the naming convention used here, in particular t_9 corresponds to one year maturity, and t_7 to six months).

$$B(t_0, t_9) = \frac{1 - Sr(t_0, t_9)\varphi_K(t_0, t_7)B(t_0, t_7)}{1 + Sr(t_0, t_9)\varphi_K(t_7, t_9)} \quad (43)$$

To find the corresponding yield, we apply Definition 2.1.9, equation (18) and our previous result to obtain:

$$y(t_0, t_9) = \frac{1 - B(t_0, t_9)}{\varphi(t_0, t_9)B(t_0, t_9)} \quad (44)$$

For all other maturities we are not in the same advantageous situation, since swap rates are quoted for annual terms (two years to twelve years) and then every five years, several swap rates will be missing when solving for zcb prices and yields. Thus, in accordance with market practice, we will assume that intermediate yields are linearly interpolated, this will let us solve for the final zcb price corresponding to every quoted swap rate.

Let T_i , $i = 1, 2, \dots, 60 \in \mathbb{N}_{60}$ represent the semiannual fixed rate payment dates in our swaps, with $T_2 = t_9$ representing one calendar year²⁶, $T_4 = t_{10}, \dots, T_{60} = t_{24}$. Table 3

Table 3: Relation between semiannual dates and available maturities

semiannual dates vs Maturities			
Short	Long		
$T_2 = t_9$	$T_4 = t_{10}$	$T_{14} = t_{15}$	$T_{24} = t_{20}$
	$T_6 = t_{11}$	$T_{16} = t_{16}$	$T_{30} = t_{21}$
	$T_8 = t_{12}$	$T_{18} = t_{17}$	$T_{40} = t_{22}$
	$T_{10} = t_{13}$	$T_{20} = t_{18}$	$T_{50} = t_{23}$
	$T_{12} = t_{14}$	$T_{22} = t_{19}$	$T_{60} = t_{24}$

shows the relation between semiannual dates $\{T_i\}_{i \in \mathbb{N}_{60}}$ and available maturities $\{t_j\}_{j \in \mathbb{N}_{24}}$.

²⁶In practice dates must be corrected according to next business day convention and market observed

Assuming all implied zcb prices are known up to maturity $t_j = T_m$, $j \geq 9$, we can use (40) and Definition 2.1.10 to solve for the yield rate corresponding to the next maturity $t_{j+1} = T_k$, $k > m \geq 2$ as follows

$$y(t_0, T_{m+l}) = \frac{1}{k-m} \{y(t_0, T_m)(k-m-l) + y(t_0, T_k)l\} \quad l = 1, \dots, k-m-1 \quad (45a)$$

$$B(t_0, T_{m+l}) = \frac{1}{(1 + y(t_0, T_{m+l}))^{\varphi(t_0, T_{m+l})}} \quad l = 1, \dots, k-m \quad (45b)$$

$$\begin{aligned} B(t_0, T_k) + Sr(t_0, T_k) \sum_{i=m+1}^k \varphi(T_{i-1}, T_i) B(t_0, T_i) \\ = 1 - Sr(t_0, T_k) \sum_{i=1}^m \varphi(T_{i-1}, T_i) B(t_0, T_i) \end{aligned} \quad (45c)$$

in the previous system of equations, the only unknown is $y(t_0, T_k)$. Notice that the solution of those equations will not only give us the unknown yield, $y(t_0, T_k)$, but also the corresponding zcb price, $B(t_0, T_k)$.

Following the procedure outlined in this section, Figure 1 and Figure 2 were obtained from swap rate data (namely US Dollar swap Semiannual/Quarter–Quarter Actual/360 30/360)²⁷ collected from Bloomberg[©] (pages SWDF and BCSW²⁸). Table 4 shows the data used and the implied yields and zcb prices obtained. Only those yields that are not linearly interpolated (namely, yields corresponding to the annual maturities t_9 to t_{24} , including all short maturities t_1 to t_8) are shown in our table. The remaining 43 yields and zcb prices, corresponding to maturities we are forced to interpolate) can be reconstructed by the reader using the first two expressions in (45). Implied yields have been rounded to the same precision used by Bloomberg[©] in reporting swap rates.

holidays. The procedure followed is this, all sixty dates are initially marked at exactly six calendar month intervals, then in case a particular $T_i, i \in \mathbb{N}_{60}$ falls in a holiday or a non business day, such T_i is corrected according to the business and holiday convention in use. For example T_3 may be a date a little after (in case of next business day) or a little before (in case of prior business day) one and a half calendar years. See §2.1.1 for more details.

²⁷Additional information regarding this swap curve can be retrieved from Bloomberg[©]'s help system.

²⁸Bloomberg[©] page BCSW was about to be decommissioned at the time we collected our data (May 12th 2003, 5:55:04pm EST) and in the process of being substituted by page SWPM. The advantage of the new page (SWPM) is that, at a difference from the old BCSW, it offered limited download of data into an Excel[©] spreadsheet. That extra feature could be used to extra automate the bootstrapping procedure here outlined.

Table 4: US Dollar Swap rate data as of 05/12/2003 and the corresponding yields and zcb prices

Initial swap rate and bootstrapped yield				
Maturity	Swap rate (%)		zcb yield (%)	zcb price
	Bid	ask		
t_1	1.3188	1.3188	1.3188	0.99974
t_2	1.3100	1.3100	1.3100	0.99887
t_3	1.3000	1.3000	1.3000	0.99780
t_4	1.2900	1.2900	1.2900	0.99671
t_5	1.2700	1.2700	1.2700	0.99568
t_6	1.2600	1.2600	1.2600	0.99467
t_7	1.2500	1.2500	1.2500	0.99365
t_8	1.2413	1.2413	1.2413	0.99057
t_9	1.2600	1.2600	1.2432	0.98751
t_{10}	1.6000	1.6330	1.5851	0.96849
t_{11}	2.0600	2.0900	2.0527	0.93996
t_{12}	2.4800	2.5100	2.4870	0.90511
t_{13}	2.8450	2.8850	2.8703	0.86622
t_{14}	3.1480	3.1700	3.1964	0.82565
t_{15}	3.3960	3.4360	3.4685	0.78491
t_{16}	3.6050	3.6450	3.7023	0.74433
t_{17}	3.7750	3.8150	3.8946	0.70542
t_{18}	3.9270	3.9680	4.0717	0.66699
t_{19}	4.0680	4.1080	4.2405	0.62906
t_{20}	4.1920	4.2320	4.3922	0.59254
t_{21}	4.4890	4.5290	4.7726	0.49186
t_{22}	4.7500	4.7860	5.1228	0.36280
t_{23}	4.8400	4.8800	5.2203	0.27505
t_{24}	4.8710	4.9110	5.2248	0.21214

Compared with other bootstrapping procedures, the process we outline in this section is simpler, and provides data one can use to plot a yield curve much like the curves shown in many sources.

At a difference with methods like Fisher–Nychka–Zervos [59], or Nelson–Siegel [136] we do not have to deal with liquidity or seniority of bond issues, no extra study is required to accept or reject data, etc.. Instead we rely in the ability of a much bigger market, that of swaps, to produce no arbitrage data which we can use to infer the yield curve implied by the Swaps market.

Still a big investment in programming must be payed. In addition to the procedures

required to solve the systems of equations, calendar, day counting and time fraction procedures must be implemented as well as different procedures to compute zcb prices, etc..

We programmed all required procedures in detail using Maple[©]. Maple[©] not only offers an environment whose didactic capabilities are evident but it also possesses a very easy to use interface and strong enough programming abilities. Another great advantage of Maple[©] is that one can easily and smoothly combine text, numerical experiments and code to create nice interactive documents. The resulting Maple[©] worksheet, [78] containing our interest rate bootstrapping and spot volatility stripping (see Chapter 4 for details) implementation plus our comments and calendar and day counting procedures, etc. can be obtained from the author.

Extra details about the outlined bootstrapping procedure as well as explanations about market uses, etc. plus examples and source code can be found in the above mentioned Maple[©] document ([78]) which can be obtained from the author.

CHAPTER III

GAME OPTIONS

Abstract

We study Game Options in financial markets with non constant parameter. Generalizations to the work by Yuri Kifer [101] are given which allow interest rates to be random. Game Options lead to a variety of interesting topics of study and applications, where techniques and ideas from different fields can be used.

3.1 Introduction

Given a contract (an option, or a bond, for example) between two parties, an issuer and a holder of the contract, several things could happen that might lead to risk for the holder. The most catastrophic of these might be the bankruptcy of the issuer. Such a credit event will force the issuer to default on the contract. Several other situations could also lead to forms of financial distress for the issuer. Such distress will influence the holder's perception of the issuer's ability to fulfill the conditions of the contract in the future. That distress might also force the issuer to look for ways to reduce its exposure or ways to get out of the contract, 'canceling' it or transferring the contract to a third party.¹

But financial distress is not the only reason why the issuer may want to transfer or cancel a contract. Consider bonds, for example. Both financial theory and 'Market' practice show us that there are several situations, not necessarily associated with default or financial distress, that may lead the issuer to desire to cancel the contract (in part or in full). For example, a bond contract may include in its indenture² a clause forbidding the issuer from

¹Obviously any such solution will involve costs to the issuer, and will likely reduce its profits and/or produce actual loss.

²Please see [17] section § 14.2 and [152] sections § 16.3 and § 20.2 for discussions on bond indentures and in particular on protective covenants. See Richard Wilson's chapter 14 in [55] for a good description of

engaging in certain kinds of transactions or redistributions. Therefore, if the issuer desires to undertake such prohibited actions, it will need to, somehow, cancel the contract³.

Thus it makes sense to study contracts in which cancellation from the issuer's part is allowed in addition to exercise or cancellation on the holder's part⁴. In fact one can easily conceive of a contract in which the issuer may want to cancel (paying a penalty) with the sole intention of raising its profit, of producing a better hedge, etc. This is the case of a game option⁵.

3.2 *Game Options*

In a paper published at the end of the year 2000 ([101]), Yuri Kifer formally introduced Game Options⁶. Roughly speaking, Game Options are contracts between two parties (a seller —issuer or writer—, and a buyer —holder or investor—), according to which the seller can cancel the contract, and the buyer can exercise rights in the contract leading to the end of the contract, at any time throughout the life of the contract. If the contract reaches its maturity, the contract ends with certain predetermined actions taken. If the writer cancels the contract, the writer will pay the buyer a sum greater than the payoff the buyer could have obtained if the buyer had exercised at that time. Otherwise, if the buyer decides to exercise, the buyer will receive a payoff corresponding to the time of exercise.

An example of such a game contract is the following. Consider an *American Game Put Option* on a certain underlying security \mathcal{S} (whose price at time t is represented by S_t), with maturity⁷ T , and strike K . This contract gives the holder the right to exercise the put at

call features.

³[17] and [152] offer instructive examples in this respect.

⁴It is assumed that there is a payment of a suitable penalty if the issuer cancels the contract.

⁵An excellent example of such a contract is a LYON (liquid–yield–option–note), first offered by Merrill–Lynch in 1985. Please see [17] section § 20.6 for a description.

⁶Note that at least a form of game option, the LYONs, have been traded since 1985. In 1986 McConnell and Schwartz [120] developed a simple model for the pricing of LYONs under constant interest rates. No more research has been done in this direction ever since.

⁷We shall assume that this maturity is lower than the possible maturity, if any, of the underlying.

any time, ξ , through the life of the contract, in which case the holder's payoff will be

$$(K - S_\xi)^+$$

where ξ is the time of exercise, and S_ξ is the price of the underlying at the time of exercise. This contract also gives the writer the right to cancel the contract at any time, κ , prior to, or at the maturity of the contract, in which case the holder will be paid the current payoff of the put, plus a certain penalty, p_κ , which we assume is non-negative. Thus, the payoff received by the holder in case of cancellation will be:

$$(K - S_\kappa)^+ + p_\kappa.$$

In a market in which the buyer of options takes on an amount of default risk, but some form of overseer is assumed, one could visualize the case of a writer's defaulting, forcing the payment of the current obligations plus a certain 'compensation' (which could be positive or negative).

3.3 *Standard market model*

In our work we first model a "Market". Such a model should include some features of real markets, but at the same time it should be as "simple as possible" allowing for analytical tractability of important market "objects" like security prices, yet still complex enough as to be of interest (mathematical, financial and/or economical). A good market model should reflect important characteristics of real markets such as sources of systematic risks and appreciation rates. It should also be a believable model, in which "securities" and the "information" available to investors is somehow modeled.

The market model that we will introduce in this section, known as *the Standard Market Model*⁸ (see Definition 3.3.13), will constitute the background for our theoretical work in this chapter.

Under the Standard Market Model, security prices are driven by Brownian motions (which constitute the sources of systematic risk), and many of the market parameters are

⁸Some authors refer to the standard market model, or to many of its particular forms, as (B, S) -model, or (B, S) -market, etc. because the model includes a Bank account and one or more Securities or stocks.

as well described in terms of suitable stochastic processes. The careful construction of the model results in security prices that follow paths that possess known regularity properties (like right continuity, etc.). This model can be seen as a “natural” extension to the renowned Black–Scholes and Merton model [15], [123], [124], [125]; and is of great importance in the field of (modern) Mathematical Finance.

3.3.1 Securities and Prices

In a real market, stocks, bonds, derivatives and a big spectrum of other financial instruments are traded according to certain rules. Some of them can be considered “basic” financial instruments or “securities”, like stocks and bonds; others can be regarded as “derivatives” in the sense that they (or their prices) are derived from basic financial instruments.

Although the number of different financial instruments traded in the different markets around the world is big, even in the biggest markets⁹ we can see that a finite number of securities are traded. Thus it makes sense to consider a market \mathcal{M} in which a finite number n of “basic” securities are traded. We will also assume that all market participants have access to a bank or *money market* account, which we can consider an extra security in our market model \mathcal{M} . We will denote by $\{\mathcal{S}_i\}_{i \in \mathbb{N}_n^*}$ the $n + 1$ securities, and will give special attention to $\mathcal{B} = \mathcal{S}_0$ which we will use to represent the *Bank account*. As explained later in this section, we will model the prices of securities in our market as real valued stochastic processes.

We will assume that our market is free of frictions and liquidity problems. We will also assume that transactions placed by market participants can not influence the market price of the securities traded (that is, transactions do not involve big volumes, and/or the “noise” or “shock” they may introduce into the market dissipates immediately). This assumption is equivalent to assuming that participants have no problem satisfying orders, no matter how big or how small (this last requirement is frequently referred as that securities are infinitely divisible).

⁹We are not trying to be extremely precise here, a “big market” could be a market in which many securities are traded, or a market with very big volumes, or in which many investors are trading, etc.

We will also assume that there is a finite time horizon $\mathcal{T} > 0$ to our market; that is, all transactions are to be placed in the time interval $[0, \mathcal{T}]$.

Notation 3.3.1. If A denotes a vector, array or matrix, we will use the notation A^\dagger to denote the transpose of A . In case needed, as an alternate notation, we will also use A' to denote A^\dagger . Whenever vector notation is needed, we will favor the representation of real vectors as columns¹⁰. Still, due to typographical issues we may list vector elements as rows of numbers. Also due to typographical issues we will favor the use of superscripts to denote vector and matrix elements.

Notation 3.3.2. If a and b are two real n -dimensional vectors, we will write $a \cdot b = a^\dagger b = \sum_{i=1}^n a_i b_i$ to denote the “dot product” of a and b .

Notation 3.3.3. If a is a real n -dimensional vector, we will denote by $\text{diag}(x)$ the $n \times n$ real diagonal matrix whose diagonal elements are the coordinates of x . Thus, the element in position (i, i) is $\text{diag}(x)_{ii} = x_i$.

We assume that uncertainty in the market is driven by $d \geq n$ independent sources of systematic risk¹¹ which we model by means of a d -dimensional Brownian Motion $W = \{W_t\}_{t \in [0, \mathcal{T}]} = \left\{ (W_t^1, W_t^2, \dots, W_t^d)^\dagger \right\}_{t \in [0, \mathcal{T}]}$, defined on a given complete filtered probability space $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, \mathcal{T}]}$ denotes the \mathcal{P} -augmentation of the natural

¹⁰An exception to this rule will be the vector processes σ^i , $i \in \mathbb{N}_n$, to be defined later in this section.

¹¹According to traditional finance theory, the risk of a given security can always be decomposed into systematic risk and unsystematic risk. We understand systematic risk, as that kind of risk that is intrinsic to the market –systematic risk is also known as systemic risk– and can not be eliminated by diversification. In principle, the expected return on an asset depends only on its systematic risk. On the other hand, unsystematic risk (also known as specific, avoidable or diversifiable) can be eliminated “at no cost” throughout diversification. Unsystematic risk is not intrinsic to the market and can be seen as due to the agent’s (corporation, company, etc.) strategies. In lieu of a simple criteria of selection, we can say that systematic risk is that risk that influences a large number of assets, while unsystematic risk influences only a reduced number of assets. The systematic risk in an asset is measured by the beta of that asset. Please see Bodie, Kane and Marcus [17] for a discussion on systematic risk; Shuetrim [164] offers an interesting study about systematic risk.

filtration of W

$$\mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \in [0, T]} = \{\sigma(W_s, 0 \leq s \leq t)\}_{t \in [0, T]}$$

(for each $t \in [0, T]$, $\mathcal{F}_t^W = \sigma(W_s, 0 \leq s \leq t)$ denotes the minimal sub σ —algebra of \mathcal{U} generated by the random variables W_s , $0 \leq s \leq t$).

Since a d —dimensional Brownian Motion is a continuous Strong Markov process, the augmentation of its natural filtration is a continuous filtration ([96] Corollary 2.7.8) which satisfies the “usual conditions”¹² ([96] Definition 1.2.25) with respect to the σ —algebra $\mathcal{F}_T \subseteq \mathcal{U}$. There will be no loss of generality if we assume that $\mathcal{U} = \mathcal{F}_T$; thus, in what follows and unless otherwise stated we will assume $\mathcal{U} = \mathcal{F}_T$.

On the other hand, the use of augmented σ —algebras provide us with several technical advantages and preserves the definition of a d —dimensional Brownian Motion, that is ([96] Theorem 2.7.9), W relative to \mathcal{F} is still a d —dimensional Brownian Motion.

Consistent with the finance dogma that says that the return on an asset’s price is driven by systematic risk, we will model the price $P^i = \{P_t^i\}_{t \in [0, T]}$, $i \in \mathbb{N}_n$, (of one share) of security \mathcal{S}_i , by means of the linear stochastic differential equation¹³

$$dP_t^i = P_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t) = P_t^i \left(\mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right) \quad (46)$$

with positive initial price P_0^i ; while the price $B = \{B_t\}_{t \in [0, T]}$ of security¹⁴ \mathcal{B} is modeled by

$$dB_t = r_t B_t dt \quad (47)$$

¹² We say that a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ of sub σ —algebras of the σ —algebra \mathcal{U} satisfies the *usual conditions* if it is a right—continuous filtration and \mathcal{F}_0 contains all \mathcal{P} —null elements of \mathcal{U} .

¹³We note that equation (46) is also consistent with the seminal work of Sharpe [157], according to which stock prices should depend on their mean rate of return (characterized in (46) by parameter μ_t^i) and their market correlation (which intuitively corresponds to the parameters σ^{ij} in (46)). At least intuitively, (46) indicates that the infinitesimal return dP_t^i/P_t^i follows a diffusion process with drift μ_t^i and dispersion coefficients σ^{ij} , $j \in \mathbb{N}_m$.

¹⁴ Due to historic and pure technical convenience we have selected \mathcal{B} and its price process B as “the” *numéraire* of our market. Such selection is indeed arbitrary, and as explained in [68], and in [163] Chapter 9, any positive, non—dividend paying process (the price process of a zcb, for example) could be used instead. More on this topic in Chapter 5.

where we assume that the process B satisfies the initial condition $B_0 = 1$.

We will write $P = \{P_t\}_{t \in [0, T]}$ to represent the n -dimensional vector process with $P_t = (P_t^1, P_t^2, \dots, P_t^n)^\dagger$, $t \in [0, T]$. We will call P the price process and P^i , $i \in \mathbb{N}_n$, the coordinate price process, or the price processes or (for a particular $i \in \mathbb{N}_n$) the price process of security \mathcal{S}_i .

The process $r = \{r_t\}_{t \in [0, T]}$ is called the *interest rate process*, and models the risk free rate observed in our market. The processes $\mu^i = \{\mu_t^i\}_{t \in [0, T]}$, and $\sigma^i = \{\sigma_t^i\}_{t \in [0, T]} = \{(\sigma_t^{i1}, \sigma_t^{i2}, \dots, \sigma_t^{id})^\dagger\}_{t \in [0, T]}$, $i \in \mathbb{N}_n$, are called the *appreciation rates* and *volatility* processes respectively. The volatility process σ^{ij} models the influence of the j^{th} , $j \in \mathbb{N}_d$, systematic risk source on the price of security \mathcal{S}_i , $i \in \mathbb{N}_n$. For convenience we will also denote by $\mu = \{\mu_t\}_{t \in [0, T]} = \{(\mu_t^1, \mu_t^2, \dots, \mu_t^n)^\dagger\}_{t \in [0, T]}$ the vector of appreciation rates, and by $\sigma = \{\sigma_t\}_{t \in [0, T]} = \{(\sigma_t^{ij}, i \in \mathbb{N}_n, j \in \mathbb{N}_d)\}_{t \in [0, T]}$ the matrix of volatilities. Thus we may rewrite (46) in vector notation as

$$\begin{aligned} dP_t &= \text{diag}(P_t)(\mu_t dt + \sigma_t dW_t) \\ P_0 &= (P_0^1, P_0^2, \dots, P_0^n) \end{aligned} \tag{48}$$

In order to consider dividends, we will assume that (at least some of) the securities \mathcal{S}_i , $i \in \mathbb{N}_n$ pay dividends continually according to *dividend rates* $\delta^i = \{\delta_t^i\}_{t \in [0, T]}$, $i \in \mathbb{N}_n$; so that dividends paid by security \mathcal{S}_i , $i \in \mathbb{N}_n$ in an interval dt amount to $P_t^i \delta_t^i dt$. Thus it makes sense to consider an additional process associated to security \mathcal{S}_i , $i \in \mathbb{N}_n$, $Y^i = \{Y_t^i\}_{t \in [0, T]}$, the *yield process of security* \mathcal{S}_i , which is defined by

$$dY_t^i = P_t^i(\mu_t^i dt + \delta_t^i dt + \sigma_t^i \cdot dW_t), \quad i \in \mathbb{N}_n \tag{49}$$

with initial condition $Y_0^i = P_0^i$. Similarly, we will define $Y = \{Y_t\}_{t \in [0, T]}$ the vector *yield process* as the n -dimensional process defined by $Y_t = (Y_t^1, Y_t^2, \dots, Y_t^n)^\dagger$, $t \in [0, T]$.

Consistent with notation previously introduced, we will denote by $\delta = \{\delta_t\}_{t \in [0, T]} = \{(\delta_t^1, \delta_t^2, \dots, \delta_t^n)^\dagger\}_{t \in [0, T]}$ the *dividend rate process*. Thus, a vector equivalent to (49) can be given

$$\begin{aligned} dY_t &= \text{diag}(P_t)((\mu_t + \delta_t) dt + \sigma_t dW_t) = dP_t + \text{diag}(P_t)\delta_t dt \\ Y_0 &= (P_0^1, P_0^2, \dots, P_0^n) \end{aligned} \tag{50}$$

The interest rate process is also known as a *parameter of the market* \mathcal{M} and collectively, r, μ, δ and σ are also known as the coefficients of market \mathcal{M} .

With the idea in mind of being able to integrate the stochastic differential equations¹⁵ (**sde**'s) (46), (47) and (49), as well as other sde's involving the Market coefficients; processes r, μ, δ and σ are assumed to be progressively measurable processes¹⁶ with respect to filtration \mathcal{F} , satisfying the integrability condition

$$\int_0^T (|r_t| + \|\mu_t\|^2 + \|\delta_t\|^2 + \sum_{i=1}^n \|\sigma_t^i\|^2) dt < \infty, \text{ } \mathcal{P}a.s. \quad (51)$$

where $\|x\|$ denotes the Euclidean norm of vector x . Therefore, P, Y and B are continuous semimartingales.

As we mentioned before, the market model we describe in this section is known as the *Standard Market Model* (**smm**), please see Karatzas [95] for a simplified version of this model. See also Shiryaev [160] chapter VII, and Karatzas and Shreve [97] (in particular chapter 1, although the whole book is an in depth study of the **smm**), and Musiela and Rutkowski [133] chapter 10 for in depth studies of other versions of this model. Additional sources are mentioned in the bibliography section at the end of this document.

The bibliography section at the end of this work cites several sources that the reader can use to, in case needed, get acquainted with many of the technical terms used here. Still we will introduce some useful definitions and provide some of the most important results (without proof) found in the literature regarding the model we present here. Standard reference for this section is found in Shiryaev's *Essentials of Stochastic Finance* [160], Karatzas' *Lectures on the Mathematics of Finance* [95], Karatzas and Shreve's *Brownian Motion and Stochastic Calculus* [96] and *Methods of Mathematical Finance* [97], Musiela and Rutkowski's *Martingale Methods in Financial Modeling* [133], and Revuz and Yor's

¹⁵See [96] §5.2.

¹⁶ Since every measurable and adapted process has a progressively measurable modification (see [96] proposition 1.12) a little extra generalization can be obtained by means of replacing our hypotheses of "progressively measurable" to "measurable and adapted" and then applying [96] proposition 1.12 to select a progressively measurable modification of the processes when needed (two processes, X and Y , defined on $(\Omega, \mathcal{U}, \mathcal{P})$ are modifications of each other if, $\forall t \geq 0, X_t = Y_t$ \mathcal{P} -a.s.).

Continuous Martingales and Brownian Motion [151], among many others. Unless otherwise stated, please refer to those sources for proofs to results mentioned in this section.

Definition 3.3.1. Let $\mathcal{B}([0, t])$ denote the Borel σ -algebra in $[0, t]$, $t \in]0, \infty[$. A real valued process $X = \{X_t\}_{t \in [0, \infty[}$ is called **progressively measurable** with respect to filtration \mathcal{F} if, for every $t \in [0, \infty[$, the associated map

$$\begin{aligned} \phi: [0, t] \times \Omega &\rightarrow \mathbb{R} \\ (s, \omega) &\rightarrow \phi(s, \omega) = X_s(\omega) \end{aligned} \tag{52}$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable.

(See also Definition 3.5.15).

Essentially, the conditions of progressive measurability and the integrability condition (51) are in place to ensure that the price processes, the bank account process and the yield processes are all Itô processes¹⁷, and the strong solutions to the sde's that define them. That is, we can explicitly write:

$$\begin{aligned} P_t^i &= P_0^i \exp \left\{ \int_0^t \mu_u^i - \frac{1}{2} \sigma_u^i \cdot \sigma_u^i du + \int_0^t \sigma_u^i \cdot dW_u \right\} \\ Y_t^i &= P_t^i + \int_0^t P_u^i \delta_u^i du \\ B_t &= \exp \left\{ \int_0^t r_u du \right\} \end{aligned} \tag{53}$$

$i \in \mathbb{N}_n$, $t \in [0, T]$.

Definition 3.3.2. A stochastic process X is **adapted** to filtration \mathcal{F} if for every t , X_t is an \mathcal{F}_t -measurable random variable.

¹⁷See for example [160]. If $a = \{a_t\}_{t \in [0, T]}$ (an n -dimensional process) and $b = \{b_t\}_{t \in [0, T]}$ (an $n \times d$ matrix process) are two adapted stochastic processes defined on $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$ such that

$$\begin{aligned} \int_0^t |a_s^i| ds &< \infty \quad \mathcal{P}\text{-a.s} \\ \int_0^t (b_s^{ij})^2 ds &< \infty \quad \mathcal{P}\text{-a.s} \end{aligned}$$

($i \in \mathbb{N}_n$, $j \in \mathbb{N}_d$) and X_0 is a \mathcal{F}_0 -measurable r.v., the process

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$$

is called an Itô process. In such a case one may also write $dX_t = a_t dt + b_t dW_t$.

Notation 3.3.4. If Y is a (real) random variable defined on a measurable space (Ω, \mathcal{U}) , and \mathcal{G} is a σ -algebra of sets of Ω , we will interpret $Y \in \mathcal{G}$ as “ Y is a \mathcal{G} -measurable random variable”.

If X is a d -dimensional stochastic process, when we say X is adapted to filtration \mathcal{F} , we understand that each component process of X is adapted to filtration \mathcal{F} ; in the case of a right continuous with left limits (RCLL)¹⁸ d -dimensional stochastic process, each component process is RCLL. Similarly, other concepts are naturally extended to vector processes in component-wise fashion.

As shown elsewhere (see [96] §1.1, 1.13):

Proposition 3.3.1. *Any adapted right or left continuous process with left (resp. right) limits (RCLL or LCRL resp.) is progressively measurable.*

Based on Proposition 3.3.1, instead of asking for progressively measurable processes, we could simply ask for adapted RCLL processes¹⁹.

RCLL processes are also known as *càdlàg* processes; *càdlàg* is the acronym from the French expression *continu à droite, limité à gauche*.

Notation 3.3.5. Let A be a random variable defined on (Ω, \mathcal{U}) and let \mathcal{P} be a probability measure defined on (Ω, \mathcal{U}) . We will use the notation $E_{\mathcal{P}}(A)$ to denote the expectation of A with respect to the probability measure \mathcal{P} , in case we need not make explicit reference to the probability measure we will write $E(A)$.

Definition 3.3.3. Let $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$ be a filtered probability space and let $\mathcal{T} > 0$ be the time horizon (that is, the set of time parameters, t , is $[0, \mathcal{T}]$). We define $\mathcal{L}^2 = \mathcal{L}_{\mathcal{P}}^2 = \mathcal{L}^2(\Omega, \mathcal{F}_{\mathcal{T}}, \mathcal{P})$ as the class of all square integrable progressively measurable processes defined on $(\Omega, \mathcal{F}_{\mathcal{T}}, \mathcal{P})$.

¹⁸A stochastic process X is called right continuous with left limits (RCLL) if $\forall \omega \in \Omega \lim_{s \uparrow t} X_s(\omega) = X_t(\omega)$ and $\lim_{s \downarrow t} X_s(\omega)$ exists for every t .

¹⁹As mentioned before; at the expense of additional technicalities (see Footnote 16) the hypothesis of progressive measurability could be relaxed to measurable and adapted plus the selection of progressively measurable modifications of the processes involved.

That is, $\phi \in \mathcal{L}^2$ if

$$E \left(\int_0^T \phi_u^2 du \right) < \infty, \quad (54)$$

where $E = E_{\mathcal{P}}$ denotes mathematical expectation with respect to probability \mathcal{P} .

Also, we define $\mathcal{L} = \mathcal{L}_{\mathcal{P}} = \mathcal{L}(\Omega, \mathcal{F}_T, \mathcal{P})$, the class of \mathcal{P} -a.s. integrable progressively measurable processes defined on $(\Omega, \mathcal{F}_T, \mathcal{P})$; thus a progressively measurable process ϕ defined on $(\Omega, \mathcal{F}_T, \mathcal{P})$ is in \mathcal{L} if it satisfies

$$\mathcal{P} \left(\int_0^T \phi_u^2 du < \infty \right) = 1. \quad (55)$$

Similarly, if Y is a \mathcal{G} -measurable random variable (\mathcal{G} a suitable σ -algebra of sets of Ω), we will write $Y \in L^2 = L_{\mathcal{P}}^2 = L^2(\Omega, \mathcal{G}, \mathcal{P})$ if

$$E(Y^2) < \infty. \quad (56)$$

$L^2(\Omega, \mathcal{G}, \mathcal{P})$ is the class of square \mathcal{P} -integrable \mathcal{G} -measurable random variables.

Finally, we define $L = L_{\mathcal{P}} = L(\Omega, \mathcal{G}, \mathcal{P})$, the class of \mathcal{P} -integrable \mathcal{G} -measurable random variables. $Y \in L(\Omega, \mathcal{G}, \mathcal{P})$ if

$$E(|Y|) < \infty \quad (57)$$

Proposition 3.3.2. *If Z is a standard one-dimensional Brownian motion defined on a filtered probability space (Ω, U, F, P) and $\phi \in \mathcal{L}_P^2$ then the Itô integral $I = \{I_t\}_{t \in [0, T]}$*

$$I_t = I(\phi)_t = \int_0^t \phi_u dZ_u, \quad t \in [0, T] \quad (58)$$

is a square integrable continuous martingale on (Ω, U, F, P) .

The quadratic variation process of $I(\phi)$, denoted $\langle I \rangle = \langle I(\phi) \rangle$, is

$$\langle I \rangle_t = \int_0^t \phi_u^2 du, \quad \forall t \in [0, T] \quad (59)$$

and, the process

$$(I(\phi)_t)^2 - \langle I(\phi) \rangle_t, \quad t \in [0, T] \quad (60)$$

is a continuous martingale on (Ω, U, P, F) .

If $\phi \in \mathcal{L}_P$ then the Itô integral $I(\phi)$ is a continuous local martingale on (Ω, P, F) .

(For a proof of this result please see [88] chapter 2, or [96] chapter 3, or [141] chapter 3 §3.2. As discussed before, see Footnote 16, the condition of progressive measurability can be relaxed to “measurable and adapted”, defining equivalence classes by means of *modification*, and then selecting a progressively measurable modification. If such changes are introduced, the results of Proposition 3.3.2 will remain valid).

It is important to notice that, as well as in real markets, some of the securities traded in our market \mathcal{M} could be characterized as stocks while others could be used to represent zcb’s²⁰.

In some cases, we may order our securities into two groups, the first $m < n$ being stocks, and the last $n - m$ being zcb’s. In case needed, we could use the more explicit notation $S^i = P^i$, $i \in \mathbb{N}_m$ to represent the stock prices, while the notation $\{B^i(t, T_i)\}_{t \in [0, T]}$, $i \in \mathbb{N}_{n-m}$ will be preferred to denote the price of the i^{th} zcb (with maturity $T_i \leq T$). In this way our notation will be consistent with that used elsewhere in this document.

When referring to bonds, the notation used to represent appreciation rates and volatilities will be slightly modified. When explicitly dealing with zcb’s we will use $\{a^i(t, T_i)\}_{t \in [0, T]}$ and $\{b^i(t, T_i)\}_{t \in [0, T]}$ respectively. Since we can distinguish between zcb’s based on their maturity, we may relax the notation a little and write $B(t, T_i)$ instead $B^i(t, T_i)$, $a(t, T_i)$ instead of $a^i(t, T_i)$, and $b(t, T_i)$ instead of $b^i(t, T_i)$. In the case security \mathcal{S}_i represents a bond, we will assume its principal to be one unit of currency. That is, if $T \leq T$ is the maturity of a zcb (security \mathcal{S}_i , for instance); we assume the contract pays one unit of currency at time T .

Investors are usually interested in the prices of the securities traded. Proposition 3.3.2, in combination with (51), (46) and our assumption that the coefficients of market \mathcal{M} are adapted progressively measurable processes, imply that we are modeling the prices of our

²⁰Still, the market model we present here is mostly a “stock based” model. According to our description only a finite number of zcb’s can be represented. Therefore, the market model here presented can not reflect a typical and very important hypothesis of bond market models, namely the assumption of “bonds of all maturities”. For a description of bond market models please see [160] chapter VII §5 and [133]. Considering the use we will make of this model, the restriction to finitely many zcb’s will not be a problem. In particular, with just a few zcb’s we can consider forward prices.

securities as log normal continuous local martingales.

3.3.2 Portfolios and Wealth

Investors trading in real markets deal with unsystematic risk through diversification, that is, a rational investor should acquire a balanced *portfolio* of securities²¹.

Definition 3.3.4. We model a **portfolio strategy** (the way in which the investor decides to distribute his funds at any time) by means of a real $1 + n$ -dimensional progressively measurable process $\Pi = \{(\pi_t^0, \pi_t)\}_{t \in [0, T]} = \{(\pi_t^0, (\pi_t^1, \pi_t^2, \dots, \pi_t^n)^\dagger)\}_{t \in [0, T]}$, for convenience we will use the symbol π to represent the real n -dimensional portfolio process $\pi = \{(\pi_t^1, \pi_t^2, \dots, \pi_t^n)^\dagger\}_{t \in [0, T]}$, and $\pi^0 = \{\pi_t^0\}_{t \in [0, T]}$ to denote the position on the bank account. Thus, we will also write $\Pi = (\pi^0, \pi) = \{(\pi_t^0, \pi_t)\}_{t \in [0, T]}$.

We interpret π_t^i , $i \in \mathbb{N}_n^*$ as the number of shares of security \mathcal{S}_i held (if $\pi_t^i > 0$) or short²² (if $\pi_t^i < 0$) at time t , $t \in [0, T]$. In the particular case of \mathcal{S}_0 , we see π_t^0 as the amount of money kept (if $\pi_t^0 > 0$) in the bank account or borrowed (if $\pi_t^0 < 0$) from the bank account at time t . Process π is also known as a *portfolio process* or simply as a *portfolio* (please see Definition 3.3.5 for additional conditions). The processes $\pi^i = \{\pi_t^i\}_{t \in [0, T]}$, $i \in \mathbb{N}_n^*$ are also known as *portfolio coefficients*. If a portfolio coefficient is null, at time $t \in [0, T]$, we understand that at that time there is no active position in the corresponding security.

Notation 3.3.6. We will use the notation $\vec{\mathbf{1}}_n$ to represent the n -dimensional real vector whose entries are all equal to 1. This notation will naturally extend to $\alpha_n = \alpha \vec{\mathbf{1}}_n$, for $\alpha \in \mathbb{R}$.

Definition 3.3.5. We define a **portfolio process** as a real n -dimensional progressively measurable process $\pi = \{(\pi_t^1, \pi_t^2, \dots, \pi_t^n)\}_{t \in [0, T]}$, such that

$$\int_0^T |\pi_t^\dagger \text{diag}(P_t)(\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t)| dt + \int_0^T \|\pi_t^\dagger \text{diag}(P_t)\sigma_t\|^2 dt < \infty \quad \mathcal{P}a.s. \quad (61)$$

²¹In Costa Rica we say that “*no es bueno cargar todos los huevos en la misma canasta*” (it is not a good idea to carry all your eggs in the same basket). An investor with a balanced portfolio of securities defends himself from unsystematic risk by dividing his funds among securities related to different industries and or different sectors —energy, technology, etc.—.

²²See Footnote 26.

In real life, investors will adopt portfolio strategies, depending on their particular needs and motivations, with the intention to accumulate wealth; wealth that could be later used to achieve some predetermined goal. In real life more complex situations are also possible but will not be considered here. In particular we will not consider “income” (the infusion of additional cash into the portfolio throughout the life of the portfolio strategy) and “consumption” (the use of part of the wealth throughout the life of the portfolio strategy).

Definition 3.3.6. Assuming w_0 is the **initial wealth** of our investor, we define the **wealth process** \mathcal{W} associated with portfolio strategy $\Pi = (\pi^0, \pi)$ as

$$\begin{cases} \mathcal{W}_0 = w_0 \\ \mathcal{W}_t = \pi_t^0 B_t + \pi_t \cdot P_t \quad t \in]0, T], \end{cases} \quad (62)$$

where $P = \{P_t\}_{t \in [0, T]} = \{(P_t^1, P_t^2, \dots, P_t^n)^\dagger\}_{t \in [0, T]}$ denotes the vector process of prices P^i , $i \in \mathbb{N}_n$. In case we need to make explicit reference to the selected portfolio strategy, we will write $\mathcal{W} = \mathcal{W}^\Pi$, if explicit reference to the initial wealth is also required we will write, $\mathcal{W} = \mathcal{W}^{\Pi, w_0}$.

With π_t^i being the number of shares of security \mathcal{S}_i , $i \in \mathbb{N}_n$ allocated in a portfolio at time t , it is clear that $\pi_t^i P_t^i$ is the (dollar) amount invested in security \mathcal{S}_i at time t according to the portfolio strategy Π .

Definition 3.3.7. We will define the **active money process** as the n -dimensional real process $\pi P = \{\pi P_t\}_{t \in [0, T]} = \{(\pi P_t^1, \pi P_t^2, \dots, \pi P_t^n)\}_{t \in [0, T]}$ with components

$$\pi P_t^i = \pi_t^i P_t^i \quad i \in \mathbb{N}_n, \quad (63)$$

that is,

$$\pi P_t = \text{diag}(P_t) \pi_t. \quad (64)$$

Using Definition 3.3.7, we can rewrite the conditions (61) in Definition 3.3.5 as

$$\int_0^T |\pi P_t \cdot (\mu_t - r_t \mathbf{1}_n + \delta_t)| dt + \int_0^T \|\pi P_t^\dagger \sigma_t\|^2 dt < \infty \quad \mathcal{P}a.s. \quad (65)$$

We have assumed that our market, \mathcal{M} , has no frictions, liquidity problems, etc. Changes in the wealth process should be caused not by the changes in the portfolio strategy (in other

words, changes in the wealth process should not be caused by changes in the coefficient processes π^i), but by changes in the yield process and/or bank account. That is, the portfolio strategy should be *self-financing*.

Definition 3.3.8. A portfolio strategy Π is called **self-financing** if the corresponding wealth process \mathcal{W}^Π satisfies

$$d\mathcal{W}_t^\Pi = \pi_t^0 dB_t + \sum_{i=1}^n \pi_t^i dY_t^i = \pi_t^0 dB_t + \pi_t \cdot dY_t. \quad (66)$$

Definition 3.3.8 and in particular (66) reflect our last comment prior to Definition 3.3.8. Intuitively, Definition 3.3.8 is equivalent to:

$$B_t d\pi_t^0 + \sum_{i=1}^n Y_t^i d\pi_t^i = 0. \quad (67)$$

If portfolio strategy Π is self-financing, we can combine (66) with (47), solve for $\pi_t^0 B_t$ from (62) and substitute into the resulting equation to eliminate the dependency on π^0 ; then, using (49) we can construct an sde for the wealth process \mathcal{W}^Π in which π^0 is not explicitly mentioned

$$\begin{aligned} d\mathcal{W}_t^\Pi &= \pi_t^0 B_t r_t dt + \pi_t \cdot dY_t \\ &= (\mathcal{W}_t^\Pi - \pi_t \cdot P_t) r_t dt + \pi_t^\dagger \text{diag}(P_t)((\mu_t + \delta_t) dt + \sigma_t dW_t) \\ &= \left(\mathcal{W}_t^\Pi - \pi_t^\dagger \text{diag}(P_t) \vec{\mathbf{1}}_n \right) r_t dt + \pi_t^\dagger \text{diag}(P_t)((\mu_t + \delta_t) dt + \sigma_t dW_t) \\ &= \mathcal{W}_t^\Pi r_t dt + \pi P_t^\dagger((\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t) dt + \sigma_t dW_t). \end{aligned} \quad (68)$$

Clearly, our last computations show that under conditions (61) from Definition 3.3.5 we can solve the sde (66) for a given initial wealth w_0 .

Note that Definition 3.3.5 makes no explicit mention of the portfolio coefficient π^0 . This is because we can always use (62) to express π^0 in terms of the bank account process, the wealth process and the corresponding portfolio process. If portfolio strategy Π is self-financing, our previous computations show that we can find an sde for the wealth process that makes no explicit use of the portfolio coefficient π^0 .

In what follows, unless otherwise stated, we will assume that we are dealing with self-financing portfolio strategies whose respective portfolio processes satisfy Definition 3.3.5, and in particular the integrability condition (61).

Definition 3.3.9. If X is an adapted process (with respect to our underlying filtered probability space $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$), we define²³ $X^* = \{X_t^*\}_{t \in [0, T]} = \{X_t/B_t\}_{t \in [0, T]}$. X^* is called the **discounted process** corresponding to X .

We refer to the process $\{1/B_t\}_{t \in [0, T]}$ as the **discount factor** of market \mathcal{M} .²⁴

²³ As it was mentioned before (see Footnote 14), we use the price process of the Bank account to discount prices. The reason this is both historical and of simple convenience, B is a “degenerate” diffusion, with no dispersion term; from the theoretical and practical point of view, to discount “general” —unspecified— processes with B is easier in the sense that we have not to worry about changes to the dispersion terms of the processes being discounted; on the other hand, it is natural to compare the returns on different assets with the return on a “benchmark” asset, and the common choice in that regard is the bank account (money in a bank account is “just sitting over there”), so historically one discounts with the bank account to obtain the price, in units of bank account price, of the asset being discounted. Both Financial wisdom and Mathematical theory allow for different processes to be used in constructing a discount factor, one could use any, strictly, positive process (a zcb for example) to discount prices. In such a case the chosen discounting process is called “*the market model numéraire*” or simple “*the numéraire*” or a numéraire (that is, the numerator in the quotient used to define a discount). In specific situations, the use of one numéraire could be more advantageous than to use another (see Chapter 5 §5.3). For example, one may not want to introduce an additional asset/stochastic process into the description of the problem (to find problems that do not require of a bank account in their description is perfectly plausible, for example the trading of a basic security by another), or it may be that the proper selection of a numéraire with a given dispersion term can facilitate somehow a particular situation. Thus, in practice, the numéraire to be used should be suitably chosen so that the discounted processes turn out to be easy to manage (in accordance to some kind of criteria) or to facilitate the description and study of one’s particular problem (for example, one may want to introduce not one but possibly several different numéraires in the study of problems involving more than one currency). See [68] for a discussion on changes of numéraire/measure and Finance, [160] Chapter VII§1.b and/or [163] Chapter 9 also offer detailed descriptions and results regarding changes of numéraire; [73] Chapter III §2 offer an study on changes of measure and diffusions; see also Chapter 5 for a particular example.

²⁴ Naturally, this assumes we are interested in discounting back to time $t = 0$. If on the contrary one wants to discount to another time, say time $s \leq t$ (fixed), we will discount to time $t = 0$ and then let the money accrue to time s , that is, we will have to divide by B_t and multiply by B_s . As a result, if X is an adapted process (with respect to our underlying filtered probability space $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$), $B_s X_t^* = \frac{B_s}{B_t} X_t$ corresponds to a discount to time $s \leq t$.

Let's go back to the sde satisfied by a self-financing portfolio's wealth process

$$d\mathcal{W}_t = \mathcal{W}_t r_t dt + \pi P_t^\dagger ((\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t) dt + \sigma_t dW_t),$$

multiplying both sides by the discount factor and rearranging we obtain

$$\frac{d\mathcal{W}_t}{B_t} - \mathcal{W}_t^* r_t dt = (\pi P_t^*)^\dagger ((\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t) dt + \sigma_t dW_t), \quad (69)$$

which by Itô's formula²⁵ implies

$$d\mathcal{W}_t^* = (\pi P_t^*)^\dagger ((\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t) dt + \sigma_t dW_t). \quad (70)$$

Since a self-financing portfolio process has to satisfy the integrability conditions (61) we know that the sde (70) satisfied by the corresponding discounted wealth process can be integrated, and that \mathcal{W}^* is a local martingale. On the other hand, (70) in conjunction with (61) implies that the discounted wealth process is an Itô process.

In fact, integrability conditions (51) and (61) are in place to ensure that security prices and discounted wealth processes are Itô processes. See [96] Chapter 3 and [151] Chapter 7 §2.

Based on our computations leading to (70), we can write explicitly

$$\mathcal{W}_t^{*\Pi, w_0} = w_0 + \int_0^t (\pi P_u^*) \cdot (\mu_u - r_u \vec{\mathbf{1}}_n + \delta_u) dt + \int_0^t (\pi P_u^*)^\dagger \sigma_u dW_u, \quad (71)$$

and,

$$\mathcal{W}_t^{\Pi, w_0} = B_t \left(w_0 + \int_0^t (\pi P_u^*) \cdot (\mu_u - r_u \vec{\mathbf{1}}_n + \delta_u) dt + \int_0^t (\pi P_u^*)^\dagger \sigma_u dW_u \right), \quad (72)$$

for some initial wealth w_0 . Notice that the process $\mathcal{W}^{*\Pi, w_0} - w_0$ can be regarded as the discounted gain due to portfolio strategy Π .

²⁵From (47) we know the bank account process is non-null. From (66) we know that the cross variation of the bank process and the wealth process satisfies $\langle B, \mathcal{W} \rangle = 0$. On the other hand, the function $g: \mathbb{R} \times (\mathbb{R} \setminus 0) \rightarrow \mathbb{R}$ defined as $g(x, y) = x/y$ is of class $C^2(\mathbb{R} \times (\mathbb{R} \setminus 0))$. Thus conditions for Itô's rule are satisfied. See [96] §3.3 .

Definition 3.3.10. The process ρ defined as

$$\rho = \{\mu_t - \vec{1}_n r_t + \delta_t\}_{t \in [0, T]} \quad (73)$$

is called the **risk premium** of market \mathcal{M} .

3.3.3 Absence of Arbitrage and Standard Markets

As described, portfolio strategies may rely on short selling²⁶ (that is, portfolio coefficients π^i , $i \in \mathbb{N}_n^*$ can take negative values). The problem is that we are assuming a market with no liquidity problems and no frictions, and lending and borrowing relies on the same interest rate. Unless we impose some restrictions to how deep in debt an investor can go, we face the possibility of doubling strategies²⁷. To protect against doubling strategies we need to impose some kind of restriction to allowed portfolios.

Definition 3.3.11. A portfolio process π , and by extension a portfolio strategy $\Pi = (\pi^0, \pi)$, is called **tame** if the corresponding discounted wealth process, $\mathcal{W}^{*\Pi}$, is \mathcal{P} a.s. bounded from below by some real constant. That is, π is tame if there exist $\alpha \in \mathbb{R}$ such that

$$\mathcal{P}(\mathcal{W}_t^{*\Pi} \geq \alpha, \forall t \in [0, T]) = 1 \quad (74)$$

²⁶Short selling refers to a valid trading strategy according to which an investor may sell a security he does not own—it means he will have to borrow it from someone, to whom he will have to pay back by returning the borrowed security plus some possible fees in a future time—. A short seller tries to profit from a declining market assuming he will be able to buy the security—at a future date— paying a price lower than the price at which he sold short. In this regard *to go short* or *to sell short* is the opposite *to go long*.

To assume a short position means to go short in a security. Most investors tend *to go long*, that is they tend *to assume long positions* in their portfolios by buying a security in the hopes it will increase its value with time. Notice that to assume a short position in a security is not the same as *to close a position*. When an investor closes a position he is either selling shares of security he owns or is buying shares of a security he is short on. In the later case, when an investor buys the same number of shares of a security he is short on it is said that the investor is *covering* his position.

²⁷An example (a very famous one) of a doubling strategy (in a discrete time setting) is discussed by Harrison and Kreps [77]; a “sure win” at roulette “is possible” if one bets on red and keeps doubling the bet until red comes out (provided one can borrow unlimited amounts of money, and assuming that there is no limit on that roulette). An investor applying a doubling strategy relies on his ability to borrow arbitrarily large amounts of money to attain arbitrarily large gains with a null initial investment.

If only tame portfolios are allowed, investors can not try to attain arbitrarily large amounts of wealth by means of falling deep in debt until luck strikes, see [43].

Doubling strategies are an example of *arbitrage opportunities*. In general an arbitrage opportunity is the possibility to make a positive profit, with probability one, starting with a null initial investment.

Definition 3.3.12. *We define an **arbitrage opportunity** as a tame portfolio strategy Π with null initial wealth that satisfies*

$$\mathcal{P}(\mathcal{W}_T^\Pi \geq 0) = 1, \quad \mathcal{P}(\mathcal{W}_T^\Pi > 0) > 0. \quad (75)$$

A market model is arbitrage-free if no tame portfolios are arbitrage opportunities.

Although tame portfolios only protect against a particular case of arbitrage opportunity²⁸, in what follows (unless the contrary is stated) we will restrict ourselves to the use of tame portfolios.

To rule out arbitrage opportunities more conditions are needed.

Theorem 3.3.3. *If market \mathcal{M} is arbitrage-free, then there exists a progressively measurable process, $\theta: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, called a **market price of risk** (or “relative risk”) process, such that*

$$\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t = \sigma_t \theta_t, \quad 0 \leq t \leq T \quad \mathcal{P}a.s. \quad (76)$$

Conversely, if such a process θ exists and satisfies, in addition to the above requirements,

$$\mathcal{P} \left(\int_0^T \|\theta_t\|^2 dt < \infty \right) = 1 \quad (77)$$

and

$$E_{\mathcal{P}} \left(\exp \left\{ - \int_0^T \theta_t^\dagger dW_t - \frac{1}{2} \int_0^T \|\theta_t\|^2 dt \right\} \right) = 1 \quad (78)$$

then \mathcal{M} is arbitrage-free.

²⁸Hur [87] discusses yet another arbitrage opportunity, namely “survival strategy”, not ruled out by imposing lower bounds to the wealth process. Survival strategies are “buy and hold” strategies that allow an investor to double (with probability one) his wealth in finite time.

The interested reader can find the proof to this result in Karatzas and Shreve's [97], Chapter 1 §2.

Let's define the process Z as

$$Z_t = Z_t(\theta) = \left\{ - \int_0^t \theta_s^\dagger dW_s \right\}_{t \in [0, T]} \quad (79)$$

It is easy to see that

$$\langle Z - \frac{1}{2} \langle Z \rangle \rangle_t = \langle Z \rangle_t = \int_0^t \theta_s^\dagger \theta_s ds \quad (80)$$

defining the process $\mathcal{E}(Z)$ as

$$\mathcal{E}(Z) = \left\{ \exp \left(Z_t - \frac{1}{2} \langle Z \rangle_t \right) \right\}_{t \in [0, T]} \quad (81)$$

and applying Itô's rule to $\mathcal{E}(Z)$ we see that

$$d\mathcal{E}_t(Z) = \mathcal{E}_t(Z) d\left\{ Z_t - \frac{1}{2} \langle Z \rangle_t \right\} + \frac{1}{2} d\langle Z - \frac{1}{2} \langle Z \rangle \rangle_t = \mathcal{E}_t(Z) dZ_t = -\mathcal{E}_t(Z) \theta_t^\dagger dW_t \quad (82)$$

which shows that the process $\mathcal{E}(Z)$ is a local martingale and satisfies the sde $d\mathcal{E}_t(Z) = \mathcal{E}_t(Z) dZ_t$, $\mathcal{E}_0(Z) = 1$ *P.a.s.* Process $\mathcal{E}(Z)$ is known as the *Doléans exponential* of process Z .

By Novikov's condition ([151] Chapter 8 §1, Proposition 1.15 or [96] §3.5.D) process $\mathcal{E}(Z)$ is a martingale and conditions (77) and (78) are satisfied if the market price of risk process, θ , satisfies

$$E_{\mathcal{P}} \left(\exp \left\{ \frac{1}{2} \int_0^s \|\theta_t\|^2 dt \right\} \right) < \infty \quad s \in [0, T] \quad (83)$$

Definition 3.3.13. A market model \mathcal{M} for which there exists a progressively measurable process (w.r.t filtration \mathcal{F}) θ that satisfies the conditions of Theorem 3.3.3 and such that the Doléans exponential given by (79), (81) is a martingale is called a **Standard Market Model**.

If market \mathcal{M} is a standard market model, $\mathcal{E}(Z(\theta))$ is a martingale, with $E(\mathcal{E}_0(Z(\theta))) = 1$. In such a case, martingale $\mathcal{E}(Z(\theta))$ defines a probability measure $\mathcal{P}^{\mathcal{E}}$ equivalent to probability \mathcal{P} . At every set $A \in \mathcal{F}_T$, $\mathcal{P}^{\mathcal{E}}(A)$ is defined as follows²⁹

$$\mathcal{P}^{\mathcal{E}}(A) = E_{\mathcal{P}}(\mathcal{E}_T(Z(\theta)) \mathbb{1}_A) \quad (84)$$

²⁹Indeed, not one, but a whole family of equivalent measures is defined by the martingale $\mathcal{E}(Z(\theta))$. $\forall t \in [0, T]$, $\mathcal{P}_t^{\mathcal{E}}(A) = E_{\mathcal{P}}(\mathcal{E}_t(Z(\theta)) \mathbb{1}_A) \quad A \in \mathcal{F}_t$. Then, if $A \in \mathcal{F}_t$, $t \in [0, T]$, we have $\mathcal{P}_t^{\mathcal{E}}(A) = E_{\mathcal{P}}(\mathcal{E}_t(Z(\theta)) \mathbb{1}_A) = E_{\mathcal{P}}(E(\mathcal{E}_T(Z(\theta)) | \mathcal{F}_t) \mathbb{1}_A) = E_{\mathcal{P}}(E(\mathcal{E}_T(Z(\theta)) \mathbb{1}_A | \mathcal{F}_t)) = E_{\mathcal{P}}(\mathcal{E}_t(Z(\theta)) \mathbb{1}_A) = \mathcal{P}_t^{\mathcal{E}}(A) = \mathcal{P}^{\mathcal{E}}(A)$.

where $\mathbb{1}_A$ represents the *indicator function* of set $A \subset \Omega$ defined as

$$\mathbb{1}_A(\omega) = \begin{cases} 1; & \text{if } \omega \in A \\ 0; & \text{if } \omega \notin A \end{cases} \quad (85)$$

The amazingly good thing about all this is that now we can apply Girsanov's theorem (see [151] Chapter 8 §1, or [96] §3.5.A). That is, the process $W^\mathcal{E} = \{W_t^\mathcal{E}\}_{t \in [0, T]} = \left\{ \left(W_t^{\mathcal{E}^1}, W_t^{\mathcal{E}^2}, \dots, W_t^{\mathcal{E}^d} \right)^\dagger \right\}_{t \in [0, T]}$ defined by

$$W^\mathcal{E} = \{W_t^\mathcal{E}\}_{t \in [0, T]} = \left\{ W_t + \int_0^t \theta_s ds \right\}_{t \in [0, T]} \quad (86)$$

is adapted to filtration \mathcal{F} , and it is also a Brownian Motion. The integral in (86) is defined component by component, in other words,

$$W_t^{\mathcal{E}^i} = W_t^i + \int_0^t \theta_s^i ds \quad (87)$$

$i \in \mathbb{N}_d, t \in [0, T]$.

Under *martingale measure* $\mathcal{P}^\mathcal{E}$ the dynamics of security prices, wealth processes, and the corresponding discounted processes change favorably. From (86) we obtain:

$$dW_t^\mathcal{E} = dW_t + \theta_t dt \quad (88)$$

multiplying by the volatility matrix σ

$$\sigma_t dW_t^\mathcal{E} = \sigma_t dW_t + \sigma_t \theta_t dt = \sigma_t dW_t + \rho_t dt \quad (89)$$

combining the last result with (70) we obtain:

$$d\mathcal{W}_t^* = (\pi P_t^*)^\dagger ((\mu_t - r_t \vec{\mathbf{1}}_n + \delta_t) dt + \sigma_t dW_t) = (\pi P_t^*)^\dagger (\rho_t dt + \sigma_t dW_t) = (\pi P_t^*)^\dagger \sigma_t dW_t^\mathcal{E} \quad (90)$$

which gives proof to the first part of the following:

Proposition 3.3.4. *Under a standard market model \mathcal{M} , the discounted wealth process associated to a self-financing portfolio strategy Π , is a local martingale with respect to the martingale measure $\mathcal{P}^\mathcal{E}$ and satisfies the sde*

$$d\mathcal{W}_t^* = (\pi P_t^*)^\dagger \sigma_t dW_t^\mathcal{E} \quad (91)$$

where σ is the market's volatility matrix and πP_t^* is the corresponding discounted active money process. If portfolio strategy Π is tame, then the discounted wealth process \mathcal{W}^* is a supermartingale and satisfies

$$E_{\mathcal{E}} \left(\mathcal{W}_T^{*\Pi, w_0} \right) \leq w_0 \quad (92)$$

where w_0 is the initial investment.

If Π is tame, the second part of the proof follows from Fatou's lemma for conditional expectations.

Remark 3.3.1. If the discounted wealth process has constant expectations (with respect to $\mathcal{P}^{\mathcal{E}}$), then it is a martingale (see [96] §1.5.19).

Definition 3.3.14. A self-financing portfolio (strategy) Π , is called a martingale generating portfolio (strategy) if the corresponding **discounted gains process** $\mathcal{W}^{*\Pi, x} - x$ (here x is the initial investment) is a martingale under the equivalent martingale measure $\mathcal{P}^{\mathcal{E}}$.

The following propositions also follow from the existence of a market price of risk process θ .

Proposition 3.3.5. Under martingale measure $\mathcal{P}^{\mathcal{E}}$, the wealth process associated to a self-financing portfolio strategy Π , satisfies the following sde

$$d\mathcal{W}^{\Pi} = \mathcal{W}^{\Pi} r_t dt + \pi P_t^{\dagger} \sigma_t dW_t^{\mathcal{E}} \quad (93)$$

where σ is the market's volatility matrix, r is the market's risk free rate and πP is the corresponding active money process.

Proposition 3.3.5 follows directly from (66), Definition 3.3.10 and (86).

Proposition 3.3.6. Under the martingale measure $\mathcal{P}^{\mathcal{E}}$, the security price processes, and the discounted security price processes follow the dynamics:

$$\begin{aligned} dP_t^i &= P_t^i \left((r_t - \delta_t^i) dt + \sigma_t^i \cdot dW_t^{\mathcal{E}} \right), \quad i \in \mathbb{N}_n \\ dP_t^{*i} &= P_t^{*i} \left(-\delta_t^i dt + \sigma_t^i \cdot dW_t^{\mathcal{E}} \right), \quad i \in \mathbb{N}_n \end{aligned} \quad (94)$$

where r and σ are coefficients of the market \mathcal{M} . In the particular in which there are no dividends, assuming that σ satisfies Novikov's condition, this shows that the discounted price processes are also martingales under the martingale measure. In general if σ satisfies Novikov's condition, (94) implies that

$$dP_t^{*i} + P_t^{*i} \delta_t^i dt = P_t^{*i} (\sigma_t^i \cdot dW_t^\mathcal{E}) , \quad i \in \mathbb{N}_n \quad (95)$$

thus showing that the processes $\left\{ P_t^{*i} \exp \left(\int_0^t \delta_u^i du \right) \right\}_{t \in [0, T]}$, $i \in \mathbb{N}_n$ are martingales.

Proposition 3.3.6 follows from (46), Definition 3.3.10, (86) and Itô's rule, from which (94) is obtained, showing that both, the logarithm of P^i and the logarithm of P^{*i} , $i \in \mathbb{N}_n$ are local martingales (that is, P^i and P^{*i} , $i \in \mathbb{N}_n$ are exponential local martingales), so P^i and P^{*i} are supermartingales. If Novikov's condition holds (see [96] §3.5.D) for the volatility matrix process σ , then P^{*i} is also a martingale.

Under a standard market model we can write:

$$\begin{aligned} \mathcal{W}_t^{*\Pi, w_0} &= w_0 + \int_0^t (\pi P_u^*)^\dagger \sigma_u dW_u^\mathcal{E} \\ \mathcal{W}_t^{\Pi, w_0} &= B_t \left(w_0 + \int_0^t (\pi P_u^*)^\dagger \sigma_u dW_u^\mathcal{E} \right) \\ P_t^{*i} &= P_0^i \exp \left\{ \int_0^t -\delta_u^i - \frac{1}{2} \sigma_u^i \cdot \sigma_u^i du + \int_0^t \sigma_u^i \cdot dW_u^\mathcal{E} \right\} \\ P_t^i &= P_0^i \exp \left\{ \int_0^t r_u - \delta_u^i - \frac{1}{2} \sigma_u^i \cdot \sigma_u^i du + \int_0^t \sigma_u^i \cdot dW_u^\mathcal{E} \right\} \end{aligned} \quad (96)$$

3.3.4 Financeable Goals and Market Completeness

Many are the reasons why an investor may decide to start investing, to create a portfolio of securities and to develop a portfolio strategy. One such reason (and a very important one indeed) is to satisfy some kind of future financial requirement. An investor may desire to, throughout a portfolio strategy, achieve or amass a sum of money by time \mathcal{T} .

For example, an investor may want to, by the time he reaches retirement age, amass a sizable amount of money that he could use to finance his needs after his retirement more or less in the same way he has been able to do before retirement age. In this case, the goal of the investor is to have enough money to finance his way of living for an unknown number of years after retirement.

Or, after an initial investment of w_0 , an investor may want to have exactly $\$A$ by time \mathcal{T} . That is, his goal is to have a final wealth of $\$A$.

It may be that our investor is already (or is about to be) embarked in other financial venues which he has to hedge against. Those other venues may require from him the future payment (at time \mathcal{T}) of a sum that is not completely known at time $t = t_0 \geq 0$.

In general, we may think of a final financial goal as a random variable G which is $\mathcal{F}_{\mathcal{T}}$ measurable. But, is such a goal attainable? Under which conditions it is possible to find a portfolio strategy such that its corresponding wealth process \mathcal{W} satisfies such a goal (that is, $\mathcal{W}_{\mathcal{T}} = G$)? What should be the initial investment w_0 ?

Let's assume we are working on a Standard Market model. Assume also that G is a $\mathcal{F}_{\mathcal{T}}$ measurable random variable and that w_0 is a real number and Π is a tame portfolio strategy, such that $\mathcal{W}_{\mathcal{T}}^{\Pi, w_0} = G$. From Proposition 3.3.4 we know

$$G = \mathcal{W}_{\mathcal{T}}^{\Pi, w_0} = B_{\mathcal{T}} \left(w_0 + \int_0^{\mathcal{T}} (\pi P_u^*)^\dagger \sigma_u dW_u^{\mathcal{E}} \right) \quad (97)$$

in other words

$$G/B_{\mathcal{T}} = \mathcal{W}_{\mathcal{T}}^{*\Pi, w_0} = w_0 + \int_0^{\mathcal{T}} (\pi P_u^*)^\dagger \sigma_u dW_u^{\mathcal{E}} \quad (98)$$

but Π is tame, by Definition 3.3.11 this means that the discounted wealth process $\mathcal{W}^{*\Pi, w_0}$ is bounded below \mathcal{P} -a.s. (or which is the same $\mathcal{P}^{\mathcal{E}}$ -a.s.). Therefore, if such a tame portfolio strategy exists, it will be required that the random variable $G/B_{\mathcal{T}}$ be $\mathcal{P}^{\mathcal{E}}$ -a.s. bounded below.

On the other hand, if such a portfolio strategy exists, one could expect (at least intuitively) that both $G/B_{\mathcal{T}}$ and $\mathcal{W}_{\mathcal{T}}^{*\Pi, w_0}$ should have the same mathematical expectation. Therefore

$$E_{\mathcal{E}}(G/B_{\mathcal{T}}) = E_{\mathcal{E}}(\mathcal{W}_{\mathcal{T}}^{*\Pi, w_0}) = E_{\mathcal{E}} \left(w_0 + \int_0^{\mathcal{T}} (\pi P_u^*)^\dagger \sigma_u dW_u^{\mathcal{E}} \right) \leq w_0 \quad (99)$$

(the last inequality coming from Proposition 3.3.4 and the assumption that Π is tame) this means that if there exists a tame portfolio strategy Π such that $\mathcal{W}_{\mathcal{T}}^{\Pi, w_0} = G$, the initial investment w_0 has to be bigger than or equal to the expectation of $G/B_{\mathcal{T}}$. If instead, one assumes that Π is martingale generating, similar conclusions are obtained.

We collect our intuitive requirements in the following definition.

Definition 3.3.15. Given that market \mathcal{M} is a standard market, and G is a \mathcal{F}_T -measurable random variable such that

$$\begin{aligned} \mathcal{P}^\mathcal{E} (G/B_T \geq \alpha) &= 1 \\ w_0 &:= E_\mathcal{E} (G/B_T) < \infty \end{aligned} \tag{100}$$

for some real constant α . We will say that G is a **financeable goal** if there exists a tame portfolio strategy Π such that

$$G/B_T = \mathcal{W}_T^{*\Pi, w_0} \quad \mathcal{P}^\mathcal{E}\text{-a.s.} \tag{101}$$

On the other hand, if all \mathcal{F}_T -measurable random variables that satisfy (100) are financeable goals, we will say that market \mathcal{M} is **complete**. Naturally a standard market that is not complete is called **incomplete**.

The following two well known results, see [95] and [97], offer a characterization of a complete market.

Proposition 3.3.7. • An smm \mathcal{M} is complete if and only if for every \mathcal{F}_T -measurable, r.v. G such that

$$E_\mathcal{E} (|G|/B_T) < \infty \tag{102}$$

and $w_0 := E_\mathcal{E} (G/B_T)$, there exists a martingale generating portfolio strategy Π such that $G/B_T = \mathcal{W}_T^{*\Pi, w_0} \quad \mathcal{P}^\mathcal{E}\text{-a.s.}$

- If $x < w_0$, there can not be a tame portfolio strategy Π_x such that its corresponding wealth process satisfies $\mathcal{W}_T^{\Pi_x, x} \geq G \quad \mathcal{P}^\mathcal{E}\text{-a.s.}$.
- If $x > w_0$, there can not be a tame portfolio strategy Π_x such that its corresponding wealth process satisfies $\mathcal{W}_T^{\Pi_x, x} = G \quad \mathcal{P}^\mathcal{E}\text{-a.s.}$ and is a martingale.

Theorem 3.3.8. An smm \mathcal{M} is complete if and only if $n = d$ (that is, the number of securities minus the bank account is equal to the number of sources of systematic risk) and the volatility (matrix) process is nonsingular $\lambda \otimes \mathcal{P}$ -a.s. (λ being the Lebesgue measure on $[0, T]$).

Notice that Theorem 3.3.8 implies that in the case of a complete market there is a unique market price of risk process θ . If \mathcal{M} is complete, the volatility matrix process σ is nonsingular $\lambda \otimes \mathcal{P}$ -a.s., thus, if we multiply (76) by the inverse of σ we will have

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \vec{1}_n + \delta_t), \quad 0 \leq t \leq T \text{ } \mathcal{P}a.s.. \quad (103)$$

3.4 Valuation of Game Options in a non-Constant Coefficient Market

Consider two parties, a buyer (investor) and a seller (or writer) involved in a contract. The buyer pays to the seller and buys the contract that the seller sells.

In the case the contract traded is an European option the buyer is “inert”, his actions (after the contract is traded) are reduced to waiting until maturity to see the final outcome of his investment, while the writer’s actions are directed to finding a hedging strategy. The writer is less inert than the buyer in the sense that he/she must hedge his/her position, yet both must wait until maturity.

In the case of an American option the buyer is actively searching for the best instant to exercise his/her option while the seller hedges his/her position. Both participants in an American option contract adopt adversary (opposing) roles, but, since it is only the buyer who can select an exercise strategy, both participants have an equal perception of the contract and the value of it. The buyer will attempt to maximize his/her gains by selecting an “optimal” exercise strategy and the seller is aware of that fact. To find the value/price at time t of an American option one needs to solve an optimal stopping problem³⁰.

If we call $\mathcal{O} = \{\mathcal{O}_t\}_{t \in [0, T]}$ the discounted payoff process of an American option, the problem of pricing such an option “reduces” to find

$$\sup_{t \in \mathfrak{S}_{0, T}} E(\mathcal{O}_t), \quad (104)$$

the “optimal discounted expected payoff”, where $\mathfrak{S}_{0, T}$ is the set of all stopping times with values in $[0, T]$ (see Notation 3.4.1 below) and the expectation (see Notation 3.3.5), E , is

³⁰Please see [52] for a detailed study of the problem of optimal stopping. [97] offers a brief study in its appendix.

computed with respect to a suitable equivalent martingale measure³¹.

In the case of a Game option (see Definition 3.4.2) these adversary roles are enhanced by the right of the writer to stop the contract at any time (paying a suitable penalization).

To find the value/price at time t of a Game option we must solve a double optimal stopping problem. The value of a Game option is the value achieved by exercising and canceling the option optimally. Solving the double optimal stopping problem one finds optimal exercise strategies (in the form of stopping times) and the corresponding payoff under such strategies (much like in the case of an American option).

Obviously, in the case of a Game option the problem at hand is much more complex since both participants, buyer and seller, can “act” by selecting an exercise and a cancellation time, respectively. But that is not the only problem. Buyer and seller are adversaries. One is looking for a strategy that will maximize his/her gains (the buyer in this case) while the other (the seller) is looking for a strategy to minimize his/her losses. This also creates opposing views regarding the contract and its value.

Let $\{\mathcal{R}^*(s, t)\}_{s, t \in [0, T]}$ (see Definition 3.4.1, Definition 3.4.2 and Notation 3.4.3 below) represent the discounted payoff of a Game option.

The buyer knows that the seller of the Game option will try to find a strategy to minimize his/her losses, still the buyer’s goal is to maximize his/her gains. Thus the buyer perceives an optimal expected discounted payoff of the form

$$\underline{V} = \sup_{t \in \mathfrak{S}_{0, T}} \inf_{s \in \mathfrak{S}_{0, T}} E(\mathcal{R}^*(s, t)). \quad (105)$$

Similarly, the seller knows the buyer’s intentions, he/she knows that the buyer will attempt to find a strategy (a stopping time) to maximize his/her gains, still, while hedging his/her position, the seller must attempt to find an opposing strategy that will allow him/her to reduce his/her losses. Thus, the seller perceives an optimal expected discounted payoff of the form

$$\bar{V} = \inf_{s \in \mathfrak{S}_{0, T}} \sup_{t \in \mathfrak{S}_{0, T}} E(\mathcal{R}^*(s, t)). \quad (106)$$

³¹For further details about the pricing of American options we refer the reader to [134], [97] and [95].

At a difference from the case of the American option (or even more, from the case of the European option) where both parties perceive the same expected value, in the case of the Game option we find two.

Thus, in order to show that a value for the Game option exists we need not only to find optimal strategies for both parties but also, we need to show that both expected discounted values are the same:

$$\underline{V} = \overline{V}.$$

A game theoretical argument based in the well-known results by Lepeltier and Maingueneau, [109], regarding the value of a zero-sum Dynkin game will show that $\underline{V} = \overline{V}$; then an arbitrage argument will show that the common value

$$V = \underline{V} = \overline{V},$$

is indeed the option's price.

Such a value, we will see, is attained at a saddle point (κ^*, ξ^*) , where κ^* and ξ^* are two special stopping times; κ^* is called an optimal cancellation time and ξ^* is called an optimal execution time. (κ^*, ξ^*) is a saddle point for the associated Dynkin game.

In fact, based in the results by Lepeltier and Maingueneau, [109], we have not only a saddle point but also ε -optimal strategies (see next sections for explanations) $(\kappa^\varepsilon, \xi^\varepsilon)$ such that $(\kappa^\varepsilon, \xi^\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (\kappa^*, \xi^*)$.

3.4.1 Game contingent claims

In order to clearly define the object of our study, we will introduce some additional definitions.

We will set ourselves in the general framework of Section §3.3, that is, of a complete smm \mathcal{M} under $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$, a complete filtered probability space, where \mathcal{F} denotes the \mathcal{P} -augmentation of the natural filtration \mathcal{F}^W of a d -dimensional Brownian Motion W . Please see Section §3.3 for details.

Notation 3.4.1. We denote by $\mathfrak{S}_{\mathcal{T}} = \mathfrak{S}_{0, \mathcal{T}}$ the set of stopping times with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, \mathcal{T}]}$ with values in $[0, \mathcal{T}]$. Similarly, for $t, u \in [0, \mathcal{T}]$, $t \leq u$, we will denote

by $\mathfrak{S}_{t,u}$ the set of stopping times with respect to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ and with values in $[t, u] \subset [0, T]$. In general, if \mathfrak{s} and \mathfrak{t} are two stopping times with values in $[0, T]$, we define $\mathfrak{S}_{\mathfrak{s}, \mathfrak{t}}$ as the set $\mathfrak{S}_{\mathfrak{s}, \mathfrak{t}} = \{\mathfrak{w} \in \mathfrak{S}_T : \mathfrak{s} \leq \mathfrak{w} \leq \mathfrak{t}\}$.

Recall that \mathcal{F} is a continuous filtration, therefore, optional times of \mathcal{F} are also stopping times of filtration \mathcal{F} ([96] Proposition 1.2.3).

Definition 3.4.1. Let $X = \{X_t\}_{t \in [0,T]}$ and $Y = \{Y_t\}_{t \in [0,T]}$ be two stochastic processes defined on $(\Omega, \mathcal{U}, \mathcal{P})$ adapted to filtration \mathcal{F} . For any pair $(s, t) \in [0, T] \times [0, T]$ we will define

$$\mathcal{R}(s, t) = X_s \mathbb{1}_{s < t} + Y_t \mathbb{1}_{t \leq s} = \begin{cases} X_s & s < t, \\ Y_t & t \leq s, \end{cases} \quad (107)$$

and $\mathcal{R} = \{\mathcal{R}(s, t)\}_{(s,t) \in [0,T] \times [0,T]}$. In case explicit reference to the processes X and Y is required, we will write $\mathcal{R}^{X,Y}$ instead of \mathcal{R} and $\mathcal{R}^{X,Y}(s, t)$ instead of $\mathcal{R}(s, t)$.

In agreement with our discussion in Section §3.2 we will define a game contingent claim as follows:

Definition 3.4.2. A game contingent claim (**gcc**) of maturity T , is a contract between two parties, a **writer** and a **holder**, consisting of two \mathcal{F} -adapted RCLL³² payoff processes \mathfrak{X} and \mathfrak{Y} of class (D)³³ such that $0 \leq \mathfrak{Y}_t \leq \mathfrak{X}_t \quad \forall t \in [0, T]$ and

$$E_{\mathcal{E}} \left(\sup_{0 \leq t \leq T} \mathfrak{X}_t^* \right) < \infty, \quad (108)$$

and two stopping times $\kappa \in \mathfrak{S}_T$, called the **cancellation time** (selected by the writer), and $\xi \in \mathfrak{S}_T$, called the **exercise time** (selected by the holder). At time $t = 0$, the holder (also known as the **buyer**, who assumes the **long position** in the contract) will pay to

³²As we have commented before, a very desirable property is that of progressive measurability; in fact we could have required both \mathfrak{X} and \mathfrak{Y} to be progressively measurable, but with the coming proof in mind, we decided to ask for it in an indirect way. Since \mathfrak{X} and \mathfrak{Y} are \mathcal{F} -adapted and RCLL we know they are also progressively measurable. See [96]. Another approach could have been to require the two processes to be measurable and adapted, see Footnote 16, which will ensure the existence of progressively measurable modifications. See [96].

³³See Footnote 56.

the writer (also known as the **seller**, who assumes the **short position** in the contract) a non random amount γ entitling him to receive from the writer, at time $\kappa \wedge \xi = \min(\kappa, \xi)$, a payoff equal to

$$\mathcal{R}(\kappa, \xi) = \mathfrak{X}_\kappa \mathbb{1}_{\kappa < \xi} + \mathfrak{Y}_\xi \mathbb{1}_{\xi \leq \kappa} . \quad (109)$$

On the other hand, at time $t = 0$ the writer receives from the holder the amount γ and agrees to pay to the holder, at time $\kappa \wedge \xi$, the amount $\mathcal{R}(\kappa, \xi)$ given in (109)³⁴. Both parties agree that after time $\kappa \wedge \xi$, and the payment of payoff $\mathcal{R}(\kappa, \xi)$, their contract is dissolved and their mutual obligations will cease. Just for convenience, we will call $\kappa \wedge \xi$ the **end of the game** or the **end of the contract**. Process \mathfrak{X} will be called the **cancellation payoff process** while \mathfrak{Y} will be called the **exercise payoff**.

The main goal of this chapter is, given stochastic processes \mathfrak{X} and \mathfrak{Y} as in Definition 3.4.2, to characterize the non random amount γ (the initial price of the game option), see Definition 3.4.11 and Theorem 3.4.26.

In what follows we will provide a way to price this game contingent claim and to find optimal strategies for the game underlying it. We will show also that there is a proper way to construct approximate hedges against this claim. See sections §3.4.2 and §3.4.3.

As it is apparent from Definition 3.4.2, this contract allows the writer to cancel the contract at any time throughout the life of the contract (by means of selecting the stopping time κ). At the same time, it allows the holder (by means of selecting the stopping time ξ) to exercise his option at any time throughout the life of the contract. From (109) we see that in case both parties decide to act at the same time (that is, in the event they chose cancellation and exercise times with same value), the payoff that the writer pays to the holder is $\mathcal{R}(\xi, \xi) = \mathfrak{Y}_\xi$. Clearly, the decision to wait until maturity (by any of the parties) is represented by making κ and/or ξ identically equal to \mathcal{T} .

³⁴A more general setting could include more than just two process. For example we could consider consumption and the continuous payment of a fee from the holder to the writer in order to keep “playing”. Also, another process could be used to determine the payoff in case both seller and buyer choose to act at the same time. Yet another generalization is to relax the condition $\mathfrak{Y}_t \leq \mathfrak{X}_t$, etc. These are generalizations that we could study in the future.

Since Definition 3.4.2 requires processes \mathfrak{X} and \mathfrak{Y} to satisfy $0 \leq \mathfrak{Y} \leq \mathfrak{X}$, we see that the payoff process \mathcal{R} (as defined in Definition 3.4.1) is bounded below. In case neither the writer nor the seller decide to act prior to maturity, the payoff received by the buyer is $\mathcal{R}(\mathcal{T}, \mathcal{T}) = \mathfrak{Y}_{\mathcal{T}}$. Due to (108), to our assumption that market \mathcal{M} is complete, and the fact that \mathcal{R} is bounded below, it is clear that $\mathcal{R}(\mathcal{T}, \mathcal{T}) = \mathfrak{Y}_{\mathcal{T}}$ is a financeable goal (see Definition 3.3.15). Therefore, there exist a martingale generating portfolio $\tilde{\Pi}$, with initial investment $\tilde{w}_0 = E_{\mathcal{E}}(\mathfrak{Y}_{\mathcal{T}}^*)$ such that

$$\mathfrak{Y}_{\mathcal{T}}^* = \mathcal{W}_{\mathcal{T}}^{*, \tilde{\Pi}, \tilde{w}_0} = \tilde{w}_0 + \int_0^{\mathcal{T}} (\pi \widetilde{P}_u^*)^{\dagger} \sigma_u dW_u^{\mathcal{E}}. \quad (110)$$

At any other time $t \in]0, \mathcal{T}]$ we will have

$$\mathcal{W}_t^{*, \tilde{\Pi}, \tilde{w}_0} = \tilde{\pi}_t^0 + \pi \widetilde{P}_t = \tilde{w}_0 + \int_0^t (\pi \widetilde{P}_u^*)^{\dagger} \sigma_u dW_u^{\mathcal{E}}, \quad (111)$$

from which $\tilde{\pi}^0$ can be obtained. Not only that; since the discounted gains process is a martingale, we can obtain the discounted wealth process by conditional expectation

$$E_{\mathcal{E}}(\mathfrak{Y}_{\mathcal{T}}^* | \mathcal{F}_t) = E_{\mathcal{E}} \left(\tilde{w}_0 + \int_0^{\mathcal{T}} (\pi \widetilde{P}_u^*)^{\dagger} \sigma_u dW_u^{\mathcal{E}} \middle| \mathcal{F}_t \right) = \tilde{w}_0 + \int_0^t (\pi \widetilde{P}_u^*)^{\dagger} \sigma_u dW_u^{\mathcal{E}} = \mathcal{W}_t^{*, \tilde{\Pi}, \tilde{w}_0}. \quad (112)$$

Obviously the pricing of a gcc is a lot more complex than this. In general Seller and Buyer are not restricted to choose cancellation and exercise times both equal to maturity.

Indeed, assuming both Seller and Buyer are rational players, they will be choosing the game strategies (that is, cancellation and execution times) that are more favorable to them. The objective of the Seller is to find the strategy (time of cancellation) that will reduce the future payment he/she has to do to the buyer, on the other hand the objective of the Buyer is to maximize the payment he/she will receive. Knowing this, the Seller wants to find a portfolio strategy that will allow him to hedge against all the involved unknowns and still will let him end up with a non negative wealth after paying $\mathcal{R}(\kappa, \xi)$ to the Buyer at the end of the contract. He/She also wants to find, if possible, the optimal time to stop (cancel) the contract. That is, (given the buyer strategy ξ) a stopping time κ that will reduce as much as possible the payment $\mathcal{R}(\kappa, \xi)$. On the other hand, the buyer will like to find the optimal stopping (execution time) strategy ξ that will maximize (given the seller's strategy κ) the payment $\mathcal{R}(\kappa, \xi)$ that the seller will make to him at the end of the contract

We can visualize the situation of Buyer and Seller with the aid of a continuous time game.

We digress to give results on Dynkin games. This involves material from here through Theorem 3.4.1. Then we apply this material to the gcc context after Theorem 3.4.1 and until the end of this section.

Assume two payers, *Player A* and *Player B*, engage in the following zero-sum³⁵ continuous time game. They observe the realization of two payoffs, RCLL processes of class (D), $\tilde{\mathfrak{X}} = \{\tilde{\mathfrak{X}}_t\}_{t \geq 0}$ and $\tilde{\mathfrak{Y}} = \{\tilde{\mathfrak{Y}}_t\}_{t \geq 0}$,

$$0 \leq \tilde{\mathfrak{Y}}_t \leq \tilde{\mathfrak{X}}_t, \quad (113)$$

$$E \left(\sup_{t \geq 0} \tilde{\mathfrak{X}}_t \right) < \infty, \quad (114)$$

defined on a complete filtered probability space $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P})$, adapted to filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ (which is assumed to satisfy the “usual conditions”³⁶). *Player A* chooses the stopping time ξ while *Player B* chooses stopping time κ . We assume there is no pre-play or play time communication between the players, and that they both arrive at their choices with no direct coercion from the other (no communication between the players is equivalent to them choosing their times/strategies simultaneously). Then, *Player B* pays to *Player A* the amount

$$J(\kappa, \xi) = E \left(\tilde{\mathfrak{X}}_\kappa \mathbb{1}_{\kappa < \xi} + \tilde{\mathfrak{Y}}_\xi \mathbb{1}_{\xi \leq \kappa} \right) = E \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa, \xi) \right). \quad (115)$$

The stopping times ξ and κ are called the *game strategies* of *Player A* and *Player B* respectively (in the Game Theoretical jargon, when a game strategy is a stopping time it is also called a *pure strategy*). In general, the set \mathfrak{S} of all stopping times relative to filtration \mathcal{G} is also called the *set of pure strategies* of the game. Since the only way in which the players can influence the value of the payoff they will pay/receive is through these chosen strategies, the strategies ξ and κ are also known as *game controls*. The quantity J defined in (115) is also known as the expected gain of the game, and as the expected payoff, etc.

³⁵ A game is said to be *zero sum* if a gain for one side entails a corresponding loss for the other side. That is, in a zero sum game the total gain to all players participating in the game adds to zero.

³⁶See Footnote 12.

It is clear that the objectives of *Player A* and *Player B* are adverse to one another. Intuitively, *Player B* (whom *Player A* assumes is a rational player) will try to minimize its payment to *Player A*. Assuming that *Player A* (whom *Player B* assumes is a rational player) will try, at the same time, to maximize such a payment, *Player B* will realize that his/her payment to *Player A* should be no larger than

$$\overline{V} = \inf_{\mathfrak{s} \in \mathfrak{S}} \sup_{\mathfrak{t} \in \mathfrak{S}} J(\mathfrak{s}, \mathfrak{t}), \quad (116)$$

(where the infimum and supremum are to be taken with respect to the collection of all stopping times relative to filtration \mathcal{G}). Similarly, *Player A*, assuming that *Player B* is also a rational player, will see that the least he/she can expect to receive is

$$\underline{V} = \sup_{\mathfrak{t} \in \mathfrak{S}} \inf_{\mathfrak{s} \in \mathfrak{S}} J(\mathfrak{s}, \mathfrak{t}) \quad (117)$$

(intuitively, *Player A* knows *Player B* —if he/she is a rational player— will attempt to minimize his/her loses, in such a case the best *Player B* could do is to somehow attain the infimum of $J(\mathfrak{s}, \cdot)$ — \mathfrak{s} being his/her perception of what *Player A* strategy could be—. Under such assumptions, *Player A* should attempt to maximize his/her gains. If *Player B* does not minimize, he/she will end up giving more to *Player A*. Thus *Player A* knows that \underline{V} as defined in (117) is the least he/she can get).

Clearly, $\underline{V} \leq \overline{V}$. \underline{V} and \overline{V} are called the *lower value* and *upper value*, respectively. The game has a *value* when $\underline{V} = \overline{V}$.

The game here described falls into the category of continuous time zero sum³⁷ games known as Dynkin Games. Such games are generalizations of the discrete time games of stopping introduced by Dynkin in 1969. Dynkin's original idea consisted in a variation of the optimal stopping problem. Later, such games were studied and generalized by several authors. Lepeltier and Maingueneau [109] studied a game very similar to the game we are considering here and showed, under very mild conditions, that such a game has a value in pure strategies (that is, the player's strategies are stopping times with respect to the filtration underlying the probability space on which the game is defined).

³⁷See Footnote 35.

Formally, Lepeltier and Maingueneau consider the following continuous time Dynkin game:

Definition 3.4.3. We denote by $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}, \mathcal{I}, J, X, Y)$ a two person continuous time Dynkin game, that is, a zero-sum game such that:

- $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P})$ is a filtered probability space and $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ satisfies the usual conditions and \mathcal{G}_0 is a trivial σ -algebra.
- \mathcal{I} , the set of strategies of the players, is the set of all stopping times of filtration \mathcal{G} .
- The criterion, expected payoff or expected gain (and sometimes, simply, payoff) of the game is of the form³⁸:

$$J(\kappa, \xi) = E \left(\mathcal{R}^{X,Y}(\kappa, \xi) \right), \quad (118)$$

where X and Y are bounded optional right continuous processes whose value at infinity is zero, that is $X_\infty = Y_\infty = 0$ and such that $Y_t \leq X_t, \forall t \in [0, \infty[$.

In this game, Player A chooses a \mathcal{G} -stopping time ξ and attempts to maximize his/her gain. Assuming an adversary role, Player B chooses a \mathcal{G} -stopping time κ and attempts to minimize his/her payment to Player A.

Remark 3.4.1. Lepeltier and Maingueneau's assumptions, [109], on processes X and Y are stronger than required by the general theory of optimal stopping (see [56], [52], [138], [158] and [97] for example); still, as Laraki and Solan argue, see [108], their results remain valid for RCLL class (D) processes, and in particular for our case of bounded below, RCLL, class (D) processes. The idea is based on results from Dellacherie and Meyer [35] regarding uniformly integrable processes, please see [108].

³⁸Here we are abusing of the notation introduced in Definition 3.4.1 in the sense that we are allowing the arguments of $\mathcal{R}^{X,Y}(\cdot, \cdot)$ to be stopping times, and not only that, but also such stopping times are allowed to take in values in $[0, \infty[$. Our use of this notation is still consistent in the sense that

$$\mathcal{R}^{X,Y}(\mathfrak{s}, \mathfrak{t}) = X_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + Y_{\mathfrak{t}} \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}}$$

for all $\mathfrak{s} \in \mathcal{I}$ and $\mathfrak{t} \in \mathcal{I}$.

By Theorem 65 of [35] Chapter IV, under the usual conditions X , and Y can be taken RCLL instead of optional (see also [126] Chapter 1 §5).

Thus, when applying Lepeltier and Maingueneau's results we will assume our processes X and Y are bounded below (non-negative actually), RCLL and of the class (D) .

As usual, saddle points, upper value, lower value and a value are defined.

Definition 3.4.4. $(\bar{\mathfrak{s}}, \bar{\mathfrak{t}}) \in \mathcal{I} \times \mathcal{I}$ is called a saddle point of the Dynkin game of Definition 3.4.3 if, $\forall(\mathfrak{s}, \mathfrak{t}) \in \mathcal{I} \times \mathcal{I}$ we have:

$$J(\bar{\mathfrak{s}}, \mathfrak{t}) \leq J(\bar{\mathfrak{s}}, \bar{\mathfrak{t}}) \leq J(\mathfrak{s}, \bar{\mathfrak{t}}). \quad (119)$$

Definition 3.4.5. We define the upper, $\bar{\mathbf{V}}$, and lower, $\underline{\mathbf{V}}$, values of the Dynkin game of Definition 3.4.3 as usual:

$$\begin{aligned} \underline{\mathbf{V}} &= \sup_{\mathfrak{t} \in \mathcal{I}} \inf_{\mathfrak{s} \in \mathcal{I}} J(\mathfrak{s}, \mathfrak{t}), \\ \bar{\mathbf{V}} &= \inf_{\mathfrak{s} \in \mathcal{I}} \sup_{\mathfrak{t} \in \mathcal{I}} J(\mathfrak{s}, \mathfrak{t}). \end{aligned} \quad (120)$$

Then, the Dynkin game of Definition 3.4.3 has a value \mathbf{V} if:

$$\underline{\mathbf{V}} = \bar{\mathbf{V}}, \quad (121)$$

in such a case the value of the game, \mathbf{V} , is equal to the common value of $\underline{\mathbf{V}}$ and $\bar{\mathbf{V}}$, that is:

$$\underline{\mathbf{V}} = \mathbf{V} = \bar{\mathbf{V}}. \quad (122)$$

Based on Definition 3.4.5 and Definition 3.4.4 it is not hard to show that if a saddle point exists then the game has a value.

Lepeltier and Maingueneau [109] show that the game of Definition 3.4.3 has a value, they also show that a saddle point exists and that there exists ε -optimal strategies for the game.

Definition 3.4.6. Let $\varepsilon > 0$, assume \mathbf{V} is the value of the Dynkin game of Definition 3.4.3. $\xi^\varepsilon \in \mathcal{I}$ is called an ε -optimal strategy for Player A if:

$$J(\mathfrak{s}, \xi^\varepsilon) + \varepsilon \geq \mathbf{V} \quad \forall \mathfrak{s} \in \mathcal{I}. \quad (123)$$

Similarly, $\kappa^\varepsilon \in \mathcal{I}$ is called an ε -optimal strategy for Player B if:

$$J(\kappa^\varepsilon, \mathbf{t}) - \varepsilon \leq \mathbf{V} \quad \forall \mathbf{t} \in \mathcal{I}. \quad (124)$$

In the spirit of Lepeltier and Maingueneau [109]:

Definition 3.4.7. $\forall \mathbf{u} \in \mathcal{I}$, we define the conditional upper value, after time \mathbf{u} , of the Dynkin game of Definition 3.4.3 as

$$\overline{\mathbf{V}}_{\mathbf{u}} = \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} E \left(\mathcal{R}^{X,Y}(\mathbf{s}, \mathbf{t}) \middle| \mathcal{G}_{\mathbf{u}} \right). \quad (125)$$

Similarly, we define the conditional lower value, after time \mathbf{u} , as

$$\underline{\mathbf{V}}_{\mathbf{u}} = \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} E \left(\mathcal{R}^{X,Y}(\mathbf{s}, \mathbf{t}) \middle| \mathcal{G}_{\mathbf{u}} \right). \quad (126)$$

Using the condition on the value at infinity³⁹ of the processes X and Y , that is that $X_\infty = Y_\infty = 0$, Lepeltier and Maingueneau show, see [109] Lemma 5, that $\forall \mathbf{u} \in \mathcal{I}$,

$$\overline{\mathbf{V}}_{\mathbf{u}} = \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} E \left(X_{\mathbf{s}} \mathbb{1}_{\mathbf{s} \leq \mathbf{t}} + Y_{\mathbf{t}} \mathbb{1}_{\mathbf{t} < \mathbf{s}} \middle| \mathcal{G}_{\mathbf{u}} \right) = \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} E \left(\tilde{\mathcal{R}}^{X,Y}(\mathbf{s}, \mathbf{t}) \middle| \mathcal{G}_{\mathbf{u}} \right), \quad (127)$$

and

$$\underline{\mathbf{V}}_{\mathbf{u}} = \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} E \left(X_{\mathbf{s}} \mathbb{1}_{\mathbf{s} \leq \mathbf{t}} + Y_{\mathbf{t}} \mathbb{1}_{\mathbf{t} < \mathbf{s}} \middle| \mathcal{G}_{\mathbf{u}} \right) = \operatorname{ess\,sup}_{\substack{\mathbf{t} \in \mathcal{I} \\ \mathbf{t} \geq \mathbf{u}}} \operatorname{ess\,inf}_{\substack{\mathbf{s} \in \mathcal{I} \\ \mathbf{s} \geq \mathbf{u}}} E \left(\tilde{\mathcal{R}}^{X,Y}(\mathbf{s}, \mathbf{t}) \middle| \mathcal{G}_{\mathbf{u}} \right). \quad (128)$$

As we implicitly identify in the previous two equations we define the alternate payoff function $\tilde{\mathcal{R}}^{X,Y}$, or simply $\tilde{\mathcal{R}}$ as

$$\tilde{\mathcal{R}}^{X,Y}(s, t) = X_s \mathbb{1}_{s \leq t} + Y_t \mathbb{1}_{t < s}, \quad (129)$$

$s, t \in [0, \infty[$, which we can extend to stopping times $\mathbf{s}, \mathbf{t} \in \mathcal{I}$

$$\tilde{\mathcal{R}}^{X,Y}(\mathbf{s}, \mathbf{t}) = X_{\mathbf{s}} \mathbb{1}_{\mathbf{s} \leq \mathbf{t}} + Y_{\mathbf{t}} \mathbb{1}_{\mathbf{t} < \mathbf{s}}, \quad (130)$$

This means that the value of the Dynkin game of Definition 3.4.3 is blind to a change in the inequality found in the definition of $\mathcal{R}^{X,Y}$, and consequently in the definition of J

³⁹This condition could be changed by $X_\infty = Y_\infty = A$, A a finite \mathcal{U} -r.v. and the results of Lepeltier and Maingueneau [109] will be preserved.

(provided that $X_\infty = Y_\infty = 0$)⁴⁰. Said in another way, [109] Lemma 5 means that the value of the Dynkin game of Definition 3.4.3 is blind to a change from \mathcal{R} to $\tilde{\mathcal{R}}$ in Definition 3.4.3, equation (118).

Apart from the little extra flexibility in the definition of the game, [109] Lemma 5 offers the authors some technical advantages they will use later to show the existence of the game.

[109] Lemma 5 also offers hints into more general results regarding this kind of game; for example the Dynkin game of Definition 3.4.3 belongs to the class of two person continuous time zero-sum games whose expected payoff is of the form:

$$E(a_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + b_{\mathfrak{t}} \mathbb{1}_{\mathfrak{t} < \mathfrak{s}} + c_{\mathfrak{t}} \mathbb{1}_{\mathfrak{s} = \mathfrak{t}}), \quad (131)$$

[109] Lemma 5 implies that if $a_\infty = b_\infty$, choosing $c \equiv a$ or $c \equiv b$ does not change the value of the game (if it exists)⁴¹.

Corollary 12 of [109] show that the Dynkin game of Definition 3.4.3 has a value, \mathbf{V} ,

$$\underline{\mathbf{V}} = \sup_{\mathfrak{t} \in \mathcal{I}} \inf_{\mathfrak{s} \in \mathcal{I}} J(\mathfrak{s}, \mathfrak{t}) = \mathbf{V} = \inf_{\mathfrak{s} \in \mathcal{I}} \sup_{\mathfrak{t} \in \mathcal{I}} J(\mathfrak{s}, \mathfrak{t}) = \overline{\mathbf{V}}, \quad (132)$$

and more generally, that the conditional upper and lower values are equal for any stopping time \mathfrak{u} of the filtration \mathcal{G} ; that is:

$$\begin{aligned} \overline{\mathbf{V}}_{\mathfrak{u}} &= \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathcal{I} \\ \mathfrak{s} \geq \mathfrak{u}}} \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathcal{I} \\ \mathfrak{t} \geq \mathfrak{u}}} E(\mathcal{R}^{X,Y}(\mathfrak{s}, \mathfrak{t}) | \mathcal{G}_{\mathfrak{u}}) \\ &= \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathcal{I} \\ \mathfrak{s} \geq \mathfrak{u}}} \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathcal{I} \\ \mathfrak{t} \geq \mathfrak{u}}} E(\tilde{\mathcal{R}}^{X,Y}(\mathfrak{s}, \mathfrak{t}) | \mathcal{G}_{\mathfrak{u}}) = \mathbf{V}_{\mathfrak{u}} = \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathcal{I} \\ \mathfrak{t} \geq \mathfrak{u}}} \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathcal{I} \\ \mathfrak{s} \geq \mathfrak{u}}} E(\tilde{\mathcal{R}}^{X,Y}(\mathfrak{s}, \mathfrak{t}) | \mathcal{G}_{\mathfrak{u}}) \\ &= \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathcal{I} \\ \mathfrak{t} \geq \mathfrak{u}}} \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathcal{I} \\ \mathfrak{s} \geq \mathfrak{u}}} E(\mathcal{R}^{X,Y}(\mathfrak{s}, \mathfrak{t}) | \mathcal{G}_{\mathfrak{u}}) = \underline{\mathbf{V}}_{\mathfrak{u}}. \end{aligned} \quad (133)$$

⁴⁰As we mentioned before, it is not hard to show that under the slightly more general assumption of $X_\infty = Y_\infty = A$, with A a finite random variable, the results of Lepeltier and Maingueneau [109] are preserved.

⁴¹In a recent paper, Laraki and Solan (see [108] Theorem 3 and Proposition 6) show that two person continuous time zero-sum games whose payoff is given by (131), (where a , b and c are \mathcal{G} adapted uniformly bounded stochastic processes, a and b right continuous such that $a \leq b$, c progressively measurable) have a value in randomized stopping times and that such a value is independent of the process c . They also show that if $a \leq c \leq b$, the value in randomized times is the same as the value in stopping times.

Definition 3.4.8. Define, for every stopping time $\mathfrak{t} \in \mathcal{I}$ and any $\varepsilon > 0$

$$\begin{aligned}\kappa_{\mathfrak{t}}^{\varepsilon} &= \inf\{t \geq \mathfrak{t} : \underline{\mathbf{V}}_t \geq X_t - \varepsilon\}, \\ \xi_{\mathfrak{t}}^{\varepsilon} &= \inf\{t \geq \mathfrak{t} : \overline{\mathbf{V}}_t \leq Y_t + \varepsilon\}.\end{aligned}\tag{134}$$

By hypotheses, X and Y are right continuous. [109] Theorem 9 shows that $\underline{\mathbf{V}}$ and $\overline{\mathbf{V}}$ are right continuous⁴². Thus, both $\underline{\mathbf{V}}_t - X_t$ and $\overline{\mathbf{V}}_t - Y_t$ are right continuous and $\kappa_{\mathfrak{t}}^{\varepsilon}$ and $\xi_{\mathfrak{t}}^{\varepsilon}$ define stopping times⁴³ $\forall \varepsilon \geq 0$ and $\forall \mathfrak{t} \in \mathcal{I}$. In particular $\kappa_{\mathfrak{t}}^0$ and $\xi_{\mathfrak{t}}^0$ are stopping times. Not only that but

$$\begin{aligned}\kappa_{\mathfrak{t}}^{0+} &= \lim_{\varepsilon \rightarrow 0^+} \kappa_{\mathfrak{t}}^{\varepsilon}, \\ \text{and} \\ \xi_{\mathfrak{t}}^{0+} &= \lim_{\varepsilon \rightarrow 0^+} \xi_{\mathfrak{t}}^{\varepsilon}.\end{aligned}\tag{135}$$

are both stopping times⁴⁴.

With the help of the above defined stopping times and a well known result from El Karoui [52] (see Theorem 10 in [109]) Lepeltier and Maingueneau, see Theorem 11, [109], show that

$$X_{\mathfrak{t}} \geq \overline{\mathbf{V}}_{\mathfrak{t}} \geq \underline{\mathbf{V}}_{\mathfrak{t}} \geq Y_{\mathfrak{t}} \quad \forall \mathfrak{t} \in \mathcal{I},\tag{136}$$

and that

$$E(\underline{\mathbf{V}}_{\kappa_{\mathfrak{t}}^{\varepsilon} \wedge \mathfrak{s}'} | \mathcal{G}_{\mathfrak{t}}) \leq \underline{\mathbf{V}}_{\mathfrak{t}} \leq \overline{\mathbf{V}}_{\mathfrak{t}} \leq E(\overline{\mathbf{V}}_{\xi_{\mathfrak{t}}^{\varepsilon} \wedge \mathfrak{s}'} | \mathcal{G}_{\mathfrak{t}}),\tag{137}$$

$\forall \mathfrak{t} \in \mathcal{I}$ and all $\mathfrak{s}, \mathfrak{s}' \in \mathcal{I}$ such that $\mathfrak{s} \geq \mathfrak{t}$ and $\mathfrak{s}' \geq \mathfrak{t}$.

As a Corollary of Theorem 11 and with the aid of a result by Stettner [167] (Lemma 2 in Lepeltier and Maingueneau's [109]), Lepeltier and Maingueneau show that the conditional upper and lower values are equal, that is that $\underline{\mathbf{V}}_{\mathfrak{t}} = \overline{\mathbf{V}}_{\mathfrak{t}}$, $\forall \mathfrak{t} \in \mathcal{I}$ and that the stopping times $\kappa_{\mathfrak{t}}^{\varepsilon}$ and $\xi_{\mathfrak{t}}^{\varepsilon}$ defined in Definition 3.4.8 are ε -optimal strategies since

⁴²Or, more precisely, that such processes have right continuous modifications, modifications that could also be selected to be rcll (see [151] Chapter II section 2 or [96] section 1.3 Theorem 3.13).

⁴³See [96] section 1.2, specifically Definition 2.1, Proposition 2.3, Exercise 2.5 and Proposition 2.6, or [151] Proposition 4.6 or [177] Proposition 3.5, etc.

⁴⁴See [177] Theorem 3.19.

$$\begin{aligned}
& E \left(X_{\kappa_t^\varepsilon} \mathbb{1}_{\kappa_t^\varepsilon < s'} + Y_{s'} \mathbb{1}_{s' \leq \kappa_t^\varepsilon} \middle| \mathcal{G}_t \right) - \varepsilon \\
& \leq E \left(\underline{\mathbf{V}}_{\kappa_t^\varepsilon \wedge s'} \middle| \mathcal{G}_t \right) \leq \underline{\mathbf{V}}_t \leq \overline{\mathbf{V}}_t \leq E \left(\overline{\mathbf{V}}_{\xi_t^\varepsilon \wedge s} \middle| \mathcal{G}_t \right) \\
& \leq E \left(X_s \mathbb{1}_{s < \xi_t^\varepsilon} + Y_{\xi_t^\varepsilon} \mathbb{1}_{\xi_t^\varepsilon \leq s} \middle| \mathcal{G}_t \right) + \varepsilon, \quad (138)
\end{aligned}$$

$\forall t \in \mathcal{I}$ and all $s, s' \in \mathcal{I}$ such that $s \geq t$ and $s' \geq t$.

Finally, Theorem 13 in [109] establishes the right continuity of the value process $\{\mathbf{V}_t\}_{t \geq 0}$, which aggregates the collection $\{\mathbf{V}_t\}_{t \in \mathcal{I}}$, while Theorem 15 in [109] shows that under the assumption that X and Y are right continuous and $-X$ and Y are left upper semicontinuous (**l.u.s.c.** or simply **lusc**) then

$$\xi_t^0 \wedge \kappa_t^0 = \lim_{\varepsilon \rightarrow 0^+} \xi_t^\varepsilon \wedge \lim_{\varepsilon \rightarrow 0^+} \kappa_t^\varepsilon = \xi_t^{0+} \wedge \kappa_t^{0+}, \quad (139)$$

and the pair (ξ_t^0, κ_t^0) is a saddle point. That is, $\forall t, s, s' \in \mathcal{I}$, $s \wedge s' \geq t$

$$E \left(\mathcal{R}^{X,Y}(s, \xi_t^0) \middle| \mathcal{G}_t \right) \geq \mathbf{V}_t = E \left(\mathcal{R}^{X,Y}(\kappa_t^0, \xi_t^0) \middle| \mathcal{G}_t \right) \geq E \left(\mathcal{R}^{X,Y}(\kappa_t^0, s') \middle| \mathcal{G}_t \right), \quad (140)$$

and in particular, the initial value of the game, \mathbf{V}_0 , satisfies:

$$\begin{aligned}
J(s, \xi_0^0) &= E \left(\mathcal{R}^{X,Y}(s, \xi_0^0) \right) \\
&\geq \mathbf{V}_0 = E \left(\mathcal{R}^{X,Y}(\kappa_0^0, \xi_0^0) \right) = J(\kappa_0^0, \xi_0^0) \\
&\geq E \left(\mathcal{R}^{X,Y}(\kappa_0^0, s') \right) = J(\kappa_0^0, s') \quad (141)
\end{aligned}$$

$s, s' \in \mathcal{I}$.

We can summarize Lepeltier and Maingueneau's results in the following Theorem:

Theorem 3.4.1. *The Dynkin game, $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}, \mathcal{I}, J, X, Y)$, of Definition 3.4.3 has a value in pure strategies, $\mathbf{V} = \mathbf{V}_0$, where $\{\mathbf{V}_t\}_{t \geq 0}$ is the right continuous process that satisfies:*

$$\mathbf{V}_u = \operatorname{essinf}_{\substack{s \in \mathcal{I} \\ s \geq u}} \operatorname{esssup}_{\substack{t \in \mathcal{I} \\ t \geq u}} E \left(\mathcal{R}^{X,Y}(s, t) \middle| \mathcal{G}_u \right) = \operatorname{esssup}_{\substack{t \in \mathcal{I} \\ t \geq u}} \operatorname{essinf}_{\substack{s \in \mathcal{I} \\ s \geq u}} E \left(\mathcal{R}^{X,Y}(s, t) \middle| \mathcal{G}_u \right) \quad \mathcal{P}\text{-a.e.} \quad (142)$$

$\forall t \in \mathcal{I}$ and $\forall \varepsilon \geq 0$ both players have pure ε -optimal strategies, κ_t^ε and ξ_t^ε , given by:

$$\begin{aligned}
\kappa_t^\varepsilon &= \inf \{ u \geq t : \mathbf{V}_u \geq X_u - \varepsilon \}, \\
\xi_t^\varepsilon &= \inf \{ u \geq t : \mathbf{V}_u \leq Y_u + \varepsilon \}.
\end{aligned} \quad (143)$$

that satisfy

$$E(\mathcal{R}^{X,Y}(\kappa_t^\varepsilon, \mathbf{u}) | \mathcal{G}_t) - \varepsilon \leq \mathbf{V}_t \leq E(\mathcal{R}^{X,Y}(\mathfrak{s}, \xi_t^\varepsilon) | \mathcal{G}_t) + \varepsilon \quad (144)$$

$\forall \mathbf{t} \in \mathcal{I}$ and all $\mathfrak{s}, \mathbf{u} \in \mathcal{I}$ such that $\mathfrak{s} \geq \mathbf{t}$ and $\mathbf{u} \geq \mathbf{t}$. If in addition the processes $-X$ and Y are l.u.s.c. then the game attains a saddle point. Defining

$$\tilde{\kappa}_t = \lim_{\varepsilon \rightarrow 0} \kappa_t^\varepsilon \text{ and } \tilde{\xi}_t = \lim_{\varepsilon \rightarrow 0} \xi_t^\varepsilon, \quad (145)$$

for $\mathbf{t} \in \mathcal{I}$ then

$$\tilde{\kappa}_t \wedge \tilde{\xi}_t = \kappa_t^0 \wedge \xi_t^0, \quad (146)$$

and

$$E(\mathcal{R}^{X,Y}(\tilde{\kappa}_t, \mathbf{u}) | \mathcal{G}_t) \leq \mathbf{V}_t \leq E(\mathcal{R}^{X,Y}(\mathfrak{s}, \tilde{\xi}_t) | \mathcal{G}_t) \quad (147)$$

$\forall \mathbf{t} \in \mathcal{I}$ and all $\mathfrak{s}, \mathbf{u} \in \mathcal{I}$ such that $\mathfrak{s} \geq \mathbf{t}$ and $\mathbf{u} \geq \mathbf{t}$. Furthermore

$$\mathbf{V}_t = E(\mathcal{R}^{X,Y}(\kappa_t^0, \xi_t^0) | \mathcal{G}_t) \quad (148)$$

$\forall \mathbf{t} \in \mathcal{I}$.

The above version of Lepeltier and Maingueneau's theorem encompasses the results of [109] Corollary 12, and Theorem 13 to Theorem 15.

For several years, Lepeltier and Maingueneau's paper has provided the most general conditions for the existence of a value of the game. Recently, Laraki and Solan [108] found several extensions to Lepeltier and Maingueneau's work. In particular, they argue that the condition of uniform boundedness of the processes $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$ can be safely changed by the condition of RCLL and class (D); they (see [108]) also show that the condition $\tilde{\mathfrak{X}}_\infty = \tilde{\mathfrak{Y}}_\infty = 0$ can be relaxed by means of adding to the game payoff a term of the form $\chi \mathbb{1}_{\xi=\kappa=\infty}$ where χ is a \mathcal{U} -measurable and integrable function [108] section 4.2. They also show that under this different condition at infinity, [108] Proposition 11, if the payoff of the game is altered to be

$$\tilde{\mathfrak{X}}_\kappa \mathbb{1}_{\kappa \leq \xi} + \tilde{\mathfrak{Y}}_\xi \mathbb{1}_{\xi < \kappa} \quad (149)$$

the resulting game still has a value in pure strategies with ε -optimal strategies, and that such strategies do not depend on that small change on the payoff function.

After such a long digression we must recall what was our motivation to get into it and what is our goal. Lepeltier and Maingueneau's paper, [109], with the extensions by Laraki and Solan, [108], represent the “state of the art” in the study of the particular form of Dynkin game that we have been discussing along the last few pages. Lepeltier and Maingueneau's results, with the extensions by Laraki and Solan, show that a value for such games exists under very general conditions. We will apply such results to a slight variation of the game, to a game with finite horizon, to justify the existence of a value for the corresponding gcc, then we will only need to show that a gcc is financeable and that a hedge exists.

In the next few pages we will see what happens when a finite horizon Dynkin game is considered, then in the next section, §3.4.2, we will study the hedging of a gcc. Our work will culminate in section §3.4.3.

Define now

$$\begin{aligned}\tilde{\mathfrak{X}}_t &= \begin{cases} \mathfrak{X}_t^* & \text{if } t \in [0, T] \\ e^{T-t}\mathfrak{X}_T^* & \text{if } t > T \end{cases} \\ \tilde{\mathfrak{Y}}_t &= \begin{cases} \mathfrak{Y}_t^* & \text{if } t \in [0, T] \\ e^{T-t}\mathfrak{Y}_T^* & \text{if } t > T \end{cases} \\ \mathcal{G}_t &= \begin{cases} \mathcal{F}_t & \text{if } t \in [0, T] \\ \mathcal{F}_T & \text{if } t > T \end{cases} \\ \mathcal{P} &= \mathcal{P}^{\mathcal{E}}\end{aligned}\tag{150}$$

where \mathfrak{X} , \mathfrak{Y} and \mathcal{F} are as in Definition 3.4.2. It is clear that the processes defined in (150) are RCLL of class (D) and that \mathcal{G} is a right-continuous filtration that satisfies the “usual conditions”. Also clear from our definition is that $\lim_{t \rightarrow \infty} \tilde{\mathfrak{X}}_t = 0 = \lim_{t \rightarrow \infty} \tilde{\mathfrak{Y}}_t$ $\mathcal{P}^{\mathcal{E}}$ -a.s. So we can safely define $\tilde{\mathfrak{X}}_{\infty} = 0 = \tilde{\mathfrak{Y}}_{\infty}$ and naturally extend the filtration $\{\mathcal{G}_t\}_{t \geq 0}$ to ∞ .

Thus, the resulting game satisfies all conditions of Lepeltier and Maingueneau with the extension of Laraki and Solan.

Notation 3.4.2. We will extend Notation 3.4.1 using \mathfrak{S} to represent the set of all stopping times with respect to the (extended) filtration \mathcal{G} . Similarly, if $t \geq 0$ we will write $\mathfrak{S}_{t,\infty}$ to

denote the subset of \mathfrak{S} that contains all stopping times whose values are bigger or equal to t (notice that the value ∞ is allowed). Finally, if $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}, \infty}$, then the stopping time \mathfrak{s} must be an $\mathcal{F}_{\mathcal{T}}$ -measurable r.v..

Definition 3.4.9 (Auxiliary Game). *The Dynkin game described above (see formulas (113), (114), (115), (116), (117), (150)) will be represented as $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}^{\mathcal{E}}, \mathfrak{S}, J, \tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}})$ and will be called the **auxiliary game**.*

Corollary 3.4.2. *The auxiliary game $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}^{\mathcal{E}}, \mathfrak{S}, J, \tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}})$ satisfy the conditions of Definition 3.4.3, that is, it is an Dynkin game, and, by Theorem 3.4.1 it has a value in pure strategies $V = V_0$, where $\{V_t\}_{t \geq 0}$ is the right continuous process that satisfies:*

$$V_t = \operatorname{ess\,inf}_{\substack{\mathfrak{s} \in \mathfrak{S} \\ \mathfrak{s} \geq t}} \operatorname{ess\,sup}_{\substack{\mathfrak{t} \in \mathfrak{S} \\ \mathfrak{t} \geq t}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_t \right) = \operatorname{ess\,sup}_{\substack{\mathfrak{t} \in \mathfrak{S} \\ \mathfrak{t} \geq t}} \operatorname{ess\,inf}_{\substack{\mathfrak{s} \in \mathfrak{S} \\ \mathfrak{s} \geq t}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_t \right) \quad \mathcal{P}^{\mathcal{E}}\text{-a.e.} \quad (151)$$

$\forall \mathfrak{t} \in \mathfrak{S}$ and $\forall \varepsilon \geq 0$ both players have pure ε -optimal strategies, $\kappa_{\mathfrak{t}}^{\varepsilon}$ and $\xi_{\mathfrak{t}}^{\varepsilon}$, given by:

$$\begin{aligned} \kappa_{\mathfrak{t}}^{\varepsilon} &= \inf \left\{ u \geq \mathfrak{t} : V_u \geq \tilde{\mathfrak{X}}_u - \varepsilon \right\}, \\ \xi_{\mathfrak{t}}^{\varepsilon} &= \inf \left\{ u \geq \mathfrak{t} : V_u \leq \tilde{\mathfrak{Y}}_u + \varepsilon \right\}. \end{aligned} \quad (152)$$

that satisfy

$$E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa_{\mathfrak{t}}^{\varepsilon}, u) \middle| \mathcal{G}_t \right) - \varepsilon \leq V_t \leq E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \xi_{\mathfrak{t}}^{\varepsilon}) \middle| \mathcal{G}_t \right) + \varepsilon \quad (153)$$

$\forall \mathfrak{t} \in \mathfrak{S}$ and all $\mathfrak{s}, u \in \mathfrak{S}$ such that $\mathfrak{s} \geq \mathfrak{t}$ and $u \geq \mathfrak{t}$. If in addition the processes $-\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$ are l.u.s.c. then the game attains a saddle point. Defining

$$\tilde{\kappa}_{\mathfrak{t}} = \lim_{\varepsilon \rightarrow 0} \kappa_{\mathfrak{t}}^{\varepsilon} \text{ and } \tilde{\xi}_{\mathfrak{t}} = \lim_{\varepsilon \rightarrow 0} \xi_{\mathfrak{t}}^{\varepsilon}, \quad (154)$$

for $\mathfrak{t} \in \mathfrak{S}$ then

$$\tilde{\kappa}_{\mathfrak{t}} \wedge \tilde{\xi}_{\mathfrak{t}} = \kappa_{\mathfrak{t}}^0 \wedge \xi_{\mathfrak{t}}^0, \quad (155)$$

and

$$E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\tilde{\kappa}_{\mathfrak{t}}, u) \middle| \mathcal{G}_t \right) \leq V_t \leq E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \tilde{\xi}_{\mathfrak{t}}) \middle| \mathcal{G}_t \right) \quad (156)$$

$\forall \mathfrak{t} \in \mathfrak{S}$ and all $\mathfrak{s}, u \in \mathfrak{S}$ such that $\mathfrak{s} \geq \mathfrak{t}$ and $u \geq \mathfrak{t}$. Furthermore

$$V_{\mathfrak{t}} = E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa_{\mathfrak{t}}^0, \xi_{\mathfrak{t}}^0) \middle| \mathcal{G}_t \right) \quad (157)$$

$\forall \mathfrak{t} \in \mathfrak{S}$.

Now recall from (150) that $\tilde{\mathfrak{X}}_t = e^{T-t}\mathfrak{X}_T^*$ and $\tilde{\mathfrak{Y}}_t = e^{T-t}\mathfrak{Y}_T^* \forall t \geq T$. That is, the values of $\tilde{\mathfrak{X}}_t$ and $\tilde{\mathfrak{Y}}_t$ diminish to zero, $\mathcal{P}^\mathcal{E}$ -a.s., as $t \rightarrow \infty$. In this case it is clear that to wait past time T to stop the game is suboptimal for *Player A*. Similarly, for *Player B* there is no advantage in considering strategies whose values surpass T .

The previous corollary speaks of a game with infinite horizon. In order to use those results in our study of game contingent claims we must be able to apply the results of Corollary 3.4.2 to a game with finite horizon. The following results will help us in that direction.

Assertion 3.4.3. $\forall \mathfrak{u} \in \mathfrak{S}_{T,\infty}$, $V_{\mathfrak{u}} = \tilde{\mathfrak{Y}}_{\mathfrak{u}}$. That is, after time T the value of the auxiliary game of definition Definition 3.4.9 is equal to the value of process $\tilde{\mathfrak{Y}}$.

Proof. Recall the definition of processes $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$ in equation (150). Let $\mathfrak{u} \in \mathfrak{S}_{T,\infty}$ be fixed. We observe that, $\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ such that $\mathfrak{s} \geq \mathfrak{u}$ and $\mathfrak{t} \geq \mathfrak{u}$ we have:

$$\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) = \tilde{\mathfrak{X}}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \tilde{\mathfrak{Y}}_{\mathfrak{t}} \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}} = e^{T-\mathfrak{s} \wedge \mathfrak{t}} (\mathfrak{X}_T^* \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \mathfrak{Y}_T^* \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}}) \geq e^{T-\mathfrak{s} \wedge \mathfrak{t}} \mathfrak{Y}_T^* = \tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}}, \quad (158)$$

with equality whenever $\mathfrak{s} = \mathfrak{t}$.

Since $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ we know $\mathfrak{s} \wedge \mathfrak{t} \in \mathfrak{S}$, and since $\mathfrak{s} \geq \mathfrak{u}$ and $\mathfrak{t} \geq \mathfrak{u}$ we clearly have $\mathfrak{s} \wedge \mathfrak{t} \geq \mathfrak{u} \geq T$ $\mathcal{P}^\mathcal{E}$ -a.s.. By definition of a stopping time we know $\{\mathfrak{s} \wedge \mathfrak{t} \leq t\} \in \mathcal{G}_t$. By definition of \mathcal{G}_t for $t \geq T$ we know, $\{\mathfrak{s} \wedge \mathfrak{t} \leq t\} \in \mathcal{G}_t = \mathcal{G}_T = \mathcal{F}_T$, $\forall t \geq T$, thus $\mathfrak{s} \wedge \mathfrak{t}$ and consequently $e^{T-\mathfrak{s} \wedge \mathfrak{t}}$ is an \mathcal{F}_T measurable r.v.. Since \mathfrak{Y}_T^* is \mathcal{F}_T measurable we see:

$$\tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} = E_{\mathcal{E}} \left(\tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} \middle| \mathcal{G}_{\mathfrak{u}} \right) \leq E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_{\mathfrak{u}} \right) \quad (159)$$

Fix $\mathfrak{t} \in \mathfrak{S}$ such that $\mathfrak{t} \geq \mathfrak{u}$ and consider the families of random variables:

$$\left\{ \tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} \right\}_{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s} \geq \mathfrak{u}} \quad \text{and} \quad \left\{ E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_{\mathfrak{u}} \right) \right\}_{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s} \geq \mathfrak{u}} \quad (160)$$

Notice that, $\forall \mathfrak{s} \in \mathfrak{S}$ such that $\mathfrak{s} \geq \mathfrak{u}$

$$\tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} = \tilde{\mathfrak{Y}}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \tilde{\mathfrak{Y}}_{\mathfrak{t}} \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}} \geq \tilde{\mathfrak{Y}}_{\mathfrak{t}} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \tilde{\mathfrak{Y}}_{\mathfrak{t}} \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}} = \tilde{\mathfrak{Y}}_{\mathfrak{t}}. \quad (161)$$

It is clear that

$$\tilde{\mathfrak{Y}}_{\mathfrak{t}} \leq \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s} \geq \mathfrak{u}} \tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} \leq \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s} \geq \mathfrak{u}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_{\mathfrak{u}} \right), \quad (162)$$

so,

$$\tilde{\mathfrak{Y}}_u = \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathfrak{G}; \\ \mathfrak{t} \geq u}} \tilde{\mathfrak{Y}}_{\mathfrak{t}} \leq \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathfrak{G}; \\ \mathfrak{t} \geq u}} \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathfrak{G}; \\ \mathfrak{s} \geq u}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_u \right) = V_u. \quad (163)$$

But, we are assuming $\mathfrak{t} \in \mathfrak{G}$ is such that $\mathfrak{t} \geq u$, this means that $\tilde{\mathfrak{Y}}_{\mathfrak{t}}$ is an element of both families, thus

$$\tilde{\mathfrak{Y}}_{\mathfrak{t}} \geq \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathfrak{G}; \\ \mathfrak{s} \geq u}} \tilde{\mathfrak{Y}}_{\mathfrak{s} \wedge \mathfrak{t}} \quad \text{and} \quad \tilde{\mathfrak{Y}}_{\mathfrak{t}} \geq \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathfrak{G}; \\ \mathfrak{s} \geq u}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_u \right). \quad (164)$$

Therefore

$$\tilde{\mathfrak{Y}}_{\mathfrak{t}} = \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathfrak{G}; \\ \mathfrak{s} \geq u}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_u \right); \quad (165)$$

which implies that

$$\tilde{\mathfrak{Y}}_u = \operatorname{esssup}_{\substack{\mathfrak{t} \in \mathfrak{G}; \\ \mathfrak{t} \geq u}} \operatorname{essinf}_{\substack{\mathfrak{s} \in \mathfrak{G}; \\ \mathfrak{s} \geq u}} E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\mathfrak{s}, \mathfrak{t}) \middle| \mathcal{G}_u \right) = V_u, \quad (166)$$

since $\tilde{\mathfrak{Y}}_{\mathfrak{t}} \leq \tilde{\mathfrak{Y}}_u \forall \mathfrak{t} \in \mathfrak{G}$ is such that $\mathfrak{t} \geq u$. \square

The previous Assertion is of much importance since it characterizes the value of the game after time \mathcal{T} . This Assertion will be used in Proposition 3.4.5 to characterize the ε -optimal strategies of *Player A*. In the case of *Player B*, given $\varepsilon \geq 0$ and $\mathfrak{t} \in \mathfrak{G}_{0, \mathcal{T}}$, its ε -optimal strategies are not so simple. In fact, depending on the details on the processes $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$ it is possible to have $\kappa_{\mathfrak{t}}^{\varepsilon} \in]\mathcal{T}, \infty[$ with non null probability. A detailed analysis could provide with more properties of such strategies, but we do not need to go further. Our interest in the Auxiliary game is totally subsidiary, instead we can easily show that under favorable circumstances we can cut off the high values of $\kappa_{\mathfrak{t}}^{\varepsilon}$ and retain its main properties under such conditions.

Let $\varepsilon \geq 0$ and let $\mathfrak{t} \in \mathfrak{G}_{0, \mathcal{T}}$ be given. Let also $u \in \mathfrak{G}_{0, \mathcal{T}}$ be such that $u \geq \mathfrak{t}$. On the event $\{\kappa_{\mathfrak{t}}^{\varepsilon} > \mathcal{T}\}$, we use the definition of the payoff function \mathcal{R} to see that:

$$\begin{aligned} \mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa_{\mathfrak{t}}^{\varepsilon}, u) &= \tilde{\mathfrak{X}}_{\kappa_{\mathfrak{t}}^{\varepsilon}} \mathbb{1}_{\kappa_{\mathfrak{t}}^{\varepsilon} < u} + \tilde{\mathfrak{Y}}_u \mathbb{1}_{u \leq \kappa_{\mathfrak{t}}^{\varepsilon}} = \tilde{\mathfrak{Y}}_u \\ &= \tilde{\mathfrak{X}}_{\kappa_{\mathfrak{t}}^{\varepsilon} \wedge \mathcal{T}} \mathbb{1}_{\kappa_{\mathfrak{t}}^{\varepsilon} \wedge \mathcal{T} < u} + \tilde{\mathfrak{Y}}_u \mathbb{1}_{u \leq \kappa_{\mathfrak{t}}^{\varepsilon} \wedge \mathcal{T}} = \mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa_{\mathfrak{t}}^{\varepsilon} \wedge \mathcal{T}, u). \end{aligned} \quad (167)$$

By Corollary 3.4.2, (153), $\forall \varepsilon \geq 0$, $\forall \mathfrak{t} \in \mathfrak{G}$ and $\forall u \in \mathfrak{G}$ such that $u \geq \mathfrak{t}$, $\kappa_{\mathfrak{t}}^{\varepsilon}$ satisfies,

$$E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}}(\kappa_{\mathfrak{t}}^{\varepsilon}, u) \middle| \mathcal{G}_{\mathfrak{t}} \right) - \varepsilon \leq V_{\mathfrak{t}}; \quad (168)$$

in particular, $\forall \mathbf{u} \in \mathfrak{S}_{0,T}$ we have

$$E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}}}(\kappa_t^\varepsilon \wedge T, \mathbf{u}) \middle| \mathcal{G}_t \right) - \varepsilon = E_{\mathcal{E}} \left(\mathcal{R}^{\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}}}(\kappa_t^\varepsilon, \mathbf{u}) \middle| \mathcal{G}_t \right) - \varepsilon \leq V_t, \quad (169)$$

where we have used (167) to justify the equality in (169) in the event $\{\kappa_t^\varepsilon > T\}$ (in the event $\{\kappa_t^\varepsilon \leq T\}$ we have equality anyways). This proves the following proposition.

Proposition 3.4.4. *$\forall \varepsilon \geq 0$ and $\forall t \in \mathfrak{S}$ with values in $[0, T]$, if Player A is restricted to use strategies $\mathbf{u} \in \mathfrak{S}_{0,T}$ whose values lie in $[0, T]$ then the stopping times $\kappa_t^\varepsilon \wedge T$ are also ε -optimal strategies corresponding to Player B.*

The next proposition, which is an immediate consequence of Assertion 3.4.3, shows that some of the ε -optimal strategies of Player A take values in $[0, T]$.

Proposition 3.4.5. *$\forall \varepsilon \geq 0$ and $\forall t \in \mathfrak{S}$ with values in $[0, T]$ the ε -optimal strategies of the auxiliary game of Definition 3.4.9 corresponding to Player A, ξ_t^ε , take values in $[0, T]$.*

Proof. Let $\varepsilon \geq 0$ and let $t \in \mathfrak{S}$ such that $t \in [0, T]$. Assume that $\mathcal{P}\{\xi_t^\varepsilon > T\} > 0$. By definition of the infimum, on $\{\xi_t^\varepsilon > T\}$, $V_u > \tilde{\mathfrak{y}}_u + \varepsilon$, $\forall u$ such that $T \leq u < \xi_t^\varepsilon$. On the other hand, Assertion 3.4.3 tell us that $V_u = \tilde{\mathfrak{y}}_u$. Thus we face a contradiction. \square

The meaning of our discussion here is that for Player A, involved in our auxiliary game $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}^\varepsilon, \mathfrak{S}, J, \tilde{\mathfrak{x}}, \tilde{\mathfrak{y}})$, it is in his/her best interest to stop the game at or before time T . Stopping the game after time T will almost surely reduce his/her gain. If $t \leq T$, it is in Player A's best interest to reduce the set from which he/she will select strategies from $\mathfrak{S}_{t,\infty}$ to $\mathfrak{S}_{t,T}$.

And what happens to Player B? By Lepeltier and Maingueneau we know that, $\forall \varepsilon \geq 0$, κ_t^ε (see (153)) is a pure ε -strategy. But, even if $t \in [0, T]$, it is not so clear that κ_t^ε is bounded. What Proposition 3.4.4 is telling us is that if $\kappa_t^\varepsilon > T$, Player B will accomplish the same by stopping at time T as what she/he will do by stopping at time κ_t^ε . That is, if $t \in [0, T]$, $\kappa_t^\varepsilon \wedge T \in \mathfrak{S}_{t,T}$ is also a pure ε -strategy for Player B. For Player B there is no advantage in considering strategies whose values surpass T .

Using the previous two results (Assertion 3.4.3, Proposition 3.4.4 and Proposition 3.4.5), Definition 3.4.9 and Corollary 3.4.2 we can write:

Corollary 3.4.6. *In the case of the auxiliary game $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}^\varepsilon, \mathfrak{S}, J, \tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}})$, $\forall t \in [0, T]$: i) pure ε -strategies, κ_t^ε and ξ_t^ε , can be chosen within the collection $\mathfrak{S}_{t,T}$ of stopping times with values in $[t, T]$.*

$$\begin{aligned}\kappa_t^\varepsilon &= \inf_{u \geq t} \{V_u \geq \mathfrak{X}_u^* - \varepsilon\} \wedge T \\ \xi_t^\varepsilon &= \inf_{u \geq t} \{V_u \leq \mathfrak{Y}_u^* + \varepsilon\}\end{aligned}\tag{170}$$

ii) *The value of the auxiliary game satisfies:*

$$V_t = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t,T}} \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}} (\mathfrak{X}_{\mathfrak{s}}^* \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \mathfrak{Y}_{\mathfrak{t}}^* \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}} | \mathcal{F}_t) = \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t,T}} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}} (\mathfrak{X}_{\mathfrak{s}}^* \mathbb{1}_{\mathfrak{s} < \mathfrak{t}} + \mathfrak{Y}_{\mathfrak{t}}^* \mathbb{1}_{\mathfrak{t} \leq \mathfrak{s}} | \mathcal{F}_t)\tag{171}$$

and

$$E_{\mathcal{E}} \left(\mathfrak{X}_{\kappa_t^\varepsilon}^* \mathbb{1}_{\kappa_t^\varepsilon < \mathfrak{t}} + \mathfrak{Y}_{\mathfrak{t}}^* \mathbb{1}_{\mathfrak{t} \leq \kappa_t^\varepsilon} \middle| \mathcal{F}_t \right) - \varepsilon \leq V_t \leq E_{\mathcal{E}} \left(\mathfrak{X}_{\mathfrak{s}}^* \mathbb{1}_{\mathfrak{s} < \xi_t^\varepsilon} + \mathfrak{Y}_{\xi_t^\varepsilon}^* \mathbb{1}_{\xi_t^\varepsilon \leq \mathfrak{s}} \middle| \mathcal{F}_t \right) + \varepsilon\tag{172}$$

$\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{t,T}$. iii) *If additionally the processes $-\mathfrak{X}^*$ and \mathfrak{Y}^* are upper semicontinuous from the left*

$$\begin{aligned}\tilde{\kappa}_t &= \lim_{\varepsilon \rightarrow 0} \kappa_t^\varepsilon \\ \tilde{\xi}_t &= \lim_{\varepsilon \rightarrow 0} \xi_t^\varepsilon \\ \tilde{\kappa}_t \wedge \tilde{\xi}_t &= \kappa_t^0 \wedge \xi_t^0\end{aligned}\tag{173}$$

$$E_{\mathcal{E}} (\mathfrak{X}_{\tilde{\kappa}_t}^* \mathbb{1}_{\tilde{\kappa}_t < \mathfrak{t}} + \mathfrak{Y}_{\mathfrak{t}}^* \mathbb{1}_{\mathfrak{t} \leq \tilde{\kappa}_t} | \mathcal{F}_t) \leq V_t \leq E_{\mathcal{E}} (\mathfrak{X}_{\mathfrak{s}}^* \mathbb{1}_{\mathfrak{s} < \tilde{\xi}_t} + \mathfrak{Y}_{\tilde{\xi}_t}^* \mathbb{1}_{\tilde{\xi}_t \leq \mathfrak{s}} | \mathcal{F}_t)\tag{174}$$

$\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{t,T}$. And

$$V_t = E_{\mathcal{E}} \left(\mathfrak{X}_{\kappa_t^0}^* \mathbb{1}_{\kappa_t^0 < \xi_t^0} + \mathfrak{Y}_{\xi_t^0}^* \mathbb{1}_{\xi_t^0 \leq \kappa_t^0} \middle| \mathcal{F}_t \right)\tag{175}$$

3.4.2 Hedging against a Game contingent claim

In order to price the gcc contract, we establish a way to hedge our position in one of such contracts by means of a replicating portfolio. To do so, we will follow a game theoretic approach based on the idea of generalized backward induction.

Definition 3.4.10. *A hedge against a gcc will consist of a martingale generating, self-financing portfolio strategy Π with initial investment (initial wealth) w_0 and a stopping*

time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ so that, with probability one, the corresponding wealth process \mathcal{W}^{Π, w_0} at time $\mathfrak{s} \wedge t$ is larger than the claim's payoff at that time, for each $t \geq 0$, that is,

$$\mathcal{W}_{\mathfrak{s} \wedge t}^{\Pi, w_0} \geq \mathcal{R}(\mathfrak{s}, t) \quad \text{a.e.} - \mathcal{P}^{\mathcal{E}} \quad (176)$$

To represent that hedge we will use the notation (Π, w_0, \mathfrak{s}) .

Definition 3.4.11. *The fair price \mathfrak{V} of a gcc is defined by*

$$\mathfrak{V} = \inf \{ w : w = \mathcal{W}_0^{\Pi, w_0} \text{ for some hedge } (\Pi, w_0, \mathfrak{s}) \text{ against the gcc} \} \quad (177)$$

One of the first technical difficulties we face pricing a gcc is due to the form of the payoff process. The following result will provide us with some of the properties of this process.

Assertion 3.4.7. *The following statements are true:*

1. *For any pair of times $(s, t) \in [0, \mathcal{T}] \times [0, \mathcal{T}]$, $\mathcal{R}(s, t)$ is $\mathcal{F}_{s \wedge t}$ -measurable.*
2. *The payoff process \mathcal{R} is RCLL and adapted in its first index.*
3. *The payoff process \mathcal{R} is adapted but not necessarily RCLL in its second index.*

Proof. The first statement is clear from the definition of \mathcal{R} , Definition 3.4.1, equation (107). Since \mathfrak{X} and \mathfrak{Y} are RCLL and adapted processes, we know \mathfrak{X}_s is \mathcal{F}_s -measurable, for all $s \in [0, \mathcal{T}]$; similarly, \mathfrak{Y}_t is \mathcal{F}_t -measurable, for all $t \in [0, \mathcal{T}]$. Hence, if $t \leq s$, $\mathcal{R}(s, t) = \mathfrak{Y}_t$ which is \mathcal{F}_t -measurable, and in such a case $t = s \wedge t$; the other case is similar. The second statement should also be clear from definition Definition 3.4.1, and statement 1) in this assertion. Since $\forall (s, t) \in [0, \mathcal{T}] \times [0, \mathcal{T}]$, $\mathcal{R}(s, t)$ is $\mathcal{F}_{s \wedge t}$ -measurable, it is also clear that $\mathcal{R}(s, t)$ is also \mathcal{F}_t -measurable, since $s \wedge t \leq t \quad \forall (s, t) \in [0, \mathcal{T}] \times [0, \mathcal{T}]$ to show that \mathcal{R} is RCLL in its first index will require only the simple computation of some limits.

The whole proof that \mathcal{R} is RCLL in its first index goes in three steps, depending on i) $s_0 < t$, ii) $s_0 > t$ or iii) $s_0 = t$. Let $t \in]0, \mathcal{T}[$ be fixed. Let $s_0 \in]0, \mathcal{T}[$

- Assume $s_0 < t$.

$$\lim_{s \rightarrow s_0^+} \mathcal{R}(s, t) = \lim_{\substack{s \rightarrow s_0^+ \\ s < t}} \mathfrak{X}_s = \mathfrak{X}_{s_0} = \mathcal{R}(s_0, t)$$

and

$$\lim_{s \rightarrow s_0^-} \mathcal{R}(s, t) = \lim_{s \rightarrow s_0^-} \mathfrak{X}_s \quad \text{exists and is finite.}$$

- Assume now that $s_0 > t$.

$$\lim_{s \rightarrow s_0^+} \mathcal{R}(s, t) = \lim_{s \rightarrow s_0^+} \mathfrak{Y}_t = \mathfrak{Y}_t = \mathcal{R}(s_0, t)$$

and

$$\lim_{s \rightarrow s_0^-} \mathcal{R}(s, t) = \lim_{\substack{s \rightarrow s_0^- \\ s > t}} \mathcal{R}(s, t) = \mathfrak{Y}_t < \infty$$

- Finally, assume $s_0 = t$.

$$\lim_{s \rightarrow s_0^+} \mathcal{R}(s, t) = \lim_{s \rightarrow s_0^+} \mathfrak{Y}_t = \mathfrak{Y}_t = \mathcal{R}(s_0, t)$$

and

$$\lim_{s \rightarrow s_0^-} \mathcal{R}(s, t) = \lim_{s \rightarrow s_0^-} \mathfrak{X}_s \quad \text{exists and is finite.}$$

Note that if $t \in [0, \mathcal{T}]$ the result is still valid. If $t = 0$, $\mathcal{R}(s, 0) = \mathfrak{Y}_t \forall s \in [0, \mathcal{T}]$ which is clearly RCLL, if $t = \mathcal{T}$, $\mathcal{R}(s, \mathcal{T}) = \mathfrak{X}_s$ for $s < \mathcal{T}$ while $\mathcal{R}(\mathcal{T}, \mathcal{T}) = \mathfrak{Y}_{\mathcal{T}}$; since \mathfrak{X} is RCLL, we see that \mathcal{R} is RCLL for $s < \mathcal{T}$ and $t = \mathcal{T}$. At $s = \mathcal{T}$, $t = \mathcal{T}$, we could only take left limits, and since \mathfrak{X} is RCLL we see that \mathcal{R} also has finite left limits at $s = \mathcal{T}$, $t = \mathcal{T}$.

Therefore, in its first index \mathcal{R} is right continuous with finite left hand side limits.

For the last statement, assuming that $s \in]0, \mathcal{T}[$ and $t_0 = s$ we can see that

$$\lim_{t \rightarrow t_0^-} \mathcal{R}(s, t) = \lim_{t \rightarrow t_0^-} \mathfrak{Y}_t \quad \text{exists and is finite;}$$

however,

$$\lim_{t \rightarrow t_0^+} \mathcal{R}(s, t) = \lim_{t \rightarrow t_0^+} \mathfrak{X}_s = \mathfrak{X}_s \quad \text{and} \quad \mathcal{R}(s, t_0) = \mathfrak{Y}_{t_0} = \mathfrak{Y}_s$$

and so, \mathcal{R} is not necessarily RCLL in its second index. That \mathcal{R} is adapted in its second index is clear from the first statement in this assertion⁴⁵. \square

Proposition 3.4.8. *For any stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ and any time $t \in [0, \mathcal{T}]$, $\mathcal{R}(\mathfrak{s}, t)$ is $\mathcal{F}_{\mathfrak{s} \wedge t}$ -measurable.*

⁴⁵Note that although \mathcal{R} is not RCLL in its second index $\forall t$, it is RCLL $\forall t \neq s$. In the case of $t = s$ what fails is the right continuity, but the limits are still finite.

Proof. Let $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ be a stopping time. From Assertion 3.4.7 we know that \mathcal{R} is an adapted RCLL process in its first component, so, for any given time $t \in [0, T]$ the process

$$\{\mathcal{R}(s, t)\}_{s \in [0, T]}$$

is progressively measurable⁴⁶. Moreover, by [96] proposition 2.18 the variable $\mathcal{R}(\mathfrak{s}, t)$ is $\mathcal{F}_{\mathfrak{s}}$ -measurable. Similarly, we assumed \mathfrak{X} to be an RCLL and adapted process, therefore, \mathfrak{X} is also progressively measurable, and $\mathfrak{X}_{\mathfrak{s}}$ is $\mathcal{F}_{\mathfrak{s}}$ -measurable. On the other hand,

$$\mathcal{R}(\mathfrak{s}, t) = \mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t} + \mathfrak{Y}_t \mathbb{1}_{\mathfrak{s} \geq t}$$

Since every stopping time is optional⁴⁷ we know that the events $\{\omega \in \Omega: \mathfrak{s}(\omega) < t\}$ and its complement $\{\omega \in \Omega: \mathfrak{s}(\omega) \geq t\}$ are elements of \mathcal{F}_t , which in turn shows that the functions $\mathbb{1}_{\mathfrak{s} < t}$ and $\mathbb{1}_{\mathfrak{s} \geq t}$ are \mathcal{F}_t -measurable. Since \mathfrak{Y} is an adapted process, it is also clear that \mathfrak{Y}_t is \mathcal{F}_t -measurable. Therefore, $\mathfrak{Y}_t \mathbb{1}_{\mathfrak{s} \geq t}$ is \mathcal{F}_t -measurable, and remains only to show that $\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t}$ is also \mathcal{F}_t -measurable.

In accordance with the definition of $\mathcal{F}_{\mathfrak{s}}$, since $\mathfrak{X}_{\mathfrak{s}}$ is $\mathcal{F}_{\mathfrak{s}}$ -measurable we know that $\forall t \in [0, T]$ and all $\alpha \in \mathbb{R}$ the event⁴⁸ $\{\omega \in \Omega: (\mathfrak{X}_{\mathfrak{s}})(\omega) \leq \alpha\} \cap \{\omega \in \Omega: \mathfrak{s}(\omega) < t\} \in \mathcal{F}_t$ which implies that the event $\{\omega \in \Omega: (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t})(\omega) \leq \alpha\} \in \mathcal{F}_t$. So $\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t}$ is \mathcal{F}_t -measurable, thus $\mathcal{R}(\mathfrak{s}, t)$ is also \mathcal{F}_t -measurable. Therefore $\mathcal{R}(\mathfrak{s}, t)$ is $\mathcal{F}_{\mathfrak{s}} \cap \mathcal{F}_t = \mathcal{F}_{\mathfrak{s} \wedge t}$ -measurable⁴⁹. \square

Claim 3.4.9. $\forall (s, t) \in [0, T] \times [0, T]$, $\mathfrak{X}_s^* \mathbb{1}_{s < t} + \mathfrak{Y}_t^* \mathbb{1}_{t \leq s} = (\mathfrak{X}_s \mathbb{1}_{s < t} + \mathfrak{Y}_t \mathbb{1}_{t \leq s}) / B_{s \wedge t}$.

Thus, it makes sense to extend our notation for discounted processes to include the payoff process \mathcal{R} .

Notation 3.4.3. In the following we will write $\mathcal{R}^*(s, t) = \mathcal{R}(s, t) / B_{s \wedge t}$.

Proposition 3.4.10. For any stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ and any time $t \in [0, T]$, $\mathcal{R}^*(\mathfrak{s}, t) = \mathcal{R}(\mathfrak{s}, t) / B_{\mathfrak{s} \wedge t}$ is $\mathcal{F}_{\mathfrak{s} \wedge t}$ -measurable.

⁴⁶See [96], proposition 1.13, if X is an RCLL and adapted process, then it is also progressively measurable.

⁴⁷See [96], proposition 2.3.

⁴⁸The original definition is $\{\omega \in \Omega: (\mathfrak{X}_{\mathfrak{s}})(\omega) \leq \alpha\} \cap \{\omega \in \Omega: \mathfrak{s}(\omega) \leq t\} \in \mathcal{F}_t$ but we can safely change \leq into $<$ because $\{\mathfrak{s}(\omega) < t\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega: \mathfrak{s}(\omega) \leq t - 1/n\}$

⁴⁹This follows from the fact that $\eta \equiv t$ is a stopping time $\forall t \in [0, T]$ and that $\mathcal{F}_{\mathfrak{s}} \cap \mathcal{F}_t = \mathcal{F}_{\mathfrak{s} \wedge t} \forall \mathfrak{s}, t$ stopping times with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ (see Lemma 2.16 in [96]).

Proof. Based on Proposition 3.4.8 given a stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ and a time $t \in [0, T]$, it is enough to see that $B_{\mathfrak{s} \wedge t}$ is $\mathcal{F}_{\mathfrak{s} \wedge t}$ -measurable, this follows from [96] proposition 2.18 and our definition⁵⁰ of $B = \{B_t\}_{t \in [0, T]}$ on page 44. \square

Let $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, from Proposition 3.4.8 we know that $\mathcal{I}^{\mathfrak{s}} = \{\mathcal{R}(\mathfrak{s}, t)\}_{t \in [0, T]}$ is $\{\mathcal{F}_{\mathfrak{s} \wedge t}\}_{t \in [0, T]}$ adapted, and \mathfrak{s} being a stopping time relative to filtration \mathcal{F} we can see that $\mathcal{I}^{\mathfrak{s}}$ is also \mathcal{F} adapted since $\mathcal{F}_{\mathfrak{s} \wedge t} \subseteq \mathcal{F}_t$, $\forall t \in [0, T]$. But, due to Assertion 3.4.7, it is not so clear if the process $\mathcal{I}^{\mathfrak{s}}$ is also progressively measurable.

Assertion 3.4.11. *Let $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, the processes $X = \{\mathbb{1}_{\mathfrak{s} < t}\}_{t \in [0, T]}$ and $Y = \{\mathbb{1}_{t \leq \mathfrak{s}}\}_{t \in [0, T]}$ are progressively measurable.*

Proof. Since, $\forall t \in [0, T]$ $Y_t = 1 - X_t$, it is enough to show that X is progressively measurable. On the other hand, since \mathfrak{s} is a stopping time relative to filtration \mathcal{F} , for every $t \in [0, T]$, the event $\{\mathfrak{s} < t\}$ is in \mathcal{F}_t . Therefore, if $\alpha \in \mathbb{R}$ we have

$$\{X_t < \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \leq 0 \\ \Omega & \text{if } \alpha > 1 \\ \{\mathfrak{s} \geq t\} & \text{if } \alpha \in]0, 1] \end{cases}$$

which implies that $\{X_t < \alpha\} \in \mathcal{F}_t$, $\forall \alpha \in \mathbb{R}$, and consequently that X is \mathcal{F} -adapted.

Let $\omega \in \Omega$ be fixed, and let $t \in]0, T]$

- If $\omega \in \{\mathfrak{s} \geq t\}$. Let $s < t$, $s \in [0, T]$; obviously $\omega \in \{\mathfrak{s} \geq t\}$ implies $\omega \notin \{\mathfrak{s} < s\}$, thus $X_s(\omega) = 0$ ($\forall s \in [0, T]$ such that $s < t$). Hence

$$\lim_{s \rightarrow t^-} X_s(\omega) = 0 = X_t(\omega)$$

- If $\omega \in \{\mathfrak{s} < t\}$. $\exists r \in \mathbb{R}^+$ such that $\omega \in \{\mathfrak{s} < t - r\}$. Thus, $\forall s \in [t - r, t[$ $\omega \in \{\mathfrak{s} < s\}$, from which

$$\lim_{s \rightarrow t^-} X_s(\omega) = 1 = X_t(\omega)$$

⁵⁰From our definition of the process B , we know B is adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ (see Lemma 3.8 in [26] or Chapter 1 in [96]). On the other hand, since $r = \{r_t\}_{t \in [0, T]}$ is assumed to be a progressively measurable process with respect to the filtration \mathcal{F} we know that its time integral also defines a progressively measurable process with respect to the same filtration.

If $t = 0$, $X_0(\omega) = 0$ since \mathfrak{s} takes values in $[0, T]$. In any case, $X(\omega)$ has right limits.

Therefore, X is LCRL and adapted, which implies (see Proposition 3.3.1) X is progressively measurable. \square

Proposition 3.4.12. *For any stopping time $\mathfrak{s} \in \mathfrak{S}_T$ the process $\mathcal{I}^\mathfrak{s} = \{\mathcal{R}(\mathfrak{s}, t)\}_{t \in [0, T]}$ is progressively measurable with respect to filtration \mathcal{F} .*

Proof. Let $\mathfrak{s} \in \mathfrak{S}_T$, Assertion 3.4.11 shows that the processes $\{\mathbb{1}_{\mathfrak{s} < t}\}_{t \in [0, T]}$ and $\{\mathbb{1}_{t \leq \mathfrak{s}}\}_{t \in [0, T]}$ are progressively measurable. By hypotheses, \mathfrak{X} and \mathfrak{Y} are RCLL and adapted, and consequently (by Proposition 3.3.1) progressively measurable.

Let $t \in [0, T]$,

$$\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t} = \begin{cases} \mathfrak{X}_{\mathfrak{s}} & \text{on } \{\mathfrak{s} < t\} \\ 0 & \text{on } \{\mathfrak{s} \geq t\} \end{cases} = \begin{cases} \mathfrak{X}_{\mathfrak{s} \wedge t} & \text{on } \{\mathfrak{s} < t\} \\ 0 & \text{on } \{\mathfrak{s} \geq t\} \end{cases} = \mathfrak{X}_{\mathfrak{s} \wedge t} \mathbb{1}_{\mathfrak{s} < t}$$

by [96] Proposition 2.18, the stopped process $\{\mathfrak{X}_{\mathfrak{s} \wedge t}\}_{t \in [0, T]}$ is progressively measurable relative to filtration \mathcal{F} .

Since, $\forall t \in [0, T]$, the sum and product of $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable functions is also $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable, we see that $\mathcal{I}^\mathfrak{s}$ is progressively measurable. \square

Assertion 3.4.13. *Let $t \in [0, T]$, $\forall A \in \mathcal{F}_t$ and $\forall \mathfrak{u}, \mathfrak{v} \in \mathfrak{S}_{t, T}$ $\mathfrak{t} = \mathfrak{u} \mathbb{1}_A + \mathfrak{v} \mathbb{1}_{A^c} \in \mathfrak{S}_{t, T}$.*

Proof. Let $t \in [0, T]$ and $A \in \mathcal{F}_t$, from $\mathfrak{u} \in [t, T]$ and $\mathfrak{v} \in [t, T]$, $\mathcal{P}^\mathcal{E}$ -a.s we obtain $t \leq \mathfrak{t} \leq T$, $\mathcal{P}^\mathcal{E}$ -a.s. . On the other hand, given $s \in [0, T]$ we know that $\{\mathfrak{u} \leq s\} \in \mathcal{F}_s$ and $\{\mathfrak{v} \leq s\} \in \mathcal{F}_s$; $\forall \mathfrak{u}, \mathfrak{v} \in \mathfrak{S}_{t, T}$. Then, if $s \geq t$, $\{\mathfrak{t} \leq s\} = \{\mathfrak{u} \mathbb{1}_A + \mathfrak{v} \mathbb{1}_{A^c} \leq s\} = (\{\mathfrak{u} \leq s\} \cap A) \cup (\{\mathfrak{v} \leq s\} \cap A^c) \in \mathcal{F}_s$. The conclusion follows naturally. \square

Proposition 3.4.14. *Given a stopping time $\mathfrak{s} \in \mathfrak{S}_T$, for any time $t \in [0, T]$, let $U_t^\mathfrak{s}$ be defined as $U_t^\mathfrak{s} = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} E_\mathcal{E}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t)$. Then the resulting stochastic process $\{U_t^\mathfrak{s}\}_{t \in [0, T]}$ is a supermartingale.*

Proof. If \mathcal{R} were an RCLL (or LCRL) process in its second index, this result will follow straight from Lemma 2 of [162] section §6, but due to Assertion 3.4.7 we will have to take an alternate path here.

First we will show that the essential supremum exists, and that the family of conditional expectations is closed under pairwise maximization then we will use that property to show that the resulting stochastic process is a supermartingale.

By hypotheses $\mathfrak{X} \geq \mathfrak{Y} \geq 0 \implies \mathcal{R}(s, t) \geq 0 \ \forall s, t \in [0, T]$, we also know that the bank account process B is nonnegative, thus

$$\mathcal{R}^*(s, t) \geq 0 \quad \mathcal{P}^{\mathcal{E}}\text{-a. s.} \quad \forall s, t \in [0, T] \quad (178)$$

and, given $t \in [0, T]$

$$E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \geq 0 \quad \mathcal{P}^{\mathcal{E}}\text{-a. s.} \quad \forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}. \quad (179)$$

Now, fix $t \in [0, T]$ and $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, and consider the family of nonnegative random variables

$$\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t)\}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} \quad (180)$$

by [97] Appendix A, Theorem A.3 we know that the essential supremum of that family exists⁵¹, so we can define

$$U_t^{\mathfrak{s}} = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \quad (181)$$

Thanks to Assertion 3.4.13 we can show that the family (180) is closed under pairwise maximization.

Let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{S}_{t, \mathcal{T}}$, and define

$$A = \{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{u}) | \mathcal{F}_t) \geq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{v}) | \mathcal{F}_t)\} \quad (182)$$

clearly $A \in \mathcal{F}_t$ and $\mathfrak{a} = \mathfrak{u} \mathbb{1}_A + \mathfrak{v} \mathbb{1}_{A^c} \in \mathfrak{S}_{t, \mathcal{T}}$ and

$$\begin{aligned} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{a}) | \mathcal{F}_t) &= E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{u}) | \mathcal{F}_t) \mathbb{1}_A + E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{v}) | \mathcal{F}_t) \mathbb{1}_{A^c} \\ &= E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{u}) | \mathcal{F}_t) \vee E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{v}) | \mathcal{F}_t) \\ &\in \{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t)\}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} \end{aligned} \quad (183)$$

⁵¹We recognize that Karatzas and Shreve's ([97]) conditions are restrictive since the essential supremum of a family of r.v.'s exists under more general conditions (see [52], [138]), but in our case their result is enough.

Therefore, Theorem A.3 from [97] Appendix A, implies that there exists a sequence of stopping times $\{\mathfrak{t}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{S}_{t, \mathcal{T}}$ such that the sequence

$$\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_t)\}_{n \in \mathbb{N}} \quad (184)$$

is non-decreasing and

$$\lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_t) = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_t) = U_t^{\mathfrak{s}} \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \quad (185)$$

Let $s \in [0, \mathcal{T}]$ such that $0 \leq s < t \leq \mathcal{T}$, by the monotone convergence theorem for conditional expectations and the tower property of conditional expectations we have

$$\begin{aligned} E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) &= E_{\mathcal{E}}\left(\lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_t)|\mathcal{F}_s\right) \\ &= \lim_{n \rightarrow \infty} E_{\mathcal{E}}(E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_t)|\mathcal{F}_s) \\ &= \lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_s) \leq U_s^{\mathfrak{s}} \end{aligned} \quad (186)$$

(where the last inequality follows from the definition of the essential supremum of a family of random variables, and because $\{\mathfrak{t}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{S}_{t, \mathcal{T}} \subseteq \mathfrak{S}_{s, \mathcal{T}}$, for $0 \leq s \leq t \leq \mathcal{T}$).

Hence $\{U_t^{\mathfrak{s}}\}_{t \in [0, \mathcal{T}]}$ is a supermartingale $\forall \mathfrak{s} \in \mathfrak{S}_{t, \mathcal{T}}$. \square

Notice that Assertion 3.4.13 can be easily extended in the following way:

Assertion 3.4.15. *Let $\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$, $\forall A \in \mathcal{F}_{\mathfrak{w}}$ and $\forall \mathfrak{u}, \mathfrak{v} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}$ $\mathfrak{t} = \mathfrak{u} \mathbb{1}_A + \mathfrak{v} \mathbb{1}_{A^c} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}$.*

In fact, if $\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$ and $A \in \mathcal{F}_{\mathfrak{w}}$ then $\{\mathfrak{u} \mathbb{1}_A + \mathfrak{v} \mathbb{1}_{A^c} \leq s\} = (\{\mathfrak{u} \leq s\} \cap A) \cup (\{\mathfrak{v} \leq s\} \cap A^c) \in \mathcal{F}_s$ as before.

With the help of Assertion 3.4.15 it is easy to modify the first part of the proof of Proposition 3.4.14 to show that, $\forall \mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$, the family $\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_{\mathfrak{w}})\}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}}$, of nonnegative random variables, is closed under pairwise maximization allowing us not only to define, $\forall \mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$, $U_{\mathfrak{w}}^{\mathfrak{s}} = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_{\mathfrak{w}})$, but also to claim the existence of a sequence of stopping times $\{\mathfrak{t}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}$ such that the sequence

$$\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_{\mathfrak{w}})\}_{n \in \mathbb{N}} \quad (187)$$

is non-decreasing and

$$\lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_{\mathfrak{w}}) = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_{\mathfrak{w}}) = U_{\mathfrak{w}}^{\mathfrak{s}} \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \quad (188)$$

In fact, since every deterministic time is a stopping time, we see that $U_t^{\mathfrak{s}} \in \{U_{\mathfrak{w}}^{\mathfrak{s}}\}_{\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}}$, $\forall t \in [0, T]$.

Proposition 3.4.16. *Given stopping times $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ and $\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}$, the family of random variables*

$$\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_{\mathfrak{w}})\}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}} \quad (189)$$

admits an essential supremum, which we denote as

$$U_{\mathfrak{w}}^{\mathfrak{s}} = \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{w}, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_{\mathfrak{w}}) \quad (190)$$

The resulting family of random variables $\{U_{\mathfrak{w}}^{\mathfrak{s}}\}_{\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}}$ contains the supermartingale of Proposition 3.4.14.

Assertion 3.4.17. $\forall 0 \leq s < t \leq T$

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) = \text{esssup}_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s) \quad (191)$$

Proof. Let $s, t \in [0, T]$, $0 \leq s < t \leq T$ and $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ be fixed. From Proposition 3.4.14's proof we know there exists a sequence of stopping times $\{\mathfrak{t}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{S}_{t, \mathcal{T}}$ such that the sequence

$$\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_t)\}_{n \in \mathbb{N}} \quad (192)$$

is non-decreasing and

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) = \lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}_n)|\mathcal{F}_s) \leq \text{esssup}_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s) \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \quad (193)$$

(we know that the family $\{E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s)\}_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}}$ is non-negative, hence, by [97] Appendix A, Theorem A.3, that family possesses an essential supremum).

On the other hand, $\forall \mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}$, $U_t^{\mathfrak{s}} \geq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_t)$. Taking conditional expectations with respect to \mathcal{F}_s we have:

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) \geq E_{\mathcal{E}}(E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_t)|\mathcal{F}_s) = E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s) \quad \forall \mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}} \quad (194)$$

this implies, by definition of the essential supremum that

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) \geq \text{esssup}_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s) \quad (195)$$

\implies

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s) = \operatorname{esssup}_{\mathfrak{w} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})|\mathcal{F}_s) \quad (196)$$

□

Claim 3.4.18. *For every stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ and every strictly decreasing sequence⁵² $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, T]$ decreasing to $t \in [0, T]$*

$$\bigcap_{n \in \mathbb{N}} \{\mathfrak{s} < t_n\} = \{\mathfrak{s} \leq t\} \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} \{\mathfrak{s} \geq t_n\} = \{\mathfrak{s} > t\} \quad (197)$$

thus

$$\lim_{w \rightarrow t^+} \mathbb{1}_{\mathfrak{s} < w} = \mathbb{1}_{\mathfrak{s} \leq t} \quad \text{and} \quad \lim_{w \rightarrow t^+} \mathbb{1}_{\mathfrak{s} \geq w} = \mathbb{1}_{\mathfrak{s} > t} \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \quad (198)$$

Similarly, if $\{\mathfrak{t}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{S}_{\mathcal{T}}$ is a strictly decreasing sequence of stopping times decreasing to $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$ we have:

$$\lim_{n \rightarrow \infty} \mathbb{1}_{\mathfrak{s} < \mathfrak{t}_n} = \mathbb{1}_{\mathfrak{s} \leq \mathfrak{t}} \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{1}_{\mathfrak{s} \geq \mathfrak{t}_n} = \mathbb{1}_{\mathfrak{s} > \mathfrak{t}} \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \quad (199)$$

Proposition 3.4.19. *Given a stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, for $t \in [0, T]$*

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}) = \sup_{\mathfrak{t} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (200)$$

And this function of t is right-continuous. Thus, the process $\{U_t^{\mathfrak{s}}\}_{t \in [0, T]}$ has a right-continuous modification $\{\tilde{U}_t^{\mathfrak{s}}\}_{t \in [0, T]}$, which can be chosen so that $\{\tilde{U}_t^{\mathfrak{s}}\}_{t \in [0, T]}$ is RCLL, adapted to filtration \mathcal{F} , and a supermartingale.

Proof. The conclusion, that $\{\tilde{U}_t^{\mathfrak{s}}\}_{t \in [0, T]}$ exists and can be chosen RCLL, follows immediately from [114] Chapter 3 Theorem 3.1 (or similarly from [96] Chapter 1 Section 3.A Theorem 3.13), once the right continuity of $E_{\mathcal{E}}(U_t^{\mathfrak{s}})$ is proven. (200) follows from Proposition 3.4.14 and Assertion 3.4.17.

⁵²Notice that the results are still valid if the sequence is monotonically decreasing and $t \notin \{t_n\}_{n \in \mathbb{N}}$. The problem arises when the value t is allowed in the sequence, since in that case the event $\{\mathfrak{s} < t\}$ (resp. $\{\mathfrak{s} \geq t\}$) will be included in the intersection $\bigcap_{n \in \mathbb{N}} \{\mathfrak{s} < t_n\}$ (resp. $\bigcap_{n \in \mathbb{N}} \{\mathfrak{s} \geq t_n\}$). To avoid this kind of inconvenience we will assume that the limit of our decreasing sequence is not included in the sequence.

The proof of $E_{\mathcal{E}}(U_t^{\mathfrak{s}})$'s right-continuity goes similar to [162] Section 2. which contains Shiryayev et al. version of the pricing of an American Option (a now classical optimal stopping problem).

Let $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ be fixed and let $t \in [0, \mathcal{T}]$. Since \mathcal{F} satisfies the usual conditions we know that \mathcal{F}_0 contains only sets of measure 0 or 1. Thus, for every random variable X defined on $(\Omega, \mathcal{U}, \mathcal{P})$, $E_{\mathcal{E}}(X|\mathcal{F}_0) = E_{\mathcal{E}}(X)$.

From Proposition 3.4.14 we have:

$$U_0^{\mathfrak{s}} = \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{0, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})|\mathcal{F}_0) = \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (201)$$

By Assertion 3.4.17, taking $s = 0$ in (191) we obtain:

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}) = \sup_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) \quad (202)$$

hence

$$\sup_{\mathfrak{w} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) = U_0^{\mathfrak{s}} \geq E_{\mathcal{E}}(U_t^{\mathfrak{s}}) = \sup_{\mathfrak{w} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) \quad (203)$$

Now, let $f : [0, \mathcal{T}] \rightarrow \mathbb{R}$ be the function defined by $f(t) = E_{\mathcal{E}}(U_t^{\mathfrak{s}})$. From Proposition 3.4.14 we know that the process, $\forall \mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, $\{U_t^{\mathfrak{s}}\}_{t \in [0, \mathcal{T}]}$ is a supermartingale, therefore, $\forall s \in [0, \mathcal{T}]$, such that $0 \leq s < t \leq \mathcal{T}$ we have

$$E_{\mathcal{E}}(U_s^{\mathfrak{s}}) \geq E_{\mathcal{E}}(E_{\mathcal{E}}(U_t^{\mathfrak{s}}|\mathcal{F}_s)) = E_{\mathcal{E}}(U_t^{\mathfrak{s}}) \quad (204)$$

thus $f(s) \geq f(t) \forall 0 \leq s < t \leq \mathcal{T}$.

For every decreasing sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, \mathcal{T}]$ decreasing to t we will have $f(t) \geq f(t_n)$, $\forall n \in \mathbb{N}$ and $f(t) \geq \lim_{n \rightarrow \infty} f(t_n)$ thus

$$f(t) \geq \lim_{s \rightarrow t^+} f(s) \quad (205)$$

Now we need to show that the other inequality ($f(t) \leq \lim_{s \rightarrow t^+} f(s)$) is valid.

Thanks to Claim 3.4.18 and to our hypotheses on \mathfrak{X} and \mathfrak{Y} we see that, $\forall \{t_n\}_{n \in \mathbb{N}} \subseteq [0, \mathcal{T}]$

strictly decreasing to t

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathcal{R}^*(\mathfrak{s}, t_n) &= \lim_{n \rightarrow \infty} (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t_n} + \mathfrak{Y}_{t_n} \mathbb{1}_{t_n \leq \mathfrak{s}}) / B_{\mathfrak{s} \wedge t_n} \\
&= (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} \leq t} + \mathfrak{Y}_t \mathbb{1}_{t < \mathfrak{s}}) / B_{\mathfrak{s} \wedge t} \\
&= (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t} + \mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} = t} + \mathfrak{Y}_t \mathbb{1}_{t < \mathfrak{s}}) / B_{\mathfrak{s} \wedge t} \\
&\geq (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t} + \mathfrak{Y}_t \mathbb{1}_{\mathfrak{s} = t} + \mathfrak{Y}_t \mathbb{1}_{t < \mathfrak{s}}) / B_{\mathfrak{s} \wedge t} \\
&= (\mathfrak{X}_{\mathfrak{s}} \mathbb{1}_{\mathfrak{s} < t} + \mathfrak{Y}_t \mathbb{1}_{t \leq \mathfrak{s}}) / B_{\mathfrak{s} \wedge t} = \mathcal{R}^*(\mathfrak{s}, t) \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.}
\end{aligned} \tag{206}$$

hence, although we know (Assertion 3.4.7) that $\{\mathcal{R}^*(\mathfrak{s}, t)\}_{t \in [0, T]}$ is not necessarily RCLL, its right limits at t are bounded below by $\mathcal{R}^*(\mathfrak{s}, t)$:

$$\lim_{w \rightarrow t^+} \mathcal{R}^*(\mathfrak{s}, w) \geq \mathcal{R}^*(\mathfrak{s}, t) \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \tag{207}$$

Now, let $\varepsilon > 0$, by definition of supremum there exists $\mathfrak{w} \in \mathfrak{S}_{t, T}$ such that

$$f(t) \leq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) + \varepsilon \tag{208}$$

and thanks to (207) we can select \mathfrak{w} so that $\mathcal{P}^{\mathcal{E}}(\mathfrak{w} > t) = 1$. Again, let $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, T]$ be a decreasing sequence decreasing to t ; $\forall n \in \mathbb{N}$ define

$$\mathfrak{w}_n = \begin{cases} \mathfrak{w} & \text{if } \mathfrak{w} \geq t_n \\ T & \text{if } \mathfrak{w} < t_n \end{cases} \tag{209}$$

it is clear that $\mathfrak{w}_n \in \mathfrak{S}_{t_n, T} \subseteq \mathfrak{S}_{t, T}$, $\forall n \in \mathbb{N}$. It also clear that $\lim_{n \rightarrow \infty} \mathfrak{w}_n = \mathfrak{w}$, $\mathcal{P}^{\mathcal{E}}\text{-a.s.}$

Now, consider $|E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) - E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}_n))|$

$$\begin{aligned}
&|E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) - E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}_n))| \\
&= |E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w})) - E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}) \mathbb{1}_{\mathfrak{w} \geq t_n} + \mathcal{R}^*(\mathfrak{s}, T) \mathbb{1}_{\mathfrak{w} < t_n})| \\
&= |E_{\mathcal{E}}(\{\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}) - \mathcal{R}^*(\mathfrak{s}, T)\} \mathbb{1}_{\mathfrak{w} < t_n})| \\
&\leq E_{\mathcal{E}}(\{\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}) + \mathcal{R}^*(\mathfrak{s}, T)\} \mathbb{1}_{\mathfrak{w} < t_n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty
\end{aligned} \tag{210}$$

which implies that

$$f(t) \leq \lim_{n \rightarrow \infty} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}_n)) + \varepsilon \leq \lim_{n \rightarrow \infty} f(t_n) + \varepsilon \tag{211}$$

which is valid $\forall \varepsilon > 0$ and for every decreasing sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, T]$ decreasing to t , thus $f(t) \leq \lim_{n \rightarrow \infty} f(t_n)$ or in other words

$$f(t) \leq \lim_{s \rightarrow t^+} f(s) \quad (212)$$

Therefore

$$f(t) = \lim_{s \rightarrow t^+} f(s) \quad (213)$$

which shows that the function defined as $f(t) = E_{\mathcal{E}}(U_t^{\mathfrak{s}})$ is right-continuous.

Hence, by [114] Chapter 3, Theorem 3.1, there exists a right-continuous modification to the process $\{U_t^{\mathfrak{s}}\}_{t \in [0, T]}$ and that right-continuous modification can be chosen so as to be an RCLL supermartingale with respect to filtration \mathcal{F} . \square

As part of the proof of Proposition 3.4.19 we have shown (see (206) and (207)) that the right limits of process $\mathcal{R}^*(\mathfrak{s}, \cdot) = \{\mathcal{R}^*(\mathfrak{s}, t)\}_{t \in [0, T]}$, $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, at time t are bounded below by $\mathcal{R}^*(\mathfrak{s}, t)$. This is an interesting property of the payoff process that we can extend to stopping times. The resulting property is somewhat akin to regularity⁵³ and states that although the process $\mathcal{R}^*(\mathfrak{s}, \cdot) = \{\mathcal{R}^*(\mathfrak{s}, t)\}_{t \in [0, T]}$, $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$, is not necessarily RCLL, its right limits at t are bounded below by $\mathcal{R}^*(\mathfrak{s}, t)$ and that this property is preserved when the “common” time t is exchanged by a stopping time.

Assertion 3.4.20. *Let $\{\mathfrak{s}_n\}_{n \in \mathbb{N}} \subset \mathfrak{S}_{\mathcal{T}}$ be a decreasing sequence of stopping times, decreasing to $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$. Let $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$. Then, $\lim_{n \rightarrow \infty} \mathcal{R}^*(\mathfrak{s}, \mathfrak{s}_n) \geq \mathcal{R}^*(\mathfrak{s}, \mathfrak{t})$.*

⁵³ A process $X = \{X_t\}_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P})$ is called *regular* if $\forall \alpha > 0$ and every increasing sequence $\{\mathfrak{t}_n\}_{n \in \mathbb{N}}$ of stopping times bounded by α and converging to \mathfrak{t} (a stopping time also bounded by α) we have $\lim_{n \rightarrow \infty} E(X_{\mathfrak{t}_n}) = E(X_{\mathfrak{t}})$. See [52], see also [177]. The requirement of boundedness on the sequence of stopping times could be dropped, while the limit is still assumed to be bounded. If the process being considered is uniformly bounded, the condition of boundedness on the stopping times could be dropped altogether. A process with this property is also called *lower regular* or *left continuous in expectation* or **l.c.e** for short. In a similar way we can define an *upper regular* process (also known as a process *right continuous in expectation* or **r.c.e.** for short); $X = \{X_t\}_{t \geq 0}$ is an r.c.e. process if for every uniformly bounded decreasing sequence $\{\mathfrak{t}_n\}_{n \in \mathbb{N}}$ of stopping times converging to \mathfrak{t} (a bounded stopping time) we have $\lim_{n \rightarrow \infty} E(X_{\mathfrak{t}_n}) = E(X_{\mathfrak{t}})$.

Proof. By [151] Chapter I, Proposition 4.11, every stopping time is the decreasing limit of a sequence of stopping times each of them taking only finitely many values. For every $n \in \mathbb{N}$ and $k \in \mathbb{N}_{2^n-1}$ define $A_{n,k}^t = \{\omega \in \Omega : t(\omega) \in [\frac{k-1}{2^n}, \frac{k}{2^n}[[$ and $A_{n,n2^n}^t = \{\omega \in \Omega : t(\omega) \geq n\}$ and define the sequence $\{t_n\}_{n \in \mathbb{N}}$ as follows⁵⁴:

$$\begin{cases} t_n = T & \text{if } t \geq n \\ t_n = \frac{k}{2^n} & \text{if } \frac{k-1}{2^n} \leq t < \frac{k}{2^n} \end{cases} \quad (214)$$

that is

$$t_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}^t} + T \mathbb{1}_{A_{n,n2^n}^t} \quad (215)$$

Then $\forall n \in \mathbb{N}$, $t_n \geq t_{n+1}$ and $t_n \xrightarrow{n \rightarrow \infty} t$. By definition, $\forall n \in \mathbb{N}$ and $k \in \mathbb{N}_{n2^n}$, t_n is constant on $A_{n,k}^t$. Since \mathfrak{Y} is an rcll process we know that

$$\lim_{n \rightarrow \infty} \mathfrak{Y}_{t_n}(\omega) = \lim_{n \rightarrow \infty} \mathfrak{Y}_{t_n(\omega)}(\omega) = \mathfrak{Y}_{t(\omega)}(\omega) = \mathfrak{Y}_t(\omega)$$

Similarly, $B_{s \wedge t_n} \xrightarrow{n \rightarrow \infty} B_{s \wedge t}$, thus, by Claim 3.4.18

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{R}^*(s, t_n) &= \lim_{n \rightarrow \infty} (\mathfrak{X}_s \mathbb{1}_{s < t_n} + \mathfrak{Y}_{t_n} \mathbb{1}_{t_n \leq s}) / B_{s \wedge t_n} \\ &= (\mathfrak{X}_s \mathbb{1}_{s \leq t} + \mathfrak{Y}_t \mathbb{1}_{t < s}) / B_{s \wedge t} \\ &= (\mathfrak{X}_s \mathbb{1}_{s < t} + \mathfrak{X}_s \mathbb{1}_{s=t} + \mathfrak{Y}_t \mathbb{1}_{t < s}) / B_{s \wedge t} \\ &\geq (\mathfrak{X}_s \mathbb{1}_{s < t} + \mathfrak{Y}_t \mathbb{1}_{t \leq s}) / B_{s \wedge t} = \mathcal{R}^*(s, t) \end{aligned} \quad (216)$$

which proves our assertion. \square

In view of Proposition 3.4.19, in what follows we will assume that $\{\tilde{U}_t^s\}_{t \in [0, T]}$ is an RCLL supermartingale that satisfies

$$\tilde{U}_t^s = U_t^s \quad \mathcal{P}^\mathcal{E}\text{-a.s.} \quad (217)$$

It seems that this supermartingale is a suitable candidate to apply the Doob–Meyer decomposition theorem, but one detail is still missing, the next propositions take care of that detail.

⁵⁴Recall that $t \in \mathfrak{S}_T$ is bounded, otherwise one should define $t_n = \infty$ if $t \geq n$. This point is of no major significance if the stopping time is bounded since after a finite number of elements, the sequence $\{s_n\}_{n \in \mathbb{N}}$ will be bounded even if $t_n = \infty$ whenever $t \geq n$.

First, we will borrow a result from [96], see Appendix D, Corollary D.4 and Theorem D.7.

Proposition 3.4.21. *For every $\mathfrak{s} \in \mathfrak{S}_T$ and $\mathfrak{w} \in \mathfrak{S}_T$ $\tilde{U}_{\mathfrak{w}}^{\mathfrak{s}} = U_{\mathfrak{w}}^{\mathfrak{s}}$ $\mathcal{P}^{\mathcal{E}}$ -a.s. .*

Based on the previous result, in what follows we will make no difference between $\tilde{U}_{\mathfrak{w}}^{\mathfrak{s}}$ and $U_{\mathfrak{w}}^{\mathfrak{s}}$.

Proposition 3.4.22. *Let $\mathfrak{s} \in \mathfrak{S}_T$, the family of random variables $\{U_{\mathfrak{w}}^{\mathfrak{s}}\}_{\mathfrak{w} \in \mathfrak{S}_T}$ is uniformly integrable⁵⁵ with respect to $\mathcal{P}^{\mathcal{E}}$.*

Proof. We already know that the process $\{U_t^{\mathfrak{s}}\}_{t \in [0, T]}$ is nonnegative, from which we can see that the family of random variables $\{U_{\mathfrak{w}}^{\mathfrak{s}}\}_{\mathfrak{w} \in \mathfrak{S}_T}$ is also nonnegative. By hypotheses (see Definition 3.4.2) $\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}) \leq \sup_{0 \leq t \leq T} \mathfrak{X}_t^*, \forall \mathfrak{s}, \mathfrak{w} \in \mathfrak{S}_T$, $\mathcal{P}^{\mathcal{E}}$ -a.s. therefore $E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{w}) | \mathcal{F}_{\mathfrak{u}}) \leq E_{\mathcal{E}}(\sup_{0 \leq l \leq T} \mathfrak{X}_l^* | \mathcal{F}_{\mathfrak{u}})$, for every $\mathfrak{u} \in \mathfrak{S}_T$. Thus

$$U_{\mathfrak{u}}^{\mathfrak{s}} \leq E_{\mathcal{E}} \left(\sup_{0 \leq l \leq T} \mathfrak{X}_l^* \middle| \mathcal{F}_{\mathfrak{u}} \right) \quad (218)$$

Since $\{U_t^{\mathfrak{s}}\}_{t \in [0, T]}$ is a supermartingale, by (203) and the definition of essential supremum,

$$E_{\mathcal{E}}(U_t^{\mathfrak{s}}) = E_{\mathcal{E}}(U_t^{\mathfrak{s}} | \mathcal{F}_0) \leq U_0^{\mathfrak{s}} = \sup_{\mathfrak{t} \in \mathfrak{S}_T} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \leq E_{\mathcal{E}} \left(\sup_{0 \leq l \leq T} \mathfrak{X}_l^* \right) < \infty \quad (219)$$

⁵⁵ A family of random variables $\{\xi_a; a \in \mathcal{A}\}$ defined on a probability space $(\Omega, \mathcal{U}, \mathcal{P})$ is called uniformly integrable with respect to probability measure \mathcal{P} if

$$\lim_{x \rightarrow \infty} \sup_{a \in \mathcal{A}} \int_{\{|\xi_a| > x\}} |\xi_a| d\mathcal{P} = 0$$

The previous condition can be substituted by the following conditions

$$\sup_{a \in \mathcal{A}} E|\xi_a| < \infty$$

and

$$\lim_{\substack{\mathcal{P}(A) \rightarrow 0 \\ A \in \mathcal{U}}} \sup_{a \in \mathcal{A}} \int_A |\xi_a| d\mathcal{P} = 0$$

Or, equivalently, if

$$E|\xi_a| \text{ is bounded in } a \in \mathcal{A}$$

and, for every $\varepsilon > 0$, there exists $\delta_{\varepsilon} > 0$ such that, for every $A \in \mathcal{U}$:

$$\mathcal{P}(A) < \delta_{\varepsilon} \implies \int_A |\xi_a| d\mathcal{P} < \varepsilon \text{ for every } a \in \mathcal{A}$$

See [114] Chapter 1 and [25] Chapter 4, §5.

Similarly, by the Optional Sampling Theorem ([96] Chapter 1 Section 3C), $\forall \mathfrak{w} \in \mathfrak{S}_T$

$$E_{\mathcal{E}}(U_{\mathfrak{w}}^{\mathfrak{s}}) = E_{\mathcal{E}}(U_{\mathfrak{w}}^{\mathfrak{s}} | \mathcal{F}_0) \leq U_0^{\mathfrak{s}} \leq E_{\mathcal{E}} \left(\sup_{0 \leq l \leq T} \mathfrak{X}_l^* \right) < \infty \quad (220)$$

This last inequality implies that $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$A \in \mathcal{U} \text{ and } \mathcal{P}^{\mathcal{E}}(A) < \delta \implies \int_A \sup_{0 \leq t \leq T} \mathfrak{X}_t^* d\mathcal{P}^{\mathcal{E}} < \varepsilon \quad (221)$$

Let $\alpha > (1/\delta)E_{\mathcal{E}}(\sup_{0 \leq l \leq T} \mathfrak{X}_l^*)$ and $\mathfrak{w} \in \mathfrak{S}_T$, by Chebyshev's inequality

$$\mathcal{P}^{\mathcal{E}}(U_{\mathfrak{w}}^{\mathfrak{s}} > \alpha) \leq \frac{E_{\mathcal{E}}(U_{\mathfrak{w}}^{\mathfrak{s}})}{\alpha} \leq \frac{E_{\mathcal{E}}(\sup_{0 \leq t \leq T} \mathfrak{X}_t^*)}{\alpha} < \delta \quad (222)$$

and

$$\int_{\{U_{\mathfrak{w}}^{\mathfrak{s}} > \alpha\}} U_{\mathfrak{w}}^{\mathfrak{s}} d\mathcal{P}^{\mathcal{E}} \leq \int_{\{U_{\mathfrak{w}}^{\mathfrak{s}} > \alpha\}} E_{\mathcal{E}} \left(\sup_{0 \leq t \leq T} \mathfrak{X}_t^* \middle| \mathcal{F}_{\mathfrak{w}} \right) d\mathcal{P}^{\mathcal{E}} = \int_{\{U_{\mathfrak{w}}^{\mathfrak{s}} > \alpha\}} \sup_{0 \leq t \leq T} \mathfrak{X}_t^* d\mathcal{P}^{\mathcal{E}} < \varepsilon \quad (223)$$

Hence we can conclude

$$\lim_{x \rightarrow \infty} \sup_{\mathfrak{w} \in \mathfrak{S}_T} \int_{\{|U_{\mathfrak{w}}^{\mathfrak{s}}| > x\}} |U_{\mathfrak{w}}^{\mathfrak{s}}| d\mathcal{P}^{\mathcal{E}} = 0 \quad (224)$$

that is, $\forall \mathfrak{s} \in \mathfrak{S}_T$ $\{U_{\mathfrak{w}}^{\mathfrak{s}}\}_{\mathfrak{w} \in \mathfrak{S}_T}$ is uniformly integrable. \square

So, for every $\mathfrak{s} \in \mathfrak{S}_T$, we have constructed the Snell envelope, $\{\tilde{U}_t^{\mathfrak{s}}\}_{t \in [0, T]}$, of process $\{\mathcal{R}^*(\mathfrak{s}, t)\}_{t \in [0, T]}$; that is, the smallest right continuous with left hand side limits (RCLL) supermartingale that dominates $\{\mathcal{R}^*(\mathfrak{s}, t)\}_{t \in [0, T]}$. Proposition 3.4.22 shows that $\{\tilde{U}_t^{\mathfrak{s}}\}_{t \in [0, T]}$ is of *Dirichlet class*⁵⁶ (or, of class (D))⁵⁷. Thus, by the Doob–Meyer decomposition Theorem (see [96] Chapter 1 section 4) there exists (unique) processes $\mathfrak{M}^{\mathfrak{s}}$ and $\mathfrak{A}^{\mathfrak{s}}$ such that

- $\mathfrak{M}^{\mathfrak{s}}$ is a uniformly integrable, right–continuous martingale adapted to \mathcal{F} ,
- $\mathfrak{A}^{\mathfrak{s}}$ is an increasing, integrable, natural process adapted to \mathcal{F} , such that⁵⁸ $\mathfrak{A}_0^{\mathfrak{s}} = 0$,

⁵⁶ Let $X = \{X_t, \mathcal{G}_t; t \in [0, \infty[\}$ be a right–continuous process and S the set of all stopping times \mathfrak{s} with respect to filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ that satisfy $\mathcal{P}(\mathfrak{s} < \infty) = 1$. X is said to be of class (D) or of *Dirichlet class* if the family $\{X_{\mathfrak{s}}\}_{\mathfrak{s} \in S}$ is uniformly integrable.

⁵⁷ According to Dellacherie, the name class (D) was coined by Doob, but there seems to be not clear consensus regarding the meaning of “ (D) ”.

⁵⁸ Under additional conditions it could be possible to show that $\mathfrak{A}^{\mathfrak{s}}$ is “flat” away from the set $\{(t, \omega) \in [0, T] \times \Omega : \mathfrak{A}_t^{\mathfrak{s}}(\omega) = \mathcal{R}^*(\mathfrak{s}, t)(\omega)\} = A$ and that $\mathfrak{A}_{\theta^*}^{\mathfrak{s}} = 0$ where θ^* is the debut of A , that is, where $\theta^* = \inf_{t \geq 0} \{(t, \cdot) \in A\}$ (as usual $\theta^* = \infty$ if $(t, \omega) \notin A \forall t$). In particular, such a thing is not difficult to show if $U^{\mathfrak{s}}$ is regular (see Footnote 53) and $\mathcal{R}^*(\mathfrak{s}, \cdot)$ is continuous, which is the case of American game options see Chapter 5.

and

$$\tilde{U}_t^s = \mathfrak{M}_t^s - \mathfrak{A}_t^s \quad t \in [0, T] \quad (225)$$

Furthermore, by the Martingale Representation Theorem ([160], Chapter III section 3.c; [96], Chapter 3 section 4; [114], Chapter 5 section 2; [151], Chapter V) there exists a progressively measurable vector n -dimensional process $\gamma = \{\gamma_t\}_{t \in [0, T]}$ such that

$$\begin{aligned} \int_0^T \|\gamma_t\|^2 dt &< \infty \quad \mathcal{P}^{\mathcal{E}}\text{-a.s.} \\ \mathfrak{M}_t^s &= \mathfrak{M}_0^s + \int_0^t \gamma_u \cdot dW_u^{\mathcal{E}} \end{aligned} \quad (226)$$

Since we have assumed that market \mathcal{M} is standard and complete, by Theorem 3.3.8 the volatility process σ must be nonsingular ($\lambda \otimes \mathcal{P}$ -a.s.) so we can define π so that

$$\widetilde{\pi P_t^*}^\dagger \sigma_t = \gamma_t \quad (227)$$

where $\widetilde{\pi P_t^*}^i = \tilde{\pi}_t^i P_t^{*i}$, $i \in \mathbb{N}_n$ (by the definition of the active money process Definition 3.3.7 $\widetilde{\pi P_t^*} = \tilde{\pi}_t^\dagger \text{diag}(P_t^*)$). Thus we can find the n -dimensional process $\tilde{\pi}$ entry by entry. Therefore we can rewrite (226) as

$$\mathfrak{M}_t^s = \mathfrak{M}_0^s + \int_0^t \widetilde{\pi P_u^*}^\dagger \sigma_u dW_u^{\mathcal{E}} \quad (228)$$

As well, we can define

$$\tilde{\pi}_t^0 = \mathfrak{M}_t^s - \widetilde{\pi P_t^*} \cdot \vec{1}_n \quad (229)$$

Thus we have constructed a portfolio strategy $\Pi^s = (\tilde{\pi}^0, \tilde{\pi})$ which is both self-financing and martingale generating.

That Π^s is martingale generating portfolio strategy is obvious from our construction (see Definition 3.3.4, Definition 3.3.5 and Definition 3.3.14). To see that Π^s is self-financing, first we notice that (49) and (50) in combination with (94) imply that, under the equivalent martingale measure $\mathcal{P}^{\mathcal{E}}$,

$$dY_t = \text{diag}(P_t) \left(\vec{1}_n r_t dt + \sigma_t dW_t^{\mathcal{E}} \right) \quad (230)$$

Now, define $\mathcal{W}_t^{\Pi^s, \mathfrak{M}_0^s} = B_t \mathfrak{M}_t^s$, $t \in [0, T]$. $\mathcal{W}^{\Pi^s, \mathfrak{M}_0^s}$ is the wealth process associated with

$\Pi^{\mathfrak{s}}$. Then

$$\begin{aligned}
d\mathcal{W}_t^{\Pi^{\mathfrak{s}}} &= \mathfrak{M}_t^{\mathfrak{s}} dB_t + B_t \widetilde{\pi P_t^*}^\dagger \sigma_t dW_t^\mathcal{E} = \left(\tilde{\pi}_t^0 + \widetilde{\pi P_t^*} \cdot \vec{\mathbf{1}}_n \right) dB_t + \widetilde{\pi P_t^*}^\dagger \sigma_t dW_t^\mathcal{E} \\
&= \tilde{\pi}_t^0 dB_t + \widetilde{\pi P_t^*}^\dagger \vec{\mathbf{1}}_n r_t dt + \widetilde{\pi P_t^*}^\dagger \sigma_t dW_t^\mathcal{E} = \tilde{\pi}_t^0 dB_t + \widetilde{\pi P_t^*}^\dagger \left(\vec{\mathbf{1}}_n r_t dt + \sigma_t dW_t^\mathcal{E} \right) \quad (231) \\
&= \tilde{\pi}_t^0 dB_t + \tilde{\pi}_t^\dagger \text{diag}(P_t) \left(\vec{\mathbf{1}}_n r_t dt + \sigma_t dW_t^\mathcal{E} \right) = \tilde{\pi}_t^0 dB_t + \tilde{\pi}_t \cdot dY_t
\end{aligned}$$

On the other hand, by Proposition 3.4.21

$$\begin{aligned}
\mathcal{W}_{\mathfrak{s} \wedge t}^{\Pi^{\mathfrak{s}}} &= B_{\mathfrak{s} \wedge t} \mathfrak{M}_{\mathfrak{s} \wedge t}^{\mathfrak{s}} = B_{\mathfrak{s} \wedge t} (\tilde{U}_{\mathfrak{s} \wedge t}^{\mathfrak{s}} + \mathfrak{A}_{\mathfrak{s} \wedge t}^{\mathfrak{s}}) \\
&\geq B_{\mathfrak{s} \wedge t} \tilde{U}_{\mathfrak{s} \wedge t}^{\mathfrak{s}} = B_{\mathfrak{s} \wedge t} \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{\mathfrak{s} \wedge t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_{\mathfrak{s} \wedge t}) \\
&\geq B_{\mathfrak{s} \wedge t} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, t) | \mathcal{F}_t) = B_{\mathfrak{s} \wedge t} \mathcal{R}^*(\mathfrak{s}, t) \\
&= \mathcal{R}(\mathfrak{s}, t)
\end{aligned} \quad (232)$$

which shows that $(\Pi^{\mathfrak{s}}, \mathfrak{M}_0^{\mathfrak{s}}, \mathfrak{s})$ is a hedge against the gcc as in Definition 3.4.10.

The previous discussion provides a proof for the following result:

Proposition 3.4.23. *For every stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ there exists a hedge against the gcc as in Definition 3.4.10.*

The importance of this result is evident, once the seller selects a cancellation time \mathfrak{s} , he/she can always find a portfolio strategy to cover his/her position at time $\mathfrak{s} \wedge t$, $t \in [0, \mathcal{T}]$. But, what happens if the buyer selects stopping time $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$ as his/her exercise strategy?

The following proposition tell us that any given hedge should be good enough as to cover the position of the seller no matter what exercise strategy is followed by the buyer.

Proposition 3.4.24. *Given a hedge (Π, w_0, \mathfrak{s}) , and a stopping time $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$, the corresponding discounted wealth process, $\mathcal{W}^{*\Pi, w_0}$, satisfies*

$$w_0 = E_{\mathcal{E}} \left(\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{*\Pi, w_0} \right) \geq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (233)$$

Proof. Let (Π, w_0, \mathfrak{s}) be a hedge against the gcc of Definition 3.4.2, by Definition 3.4.10 we know that the corresponding wealth process associated to that hedge satisfies $\mathcal{W}_{\mathfrak{s} \wedge t}^{\Pi, w_0} \geq \mathcal{R}(\mathfrak{s}, t) \geq \mathfrak{Y}_t$ for every $t \in [0, \mathcal{T}]$. Since Π is a self-financing martingale generating portfolio

strategy we know that $\mathcal{W}^{*\Pi, w_0}$ is a martingale. Let $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$, by the Optional Sampling Theorem ([96] Chapter 1 section 3C) we have:

$$w_0 = \mathcal{W}_0^{*\Pi, w_0} = E_{\mathcal{E}} \left(\mathcal{W}_0^{*\Pi, w_0} \right) = E_{\mathcal{E}} \left(\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{*\Pi, w_0} \right) \quad (234)$$

Let $\{\mathfrak{t}_n\}_{n \in \mathbb{N}}$ be a decreasing sequence taking finitely many values (see proof of Assertion 3.4.20) decreasing to \mathfrak{t} . Then, by Assertion 3.4.20 we have⁵⁹

$$\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{\Pi, w_0} = \lim_{n \rightarrow \infty} \mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}_n}^{\Pi, w_0} \geq \lim_{n \rightarrow \infty} \mathcal{R}(\mathfrak{s}, \mathfrak{t}_n) \geq \mathcal{R}(\mathfrak{s}, \mathfrak{t}), \quad (235)$$

from where we see that

$$\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{*\Pi, w_0} \geq \mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) \quad (236)$$

and

$$w_0 = \mathcal{W}_0^{*\Pi, w_0} = E_{\mathcal{E}} \left(\mathcal{W}_0^{*\Pi, w_0} \right) = E_{\mathcal{E}} \left(\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{*\Pi, w_0} \right) \geq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (237)$$

□

3.4.3 The pricing of Game contingent claims

Recall that the fair price, \mathfrak{V} , Definition 3.4.11 of a gcc is defined as

$$\mathfrak{V} = \inf \{ w : w = \mathcal{W}_0^{\Pi, w_0} \text{ for some hedge } (\Pi, w_0, \mathfrak{s}) \text{ against the gcc} \} \quad (177)$$

Proposition 3.4.25. *The fair price \mathfrak{V} satisfies:*

$$\mathfrak{V} \geq \inf_{\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}} \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (238)$$

Proof. By Proposition 3.4.24 given a hedge (Π, w_0, \mathfrak{s}) and a stopping time $\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}$

$$w_0 = \mathcal{W}_0^{*\Pi, w_0} = E_{\mathcal{E}} \left(\mathcal{W}_0^{*\Pi, w_0} \right) = E_{\mathcal{E}} \left(\mathcal{W}_{\mathfrak{s} \wedge \mathfrak{t}}^{*\Pi, w_0} \right) \geq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (239)$$

thus

$$w_0 = \mathcal{W}_0^{*\Pi, w_0} = E_{\mathcal{E}} \left(\mathcal{W}_0^{*\Pi, w_0} \right) \geq \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (240)$$

then, by Definition 3.4.11

$$\mathfrak{V} \geq \inf_{\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}} \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \quad (241)$$

□

⁵⁹If \mathfrak{t} already takes finitely many values, we do not require the step below since in such a case it could be possible to write $\mathfrak{t} = \sum_{n=1}^N a_n \mathbb{1}_{\{\mathfrak{t}=a_n\}}$, where $N \in \mathbb{N}$, and $\{a_n\}_{n \in \mathbb{N}_N}$ is a finite set.

We are now in position to prove that Kifer's result [101] remains valid under the smm \mathcal{M} .

Theorem 3.4.26. *The fair price \mathfrak{V} of the gcc of Definition 3.4.2, as stated in Definition 3.4.11, equals $\mathcal{V}_0 = \mathcal{V}_0^*$ where $\mathcal{V}^* = \{\mathcal{V}_t^*\}_{t \in [0, T]}$ is the right continuous process such that $\mathcal{P}^\mathcal{E}$ -a.s.*

$$\mathcal{V}_t^* = \operatorname{ess\,inf}_{\mathfrak{s} \in \mathfrak{S}_{t, T}} \operatorname{ess\,sup}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) = \operatorname{ess\,sup}_{\mathfrak{s} \in \mathfrak{S}_{t, T}} \operatorname{ess\,inf}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \quad (242)$$

Moreover, for each $t \in [0, T]$ and $\varepsilon > 0$ the stopping times

$$\kappa_t^\varepsilon = \inf\{u \geq t : \mathfrak{X}_u^* \leq \mathcal{V}_u^* + \varepsilon\} \wedge T \quad (243a)$$

and

$$\xi_t^\varepsilon = \inf\{u \geq t : \mathfrak{Y}_u^* \geq \mathcal{V}_u^* - \varepsilon\} \quad (243b)$$

belong to $\mathfrak{S}_{t, T}$ and satisfy

$$E_{\mathcal{E}}(\mathcal{R}^*(\kappa_t^\varepsilon, \mathfrak{t}) | \mathcal{F}_t) - \varepsilon \leq \mathcal{V}_t^* \leq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \xi_t^\varepsilon) | \mathcal{F}_t) + \varepsilon \quad (244)$$

for any $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{t, T}$. Furthermore, for each $\varepsilon > 0$ there exists a self-financing, martingale generating portfolio strategy $\Pi^{\kappa_0^\varepsilon} = (\pi^0, \kappa_0^\varepsilon, \pi^{\kappa_0^\varepsilon})$ such that the pair κ_0^ε and $\Pi^{\kappa_0^\varepsilon}$ defines a hedge against the gcc of Definition 3.4.2 (as stated in Definition 3.4.10) with initial investment $w_0 = \mathcal{W}_0^{\Pi^\varepsilon} \leq \mathcal{V}_0^* + \varepsilon$.

If, in addition, the processes \mathfrak{Y} and $-\mathfrak{X}$ are upper semicontinuous from the left, i.e. that they may have only positive jumps at points of discontinuity. Then, the stopping times

$$\tilde{\kappa}_t = \lim_{\varepsilon \rightarrow 0^+} \kappa_t^\varepsilon \quad (245a)$$

and

$$\tilde{\xi}_t = \lim_{\varepsilon \rightarrow 0^+} \xi_t^\varepsilon \quad (245b)$$

satisfy, $\mathcal{P}^\mathcal{E}$ -a.s.,

$$E_{\mathcal{E}}(\mathcal{R}^*(\tilde{\kappa}_t, \mathfrak{t}) | \mathcal{F}_t) \leq \mathcal{V}_t^* \leq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \tilde{\xi}_t) | \mathcal{F}_t) \quad (246)$$

for any $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}$. Moreover, $\widetilde{\kappa}_t \wedge \widetilde{\xi}_t = \kappa_t^0 \wedge \xi_t^0$, where κ_t^0 and ξ_t^0 are defined by (243) with $\varepsilon = 0$, and so, \mathcal{P}^ε -a.s.,

$$\mathcal{V}_t^* = E_{\mathcal{E}}(\mathcal{R}^*(\kappa_t^0, \xi_t^0) | \mathcal{F}_t) \quad (247)$$

Furthermore, there exists a self-financing, martingale generating, portfolio strategy $\Pi^{\widetilde{\kappa}_0}$ such that the pair $\widetilde{\kappa}_0$ and $\Pi^{\widetilde{\kappa}_0}$ defines a hedge against the gcc of Definition 3.4.2 with initial investment $\mathcal{V}_0^* = \mathcal{W}_0^{\Pi^{\widetilde{\kappa}_0}}$ and such a strategy is unique (\mathcal{P}^ε -a.s.) up to time $\kappa_t^0 \wedge \xi_t^0$.

Given $t \in [0, \mathcal{T}]$ and $\varepsilon > 0$, the stopping times κ_t^ε and ξ_t^ε represent the ε -strategies of Seller and Buyer, respectively. The family of stopping times $\widetilde{\kappa}_t$ offers the Seller, via Proposition 3.4.23, a perfect hedge against the gcc, a hedge whose initial investment is equal to the fair value \mathfrak{V} of the gcc. As in our discussion at the beginning of this section, the variable $\mathcal{R}^*(\kappa_t^0, \xi_t^0)$ is financeable, players able to find the stopping times κ_t^0 and ξ_t^0 will be able to find the initial investment (and the corresponding self-financing martingale generating portfolio strategy) that will allow him/her to cover his position up to time $\kappa_t^0 \wedge \xi_t^0$ when the gcc should be canceled or exercised. Process \mathcal{V}^* can be regarded as the discounted price process, $\mathcal{V}_t^* = \mathcal{V}_t / B_t$.

Proof of Theorem 3.4.26. Recall from §3.4.1 the definition of the auxiliary game (Definition 3.4.9) $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P}^\varepsilon, \mathfrak{S}, J, \widetilde{\mathfrak{X}}, \widetilde{\mathfrak{Y}})$. By Corollary 3.4.6 we know that the auxiliary game satisfies (170) to (175) which correspond to (242) to (247).

Proposition 3.4.23 shows that for every stopping time $\mathfrak{s} \in \mathfrak{S}_{\mathcal{T}}$ there is a hedge against the gcc, in particular if $\mathfrak{s} = \kappa_0^\varepsilon$, $\varepsilon > 0$, there exists $\Pi^{\kappa_0^\varepsilon} = (\pi^{0, \kappa_0^\varepsilon}, \pi^{\kappa_0^\varepsilon})$, a self-financing martingale generating portfolio, and, by Proposition 3.4.14, (203), and Assertion 3.4.17 (if $s = t = 0$ we obtain the first part of (191) from Assertion 3.4.17)

$$w_0^{\kappa_0^\varepsilon} = \mathfrak{M}_0^{\kappa_0^\varepsilon} = U_0^{\kappa_0^\varepsilon} = \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\kappa_0^\varepsilon, \mathfrak{t})) \quad (248)$$

(\mathfrak{M}^s as in the proof of Proposition 3.4.23), such that $(\Pi^{\kappa_0^\varepsilon}, w_0^{\kappa_0^\varepsilon}, \kappa_0^\varepsilon)$ is a hedge against the gcc.

Similarly, Proposition 3.4.23 shows that the stopping time $\widetilde{\kappa}_0$ defines a self-financing martingale generating portfolio strategy $\Pi^{\widetilde{\kappa}_0} = (\pi^{0, \widetilde{\kappa}_0}, \pi^{\widetilde{\kappa}_0})$ with initial investment $w_0^{\widetilde{\kappa}_0} = \sup_{\mathfrak{t} \in \mathfrak{S}_{\mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\widetilde{\kappa}_0, \mathfrak{t}))$ such that $(\Pi^{\widetilde{\kappa}_0}, w_0^{\widetilde{\kappa}_0}, \widetilde{\kappa}_0)$ is a hedge against the gcc.

By (172) and Definition 3.4.11

$$\mathfrak{V} \leq w_0^{\kappa_0^\varepsilon} = \sup_{\mathfrak{t} \in \mathfrak{S}_T} E_{\mathcal{E}}(\mathcal{R}^*(\kappa_0^\varepsilon, \mathfrak{t})) \leq \mathcal{V}_0^* + \varepsilon \quad (249)$$

(\mathcal{V}^* the value of the auxiliary game), from where we obtain $\mathfrak{V} \leq \mathcal{V}_0^*$. On the other hand, Proposition 3.4.25 implies that

$$\mathcal{V}_0^* = \inf_{\mathfrak{s} \in \mathfrak{S}_T} \sup_{\mathfrak{t} \in \mathfrak{S}_T} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t})) \leq \mathfrak{V} \leq \mathcal{V}_0^* \quad (250)$$

Therefore $\mathfrak{V} = \mathcal{V}_0^*$, that is, the initial value of the contract equals the initial value of the auxiliary game.

Note that (174) and Definition 3.4.11 implies that

$$\mathfrak{V} \leq w_0^{\widetilde{\kappa}_0} = \sup_{\mathfrak{t} \in \mathfrak{S}_T} E_{\mathcal{E}}(\mathcal{R}^*(\widetilde{\kappa}_0, \mathfrak{t})) \leq \mathcal{V}_0^* = \mathfrak{V} \quad (251)$$

that is, the hedge $(\Pi^{\widetilde{\kappa}_0}, w_0^{\widetilde{\kappa}_0}, \widetilde{\kappa}_0)$ is a perfect hedge, in the sense that its initial investment is equal to the fair value of the gcc. \square

Our work in this chapter offers a very general model under which the pricing of some Game contingent claims is possible, showing that the price exists under very general assumptions.

Still, there is much to do. The work shown here constitutes a good starting point for further development. In particular in the light of the results shown here the study of more complex Game contingent claims involving a cumulative income process is not too difficult. That is, the study of gcc's whose payoff process is of the form

$$\mathcal{R}(s, t) = \mathfrak{X}_s \mathbb{1}_{s < t} + \mathfrak{Y}_t \mathbb{1}_{t \leq s} + \int_{]0, s \wedge t[} \frac{dC_u}{B_u} \quad (252)$$

or maybe

$$\mathcal{R}(s, t) = \mathfrak{X}_s \mathbb{1}_{s < t} + \mathfrak{Z}_s \mathbb{1}_{s = t} + \mathfrak{Y}_t \mathbb{1}_{t < s} + \int_{]0, s \wedge t[} \frac{dC_u}{B_u} \quad (253)$$

where C is the cumulative income process and \mathcal{R} is assumed bounded below could be included into the framework we have presented here.

The following are possible extensions to our work:

- Our approach to hedging is “seller centered”, in fact one may argue that our notion of hedge is very similar to the notion of “upper hedge”. Can we define “lower hedges” for the buyer? What is their relationship with the price? Naturally, in the case of a complete market, the price found by means of upper hedging is the same as the one we could find by means of lower hedging. This problem turns more interesting in the case of incomplete markets and general Game contingent claims of the form (253).
- In our study we have assumed that $\mathfrak{Y} \leq \mathfrak{X}$. Can we work without that condition? Intuitively (and assuming a payoff function of the form (107)), the contract will make sense if $\mathfrak{Y}_0 \leq \mathfrak{X}_0$ and if $\inf_t \{\mathfrak{Y}_t \geq \mathfrak{X}_t\} > 0$. What other conditions are required?
- In Chapter 5 we suggest an example of Game option and offer a numerical method for its approximate valuation. It is clear that there is much to be done in this regard. What kinds of Game options may be of practical interest? Which numerical methods can be applied to the valuation of such examples? In some situations this can give rise to new problems in the form of variational inequalities and viscosity problems.
- In Chapter 5 we show that the time t value of a gcc is always lower than the time t value of the corresponding American contingent claim (**acc**) defined by the execution payoff \mathfrak{Y} . If we denote by \mathcal{V}^{gcc} and \mathcal{V}^{acc} the price processes of a gcc and its corresponding acc (resp.), Proposition 5.1.2 shows that $\mathcal{V}_t^{acc} - \mathcal{V}_t^{gcc} \geq 0$ a.e.. Proposition 5.1.1 shows that the value of a gcc increases as its cancellation payoff increases; still, a detailed study of the “Game–American” premium, $\mathcal{V}^{acc} - \mathcal{V}^{gcc}$ is needed.

CHAPTER IV

CALIBRATION OF THE HULL–WHITE MODEL

Throughout the years several interest rate models have been proposed in an attempt to simulate the observed behavior of interest rates, to better hedge one’s position in the future, to properly price long-term, mid-term or short-term contracts that are sensitive to interest rates, etc. Models proposed are as simple as “assume rates are constant for a given period of time” or as complex as multi-factor stochastic models. Obviously, different arguments have been given in favor or against all of such models, but still due to several different reasons some of them have gained acceptance amongst practitioners. One such model is the one-factor Hull–White model¹.

The Hull–White model has gone through an evolution of sorts resulting in a model that is more general now than it was when it was first proposed by John Hull and Alan White in 1990².

The Hull–White model belongs to a class of interest rate models called affine models³(see [133], [144]). It features mean reversion and normal interest rates. The Hull–White model extends the Ho–Lee model [80]; in fact, at the moment of its inception, the Hull–White model was general enough as to include as sub-cases most of the models previously defined. At least in some of its least general forms, the Hull–White model is analytically tractable, leading to discount bond prices and options-on-bond prices that can be valued analytically. Other contracts can be reduced to portfolios on zero coupon bonds (zcb’s), and thus be analytically valued as well. The Hull–White model can be fitted to an initial term structure of interest rates; and although interest rates under this model are still normal, the probability

¹There are three major approaches to the modeling of interest rates which are: spot rate modeling, forward rate modeling and the so called market models. The Hull White model of interest rates belongs to the first approach.

²See [83].

³Please see Footnote 11.

of them becoming negative is smaller than in previous models⁴.

4.1 The Hull–White model

According to the Hull–White model, short-term interest rates r_t are modeled as⁵

$$dr_t = (a(t) - br_t) dt + c dW_t, \quad (254)$$

where b and c are positive constants, $a(t)$ is a deterministic function of t and W is a standard Brownian motion.

A more general version of the Hull–White model assumes the other two parameters b and c are not constant, but positive deterministic functions of time.

$$dr_t = (\alpha(t) - \beta(t)r_t) dt + \gamma(t) dW_t. \quad (255)$$

Although some characteristics of the model are apparent from the stochastic differential equation (sde) (255), from the practical point of view, the model can not be used directly from (255). To properly use an interest rate model, one needs to deeply familiarize oneself with all its details, and practical applications of a model will require *calibration*.

When talking about calibration, one is not referring to a single entity but to a collection of techniques tightly related to the model one wants to work with. Thus, different models may require different calibration methods, and in most cases different calibration procedures can be followed to calibrate a given model. Calibration is highly subjective and demands a deep understanding of the details of the particular model one is working with. It also requires a good knowledge of the financial instruments found in the market, as it also depends on the the quality and quantity of data one can recover. The calibration procedure we present here is not unique in the sense that it is not the only approach one may follow⁶. On the

⁴Although this can be seen as a problem, one may also remember that these models are designed to model real interest rates and not nominal or “quoted” rates. Real interest rates should reflect the effects of inflation and other economic processes that could make the real interest rates negative. Thus, what could be seen by some as a problem could also be seen by others as a feature.

⁵There exist several versions of this model, some of them introduced by Hull and White in their papers, some introduced by other authors. We will use [83]

⁶See for example [142] for a calibration procedure based in a linear programming approach. [144], [19] offer sketches of a possible calibration approach similar to the one shown here.

contrary, the procedure shown here is the result of our studies, and contains subjective judgments influenced by the literature studied.

In order to get started, we will invest some time in the study of equation (255).

First, to ensure the existence of strong solutions of equation (255) we need to impose additional conditions on functions α , β , and γ and the initial value r_0 .

We assume that the Brownian motion W is defined on a p.s. $(\Omega, \mathcal{U}, \mathcal{P})$ and denote by $\mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \in [0, T]}$ the natural filtration of W . Let η be a finite a.e. r.v. independent⁷ of the σ -algebra $\bigvee \mathcal{F}^W = \sigma(\mathcal{F}_t^W; 0 \leq t \leq T)$. Then denote by $\{\mathcal{F}_t\}_{t \in [0, T]}$ the augmentation of the filtration $\{(\sigma(\eta) \vee \mathcal{F}_t^W)\}_{t \in [0, T]}$.

Since functions α , β , and γ are assumed to be deterministic, by [141], section 5.2, Theorem 5.2.1. asking them to be measurable and bounded will automatically satisfy all other conditions necessary for the existence of unique strong solutions with initial condition $r_0 = \eta$. For more details regarding the existence of strong solutions of general diffusion type equations please see [114], chapter 4, Theorem 4.6 and its Corollary. A similar result can be found elsewhere, for example [141], section 5.2 and [96], chapter 5 section 2.

Thus, assuming that α , β , and γ are measurable positive bounded functions of time, and provided that the random variable η has finite second moment, equation (255) has a unique (up to modification) continuous solution $r = \{r_t\}_{t \in [0, T]}$ with initial value $r_0 = \eta$ adapted to the filtration \mathcal{F} , generated by η and the Brownian Motion W , and such that

$$E \left[\int_0^T |r_t|^2 dt \right] < \infty. \quad (256)$$

Even in this more general version of the Hull–White model, the associated stochastic differential equation, (255), is not hard to solve and its solution can be obtained through the standard method of integrating factors.

Denote

$$\eta(t) = \exp \left(\int_0^t \beta(u) du \right). \quad (257)$$

⁷Instead of this very technical setting, we could simply assume that our Brownian motion is adapted to a filtration, $\{\mathcal{F}_t\}_{t \in [0, T]}$, finer than the augmentation of its natural filtration and that η is a finite a.e. \mathcal{F}_0 -measurable random variable.

It is straightforward that

$$\begin{aligned} d(\eta(t)r_t) &= d\eta(t)r_t + \eta(t) dr_t = \eta(t)\{\beta(t)r_t dt + (\alpha(t) - \beta(t)r_t) dt + \gamma(t) dW_t\} \\ &= \eta(t)\{\alpha(t) dt + \gamma(t) dW_t\}. \end{aligned} \quad (258)$$

The last sde can be integrated directly, yielding:

$$r_t = (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u)\alpha(u) du + \int_0^t \eta(u)\gamma(u) dW_u \right\}, \quad (259)$$

or, similarly, assuming $s \leq t$

$$r_t = (1/\eta(t)) \left\{ \eta(s)r_s + \int_s^t \eta(u)\alpha(u) du + \int_s^t \eta(u)\gamma(u) dW_u \right\} \quad s \leq t. \quad (260)$$

Clearly, (259) can be seen as a sub-case of (260), but both formulas can be useful.

A process of the form (259) is Gaussian (see [96], Chapter 5; also [151] or [88]) and interest rate models of the form (255) are known as Gaussian models⁸. Using (259), we may obtain the unconditional expectation of r_t as:

$$\mathbf{m}_t = E(r_t) = (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u)\alpha(u) du \right\} \quad (261)$$

and in case $s \leq t$

$$E(r_t | \mathcal{F}_s) = (1/\eta(t)) \left\{ \eta(s)r_s + \int_s^t \eta(u)\alpha(u) du \right\} \quad s \leq t \quad (262)$$

We can obtain as well the variance (conditional and unconditional) of process r from (259) and (260)

$$\mathbf{v}_t = \text{Var}(r_t) = \frac{1}{\eta^2(t)} \int_0^t \eta^2(u)\gamma^2(u) du \quad (263)$$

⁸ An interest rate model belongs to the class of Gaussian models if it can be expressed in the form

$$dr_t = (a(t) + b(t)r_t)dt + c(t) dW_t$$

where a , b , and c are deterministic functions of t . This means that the generalized Hull–White model, (255) is the archetypical Gaussian model, it also means that the work we are doing here regarding the calibration of the Hull White model is valid for all Gaussian models. Gaussian models are a sub class of the class of affine models. As we will show later, under Gaussian models, bond prices are log-normally distributed which in our case agrees with our market setting in Chapter 3. We will use “Hull–White” and “Gaussian” interchangeably.

and

$$\text{Var}(r_t | \mathcal{F}_s) = \frac{1}{\eta^2(t)} \int_s^t \eta^2(u) \gamma^2(u) du \quad s \leq t, \quad (264)$$

and covariance function

$$\text{Cov}(r_t, r_s) = \frac{1}{\eta(t)\eta(s)} \int_0^{t \wedge s} \eta^2(u) \gamma^2(u) du. \quad (265)$$

As usual, we define B as the bank account (price) process (or money market process) (47):

$$B_t = \exp \left\{ \int_0^t r_u du \right\} \quad (266)$$

Under usual market hypothesis⁹, and assuming risk-neutral expectations hypothesis¹⁰, the time t prices of (risk free) zero coupon bonds of maturity T are:

$$B(t, T) = E \left(\exp \left\{ - \int_t^T r_u du \right\} \middle| \mathcal{F}_t \right) \quad t \in [0, T]. \quad (267)$$

Thus, it also makes sense to devote some time to the analysis of the process defined as

$$\begin{aligned} \nu_t &= \int_0^t r_u du \\ &= r_0 \int_0^t (1/\eta(u)) du + \int_0^t (1/\eta(u)) \int_0^u \eta(v) \alpha(v) dv du \\ &\quad + \int_0^t (1/\eta(u)) \int_0^u \eta(v) \gamma(v) dW_v du. \end{aligned} \quad (268)$$

Process $\nu = \{\nu_t\}_{t \in [0, T]}$ is also Gaussian, and so ν_t is normal, for all $t \in]0, T]$, with mean:

$$E \left(\int_0^t r_u du \right) = \int_0^t (1/\eta(u)) \left\{ r_0 + \int_0^u \eta(v) \alpha(v) dv \right\} du; \quad (269)$$

from (261) we may write

$$E\nu_t = E \left(\int_0^t r_u du \right) = \int_0^t \mathbf{m}_u du. \quad (270)$$

⁹As in §3.3 we assume that our market is free of frictions and liquidity problems, that transactions placed by market participants can not influence the market price of the securities traded. Our assumption of no frictions include the assumption that securities (prices) are divisible, that is that a market participant could trade any fraction of a security. See §3.3 for more details. See also [133] Chapter 12.

¹⁰That is, we assume that the formula (267) is valid under probability measure \mathcal{P} , or in other words \mathcal{P} here is the equivalent martingale measure of Chapter 3.

Similarly, using (260) and (262) we can compute the conditional expectation of ν_t as:

$$\begin{aligned} E(\nu_t - \nu_s | \mathcal{F}_s) &= \int_s^t (1/\eta(u)) \left\{ \eta(s)r_s + \int_s^u \eta(v)\alpha(v) dv \right\} du \\ &= \int_s^t E(r_u | \mathcal{F}_s) du \quad s \leq t. \end{aligned} \quad (271)$$

To compute the covariance function, we first observe that

$$\nu_t - E(\nu_t) = \int_0^t \frac{1}{\eta(u)} \int_0^u \eta(v)\gamma(v) dW_v du = \int_0^t \eta(v)\gamma(v) \int_v^t \frac{1}{\eta(u)} du dW_v. \quad (272)$$

Thus:

$$\begin{aligned} \text{Cov}(\nu_t, \nu_s) &= E \left(\left\{ \int_0^{t \wedge s} \eta(v)\gamma(v) \int_v^{t \wedge s} \frac{1}{\eta(u)} du dW_v \right\} \times \right. \\ &\quad \left. \times \left\{ \int_0^{t \vee s} \eta(v)\gamma(v) \int_v^{t \vee s} \frac{1}{\eta(u)} du dW_v \right\} \right) \\ &= E \left(\left\{ \int_0^{t \wedge s} \eta(v)\gamma(v) \int_v^{t \wedge s} \frac{1}{\eta(u)} du dW_v \right\}^2 + \right. \end{aligned} \quad (273)$$

$$\begin{aligned} &\quad + \left\{ \int_0^{t \wedge s} \eta(v)\gamma(v) \int_v^{t \wedge s} \frac{1}{\eta(u)} du dW_v \right\} \times \\ &\quad \times \left\{ \int_0^{t \wedge s} \eta(v)\gamma(v) \int_{t \wedge s}^{t \vee s} \frac{1}{\eta(u)} du dW_v \right. \\ &\quad \left. + \int_{t \wedge s}^{t \vee s} \eta(v)\gamma(v) \int_v^{t \vee s} \frac{1}{\eta(u)} du dW_v \right\} \Big) \\ \text{Cov}(\nu_t, \nu_s) &= \int_0^{t \wedge s} \eta^2(v)\gamma^2(v) \left(\int_v^{t \wedge s} \frac{1}{\eta(u)} du \right)^2 dv \\ &\quad + \left[\int_{t \wedge s}^{t \vee s} \frac{1}{\eta(u)} du \right] \int_0^{t \wedge s} \eta^2(v)\gamma^2(v) \int_v^{t \wedge s} \frac{1}{\eta(u)} du dv. \end{aligned} \quad (274)$$

The unconditional variance will be

$$\text{Var}(\nu_t) = \int_0^t \eta^2(v)\gamma^2(v) \left(\int_v^t \frac{1}{\eta(u)} du \right)^2 dv. \quad (275)$$

We can also compute the conditional variance of $\nu_t - \nu_s$; from (260), (262) and (269),

$$\begin{aligned} \text{Var}(\nu_t - \nu_s | \mathcal{F}_s) &= \text{Var} \left(\int_s^t r_u du \middle| \mathcal{F}_s \right) \quad s \leq t \\ &= E \left(\left\{ \int_s^t \frac{1}{\eta(u)} \int_s^u \eta(v)\gamma(v) dW_v du \right\}^2 \middle| \mathcal{F}_s \right) \\ &= E \left(\left\{ \int_s^t \eta(v)\gamma(v) \int_v^t \frac{1}{\eta(u)} du dW_v \right\}^2 \middle| \mathcal{F}_s \right) \\ &= \int_s^t \eta^2(v)\gamma^2(v) \left\{ \int_v^t \frac{1}{\eta(u)} du \right\}^2 dv \quad s \leq t. \end{aligned} \quad (276)$$

Thus we have described completely the dynamics of process $\nu = \{\nu_t\}_{t \in [0, T]}$ as a Gaussian process, each ν_t normal, with mean and variance given by (269), (271), and covariances and variances as in (274), (275), (276).

Our study of the process $\nu = \{\nu_t\}_{t \in [0, T]}$ allows us to obtain a more explicit result than (267) in the pricing of a zero coupon bond (zcb). In fact we notice that (267) transforms into the following:

$$\begin{aligned} B(t, T) &= E \left(\exp \left\{ - \int_t^T r_u du \right\} \middle| \mathcal{F}_t \right) \quad t \in [0, T] \\ &= E \left(\exp(-\nu_T + \nu_t | \mathcal{F}_t) \right) = E \left(\exp \{ \mathbf{i}[\mathbf{i}(\nu_T - \nu_t)] \} | \mathcal{F}_t \right) \\ &= Ch_{(\nu_T - \nu_t) | \mathcal{F}_t}(\mathbf{i}) \quad t \in [0, T] \end{aligned} \quad (277)$$

where Ch denotes the conditional characteristic function of $\nu_T - \nu_t$, conditional to \mathcal{F}_t , $t \in [0, T]$; and

$$\begin{aligned} B(t, T) &= Ch_{(\nu_T - \nu_t) | \mathcal{F}_t}(\mathbf{i}) \quad t \in [0, T] \\ &= \exp \left(-E(\nu_T - \nu_t | \mathcal{F}_t) + (1/2) \text{Var}(\nu_T - \nu_t | \mathcal{F}_t) \right) \\ &= \exp \left(- \int_t^T (1/\eta(u)) \left\{ \eta(t)r_t + \int_t^u \eta(v)\alpha(v) dv \right\} du \right. \\ &\quad \left. + (1/2) \int_t^T \eta^2(v)\gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \right) \quad t \in [0, T]. \end{aligned} \quad (278)$$

For $t = 0$, we obtain

$$\begin{aligned} B(0, T) &= \exp \left(- \int_0^T (1/\eta(u)) \left\{ r_0 + \int_0^u \eta(v)\alpha(v) dv \right\} du \right. \\ &\quad \left. + (1/2) \int_0^T \eta^2(v)\gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \right) \\ &= \exp \left(- \int_0^T \mathbf{m}_u du + (1/2) \int_0^T \eta^2(v)\gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \right). \end{aligned} \quad (279)$$

It is clear from (278) (and in the particular case of $t = 0$, from (279)) that the price $B(t, T)$ of a zcb is the exponential of a linear expression of r_t ; that is, that the Hull–White model is an *affine* model¹¹. Or in symbols, we have that

$$B(t, T) = \exp(-r_t \mathcal{S}(t, T) - \mathcal{I}(t, T)) \quad t \in [0, T] \quad (280)$$

¹¹ In accordance to [42] an interest rate model is called *affine* if the yield to maturity

$$\mathcal{Y}(t, T) = \frac{1}{T-t} E \left(\int_t^T r_u du \middle| \mathcal{F}_t \right)$$

where the slope $\mathcal{S}(t, T)$ is given by:

$$\mathcal{S}(t, T) = \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \quad (281)$$

and the intercept $\mathcal{I}(t, T)$ is given by:

$$\begin{aligned} \mathcal{I}(t, T) = & \int_t^T \frac{1}{\eta(u)} \int_t^u \eta(v) \alpha(v) dv du \\ & - (1/2) \int_t^T \eta^2(v) \gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \quad t \in [0, T] \end{aligned} \quad (282)$$

We notice that we can write $\mathcal{I}(t, T)$ in terms of $\mathcal{S}(t, T)$, but first we need to play a little with the integrals; for example,

$$\int_t^T \frac{1}{\eta(u)} \int_t^u \eta(v) \alpha(v) dv du = \int_t^T \eta(v) \alpha(v) \int_v^T \frac{1}{\eta(u)} du dv \quad (283)$$

Thus, we may write (282) as:

$$\begin{aligned} \mathcal{I}(t, T) = & \int_t^T \eta(v) \alpha(v) \int_v^T \frac{1}{\eta(u)} du dv \\ & - (1/2) \int_t^T \eta^2(v) \gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \\ = & \int_t^T \alpha(v) \mathcal{S}(v, T) - (1/2) \gamma^2(v) \mathcal{S}^2(v, T) dv \quad t \in [0, T] \end{aligned} \quad (284)$$

Hence, we have been able to completely describe the time t price of a zcb of maturity T under the Hull–White model as:

$$B(t, T) = \exp(-(r_t \mathcal{S}(t, T) + \mathcal{I}(t, T))) \quad t \in [0, T] \quad (285)$$

$$\mathcal{S}(t, T) = \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \quad (286)$$

$$\mathcal{I}(t, T) = \int_t^T \alpha(v) \mathcal{S}(v, T) - (1/2) \gamma^2(v) \mathcal{S}^2(v, T) dv \quad t \in [0, T] \quad (287)$$

of zcb's under such a model can be written as an affine function of the short-term rate r_t . That is, if there exists continuously differentiable deterministic functions $f(t, T)$ and $g(t, T)$ such that

$$B(t, T) = \exp(f(t, T) + g(t, T)r_t).$$

Alternatively, taking derivatives of the expressions above, we can obtain differential, instead of integral, representations for the slope and intercept functions:

$$\left. \frac{\partial \mathcal{S}}{\partial t} \right|_{(t,T)} = \beta(t) \mathcal{S}(t, T) - 1 \quad \mathcal{S}(T, T) = 0 \quad (288)$$

$$\left. \frac{\partial \mathcal{I}}{\partial t} \right|_{(t,T)} = -\alpha(t) \mathcal{S}(t, T) + (1/2) \gamma^2(t) \mathcal{S}^2(t, T) \quad \mathcal{I}(T, T) = 0 \quad (289)$$

We can also test our results applying Itô's formula directly to (285) (this will lead to the corresponding *bond pricing pde*¹²). From (285) and (255) we obtain by Itô's formula:

$$\begin{aligned} dB(t, T) &= \left(-r_t \frac{\partial \mathcal{S}}{\partial t} - \frac{\partial \mathcal{I}}{\partial t} \right) B(t, T) dt - \mathcal{S}(t, T) B(t, T) dr_t \\ &\quad + \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) B(t, T) dt \\ &= B(t, T) \left\{ \left(-r_t \frac{\partial \mathcal{S}}{\partial t} - \frac{\partial \mathcal{I}}{\partial t} - \mathcal{S}(t, T)(\alpha(t) - \beta(t)r_t) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) \right) dt - \gamma(t) \mathcal{S}(t, T) dW_t \right\} \end{aligned} \quad (290)$$

As expected $\{B(t, T)\}_{t \in [0, T]}$ is a log-normal process with return drift:

$$-r_t \frac{\partial \mathcal{S}}{\partial t} - \frac{\partial \mathcal{I}}{\partial t} - \mathcal{S}(t, T)(\alpha(t) - \beta(t)r_t) + \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) \quad (291)$$

and volatility

$$-\gamma(t) \mathcal{S}(t, T) \quad (292)$$

Since we are assuming the risk neutral expectations hypothesis and a single factor interest rate model of the form

$$dr_t = \mu_t dt + \sigma_t dW_t \quad (293)$$

we can apply [133] Proposition 12.2.1; which implies that there must be a process $b = \{b(t, T)\}_{t \in [0, T]}$ such that

$$dB(t, T) = B(t, T)(r_t dt + b(t, T) dW_t) \quad (294)$$

it is clear, from (290) that such process is the bond volatility we found above, thus

$$b(t, T) = -\gamma(t) \mathcal{S}(t, T) \quad (295)$$

¹²We use the acronym **pde** to refer to Partial Differential Equations, we may also use PDE to refer to them.

that is, the zcb's volatility, which according to (255) and (281) is a deterministic process. But this is not all we obtain from [133] Proposition 12.2.1, we also have that the drift (291) should reduce to r_t , that is:

$$-r_t \frac{\partial \mathcal{S}}{\partial t} - \frac{\partial \mathcal{I}}{\partial t} - \mathcal{S}(t, T)(\alpha(t) - \beta(t)r_t) + \frac{1}{2}\gamma^2(t)\mathcal{S}^2(t, T) = r_t \quad (296)$$

which implies

$$\frac{\partial \mathcal{S}}{\partial t} = \beta(t)\mathcal{S}(t, T) - 1 \quad (297)$$

$$\frac{\partial \mathcal{I}}{\partial t} = (1/2)\gamma^2(t)\mathcal{S}^2(t, T) - \alpha(t)\mathcal{S}(t, T) \quad (298)$$

both equations above agree with (288) and (289). Thus, we can write (290) as

$$dB(t, T) = B(t, T)(r_t dt - \gamma(t)\mathcal{S}(t, T) dW_t) \quad (299)$$

Applying again [133] Proposition 12.2.1 we can write the time t price of a zcb of maturity T in terms of the bank account process (47), the time 0 price of a zcb of maturity T and the bond volatility (295):

$$B(t, T) = B(0, T)B_t \exp \left\{ - \int_0^t \gamma(u)\mathcal{S}(u, T) dW_u - \frac{1}{2} \int_0^t \gamma^2(u)\mathcal{S}^2(u, T) du \right\} \quad (300)$$

4.1.1 A few more formulas

One might think that after learning so much about the Hull–White model and the pricing of a zcb of that model, to calibrate the model to fit to initial (market) data should be simple and immediate. But the truth is that although we have a very detailed solution, to calibrate to initial data still requires some juggling with the formulas.

But first we should consider what kind of initial data is available, how that data will be obtained, and a few other points.

In fact, the act of finding initial data may be a problem by itself. Here we list some possible sources:

Dr. J. Huston McCulloch, Professor of Economics and Finance at The Ohio State University maintains the web site called “The US Real Term Structure of Interest Rates with Implicit Inflation Premium” in which series of “clean” US term structure data (yield

curve, forward rate, and zcb prices) including inflation effect (that will be “*real*” *term structure data* and not *nominal term structure data*¹³) can be found at the web site:

<http://economics.sbs.ohio-state.edu/jhm/ts/ts.html>

this data has been already ‘preprocessed’ from other sources via bootstrapping, inflation analysis, etc.

The Internet web site of the Bank of England is a very good source of financial information regarding the United Kingdom. Historical yield curve data for the English market can be found at the web site:

<http://www.bankofengland.co.uk/statistics/yieldcurve/main.htm>

Some US Treasuries data can be found at the US Federal Reserve web site

<http://www.federalreserve.gov/releases/>

and at the US Treasury web site

<http://www.publicdebt.treas.gov/of/ofaicqry.htm>

Data can also be obtained (for a price) from different private sources such as Telerate[©], Bloomberg[©], The WSJ[©], CME[©], etc.

Data can also be obtained indirectly from other markets that closely react to changes in the yield curve (for example the swaps market). Such data is also available (for a price)

¹³Nominal interest rates are those rates advertised by the borrowers (lenders), such rates inform you of the interest that the borrower will pay (lender will charge) for the money the lender (borrower) is giving (receiving). A real interest rate reflects the effects of inflation and other factors that may alter the acquisitive value of the interest received. For example (assuming that inflation is the only important source of loss in acquisitive value), an investment of \$100.00 at a nominal rate of 6% per annum (to simplify the reasoning, we will assume all rates in this example are simple compounded) for a maturity of one year will pay \$6.00. But those \$6.00 will be received a year from today, and due to inflation those —future— \$6.00 will not be as strong as they could be today. Assuming an inflation of 0.5% per annum, \$6.00 a year from today will be worth

$$\$5.47 \simeq \$100 \times \left(\frac{1 + 0.06}{1 + 0.005} - 1 \right)$$

\$5.47 dollars today. That is, the real interest rate paid was not 6% as advertised but only 5.47%. Real interest rates diminish as inflation rates increase, in our previous example an inflation of 6% per annum will completely offset the nominal rate. For a basic introduction to the notion of nominal and real interest rates please see [17] Chapter 5 and [152] Chapter 7.

from different vendors, like Bloomberg[©].

Nominal term structure data (and corresponding forward rates and zcb prices) can be bootstrapped from market Treasurys prices. Following this approach all available data is collected and analyzed; prices corresponding to very “young” bonds as well as very “old”, low liquidity bonds must be taken out to later apply one of the available bootstrapping procedures, see for example [59]¹⁴, and [136], which according to [1] are two of the most used methods in the world. The bootstrapping procedure can be improved and simplified if STRIPS data is considered.

The problem with this approach is its extreme subjectivity, but as we have learned from our studies, subjectivity is something we can not escape when a calibration is needed. In fact, the whole process is more of an art than a science, and to obtain good results the practitioner must develop practice skills and a deep understanding of the market and the model.

In an attempt to cope with the complications found in the usual bootstrapping procedures, simpler methods have been suggested. For example according to Sack, [153], term structure information recovered from STRIP prices is very reliable. Since STRIPS are already zcb’s (when compared with the other procedures), to extract term structure information from STRIPS is easier.

In our case, initial term structure information will be bootstrapped, using a different procedure, from other liquid instruments that reflect the market perception of the yield curve and its volatility.

In the previous pages we have provided an in-depth study of the model that led us to the bond pricing formulas (285), (281), (287), (295), (291), (299), etc. The labor that is at hand now is to use all those formulas to solve for the different model parameters, and to rewrite some of them to a more usable form. The general idea is to be able to fit a given initial term structure to the model, thus we must find ways to represent future values in terms of time zero values.

¹⁴A slightly earlier version [58] can be downloaded from Mark Fisher’s web site

Before we begin, we make a few additional preliminary remarks. Most of the problem here comes from the fact that we are working with the model (255). Assuming one or more parameters are constant will greatly simplify the task ahead (please see §4.2). The reader should quickly notice that most of our formulas, and this includes the ones we are about to write, will simplify a lot if one uses the original version of the Hull–White’s model (254). But this not only applies to the formulas, it also applies to the calibration process, as we will see later (please see §4.5).

Recall that, as we mentioned before, our goal is to fit a given initial term structure to the model of (255). In order to do that we need to find formulas allowing us to represent the parameters of the model (or known functions of those parameters) in terms of “time zero” values. That is, in terms of the data available to us at the beginning. We may assume that we will have access to a given initial yield curve (obtained as outlined in Chapter 2 §2.2). It is clear that such a little initial information is not enough to determine all parameters in (255), thus we need to assume as well that some information regarding the “initial” volatility¹⁵ of bond prices is given and that such information is available in the form of continuously differentiable functions of time. Later in this chapter (see §4.3) we will return to this topic and explain how to obtain that last piece of information.

Nothing more will be assumed in the construction of the following formulas. We start with (281), the slope function in our bond price formula,

$$\mathcal{S}(t, T) = \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \quad (281)$$

since $\eta(0) = 1$ we see that

$$\mathcal{S}(0, T) = \eta(0) \int_0^T \frac{1}{\eta(u)} du = \int_0^T \frac{1}{\eta(u)} du \quad (301)$$

Therefore

$$\int_t^T \frac{1}{\eta(u)} du = \mathcal{S}(0, T) - \mathcal{S}(0, t). \quad (302)$$

¹⁵As we will see in this section, the parameters $\mathcal{S}(0, t)$ and $\gamma(t)$ can not be recovered from the initial yield curve, but according to (295) under the Hull–White model zcb volatility depends on those two unknowns. Thus, at least theoretically, one could recover $\mathcal{S}(0, t)$ and $\gamma(t)$ from initial bond volatility data if such information is abundant.

On the other hand,

$$\frac{d}{dt}\mathcal{S}(0, t) = \frac{d}{dt} \int_0^t \frac{1}{\eta(u)} du = \frac{1}{\eta(t)} \quad (303)$$

Thus

$$\mathcal{S}(t, T) = \frac{\mathcal{S}(0, T) - \mathcal{S}(0, t)}{\frac{d}{dt}\mathcal{S}(0, t)} \quad (304)$$

hence, if $\mathcal{S}(0, t)$ is known, we have a way to compute $\mathcal{S}(t, T)$ at any time t and for any maturity T .

Similarly, consider now (287), the intercept function in our bond pricing formula,

$$\mathcal{I}(t, T) = \int_t^T \alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) dv \quad t \in [0, T]. \quad (287)$$

Differentiating with respect to t , we found that

$$\left. \frac{\partial \mathcal{I}}{\partial t} \right|_{(t, T)} = -\alpha(t)\mathcal{S}(t, T) + (1/2)\gamma^2(t)\mathcal{S}^2(t, T) \quad \mathcal{I}(T, T) = 0. \quad (289)$$

Taking derivatives with respect to T , we obtain

$$\frac{\partial^2 \mathcal{I}}{\partial T \partial t} = \{\gamma^2(t)\mathcal{S}(t, T) - \alpha(t)\} \frac{\partial \mathcal{S}}{\partial T}. \quad (305)$$

We can get rid of the dependency on α if we multiply by S , since we can solve for αS from (289). Thus

$$\mathcal{S}(t, T) \frac{\partial^2 \mathcal{I}}{\partial T \partial t} = \left\{ \frac{1}{2}\gamma^2(t)\mathcal{S}^2(t, T) + \frac{\partial \mathcal{I}}{\partial t} \right\} \frac{\partial \mathcal{S}}{\partial T} \quad (306)$$

Since we can obtain $\mathcal{S}(t, T)$ from $\mathcal{S}(0, t)$, knowing γ we can solve for I from PDE (306) we just found. Of course, one of the problems of such an approach is clearly apparent from the fact that differentiation is not a numerically stable procedure, thus one may want to avoid paths leading to higher order differentiations.

From (287), we have

$$\begin{aligned} \mathcal{I}(t, T) &= \int_t^T \alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) dv \\ &= \int_0^T \alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) dv \\ &\quad - \int_0^t \alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) dv \\ &= \mathcal{I}(0, T) - \int_0^t \alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) dv. \end{aligned} \quad (307)$$

In order to rewrite the second integral in terms of data known at time zero, we replace the $\mathcal{S}(v, T)$ terms in this integral. From $\mathcal{S}(t, T)$'s definition, (281), we see that

$$\begin{aligned}\mathcal{S}(v, T) &= \mathcal{S}(v, t) + \mathcal{S}(v, T) - \mathcal{S}(v, t) \\ &= \mathcal{S}(v, t) + \eta(v) \int_t^T \frac{1}{\eta(u)} du \\ &= \mathcal{S}(v, t) + \frac{\eta(v)}{\eta(t)} \mathcal{S}(t, T).\end{aligned}\tag{308}$$

Thus,

$$\begin{aligned}\alpha(v)\mathcal{S}(v, T) - (1/2)\gamma^2(v)\mathcal{S}^2(v, T) &= \alpha(v)\mathcal{S}(v, t) - (1/2)\gamma^2(v)\mathcal{S}^2(v, t) \\ &\quad + \frac{\eta(v)}{\eta(t)} \mathcal{S}(t, T) (\alpha(v) - \gamma^2(v)\mathcal{S}(v, t)) \\ &\quad - (1/2)\gamma^2(v) \frac{\eta^2(v)}{\eta^2(t)} \mathcal{S}^2(t, T).\end{aligned}\tag{309}$$

This will let us write $\mathcal{I}(t, T)$ as:

$$\begin{aligned}\mathcal{I}(t, T) &= \mathcal{I}(0, T) - \mathcal{I}(0, t) + \frac{1}{2} \frac{1}{\eta^2(t)} \mathcal{S}^2(t, T) \int_0^t \gamma^2(v) \eta^2(v) dv \\ &\quad - \frac{\mathcal{S}(t, T)}{\eta(t)} \int_0^t \alpha(v) \eta(v) - \gamma^2(v) \eta(v) \mathcal{S}(v, t) dv\end{aligned}\tag{310}$$

If g is a function of two variables, well defined in its diagonal, integrable with respect to the first and partially differentiable with respect to the second, then we know that F defined as

$$F(t) = \int_0^t g(v, t) dv\tag{311}$$

satisfies

$$\frac{dF}{dt} = g(t, t) + \int_0^t \frac{\partial}{\partial t} g(v, t) dv.\tag{312}$$

From (281) we notice that

$$\frac{\partial}{\partial T} \mathcal{S}(t, T) = \frac{\eta(t)}{\eta(T)}.\tag{313}$$

Combining the previous observations with (312), (281), and (303), we find that

$$\begin{aligned}\mathcal{I}(t, T) &= \mathcal{I}(0, T) - \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) \\ &\quad + \frac{1}{2} \mathcal{S}^2(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv.\end{aligned}\tag{314}$$

Thus, assuming $\mathcal{I}(0, t)$, $t \in [0, T]$ is known, as well as $\gamma(t)$ and $\mathcal{S}(0, t)$, $t \in [0, T]$; all terms in (314) are known from initial data¹⁶. This approach is superior to the PDE, (306), we found earlier in the sense that only first order differentiations are required.

The problem then reduces to finding $\mathcal{I}(0, t)$, $\mathcal{S}(0, t)$ and $\gamma(t)$, $t \in [0, T]$, from initial data; that is, because we know $\mathcal{S}(t, T)$, we obtain $\beta(t)$ from (288) using (304)

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{S}(t, T) &= \frac{\partial}{\partial t} \left(\frac{\mathcal{S}(0, T) - \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)} \right) \\ &= \frac{\left\{ \frac{d}{dt} (\mathcal{S}(0, T) - \mathcal{S}(0, t)) \frac{d}{dt} \mathcal{S}(0, t) - (\mathcal{S}(0, T) - \mathcal{S}(0, t)) \frac{d^2}{dt^2} \mathcal{S}(0, t) \right\}}{\left(\frac{d}{dt} \mathcal{S}(0, t) \right)^2} \\ &= -1 - \mathcal{S}(t, T) \frac{\frac{d^2}{dt^2} \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)} \\ &= \beta(t) \mathcal{S}(t, T) - 1 \end{aligned} \quad (315)$$

which implies that

$$\beta(t) = - \frac{\frac{d^2}{dt^2} \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)}. \quad (316)$$

Similarly, we can also find $\alpha(t)$ from (288), (289) and (314). From (313) it is clear that

$$\left. \frac{\partial}{\partial t} \mathcal{S}(v, t) \right|_{v=t} = 1 \quad (317)$$

and

$$\frac{\partial^2}{\partial t^2} \mathcal{S}(v, t) = -\beta(t) \frac{\partial}{\partial t} \mathcal{S}(v, t) \quad (318)$$

thus

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{I}(t, T) &= \frac{\partial}{\partial t} \left(\mathcal{I}(0, T) - \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) \right. \\ &\quad \left. + \frac{1}{2} \mathcal{S}^2(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \right) \\ &= -\frac{d}{dt} \mathcal{I}(0, t) - \frac{\partial}{\partial t} \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d^2}{dt^2} \mathcal{I}(0, t) \\ &\quad + \mathcal{S}(t, T) \frac{\partial}{\partial t} \mathcal{S}(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \end{aligned}$$

¹⁶As it is implied by (280) (see also (325) and (326)); if one knows the initial term structure, that is, the initial yield curve, one of $\mathcal{S}(0, t)$ or $\mathcal{I}(0, t)$, $t \in [0, T]$ is known. γ should be recovered from the term structure of volatilities known at time $t = 0$.

$$\begin{aligned}
& + \frac{1}{2} \mathcal{S}^2(t, T) \left[\gamma^2(t) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 \Big|_{v=t} \right. \\
& \quad \left. + 2 \int_0^t \gamma^2(v) \frac{\partial}{\partial t} \mathcal{S}(v, t) \frac{\partial^2}{\partial t^2} \mathcal{S}(v, t) dv \right] \\
& = -\{1 + \beta(t) \mathcal{S}(t, T) - 1\} \frac{d}{dt} \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d^2}{dt^2} \mathcal{I}(0, t) \\
& \quad + \mathcal{S}(t, T) \{\beta(t) \mathcal{S}(t, T) - 1\} \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
& \quad + \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) + \mathcal{S}^2(t, T) \int_0^t \gamma^2(v) [-\beta(t)] \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
& = -\beta(t) \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d^2}{dt^2} \mathcal{I}(0, t) + \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) \\
& \quad - \mathcal{S}(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv
\end{aligned} \tag{319}$$

combining the last computation with (289) we obtain

$$\begin{aligned}
\alpha(t) \mathcal{S}(t, T) & = -\frac{\partial}{\partial t} \mathcal{I}(t, T) + (1/2) \gamma^2(t) \mathcal{S}^2(t, T) \\
& = \beta(t) \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) + \mathcal{S}(t, T) \frac{d^2}{dt^2} \mathcal{I}(0, t) - \frac{1}{2} \gamma^2(t) \mathcal{S}^2(t, T) \\
& \quad + \mathcal{S}(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
& \quad + (1/2) \gamma^2(t) \mathcal{S}^2(t, T) \\
& = \mathcal{S}(t, T) \left[\beta(t) \frac{d}{dt} \mathcal{I}(0, t) + \frac{d^2}{dt^2} \mathcal{I}(0, t) + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \right]
\end{aligned} \tag{320}$$

Hence

$$\begin{aligned}
\alpha(t) & = \beta(t) \frac{d}{dt} \mathcal{I}(0, t) + \frac{d^2}{dt^2} \mathcal{I}(0, t) + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
& = -\frac{\frac{d^2}{dt^2} \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)} \frac{d}{dt} \mathcal{I}(0, t) + \frac{d^2}{dt^2} \mathcal{I}(0, t) + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv.
\end{aligned} \tag{321}$$

So, assuming we know $\mathcal{S}(0, t)$, $\mathcal{I}(0, t)$ and $\gamma(t)$, $t \in [0, T]$, then (321), (316), (304), and (314), will let us completely determine $\alpha(t)$, $\beta(t)$, $\mathcal{S}(t, T)$, and $\mathcal{I}(t, T)$, $t \in [0, T]$. If, on the other hand, we know $\mathcal{S}(0, t)$, $\mathcal{I}(0, t)$ and $\alpha(t)$, $t \in [0, T]$, we can use (321) to determine $\gamma(t)$ in the following way. From (313) and (321) we have:

$$\frac{1}{\eta^2(t)} \int_0^t \gamma^2(v) \eta^2(v) dv = \alpha(t) + \frac{\frac{d^2}{dt^2} \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)} \frac{d}{dt} \mathcal{I}(0, t) - \frac{d^2}{dt^2} \mathcal{I}(0, t), \tag{322}$$

thus

$$\begin{aligned}\gamma^2(t) &= \frac{1}{\eta^2(t)} \frac{d}{dt} \left\{ \eta^2(t) \left[\alpha(t) + \frac{\frac{d^2}{dt^2} \mathcal{S}(0,t)}{\frac{d}{dt} \mathcal{S}(0,t)} \frac{d}{dt} \mathcal{I}(0,t) - \frac{d^2}{dt^2} \mathcal{I}(0,t) \right] \right\} \\ &= \frac{1}{\eta^2(t)} \frac{d}{dt} \left\{ \eta^2(t) \left[\alpha(t) - \beta(t) \frac{d}{dt} \mathcal{I}(0,t) - \frac{d^2}{dt^2} \mathcal{I}(0,t) \right] \right\} ;\end{aligned}\tag{323}$$

hence, we can write

$$\begin{aligned}\gamma^2(t) &= 2\beta(t) \left\{ \alpha(t) - \beta(t) \frac{d}{dt} \mathcal{I}(0,t) - \frac{d^2}{dt^2} \mathcal{I}(0,t) \right\} \\ &\quad + \frac{d}{dt} \left\{ \alpha(t) - \beta(t) \frac{d}{dt} \mathcal{I}(0,t) - \frac{d^2}{dt^2} \mathcal{I}(0,t) \right\}\end{aligned}\tag{324}$$

((316) can be used to obtain $\beta(t)$ from $\mathcal{S}(0,t)$).

We collect all our findings in Table 15.

Now we need to find a simple way to recover $\mathcal{S}(0,t)$, $\mathcal{I}(0,t)$ and $\gamma(t)$, (or $\mathcal{S}(0,t)$, $\mathcal{I}(0,t)$ and $\alpha(t)$ ¹⁷) $t \in [0, \mathcal{T}]$, from initial data.

From (280) we have

$$B(0,t) = \exp(-r_0 \mathcal{S}(0,t) - \mathcal{I}(0,t))\tag{325}$$

which implies that

$$\mathcal{I}(0,t) = -r_0 \mathcal{S}(0,t) - \ln[B(0,t)]\tag{326}$$

this means that we can obtain $\mathcal{I}(0,t)$, $t \in [0, \mathcal{T}]$ from (326)¹⁸ if we know $\mathcal{S}(0,t)$, and the initial term structure¹⁹.

As we saw in Chapter 2, §2.2, the initial term structure can be recovered from Swaps market data. In fact, the procedure outlined in Chapter 2, §2.2 gives us also the initial zcb price curve.

¹⁷We have studied the calibration problem under the implicit assumption that $\gamma(t)$, as well as $\mathcal{S}(0,t)$ and $\mathcal{I}(0,t)$, could be determined from additional data. This assumption of course is not the only one available to us. One could otherwise assume that $\alpha(t)$, and $\mathcal{S}(0,t)$ and $\mathcal{I}(0,t)$, could be determined from additional data, and use (324) to determine $\gamma(t)$.

¹⁸Here we are not using anything else but the fact that the Hull–White model is affine. Thus this is a valid calibration step for any affine method.

¹⁹From the initial term structure, the initial zcb price curve can be derived.

Assuming that $B(0, t)$ is smooth enough we can combine (321) with (314) and (316) to obtain an expression for $\alpha(\cdot)$ that does not explicitly contains $\mathcal{I}(0, t)$.

$$\begin{aligned}
\alpha(t) &= \beta(t) \left\{ -r_0 \frac{d}{dt} \mathcal{S}(0, t) - \frac{d}{dt} \{\ln[B(0, t)]\} \right\} \\
&\quad - r_0 \frac{d^2}{dt^2} \mathcal{S}(0, t) - \frac{d^2}{dt^2} \{\ln[B(0, t)]\} + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
&= \frac{\frac{d^2}{dt^2} \mathcal{S}(0, t)}{\frac{d}{dt} \mathcal{S}(0, t)} \frac{d}{dt} \{\ln[B(0, t)]\} - \frac{d^2}{dt^2} \{\ln[B(0, t)]\} + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv \\
&= -\beta(t) \frac{d}{dt} \{\ln[B(0, t)]\} - \frac{d^2}{dt^2} \{\ln[B(0, t)]\} + \int_0^t \gamma^2(v) \left\{ \frac{\eta(v)}{\eta(t)} \right\}^2 dv
\end{aligned} \tag{327}$$

The previous expressions may come in handy if we have the means to obtain $\beta(t)$ and $\gamma(t)$ by alternate methods that do not involve the knowledge of $\alpha(t)$; or to find $\alpha(t)$ directly from the initial zcb price curve and $\gamma(t)$ and $\mathcal{S}(0, t)$, which we may have found by alternate methods.

Therefore, our problem has been reduced to finding a way to obtain $\mathcal{S}(0, t)$, and $\gamma(t)$ from market data. One look at (295) will give us the hint we need. Since $\gamma(t)$ and $\mathcal{S}(t, T)$ both show up in the expression for bond volatility, it does make sense to look for a market contract (or contracts) that depends on bond volatility, collect such data and use it to find our two unknowns.

Notice that this problem (the need of more data to figure out $\mathcal{S}(0, t)$ and $\gamma(t)$) arises due to the fact that the general Hull–White model, (255), over-specifies the short rate by introducing three deterministic parameters; while based on (326) one may argue that only one unknown functional parameter ($\alpha(\cdot)$, for example) should be enough for that effect. This feature of the Hull–White model can be seen as an advantage (it lets you put more information into your model), and also as a problem since in practical applications volatility data has to be carefully selected together with “realistic” (and that is very subjective) forms of volatility functions and/or volatility surfaces.

On the other hand, market data associated with the initial term structure and/or the initial term structure of volatilities could be not rich enough (for example, one of our implicit assumptions is that the initial yield curve is differentiable, or equivalently, that the zcb price curve is differentiable, but in practice one only has a finite number of points on that curve; a

similar problem is faced when volatility data is needed). In the general case of a calibration to market data, one is usually forced to introduce additional assumptions regarding the structure of volatilities, and/or the analytical form of γ and \mathcal{S} .

In order to acquire bond volatility data, one requires a bond volatility dependent contract. The natural choice of a bond volatility dependent contract is a bond option, for example a call on a bond.

At least from the theoretical standpoint, the pricing of an European Call on a zcb under Gaussian models is well known, but European Calls on zcb's are not common in the market. Thus one must look for different contracts. Such contracts could be, for example, Caps and/or Swaptions.

Caps can be seen as portfolios of European Calls on zcb's, in this case called caplets. The valuation of Caps is still a topic of research, but in the case of markets under Gaussian Models pricing formulas are known. This topic is very well presented by Musiela and Rutkowski, [133] Chapter 16 and Bielecki and Rutkowski, [12] Chapter 15. Swaptions, on the other hand, are options on swaps. These are also very well presented by Musiela and Rutkowski, [133] Chapter 16 and Bielecki and Rutkowski, [12] Chapter 15.

Due to our particular needs and in order to avoid unnecessary and additional complications, we will restrict our calibration efforts to the use of Caps data. In the next section we will give a brief introduction to Caps and Floors and their valuation; but first we will study in detail a few particular cases of the Hull–White model.

Although we have found many useful formulas in this chapter, more manipulations may be needed once particular analytical forms of $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are selected. On the other hand, if the particular model selected is simple enough (please see the first example in the next subsection), the calibration may be accomplished using just a few of the formulas presented here.

4.2 A few calibration examples

Before we continue with our study, we will see how is all this related to a few concrete examples, all of those sub-cases of (255).

4.2.1 All parameters constant

If all parameters α , β , and γ in (255) are constant, the Hull-White model (255) reduces to the Vasicek model²⁰ according to which interest rates are modeled by means of a, shifted, Ornstein–Uhlenbeck process. Let $\alpha(t) \equiv a_1$, $\beta(t) \equiv a_2$ and $\gamma(t) \equiv a_3$, we can rewrite (255) as

$$dr_t = (a_1 - a_2 r_t) dt + a_3 dW_t, \quad (328)$$

where a_1 , a_2 and a_3 are unknown constants that we must determine using initial data.

Clearly, since all parameters are now constant all our formulas will simplify noticeably. In particular, all important quantities ((261), (263), (281) and (282), for example) can be solved for explicitly in terms of the model's parameters. Without much ado, carrying over all the required substitutions and simplifications we obtain:

$$\eta(t) = \exp \left(\int_0^t \beta(u) du \right) = \exp(a_2 t). \quad (329)$$

$$\begin{aligned} r_t &= (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du + \int_0^t \eta(u) \gamma(u) dW_u \right\} \\ &= \exp(-a_2 t) \left\{ r_0 + \frac{a_1}{a_2} (\exp(a_2 t) - 1) + a_3 \int_0^t \exp(a_2 u) dW_u \right\} \\ &= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \frac{a_1}{a_2} (e^{a_2 t} - e^{a_2 s}) + a_3 \int_s^t e^{a_2 u} dW_u \right\} \quad s \leq t. \end{aligned} \quad (330)$$

$$\begin{aligned} m_t &= E(r_t) = (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du \right\} \\ &= e^{-a_2 t} \left\{ r_0 + \frac{a_1}{a_2} (e^{a_2 t} - 1) \right\}. \end{aligned} \quad (331)$$

We can see from (331) and (332) that the long term mean of the interest rate process is a_1/a_2 .

$$\begin{aligned} E(r_t | \mathcal{F}_s) &= (1/\eta(t)) \left\{ \eta(s) r_s + \int_s^t \eta(u) \alpha(u) du \right\} \quad s \leq t \\ &= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \frac{a_1}{a_2} (e^{a_2 t} - e^{a_2 s}) \right\} \quad s \leq t. \end{aligned} \quad (332)$$

²⁰The Vasicek Model of interest rates was introduced by Oldrich Vasicek, [174] in 1977. The Vasicek Model was the first interest rate model to exhibit mean reversion. Traditionally, the Vasicek model is represented by an sde of the form

$$dr_t = a(b - r_t) dt + c dW_t$$

where a is the speed of mean reversion, b is the long term mean and c is the volatility of the model.

$$\begin{aligned}\mathcal{V}_t &= \text{Var}(r_t) = \frac{1}{\eta^2(t)} \int_0^t \eta^2(u) \gamma^2(u) du \\ &= \frac{a_3^2}{2a_2} (1 - e^{-2a_2 t}).\end{aligned}\tag{333}$$

Similarly, we see from (333) that the interest rate process will exhibit a long term variance equal to $a_3^2/(2a_2)$.

$$\begin{aligned}\text{Var}(r_t | \mathcal{F}_s) &= \frac{1}{\eta^2(t)} \int_s^t \eta^2(u) \gamma^2(u) du \quad s \leq t \\ &= \frac{a_3^2}{2a_2} (1 - e^{-2a_2(t-s)}).\end{aligned}\tag{334}$$

$$\begin{aligned}\mathcal{S}(t, T) &= \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \\ &= \frac{1}{a_2} (1 - e^{-a_2(T-t)}).\end{aligned}\tag{335}$$

$$\begin{aligned}\mathcal{I}(t, T) &= \int_t^T \frac{1}{\eta(u)} \int_t^u \eta(v) \alpha(v) dv du \\ &\quad - (1/2) \int_t^T \eta^2(v) \gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \quad t \in [0, T] \\ &= \frac{3a_3^2 - 4a_1a_2}{4a_2^3} + \frac{a_3^2 - 2a_1a_2}{2a_2^2} (t - T) \\ &\quad + \frac{a_1a_2 - a_3^2}{a_2^3} e^{a_2(t-T)} + \frac{a_3^2}{4a_2^3} e^{2a_2(t-T)} \quad t \in [0, T].\end{aligned}\tag{336}$$

Since both (335) and (336) can be solved explicitly in terms of the model's parameters, we do not need to worry about how to find functions of the model's parameters depending only on initial data. Instead we can use (335) and (336) in combination with (326) to obtain a_1 , a_2 , a_3 and r_0 from initial (yield curve) data. From (335) and (336) we have

$$\mathcal{S}(0, t) = \frac{1}{a_2} (1 - e^{-a_2 t}) \quad t \in [0, T], \tag{337}$$

and

$$\mathcal{I}(0, t) = \frac{3a_3^2 - 4a_1a_2}{4a_2^3} - \frac{a_3^2 - 2a_1a_2}{2a_2^2} t + \frac{a_1a_2 - a_3^2}{a_2^3} e^{-a_2 t} + \frac{a_3^2}{4a_2^3} e^{-2a_2 t} \quad t \in [0, T], \tag{338}$$

which we can now combine with (326) to write:

$$\begin{aligned}-\ln[B(0, t)] &= \mathcal{I}(0, t) + r_0 \mathcal{S}(0, t) \\ &= \frac{r_0}{a_2} + \frac{3a_3^2}{4a_2^3} - \frac{a_1}{a_2^2} + \left(\frac{a_1}{a_2} - \frac{a_3^2}{2a_2^2} \right) t \\ &\quad + \left(-\frac{r_0}{a_2} + \frac{a_1}{a_2^2} - \frac{a_3^2}{a_2^3} \right) e^{-a_2 t} + \frac{a_3^2}{4a_2^3} e^{-2a_2 t} \quad t \in [0, T].\end{aligned}\tag{339}$$

As we can easily see, (339) relates all parameters in Vasicek's Model, namely a_1 , a_2 , a_3 and r_0 , to initial data (in this particular case represented by $-\ln[B(0, t)]$).

In Chapter 2 we explained how to obtain a yield curve and a zcb's initial price curve from swap rate data (please see §2.2). The procedure presented in Chapter 2 returns 24 zcb prices for 24 different maturities (as well as 24 corresponding rates), see Table 4. That data can be used to approximate values for a_1 , a_2 , a_3 and r_0 by means of a least squares regression.

Apart from the apparent complexity and size of expression (339), one of the obvious problems in finding a_1 , a_2 , a_3 and r_0 comes from the fact that (339) is not linear in those parameters. Several procedures have been developed to deal with this kind of problem (see [110], [119], [34], [38], [13], [135], and [146]; more sources are mentioned in the bibliography), amongst them we find the Newton–Gauss methods.

The Taylor–Newton–Gauss least squares method is a numeric iterative method used to fit non linear expressions to data. The method uses a truncated Taylor series (truncated after the first order) to “linearize” the equation to be fitted (in our case equation (339)). Assuming that a fairly good initial approximation to the parameters is known, the method iterates to find a good approximation to the parameters.

We used Taylor–Newton–Gauss least squares to find the parameters a_1 , a_2 , a_3 and r_0 that better (within a given tolerance) fitted (339) to the May 12th 2003 implied yield curve we obtained from swaps market data in Chapter 2. Figure 8 shows the resulting yield curve computed using Vasicek's model with the parameters we found applying the Taylor–Newton–Gauss least squares method²¹ to (339) and the data on Table 4.

²¹A reasonably good guess (namely $a_1 \sim 0.1$, $a_2 \sim 0.1$, $a_3 \sim 0.1$ and $r_0 \sim 0.01$) was found by trial and error; then we ran 7 and 100 iterations (no significant improvement was obtained by increasing the number of iterations). We also tried the Levenberg–Marquardt ([110] and [119]) method which did not provide significant improvements in our approximations. Full descriptions of these methods can be found in the bibliography, in particular the reader may consider [13], and Björck's section in [27].

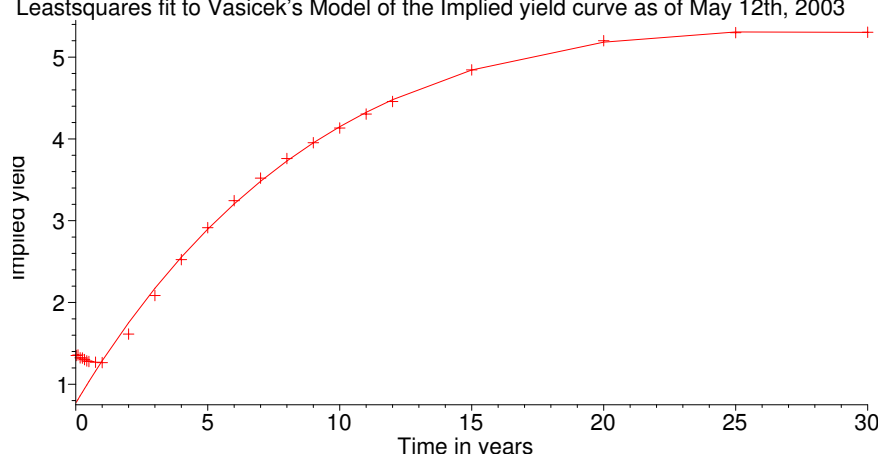


Figure 8: This plot shows the yield curve data (“+”s in the plot) extracted from Swap rate data as on May 12 2003 as well as a curve obtained as the least squares fit of equation (339) (which in turn is obtained from the Vasicek’s Model and contains all parameters of the Vasicek model) to the data from Table 4. The numerical least squares method used to obtain the parameters of the Vasicek Model is known as the Taylor–Newton–Gauss method. The Taylor–Newton–Gauss method uses truncated Taylor series and Newton differentiation to construct an approximate solution to a least squares problem that is nonlinear in its parameters. As we mentioned before, May 12 2003 data was used (see Table 4). May 12 2003 was used as the reference date, and time is counted from that date on (thus $t = 1$ corresponds to Wed. May 12 2004, etc.).

As a result of our computations, after only 7 iterations, we found the parameters:

$$\begin{aligned} a_1 &\simeq 0.01112394960, & a_2 &\simeq 0.05400807283, \\ a_3 &\simeq 0.03269603106, & r_0 &\simeq 0.007695335648, \end{aligned} \tag{340}$$

rounded to six significant figures, the sum of square residuals is 0.0000693540, the curve fitting variance is 0.00000299181, and the correlation is 0.999992, the root mean square error²² (rms) is 0.00170060 and the rms percentage average response is 0.550607. As Figure

²²If we denote by y_i^{new} the i^{th} approximate value (found after the evaluation of the fitted equation, say $f(t; \vec{a})$ —where \vec{a} is an n -dimensional vector of parameters—on the i^{th} time t_i) and denote by y_i the original i^{th} value (from data), then we can define $\delta_i = y_i^{new} - y_i$, the i^{th} residual. We can define the following useful terms that can be used to judge how good the fit is:

- *Sum of square residuals:* $\sum_{i=1}^N \delta_i^2$. With least squares one wants to find the vector of parameters \vec{a} that minimizes $m(\vec{a}) = \sum_{i=1}^N (f(t_i; \vec{a}) - y_i)^2$; thus, for a good fit one expects to get a very small sum of square residuals.

9 shows, zcb prices obtained from the Vasicek model are very close to our data, still we can see in Figure 8 that the fitting of Table 4 to equation (339) is far from perfect. The fitted curve gets close to the mid and long maturity rates, but fails to follow the form of Figure 1 at short maturities.

We must remember that this procedure attempts to obtain a calibration to the initial yield curve; that is, no volatility information is explicitly used. Still, this calibration will imply a bond volatility in accordance with (337) and (295).

In our first numeric experiment we did nothing to ensure positive approximations to the coefficients. To ensure non negative values of r_0 , a_1 , etc. we changed r_0 in (339) by b_4^2 , a_1 by b_1^2 , a_2 by b_2^2 and a_3 by b_3^2 to obtain

$$\begin{aligned} -\ln[B(0, t)] &= \mathcal{I}(0, t) + b_4^2 \mathcal{S}(0, t) \\ &= \frac{b_4^2}{b_2^2} + \frac{3b_3^4}{4b_2^6} - \frac{b_1^2}{b_2^4} + \left(\frac{b_1^2}{b_2^2} - \frac{b_3^4}{2b_2^4} \right) t \\ &\quad + \left(-\frac{b_4^2}{b_2^2} + \frac{b_1^2}{b_2^4} - \frac{b_3^4}{b_2^6} \right) e^{-b_2^2 t} + \frac{b_3^4}{4b_2^6} e^{-2b_2^2 t} \quad t \in [0, T]. \end{aligned} \quad (341)$$

Then we tried again to fit the initial data from Table 4 to the Vasicek model, this time

-
- *Root mean square (rms) error:* $\sqrt{\frac{1}{N} \sum_{i=1}^N \delta_i^2}$. This number is closely related to the sum of square residuals and measures the average vertical distance between the data and the corresponding values on the fitted curve. Again, we will like to have this small for a good fit.
 - *Curve fitting variance:* $\frac{1}{N-1} \sum_{i=1}^N (\delta_i - \bar{\delta})^2$, where $\bar{\delta}$ represents the mean of the residuals $\bar{\delta} = \frac{1}{N} \sum_{i=1}^N \delta_i$.
 - *rms percentage average response:* $100 * rms / \bar{y}^{new}$. If all involved numbers are small (smaller than one as is our case), their differences will also be small. Therefore the value of the sum of square residuals or the rms may be deceiving, thus we will like to measure such errors relative to the sizes of the y 's. Again, in a good fit we will like to have a small rms percentage average response.
 - *Correlation:* $\frac{\text{Cov}(y, y^{new})}{\sqrt{\text{Var}(y) \text{Var}(y^{new})}}$, where $\text{Var}(y) = \frac{1}{N-1} \sum_{i=1}^N (y_i - \bar{y})^2$ and $\text{Cov}(y, y^{new}) = \frac{1}{N-1} \sum_{i=1}^N (y_i - \bar{y})(y_i^{new} - \bar{y}^{new})$.

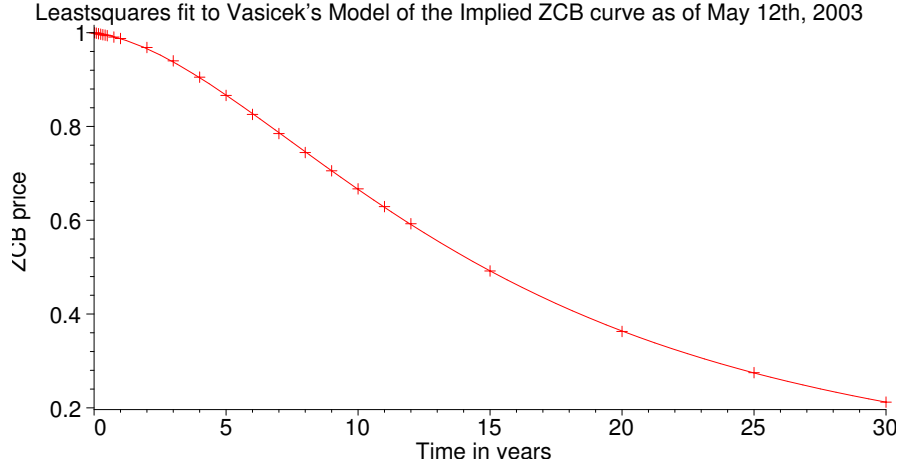


Figure 9: This plot shows the implied ZCB curve data extracted from Swap rate data as on May 12 2003 as well as the ZCB curve obtained as the least squares fit of the Vasicek's Model to that data. A quick comparison with Figure 9 shows that although the fit to the yield curve is not that perfect, ZCB prices from Vasicek's model are very close to those obtained from the implied yield curve.

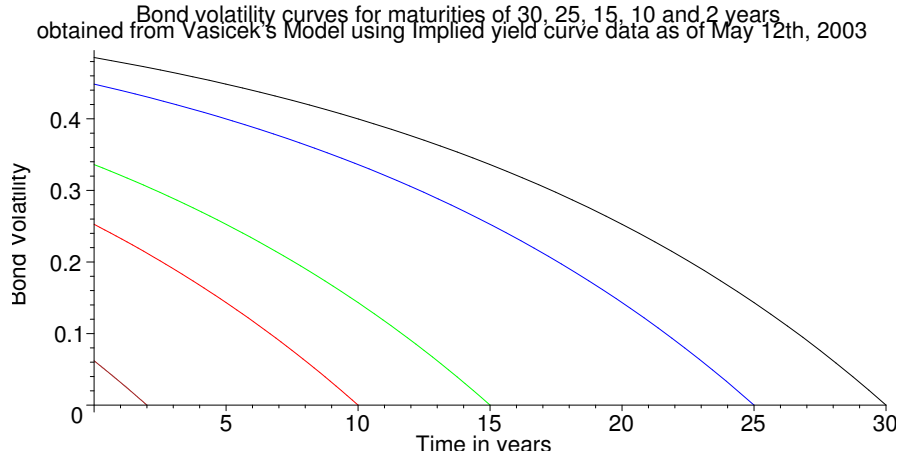


Figure 10: This plot shows some bond volatility curves (for maturities of 30, 25, 15, 10 and 2 years), see (295) and (335), obtained from our least squares fit of Vasicek's model to the implied yield curve data extracted from Swap rate data as on May 12 2003, Table 4. It is apparent that although our approximation to the yield curve is reasonably good, the resulting volatility curves may not be realistic. Clearly a more complex model is required if one wants to obtain not only a good fit to the initial yield curve but also a good approximation to bond volatilities. Notice that we are plotting the negative of the volatility curve (from (295) and (335) we see that bond volatility should be negative since the affine bond slope function and the Hull–White volatility parameter are positive). Compare with Figure 44 which depicts implied bond volatility curves obtained after a perfect fit of a Hull–White model to initial spot volatility using the methods presented in §4.5.

using equation (341); again, after 7 iterations²³ we obtained:

$$\begin{aligned} a_1 &\simeq 0.01112394997, & a_2 &\simeq 0.05400807604, \\ a_3 &\simeq 0.03269603203, & r_0 &\simeq 0.00769533529, \end{aligned} \tag{342}$$

rounded to six significant figures, the correlation is 0.999992, sum of square residuals is 0.0000694865, the rms is 0.00170046, the rms percent average response is 0.550539 and the curve fitting variance is 0.00000299181. By themselves those numbers do not say too much, but a quick comparison with the corresponding measurements for the previous fit shows us that we did not obtained an improved model, in average, results obtained in this second numerical experiment differ from those displayed in (340) after the seventh significant figure. This second approach ensures non-negative values for r_0 , a_1 , a_2 and a_3 but, due to the very small difference between the results shown in (340) and (342) the resulting curves are almost indistinguishable. We collect the results of our numerical experiment in Figure 11, Figure 12, and Figure 13.

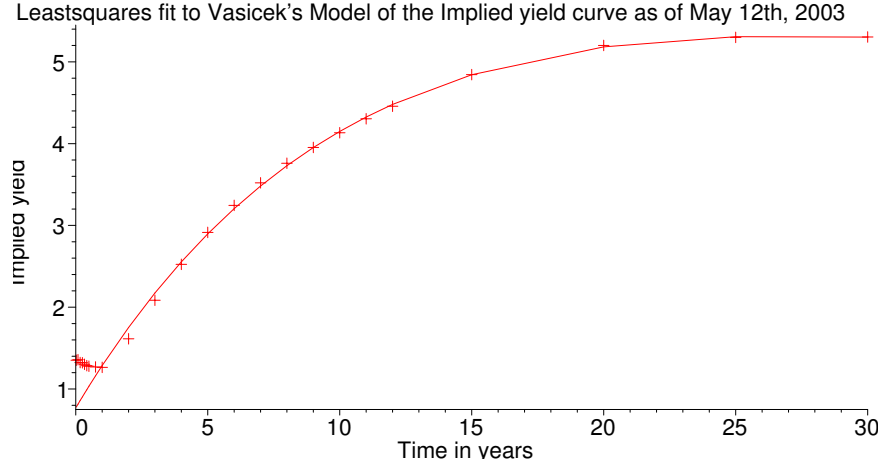


Figure 11: Compared with Figure 8 we see that forcing r_0 , a_1 , a_2 and a_3 to be positive offers no particular advantage. In our previous numerical experiment the yield curve (see Figure 8) was always positive and followed (in a non so loose a way) the shape of Table 4 data. This plot shows a curve with the same characteristics.

²³As before, a reasonably good guess (namely $b_1 \sim 0.1$, $b_2 \sim 0.2$, $b_3 \sim 0.15$ and $\sim b_4 \sim 0.1$) was found by trial and error; then we ran 7, 50 and 100, etc. iterations and compared our results. After 8 iterations there was no significant improvement in our approximations.

We must call the reader's attention to the fact that the model considered in this example (see (328)) contains only a few parameters. Even if we obtain a “close” fit, some features of the data may not be recoverable (see for example Figure 8 and Figure 11, pay special attention to short maturities), even more; there may be yield curve shapes that can not be approximated in any satisfactory manner. This comment will remain valid for most of the examples shown in this chapter.

Also of importance is the following observation, the calibration consider here uses no information regarding volatility data. This means that volatility related results implied by the calibrated model could not be realistic at all. One should be cautious when making inferences regarding contracts that may be tied to interest rate volatility or equivalently to Bond volatility. Figures Figure 10 and Figure 13 show bond volatility implied by our calibration of (328) to the data in Table 4.

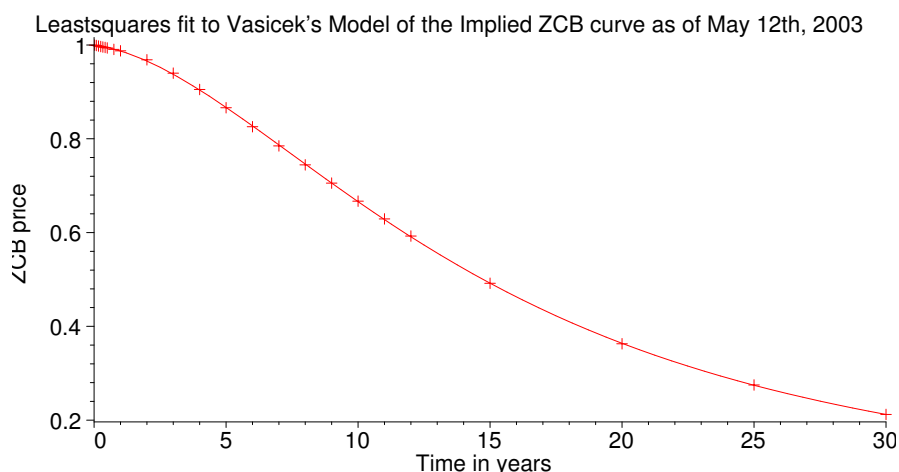


Figure 12: This plot shows the implied zcb curve obtained after fitting the Vasicek Model to data from Table 4, to ensure a non negative parameters (341) was used instead of (339). As in Figure 11 we observe that forcing the parameters to be positive offers no noticeable advantage. The error in the fit is of roughly the same order as in the previous numerical experiment (see (342) and comments afterward).

Bond volatility curves for maturities of 30, 25, 15, 10 and 2 years, obtained from Vasicek's Model using Implied yield curve data as of May 12th, 2003

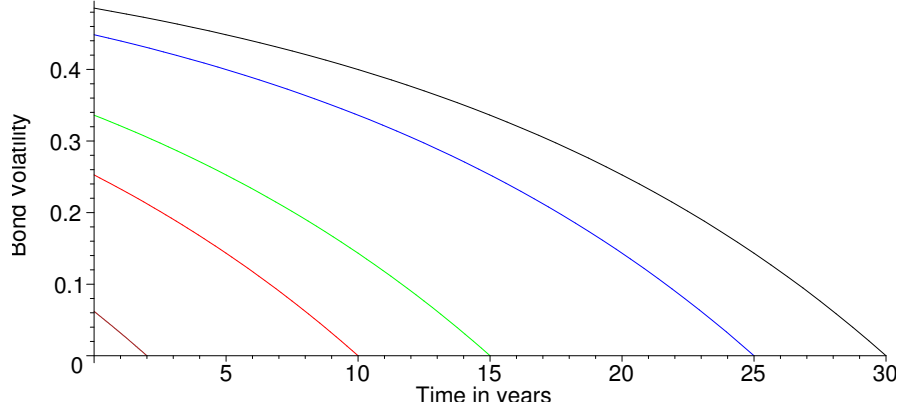


Figure 13: Bond volatility curves (for maturities of 2, 10, 15, 20, 25, and 30 years) obtained from our fit to the Vasicek model using (341) are shown in this figure. The reader can compare with Figure 10. Notice that we are plotting the negative of the volatility curve (from (295) and (335) we see that bond volatility should be negative since the affine bond slope function and the Hull–White volatility parameter are positive).

4.2.2 Constant β and γ

When α is a function of time and β and γ are constant one obtains what we call the classic Hull–White model. Let $\alpha = \alpha(t)$, $\beta(t) = a_2$ and $\gamma(t) = a_3$, we can rewrite (255) as

$$dr_t = (\alpha(t) - a_2 r_t) dt + a_3 dW_t, \quad (343)$$

(343) is also known as the extended Vasicek model.

(343) offers more generality than (328) since the parameter $\alpha(t)$ will let us fit exactly the initially observed yield curve. Still, some of our results in this case are like those found in the previous example. In particular, we obtain the same expressions for $\eta(t)$, \mathbf{V}_t and $\mathcal{S}(t, T)$ and consequently the same expression for bond volatility.

$$\eta(t) = \exp \left(\int_0^t \beta(u) du \right) = \exp(a_2 t). \quad (344)$$

$$\begin{aligned} r_t &= (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du + \int_0^t \eta(u) \gamma(u) dW_u \right\} \\ &= \exp(-a_2 t) \left\{ r_0 + \int_0^t \exp(a_2 u) \alpha(u) du + a_3 \int_0^t \exp(a_2 u) dW_u \right\} \\ &= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \int_s^t e^{a_2 u} \alpha(u) du + a_3 \int_s^t e^{a_2 u} dW_u \right\} \quad s \leq t. \end{aligned} \quad (345)$$

$$\begin{aligned}
\mathbf{m}_t = E(r_t) &= (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du \right\} \\
&= e^{-a_2 t} \left\{ r_0 + \int_0^t e^{a_2 u} \alpha(u) du \right\}.
\end{aligned} \tag{346}$$

$$\begin{aligned}
E(r_t | \mathcal{F}_s) &= (1/\eta(t)) \left\{ \eta(s) r_s + \int_s^t \eta(u) \alpha(u) du \right\} \quad s \leq t \\
&= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \int_s^t e^{a_2 u} \alpha(u) du \right\} \quad s \leq t.
\end{aligned} \tag{347}$$

$$\begin{aligned}
\mathbf{v}_t = \text{Var}(r_t) &= \frac{1}{\eta^2(t)} \int_0^t \eta^2(u) \gamma^2(u) du \\
&= \frac{a_3^2}{2a_2} (1 - e^{-2a_2 t}).
\end{aligned} \tag{348}$$

$$\begin{aligned}
\text{Var}(r_t | \mathcal{F}_s) &= \frac{1}{\eta^2(t)} \int_s^t \eta^2(u) \gamma^2(u) du \quad s \leq t \\
&= \frac{a_3^2}{2a_2} (1 - e^{-2a_2(t-s)}).
\end{aligned} \tag{349}$$

$$\begin{aligned}
\mathcal{S}(t, T) &= \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \\
&= \frac{1}{a_2} (1 - e^{-a_2(T-t)}).
\end{aligned} \tag{350}$$

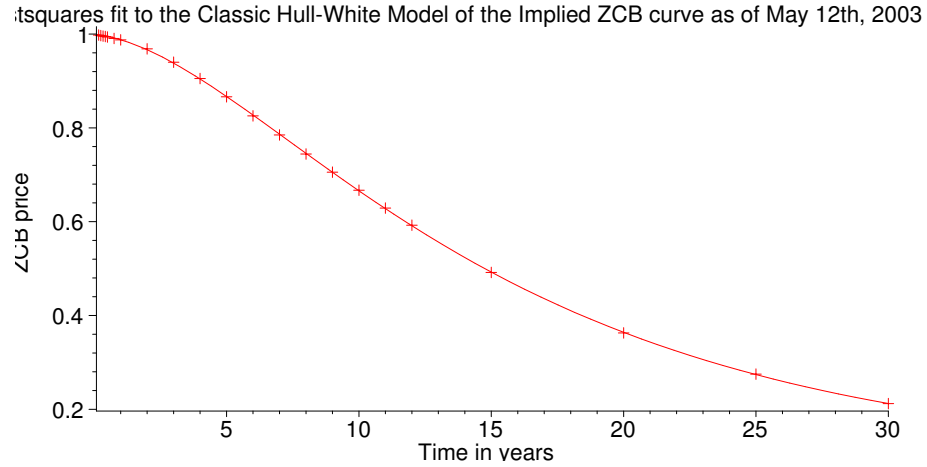


Figure 14: This plot shows the implied ZCB curve data extracted from Swap rate data as on May 12 2003 as well as the ZCB curve obtained as the least squares fit of the Classic Hull–White Model, (343), to that data assuming a mean reversion parameter of polynomial form (quadratic in this example).

As in the previous example, \mathcal{S} can be solved explicitly in terms of a_2 . Thus, we may express \mathcal{I} using (350) and (314) which describes \mathcal{I} in terms of $\mathcal{I}(0, t)$ and \mathcal{S} . Or, we may

use (287) instead and rely in our ability to express α in terms of initial data.

$$\begin{aligned}
\mathcal{I}(t, T) &= \int_t^T \alpha(v) \mathcal{S}(v, T) - (1/2) \gamma^2(v) \mathcal{S}^2(v, T) dv \quad t \in [0, T] \\
&= \frac{1}{a_2} \int_t^T \alpha(v) \{1 - \exp(-a_2(T - v))\} dv \\
&\quad + \frac{3a_3^2}{4a_2^3} - \frac{a_3^2}{2a_2^2}(T - t) - \frac{a_3^2}{a_2^3} e^{-a_2(T-t)} + \frac{a_3^2}{4a_2^3} e^{-2a_2(T-t)} \\
&\quad t \in [0, T].
\end{aligned} \tag{351}$$

Clearly, we still need to find the parameters of the model: r_0 , $\alpha(t)$, a_2 and a_3 .

Assuming a_2 and a_3 are known and that the initial data (in this case $B_{Market}(0, t)$) is smooth enough, we can use (327) to express $\alpha(t)$ in terms of initial data, thus fitting exactly the model to initial data (the initial zcb price curve):

$$\begin{aligned}
\alpha(t) &= -\beta(t) \frac{d}{dt} \{\ln[B(0, t)]\} - \frac{d^2}{dt^2} \{\ln[B(0, t)]\} + \int_0^t \gamma^2(v) \left\{ \frac{\eta(v)}{\eta(t)} \right\}^2 dv \\
&= -a_2 \frac{d}{dt} \{\ln[B(0, t)]\} - \frac{d^2}{dt^2} \{\ln[B(0, t)]\} + \frac{a_3^2}{2a_2} \{1 - \exp(-2a_2 t)\}.
\end{aligned} \tag{352}$$

To actually fit the model to initial data, one could use Table 4 to obtain a cubic or fifth order spline interpolating curve²⁴, we may call that curve $B_{Market}(0, t)$, then we will use $B_{Market}(0, t)$ instead of the theoretical curve $B(0, t)$ in (352). This procedure assumes that we can find the values of r_0 , a_2 and a_3 by other means.

Thus, the only problem remaining is to determine the values of r_0 , a_2 and a_3 . If additional initial data is available (bond volatility data or Caps volatility data, for example), we could use the results of the next section to build a least squares regression from which those parameters will be obtained. Then $\alpha(t)$ will be completely determined by (352), thus solving the calibration problem.

²⁴Market practice is to linearly interpolate the data, but such practice introduces problems with respect to the required second order differentiability of the zcb price curve. A fifth order spline curve will reproduce closely some important features –concavity in particular– of the theoretical yield curve at long maturities, but it will require much more computing power than a cubic spline interpolation, a fifth order spline will also require many more internal parameters to be updated every time the model is calibrated. Our numerical experiments show that at short to mid maturities both cubic and fifth order splines give similar results. In both cases an analytical form of (352) can be constructed.

On the other hand, the values of r_0 , a_2 and a_3 could be determined by other means. For example we may know that $r_0 = 0.0125$, $a_2 = 0.8$ and $a_3 = 0.013$ (according to our previous calibration of the Vasicek model, those values seem to be reasonable choices to be used in this example). Then, by (352) we will obtain a perfect fit to the initial yield curve. Figure 15 shows a plot of $\alpha(t)$ found using the data of Table 4 and the values $r_0 = 0.0125$, $a_2 = 0.8$ and $a_3 = 0.013$.

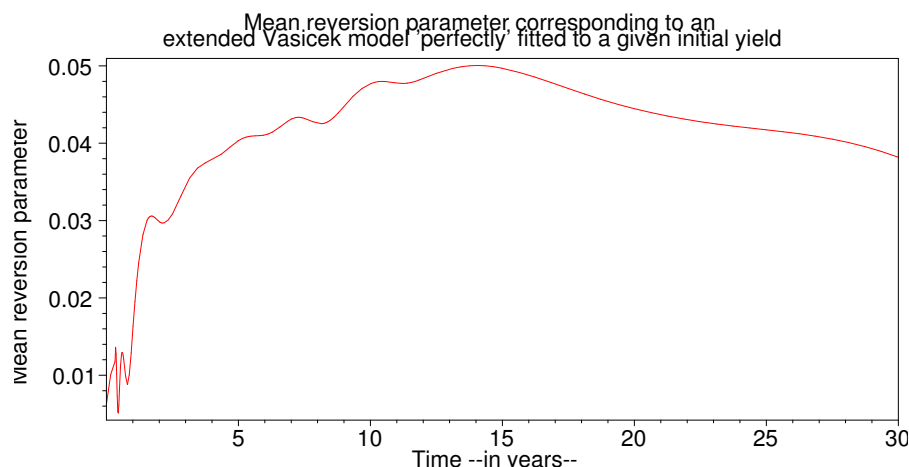


Figure 15: We present here the plot of mean reversion parameter function found using (352), the data in Table 4 and the values $r_0 = 0.0125$, $a_2 = 0.8$ and $a_3 = 0.013$. Using initial data, a smooth enough ZCB price curve was constructed (we used quintic splines), then such a curve was used in combination with (352). This function will allow us to obtain a perfect fit to initial data using the model (343).

If one is offered the choice, to deduce the values of r_0 , a_2 and a_3 from volatility related data should be preferred, otherwise, as we commented at the end of the preceding example, extreme caution should be used if one wants to use the calibrated model to infer about contracts that may be related to interest rate volatility.

If no more initial data is available, or if for some reason to assign values to r_0 , a_2 and a_3 is not viable (either by using an alternate method or by any other criteria), we will require to play a little more with formulas in order to achieve a calibration to initial yield data.

A way out of this is to assume a particular analytical form for $\alpha(t)$, for example that $\alpha(t)$ is a polynomial of a known degree, $\alpha(t) = p_0 + p_1t + p_2t^2 + \dots + p_Nt^N$. Of course we can still use (326) to write an expression for $\mathcal{I}(0, t)$ in terms of initial data. Combining

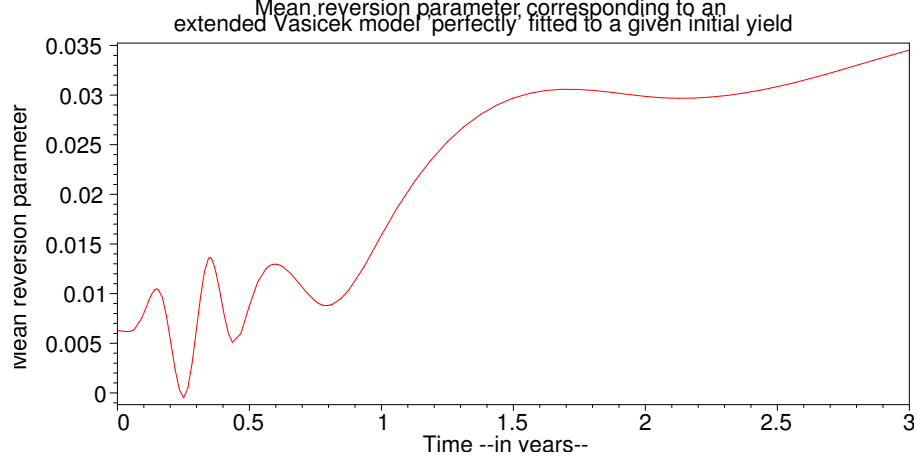


Figure 16: This plots shows a detail of Figure 15 for maturities shorter than three years..

(351), (350) and (326) we obtain:

$$\begin{aligned}
 \mathcal{I}(0, t) &= \frac{1}{a_2} \int_0^t \alpha(v) \{1 - \exp(-a_2(t - v))\} dv \\
 &\quad + \frac{3a_3^2}{4a_2^3} - \frac{a_3^2}{2a_2^2}t - \frac{a_3^2}{a_2^3}e^{-a_2t} + \frac{a_3^2}{4a_2^3}e^{-2a_2t} \\
 &= \frac{r_0}{a_2} \{1 - e^{-a_2t}\} - \ln[B(0, t)] \quad t \in [0, \mathcal{T}].
 \end{aligned} \tag{353}$$

Then we may plug the analytical form we have selected for $\alpha(t)$, $\alpha(t) = p_0 + p_1t + p_2t^2 + \dots + p_Nt^N$ for example, into the integral of (353), and formally solve that integral. As in our previous example, the resulting formula (353) + $\alpha(t) = p_0 + p_1t + p_2t^2 + \dots + p_Nt^N$ involves all the parameters of our model and can be used to obtain, by least squares, a collection of values for such parameters.

For example, if we adopt the analytical form

$$\alpha(t) = p_0 + p_1t + p_2t^2, \tag{354}$$

the integral in (353) will be

$$\begin{aligned}
 -\frac{p_0}{a_2} + \frac{p_1}{a_2^2} - 2\frac{p_2}{a_2^3} + \left(p_0 - \frac{p_1}{a_2} + 2\frac{p_2}{a_2^2}\right)t + \left(\frac{1}{2}p_1 - \frac{p_2}{a_2}\right)t^2 + \frac{1}{3}p_2t^3 \\
 + \left(\frac{p_0}{a_2} - \frac{p_1}{a_2^2} + 2\frac{p_2}{a_2^3}\right)\exp(-a_2t); \tag{355}
 \end{aligned}$$

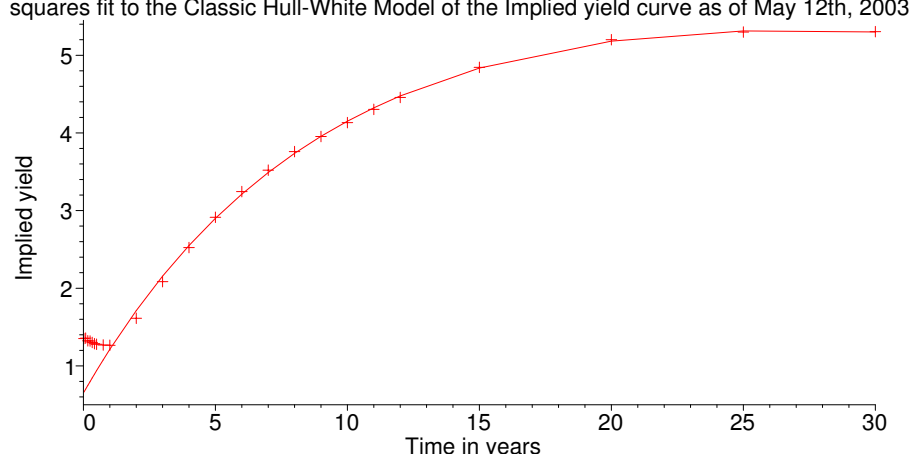


Figure 17: This plot shows the implied yield curve data extracted from Swap rate data as on May 12 2003 as well as the yield curve obtained as the least squares fit of the Classic Hull–White Model, (343), to that data assuming a mean reversion parameter of polynomial form (quadratic in this example). To obtain the model parameters we applied the Levenberg–Marquardt method to (356) and the data from Table 4.

plugging the last result into (353) we will obtain

$$\begin{aligned}
& -\frac{p_0}{a_2^2} + \frac{p_1}{a_2^3} - 2\frac{p_2}{a_2^4} + \frac{3a_3^2}{4a_2^3} + \frac{r_0}{a_2} + \left(\frac{p_0}{a_2} - \frac{p_1}{a_2^2} + 2\frac{p_2}{a_2^3} - \frac{a_3^2}{2a_2^2} \right) t + \left(\frac{p_1}{2a_2} - \frac{p_2}{a_2^2} \right) t^2 \\
& + \frac{p_2}{3a_2} t^3 + \left(\frac{p_0}{a_2^2} - \frac{p_1}{a_2^3} + 2\frac{p_2}{a_2^4} - \frac{a_3^2}{a_2^3} - \frac{r_0}{a_2} \right) e^{-a_2 t} + \frac{a_3^2}{4a_2^3} e^{-2a_2 t} = -\ln[B(0, t)] \quad (356) \\
& t \in [0, T] ,
\end{aligned}$$

which we can use in combination with Table 4 to obtain, by means of a nonlinear least squares regression, the implied values of the parameters p_0 , p_1 , p_2 , r_0 , a_2 and a_3 .

Figure 17 and Figure 14 show our results in the calibration of (343), using the Levenberg–Marquardt method, under the assumption that the mean reversion parameter $\alpha(t)$ is of the form (354). The resulting parameters found, with a standard deviation of residuals of 0.0020231342 and an R^2 of 1.00, are:

$$\begin{aligned}
a_2 & \sim 0.179084373 , & a_3 & \sim 3.999277 \times 10^{-12} , \\
r_0 & \sim 0.006507327 , & p_0 & \sim 0.0129721181 , \\
p_1 & \sim -0.0000752610236 , & p_2 & \sim -0.00000408742152 ,
\end{aligned} \quad (357)$$

(as before, we did tried a numerical experiment with a modified model function —to force a positive r_0 —, see Figure 14 and Figure 17).

Notice that the resulting bond volatility formula corresponding to this example is of the same form as that we obtained in the previous example, and although the values of the parameters involved are different we expect curves very similar to those shown in Figure 13 and Figure 10. As we have mentioned before, one should be cautious if a model calibrated (only) to initial yield data is to be used to make assumptions about contracts that may not only be related to the interest rate but also to interest rate volatility not considered into the calibration process.

4.2.3 Constant α and β

We consider now the case of α and β constant and equal to a_1 and a_2 respectively while $\gamma = \gamma(t)$. Under these assumptions we can rewrite (255) as follows:

$$dr_t = (a_1 - a_2 r_t)dt + \gamma(t) dW_t. \quad (358)$$

This version of the Hull–White model differs from our previous two examples in the fact that it allows for a more complex theoretical bond volatility. We can use this quality to model an initial term structure of volatilities, for example volatilities extracted from “flat” Cap volatilities (please see §4.3).

As before, we offer some of the properties of this model. As expected some of the formulas remain the same, in particular, the expressions for η and \mathcal{S} .

$$\eta(t) = \exp\left(\int_0^t \beta(u) du\right) = \exp(a_2 t). \quad (359)$$

$$\begin{aligned} r_t &= (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du + \int_0^t \eta(u) \gamma(u) dW_u \right\} \\ &= \exp(-a_2 t) \left\{ r_0 + \frac{a_1}{a_2} (\exp(a_2 t) - 1) + \int_0^t \exp(a_2 u) \gamma(u) dW_u \right\} \\ &= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \frac{a_1}{a_2} (e^{a_2 t} - e^{a_2 s}) + \int_s^t e^{a_2 u} \gamma(u) dW_u \right\} \quad s \leq t. \end{aligned} \quad (360)$$

$$\begin{aligned} m_t &= E(r_t) = (1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du \right\} \\ &= e^{-a_2 t} \left\{ r_0 + \frac{a_1}{a_2} (e^{a_2 t} - 1) \right\} = \frac{a_1}{a_2} + \left(r_0 - \frac{a_1}{a_2} \right) e^{-a_2 t}. \end{aligned} \quad (361)$$

As in the case of Vasicek’s model (Hull White’s model with all parameters constant as in our first example, see (331)), (361) means that the long term mean of the rate is given by

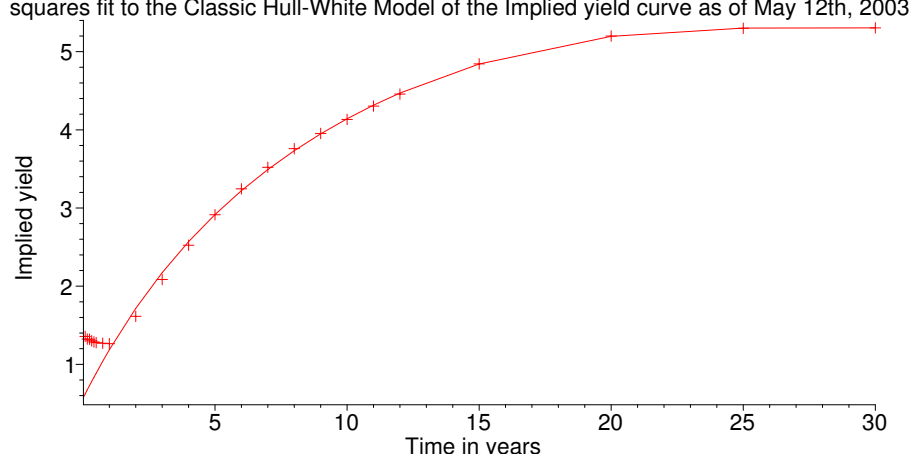


Figure 18: This plot shows the implied yield curve data extracted from Swap rate data as on May 12 2003 as well as the yield curve obtained as the least squares fit of the Hull–White Model (358) to that data assuming constant mean reversion and speed of mean reversion parameters and a volatility parameter of polynomial form (quadratic in this example). We applied the Levenberg–Marquardt method of nonlinear least squares to equation (369) which contains all the parameters of the model.

the ratio $\frac{a_1}{a_2}$. A similar observation applies to (362) below, and to (332) above.

$$\begin{aligned}
 E(r_t | \mathcal{F}_s) &= (1/\eta(t)) \left\{ \eta(s)r_s + \int_s^t \eta(u)\alpha(u) du \right\} \quad s \leq t \\
 &= e^{-a_2 t} \left\{ e^{a_2 s} r_s + \frac{a_1}{a_2} (e^{a_2 t} - e^{a_2 s}) \right\} \\
 &= \frac{a_1}{a_2} + \left(r_s - \frac{a_1}{a_2} \right) e^{-a_2(t-s)} \quad s \leq t.
 \end{aligned} \tag{362}$$

Our three examples have had in common the fact that they share the same expression for $\mathcal{S}(\cdot, \cdot)$, please compare (363) with (335) and (350)

$$\begin{aligned}
 \mathcal{S}(t, T) &= \eta(t) \int_t^T \frac{1}{\eta(u)} du \quad t \in [0, T] \\
 &= \frac{1}{a_2} \left(1 - e^{-a_2(T-t)} \right).
 \end{aligned} \tag{363}$$

In this case, since we do not know the form of $\gamma(u)$ we can not explicitly evaluate the following expressions,

$$\begin{aligned}
 \mathcal{V}_t = \text{Var}(r_t) &= \frac{1}{\eta^2(t)} \int_0^t \eta^2(u) \gamma^2(u) du \\
 &= e^{-2a_2 t} \int_0^t e^{2a_2 u} \gamma^2(u) du.
 \end{aligned} \tag{364}$$

$$\begin{aligned}
\text{Var}(r_t \mid \mathcal{F}_s) &= \frac{1}{\eta^2(t)} \int_s^t \eta^2(u) \gamma^2(u) du \quad s \leq t \\
&= e^{-2a_2 t} \int_s^t e^{2a_2 u} \gamma^2(u) du.
\end{aligned} \tag{365}$$

$$\begin{aligned}
\mathcal{I}(t, T) &= \int_t^T \frac{1}{\eta(u)} \int_t^u \eta(v) \alpha(v) dv du \\
&\quad - (1/2) \int_t^T \eta^2(v) \gamma^2(v) \left\{ \int_v^T \frac{1}{\eta(u)} du \right\}^2 dv \quad t \in [0, T] \\
&= \frac{a_1}{a_2} \left\{ (T - t) + \frac{1}{a_2} [\exp(-a_2(T - t)) - 1] \right\} \\
&\quad - \frac{1}{2a_2^2} \int_t^T \left\{ 1 + e^{-2a_2(T-v)} - 2e^{-a_2(T-v)} \right\} \gamma^2(v) dv \quad t \in [0, T].
\end{aligned} \tag{366}$$

Instead, we can write:

$$\begin{aligned}
\mathcal{I}(0, t) &= \frac{a_1}{a_2} \left\{ t + \frac{1}{a_2} [\exp(-a_2 t) - 1] \right\} \\
&\quad - \frac{1}{2a_2^2} \int_0^t \left\{ 1 - e^{-a_2(t-v)} \right\}^2 \gamma^2(v) dv \quad t \in [0, T],
\end{aligned} \tag{367}$$

and

$$\mathcal{S}(0, t) = \frac{1}{a_2} (1 - e^{-a_2 t}). \tag{368}$$

squares fit to the Hull-White Model (with non constant volatility) of the Implied ZCB curve as of May 12th, 2003

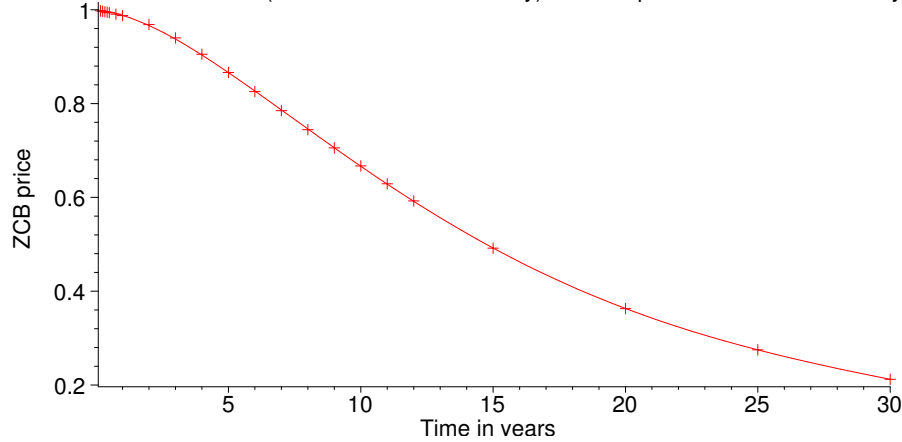


Figure 19: Implied zcb curve data extracted from Swap rate data as on May 12 2003 as well as the zcb curve obtained from our least squares fit of the Hull-White Model (358) to Table 4 data assuming constant mean reversion and speed of mean reversion parameters and a volatility parameter of polynomial form (quadratic in this example).

This version of Hull-White's model is usually used in the calibration to interest rate volatility data. Such use should be obvious from the structure of the model. Since both γ

and a_2 appear in the bond volatility formula (see (295) and (363)), model (358) is (out of the three examples considered up to now) the one that allows for a richer family of bond volatilities. This feature is exploited usually in the form of a calibration to initial spot volatility data (in fact, as we will see later, one can achieve a “perfect fit” to initial spot volatility following a very simple scheme).

Still, one could use model (358) to fit an initial term structure of interest rates.

Notice that such application of the model is highly unusual and that the resulting volatility structure we may obtain from this example will be, most likely, unrealistic; and that that will be the case even if the fit to initial yield is “perfect”. Please see Figure 20 where we show some bond volatility curves obtained as a result to the calibration of model (358) to the data from Table 4 under the assumption of a polynomial γ .

As it has happened before, even after combining (367) and (368) with (326) we can not, directly, find implied values for a_2 , a_1 and γ . If instead, one looks at (324) an expression for γ will be found that could be used to perfectly fit the model to an initial yield curve. In our case (324), plus (326), reduces to:

$$\gamma^2(t) = 2a_1a_2 + 2a_2^2 \frac{d}{dt}(\ln B(0, t)) + 3a_2 \frac{d^2}{dt^2}(\ln B(0, t)) + \frac{d^3}{dt^3}(\ln B(0, t)) \quad (369)$$

which requires the log of the initial zcb curve to be at least three times differentiable (also a_1 and a_2 should be selected in such a way as to ensure the non-negativity of the right hand side of (369)).

On the other hand, (367), (368) and (326) could still be used to calibrate the model if we assume some explicit form for γ . For example

$$\gamma(t) = p_0 + p_1t + \cdots + p_k t^k \quad (370a)$$

or

$$\gamma(t) = f(t; p_0, \dots, p_k) \quad (370b)$$

where p_0, p_1, \dots, p_k , are k coefficients (k a finite natural number selected by us using our “best knowledge”) to be determined and f is a known function of t and the coefficients p_0, p_1, \dots, p_k . In such a case we could attempt a nonlinear least squares regression as we

have done in our previous examples, from where a_1 , a_2 and γ will be approximated using our initial data, namely data like that displayed in Table 4.

For example, if we assume that γ can be represented by a second degree polynomial of the form (370), we can combine (367) and (368) with (326) and (370) to obtain:

$$\begin{aligned}
& \frac{1}{10} \frac{p_2^2}{a_2^2} t^5 + \frac{1}{4a_2^2} \left(p_1 p_2 - 3 \frac{p_2^2}{a_2} \right) t^4 + \frac{1}{a_2^2} \left(\frac{1}{3} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] - \frac{3}{2} \frac{p_1 p_2}{a_2} + \frac{7}{2} \frac{p_2^2}{a_2^2} \right) t^3 \\
& + \frac{1}{2a_2^2} \left(p_0 p_1 - \frac{3}{a_2} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] + \frac{21}{2} \frac{p_1 p_2}{a_2^2} - \frac{45}{2} \frac{p_2^2}{a_2^3} \right) t^2 \\
& + \frac{1}{a_2} \left(a_1 + \frac{1}{2} \frac{p_0^2}{a_2} - \frac{3}{2} \frac{p_0 p_1}{a_2^2} + \frac{7}{2a_2^3} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] - \frac{45}{4} \frac{p_1 p_2}{a_2^4} + \frac{93}{4} \frac{p_2^2}{a_2^5} \right) t \\
& - \frac{1}{a_2} \left(r_0 - \frac{a_1}{a_2} - \frac{p_0^2}{a_2^2} + 2 \frac{p_0 p_1}{a_2^3} - \frac{4}{a_2^4} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] + 12 \frac{p_1 p_2}{a_2^5} - 24 \frac{p_2^2}{a_2^6} \right) e^{-a_2 t} \quad (371) \\
& - \frac{1}{4a_2^3} \left(p_0^2 - \frac{p_0 p_1}{a_2} + \frac{1}{a_2^2} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] - \frac{3}{2} \frac{p_1 p_2}{a_2^3} + \frac{3}{2} \frac{p_2^2}{a_2^4} \right) e^{-2a_2 t} \\
& + \frac{r_0}{a_2} - \frac{a_1}{a_2^2} - \frac{3}{4} \frac{p_0^2}{a_2^3} + \frac{7}{4} \frac{p_0 p_1}{a_2^4} - \frac{15}{4a_2^5} \left[\frac{1}{2} p_1^2 + p_0 p_2 \right] + \frac{93}{8} \frac{p_1 p_2}{a_2^6} - \frac{189}{8} \frac{p_2^2}{a_2^7} \\
& = -\ln B(0, t)
\end{aligned}$$

which, although more complex than (341) and (356), could still be used in a least squares regression.

As in the previous example, we used the Levenberg–Marquardt method to obtain approximations to the parameters of (358) under the hypothesis of a quadratic, (370), γ function. With a standard deviation of residuals of 0.00161780368 we found

$$\begin{aligned}
a_1 & \sim 0.01502679329, & a_2 & \sim 0.3066419325, \\
r_0 & \sim 0.005652634312, & p_0 & \sim -0.02827933947, \\
p_1 & \sim -0.005063641306, & p_2 & \sim 0.0002591939827.
\end{aligned} \quad (372)$$

please see Figure 18 and Figure 19 for depictions of our findings.

Another possibility is to make use of the results of the next section, which will allow us to introduce a second set of initial data. That new data will come from Caps and will convey information regarding bond volatility at initial time. With the help of (295), data related to initial bond volatility can be used to determine (or more properly, to approximate) γ and \mathcal{S} .

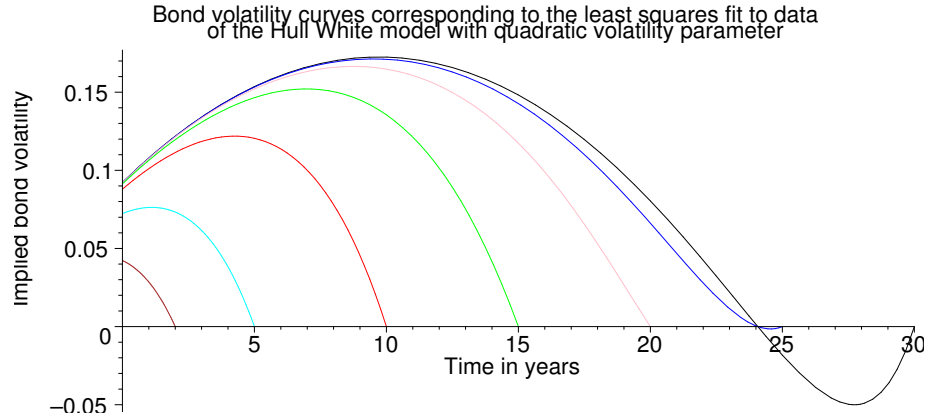


Figure 20: Bond volatility curves for maturities of 2, 5, 10, 15, 20, 25, and 30 years are shown (as before, we are plotting the negative of the bond volatility function). These curves are obtained from the least squares fit of the Hull–White model with non constant volatility parameter, (358), under the assumption of a quadratic γ function. As it may be expected, the curves we obtain do not need to be realistic since only initial yield curve information has been used to obtain them. Compare with Figure 44 which depicts implied bond volatility curves obtained after a perfect fit of a Hull–White model to initial spot volatility using the methods presented in §4.5.

4.3 Caps

As we saw in the subsection of examples (see §4.2, and in particular the second example), it is always possible to calibrate (255) once some assumptions, regarding the model parameters α, β and γ , are made. In fact, if the right kind of assumptions are made, it will be possible to replicate the initial yield curve using (255) or a reduced version of it like (343). The problem with that approach is that the resulting bond volatility curves are not necessarily realistic representations of the bond volatility observed.

In case a “good” fit to volatility data is also needed it is necessary to introduce some additional steps to the calibration process.

In this section we will be discussing a possible approach in that regard. This idea uses (295) and its relation with both, market practice in the pricing of *Cap* [*Floor*] contracts and the theoretical valuation of such contracts. Our presentation of this topic is not complete and many extensions are possible (the first extension that comes to mind is to also introduce *Swaptions* data, for example). As we mentioned earlier, additional information regarding the contracts discussed in this section may be found in the books by Musiela and Rutkowski,

Martingale Methods in Financial Modeling, [133] and Bielecki and Rutkowski, *Credit Risk: Modeling, valuation and hedging* [12]. On the other hand, books like Hull's *Options, Futures, & Other Derivatives*, [82], provide also a view to market practice.

Throughout this section we will be using notation and terms defined in Chapter 2. In particular: \mathcal{N} is used to represent a nominal, principal or notional amount, $\varphi(t, s)$ is the time fraction between dates t and s (with respect to known conventions), $B(t, T)$ is the time t price of a zcb of maturity T , B_t is the value of the bank account (money market account) and $T_i, i \in \mathbb{N}_n$ are used to represent n (reset or payment) dates, such that $0 \leq t \leq T_1 \leq \dots \leq T_n$.

In what follows we assume (unless otherwise specified) that *Cap* [*Floor*] contracts are settled at a settlement time (date) $t, t \geq 0$. To be consistent, cash flows will be explicitly discounted to time t .

4.3.1 Theoretical price of Caps

Definition 4.3.1. *An interest rate Cap [Floor] is a contract between two parties in which the seller agrees (for a price) to pay the holder a cash amount (based on a nominal) if a given interest rate exceeds [falls below] a previously agreed value at some future reset dates.*



Figure 21: Cap time schedule

Payments, if they are positive, will be given at some previously agreed payment dates. If we call \mathcal{N} the nominal, \mathcal{K} the strike, Vr the floating rate, φ the day counting convention, t the settlement date, and $T_i, i \in \mathbb{N}_n^$ the reset ($i \in \mathbb{N}_{n-1}^*$), and installment (payment) dates ($i \in \mathbb{N}_n$), we can write the payment at date T_i as:*

$$\mathcal{N}\varphi(T_{i-1}, T_i)(Vr(T_{i-1}, T_i) - \mathcal{K})^+ \quad (373)$$

in the case of a cap, or as:

$$\mathcal{N}\varphi(T_{i-1}, T_i)(\mathcal{K} - Vr(T_{i-1}, T_i))^+ \quad (374)$$

in the case of a *Floor*. Note that those payments can be discounted back to time t by multiplying²⁵ by B_t/B_{T_i} .

Definition 4.3.2. An interest rate *Cap* (*Floor*) consisting of single reset and payment dates is called a *caplet* (*floorlet*). In such a case is clear that the discounted value (to time t) of a caplet will be

$$\mathcal{N} \frac{B_t}{B_{T_p}} \varphi(T_r, T_p) (Vr(T_r, T_p) - \mathcal{K})^+ \quad (375)$$

while in the case of a floorlet we will have

$$\mathcal{N} \frac{B_t}{B_{T_p}} \varphi(T_r, T_p) (\mathcal{K} - Vr(T_r, T_p))^+ \quad (376)$$

where t is the settlement, T_r the reset, and $T_p > T_r$ is the payment date, \mathcal{N} is the nominal,

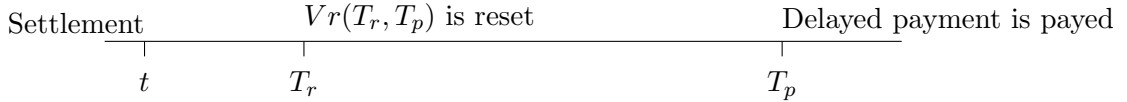


Figure 22: Caplet time schedule and payment

\mathcal{K} the strike and φ the day counting convention applicable to variable rate Vr (reset at time T_r and valid throughout the time interval $[T_r, T_p]$).

Note 4.3.1. Notice that if $T_0 = t$ the value of $Vr(T_0, T_1)$ will be known at settlement, thus the first payment will not be stochastic. For this reason, $t < T_0$ is a usual Market requirement in the definition of a *Cap* [*Floor*] contract. Otherwise, if $t = T_0$, a *Cap* should not include the deterministic payment.

Similarly, in the definition of a *Caplet*, the case $t < T_r$ is the interesting one.

It is clear that a *Cap* [*Floor*] can be seen as a portfolio of caplets [*floorlets*], each of which is a call [*put*] on the variable rate Vr .

Following similar arguments to those used in our discussion on general forward agreements in Chapter 2, we see that if rate Vr represents the yield of an arbitrage free bond

²⁵Dividing by B_{T_i} will discount the payoffs to time zero, to obtain the corresponding values at time t we must let the time zero values accrue until time t , thus multiplying the time zero discounted payoffs by B_t .

family, $Vr(T_r, T_p)$ can be seen as a forward rate corresponding to such family²⁶. Therefore, a caplet [floorlet] can be seen as a call on the forward rate Vr reset at time $T_r \leq T_p$ with expiration time T_r and payment delayed to time T_p . Similarly, a caplet [floorlet] could be seen as an European put [call] on a zcb²⁷ which intuitively match with our comments at the end of subsection §4.1.1.

Since the nominal \mathcal{N} appears in our formulas as a multiplicative factor, in order to simplify the analysis we may, and will, assume that the nominal is equal to 1, that is $\mathcal{N} = 1$. Expressions corresponding to other values of the nominal can easily be recovered by multiplying Cap and Caplet [Floor and Floorlet] prices by the appropriate amount.

In [133], Chapter 16, Musiela and Rutkowski show that under a Gaussian model²⁸, the price of a caplet is given by:

Lemma 4.3.2. *For any $0 < T_r \leq T_p \leq \mathcal{T}$, the arbitrage price at time $t \in [0, T_r[$ of a caplet*

²⁶On one hand, (36) can be used to see that

$$Vr(T_r, T_p) = \frac{1}{\varphi(T_r, T_p)}(1 + \varphi(T_r, T_p)Vr(T_r, T_p) - 1) = \frac{1}{\varphi(T_r, T_p)} \left(\frac{1}{B(T_r, T_p)} - 1 \right) = Fr(T_r, T_r, T_p)$$

which is the forward rate valid on period $[T_r, T_p]$ as seen at time T_r . On the other hand, assuming no transaction costs and the perfect divisibility of bonds, arbitrage arguments imply that $B(t, T_p) = B(t, T_r)B(T_r, T_p)$, for $t \leq T_r \leq T_p$, which implies that $Vr(T_r, T_p) = Fr(t, T_r, T_p)$.

²⁷From Definition 4.3.2 we have

$$\begin{aligned} \mathcal{N} \frac{B_t}{B_{T_p}} \varphi(T_r, T_p) (Vr(T_r, T_p) - \mathcal{K})^+ &= \mathcal{N} \frac{B_t}{B_{T_p}} (\varphi(T_r, T_p) Vr(T_r, T_p) - \varphi(T_r, T_p) \mathcal{K})^+ \\ &= \mathcal{N} \frac{B_t}{B_{T_p}} \left(\frac{1}{B(T_r, T_p)} - (1 + \varphi(T_r, T_p) \mathcal{K}) \right)^+ \\ &= \mathcal{N} \frac{B_t}{B_{T_p}} \frac{1 + \varphi(T_r, T_p) \mathcal{K}}{B(T_r, T_p)} \left(\frac{1}{1 + \varphi(T_r, T_p) \mathcal{K}} - B(T_r, T_p) \right)^+ \\ &= \mathcal{N} \frac{B_t}{B_{T_p}} \frac{1}{\kappa B(T_r, T_p)} (\kappa - B(T_r, T_p))^+ \end{aligned}$$

which apart from multiplicative factors, is the payoff of a put option with strike $\kappa = (1 + \varphi(T_r, T_p) \mathcal{K})^{-1}$ (which we can see as the price of a time T_r zcb with maturity T_p and yield \mathcal{K}) and expiry T_r on a time T_r zcb with maturity T_p . Here we have assumed that the φ -time fraction is short so that simple compounding can be used. The case of a floorlet is similar.

²⁸As we mentioned before, the generalized Hull–White model (255) is the archetypical Gaussian model, thus we use “Hull–White” and “Gaussian” interchangeably. See Footnote 8.

with expiry date T_r , settlement date t , payment (delayed to) date T_p , and strike level \mathcal{K} on a floating rate Vr is given by the formula

$$\begin{aligned} \mathbf{Cpl}(t, T_r, T_p, \varphi, Vr, \mathcal{K}, \mathbf{v}) \\ = B(t, T_r)\Phi(\mathbf{e}(t, T_r, T_p)) - (1 + \mathcal{K}\varphi(T_r, T_p))B(t, T_p)\Phi(\mathbf{e}(t, T_r, T_p) - \mathbf{v}(t, T_r, T_p)) \end{aligned} \quad (377)$$

where

$$\mathbf{e}(t, T_r, T_p) = \frac{1}{\mathbf{v}(t, T_r, T_p)} \ln \left(\frac{B(t, T_r)}{(1 + \mathcal{K}\varphi(T_r, T_p))B(t, T_p)} \right) + \frac{1}{2}\mathbf{v}(t, T_r, T_p), \quad (378)$$

and

$$\mathbf{v}^2 = \mathbf{v}^2(t, T_r, T_p) = \int_t^{T_r} |b(u, T_r) - b(u, T_p)|^2 du. \quad (379)$$

Φ is the cumulative normal distribution, $b(t, T)$ is the bond volatility and $B(t, T_r)$ and $B(t, T_p)$ are the time t prices of zcb's of maturities T_r and T_p whose yield is given by rate Vr .

In the previous lemma, $b(t, T)$ is the bond volatility at time t of a zcb of maturity T as mentioned in §4.1, (294) and in [133] Proposition 12.2.1. Since we are assuming $\mathcal{N} = 1$, we can regard \mathbf{Cpl} as the caplet price per unit of nominal.

In our particular case, we know that under the Hull–White model, bond volatility is given by (295), that is:

$$b(t, T) = -\gamma(t)\mathcal{S}(t, T),$$

therefore we can rewrite (379) as:

$$\mathbf{v}^2(t, T_r, T_p) = \int_t^{T_r} \gamma^2(u) |\mathcal{S}(u, T_r) - \mathcal{S}(u, T_p)|^2 du, \quad (380)$$

which can be used in the calibration process to recover γ and \mathcal{S} . In practice, this will still require some kind of compromise in the form of assumptions regarding the analytical form of γ .

In light of Lemma 4.3.2 and Definition 4.3.1 it is not difficult to prove the following proposition (see Musiela and Rutkowski, [133], Proposition 16.2.1):

Proposition 4.3.3. *The price at time $t < T_0$ of an interest rate Cap with strike \mathcal{K} , on an underlying floating rate Vr , settled in arrears at times $\{T_j\}_{j \in \mathbb{N}_n^*}$, equals*

$$\mathbf{Cap}(t, \{T_i\}_{i \in \mathbb{N}_n^*}, \varphi, Vr, \mathcal{K}, \{\mathbf{v}_i\}_{i \in \mathbb{N}_n}) = \sum_{j=1}^n B(t, T_{j-1}) \Phi(\mathbf{e}_j(t)) - (1 + \mathcal{K} \varphi(T_{j-1}, T_j) B(t, T_j) \Phi(\mathbf{e}_j(t) - \mathbf{v}_j(t))), \quad (381)$$

where

$$\mathbf{e}_j(t) = \mathbf{e}(t, T_{j-1}, T_j) = \frac{1}{v_j(t)} \ln \left(\frac{B(t, T_{j-1})}{(1 + \mathcal{K} \varphi(T_{j-1}, T_j) B(t, T_j))} \right) + \frac{1}{2} v_j(t), \quad (382)$$

and

$$\mathbf{v}_j^2(t) = \mathbf{v}^2(t, T_{j-1}, T_j) = \int_t^{T_{j-1}} |b(u, T_{j-1}) - b(u, T_j)|^2 du. \quad (383)$$

$\{B(t, T_i)\}_{i \in \mathbb{N}_n^*}$ are the time t prices of zcb's of maturities $\{T_i\}_{i \in \mathbb{N}_n^*}$ whose yield is given by rate Vr , and $b(\cdot, \cdot)$ is the bond volatility.

As before, since we are assuming $\mathcal{N} = 1$, we can regard \mathbf{Cap} as the Cap price per unit of nominal.

Observe that in the case interest rate Vr dynamics are as in the Hull–White model of interest rates, (255), (383) acquires a more familiar look:

$$\mathbf{v}_j^2(t) = \mathbf{v}^2(t, T_{j-1}, T_j) = \int_t^{T_{j-1}} \gamma^2(u) |\mathcal{S}(u, T_{j-1}) - \mathcal{S}(u, T_j)|^2 du. \quad (384)$$

where \mathcal{S} is given by (281) and γ is the volatility parameter of the Hull–White model (255).

4.3.2 Market price of Caps

Caps, as well as Floors, Swaps, FRA's, Swaptions, etc., are over the counter (OTC) derivatives used by many companies to hedge against their interest rate risk exposures. Ever since the introduction of interest rate options, the total notional outstanding on such contracts has been steadily increasing. According to the International Swaps and Derivatives Association (ISDA), by the second half of 1997 the total outstanding notional on interest rates options was estimated to be of \$4920.10 billion dollars²⁹ (the ISDA does not provide

²⁹Figures taken from ISDA's Surveys & Market Statistics: Historical data, ISDA Market Survey results, 1987–present (http://www.isda.org/statistics/stat_nav.html).

surveys on interest rate options after 1997; instead total market size, including interest rate swaps, cross-currency swaps and interest rate options are provided for years after 1997; in the second half of 1997, total market size was estimated to be of \$29035.00 billion dollars, by the end of 2003 that figure had grown to \$142306.92 billion). According to the Bank for International Settlements (BIS), by the end of 2003 the outstanding notional of interest rate options (this includes Caps, Floors, Swaptions, collars, etc.) had grown to \$20012 billion dollars³⁰. When we talk about the cap/Floor markets, or in general, about the interest rate options markets we refer to the many companies that enter into the over the counter transaction of interest rate options. For example, the US market includes several Commercial Banks (CB's) and Trust Companies (TC's) as JPMorgan Chase, Bank of America, State Street Bank & Trust Co, and Northern Trust Co, to name just a few³¹.

Market practice regarding the pricing of Cap and Floor contracts differs from the theoretical description we presented above. In particular, market prices for Caps and Floors are computed using Black's model [14], and those prices are not directly quoted; instead "flat volatilities" (we will describe this concept later) for at the money (ATM) Caps and Floors are quoted.

Caplets are (market) priced³² under the assumption of log-normal drift-less interest rates with piecewise constant volatility, that is Vr , the underlying of the Cap/Caplet (Floor or Floorlet) contract, is assumed to follow lognormal dynamics and the percent variation of such rate dVr/Vr has zero drift and piecewise constant volatility, for every maturity³³.

³⁰Figures taken from BIS's Regular OTC Derivatives Market Statistics: The Global OTC Derivatives Market at end-December 2003 (http://www.bis.org/publ/otc_hy0405.htm). Additional information regarding US market size can be found in the web site of the Comptroller of the Currency (<http://www.occ.treas.gov/deriv/deriv.htm>).

³¹Additional information in this regard can be found at the web site of the Comptroller of the Currency (<http://www.occ.treas.gov/deriv/deriv.htm>). The interested reader can search the above mentioned web site for the Quarterly Derivatives Fact Sheets.

³²See [116], [19] and [82].

³³In a sense the idea is extremely simple, according to market practice, each underlying forward rate $Vr(t, T_r, T_p)$ is modeled separately as a lognormal driftless process, each of them totally correlated to the previous and next rate. If there are n reset dates, then there are n forward rate processes, each of them

The floating rate Vr is always assumed to follow a pre-specified yield curve (a LIBOR rate for example) or family of zcb prices. In consistency with such assumptions regarding the dynamics of the underlying interest rate Vr , another assumption used in the market pricing of caplets is that bond volatility of the associated bond family behaves in a particular and simple way; if $b(t, T)$ is the time t bond volatility (under Black's model) for a zcb of maturity T then

$$|b(t, T_r) - b(t, T_p)| = \sigma_{r,p}, \quad (385)$$

where $\sigma_{r,p}$ is assumed to be constant on the time interval $[T_r, T_p]$, but possibly different on another time interval. All these assumptions are made to be able to apply Black's model³⁴, but they do not necessarily agree with the theoretical treatment of bonds, interest rates and interest rate options under different interest rate models. For example, in our case we assume that interest rates follow the dynamics expressed by (255), which will give rise to Gaussian rates and not to log-normal ones. Off course, if we are going to conciliate both views, the market use of log-normal (forward) rates with the theoretical assumption of (short) rates following a model like (255) (or for that matter of any other form) then we will need to impose some restrictions or assumptions with regards to the form of $b(\cdot, \cdot)$.

As we mentioned above, seen as portfolios of European options on a log-normal underlying, Caps are then priced using Black's model (roughly speaking, applying Black-Scholes' formula to the "log-normal" forward rate Vr obtained from a chosen family of zcb's).

Definition 4.3.3. *Let N be the nominal, T_r be the reset date and T_p the payment date,*

associated to its own stochastic dynamics of the form $dVr(t, T_i, T_{i+1}) = \sigma_{i,i-1} Vr(t, T_i, T_{i+1}) dW_t^i$.

³⁴Although Black's model, see [14], was developed to price options on a futures commodity contract, due to its simplicity and affinity to the Black-Scholes pricing model, it has been applied to a wide variety of derivatives preserving the "spirit" of the model's assumptions, that is that the underlying (a forward rate, a bond, etc) follows lognormal dynamics, its percent variation is drift-less and has constant or piecewise constant volatility. The result of those assumptions is a pricing formula similar to the Black-Scholes formula, but with zero "risk free" rate. In a sense, to use Black's model in the pricing of a derivative corresponds to the assumption that the underlying "exists" in the form of an observable asset in a Black-Scholes "market" with zero risk free rate.

both relative to time of settlement t , $t < T_r < T_p$, which the market usually takes as equal to zero. Let φ be the day counting convention to be used, \mathcal{K} be the strike, $Vr(T_r, T_p)$ the floating rate, and $\sigma_{r,p}$ be the caplet volatility. According to [19], [82] and [116]³⁵, the market price of a caplet with those parameters is given by:

$$\begin{aligned} \mathbf{Cpl}^M(t, T_r, T_p, \varphi, \mathcal{N}, Vr(T_r, T_p), \mathcal{K}, \sigma_{r,p}) \\ = \mathcal{N}B(t, T_p)\varphi(T_r, T_p)\{Vr(T_r, T_p)\Phi(e(t, T_r, T_p, \mathcal{K}, Vr(T_r, T_p), \sigma_{r,p})) \\ - \mathcal{K}\Phi(e(t, T_r, T_p, \mathcal{K}, Vr(T_r, T_p), \sigma_{r,p}) - \sigma_{r,p}\sqrt{T_r - t})\} \\ e(t, T_r, T_p, \mathcal{K}, Vr(T_r, T_p), \sigma_{r,p}) = \frac{1}{\sigma_{r,p}\sqrt{T_r - t}} \ln\left(\frac{Vr(T_r, T_p)}{\mathcal{K}}\right) + \frac{1}{2}\sigma_{r,p}\sqrt{T_r - t}. \end{aligned} \quad (386)$$

Writing $Vr(T_r, T_p)$ in terms of bond family³⁶ $B(s, t)$, $s \leq t \leq T$, which is assumed to follow the day counting convention φ ³⁷, we can rewrite (386) as follows:

$$\begin{aligned} \mathbf{Cpl}^M(t, T_r, T_p, \varphi, \mathcal{N}, B(t, T_r), B(t, T_p), \mathcal{K}, \sigma_{r,p}) \\ = \mathcal{N}\{(B(t, T_r) - B(t, T_p))\Phi(e(t, T_r, T_p, \varphi, \mathcal{K}, B(t, T_r), B(t, T_p), \sigma_{r,p})) \\ - \mathcal{K}\varphi(T_r, T_p)B(t, T_p)\Phi(e(t, T_r, T_p, \varphi, \mathcal{K}, B(t, T_r), B(t, T_p), \sigma_{r,p}) - \sigma_{r,p}\sqrt{T_r - t})\} \\ e(t, T_r, T_p, \varphi, \mathcal{K}, B(t, T_r), B(t, T_p), \sigma_{r,p}) \\ = \frac{1}{\sigma_{r,p}\sqrt{T_r - t}} \ln\left(\frac{B(t, T_r) - B(t, T_p)}{\mathcal{K}\varphi(T_r, T_p)B(t, T_p)}\right) + \frac{1}{2}\sigma_{r,p}\sqrt{T_r - t}, \end{aligned} \quad (387)$$

(387) will come handy when dealing with market data.

As we said before, a caplet is nothing but a Cap on three times, t , the settlement ($t = 0$ in the case of data recovered from the market), T_r , the reset, and T_p , the payment date. Caps on more times are then priced as portfolios of caplets. In the case of the American market, it is usual practice to place the first reset date three months after settlement and the first payment date three months afterward. Future resets are made to coincide with the payment date of the preceding caplet, while future dates are chosen three months after

³⁵This also agrees with Bloomberg's[©] documentation.

³⁶As we argued before, Vr is seen as the forward rate implied by the bond family, thus we can write $Vr(T_r, T_p)$ in terms of zcb's $B(0, T_r)$ and $B(0, T_p)$ and the time fraction $\varphi(T_r, T_p)$ by means of (36). That is $1 + Vr(T_r, T_p)\varphi(T_r, T_p) = B(T_r, T_p)^{-1}$.

³⁷It is not difficult to relax this requirement, but we will not need such generalization.

their corresponding reset. As a result of this practice, there is no caplet assigned to the first quarter in a Cap³⁸. In other cases, and in particular in other countries, the time distance between resets (and between payment dates) may be different. If all caplet periods are of the same length, that length is called the *tenor* of the Cap. In Europe Caps with a tenor of six months are common, while in the US market the preferred tenor is three months. Table 5 shows a relation between commonly available maturities in USA and the number of caplets per cap in case of the two most frequently used tenors, 6 and 3 months.

Table 5: This table shows the usual maturities of Caps [Floors] quoted in the US Market. Also shown in this table are the number of Caplets [Floorlets] per Cap [Floor] in the two most used tenors.

Available Maturities			
3 month tenor		6 month tenor	
Cap Mat.	# of Caplets	Cap Mat.	# of Caplets
1 year	3 caplets	1 year	1 caplet
2 years	7 caplets	2 years	3 caplets
3 years	11 caplets	3 years	5 caplets
4 years	15 caplets	4 years	7 caplets
5 years	19 caplets	5 years	9 caplets
7 years	27 caplets	7 years	13 caplets
10 years	39 caplets	10 years	19 caplets

For example, it is clear that a one year maturity Cap consists of three caplets with resets at the 3, 6, and 9 month times; and a two year Cap should include seven caplets with resets at the 3, 6, 9, 12, 15, 18, and 21 month times.

Definition 4.3.4. For T_0, T_1, \dots, T_{n-1} the reset dates, and T_1, T_2, \dots, T_n the payment dates, the time $t < T_0$ market price of a Cap on a floating rate Vr that follows the yield curve of a bond family $B(s, u)_{s \leq u \leq T}$, with a notional N , day counting convention φ , strike K and caplet volatilities $\sigma_{i-1, i} = \sigma_{T_{i-1}, T_i}$, $i \in \mathbb{N}_n$ is given by:

³⁸Since the rate Vr is reset at the beginning of each quarterly period, the “first” caplet of a Cap, that is the caplet that will correspond to the first period of the Cap, will be completely determined at the first reset date (that is at settlement or a short time afterward). That is, the first caplet is deterministic and not stochastic, as all the remaining caplets. Thus, market practice is to “drop” the deterministic caplet.

$$\begin{aligned} \mathbf{Cap}^M(t, \{T_i\}_{i \in \mathbb{N}_n^*}, \varphi, \{B(t, T_i)\}_{i \in \mathbb{N}_n^*}, \mathcal{K}, \{\sigma_{i-1, i}\}_{i \in \mathbb{N}_n}) \\ = \sum_{i=1}^n \mathbf{Cpl}^M(t, T_{i-1}, T_i, \varphi, \mathcal{N}, B(t, T_{i-1}), B(t, T_i), \mathcal{K}, \sigma_{i-1, i}). \end{aligned} \quad (388)$$

But Cap prices are not directly quoted in the market, instead an indirect approach is followed. The Market quotes “flat” cap volatilities for at the money Caps priced at time $t = 0 = T_0$ (remember, in this case the we drop the first Caplet since that one is deterministic —otherwise, no–arbitrage will require that the parties involved in the cap exchange the first cash flow, which is equivalent to drop the first Caplet—); a *flat volatility* is the number $\tilde{\sigma}_n$ that solves the equation

$$\begin{aligned} \mathbf{Cap}^M(0, \{T_i\}_{i \in \mathbb{N}_n^*}, \varphi, \{B(0, T_i)\}_{i \in \mathbb{N}_n^*}, \{\tilde{\sigma}_n\}_{i \in \mathbb{N}_n}) \\ = \mathbf{Cap}^M(0, \{T_i\}_{i \in \mathbb{N}_n^*}, \varphi, \{B(0, T_i)\}_{i \in \mathbb{N}_n^*}, \{\sigma_{i-1, i}\}_{i \in \mathbb{N}_n}). \end{aligned} \quad (389)$$

Obviously, this practice ‘destroys’ the ‘fine’ structure of caplet volatilities (at least it hides them from the “public”).

Caplet volatilities are also known as *spot volatilities*.

In order to recover spot volatilities from flat volatilities one must follow market conventions. This is done in the next subsection.

4.4 Stripping spot volatilities

The process according to which spot volatilities are extracted/recovered from flat volatility quotes is commonly known as *stripping* caplet volatility or as *stripping* spot volatility.

Definition 4.4.1. A Cap [Floor] with reset and payment times $\{T_i\}_{i \in \mathbb{N}_n^*}$ and settlement date t is said to be at the money (ATM) if and only if its strike \mathcal{K}_{ATM} is equal to the corresponding swap rate³⁹ (see §2.1.5, and in particular see (40)):

$$\mathcal{K}_{ATM} = Sr(t, \{T_i\}_{i \in \mathbb{N}_n^*}, \varphi) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^n \varphi(T_{i-1}, T_i) B(t, T_i)}. \quad (2.40)$$

Similarly, a Cap is out of the money (OTM) if its strike $\mathcal{K} > \mathcal{K}_{ATM}$. If, on the other hand its strike $\mathcal{K} < \mathcal{K}_{ATM}$ we say that the Cap is in the money (ITM).

³⁹That is, with the same underlying family of zcb’s and the same payment schedule.

In the American market, flat Cap volatilities are quoted for ATM Caps of maturities of 1, 2, 3, 4, 5, 7, and 10 years. See Table 6 for a sample of such kind of data.

Table 6: Flat Cap volatility as reported on Bloomberg[©] on May 12th 2003. This table shows the bids at 3:56pm of that day.

Flat Volatilities	
Maturity	Volatility (%)
1 year	44.200
2 years	48.300
3 years	44.300
4 years	41.600
5 years	37.100
7 years	31.700
10 years	25.700

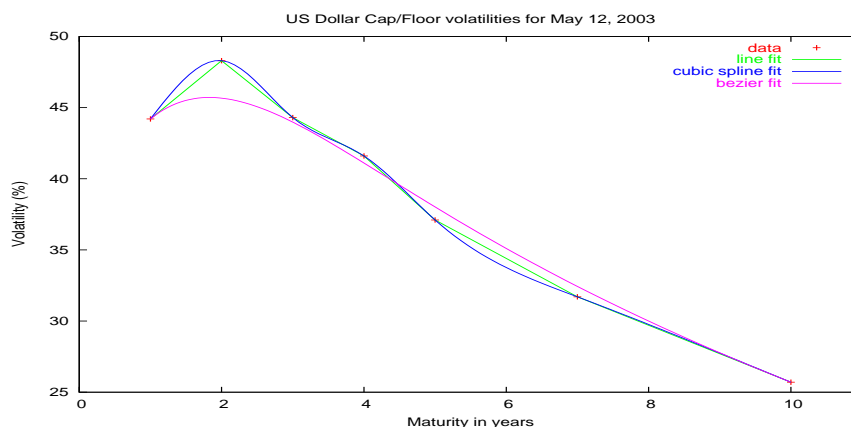


Figure 23: This plot shows “flat” volatility data for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6). The plot shows line (first order), cubic and cubic b-spline interpolations for that data. In the US markets flat volatilities are quoted for Caps/Floors of maturities 1, 2, 3, 4, 5, 7, and 10 years.

Market practice is to (linearly) interpolate⁴⁰ cap volatilities (‘flat’ Cap volatilities) and

⁴⁰Intuitively, one can see that market practice, to linearly interpolate, may not be optimal and even worse, that depending on the situation may not even be appropriate (for example, recall the bootstrapping procedure explained in Chapter 2, if swap rates are linearly interpolated, the corresponding forward rates

to use the interpolated volatilities to reconstruct, from (389), the market price of a Cap on a maturity different from those quoted.

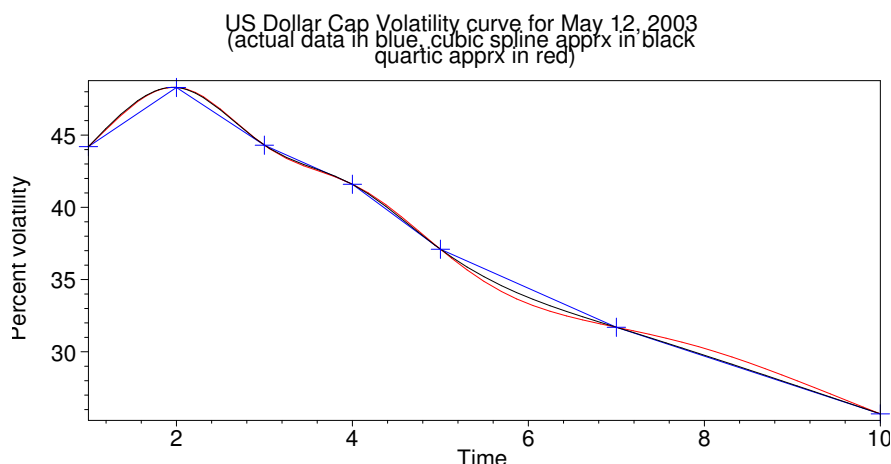


Figure 24: This plot compares the linear, cubic and quartic interpolation of “flat” volatility data for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6) using splines of appropriate orders. The plot shows the original data (crosses), line (first order), cubic and quartic interpolations for that data.

It is not clear what the procedure should be for maturities shorter than the shorter maturity quoted. For example, if information regarding Caps of short maturities is available, for example a quote on an ATM Cap of maturity of 3 months, then one could interpolate Flat volatilities to obtain the prices of ATM Caps of six and nine months maturity.

Otherwise, if no information is available one could opt to assume all Caplets of the Cap of shortest maturity have the same Spot volatility (equal to the Flat volatility), or, one could extrapolate Flat volatilities for shorter maturities.

Thus, in the case of a three month tenor (the case of the six month tenor is a lot easier), assuming all three initial caplets are of the same volatility⁴¹, or extrapolating Flat volatilities

may not be stable, the implied yield and the swap curve will not be differentiable, etc.). Instead, to use a form of higher order interpolation makes more sense. With respect to this, I have found some references in the literature, as well as in bulletin boards, regarding the “new trend” of using higher order interpolation—cubic splines to be more specific—instead of linear interpolation. Using cubic splines one can expect to obtain smoothed curves. We experimented with this idea, and show the results here.

⁴¹Unless, of course, one has the ability to find quotes on such caplets. For example, in the case of European swap rates, or LIBOR, one may use Eurodollar options (which are effectively caplets [floorlets]

for Caps of maturities of six (and nine months in the case of three month tenor), one can compute the market prices of (appropriate) Caps differing on a caplet. Then subtracting successive prices, one may find the market prices of such Caplets. Finally, using (387), we find the implied caplet volatility.

Let's denote by $\{T_i\}_{i \in \mathbb{N}_{39}^*}$ the quarter dates⁴² starting 3 months after settlement.

For example, to strip the fourth Caplet volatility from Table 6, one first (linearly) interpolate the flat Cap volatility corresponding to a Cap with maturity of 1 year and a quarter (remember, we either assume the first three spot volatilities to be equal to the first flat volatility or extrapolate Flat volatilities for ATM Caps of maturities of six and nine months):

$$\tilde{\sigma}_{1.25} = \tilde{\sigma}_{T_4} = x\tilde{\sigma}_{T_7} + (1-x)\tilde{\sigma}_{T_3} \Big|_{x=0.25} = 0.25 \times 48.300 + 0.75 \times 44.200 = 45.225 \quad (390)$$

Then, one uses Definition 4.3.4, Definition 4.3.3, the data from Table 4 and Table 6 and Definition 4.4.1 to determine the value of the ATM Caps with maturities of one year and one year and a quarter (actually, in the computation of the price of the Cap of one year maturity, the strike of the Cap of one year and a quarter **must** be used, thus the one year maturity Cap could be in or out depending on the swap rate curve). To reduce notation, let's call such Cap prices $\mathbf{Cap}^M(T_4)$, and $\mathbf{Cap}^{Mio}(T_3)$. Similarly, we can denote by $\mathbf{Cpl}^M(T_3, T_4)$ the price of the fourth Caplet in the Cap with maturity of one year and a quarter, $\mathbf{Cap}^M(T_4)$. Thus

$$\mathbf{Cap}^M(T_4) - \mathbf{Cap}^{Mio}(T_3) = \mathbf{Cpl}^M(T_3, T_4) \quad (391)$$

will give us the implied price of the fourth Caplet.

and very short –less than a year– caps [floors]).

⁴²In practice one uses usual business day conventions and actual/actual day counting as explained in Chapter 2 (numerical results shown in the following tables are consistent with this practice, that is usual business day conventions and actual/actual and actual/360 day counting was used to obtain such results). To simplify the notation in what remains of this exposition we will assume all years and months have the same length, that all days are business days and that dates start at settlement day (this is somehow equivalent to 30/360 day counting and no business day convention with the calendar starting at settlement). This way $T_0 = 0.25$, $T_3 = 1.00$, etc.

Next, we use Definition 4.3.3, Definition 4.4.1 and the value found in (391) to write an equation for $\sigma_{3,4}$ (the Caplet volatility for the period $[T_3, T_4]$) in terms of the Caplet price found in (391). That expression is of the form of (386) (or (387)) and the only unknown is $\sigma_{3,4}$. This equation is solved numerically.

The process to strip all remaining spot volatilities is the same. Table 9 shows the values of spot volatilities stripped from Table 6, Figure 27 and Figure 28 show ATM Caplet prices and Spot volatilities, respectively, found the procedure described here. To construct Table 9 we opted to extrapolate Flat volatilities for ATM Caps of maturities of six and nine months. In order to do that, we constructed a linear spline function using Table 6.

Using that function, a new table containing thirty nine Flat volatilities was obtained (see Table 7) —we denote such Flat volatilities as $\{\tilde{\sigma}_{T_i}\}_{i \in \mathbb{N}_{39}}$ —, and from that table the prices per unit of nominal of ATM Caps of all those maturities were computed using Definition 4.4.1, Definition 4.3.4, and Definition 4.3.3, see Figure 25 and Table 8. Let's denote such prices as $\mathbf{Cap}^M(T_i)$, $i \in \mathbb{N}_{39}$.

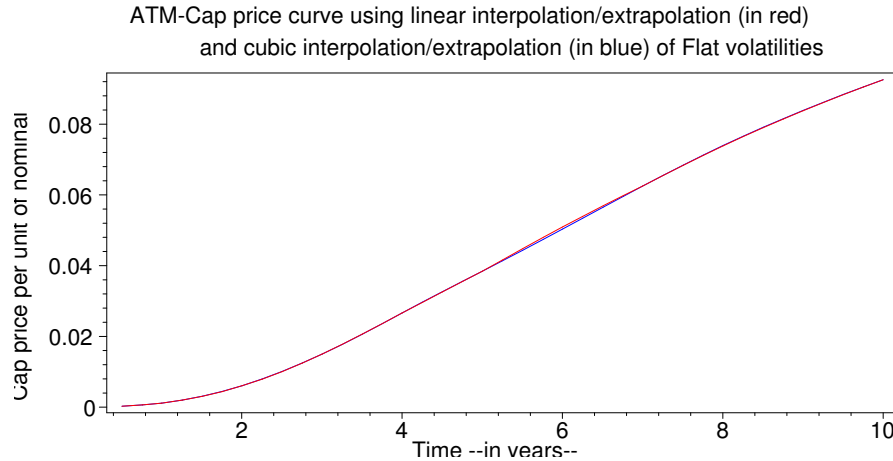


Figure 25: This plot compares ATM Cap prices obtained from linear and cubic interpolation/extrapolation of “flat” volatility data for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6) using splines of appropriate orders. The resulting curves differ in less than 0.6% at their maximal separation, please see Figure 26 for a depiction of the differences of the prices using the two indicated methods.

Clearly, the first of all those Caps (the Cap of six months maturity) contains only one Caplet, and its corresponding Flat volatility should also be equal to the first Spot

Table 7: Interpolated (extrapolated in the case of short maturities) Flat Cap volatility, see Table 6. Interpolation (extrapolation) was done using linear and cubic splines (results have been rounded to the precision of the initial data). actual/actual time to maturity in years, using next business day convention, is also shown.

Interpolated and extrapolated Flat Volatilities			
Maturity	TtM (act/act)	Linear I/E (%)	Cubic I/E (%)
3m	93/365	41.149	40.380
6m	37/73	42.179	41.314
9m	50851/66795	43.218	42.695
12m	66911/66795	44.200	44.200
15m	28022/22265	45.249	45.803
18m	201347/133590	46.265	47.136
21m	641/365	47.282	48.039
24m	732/365	48.300	48.300
27m	823/365	47.300	47.792
30m	914/365	46.300	46.744
33m	1006/365	45.289	45.460
36m	1096/365	44.300	44.300
39m	1187/365	43.625	43.457
42m	1279/365	42.943	42.841
45m	1371/365	42.260	42.275
48m	4	41.600	41.600
51m	1552/365	40.468	40.638
54m	1644/365	39.335	39.481
57m	317666/66795	38.205	38.253
60m	334091/66795	37.100	37.100
63m	350881/66795	36.421	36.065
66m	122557/22265	35.741	35.175
69m	2103/365	35.046	34.394
72m	6	34.402	33.765
75m	2282/365	33.721	33.179
78m	2376/365	33.025	32.640
81m	2467/365	32.352	32.156
84m	7	31.700	31.700
87m	2649/365	31.185	31.210
90m	548/73	30.686	30.729
93m	2831/365	30.188	30.242
96m	2922/365	29.689	29.751
99m	3013/365	29.190	29.254
102m	3104/365	28.692	28.754
105m	584846/66795	28.188	28.245
108m	601271/66795	27.697	27.746
111m	618061/66795	27.194	27.233
114m	211617/22265	26.691	26.718
117m	3561/365	26.188	26.201
120m	10	25.700	25.700

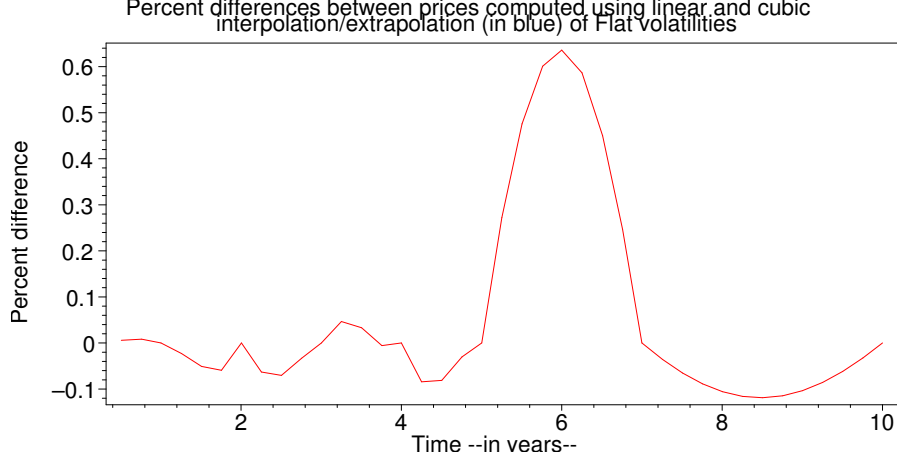


Figure 26: This plot depicts the percentage difference of ATM Cap prices obtained from linear and cubic interpolation/extrapolation of “flat” volatility data for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6) using splines of appropriate orders. The differences of the prices have been “normalized” using the highest ATM Cap Price. As it is apparent from this plot, the resulting curves differ in less than 0.6% at their maximal separation, please see Figure 25 for a depiction of the prices constructed using the two indicated methods.

volatility⁴³, that is $\mathbf{Cap}^M(T_1) = \mathbf{Cpl}^M(T_0, T_1)$ and $\sigma_{0,1} = \tilde{\sigma}_{T_1}$.

Next, we follow an iterative process. Knowing all Spot volatilities for time periods between T_1 and T_{i-1} , we use Definition 4.4.1, Definition 4.3.4, and Definition 4.3.3 and the ATM Strike of an ATM Cap of maturity T_i to obtain $\mathbf{Cap}^{Mio}(T_{i-1})$. Thus we find the price of the Caplet of reset T_{i-1} and payment date T_i as:

$$\mathbf{Cap}^M(T_i) - \mathbf{Cap}^{Mio}(T_{i-1}) = \mathbf{Cpl}^M(T_{i-1}, T_i) \quad (392)$$

With the help of Definition 4.3.3 and (386), $\sigma_{i-1,i}$ can be found using a numeric solver.

The process stops with $\sigma_{38,39}$.

Table 9 collects the results of our computations.

We also performed a Stripping of Spot volatilities using the same data, using cubic interpolation (extrapolation in the case of short maturities) of “flat” volatility instead of

⁴³Even if information regarding Caplets of maturities prior or equal to one year is known by other means, the stripping process will be similar to the description given here. If Spot volatilities for time periods prior to six months are needed, one can also take extrapolated Flat volatilities instead, or market information obtained elsewhere, etc..

Table 8: Price for ATM Caps on US Dollars. Prices (rounded to five significant figures) were computed using interpolated (extrapolated for short maturities) bids of Flat Cap volatility, as shown in Table 7. actual/actual time to maturity in years, using next business day convention, is also shown.

Cap Prices per unit of nominal			
Maturity	TtM (act/act)	Linear I/E	Cubic I/E
3m	93/365		
6m	37/73	0.00025963	0.00025433
9m	50851/66795	0.00064297	0.00063524
12m	66911/66795	0.0011254	0.0011254
15m	28022/22265	0.0019859	0.0020072
18m	201347/133590	0.0030513	0.0030983
21m	641/365	0.0043889	0.0044438
24m	732/365	0.0060615	0.0060615
27m	823/365	0.0078968	0.0079551
30m	914/365	0.010033	0.010098
33m	1006/365	0.012450	0.012480
36m	1096/365	0.014994	0.014994
39m	1187/365	0.017753	0.017710
42m	1279/365	0.020648	0.020618
45m	1371/365	0.023643	0.023648
48m	4	0.026630	0.026630
51m	1552/365	0.029597	0.029675
54m	1644/365	0.032582	0.032657
57m	317666/66795	0.035556	0.035584
60m	334091/66795	0.038432	0.038432
63m	350881/66795	0.041626	0.041374
66m	122557/22265	0.044799	0.044359
69m	2103/365	0.048002	0.047446
72m	6	0.050919	0.050330
75m	2282/365	0.053939	0.053396
78m	2376/365	0.056951	0.056535
81m	2467/365	0.059796	0.059568
84m	7	0.062478	0.062478
87m	2649/365	0.065506	0.065539
90m	548/73	0.068377	0.068437
93m	2831/365	0.071173	0.071256
96m	2922/365	0.073881	0.073979
99m	3013/365	0.076488	0.076596
102m	3104/365	0.079001	0.079111
105m	584846/66795	0.081454	0.081560
108m	601271/66795	0.083781	0.083877
111m	618061/66795	0.086093	0.086172
114m	211617/22265	0.088339	0.088396
117m	3561/365	0.090521	0.090551
120m	10	0.092571	0.092571

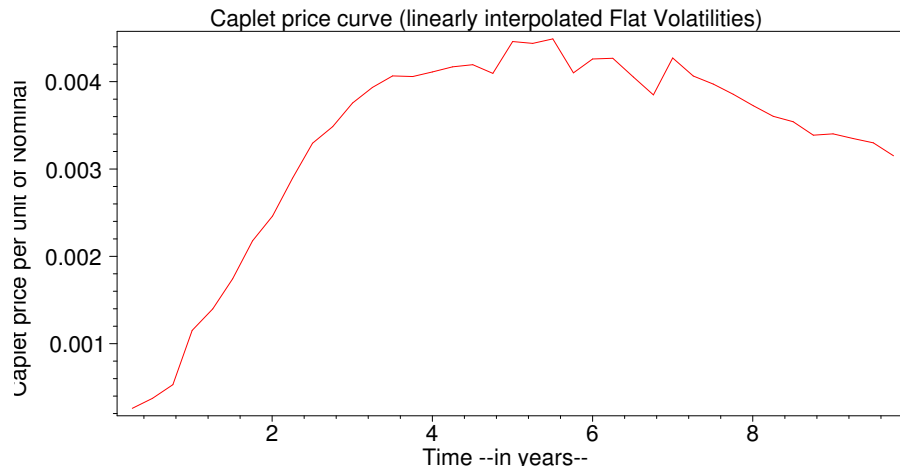


Figure 27: This plot depicts the ATM Caplet prices found using the procedure described in this section. We used linear interpolation (extrapolation for short maturities) of “flat” volatility data for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6). Compare with or results using cubic interpolation of “flat” volatilities, Figure 29.

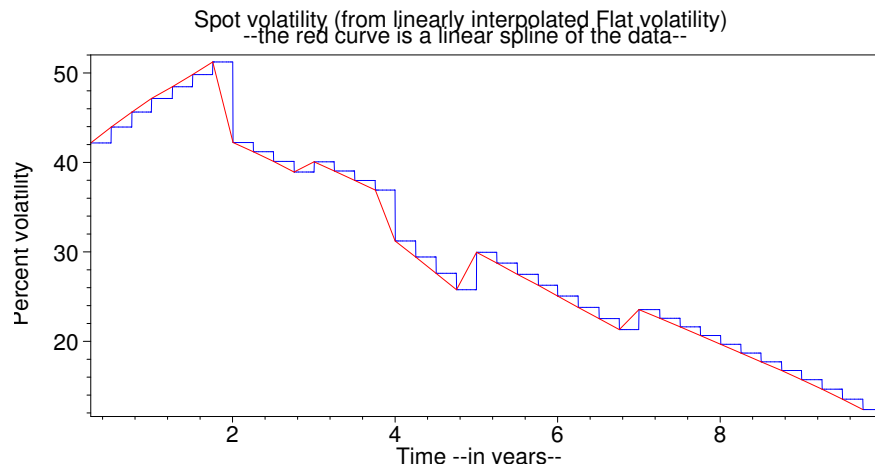


Figure 28: This “curve” was obtained following the Stripping procedure explained in this section using a linear interpolation (extrapolation in case of short maturities) of “flat” volatilities for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6). Since the “flat” volatility curve exhibits a hump, it is expected that spot volatilities will increase for a while before starting to go down with maturity. Compare with or results using cubic interpolation of “flat” volatilities, Figure 30.

Table 9: Spot (Caplet) volatility stripped from Caps data quoted on US Dollars Caps as reported on Bloomberg[©] on May 12th 2003. This table uses the bids at 3:56pm of that day. To be able to find spot volatilities for the first three Caplets we used a linear spline to extrapolate the Flat volatilities of Caps with six and nine months maturity —this way we need not to look for additional data, in practice this is not recommended—. The notation *nnnm*, *nnn* a multiple of 3 between 1 and 120, refers to the month *nnn* after settlement. Volatility is reported as percent volatility, that is spot volatility by 100, and is rounded to five significant figures.

Spot Volatilities (%)					
Term	Volatility	Term	Volatility	Term	Volatility
<i>Year 1</i>		<i>Year 5</i>		<i>Year 9</i>	
		48m–51m	31.226	96m–99m	19.675
3m–6m	42.179	51m–54m	29.429	99m–102m	18.694
6m–9m	43.951	54m–57m	27.606	102m–105m	17.715
9m–12m	45.634	57m–60m	25.771	105m–108m	16.740
<i>Year 2</i>		<i>Year 6</i>		<i>Year 10</i>	
12m–15m	47.150	60m–63m	29.962	108m–111m	15.718
15m–18m	48.454	63m–66m	28.745	111m–114m	14.660
18m–21m	49.820	66m–69m	27.492	114m–117m	13.537
21m–24m	51.225	69m–72m	26.280	117m–120m	12.374
<i>Year 3</i>		<i>Year 7</i>			
24m–27m	42.217	72m–75m	25.064		
27m–30m	41.192	75m–78m	23.799		
30m–33m	40.112	78m–81m	22.542		
33m–36m	38.924	81m–84m	21.313		
<i>Year 4</i>		<i>Year 8</i>			
36m–39m	40.071	84m–87m	23.553		
39m–42m	39.048	87m–90m	22.585		
42m–45m	37.979	90m–93m	21.622		
45m–48m	36.909	93m–96m	20.652		

linear interpolation. The results of such computations are displayed in Figure 29, Figure 30, and Table 10. As the reader may quickly notice, using cubic interpolation/extrapolation of “flat” volatilities results in smoother Caplet Prices (Figure 29) and Spot volatility (Figure 30) curves.

4.5 A few calibration examples, continued

We will continue with our calibration examples, considering the calibration to spot volatilities. Then, in the last sub-subsection we will explain briefly how to put it all together analyzing a very simple example.

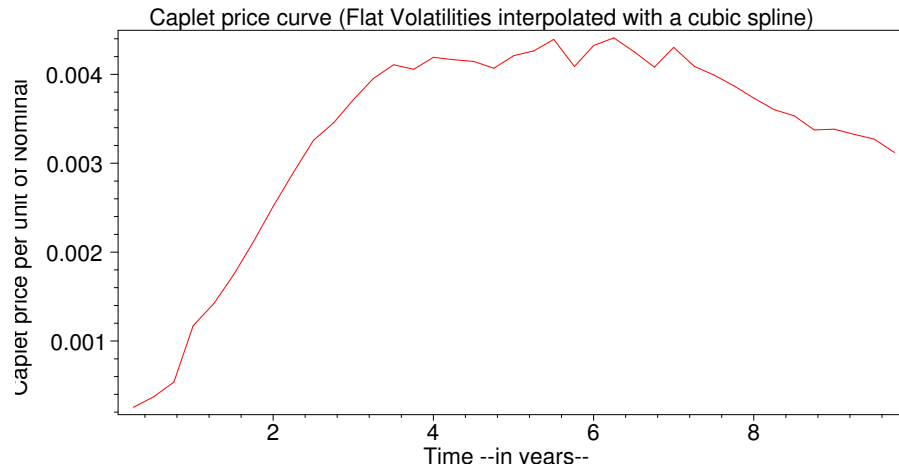


Figure 29: This plot depicts ATM Caplet prices found using the procedure described in this section. To obtain this curve we used cubic interpolation (extrapolation for short maturities) —see Table 7— of “flat” volatility data for bids on US Dollar Caps/Floors quoted on May 12th 2003 (see Table 6). Compare with or results using linear interpolation of “flat” volatilities, Figure 27.

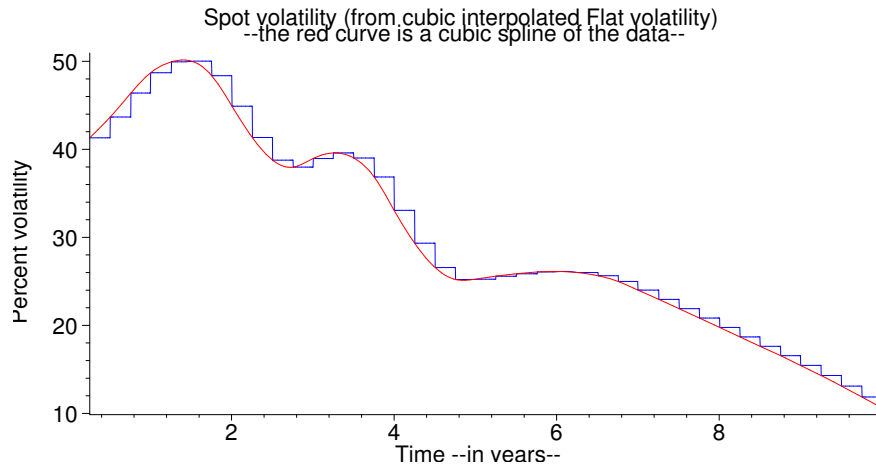


Figure 30: This “curve” was obtained following the Stripping procedure explained in this section using cubic interpolation (extrapolation in case of short maturities) of “flat” volatilities for US Dollar Caps/Floors quoted for May 12th 2003 (see Table 6). Comparing with the results obtained when a linear interpolation of “flat” volatilities was used (Figure 28) we observe that although the general behavior of the spot curve is maintained the resulting curve is smoother.

Table 10: Spot (Caplet) volatility stripped from Caps data quoted on US Dollars Caps as reported on Bloomberg[©] on May 12th 2003. This table uses the bids at 3:56pm of that day. We used a cubic spline to extrapolate the Flat volatilities of Caps with six and nine months maturity so that Spot volatility of the first three Caplets could be found —this way we need not to look for additional data, in practice this is not recommended—. As before, the notation *nnm*, *nnn* a multiple of 3 between 1 and 120, refers to the month *nnn* after settlement. Volatility is reported as percent volatility, that is spot volatility by 100, and is rounded to five significant figures.

Spot Volatilities (%)					
Term	Volatility	Term	Volatility	Term	Volatility
<i>Year 1</i>		<i>Year 5</i>		<i>Year 9</i>	
		48m–51m	33.072	96m–99m	19.766
3m–6m	41.314	51m–54m	29.338	99m–102m	18.692
6m–9m	43.668	54m–57m	26.567	102m–105m	17.622
9m–12m	46.399	57m–60m	25.211	105m–108m	16.558
<i>Year 2</i>		<i>Year 6</i>		<i>Year 10</i>	
12m–15m	48.701	60m–63m	25.247	108m–111m	15.450
15m–18m	49.941	63m–66m	25.586	111m–114m	14.309
18m–21m	50.021	66m–69m	25.863	114m–117m	13.104
21m–24m	48.369	69m–72m	26.064	117m–120m	11.859
<i>Year 3</i>		<i>Year 7</i>			
24m–27m	44.897	72m–75m	26.124		
27m–30m	41.341	75m–78m	26.003		
30m–33m	38.775	78m–81m	25.640		
33m–36m	37.977	81m–84m	24.987		
<i>Year 4</i>		<i>Year 8</i>			
36m–39m	38.959	84m–87m	24.008		
39m–42m	39.590	87m–90m	22.952		
42m–45m	39.019	90m–93m	21.899		
45m–48m	36.863	93m–96m	20.836		

Throughout this subsection we will use the same notation used earlier in §4.2.

4.5.1 All parameters are constant... with a twist

As we saw in §4.2, when all parameters are constant, the Hull–White model (255) reduces to the Vasicek model (328). Vasicek’s model is indeed the simplest case of the Hull–White model and in that case all formulas required in the calibration process reduce to their simplest forms.

In order to calibrate (328) to spot volatility we can use (380). Combining (380) with

(335) (plus the assumption that all parameters are constant) we obtain:

$$\begin{aligned}
v^2(t, T_r, T_p) &= \int_t^{T_r} \gamma^2(u) |\mathcal{S}(u, T_r) - \mathcal{S}(u, T_p)|^2 du \\
&= \frac{a_3^2}{a_2^2} \int_t^{T_r} \left(e^{-a_2(T_r-u)} - e^{-a_2(T_p-u)} \right)^2 du \\
&= \frac{a_3^2}{2a_2^3} \left\{ 1 - e^{-a_2(T_p-T_r)} \right\}^2 \left\{ 1 - e^{-2a_2(T_r-t)} \right\}, \quad (393)
\end{aligned}$$

where we have assumed that $t \leq T_r \leq T_p$.

We can combine our last result, (393), with (377) and (378) to obtain a pricing formula at time $t = 0$, see Lemma 4.3.2

From (393) we have

$$v(t, T_r, T_p) = \frac{a_3}{a_2 \sqrt{2a_2}} \sqrt{1 - e^{-2a_2(T_r-t)}} \left\{ 1 - e^{-a_2(T_p-T_r)} \right\}. \quad (394)$$

Adopting the practice of pricing Caps [Floors] and Caplets [Floorlets] at time $t = 0$, (394) reduces to

$$v(0, T_r, T_p) = \frac{a_3}{a_2 \sqrt{2a_2}} \sqrt{1 - e^{-2a_2 T_r}} \left\{ 1 - e^{-a_2(T_p-T_r)} \right\}. \quad (395)$$

From Lemma 4.3.2 we know that for any $0 < T_r \leq T_p \leq \mathcal{T}$, the arbitrage price at time $t = 0$ of a caplet with expiry date T_r , settlement date $t = 0$, payment (delayed to) date T_p , and strike level \mathcal{K} on a floating rate Vr is given by the formula

$$\begin{aligned}
\mathbf{Cpl}(0, T_r, T_p, \varphi, Vr, \mathcal{K}, v) \\
= B(0, T_r) \Phi(e(0, T_r, T_p)) - (1 + \mathcal{K} \varphi(T_r, T_p)) B(0, T_p) \Phi(e(0, T_r, T_p) - v(0, T_r, T_p)) \quad (396)
\end{aligned}$$

where

$$e(0, T_r, T_p) = \frac{1}{v(0, T_r, T_p)} \ln \left(\frac{B(0, T_r)}{(1 + \mathcal{K} \varphi(T_r, T_p)) B(0, T_p)} \right) + \frac{1}{2} v(0, T_r, T_p), \quad (397)$$

and $v = v(0, T_r, T_p)$ is given by (395). As before, Φ is the cumulative normal distribution, and $B(0, T_r)$ and $B(0, T_p)$ are the time $t = 0$ prices of zcb's of maturities T_r and T_p whose yield is given by rate Vr .

With the exception of $v(0, T_r, T_p)$, all other parameters involved in (396) and (397) are known from initial data (the strike level \mathcal{K} being taken as the corresponding ATM Swap rate \mathcal{K}_{ATM} of Definition 4.4.1).

On the other hand, we notice that (395) contains not all the parameters of models (328) or (343), but it does contain all parameters in those models that determine the volatility structure of bond options priced under such models. Thus, (395) plus (396) and (397) in combination with Definition 4.3.3 offers us an equation suitable for calibration to volatilities (we will use $t = 0$ in either (386) or (387), see (398) and (399) below).

That is, assuming a nominal of one dollar, $\mathcal{N} = 1$

$$\mathbf{Cpl}(0, T_r, T_p, \varphi, Vr(T_r, T_p), \mathcal{K}_{ATM}, \mathbf{v}) = \mathbf{Cpl}^M(0, T_r, T_p, \varphi, Vr(T_r, T_p), \mathcal{K}_{ATM}, \sigma_{r,p}) \quad (398)$$

which, by definition of Vr , can also be expressed in terms of time zero prices of zcb's as:

$$\begin{aligned} \mathbf{Cpl}(0, T_r, T_p, \varphi, B(0, T_r), B(0, T_p), \mathcal{K}_{ATM}, \mathbf{v}) \\ = \mathbf{Cpl}^M(0, T_r, T_p, \varphi, B(0, T_r), B(0, T_p), \mathcal{K}_{ATM}, \sigma_{r,p}) \end{aligned} \quad (399)$$

No matter which version we choose, (398) or (399), the right-hand-side is known to us as a sub-product of the stripping process (see Table 11).

Some of the problems we face are, amongst others, the following. i) Expression (395) contains only two parameters while we may have up to 39 pairs of dates and ATM Caplet prices (or which is the same 39 pairs of dates and spot volatilities, see Table 9 and/or Table 11). In most cases, a least squares fit to data will not be able to produce “good” volatility curves. ii) Market practice assumes piecewise constant volatilities and a model with constant coefficients will not be able to reproduce such a behavior mainly because the accepted market practice models each forward rate independently. iii) Even if a more complex version of the Hull–White model is used (which may let us perfectly fit the model to Caplet data) there is no way to know if we will be able to preserve the shape of the volatility curve, reproduce humps, etc. (some of these issues could be addressed assuming a more complex calibration process, and in particular including into the calibration process some form of volatility surface information, for example, Swaptions data).

Notice that if one is interested only in a model capable of reproducing spot volatility for a given maturity, that is, capable of reproducing a particular Caplet’s price, a model like (328) could be enough, and only one solution of (398) or (399) will be required.

Table 11: We show here the implied ATM Market prices of Caplets of all available maturities. Prices, rounded to five significant figures, computed using both methods of interpolation (linear and cubic) of Flat volatilities are shown.

ATM Caplet Prices per unit of Nominal			
Period	TtM (act/act)	Linear I/E	Cubic I/E
0m–3m	93/365		
3m–6m	37/73	0.00025963	0.00025433
6m–9m	50851/66795	0.00037397	0.00037152
9m–12m	66911/66795	0.00053049	0.00053828
12m–15m	28022/22265	0.0011516	0.0011723
15m–18m	201347/133590	0.0013978	0.0014208
18m–21m	641/365	0.0017432	0.0017468
21m–24m	732/365	0.0021775	0.0021189
24m–27m	823/365	0.0024614	0.0025218
27m–30m	914/365	0.0028944	0.0028981
30m–33m	1006/365	0.0032955	0.0032586
33m–36m	1096/365	0.0034848	0.0034566
36m–39m	1187/365	0.0037547	0.0037170
39m–42m	1279/365	0.0039338	0.0039542
42m–45m	1371/365	0.0040659	0.0041081
45m–48m	4	0.0040590	0.0040571
48m–51m	1552/365	0.0041107	0.0041922
51m–54m	1644/365	0.0041694	0.0041653
54m–57m	317666/66795	0.0041944	0.0041461
57m–60m	334091/66795	0.0040954	0.0040691
60m–63m	350881/66795	0.0044595	0.0042106
63m–66m	122557/22265	0.0044383	0.0042635
66m–69m	2103/365	0.0044897	0.0043936
69m–72m	6	0.0041010	0.0040888
72m–75m	2282/365	0.0042592	0.0043248
75m–78m	2376/365	0.0042675	0.0044109
78m–81m	2467/365	0.0040531	0.0042529
81m–84m	7	0.0038490	0.0040820
84m–87m	2649/365	0.0042715	0.0043036
87m–90m	548/73	0.0040648	0.0040902
90m–93m	2831/365	0.0039718	0.0039912
93m–96m	2922/365	0.0038560	0.0038690
96m–99m	3013/365	0.0037241	0.0037306
99m–102m	3104/365	0.0036035	0.0036033
102m–105m	584846/66795	0.0035414	0.0035347
105m–108m	601271/66795	0.0033875	0.0033747
108m–111m	618061/66795	0.0034021	0.0033832
111m–114m	211617/22265	0.0033490	0.0033249
114m–117m	3561/365	0.0032997	0.0032713
117m–120m	10	0.0031511	0.0031205

Another possibility is to proceed in accordance to market practice. Although Vasicek's model (328) is not lognormal (or one in which forward rates are lognormal) it is still evident that given one number $\sigma_{r,p}$ and a time interval $[T_r, T_p]$, one can always find numbers a_2 and a_3 such that (399) is satisfied. Finding such numbers one could calibrate a model of the form of (328) to each Caplet volatility, but, as in the case of market practice and Black's model, n models are required.

If we are to price one single instrument for a given maturity, the procedure we mentioned here will put spot volatility information regarding the maturity of our instrument into the interest rate model.

That is, at the time interval $[T_r, T_p]$, $T_r < T_p$, $T_r = 6\text{m}, 9\text{m}, \text{etc.}$, T_r , T_p , and $\sigma_{r,p}$ are known. One could fix $r_0 = 0$, $a_1 = 0$, and one of a_2 or a_3 ,⁴⁴ for example⁴⁵ $a_2 = 10$ and use the remaining parameter to satisfy (399), or in case possible a piecewise constant function that will help to fit the data from Table 6 to the resulting model

$$dr_t = -a_2 r_t dt + a_3 dW_t \quad (400)$$

This will be the “twist” we mentioned in the title of this sub-subsection.

Using the results of the Stripping process described in the previous section, we used (396) to find, for each time period $[T_r, T_p]$, the numbers $\mathbf{v}^{EV}(0, T_r, T_p)$ that, when plugged back into (396), will reproduce the implied Market prices of Caplets listed in Table 11, Table 12 lists the results of our computations. The “goodness” of the numbers in Table 12 can be easily tested using (398) or (399) to solve for new implied Spot volatilities, these numbers and those listed in Table 9 and/or in Table 10 (depending on which prices were used) should match (rounding errors considered). Figure 31 show that the accuracy of our

⁴⁴Or one could use any other values determined by other means, etc..

⁴⁵Notice that the procedures here outlined still leave at least one parameter “free”, a best choice of such parameter can be determined by comparison of the Market prices of other instruments (for example captions, or swaptions, etc.) with the prices of such instruments implied by the model. I selected $a_2 = 10$ to exaggerate the features of some curves, features that will be difficult to discern if smaller values of a_2 are used. This selection of $a_2 = 10$ may not be optimal. Again, our choice was done solely to exaggerate the features of the curves in Figure 37 with no concern about the goodness of the model.

Table 12: We show here the implied ATM Hull–White Caplet volatilities, rounded to five significant figures, for all available maturities. Prices computed using both methods of interpolation (linear and cubic) of Flat volatilities were used.

Percent $v^{EV}(0, T_r, T_p)$			
Period	TtM (act/act)	Linear I/E	Cubic I/E
0m–3m	93/365		
3m–6m	37/73	0.065294	0.063960
6m–9m	50851/66795	0.096866	0.096248
9m–12m	66911/66795	0.11976	0.12174
12m–15m	28022/22265	0.17827	0.18402
15m–18m	201347/133590	0.21585	0.22231
18m–21m	641/365	0.26420	0.26523
21m–24m	732/365	0.32078	0.30352
24m–27m	823/365	0.31286	0.33208
27m–30m	914/365	0.35398	0.35523
30m–33m	1006/365	0.39709	0.38428
33m–36m	1096/365	0.42149	0.41158
36m–39m	1187/365	0.48064	0.46781
39m–42m	1279/365	0.51537	0.52224
42m–45m	1371/365	0.54253	0.55674
45m–48m	4	0.54890	0.54824
48m–51m	1552/365	0.51669	0.54617
51m–54m	1644/365	0.52010	0.51854
54m–57m	317666/66795	0.51845	0.49945
57m–60m	334091/66795	0.50055	0.48994
60m–63m	350881/66795	0.62166	0.52665
63m–66m	122557/22265	0.62521	0.55857
66m–69m	2103/365	0.63850	0.60184
69m–72m	6	0.58861	0.58395
72m–75m	2282/365	0.61574	0.64093
75m–78m	2376/365	0.61969	0.67518
78m–81m	2467/365	0.58895	0.66723
81m–84m	7	0.55697	0.64990
84m–87m	2649/365	0.67591	0.68853
87m–90m	548/73	0.64739	0.65759
90m–93m	2831/365	0.63732	0.64525
93m–96m	2922/365	0.62443	0.62985
96m–99m	3013/365	0.60900	0.61174
99m–102m	3104/365	0.59222	0.59216
102m–105m	584846/66795	0.58098	0.57799
105m–108m	601271/66795	0.55050	0.54465
108m–111m	618061/66795	0.54175	0.53271
111m–114m	211617/22265	0.51841	0.50620
114m–117m	3561/365	0.49117	0.47567
117m–120m	10	0.44564	0.42732

Table 13: Hull–White volatility parameters implied from Market data for all available maturities. Volatility parameters were computed using both methods of interpolation (linear and cubic) of Flat volatilities assuming $a_2 = 10$, the resulting values were rounded to five significant figures.

Volatility parameters $\gamma = a_3$		
Period	Linear I/E	Cubic I/E
0m–3m		
3m–6m	0.031851	0.031201
6m–9m	0.047011	0.046712
9m–12m	0.058876	0.059849
12m–15m	0.086346	0.089131
15m–18m	0.10529	0.10844
18m–21m	0.12884	0.12935
21m–24m	0.15638	0.14797
24m–27m	0.15252	0.16189
27m–30m	0.17257	0.17318
30m–33m	0.19311	0.18689
33m–36m	0.20600	0.20115
36m–39m	0.23432	0.22806
39m–42m	0.25064	0.25398
42m–45m	0.26384	0.27075
45m–48m	0.26896	0.26863
48m–51m	0.25128	0.26561
51m–54m	0.25293	0.25218
54m–57m	0.25221	0.24297
57m–60m	0.24479	0.23960
60m–63m	0.30251	0.25627
63m–66m	0.30424	0.27181
66m–69m	0.30917	0.29142
69m–72m	0.28998	0.28768
72m–75m	0.29945	0.31170
75m–78m	0.29997	0.32683
78m–81m	0.28712	0.32528
81m–84m	0.27364	0.31930
84m–87m	0.32718	0.33329
87m–90m	0.31560	0.32058
90m–93m	0.31070	0.31456
93m–96m	0.30441	0.30706
96m–99m	0.29689	0.29823
99m–102m	0.28871	0.28868
102m–105m	0.28263	0.28117
105m–108m	0.26921	0.26635
108m–111m	0.26363	0.25922
111m–114m	0.25227	0.24633
114m–117m	0.23894	0.23140
117m–120m	0.21836	0.20938

solutions is higher than the precision of the input data.

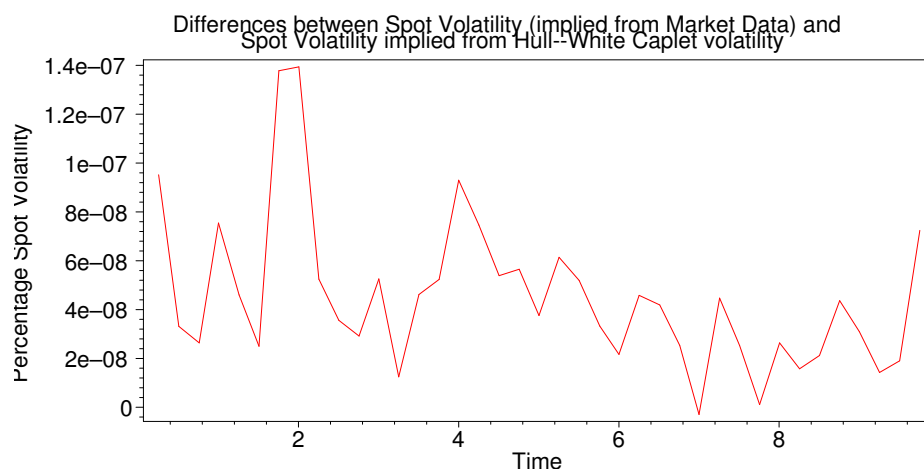


Figure 31: Using the results in Table 11 and Table 12 (that is, implied Hull–White caplet volatilities) we found the Black volatilities implied by such data and compared back with the original data (Table 9 and Table 10). This plot shows the percentage difference between the stripped spot volatility and the Black spot volatility implied by the results in Table 12 (linear case), the differences are well below the precision of the original market data (see Table 6).

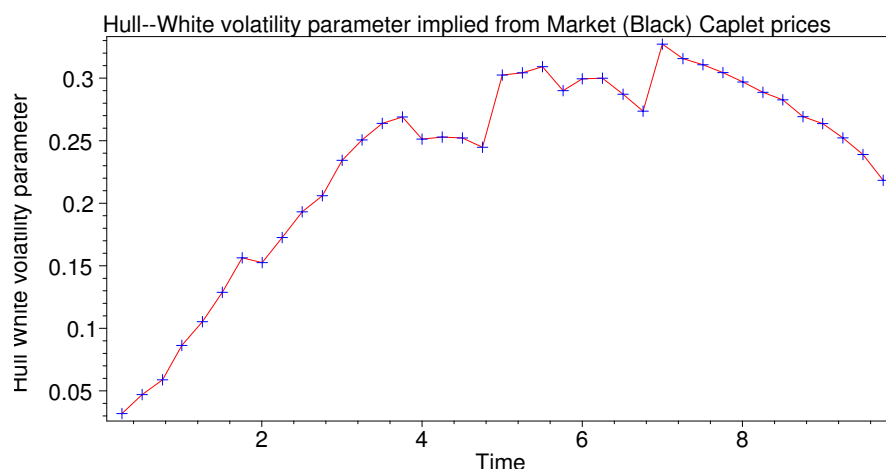


Figure 32: Hull–White volatility parameters obtained from linear approximation of Cap “flat” Volatility data, see Table 13.

Later, we used (395), assuming $a_2 = 10$, to find the volatility parameter a_2 for each maturity. See Figure 32, Figure 33 and Table 13.

No matter the path chosen, one may safely assume that the mean reversion coefficient

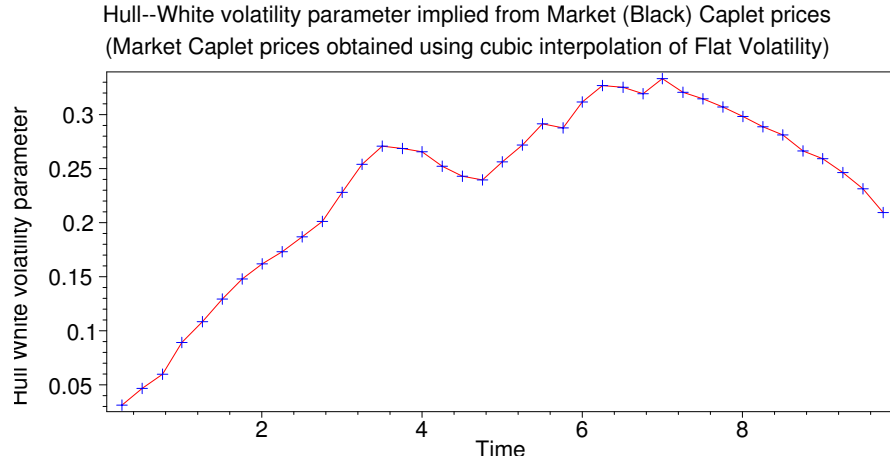


Figure 33: Hull–White volatility parameters obtained from cubic approximation of Cap “flat” Volatility data, see Table 13.

and the initial value of the rate, a_1 and r_0 , are null.

Thus, we have assumed that the coefficients of (400) are constants and possibly different for each maturity, which allows us to obtain the results displayed in Table 12 and Table 13. This kind of assumption may not be satisfactory since implicitly we are assuming a different model of the form of (400) for every maturity shown in Table 8.

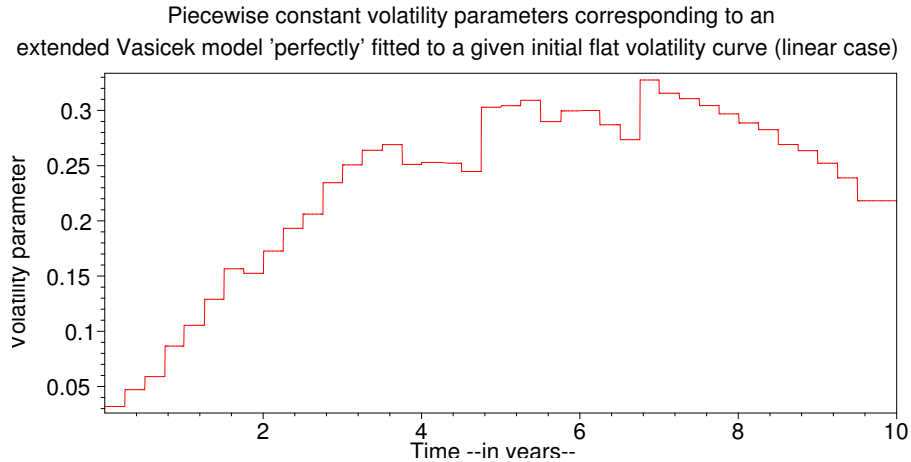


Figure 34: It is also market practice to fit spot volatility to a model of interest rates using a piecewise constant volatility parameter. Here we show the result of the solution of the system of equations (404) and (405) using Hull–White volatility from the second column of Table 12. The corresponding 39 volatility parameters are displayed on Table 14.

Another possibility is to use a model in which not all parameters are constant. For example, assuming $\alpha(t) \equiv a_1$ constant, and β and γ functions of time, one may assign particular functional forms to $\beta(t)$ and $\gamma(t)$ and use a least squares procedure to better fit such functions to the data.

Following that line of reasoning, a simple approach to consider could be to calibrate a model similar to (328), assuming piecewise constant coefficients a_2 and/or a_3 . To fix ideas, assume, as before, that a_1 and r_0 , are null and that $a_2 = 10$. Assume also that

$$\gamma(t) = k_0 \mathbb{1}_{[0, T_1[}(t) + \sum_{i=1}^{i=38} k_i \mathbb{1}_{[T_{i-1}, T_i[}(t) \quad (401)$$

where k_i , $i \in \mathbb{N}_{38}^*$, are 39 constants to be determined, and as usual the T_i , $i \in \mathbb{N}_{39}^*$ are used to represent all the reset and maturity dates we have at our disposal (see the first column of Table 8, for a list of such resets). $\mathbb{1}_A$ is the indicating function of set A . Thus, we want to calibrate the model

$$dr_t = a_2 r_t dt + \gamma(t) dW_t, \quad (402)$$

to initial spot volatility data such as that contained in Table 9 or in Table 10, which we have constructed in the previous section as a result of the stripping process applied to flat volatility data contained in Table 6 (see previous section for a description of that process).

Piecewise constant volatility parameters corresponding to an extended Vasicek model 'perfectly' fitted to a given initial flat volatility curve (cubic case)

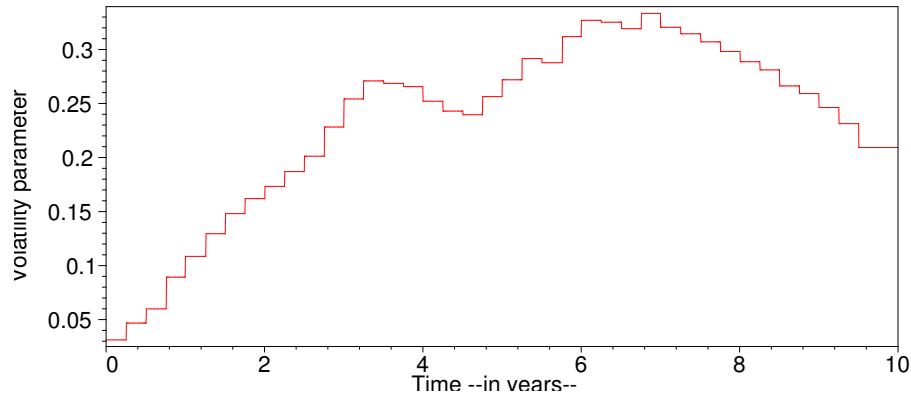


Figure 35: Here we show the result of the solution of the system of equations (404) and (405) using Hull–White volatility from the third column of Table 12. The corresponding 39 volatility parameters are displayed on Table 14.

Plugging (401) and (335) into , we obtain (for $t = 0$):

$$\begin{aligned}
\mathbf{v}^2(0, T_i, T_{i+1}) &= \int_0^{T_i} \gamma^2(u) |\mathbf{S}(u, T_i) - \mathbf{S}(u, T_{i+1})|^2 du \\
&= \frac{k_0^2}{a_2^2} \int_0^{T_0} \left(e^{-a_2(T_i-u)} - e^{-a_2(T_{i+1}-u)} \right)^2 du + \sum_{j=1}^i \frac{k_j^2}{a_2^2} \int_{T_{j-1}}^{T_j} \left(e^{-a_2(T_i-u)} - e^{-a_2(T_{i+1}-u)} \right)^2 du \\
&= \frac{1}{2a_2^3} \left(e^{-a_2 T_i} - e^{-a_2 T_{i+1}} \right)^2 \left(k_0^2 (e^{2a_2 T_0} - 1) + \sum_{j=1}^i k_j^2 (e^{2a_2 T_j} - e^{2a_2 T_{j-1}}) \right), \quad (403)
\end{aligned}$$

adopting the usual convention that a sum adds to zero when its lower limit is a unit larger than its upper limit, the above expression should be valid for $i \in \mathbb{N}_{38}^*$. Notice also that the only unknowns in (403) are the 39 parameters that determine $\gamma(t)$ according to (401) (for each $i \in \mathbb{N}_{38}^*$, the values $100 \times \mathbf{v}(0, T_i, T_{i+1})$ are listed in Table 12).

Equation (403) defines a system of 39 equations that we can solve explicitly

$$k_0^2 = \frac{2a_2^3 \mathbf{v}^2(0, T_0, T_1)}{(e^{-a_2 T_0} - e^{-a_2 T_1})^2} \quad (404)$$

$$\begin{aligned}
k_i^2 &= \frac{1}{e^{2a_2 T_i} - e^{2a_2 T_{i-1}}} \left(\frac{2a_2^3 \mathbf{v}^2(0, T_i, T_{i+1})}{(e^{-a_2 T_i} - e^{-a_2 T_{i+1}})^2} \right. \\
&\quad \left. - k_0^2 (e^{2a_2 T_0} - 1) - \sum_{j=1}^{i-1} k_j^2 (e^{2a_2 T_j} - e^{2a_2 T_{j-1}}) \right), \quad i \in \mathbb{N}_{38}. \quad (405)
\end{aligned}$$

Table 14 contains the values of the k_i , $i \in \mathbb{N}_{38}^*$ obtained from Table 12 and (404) and (405). Figure 34, and, Figure 35, show plots corresponding to those results.

4.5.2 A case of shifted rates

A very simple way to put it all together is to consider a case of *shifted rates*. Note that this “trick” can be applied not only to Hull–White models but to any interest rate model.

Consider for example the following model

$$\begin{aligned}
r_t &= \Psi(t) + \mathcal{R}_t \\
d\mathcal{R}_t &= -a_2 \mathcal{R}_t dt + a_3 dW_t \quad \mathcal{R}_0 = 0
\end{aligned} \quad (406)$$

where $\Psi(t)$ is a deterministic differentiable function of t to be determined and both a_2 and a_3 are known to be non-zero constants. The reader will quickly notice that this model is

Table 14: Piecewise constant volatility coefficients, rounded to five significant figures, were obtained from data shown in Table 12 and the solution of the system of equations given by (404) and (405). See Figure 34 and Figure 35 for plots of the corresponding deterministic piecewise volatility parameter functions.

Piecewise Volatility Coefficients		
Period	Linear I/E	Cubic I/E
0m–3m	0.031851	0.031201
3m–6m	0.047094	0.046795
6m–9m	0.058942	0.059921
9m–12m	0.086536	0.089332
12m–15m	0.10539	0.10855
15m–18m	0.12899	0.12948
18m–21m	0.15656	0.14809
21m–24m	0.15249	0.16198
24m–27m	0.17270	0.17325
27m–30m	0.19325	0.18698
30m–33m	0.20608	0.20124
33m–36m	0.23451	0.22824
36m–39m	0.25074	0.25414
39m–42m	0.26393	0.27086
42m–45m	0.26899	0.26862
45m–48m	0.25114	0.26559
48m–51m	0.25295	0.25209
51m–54m	0.25220	0.24290
54m–57m	0.24474	0.23958
57m–60m	0.30289	0.25639
60m–63m	0.30425	0.27191
63m–66m	0.30920	0.29154
66m–69m	0.28986	0.28766
69m–72m	0.29953	0.31190
72m–75m	0.29997	0.32693
75m–78m	0.28704	0.32527
78m–81m	0.27354	0.31925
81m–84m	0.32758	0.33341
84m–87m	0.31554	0.32050
87m–90m	0.31066	0.31452
90m–93m	0.30437	0.30700
93m–96m	0.29684	0.29817
96m–99m	0.28865	0.28862
99m–102m	0.28258	0.28112
102m–105m	0.26912	0.26625
105m–108m	0.26358	0.25917
108m–111m	0.25219	0.24624
111m–114m	0.23885	0.23130
114m–117m	0.21822	0.20923

nothing more than a sub-case of (255). Since we are assuming that $\Psi(t)$ is differentiable, by (406) we have $dr_t = d\Psi(t) + d\mathcal{R}_t = d\Psi(t) - a_2\mathcal{R}_t dt + a_3 dW_t = d\Psi(t) - a_2(r_t - \Psi(t)) dt + a_3 dW_t = \left(\frac{d\Psi(t)}{dt} + a_2\Psi(t) - a_2r_t \right) dt + a_3 dW_t = (\alpha(t) - a_2r_t) dt + a_3 dW_t$.

Let $B_{\mathcal{R}}(\cdot, \cdot)$ denote the prices of zcb's under "rate" \mathcal{R} . Similarly, let $B_M(\cdot, \cdot)$ denote the market prices of zcb's and $B(\cdot, \cdot)$ denote the prices of zcb's under rate r . Let $T \in [0, T]$ by (267) we have

$$B(0, T) = E \left(\exp \left\{ - \int_0^T r_u du \right\} \middle| \mathcal{F}_t \right) = \exp \left(\int_0^T \Psi(u) du \right) B_{\mathcal{R}}(0, T) \quad (407)$$

If model (406) is to fit the initial yield curve then $B(0, T) = B_M(0, T)$ when $T = t_i$, $i \in \mathbb{N}_{24}$, is one of the known maturities⁴⁶ from Table 4. Thus for each maturity t_i , $i \in \mathbb{N}_{24}$

$$\int_0^{t_i} \Psi(u) du = \ln \left(\frac{B_{\mathcal{R}}(0, t_i)}{B_M(0, t_i)} \right) \quad (408)$$

Clearly, we can interpolate the right hand side of (408) using splines, from where an explicit form for $\Psi(t)$ can be obtained.

In this way, we perfectly fit (406) to the initial yield curve while the other parameters, a_2 and a_3 can be used to introduce initial volatility structure into the model.

In most situations, the explicit knowledge of function $\Psi(t)$ is not required. In such a case we know that rate r also satisfies

$$dr_t = (\alpha(t) - a_2r_t) dt + a_3 dW_t, \quad (409)$$

where

$$\alpha(t) = \frac{d\Psi(t)}{dt} + a_2\Psi(t). \quad (410)$$

Thus, we can use the results of the previous subsection plus equation (352) to obtain a perfect fit to the initial yield that also contain information regarding initial volatility structure. Figure 36 and Figure 37 display curves obtained following this approach.

Similarly, we could use a model like:

$$\begin{aligned} r_t &= \Psi(t) + \mathcal{R}_t \\ d\mathcal{R}_t &= -a_2\mathcal{R}_t dt + \gamma(t) dW_t \quad \mathcal{R}_0 = 0 \end{aligned} \quad (411)$$

⁴⁶More properly, when T denotes the corresponding time fraction to the maturity t_i as described in Chapter 2.

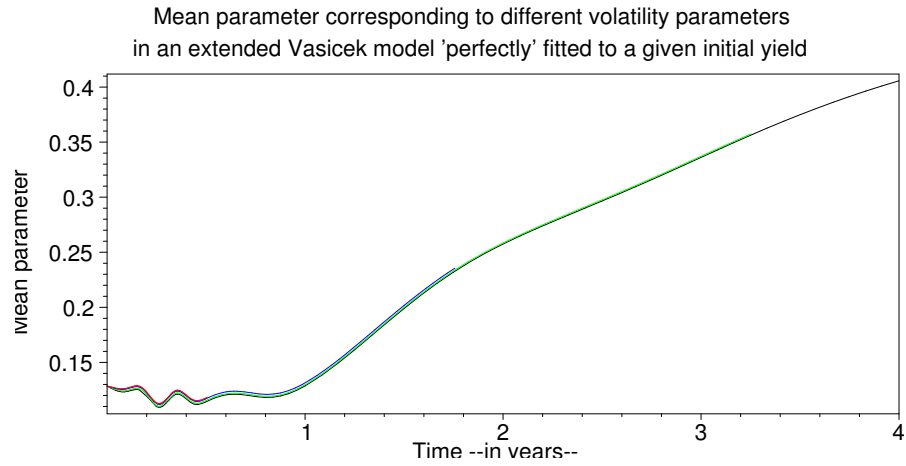


Figure 36: Several Hull–White mean parameter functions corresponding to perfect fits to initial yield data from Table 4 are displayed. Initial volatility information in the form of data displayed in Table 13 was used to construct these curves.

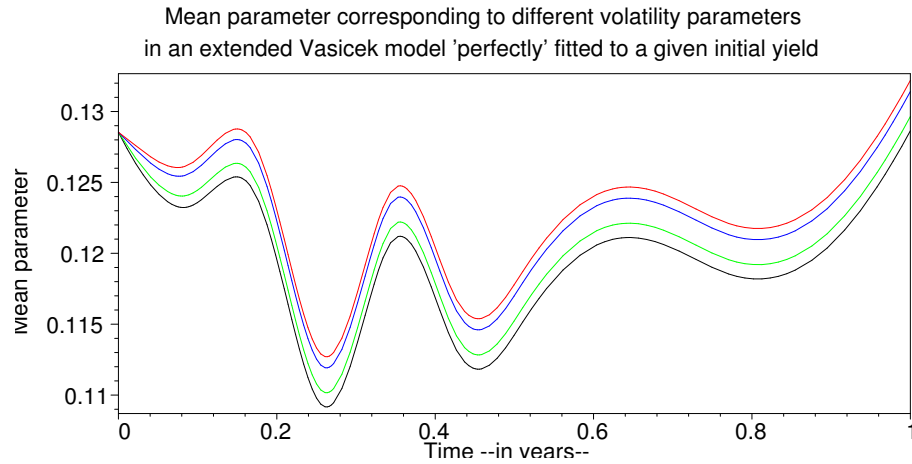


Figure 37: We show a zoom into Figure 36. Several Hull–White mean parameter functions corresponding to perfect fits to initial yield data from Table 4 are displayed. Initial volatility information in the form of data displayed in Table 13 was used to construct these curves

where a_2 is known to be constant and $\gamma(t)$ is a deterministic, integrable, function of t . Such a function $\gamma(t)$ could be used to fit to several Caplet volatilities.

Here we retake the findings of the previous sub-section where piecewise constant functions were constructed to perfectly fit model (411) with $a_2 = 10$ to initial spot volatility. At the same time the idea of shifted rates could be used to find a perfect fit to the initial yield curve. Figure 34 and Figure 35 show the piecewise constant volatility parameters, see also Figure 38, Figure 39, Figure 40, and Figure 41, for depictions of the corresponding mean parameter functions that will perfectly fit the shifted rate to initial yield data.

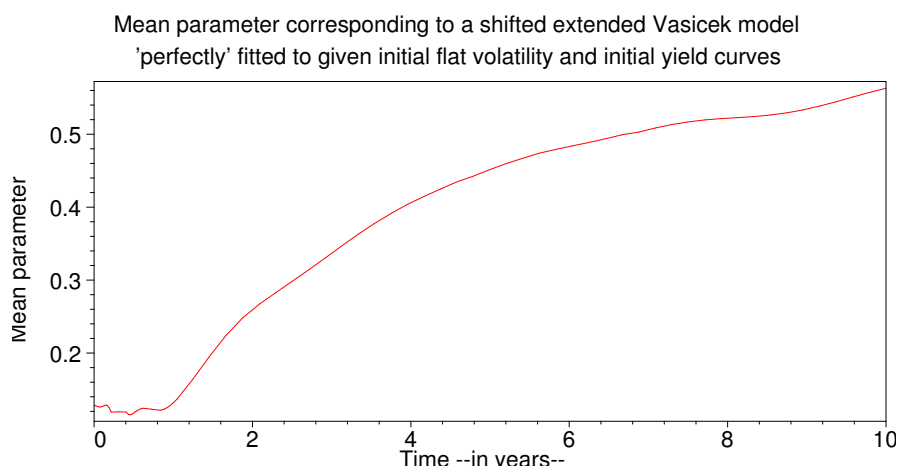


Figure 38: Hull-White mean parameter function corresponding to a perfect fit of the model (411) to initial yield (see Table 4) and a perfect fit to initial spot volatility (obtained from linearly interpolated flat volatility, see Table 9 and Table 7).

Even though we have used two different approaches to reconstruct caplet prices, and consequently spot volatility, the numerical differences between the two resulting mean parameter functions are small. To help see those differences we are including Figure 42 which shows a plot of the difference between the the mean parameter functions plotted in Figure 38 and Figure 40.

As expected, model (411) not only allows for the perfect fitting to initial yield and spot volatility but also will produce richer bond volatility curves. Figure 43 and Figure 44 show plots of bond volatility curves implied by model (411), compare with Figure 20, Figure 10, and Figure 13 that show samples of bond volatility curves corresponding to Vasicek's model

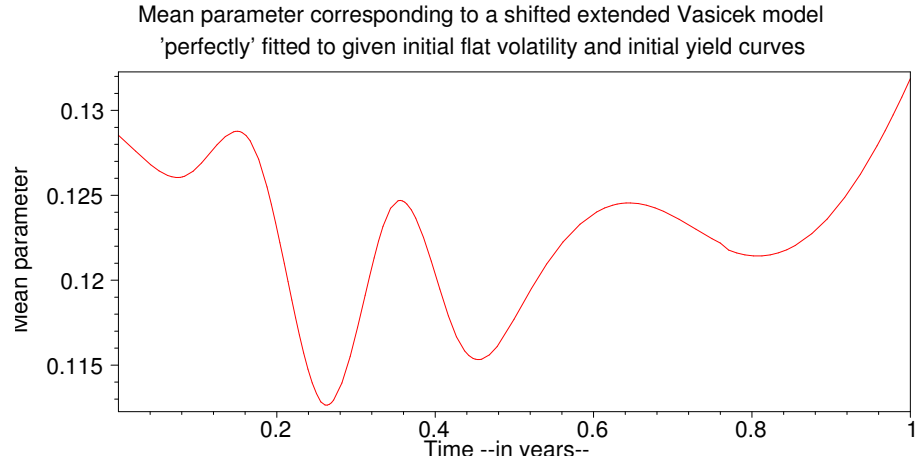


Figure 39: Detail of Figure 38 for short maturities.

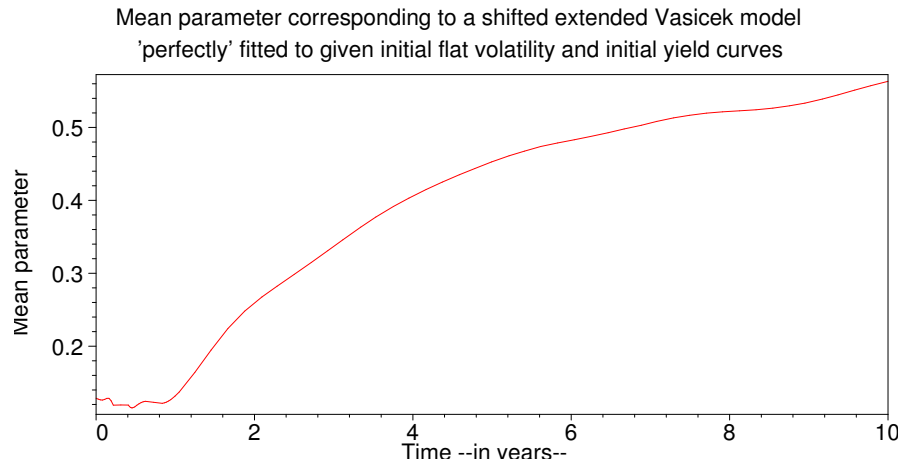


Figure 40: Hull-White mean parameter function corresponding to a perfect fit of the model (411) to initial yield (see Table 4) and a perfect fit to initial spot volatility (obtained from cubically interpolated flat volatility, see Table 9 and Table 7).

(328) and Hull-White model with constant mean and speed of mean reversion parameters and time dependent volatility parameter (we showed in a previous section how to calibrate such a model to initial yield data).

4.6 *Final remarks*

In this chapter we have presented a calibration procedure to calibrate the Hull-White model of interest rates, (255), to market data. Both, calibration to an initial yield curve and

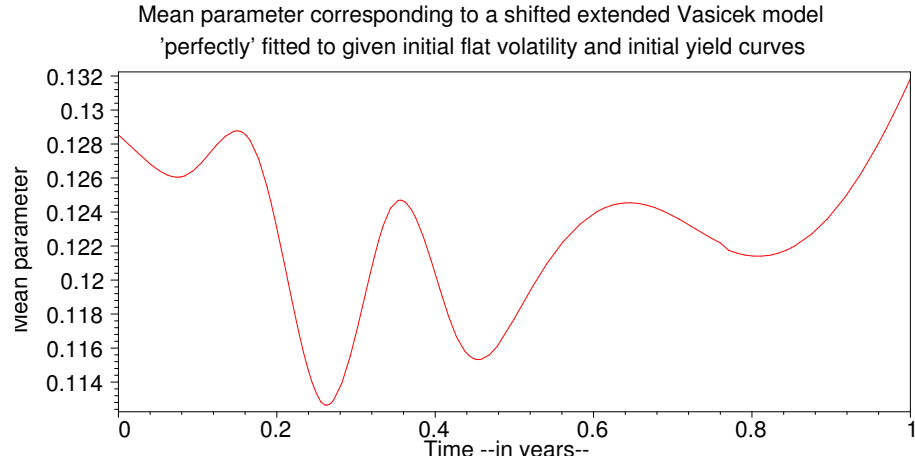


Figure 41: Detail of Figure 40 for short maturities.

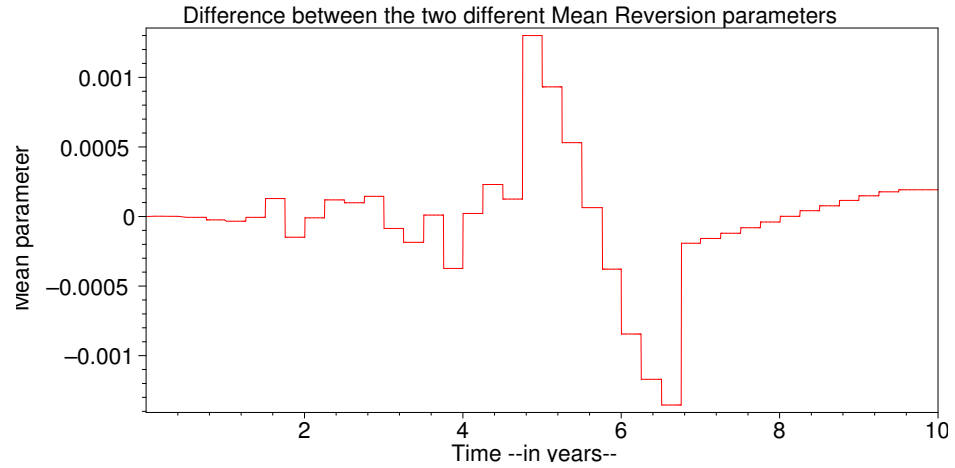


Figure 42: Numerical differences between Figure 38 and Figure 40

calibration to an initial term structure of volatilities has been considered. We have given also several examples showing how to proceed in different situations (different sub-cases of (255) and or initial information to be used in the calibration process). We have also given a short explanation showing what to do if one needs to integrate both kinds of data (initial yield and initial term structure of volatilities) into the calibration.

Our motivation in this chapter has been to explicitly present a calibration process, following accepted market conventions, that we could later use in the numerical valuation of some Game Options. Scattered through the literature we have found several times the

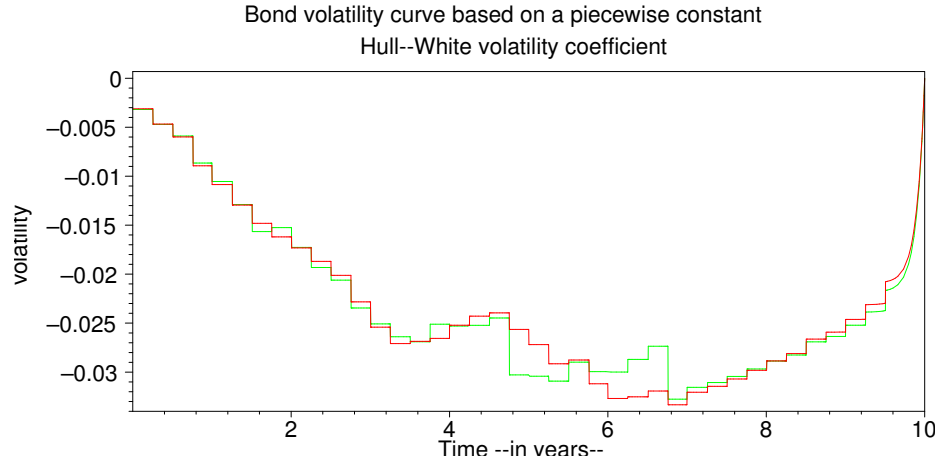


Figure 43: Bond volatility curves (see (295), (335) and (401)) when Hull–White model is fitted to initial spot volatility data. We show curves corresponding to a maturity of ten years (observe that such curves should be negative when both Hull–White volatility parameter and the corresponding affine slope functions are positive). Compare with Figure 20, Figure 10, and Figure 13 which show implied bond volatility curves obtained in previous examples. Curves were obtained using linear and cubic interpolation of flat volatility.

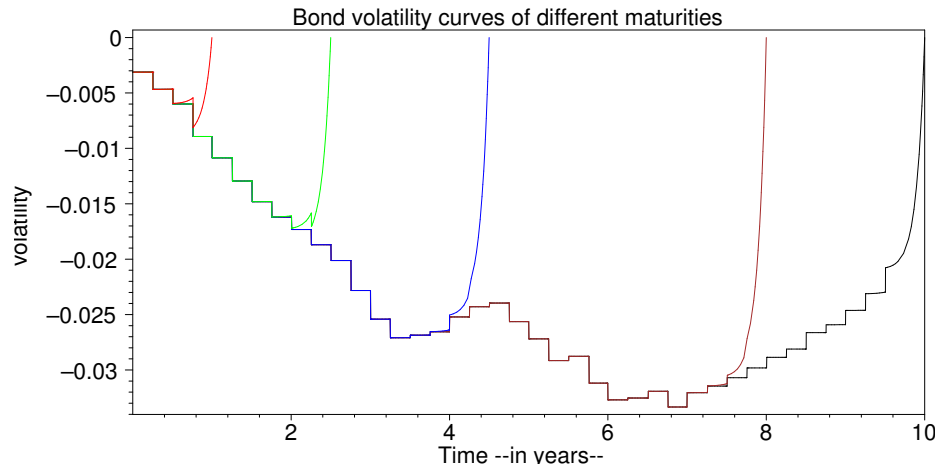


Figure 44: Bond volatility curves (see (295), (335) and (401)) when Hull–White model is fitted to initial spot volatility data. We show curves corresponding to different maturities between one and ten years. Compare with Figure 20, Figure 10, and Figure 13 which show implied bond volatility curves obtained in previous examples. Curves were obtained using cubic interpolation of flat volatility.

Table 15: Formulas used in the calibration of the Hull White model

Parameter	Formula	#	Comments
$\alpha(t)$	$-\frac{\frac{d^2}{dt^2}\mathcal{S}(0,t)}{\frac{d}{dt}\mathcal{S}(0,t)}\frac{d}{dt}\mathcal{I}(0,t) + \frac{d^2}{dt^2}\mathcal{I}(0,t) + \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t}\mathcal{S}(v,t) \right\}^2 dv$	(321)	This formula depends only on initial data, we can use it to determine α if $\mathcal{S}(0,t)$, $\mathcal{I}(0,t)$ and γ are known.
$\beta(t)$	$-\frac{\frac{d^2}{dt^2}\mathcal{S}(0,t)}{\frac{d}{dt}\mathcal{S}(0,t)}$	(316)	This formula depends only on initial data, we can use it to determine β if $\mathcal{S}(0,t)$ is known.
$\gamma(t)$	$\gamma^2(t) = 2\beta(t) \left\{ \alpha(t) - \beta(t)\frac{d}{dt}\mathcal{I}(0,t) - \frac{d^2}{dt^2}\mathcal{I}(0,t) \right\} + \frac{d}{dt} \left\{ \alpha(t) - \beta(t)\frac{d}{dt}\mathcal{I}(0,t) - \frac{d^2}{dt^2}\mathcal{I}(0,t) \right\}$	(324)	Combined with (316) and (326), this formula depends only on initial data, $\mathcal{S}(0,t)$, and α (which could be found by some other means, please see our examples).
$\mathcal{I}(0,t)$	$-r_0\mathcal{S}(0,t) - \ln[B(0,t)]$	(326)	This formula depends only on initial data, we can use it to determine $\mathcal{I}(0,t)$ $t \in [0, T]$ if $\mathcal{S}(0,t)$ and the initial term structure are known.
$\mathcal{S}(t,T)$	$\frac{\mathcal{S}(0,T) - \mathcal{S}(0,t)}{\frac{d}{dt}\mathcal{S}(0,t)}$	(304)	This formula depends only on initial data, we can use it to determine $\mathcal{S}(t,T)$ $t, T \in [0, T]$ $t \leq T$ if $\mathcal{S}(0,t)$ is known.

mention of some “widely accepted market” practices with regards to the calibration of several interest rate models, but not one single explicit calibration is shown. The present chapter should fill in that void.

Although the models we consider here are all subcases of the Hull–White model (255), it is clear that our ideas can be also applied to other interest rate models as well.

As we mentioned before, the calibration procedure presented here is not the only one available, nor it is the most general. Our calibration procedure responds both to the model

Table 16: Some additional formulas used in the calibration of the Hull White model

Parameter	Formula	#	Comments
$b(t, T)$	$-\gamma(t)\mathcal{S}(t, T)$	(295)	This formula depends only on $\mathcal{S}(t, T)$ and γ which in turn depend only on initial data, we can use it to determine $b(t, T)$ $t, T \in [0, T]$ $t \leq T$.
$\mathcal{I}(t, T)$	$\mathcal{I}(0, T) - \mathcal{I}(0, t) - \mathcal{S}(t, T) \frac{d}{dt} \mathcal{I}(0, t) + \frac{1}{2} \mathcal{S}^2(t, T) \int_0^t \gamma^2(v) \left\{ \frac{\partial}{\partial t} \mathcal{S}(v, t) \right\}^2 dv$	(314)	This formula depends only on initial data, we can use it to determine $\mathcal{I}(t, T)$ $t, T \in [0, T]$ $t \leq T$ if $\mathcal{S}(0, t)$, $\mathcal{I}(0, t)$ and γ are known.
$m_t = E(r_t)$	$(1/\eta(t)) \left\{ r_0 + \int_0^t \eta(u) \alpha(u) du \right\}$	(261)	Useful to determine parameters only if reliable historical data is available, otherwise may be used as a benchmark
$\mathcal{V}_t = \text{Var}(r_t)$	$\frac{1}{\eta^2(t)} \int_0^t \eta^2(u) \gamma^2(u) du$	(263)	ibidem.
$\text{Cov}(r_t, r_s)$	$\frac{1}{\eta(t)\eta(s)} \int_0^{t \wedge s} \eta^2(u) \gamma^2(u) du$	(265)	ibidem.

selected and to the kind of data available.

Several improvements are possible, for example one may consider the introduction of volatility data coming from Swaptions and or Captions data. Intuitively this will improve the model's ability to respond to volatility. Still much care has to be taken in this regard, in particular if the reader has in mind the commercial application of the ideas contained in this chapter and the possible extension we mention here. When calibrating to an initial term structure of volatilities one must ask if the data at hand is relevant to the instruments one wants to price using the calibrated interest rate model. One must also consider what happens with the implied volatilities at future times (typical questions one may ask are: are humps present in the initial volatility curve?, do we obtained "echoes" of those humps into the volatility structure implied by the model?, are those somehow preserved through time?, are other features of the data preserved and or explained by the curves implied by the model? etc., etc.). Recalibration might be required if implied future volatilities are not realistic, but somehow a "good model" is one you do not have to recalibrate frequently.

Following what is, according to several authors⁴⁷, considered as market practice we have used linear interpolation to interpolate flat volatilities as well as to interpolate Swap rates, etc. (see also Chapter 2). Intuition indicates that such a practice could not be optimal, and that the choice of linear interpolants (for comparison we have also used cubic interpolants to obtain “parallel” results) could have been inspired by an attempt to reach ease of computation, and some form of simplicity mimicking the form in which Black’s model is applied to the pricing of Caps and Swaptions and maybe not by serious mathematical considerations. For example in the previous section, following market conventions, we have constructed “staircase”-like volatility parameters. The construction of piecewise constant functions is a widely accepted market practice⁴⁸, but the resulting parameter, being a piecewise constant function is not continuous, much less differentiable.

A more careful calibration process will compare results obtained in the “replication” of prices of some “benchmark” instruments to determine the best interpolation procedure depending on the actual form of the yield curve and initial term structure of volatilities one is dealing with. As the reader can see, at the end of the previous section at least one parameter was being given arbitrary values, the use of “benchmark” instruments will help to find proper, non-arbitrary values for such free parameters.

There is no standard interest rate model. Some of the general ideas given in this chapter could be used to calibrate other interest rate models. We selected the Hull–White model because of its analytical properties (which we can exploit in the pricing of some Game Options), its historical importance, and the fact that it is still used by practitioners around the world. Yet, many more models exist and are being developed. In a “real world” application of interest rate models, one should consider not one but a collection of models, each of them calibrated to the given data and compared to “benchmark” instrument prices in order to select the most appropriate model for the problem/data you have.

⁴⁷Please refer to the bibliography provided at the end of this document, for example [116], [82], etc..

⁴⁸See for example [144] and [19] who mention a similar idea in the calibration of the lognormal market model.

CHAPTER V

EXAMPLES. PRICING OF GAME OPTIONS

Abstract

We present a particular example of a Game Option and suggest its numerical pricing using partial differential equations and approximations based in finite difference methods.

5.1 Background Results

The previous three chapters of this work were devoted to the development of some tools and results required in this chapter.

Our goal here is to show an example of the numerical approximation of the value of a particular Game option in a market with non-constant interest rates.

The second chapter (see Chapter 2) contains some preliminaries plus (see Chapter 2 §2.2) our rendition of a Bootstrapping method that allows us to obtain yield curve information from a given swap curve. In practice, data obtained through the method of Chapter 2 §2.2 can be used to calibrate an interest rate model.

The first part of Chapter 4 contains a detailed study of a particular interest rate model, the Hull–White interest rate model. In the second part of that chapter we show a Stripping method that can be used to obtain spot volatility data from flat volatility obtained from Caps/Floors quotes. We also show detailed examples of calibration (of the Hull–White model, although many of those ideas can be used in the calibration of other interest rate models as well) in which data coming from our Bootstrapping and Stripping methods is used.

In this chapter we retake some of those ideas and suggest a numerical approximation to the price of a particular example of Game option.

But, before we start the numeric approximations, we need to, briefly, review our results

from Chapter 3 and observe how they apply to our particular example. We also need to recall some usefull results from the general theory of diffusions and Markov processes.

Without much ado we proceed now into our short review of the results from Chapter 3.

Recall the general setting of Chapter 3 where we assumed we are given a filtered probability space $(\Omega, \mathcal{U}, \mathcal{G}, \mathcal{P})$, where \mathcal{F} is the \mathcal{P} -augmentation of the natural filtration of a d -dimensional Brownian Motion W . We assumed $\mathcal{G} = \mathcal{F}_T$. We also assumed that asset prices follow strictly positive processes which we model by means of linear stochastic differential equations (more explicitly as exponential diffusions), and that an smm (standard market model) is defined under such filtered probability space.

In Chapter 3 we showed that a game contingent claim (**gcc**) with RCLL left upper semicontinuous payoff processes $-\mathfrak{X}$ and \mathfrak{Y} , $0 \leq \mathfrak{Y} \leq \mathfrak{X}$, satisfying condition (108), not only has a value,

$$\mathcal{V}_t^* = \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t,T}} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) = \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t,T}} \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t), \quad (412)$$

Theorem 3.4.26 (242), but that $\forall t \in [0, T] \ (\exists \kappa_t \in \mathfrak{S}_{t,T}) \wedge (\exists \xi_t \in \mathfrak{S}_{t,T})$ (please see Chapter 3 for notation and definitions¹), optimal stopping times for the seller and the buyer such that

$$\mathcal{V}_t^* = E_{\mathcal{E}}(\mathcal{R}^*(\kappa_t, \xi_t) | \mathcal{F}_t), \quad (413)$$

and

$$E_{\mathcal{E}}(\mathcal{R}^*(\kappa_t, \mathfrak{t}) | \mathcal{F}_t) \leq \mathcal{V}_t^* \leq E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \xi_t) | \mathcal{F}_t), \quad (414)$$

$\mathcal{P}^{\mathcal{E}}$ -a.s., for any $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{t,T}$. That is, if the seller chooses any other strategy, $\mathfrak{s} \in \mathfrak{S}_{t,T}$, he/she may end up paying more to the buyer than if he/she chooses κ_t ; on the other hand, if the buyer uses a different strategy, $\mathfrak{t} \in \mathfrak{S}_{t,T}$, he/she may end up receiving a smaller payment. To both seller and buyer it is optimal to use their corresponding optimal stopping time (κ_t and ξ_t , respectively) since the failure of the adversary to follow their corresponding optimal strategy may result in a better situation (for the party following the optimal strategy).

¹Since $\tilde{\kappa}_t \wedge \tilde{\xi}_t = \kappa_t^0 \wedge \xi_t^0$, we will use κ_t to represent both κ_t^0 and $\tilde{\kappa}_t$, $\forall t \in [0, T]$. Similarly, $\forall t \in [0, T]$ we will use ξ_t to represent both ξ_t^0 and $\tilde{\xi}_t$. See Chapter 3 for details.

If both seller and buyer act optimally, based in the information available to them at time $t \in [0, T]$, the contract should be canceled/executed at time $\kappa_t \wedge \xi_t$.

We also showed that there exists a portfolio strategy, Π^{κ_0} , such that it defines a hedge against the gcc and that $\mathcal{V}_0^* = \mathcal{V}_0 = \mathcal{W}_0^{\Pi^{\kappa_0}}$ and that such portfolio strategy is ($\mathcal{P}^{\mathcal{E}}$ -a.s.) unique up to the end of the contract, $\kappa_t \wedge \xi_t$.

Another property of a gcc with maturity T and payoff processes \mathfrak{Y} and \mathfrak{X} is that $\forall t \in [0, T]$

$$\mathfrak{Y}_t^* \leq \mathcal{V}_t^* \leq \mathfrak{X}_t^*, \quad (415)$$

in fact, this property is valid also for all stopping times $\mathfrak{t} \in \mathfrak{S}_{0,T}$, that is

$$\mathfrak{Y}_{\mathfrak{t}}^* \leq \mathcal{V}_{\mathfrak{t}}^* \leq \mathfrak{X}_{\mathfrak{t}}^*. \quad (416)$$

By definition of ξ_t we know that

$$\mathfrak{Y}_s^* < \mathcal{V}_s^*, \quad \forall s < \xi_t, \quad (417)$$

similarly, by definition of κ_t we have

$$\mathcal{V}_s^* < \mathfrak{X}_s^*, \quad \forall s < \kappa_t. \quad (418)$$

(otherwise, the definition of these stopping times will be contradicted). Thus, we see that, $\forall s \in [0, T]$

$$\begin{aligned} \mathfrak{Y}_s^* < \mathcal{V}_s^* & \quad \underline{\text{or}} \quad \mathfrak{Y}_s^* = \mathcal{V}_s^*, \\ \mathcal{V}_s^* < \mathfrak{X}_s^* & \quad \underline{\text{or}} \quad \mathcal{V}_s^* = \mathfrak{X}_s^*. \end{aligned} \quad (419)$$

This means that there are three well defined “regions” in $[0, T] \times \Omega$, i) a *continuation region* $\mathcal{Co} = \{(t, \omega) \mid \mathfrak{Y}_t^*(\omega) < \mathcal{V}_t^*(\omega) < \mathfrak{X}_t^*(\omega)\} \subset [0, T] \times \Omega$, where neither exercise nor cancellation are optimal; ii) a *cancellation region* $\mathcal{Ka} = \{(t, \omega) \mid \mathcal{V}_t^*(\omega) \geq \mathfrak{X}_t^*(\omega)\} \cup \{T\} \times \Omega \subset [0, T] \times \Omega$, where the gcc should be canceled, and iii) an *exercise region* $\mathcal{Ex} = \{(t, \omega) \mid \mathcal{V}_t^*(\omega) \leq \mathfrak{Y}_t^*(\omega)\} \subset \mathbb{R}^+ \times \Omega$, where the gcc should be exercised. Observe that the stopping times κ_0 and ξ_0 are the first hitting times corresponding to the cancellation and exercise regions, respectively.

What happens if \mathfrak{X} is large?

Let $\{\mathfrak{X}^{(n)}\}_{n \in \mathbb{N}}$ be an increasing sequence of RCLL stochastic processes such that, for all $t \in [0, \mathcal{T}]$ $\mathfrak{X}_t^{(n)} \geq \mathfrak{X}_t$ and such that $-\mathfrak{X}^{(n)}$ is left upper semicontinuous, and satisfies condition (108), $\forall n \in \mathbb{N}$. Define now $\mathcal{R}^{(n)} = \mathcal{R}^{\mathfrak{X}^{(n)}, \mathfrak{Y}}$, see Definition 3.4.1. Then, given $n \in \mathbb{N}$, $\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ $\mathcal{R}^{(n)}(\mathfrak{s}, \mathfrak{t}) = \mathcal{R}(\mathfrak{s}, \mathfrak{t}) + (\mathfrak{X}_\mathfrak{s}^{(n)} - \mathfrak{X}_\mathfrak{s})\mathbb{1}_{\mathfrak{s} < \mathfrak{t}}$, and $\mathcal{R}^{(n)} \geq \mathcal{R} \forall n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $n \leq m$, then, $\forall \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ we have $\mathcal{R}^{(n)}(\mathfrak{s}, \mathfrak{t}) = \mathcal{R}(\mathfrak{s}, \mathfrak{t}) + (\mathfrak{X}_\mathfrak{s}^{(n)} - \mathfrak{X}_\mathfrak{s})\mathbb{1}_{\mathfrak{s} < \mathfrak{t}} \leq \mathcal{R}^{(m)}(\mathfrak{s}, \mathfrak{t})$. That is, $\{\mathcal{R}^{(n)}\}_{n \in \mathbb{N}}$ is an increasing sequence of families of random variables indexed by stopping times in \mathfrak{S} .

Fixing $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $n \leq m$, and $t \in [0, \mathcal{T}]$,

$$\begin{aligned} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) &\leq \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^{(n)*}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \\ &\leq \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^{(m)*}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t), \end{aligned}$$

thus,

$$\begin{aligned} \mathcal{V}_t^* &= \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, \mathcal{T}}} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^*(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \leq \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, \mathcal{T}}} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^{(n)*}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) = \mathcal{V}_t^{(n)*} \\ &\leq \operatorname{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, \mathcal{T}}} \operatorname{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}}(\mathcal{R}^{(m)*}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) = \mathcal{V}_t^{(m)*}, \end{aligned} \tag{420}$$

that is, the time t price of a gcc increases as \mathfrak{X} increases. Similarly, we obtain the following result

Proposition 5.1.1. *Let \mathfrak{Y} , \mathfrak{X} , \mathfrak{Z} , be three RCLL processes (defined on and adapted to the filtered probability space underlying our market model of Chapter 3). Assume $\mathfrak{Y} \leq \mathfrak{X} \leq \mathfrak{Z}$. Assume also that \mathfrak{X} and \mathfrak{Z} satisfy condition (108). The time t , $t \in [0, \mathcal{T}]$, price of the gcc defined by \mathfrak{Y} and \mathfrak{X} is lower or equal than the time t price of the gcc defined by \mathfrak{Y} and \mathfrak{Z} .*

In the limit, when $n \rightarrow \infty$, $\mathfrak{X}^{(n)} \rightarrow \infty$, and the (discounted) gcc payoff degenerates into

$$\mathcal{R}^{(\infty)*}(s, t) = \begin{cases} \infty & s < t, \\ \mathfrak{Y}_t^* & t \leq s. \end{cases} \tag{421}$$

From the point of view of a game of stopping, as the cancelation payoff increases, to cancel the gcc becomes ever more expensive for the seller. In the limit, it will be totally unthinkable

to cancel the gcc, in fact, in the limit, the seller is “not allowed” to cancel the gcc. But this is the case of an American contingent claim² (**acc**).

We observe also that, fixing $\mathbf{t} \in \mathfrak{S}_{t,T}$, $\mathcal{R}^{(\infty)*}(\mathbf{t}, \mathbf{t}) = \mathfrak{V}_{\mathbf{t}}^*$, therefore

$$\operatorname{esssup}_{\mathbf{t} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathfrak{V}_{\mathbf{t}}^* | \mathcal{F}_t) \geq \operatorname{esssup}_{\mathbf{t} \in \mathfrak{S}_{t,T}} \operatorname{essinf}_{\mathbf{s} \in \mathfrak{S}_{t,T}} E_{\mathcal{E}}(\mathcal{R}^{(\infty)*}(\mathbf{s}, \mathbf{t}) | \mathcal{F}_t) = \mathcal{V}_t^{(\infty)*} \quad (422)$$

Proposition 5.1.2. *Let \mathfrak{V} , \mathfrak{X} , be two RCLL processes (defined on and adapted to the filtered probability space underlying our market model of Chapter 3) such that \mathfrak{X} satisfy condition (108). Assume $\mathfrak{V} \leq \mathfrak{X}$. The time t , $t \in [0, T]$, price of the gcc defined by \mathfrak{V} and \mathfrak{X} is lower or equal than the time t price of the acc defined by \mathfrak{V} .*

Observe that Proposition 5.1.1 and Proposition 5.1.2 remain valid if \mathcal{R} takes the form of (253).

5.2 The Markovian Case

Recall from Chapter 3 equations (46) and/or (48), that we model our asset prices P_t using diffusion processes. In the case the coefficients μ and σ (see Chapter 3 for definitions) are deterministic functions of time t we know the solutions of (48) are Markovian (see [163] Chapter 6 for example).

We have learnt some properties of the price of a gcc. We know that the time t , $t \in [0, T]$, \mathcal{V}_t^* , price of a gcc is given by Theorem 3.4.26, and in particular by (242), that under proper conditions there is a saddle point and that the price is always bounded by the payoff processes defining the gcc (see (415) and (419)). We also know that the bigger the value of the cancelation process is at time t , $t \in [0, T]$, the bigger the time t price of the gcc; and that the price of the corresponding acc (with payoff process equal to the exercise payoff of the gcc) is always bigger than the gcc price. In this regard, Proposition 5.1.2 offers a natural upper bound on the value of a gcc.

We will study now a few additional properties we can derive from the form of the payoff processes.

²See [94] for a description of american contingent claims and their pricing, see also [97], [95], etc..

As before, we will use the notation introduced in Chapter 3. Assume that there exists real, bounded from below, and measurable functions Φ and Ψ defined on $\mathbb{R}^+ \times \mathbb{R}^n$ such that $\Phi(t, \vec{x}) \leq \Psi(t, \vec{x})$ for all $(t, \vec{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$ and such that $\mathfrak{X}_t = \Psi(t, P_t)$ and $\mathfrak{Y}_t = \Phi(t, P_t)$, then we can rewrite the gcc payoff, (107), as:

$$\mathcal{R}(s, t) = \mathcal{R}^{\Psi, \Phi}(s, t) = \Psi(s, P_s) \mathbb{1}_{s < t} + \Phi(t, P_t) \mathbb{1}_{t \leq s}. \quad (423)$$

Recall that $\text{diag}(\vec{x})$ denotes a diagonal $n \times n$ matrix whose diagonal entries are the n components of $\vec{x} \in \mathbb{R}^n$, that is $\text{diag}(\vec{x})_{ii} = x_i$. We define the *diffusion matrix* of our market model as

$$\mathbf{a}(t, \vec{x}) = \text{diag}(\vec{x}) \sigma_t (\text{diag}(\vec{x}) \sigma_t)^\dagger = \text{diag}(\vec{x}) \sigma_t \sigma_t^\dagger \text{diag}(\vec{x}) \quad (424)$$

we will assume that \mathbf{a} satisfies the ellipticity condition

$$\|\mathbf{a}(t, \vec{x})\| = \vec{x}^\dagger \sigma_t \sigma_t^\dagger \vec{x} \geq l \|\vec{x}\|^2 \quad \text{for some } l > 0, \forall (t, \vec{x}) \in \mathbb{R}^+ \times \mathbb{R}^n \quad (425)$$

Condition (425), which is a bounded-from-below condition, is imposed (see [63], [96], [88]) to ensure that not all higher order coefficients in the differential generator (see [141], [96]) of diffusion P_t vanish³.

In a very simplified way, we can introduce the differential generator of a diffusion as follows.

Theorem 5.2.1. *Let $\mu : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ be measurable functions satisfying:*

$$\|\mu(t, \vec{x})\| + \|\sigma(t, \vec{x})\| \leq C(1 + \|\vec{x}\|); \quad \vec{x} \in \mathbb{R}^n, t \in [0, T], \quad (426)$$

for some constant $C > 0$, and

$$\|\mu(t, \vec{x}) - \mu(t, \vec{y})\| + \|\sigma(t, \vec{x}) - \sigma(t, \vec{y})\| \leq D \|\vec{x} - \vec{y}\| \quad (427)$$

for some constant $D > 0$. Let $(\Omega, \mathcal{U}, \mathcal{P})$, be a probability space, let $W = \{W_t\}_{0 \leq t < \infty}$, be a d -dimensional Brownian Motion and \mathcal{F}^W the filtration generated by W . Let Ξ be an \mathbb{R}^n

³To say the truth, the situation is a little more complex than this. If nothing else is done first, there will be a singularity at the origin. This problem can be removed after a change of variables of the form $x_i = K_i e^{y_i}$ in which case the ellipticity condition we are giving (425) will work.

valued random vector independent of the d -dimensional Brownian Motion W and such that

$$E\|\Xi\|^2 < \infty \quad (428)$$

Then, there exists a continuous adapted process $X = \{X_t\}_{t \in [0, \mathcal{T}]}$ which is a strong solution to

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad X_0 = \Xi. \quad (429)$$

Moreover, this process is square integrable. There exists a constant $L > 0$ (that depends only on C , D , and \mathcal{T}) such that

$$E\|X_t\|^2 \leq L(1 + E\|\Xi\|^2)e^{Lt}, \quad t \in [0, \mathcal{T}]. \quad (430)$$

As usual if A is an $n \times d$ matrix, we write $A = (a_{ij})$, and define $\|A\| = \sum_{i=1}^n \sum_{j=1}^d a_{ij}$. See [63], Chapter 5 or [141] Chapter 5 and Chapter 7 for details, similarly [96] Chapter 5 provides with the details.

Global condition (427) in Theorem 5.2.1 can be relaxed. If instead of (427) we adopt

$$\|\mu(t, \vec{x}) - \mu(t, \vec{y})\| + \|\sigma(t, \vec{x}) - \sigma(t, \vec{y})\| \leq D_r \|\vec{x} - \vec{y}\|, \quad (431)$$

$r > 0$, $\|\vec{x}\| \leq r$, $\|\vec{y}\| \leq r$, $t \in [0, \mathcal{T}]$ and $D_r > 0$ a constant depending only on \mathcal{T} and r ; the Theorem 5.2.1 remains valid.

Let $X^{s, \vec{x}}$ be the solution of (429) for $t \geq s$ under the “initial condition” $X_s = \vec{x}$ \mathcal{P} -a.e., that is:

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad X_s = \vec{x}, \quad t \geq s. \quad (432)$$

Under the conditions of Theorem 5.2.1, with (431) instead of (427), the solutions of (432) are Markovian (see [63], see also [73] Chapter III or [163]) with transition function

$$p(s, \vec{x}, t, A) = \mathcal{P}(X_t^{s, \vec{x}} \in A), \quad \mathcal{T} \geq t \geq s \geq 0, \quad (433)$$

A a borel set in \mathbb{R}^n and satisfy the strong Markov property. Also, such solutions satisfy $E(\sup_{t \leq s} |X_t^{s, \vec{x}} - X_t^{u, \vec{y}}|^2) \leq K(\|\vec{x} - \vec{y}\|^2 + |s - u|)$, if $\|\vec{x}\| \leq r$ and $\|\vec{y}\| \leq r$, $s \leq u \leq \mathcal{T}$, where K is a constant that depends on \mathcal{T} and r . If μ and σ are piecewise continuous, $X^{u, \vec{x}}$ is a diffusion.

Usually, $\mu(t, X_t)$ is called the *drift* and $\sigma(t, X_t)$ is called the *dispersion matrix*. In general $\mathbf{a}(t, X_t) = \sigma(t, X_t)\sigma(t, X_t)^\dagger$ is called the *diffusion matrix* of diffusion (429) (or (432)), for convenience we may write $\sigma_i(t, X_t)$ to represent the i -th row of matrix $\sigma(t, X_t)$, and $\mu_j(t, X_t)$ to represent the j -th component of vector $\mu(t, X_t)$.

Let $u : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of class $C^{1,2}$ defined on a subset of $\mathbb{R}^+ \times \mathbb{R}^n$. According to Itô's formula, the process $\{u(t, X_t^{s,\vec{x}})\}_{t \in [s,T] \subseteq [0,T]}$, $\vec{x} \in \mathbb{R}^n$ satisfies

$$\begin{aligned} du(t, X_t^{s,\vec{x}}) = & \left(\frac{\partial u}{\partial t} + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n \mathbf{a}_{ik} \frac{\partial^2 u}{\partial x_i^2 \partial x_k^2} \right) \Big|_{(t, X_t^{s,\vec{x}})} dt \\ & + \left(\sum_{i=1}^n \sigma_i \frac{\partial u}{\partial x_i} \right) \Big|_{(t, X_t^{s,\vec{x}})} dW_t \end{aligned} \quad (434)$$

the “ dt ” coefficient in (434) is called the *differential generator* of diffusion (429) and (432) applied to function u . It is customary to denote by \mathcal{A} the *infinitesimal generator* of diffusion (429) and (432):

$$\mathcal{A}u = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n \mathbf{a}_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i}; \quad (435)$$

in case we need to make explicit reference to diffusion X , (429), we will use \mathcal{A}_X instead of \mathcal{A} . To simplify our notation we may “drop” the symbols s, \vec{x} from our notation of the solution of (432). It seems that there is not a standard notation to use in the case of the differential generator, in case needed we will use the symbol \mathcal{L} , thus

$$\mathcal{L}u = \frac{\partial u}{\partial t} + \mathcal{A}u, \quad (436)$$

to represent the differential generator of X applied to the $C^{1,2}$ function u ; as before, \mathcal{L}_X will be preferred over \mathcal{L} when explicit mention of the underlying diffusion is needed.

We can rewrite (434), using (435), in integral form as follows:

$$u(t, X_t) - u(0, X_0) - \int_0^t \left(\frac{\partial u}{\partial s} + \mathcal{A}u \right) \Big|_{(s, X_s)} ds = \int_0^t \left(\sum_{i=1}^n \sigma_i \frac{\partial u}{\partial x_i} \right) \Big|_{(s, X_s)} dW_s \quad (437)$$

thus, the left hand side of (437) is a continuous local martingale⁴. If u is of class $C^{1,2}$ has bounded first derivatives and σ is t -bounded (that is, if $\|\sigma(s, \vec{y})\| \leq K_t$, $0 \leq s \leq t$,

⁴Please see [96], chapter 5 for details. In particular Theorem 2.9 and Proposition 4.2. There are more sources in the bibliography, in particular, the reader may consider [88], [141] and [114] amongst many others.

$t \in [0, T]$, K_t a constant depending only on t) then the left hand side of (437) is a continuous square integrable martingale. The boundedness condition on σ may be changed by the local condition of σ being bounded on compact sets.

Recall from Chapter 3 §3.3.1 and our presentation of the standard market model (**smm**) that we denote by S a stock price process and by $B(\cdot, T)$ the price process of a zcb of maturity T . According to the smm of Chapter 3, S and $B(\cdot, T)$ satisfy

$$dS_t = S_t(\varsigma_t dt + \varrho_t \cdot dW_t), \quad (438)$$

with initial price S_0 ⁵, and

$$dB(t, T) = B(t, T)(a(t, T) dt + b(t, T) \cdot dW_t), \quad (439)$$

with initial price $B(0, T)$, maturity $T \in [0, T]$, $T > 0$, and final value $B(T, T) = 1$. Consider now the process defined by $Z = (S, B(\cdot, T))$, with initial value $(S_0, B(0, T))$. It is clear that Z 's dynamics are given by (438) and (439) (notice that $\{\varrho_t\}_{t \in [0, T]}$, $\{b(t, T)\}_{t \in [0, T]}$ and $\{W_t\}_{t \in [0, T]}$ are d -dimensional processes). If $u : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of class $C^{1,2}$ we can write:

$$\begin{aligned} \mathcal{A}_Z u(t, S_t, B(t, T)) &= \left\{ S_t \varsigma_t \frac{\partial u}{\partial x} + B(t, T) a(t, T) \frac{\partial u}{\partial y} \right. \\ &\quad \left. + \frac{1}{2} \left\{ \|\varrho_t\|^2 S_t^2 \frac{\partial^2 u}{\partial x^2} + \|b(t, T)\|^2 (B(t, T))^2 \frac{\partial^2 u}{\partial y^2} + 2\varrho_t \cdot b(t, T) S_t B(t, T) \frac{\partial^2 u}{\partial x \partial y} \right\} \right\} \Big|_{(t, Z_t)} \end{aligned} \quad (440)$$

or, in terms of t and $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} \mathcal{A}_Z u &= x \varsigma_t \frac{\partial u}{\partial x} + y a(t, T) \frac{\partial u}{\partial y} \\ &\quad + \frac{1}{2} \left\{ \|\varrho_t\|^2 x^2 \frac{\partial^2 u}{\partial x^2} + \|b(t, T)\|^2 y^2 \frac{\partial^2 u}{\partial y^2} + 2\varrho_t \cdot b(t, T) xy \frac{\partial^2 u}{\partial x \partial y} \right\} \end{aligned} \quad (441)$$

⁵To simplify the analysis we will assume that stock S pays no dividends; otherwise, depending on situation, we should be using its yield process Y which satisfies

$$dY_t = S_t(\varsigma_t dt + \delta_t dt + \varrho_t \cdot dW_t) \quad Y_0 = S_0$$

where δ represents the dividend rate process. Obviously, to change this assumption is not too difficult, roughly speaking we should only have to change ς by $\varsigma + \delta$ in the right places thereafter.

Similarly, we can write Itô's formula, (434) for process $\{u(t, Z_t)\}_{t \in [0, T]}$, $Z_t = (S_t, B(t, T))$ $t \in [0, T]$, as:

$$\begin{aligned} du(t, S_t, B(t, T)) = & \left\{ \frac{\partial u}{\partial t}(t, S_t, B(t, T)) + \mathcal{A}_Z u(t, S_t, B(t, T)) \right\} dt \\ & + \left(\varrho_t \frac{\partial u}{\partial x}(t, S_t, B(t, T)) + b(t, T) \frac{\partial u}{\partial y}(t, S_t, B(t, T)) \right) \cdot dW_t \end{aligned} \quad (442)$$

5.3 *Forward price, forward measure and a simple example of change of Numéraire*

A particularly interesting case of last example is obtained when $u(x, y) = x/y$ ($y \neq 0$). In such a case (442) reduces to:

$$d\left(\frac{S_t}{B(t, T)}\right) = \left(\frac{S_t}{B(t, T)}\right) \{(\varsigma_t - a_t + b(t, T) \cdot b(t, T) - \varrho_t \cdot b(t, T)) dt + (\varrho_t - b(t, T)) \cdot dW_t\}. \quad (443)$$

Definition 5.3.1 (Forward price). Process $\{S_t/B(t, T)\}_{t \in [0, T]}$ is known as the forward price⁶ of stock S with respect to the zcb price process $B(\cdot, T)$ and is denoted as $F_S(\cdot, T)$; that is

$$F_S(t, T) = \frac{S_t}{B(t, T)} \quad t \in [0, T]. \quad (444)$$

In terms of (444) we can write (443) as

$$dF_S(t, T) = F_S(t, T) \{(\varsigma_t - a_t + b(t, T) \cdot b(t, T) - \varrho_t \cdot b(t, T)) dt + (\varrho_t - b(t, T)) \cdot dW_t\}, \quad (445)$$

where $F_S(0, T) = S_0/B(0, T)$, the process $\{\varrho_t - b(t, T)\}_{t \in [0, T]}$ will be called *forward price volatility* or simply *forward volatility*. Our computations show that the forward price is also an Itô process, a diffusion with drift process $\{F_S(t, T)(\varsigma_t - a_t + b(t, T) \cdot b(t, T) - \varrho_t \cdot b(t, T))\}_{t \in [0, T]}$ and dispersion process $\{F_S(t, T)(\varrho_t - b(t, T))\}_{t \in [0, T]}$.

According to Definition 3.3.13, §3.3.3, Chapter 3 there exists a process⁷ θ such that

⁶Please see [133], Chapter 13 for a detailed description of forward price processes and their corresponding measures. Alternatively, [163] Chapter 5 and Chapter 9 provides also with a detailed description of, respectively, forward price processes and their corresponding associated measures.

⁷From Chapter 3 we know that σ is an $n \times d$ -dimensional (matrix) process (this is to accommodate our d sources of risk, that is a d dimensional standard Brownian motion W , and n different basic securities (S

$\rho_t = \mu_t - \vec{1}_n r_t + \delta_t = \sigma_t \theta_t$, $t \in [0, T]$. Or, using the notation of this section, such that:

$$\varrho_t \cdot \theta_t = \varsigma_t - r_t, \quad (446)$$

and,

$$b(t, T) \cdot \theta_t = a(t, T) - r_t. \quad (447)$$

Such a process (called *market price of risk*, see Chapter 3 Theorem 3.3.3) is used to define the martingale measure $\mathcal{P}^\mathcal{E}$ under which our discounted asset prices (see Definition 3.3.9 and §3.3.3) are driftless⁸ diffusions with respect to the “new” Brownian motion $W^\mathcal{E}$ defined as $dW_t^\mathcal{E} = \theta_t dt + dW_t$.

Under the risk neutral measure, (84), $\mathcal{P}^\mathcal{E}$, the discounted prices of S and $B(\cdot, T)$ satisfy the sde's:

$$dS_t^* = S_t^* \varrho_t \cdot dW_t^\mathcal{E} \quad (448a)$$

$$d(B(t, T))^* = (B(t, T))^* b(t, T) \cdot dW_t^\mathcal{E} \quad (448b)$$

with initial values S_0 and $B(0, T)$. Notice also that the forward price process $F_S(\cdot, T)$ satisfies

$$F_S(t, T) = \frac{S_t}{B(t, T)} = \frac{S_t^*}{(B(t, T))^*} \quad t \in [0, T]. \quad (449)$$

Thus, in terms of the risk neutral measure $\mathcal{P}^\mathcal{E}$ we can re-write (445) as follows:

$$\begin{aligned} dF_S(t, T) &= F_S(t, T) \{ (\|b(t, T)\|^2 - \varrho_t \cdot b(t, T)) dt + (\varrho_t - b(t, T)) \cdot dW_t^\mathcal{E} \} \\ &= F_S(t, T) \{ (\varrho_t - b(t, T)) \cdot (-b(t, T)) dt + (\varrho_t - b(t, T)) \cdot dW_t^\mathcal{E} \} \\ &= F_S(t, T) (\varrho_t - b(t, T)) \cdot \{-b(t, T) dt + dW_t^\mathcal{E}\}, \\ F_S(0, T) &= \frac{S_0}{B(0, T)}. \end{aligned} \quad (450)$$

and $B(\cdot, T)$ are the price processes of two of them), then the risk premium ρ is an n -dimensional process and the market price of risk process θ is a d -dimensional process. In general, this last process will exist if the volatility matrix process, σ , has a left inverse \mathcal{P} -a.e.. Similarly, processes ϱ and $b(\cdot, T)$ are d -dimensional.

⁸Recall we are assuming that S pays no dividends, otherwise the first equation in (448) should be changed for

$$dS_t^* = S_t^* (-\delta dt + \varrho_t \cdot dW_t^\mathcal{E})$$

which is not driftless.

In order to avoid a singularity in (450) we may want to assume that the forward volatility process is not of null length, that is, that $\|\varrho_t - b(t, T)\| \neq 0$, this assumption can be phrased in a form consistent with the ellipticity condition (425), that is we assume:

$$\|\varrho_t - b(t, T)\| > l, \quad \text{for some } l > 0, \forall t \in \mathbb{R}^+. \quad (451)$$

According to Girsanov's theorem⁹ if

$$\int_0^s \|b(t, T)\|^2 dt < \infty \quad \mathcal{P}^\mathcal{E} \text{ a.e.}; \quad s \in [0, T], \quad (452)$$

it is possible to find a change of measure under which the forward price will satisfy a driftless sde. Notice that this last requirement is already part of our general assumptions (see Chapter 3 (51)). Thus, by Girsanov's theorem

$$dW_t^F = -b(t, T) dt + dW_t^\mathcal{E} \quad (453)$$

defines a d -dimensional standard Brownian motion. Assuming process $b(\cdot, T)$ satisfies also Novikov's condition

$$E_\mathcal{E} \left(\exp \left\{ \frac{1}{2} \int_0^s \|b(t, T)\|^2 dt \right\} \right) < \infty \quad s \in [0, T] \quad (454)$$

we know that the Doléans exponential of $-b(\cdot, T)$,

$$\mathcal{E}(-b(\cdot, T)) = \left\{ \exp \left(\int_0^t b(s, T) \cdot dW_s^\mathcal{E} - \frac{1}{2} \int_0^t \|b(s, T)\|^2 ds \right) \right\}_{t \in [0, T]}, \quad (455)$$

is not only a continuous local martingale satisfying

$$d\mathcal{E}_t(-b(\cdot, T)) = \mathcal{E}_t(-b(\cdot, T)) b(t, T) \cdot dW_t^\mathcal{E}, \quad \mathcal{E}(-b(\cdot, T))_0 = 1, \quad (456)$$

but it is also a martingale such that $E_\mathcal{E}(\mathcal{E}_t(-b(\cdot, T))) = 1$; $t \in [0, T]$.

In this case a “new” equivalent martingale measure exists, defined by

$$\mathcal{P}^F(A) = E_\mathcal{E}[\mathbb{1}_A \mathcal{E}_T(-b(\cdot, T))]; \quad A \in \mathcal{F}_T \quad (457)$$

(notice that there is no problem with this definition and that the martingale property ensures that $\mathcal{P}^F(A) = E_\mathcal{E}[\mathbb{1}_A \mathcal{E}_t(-b(\cdot, T))]$; $A \in \mathcal{F}_t$, $t \in [0, T]$)

⁹ See [96] Chapter 3 section §3.5.A, and in particular Theorem 3.5.1 . See also Novikov's condition, [96] Chapter 3 section §3.5.D. Compare to our discussion in Chapter 3 after Theorem 3.3.3.

Measure \mathcal{P}^F is an equivalent martingale measure (see Chapter 3),

$$\frac{d\mathcal{P}^F}{d\mathcal{P}^\mathcal{E}} = \mathcal{E}_T(-b(\cdot, T)). \quad (458)$$

Definition 5.3.2 (Forward measure). *The equivalent martingale measure defined at (457), \mathcal{P}^F , is called the maturity T , $T \in [0, T]$, Forward measure. In case explicit reference to the maturity of the underlying zcb is required we will write \mathcal{P}_T^F instead of \mathcal{P}^F .*

Last time we constructed an equivalent martingale measure, see Chapter 3 §3.3.3, we did not pay much attention to the Doléans exponential. Compare (448) and (456). It is not hard to see that the Doléans exponential of $-b(\cdot, T)$ is equal to the discounted price process $(B(\cdot, T))^*$ divided by the initial price of the zcb $B(0, T)$, that is:

$$\mathcal{E}_t(-b(\cdot, T)) = \frac{(B(t, T))^*}{B(0, T)} = \frac{(B(t, T))^*}{(B(0, T))^*}. \quad (459)$$

This is very convenient. If X is an \mathcal{F}_T measurable r.v. we can write:

$$E_F(X) = \frac{1}{(B(0, T))^*} E_{\mathcal{E}}(X(B(T, T))^*) \quad (460a)$$

and,

$$E_F(X|\mathcal{F}_s) = \frac{1}{(B(s, T))^*} E_{\mathcal{E}}(X(B(t, T))^*|\mathcal{F}_s) \quad (460b)$$

if X is \mathcal{F}_t measurable, $0 \leq s \leq t \leq T$ (assuming the integrals in (460) exist).

Also, under measure \mathcal{P}^F , (450) reduces to

$$dF_S(t, T) = F_S(t, T)(\varrho_t - b(t, T)) \cdot dW_t^F, \quad F_S(0, T) = \frac{S_0}{B(0, T)}. \quad (461)$$

Notice that even if we allow ϱ and $b(\cdot, T)$ to be stochastic processes, but restrict them so that the forward volatility is deterministic then the forward price process will be a diffusion Markov process. If diffusion $F_S(\cdot, T)$ is started at a different time $w \geq 0$ with initial value $F_S(w, T) = x$ we can write

$$dF_S(t, T) = F_S(t, T)(\varrho_t - b(t, T)) \cdot dW_t^F, \quad t \geq w; \quad F_S(w, T) = x. \quad (462)$$

and $F_S^{w,x}(\cdot, T)$ to denote its solution. We may default to the notation $F_S(\cdot, T)$ if no explicit mention of w or x is necessary. Similarly, we may opt for the more explicit notation $F_S^x(\cdot, T)$ if we want to emphasize that the initial value at time zero is x .

Equations (460) motivate the following observation. Let $X = \{X_t\}_{t \in [0, T]}$ be an \mathcal{F} -adapted stochastic process; its forward price at time s , $0 \leq t \leq s \leq T$, for a maturity T will be:

$$F_X(s, T) = \frac{X_s}{B(s, T)} \quad (463)$$

similarly, we may define

$$X_s^* = \frac{X_s}{B_s} \quad (464)$$

by (460) we can write:

$$B(t, T)E_F(F_X(s, T)|\mathcal{F}_t) = B_tE_{\mathcal{E}}(X_s^*|\mathcal{F}_t), \quad (465)$$

$0 \leq t \leq s \leq T$, $T \in [0, T]$; assuming that both, the forward price and discounted process defined above are \mathcal{P}^F -integrable and $\mathcal{P}^{\mathcal{E}}$ -integrable, respectively. Observe that (465) is still valid if time s is changed by a stopping time $\mathfrak{s} \in \mathfrak{S}_T$.

Equation (465) shows the equivalence between discounting using the Bank account and discounting with respect to a zcb of maturity T . As we commented before, see Footnote 14 and Footnote 23, we could use any positive, non dividend paying asset to discount. In case we use another non dividend paying asset to discount, changes of measure of the type we have described in this example will be available, as well as equivalence relationships as that given by (465).

If we call Y_t the random variable in (465) we can use (465), Definition 5.3.1 and Definition 3.3.9 to write:

$$Y_t^* = E_{\mathcal{E}}(X_s^*|\mathcal{F}_t), \quad (466a)$$

$$F_Y(t, T) = E_F(F_X(s, T)|\mathcal{F}_t), \quad (466b)$$

$0 \leq t \leq s \leq T$, $T \in [0, T]$, any of the related equations above can be used to compute Y_t .

Our computations leading to (461) and (465) show that the forward price of an asset with respect to a given zcb of maturity T is a martingale with respect to the forward measure \mathcal{P}^F , and that discounting with respect to the Bank account is equivalent to discounting with respect to a zcb price process. Clearly such results could be easily generalized to “discounting” with respect to a positive martingale. The strength of (466) lies in the

fact that under favorable conditions we can, conveniently, change the frame of reference in our pricing formulas. If formulas under the forward measure are simpler we can use that framework to price a contract and then come back to the risk neutral probability measure using (465) and Definition 5.3.1.

5.4 The Markovian Case continued

We retake now the example we were working with in §5.2.

What are the infinitesimal and differential generators of diffusion $F_S(\cdot, T)$?

Let $u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, z) \rightarrow u(t, z)$ be a $C^{1,2}$ function, by Itô's formula (434), we have:

$$\begin{aligned} du(t, F_S(t, T)) = & \left\{ \frac{\partial u}{\partial t}(t, F_S(t, T)) + \frac{1}{2}(F_S(t, T))^2 \|\varrho_t - b(t, T)\|^2 \frac{\partial^2 u}{\partial z^2}(t, F_S(t, T)) \right\} dt \\ & + F_S(t, T) \left(\frac{\partial u}{\partial z}(t, F_S(t, T)) \right) (\varrho_t - b(t, T)) \cdot dW_t^F, \end{aligned} \quad (467)$$

from where we obtain

$$\mathcal{A}_{F_S(\cdot, T)} u = \frac{1}{2} z^2 \|\varrho_t - b(t, T)\|^2 \frac{\partial^2 u}{\partial z^2}, \quad (468)$$

and,

$$\mathcal{L}_{F_S(\cdot, T)} u = \frac{\partial u}{\partial t} + \mathcal{A}_{F_S(\cdot, T)} u = \frac{\partial u}{\partial t} + \frac{1}{2} z^2 \|\varrho_t - b(t, T)\|^2 \frac{\partial^2 u}{\partial z^2}, \quad (469)$$

compare with (542) and (543).

For example, equation (469) can be used to obtain the price of an European Call/Put on a Forward price. Consider an European option whose final payoff at time $M < T$ is given by:

$$B(M, T)(F_S(M, T) - \mathcal{K})^+, \quad (470)$$

\mathcal{K} a nonnegative constant. According to European contingent claim pricing theory (basically according to arbitrage free pricing, see [95] Chapter 1 sections §2 and §3, [133] Chapter 5, [96], etc.), the time t price of such an European contract will be:

$$V_t = B_t E_{\mathcal{E}} \left(\frac{B(M, T)}{B_M} (F_S(M, T) - \mathcal{K})^+ | \mathcal{F}_t \right) \quad (471)$$

Assume that $\varrho_t - b(t, T)$ is deterministic and that

$$\|\varrho_t - b(t, T)\| \leq L, \quad \forall t \in [0, M] \quad (472)$$

where $L > 0$ is a constant. Recall that we are assuming the conditions of Theorem 5.2.1 plus the ellipticity condition of (425) (or equivalently (451)).

From our computations above we know that discounting with respect to the Bank account is equivalent to discount with respect to the price process of a zcb of maturity T (equivalence given by (465) and (466)). Thus we can write:

$$\begin{aligned}
V_t &= B_t E_{\mathcal{E}} \left(\frac{B(M, T)}{B_M} (F_S(M, T) - \mathcal{K})^+ | \mathcal{F}_t \right) \\
&= B(t, T) E_F \left(B(M, T) (F_S(M, T) - \mathcal{K})^+ \frac{1}{B(M, T)} | \mathcal{F}_t \right) \\
&= B(t, T) E_F ((F_S(M, T) - \mathcal{K})^+ | \mathcal{F}_t) \\
&= B(t, T) u(t, F_S(t, T)) \quad (473)
\end{aligned}$$

where u is a solution to the equation

$$\begin{cases} \mathcal{L}_{F_S(\cdot, T)} u = \frac{\partial u}{\partial t} + \frac{1}{2} z^2 \|\varrho_t - b(t, T)\|^2 \frac{\partial^2 u}{\partial z^2} = 0, & t \in [0, M[, z \geq 0 \\ u(M, z) = (z - \mathcal{K})^+, & z > 0 \\ u(t, 0) = 0, & t \in [0, M] \end{cases} \quad (474)$$

(see [141] Chapter 12 or equivalently, [45] Chapter 5 and [46] Chapter 13, the ellipticity condition we imposed ensures that u is a $C^{1,2}$ function and that it satisfies the pde above). In view of (466), $u(t, F_S(t, T))$ is not the time t price of the option, but the time t forward price of the option. The time t price of the option can be found after multiplying $u(t, F_S(t, T))$ by the time t price of the T maturity zcb.

The power of this procedure is that we have effectively hidden the random rate inside the forward price diffusion simplifying noticeably the pricing equations.

Equation (474) can be solved explicitly.

The conditions we have imposed on $\varrho_t - b(t, T)$ (namely, the uniform ellipticity condition (451), the upper ellipticity or upper boundedness condition (472), plus the assumption that

the forward volatility is deterministic) will ensure that the initial value problem

$$\begin{cases} \frac{\partial v}{\partial \tau}(\tau, x) = \frac{1}{2}c(\tau) \left(\frac{\partial^2 v}{\partial x^2}(\tau, x) - \frac{\partial v}{\partial x}(\tau, x) \right), & r \in [0, M], x \in \mathbb{R} \\ v(0, x) = \alpha(x), & x \in \mathbb{R} \\ \lim_{x \rightarrow -\infty} v(\tau, x) = 0, & r \in [0, M] \\ c(\tau) = \|\varrho_{M-\tau} - b(M - \tau, T)\|^2 \end{cases} \quad (475)$$

has a solution whenever $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel-measurable function satisfying the integrability condition

$$\int_{-\infty}^{\infty} |\alpha(x)| e^{-ax^2} dx < \infty \quad (476)$$

for some $a > 0$.

The pde in (475) is obtained from (474) after the changes of variables $\tau = M - t$, $z = \mathcal{K}e^x$ and $u(z, \tau) = \mathcal{K}v(\tau, x)$.

In our case, the initial ($\tau = 0$) condition is $u(0, z) = \max\{z - \mathcal{K}, 0\}$, which transforms into $v(0, x) = (e^x - 1)^+ = \alpha(x)$, which clearly satisfies (476).

Under the assumption that (451), (472) and (476) are satisfied we can obtain explicit solutions for (475) as follows. Define

$$\phi(\tau) = \int_0^\tau c(u) du, \quad (477)$$

and

$$\Upsilon(\tau, x, y) = \frac{1}{\sqrt{2\pi\phi(\tau)}} \exp\left(\frac{-(2(x - y) - \phi(\tau))^2}{8\phi(\tau)}\right), \quad (478)$$

under the convention that $\Upsilon(\tau, x, y)$ reduces to the Dirac delta function concentrated at $x - y$, $\delta(x - y)$, when $\tau = 0$ (that is, when $\phi = 0$).

It is not hard to show that Υ satisfies the pde in (475).

Define now

$$v(\tau, x) = \int_{-\infty}^{\infty} \alpha(y) \Upsilon(\tau, x, y) dy \quad (479)$$

then, v is well defined for $(\tau, x) \in [0, 1/2a[\times \mathbb{R}$, has derivatives of all orders and satisfies (475).

Notice that turning back our changes of variables from (τ, x) to (t, z) we will obtain the solution to the corresponding final value problem. In the particular case of $v(0, x) =$

$(e^x - 1)^+ = \alpha(x)$, namely, when the final value is given by $\max\{z - \mathcal{K}, 0\}$, the solution to that final value problem will provide us with the arbitrage price for the European contract we are discussing here.

In fact, in the case of $v(0, x) = (e^x - 1)^+ = \alpha(x)$, (479) can be integrated explicitly.

In (z, t) variables the explicit solution of (474) is

$$u(t, z) = z\Phi(\mathbf{d}(t, z, M, T, \mathcal{K})) - \mathcal{K}\Phi(\mathbf{d}(t, z, M, T, \mathcal{K}) - \phi(t, M, T)), \quad (480)$$

where

$$\phi(t, M, T) = \int_t^M \|\varrho_t - b(t, T)\|^2 du, \quad (481)$$

and

$$\mathbf{d}(t, z, M, T, \mathcal{K}) = \frac{1}{\phi(t, M, T)} \ln\left(\frac{z}{\mathcal{K}}\right) + \frac{1}{2}\phi(t, M, T), \quad (482)$$

as usual, Φ is used to represent the Standard Normal distribution. Thus, the time t arbitrage price of an European call on the forward price of stock S , settling at time M is given by:

$$B(t, T) u\left(t, \frac{S_t}{B(t, T)}\right) \quad t \in [0, M]. \quad (483)$$

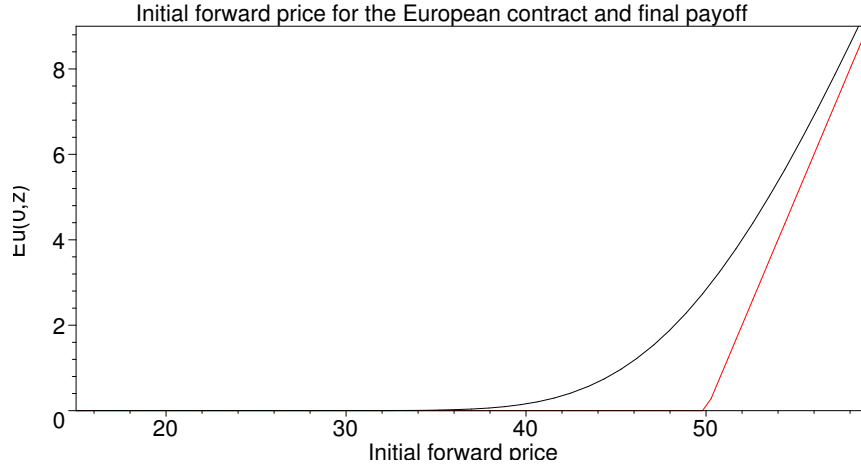


Figure 45: This plot shows the initial forward price of an European Call on a forward price computed using (480). In this particular example we assumed a Vasicek model with interest rate volatility of 0.20 and interest rate speed of mean reversion of 0.10. The other parameters used were, stock volatility 0.25, forward strike 50.0, and zcb maturity 1.00.

5.5 American Game Call

Based in our prior description of what a Game Option is, see Chapter 3, we see that a very natural extension to the concept of American Call/Put is the concept of an American Game Call/Put. An *American Game Call* is a contract between two parties in which one, *the Seller*, agrees to sell to the other party, *the Buyer*, a given asset at or before a particular time, known as the *Maturity* of the contract, for a pre-specified price, the *Strike*. The Buyer in this kind of contract can *exercise*¹⁰ his/her option at any time at or before Maturity, while the Seller (also known as the *Writer*, or *Issuer*) preserves the right to cancel/stop the contract at any time prior to Maturity, provided he/she will pay to the Buyer (also known as the *Holder* or *Investor*) a *fine* (also known as a *penalization*) plus the current value of the option. That is, in case of cancelation, the Buyer will receive the amount he/she should have received if he/she had decided to exercise at time of cancelation plus a positive amount in compensation for his/her possible losses due to the non continuation of the contract. In case the option reaches maturity without being exercised or cancelled, the Buyer is awarded the intrinsic value of the option at maturity time, that is, he/she is given the same payoff as if she/he had decided to exercise at maturity.

As we can see, an American Game Call (**agc**) is a particular case of Game Option (see Chapter 3, §3.2 and Chapter 3, §3.4) in which the processes \mathfrak{X} and \mathfrak{Y} that define the payoff $\mathcal{R}^{\mathfrak{X}, \mathfrak{Y}}$ of the option are of the form $\mathfrak{X}_t = (S_t - K_t)^+ + \mathfrak{p}_t$ and $\mathfrak{Y}_t = (S_t - K_t)^+$ where S_t represents the time t price of a given asset, and K_t represents the time t *Strike* price of the option (which could be a random process), and $\mathfrak{p}_t \geq 0$ is the time t *penalization* that the Writer should pay (in addition to the current payoff of the option) to the Buyer in case of cancelation. Naturally, $\mathfrak{p}_t = \mathfrak{X}_t - \mathfrak{Y}_t$ could also be a random process. If M is the maturity of the contract, the payoff at maturity is $\mathfrak{Y}_M = (S_M - K_M)^+$.

In this chapter we propose the numerical pricing of a further extension of the idea of an American Call, this will be an American Game Call where the contract is not written on the spot price of a given asset, but on its forward price. Alternatively, the contract

¹⁰That is, can opt to acquire the underlying asset, at strike price.

we proposed here can be seen as a Game American Call whose time t value is adjusted to reflect the time t price of a given zero coupon bond, so that the option's payoff can be expressed in units of zcb price. This way, both strike and penalization —although simple— are stochastic processes and not mere constants or deterministic functions. Considering the pricing of an American Game Call on a forward price we will be studying a contract that is not only immersed in a market where interest rates are stochastic, but that also, in a very natural way, is explicitly sensitive to the interest rate underlying our market.

All theoretical assumptions and notation are those used throughout this paper and in particular like those in Chapter 3 §3.3 and those presented earlier in this chapter. Please refer to Chapter 3 §3.3, and to sections §5.1 to §5.4 for details and definitions.

In particular, we will assume that we are dealing with a standard market \mathcal{M} with a finite time horizon $\mathcal{T} < \infty$, in which uncertainty is driven by a d -dimensional Brownian Motion W defined on a complete, filtered probability space $(\Omega, \mathcal{U}, \mathcal{F}, \mathcal{P})$, where \mathcal{F} denotes the \mathcal{P} -augmentation of the natural filtration of W and where $\mathcal{U} = \mathcal{F}_{\mathcal{T}}$, that is, the filtration \mathcal{F} of sub- σ -algebras of \mathcal{U} satisfies the usual conditions¹¹. We assume that asset prices follow strictly positive processes which we model as lognormal processes, see Chapter 3 (46) and (438) and (439) below. On the other hand, we will assume that the risk free rate r is modeled by means of a Hull–White interest rate model¹², see Chapter 4, (255). We will assume an initial term structure to be given, to which the interest rate model is fitted.

¹¹In Sub-section §3.3.1, Chapter 3, we went into the theoretical details of the definition of a standard market. See §3.3.1 and in particular Footnote 12.

¹²As it can be inferred from our discussion below, our setting does not depend on our selection of a Hull–White model of interest rates; on the contrary, this setting will be valid as long as the interest rate model implies bond dynamics in accordance with (439) and the volatility of the forward price $\varrho_t - a(t, T)$ is a deterministic process. The first condition is satisfied by any interest rate model where the dynamics of the interest rate are modeled by an Itô process.

5.6 The pricing of an American Game Call on a Forward price

Let's adopt once again the settings of sections §5.2 to §5.4. As before, we consider a stock whose price process is given by equation (438) and a zero coupon bond of maturity $T \leq \mathcal{T}$ whose price process follows the dynamics of (439). As before, we will assume that the conditions of Theorem 5.2.1 hold plus the ellipticity, (451), and boundedness of the forward volatility, (472).

Consider an American Game Call of maturity M , $M \leq T \leq \mathcal{T}$ whose payoff processes are given by the formulas

$$\begin{aligned}\mathfrak{Y}_t &= \tilde{\Phi}(S_t, B(t, T)) = (S_t - \mathcal{K}B(t, T))^+ \\ &= B(t, T)(F_S(t, T) - \mathcal{K})^+ = B(t, T)\Phi(F_S(t, T)), \quad t \in [0, M],\end{aligned}\tag{484}$$

and

$$\begin{aligned}\mathfrak{X}_t &= \tilde{\Psi}(S_t, B(t, T)) = (S_t - \mathcal{K}B(t, T))^+ + \mathfrak{p}B(t, T) \\ &= B(t, T)((F_S(t, T) - \mathcal{K})^+ + \mathfrak{p}) = B(t, T)\Psi(F_S(t, T)), \quad t \in [0, M],\end{aligned}\tag{485}$$

where \mathfrak{p} and \mathcal{K} are nonnegative constants and $\tilde{\Psi}$ and $\tilde{\Phi}$ (resp. Ψ and Φ) are two continuous, convex, functions from \mathbb{R}^{+2} (resp. \mathbb{R}^+) into \mathbb{R} defined by

$$\begin{aligned}\tilde{\Phi}(x, y) &= (x - y\mathcal{K})^+ & \tilde{\Psi}(x, y) &= \tilde{\Phi}(x, y) + y\mathfrak{p}, & (x, y) &\in [0, \infty[\times]0, \infty[\\ \Phi(z) &= (z - \mathcal{K})^+ & \Psi(z) &= \Phi(z) + \mathfrak{p} & z &\in [0, \infty[\end{aligned}\tag{486}$$

Thus, $\tilde{\Phi}(x, y) = y\Phi(x/y)$ and $\tilde{\Psi}(x, y) = y\Psi(x/y)$, $\forall (x, y) \in [0, \infty[\times]0, \infty[$. In consistency to our prior descriptions, at time of maturity, the payoff of the option will be given by $\mathfrak{Y}_M = \tilde{\Phi}(S_M, B(M, T)) = (S_M - \mathcal{K}B(M, T))^+ = B(M, T)(F_S(M, T) - \mathcal{K})^+ = B(M, T)\Phi(F_S(M, T))$.

\mathcal{K} will be called the *forward strike price* or *forward strike* of the option, \mathfrak{p} the *forward penalization premium* or simply the *forward penalization*. Similarly, $\mathcal{K}B(t, T)$ will be the time t strike price of the option while $\mathfrak{p}_t = \mathfrak{p}B(t, T)$ will be the time t penalization.

Since Φ is continuous and nonnegative, \mathfrak{p} is nonnegative and $F_S(\cdot, T)$, S , and $B(\cdot, T)$ are diffusions (satisfying (461), (438) and (439), resp.) we know that $-\mathfrak{X}$ and \mathfrak{Y} are rcll

processes (in fact they are continuous processes, not only rcll and left upper semicontinuous) satisfying the conditions of Chapter 3 while $F_S(\cdot, T)$, S , and $B(\cdot, T)$ satisfy the additional conditions imposed in this chapter. Clearly, the processes just defined satisfy $\mathfrak{X}_t \geq \mathfrak{Y}_t$ and (415), that is:

$$\begin{aligned}\mathfrak{Y}_t^* &= \tilde{\Phi}(S_t, B(t, T))/B_t = \Phi(S_t/B(t, T))B(t, T)/B_t \\ &= \Phi(F_S(t, T))B(t, T)/B_t \leq \mathcal{V}_t^* \leq \tilde{\Psi}(S_t, B(t, T))/B_t \\ &= \Psi(S_t/B(t, T))B(t, T)/B_t = \Psi(F_S(t, T))B(t, T)/B_t = \mathfrak{X}_t^*; \quad (487)\end{aligned}$$

which we can rewrite as

$$F_{\mathfrak{Y}}(t, T) = \Phi(F_S(t, T)) \leq F_{\mathcal{V}}(t, T) = \frac{\mathcal{V}_t}{B(t, T)} \leq \Psi(F_S(t, T)) = F_{\mathfrak{X}}(t, T); \quad (488)$$

as usual “*” is used to denote discounting with respect to the bank account B (see Definition 3.3.9), while \mathcal{V} stands for the price process of the game contingent claim, in this case the price process of our game American call on the forward price of stock S with respect to the zcb $B(\cdot, T)$. Equations (415) to (419) can be rewritten to reflect our current setting, in particular we may write:

$$\begin{aligned}\Phi(F_S(t, T)) &\leq F_{\mathcal{V}}(t, T) \leq \Psi(F_S(t, T)), \\ F_{\mathcal{V}}(s, T) &< \Psi(F_S(s, T)), & \forall s < \kappa_t, \\ \Phi(F_S(s, T)) &< F_{\mathcal{V}}(s, T), & \forall s < \xi_t, \\ \Phi(F_S(t, T)) &< \frac{\mathcal{V}_t}{B(t, T)} = F_{\mathcal{V}}(t, T) \quad \underline{\text{or}} \quad \Phi(F_S(t, T)) = \frac{\mathcal{V}_t}{B(t, T)} = F_{\mathcal{V}}(t, T), \\ F_{\mathcal{V}}(t, T) &= \frac{\mathcal{V}_t}{B(t, T)} < \Psi(F_S(t, T)) \quad \underline{\text{or}} \quad F_{\mathcal{V}}(t, T) = \frac{\mathcal{V}_t}{B(t, T)} = \Psi(F_S(t, T)),\end{aligned} \quad (489)$$

for all $t \in [0, M]$. Where κ_t and ξ_t are the cancellation and exercise times (see Chapter 3, in particular Theorem 3.4.26, (243))

$$\begin{aligned}\xi_t &= \inf\{s \geq t : \mathfrak{Y}_s^* \geq \mathcal{V}_s^*\}, \\ \kappa_t &= \inf\{s \geq t : \mathfrak{X}_s^* \leq \mathcal{V}_s^*\} \wedge M.\end{aligned} \quad (490)$$

As commented before, κ_t and ξ_t can be seen as the first hitting times, after time t , corresponding to the cancellation and exercise regions, while the end of the contract (after time

t) $\kappa_t \wedge \xi_t$ is the first exit time, after time t , from the continuation region (see comment right after (419)). The reader may recall that we showed such stopping times are finite, see Proposition 3.4.4 and Proposition 3.4.5.

Also, we may rewrite (423), the gcc payoff function $\mathcal{R}^{\Psi, \Phi}$, as follows:

$$\begin{aligned}\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(s, t) &= B(s \wedge t, T) \left([(S_s/B(s, T) - \mathcal{K})^+ + \mathfrak{p}] \mathbb{1}_{s < t} + (S_t/B(t, T) - \mathcal{K})^+ \mathbb{1}_{t \leq s} \right) \\ &= B(s \wedge t, T) (\Psi(F_S(s, T)) \mathbb{1}_{s < t} + \Phi(F_S(t, T)) \mathbb{1}_{t \leq s}) \\ &= B(s \wedge t, T) \mathcal{R}^{\Psi, \Phi}(s, t),\end{aligned}\tag{491}$$

where

$$\begin{aligned}\mathcal{R}^{\Psi, \Phi}(s, t) &= \Psi(F_S(s, T)) \mathbb{1}_{s < t} + \Phi(F_S(t, T)) \mathbb{1}_{t \leq s} \\ &= \Phi(F_S(s, T)) \mathbb{1}_{s < t} + \mathfrak{p} \mathbb{1}_{s < t} + \Phi(F_S(t, T)) \mathbb{1}_{t \leq s} \\ &= \Phi(F_S(s \wedge t, T)) + \mathfrak{p} \mathbb{1}_{s < t}\end{aligned}\tag{492}$$

In the end, $\mathcal{R}^{\Psi, \Phi}(s, t)$ is a measurable function of $F_S(\cdot, T)$, s and t .

Based on Definition 5.3.2, and in the same spirit of Notation 3.4.3, we can regard $\mathcal{R}^{\Psi, \Phi}(s, t)$ as the forward price of $\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(s, t)$

$$F_{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}}(s \wedge t, T) = \mathcal{R}^{\Psi, \Phi}(s, t) = \frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(s, t)}{B(s \wedge t, T)}\tag{493}$$

According to Theorem 3.4.26 the discounted price process of a game contingent claim satisfies:

$$\mathcal{V}_t^* = \operatorname{essinf}_{s \in \mathfrak{S}_{t, \mathcal{T}}} \operatorname{esssup}_{t \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}} \left(\frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(\mathfrak{s}, \mathfrak{t})}{B_{s \wedge t}} \middle| \mathcal{F}_t \right) = \operatorname{esssup}_{t \in \mathfrak{S}_{t, \mathcal{T}}} \operatorname{essinf}_{s \in \mathfrak{S}_{t, \mathcal{T}}} E_{\mathcal{E}} \left(\frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(\mathfrak{s}, \mathfrak{t})}{B_{s \wedge t}} \middle| \mathcal{F}_t \right); \tag{494}$$

we are now in position to adapt that result to our particular setting.

In view of (465), (484), (485), (491), and (491) we have the following proposition.

Proposition 5.6.1. *The price process, \mathcal{V} , of a Game American Call of maturity $M \leq T \leq \mathcal{T}$ on the forward price of a Stock S with respect to zcb $B(\cdot, T)$ defined by the payoff processes (484) and (485) satisfies:*

$$F_{\mathcal{V}}(t, T) = \operatorname{essinf}_{s \in \mathfrak{S}_{t, M}} \operatorname{esssup}_{t \in \mathfrak{S}_{t, M}} E_F(\mathcal{R}^{\Psi, \Phi}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) = \operatorname{esssup}_{s \in \mathfrak{S}_{t, M}} \operatorname{essinf}_{t \in \mathfrak{S}_{t, M}} E_F(\mathcal{R}^{\Psi, \Phi}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t). \tag{494}$$

Proof. This proposition follows immediately from the definitions of esssup , essinf and (465), (484), (485), (491), and (492). By (465) we have

$$E_{\mathcal{E}} \left(\frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(\mathfrak{s}, \mathfrak{t})}{B_{\mathfrak{s} \wedge \mathfrak{t}}} \middle| \mathcal{F}_t \right) = \frac{B(t, T)}{B_t} E_F (\mathcal{R}^{\Psi, \Phi}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t). \quad (495)$$

Therefore

$$\begin{aligned} \mathcal{V}_t^* &= \text{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, T}} \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} E_{\mathcal{E}} \left(\frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(\mathfrak{s}, \mathfrak{t})}{B_{\mathfrak{s} \wedge \mathfrak{t}}} \middle| \mathcal{F}_t \right) \\ &= \text{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, T}} \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} \left\{ E_F (\mathcal{R}^{\Psi, \Phi}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \frac{B(t, T)}{B_t} \right\} \\ &= \frac{B(t, T)}{B_t} \text{essinf}_{\mathfrak{s} \in \mathfrak{S}_{t, T}} \text{esssup}_{\mathfrak{t} \in \mathfrak{S}_{t, T}} E_F (\mathcal{R}^{\Psi, \Phi}(\mathfrak{s}, \mathfrak{t}) | \mathcal{F}_t) \\ &= \frac{\mathcal{V}_t}{B_t} = \frac{B(t, T)}{B_t} F_{\mathcal{V}}(t, T), \end{aligned} \quad (496)$$

(the other equality is obtained similarly). \square

Observe that the change of numéraire technique presented in section §5.3 allows us to translate our results from Chapter 3 (which were developed under the risk neutral measure) to equivalent results stated under the forward measure (induced by the zcb $B(\cdot, T)$ of maturity T). Intuitively, we could rework the whole Chapter 3 changing the discounting with respect to the bank account to a different form of discounting using a given nondivident paying positive asset. In doing so, we will be able to obtain a Theorem as Theorem 3.4.26 stating also the existence of ε -optimal strategies and although such strategies may be different from those found in Theorem 3.4.26 they should lead to the same saddle points since

$$\xi_t = \inf\{s \geq t : \mathfrak{V}_s^* \geq \mathcal{V}_s^*\} = \inf\{s \geq t : \mathfrak{V}_s \geq \mathcal{V}_s\}; \quad (497)$$

and, in the particular case of the current example

$$\xi_t = \inf\{s \geq t : \mathfrak{V}_s \geq \mathcal{V}_s\} = \inf\{s \geq t : F_{\mathfrak{V}}(s, T) \geq F_{\mathcal{V}}(s, T)\}, \quad (498)$$

and similarly in the case of the cancellation payoff.

According to Theorem 3.4.26, for every t there exists κ_t and $\xi_t \in \mathfrak{S}_{t, M}$, κ_t and ξ_t defined in (490), such that

$$\mathcal{V}_t^* = E_{\mathcal{E}} \left(\frac{\mathcal{R}^{\tilde{\Psi}, \tilde{\Phi}}(\kappa_t, \xi_t)}{B_{\kappa_t \wedge \xi_t}} \middle| \mathcal{F}_t \right), \quad (499)$$

our previous computations show the following proposition.

Proposition 5.6.2. *For every $t \in [0, M]$, the stopping times ξ_t and κ_t of Theorem 3.4.26 satisfy*

$$\begin{aligned}\xi_t &= \inf\{s \geq t : F_{\mathfrak{V}}(s, T) \geq F_{\mathcal{V}}(s, T)\}, \\ \kappa_t &= \inf\{s \geq t : F_{\mathfrak{X}}(s, T) \leq F_{\mathcal{V}}(s, T)\} \wedge M,\end{aligned}\tag{500}$$

and

$$F_{\mathcal{V}}(t, T) = E_F \left(\mathcal{R}^{\Psi, \Phi}(\kappa_t, \xi_t) \middle| \mathcal{F}_t \right).\tag{501}$$

That is, we can obtain the price of our Game American Call under the forward measure as well. As in the case of the European call option we studied before, we can write

$$F_{\mathcal{V}}(t, T) = u(t, F_S(t, T))\tag{502}$$

Explicitly, we can rewrite (500) as

$$\begin{aligned}\xi_t &= \inf\{s \geq t : \Phi(x) \geq u(s, x); x = F_S(s, T)\}, \\ \kappa_t &= \inf\{s \geq t : \Psi(x) \leq u(s, x); x = F_S(s, T)\} \wedge M,\end{aligned}\tag{503}$$

which are both finite. ξ_t is the first hitting time (after time t) to the exercise region

$$\mathcal{E}x = \{(t, \omega) : \Phi(x) \geq u(t, x); x = F_S(t, T)(\omega)\},\tag{504}$$

κ_t is the first hitting time (after time t) to the cancellation region

$$\mathcal{K}a = \{(t, \omega) : \Psi(x) \leq u(t, x); x = F_S(t, T)(\omega)\},\tag{505}$$

and $\kappa_t \wedge \xi_t$ is the first exit time (after time t) from

$$\mathcal{C}o = \{(t, \omega) : \Phi(x) < u(t, x) < \Psi(x); x = F_S(t, T)(\omega)\},\tag{506}$$

the continuation region.

Notice that those regions define also three regions in \mathbb{R}^{+2} , namely

$$\widehat{\mathcal{E}x} = \{(t, x) : \Phi(x) \geq u(t, x)\},\tag{507a}$$

$$\widehat{\mathcal{K}a} = \{(t, x) : \Psi(x) \leq u(t, x)\},\tag{507b}$$

and

$$\widehat{\mathcal{C}o} = \{(t, x) : \Phi(x) < u(t, x) < \Psi(x)\}, \quad (507c)$$

$\forall (t, \omega) \in \mathbb{R} \times \Omega$, $(t, F_S(t, T)(\omega))$ falls into one, and only one of such sets. In fact we know that if $(t, \omega) \in \mathcal{E}x$ or if $(t, \omega) \in \mathcal{K}a$ only equality is possible (see (489)), similarly, if $(t, F_S(t, T) \in \widehat{\mathcal{E}x}$ or if $(t, F_S(t, T) \in \widehat{\mathcal{K}a}$ then $\Phi(F_S(t, T)) = u(t, F_S(t, T))$ or $\Psi(F_S(t, T)) = u(t, F_S(t, T))$; we can express this as follows:

$$(u(t, F_S(t, T)) - \Phi(F_S(t, T))) (\Psi(F_S(t, T)) - u(t, F_S(t, T))) = 0 \quad \text{on} \quad \mathcal{K}a \bigcup \mathcal{E}x \quad (508)$$

or equivalently

$$(u(t, x) - \Phi(x)) (\Psi(x) - u(t, x)) = 0 \quad \text{on} \quad \widehat{\mathcal{K}a} \bigcup \widehat{\mathcal{E}x} \quad (509)$$

We expect u to satisfy an equation of the form

$$\frac{\partial u}{\partial \tau} - g(\tau, z, T) \frac{\partial^2 u}{\partial z^2} = 0 \quad (510)$$

on the continuation region, so, at any time at least one of the following terms should be zero

$$\frac{\partial u}{\partial \tau} - g(\tau, z, T) \frac{\partial^2 u}{\partial z^2} \quad \text{or} \quad u(\tau, z) - \max\{z - \mathcal{K}, 0\} \quad \text{or} \quad \mathfrak{p} + \max\{z - \mathcal{K}, 0\} - u(\tau, z) \quad (511)$$

thus, we can express all three conditions in the form of a triple product:

$$\begin{cases} \left(\frac{\partial u}{\partial \tau} - g(\tau, z, T) \frac{\partial^2 u}{\partial z^2} \right) \left(u(\tau, z) - (z - \mathcal{K})^+ \right) \left(\mathfrak{p} + (z - \mathcal{K})^+ - u(\tau, z) \right) = 0 \\ \mathfrak{p} + (z - \mathcal{K})^+ \geq u(z, \tau) \geq (z - \mathcal{K})^+ \\ u(\tau, 0) = 0, \quad u(0, z) = (z - \mathcal{K})^+ \end{cases} \quad (512)$$

where $0 \leq z$, $0 \leq \tau \leq M$ and $g(\tau, z, M) = \frac{1}{2}z^2|\varrho_{M-\tau} - b(M - \tau, T)|^2$.

We can obtain the pde that u satisfies in another way, namely by means of the construction of a hedge.

In what follows we will use some ideas from Musiela and Rutkowski, [133] chapter 15, and from Chapter 3. As mentioned in the previous section, will assume that we are dealing

with a standard market \mathcal{M} in which uncertainty is driven by a d -dimensional Brownian Motion W , and in which at least two securities, a stock S and a bond $B(\cdot, T)$, are traded¹³.

As we mentioned before, we are interested in the pricing of an American game call written on the forward price $F_S(t, T) = S_t/B(t, T)$ of stock S , with strike \mathcal{K} , maturity $M \leq T \leq \mathcal{T} < \infty$ and penalty $\mathfrak{p}B(t, T)$, where \mathcal{K} and \mathfrak{p} are (known) constants. At maturity (that is, if the option has not been canceled or exercised prior to that date) the option's payoff is given by

$$\begin{aligned}\mathcal{V}_M &= B(M, T)\Phi(F_S(M, T)) = B(M, T)(F_S(M, T) - \mathcal{K})^+ \\ &= \tilde{\Phi}(S_M, B(M, T)) = (S_M - KB(M, T))^+ = \mathfrak{V}_M.\end{aligned}\quad (513)$$

In case of early exercise or early cancelation, if ξ is the time at which the buyer decides to execute, and κ is the time at which the seller decided to cancel, then the payoff at the end of the game will be

$$\begin{aligned}\mathcal{V}_{\xi \wedge \kappa} &= \mathcal{R}^{\tilde{\Phi}, \tilde{\Psi}}(\kappa, \xi) \\ &= (B(\kappa, T)(F_S(\kappa, T) - \mathcal{K})^+ + \mathfrak{p}B(\kappa, T)) \mathbb{1}_{\kappa < \xi} \\ &\quad + B(\xi, T)(F_S(\xi, T) - \mathcal{K})^+ \mathbb{1}_{\xi \leq \kappa} \\ &= B(\kappa \wedge \xi, T) ((F_S(\kappa, T) - \mathcal{K})^+ + \mathfrak{p}) \mathbb{1}_{\kappa < \xi} + (F_S(\xi, T) - \mathcal{K})^+ \mathbb{1}_{\xi \leq \kappa} \\ &= B(\kappa \wedge \xi, T) \mathcal{R}^{\Phi, \Psi}(\kappa, \xi)\end{aligned}\quad (514)$$

at any other time t , it is clear that the price \mathcal{V}_t of our contract satisfies

$$\begin{aligned}\tilde{\Psi}(S_t, B(t, T)) &= B(t, T)(F_S(t, T) - \mathcal{K})^+ + \mathfrak{p}B(t, T) \\ &\geq \mathcal{V}_t \geq B(t, T)(F_S(t, T) - \mathcal{K})^+ = \tilde{\Phi}(S_t, B(t, T)).\end{aligned}\quad (515)$$

Although this feature of a game option was already discussed before (see Chapter 3), from (515) we notice that as \mathfrak{p} increases, the issuer will be less likely to cancel the contract prior to maturity. In the limit, as \mathfrak{p} goes to infinity, condition (515) resembles the corresponding condition on an American Call. It is also clear that the value of an American Call

¹³For the sake of simplicity, the reader may assume W represents a two dimensional brownian motion, and that ϱ and b (see below) are two dimensional as well.

in same terms should be higher than the value of the American Game Call, see Proposition 5.1.2. In fact, being less likely to the issuer to cancel as \mathfrak{p} increases, and also being that the holder will receive a higher pay if \mathfrak{p} increases, we can expect the value of the American game call to increase as \mathfrak{p} increases (naturally, the value of the American Game Call is always capped by the value of the American call, see Proposition 5.1.2).

With the idea of representing the price of an American game call in terms of the price of the underlying, the bond price and the time, it is assumed that the price of our American game call \mathcal{V}_t admits a representation of the form

$$\mathcal{V}_t = v(t, S_t, B(t, T)), \quad (516)$$

where

$$\begin{aligned} v: [0, M] \times \mathbb{R}_+ \times [0, 1] &\longrightarrow \mathbb{R} \\ (t, x, y) &\longmapsto v = v(t, x, y), \end{aligned} \quad (517)$$

is an unknown function that satisfies

$$v(M, x, y) = y(x/y - \mathcal{K})^+, \quad \forall (x, y) \in \mathbb{R}_+ \times]0, 1], \quad (518)$$

the condition at maturity, and

$$y(x/y - \mathcal{K})^+ \leq v(t, x, y) \leq y((x/y - \mathcal{K})^+ + \mathfrak{p}), \quad \forall (t, x, y) \in [0, M[\times \mathbb{R}_+ \times]0, 1]. \quad (519)$$

This function will be not only the solution to our problem, but also the one we will have to approximate with some numerical procedure. This function, we expect, will be the solution of a partial differential equation of order, at least, two. Then it makes sense to ask for this function to be of class $C^{1,2,2}$ on the open $]0, \infty[\times]0, 1[\times]0, M[$ ¹⁴. Of course, we will need these requirements only while the option is alive. More explicitly, we will construct a diffusion equation (one very similar to the Heat equation) whose solution u is related to v by the expression $v(t, x, y) = yu(t, x/y)$.

¹⁴Also, we will need to apply Ito's to it, thus we need v to be of class C^2 in its first two variables and of class C in its third variable, our assumption of ellipticity takes care of this, see [141] Chapter 12, or [45] Chapter 5.

Following a similar derivation process as that followed by Musiela and Rutkowski in [133] Section §15.3, we want to construct a replicating portfolio in terms of the price of the underlying stock and the price of the zcb being used to compute the forward price. Assuming continuous trading, the attainability of our American game call and the perfect divisibility of assets¹⁵, let $\pi = (\pi^1, \pi^2)$ denote a self-financing portfolio strategy based on those two assets. Thus, we define the wealth process \mathcal{W} as:

$$\mathcal{W}_t = \pi_t^1 S_t + \pi_t^2 B(t, T), \quad (520)$$

where π_t^1 and π_t^2 are the time $t \leq M$ amounts of the two assets in our portfolio. We assume that there is no consumption, or any other form of market friction. We assume that funds are thus transferred from one account to the other every time our position in one of the assets changes. Based on our assumption of π being a self-financing portfolio, and combining (438) and¹⁶ (439) we obtain

$$\begin{aligned} d\mathcal{W}_t &= \pi_t^1 dS_t + \pi_t^2 dB(t, T) \\ &= (\pi_t^1 \mu_t S_t + \pi_t^2 a(t, T) B(t, T)) dt + (\pi_t^1 \varrho_t S_t + \pi_t^2 b(t, T) B(t, T)) \cdot dW_t. \end{aligned} \quad (521)$$

Since we are using this portfolio strategy to replicate an American Game Call option, we assume that the payoff of our option is equal to the wealth process for the portfolio (in Chapter 3 we learnt we can hedge against a gcc, so we know that a portfolio as the one we propose exist)

$$\mathcal{V}_t = v(t, S_t, B(t, T)) = \mathcal{W}_t = \pi_t^1 S_t + \pi_t^2 B(t, T) \quad t \in [0, M], \quad (522)$$

therefore

$$\pi_t^2 = (v(t, S_t, B(t, T)) - \pi_t^1 S_t) / B(t, T), \quad (523)$$

this will let us rewrite (521) as (we will write v as a short hand for $v(S_t, B(t, T), t)$)

¹⁵All these assumptions are consistent with our general model and are expressed here for emphasis. Chapter 3 shows that this option is attainable. Continuous trading and the perfect divisibility of assets are both underlying assumptions of the market model in place, see Chapter 3. Notice that notation and terms used here are the same or very similar to those used in Chapter 3.

¹⁶The case of a dividend paying stock could be studied later as part of future developments.

$$\begin{aligned}
d\mathcal{W}_t &= d\mathcal{V}_t \\
&= \left(\pi_t^1(\mu_t - a(t, T))S_t + a(t, T)v \right) dt + \left(\pi_t^1(\varrho_t - b(t, T))S_t + b(t, T)v \right) \cdot dW_t. \quad (524)
\end{aligned}$$

Assuming that the option has not been executed/canceled at time t , $t \in [0, M[$ (if the option is executed/canceled we already know what function v should be, see (514)) and remembering (438) and (439) while applying Ito's lemma at (522) we obtain:

$$\begin{aligned}
d\mathcal{V}_t &= \left\{ \frac{\partial v}{\partial t} + \mu_t S_t \frac{\partial v}{\partial x} + a(t, T)B(t, T) \frac{\partial v}{\partial y} + \varrho_t \cdot b(t, T)S_t B(t, T) \frac{\partial^2 v}{\partial x \partial y} \right. \\
&\quad \left. + \frac{1}{2} \left(\|\varrho_t\|^2 S_t^2 \frac{\partial^2 v}{\partial x^2} + \|b(t, T)\|^2 B(t, T)^2 \frac{\partial^2 v}{\partial y^2} \right) \right\} dt \\
&\quad + \left(\varrho_t S_t \frac{\partial v}{\partial x} + b(t, T)B(t, T) \frac{\partial v}{\partial y} \right) \cdot dW_t, \quad (525)
\end{aligned}$$

then, comparing the last equation with (524) we find that

$$\left\{ \begin{aligned} &\left\{ \frac{\partial v}{\partial t} + \mu_t S_t \frac{\partial v}{\partial x} + a(t, T)B(t, T) \frac{\partial v}{\partial y} + \varrho_t \cdot b(t, T)S_t B(t, T) \frac{\partial^2 v}{\partial x \partial y} \right. \\ &\quad \left. + \frac{1}{2} \left(\|\varrho_t\|^2 S_t^2 \frac{\partial^2 v}{\partial x^2} + \|b(t, T)\|^2 B(t, T)^2 \frac{\partial^2 v}{\partial y^2} \right) \right\} dt, \\ &= \left(\pi_t^1(\mu_t - a(t, T))S_t + a(t, T)v \right) dt \\ &\quad \left(\varrho_t S_t \frac{\partial v}{\partial x} + b(t, T)B(t, T) \frac{\partial v}{\partial y} \right) \cdot dW_t = \left(\pi_t^1(\varrho_t - b(t, T))S_t + b(t, T)v \right) \cdot dW_t, \end{aligned} \right. \quad (526)$$

the second of such equations is equivalent to the following stochastic integral

$$\int_0^t \left(\varrho_u S_u \left(\pi_u^1 - \frac{\partial v}{\partial x} \right) + b(u, T) \left(v - \pi_u^1 S_u - B(u, T) \frac{\partial v}{\partial y} \right) \right) \cdot dW_u = 0, \quad (527)$$

which should be valid for $t \in [0, M]$ and until the option is executed or canceled. The first term in (527) will be identically zero if we assume that

$$\pi_t^1 = \frac{\partial v}{\partial x} \Big|_{(t, S_t, B(t, T))}. \quad (528)$$

Similarly, the second term in integral (527) will be identically zero if we assume that

$$v(t, S_t, B(t, T)) = S_t \frac{\partial v}{\partial x} \Big|_{(t, S_t, B(t, T))} + B(t, T) \frac{\partial v}{\partial y} \Big|_{(t, S_t, B(t, T))}. \quad (529)$$

Notice that these two assumptions (namely (528) and (529) and (522)) also imply that

$$\pi_t^2 = \frac{\partial v}{\partial y} \Big|_{(t, S_t, B(t, T))}. \quad (530)$$

The last requirements translate directly into requirements for our unknown function v , in particular we will need

$$v(t, x, y) = x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y}, \quad (531)$$

on $]0, M[\times]0, \infty[\times]0, 1[$. Interestingly, (531) also gives us a hint on the form of the unknown function v . In particular, if

$$\begin{aligned} u: [0, M] \times \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ (t, z) &\longmapsto u(t, z), \end{aligned} \quad (532)$$

is a function of class $C^{1,2}$ (more on this function later), let $f(x, y, t) = yu(t, x/y)$. For $y > 0$ we will have

$$\begin{cases} \frac{\partial f}{\partial x}(t, x, y) = \frac{\partial u}{\partial z}(t, x/y), \\ \frac{\partial f}{\partial y}(t, x, y) = u(t, x/y) - \frac{x}{y} \frac{\partial u}{\partial z}(t, x/y), \end{cases} \quad (533)$$

which shows f satisfies (531). On the other hand

$$\frac{\partial}{\partial y}(v(t, \alpha y, y)/y) = -\frac{1}{y^2} \left(v(t, \alpha y, y) - \alpha y \frac{\partial v}{\partial x}(t, \alpha y, y) - y \frac{\partial v}{\partial y}(t, \alpha y, y) \right) = 0, \quad (534)$$

which clearly implies that function $\tilde{f}(t, \alpha, y) = v(t, \alpha y, y)/y$ does not depend on y . Therefore, we see that function v is of the form $v(t, x, y) = yu(t, x/y)$, where u is an unknown real function of class $C^{1,2}$ on $[0, M] \times \mathbb{R}_+$. Similarly, we note that

$$\mathcal{V}_t = B(t, T)u \left(t, \frac{S_t}{B(t, T)} \right) = B(t, T)u(t, F_S(t, T)). \quad (535)$$

Notice as well that (484), (485) and (515) imply, as expected, that

$$\Phi(F_S(t, T)) \leq u(t, F_S(t, T)) \leq \Psi(F_S(t, T)), \quad (536)$$

which should be valid not only while the option has not been exercised or canceled but, see (514), at exercise and cancellation.

Returning to (526), when we apply assumptions (528) and (529) to the drift part of (524) we obtain

$$\begin{aligned} \pi_t^1(\mu_t - a(t, T))S_t + a(t, T)v &= \frac{\partial v}{\partial x}(\mu_t - a(t, T))S_t + a(t, T) \left(S_t \frac{\partial v}{\partial x} + B(t, T) \frac{\partial v}{\partial y} \right) \\ &= \mu_t S_t \frac{\partial v}{\partial x} + a(t, T)B(t, T) \frac{\partial v}{\partial y}. \end{aligned} \quad (537)$$

From this the first equation in (526) will reduce to

$$\left\{ \frac{\partial v}{\partial t} + \varrho_t \cdot b(t, T) S_t B(t, T) \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \left(\|\varrho_t\|^2 S_t^2 \frac{\partial^2 v}{\partial x^2} + \|b(t, T)\|^2 B(t, T)^2 \frac{\partial^2 v}{\partial y^2} \right) \right\} dt = 0. \quad (538)$$

Thus, the corresponding integral should be zero for all $t \in [0, M]$. In terms of our unknown function v we may write

$$\frac{\partial v}{\partial t} + \varrho_t \cdot b(t, T) xy \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \left(\|\varrho_t\|^2 x^2 \frac{\partial^2 v}{\partial x^2} + \|b(t, T)\|^2 y^2 \frac{\partial^2 v}{\partial y^2} \right) = 0. \quad (539)$$

Differentiating (531) with respect to x and y we obtain two additional second order partial differential equations satisfied by v

$$\begin{cases} x \frac{\partial^2 v}{\partial x^2} + y \frac{\partial^2 v}{\partial x \partial y} = 0, \\ x \frac{\partial^2 v}{\partial x \partial y} + y \frac{\partial^2 v}{\partial y^2} = 0, \end{cases} \quad (540)$$

hence,

$$\begin{cases} xy \frac{\partial^2 v}{\partial x \partial y} = -x^2 \frac{\partial^2 v}{\partial x^2}, \\ y^2 \frac{\partial^2 v}{\partial y^2} = x^2 \frac{\partial^2 v}{\partial x^2}. \end{cases} \quad (541)$$

Plugging the last two relations into (539) we obtain

$$\frac{\partial v}{\partial t} + \frac{1}{2} \|\varrho_t - b(t, T)\|^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \quad (542)$$

which does not explicitly depend on y . Plugging $v(x, y, t) = yu(x/y, t)$ and $z = x/y$ into (542) we obtain a second order partial differential equation for u ,

$$\frac{\partial u}{\partial t} + \frac{1}{2} \|\varrho_t - b(t, T)\|^2 z^2 \frac{\partial^2 u}{\partial z^2} = 0, \quad (543)$$

which should be satisfied by function u while the option is not canceled or exercised, that is, which should be valid in the continuouation region $\widehat{\mathcal{Co}}$. As before, we arrive at the equation

$$\left(\frac{\partial u}{\partial t} + \frac{1}{2} \|\varrho_t - b(t, T)\|^2 z^2 \frac{\partial^2 u}{\partial z^2} \right) (u(t, x) - \Phi(x)) (\Psi(x) - u(t, x)) = 0 \quad \text{on} \quad \widehat{\mathcal{Co}} \cup \widehat{\mathcal{Ka}} \cup \widehat{\mathcal{Ex}}. \quad (544)$$

Notice that equation (543) could also be written as

$$\mathcal{L}_{F_S(\cdot, T)} u = \frac{\partial u}{\partial t} + \mathcal{A}_{F_S(\cdot, T)} u = 0 \quad \text{where} \quad \mathcal{A}_{F_S(\cdot, T)} = \frac{1}{2} z^2 \|\varrho_t - b(t, T)\|^2 \frac{\partial^2}{\partial z^2}, \quad (545)$$

since $\frac{1}{2}z^2|\varrho_t - b(t, T)|^2 \geq 0$ we know that $\mathcal{A}_{F_S(\cdot, T)}$ is an elliptic operator, thus $\mathcal{L}_{F_S(\cdot, T)}$ is parabolic and the pde in (543) is of the parabolic type. This is not the first time we find those operators.

As we commented before, we observe that we could get rid of the z^2 in the definition of $\mathcal{L}_{F_S(\cdot, T)}$ by means of an exponential change of variables like $z = \mathcal{K}e^x$, such a change will move to $-\infty$ the degeneracy of the operator \mathcal{L} due to the term z^2 . Still, the transformed operator¹⁷ will have the term $\frac{1}{2}\|\varrho_t - b(t, T)\|^2$ as a factor. In order to avoid a degeneracy in time, we will require to add an additional condition on $|\varrho_t - b(t, T)|^2$ to ensure it is bounded away from zero. In fact, assuming that our stock is not correlated to the zcb, their volatility parameters will occupy different components of the Brownian motion process W , this clearly reduces the possibility of $\|\varrho_t - b(t, T)\|^2$ being null to the event where both processes ϱ_t and $b(t, T)$ vanish simultaneously. If additionally ϱ_t is assumed constant—a “Black–Scholes” stock—then $\|\varrho_t - b(t, T)\|^2$ is bounded away from zero. So we will assume $\|\varrho_t - b(t, T)\|^2 \geq l > 0$ (l a constant). Looking again into the simple case where ϱ is constant, if $b(t, T)$ is obtained from an interest rate model like Hull–White (see Chapter 4) we know the coefficient $\|\varrho_t - b(t, T)\|^2$ is also bounded above—this additional condition is also consistent with data obtained from market observations, where both ϱ_t and $b(t, T)$ are bounded—. Thus it is natural to assume that there exist another constant $L > 0$ such that $L \geq \|\varrho_t - b(t, T)\|^2$.

As we have mentioned before, we will assume that $\|\varrho_t - b(t, T)\|^2$ is a deterministic, integrable, function of t and that there exists constants $\infty > L > 0$ and $l > 0$ such that

$$L > \|\varrho_t - b(t, T)\|^2 > l, \quad t \in]0, M[. \quad (546)$$

Rewriting all other conditions ((513), (514), (515), (518)) in terms of the unknown

¹⁷After the change $z = \mathcal{K}e^x$, the resulting operator will be

$$\mathcal{L} = \frac{1}{2}\|\varrho_t - b(t, T)\|^2 \left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} \right).$$

function u we obtain the following pde

$$\left\{ \begin{array}{ll} \left(\frac{\partial u}{\partial t} + \frac{1}{2} \|\varrho_t - b(t, T)\|^2 z^2 \frac{\partial^2 u}{\partial z^2} \right) \times \\ \quad \times (u(t, z) - \Phi(z)) (\Psi(z) - u(t, z)) = 0 & 0 < t < M, 0 < z \\ \Psi(z) \geq u(t, z) \geq \Phi(z) & 0 < t < M, 0 \leq z \\ u(M, z) = \Phi(z) & 0 \leq z \\ u(t, 0) = 0 & 0 \leq t \leq M \end{array} \right. \quad (547)$$

Naturally, the contract is terminated once the issuer cancels or the holder exercises.

Traditionally, pde problems like (547) are expressed in the form of initial value problems and whenever possible the pde is “reduced” via changes of variables. To our advantage, pde (543) has only two terms; this will work in our advantage later on.

In order to turn (547) into an initial value problem, we will make a time change of the form:

$$\tau = M - t. \quad (548)$$

Thus, in the τz -space equation (543) we will transform into:

$$\frac{\partial u}{\partial \tau}(\tau, z) - g(\tau, z, M) \frac{\partial^2 u}{\partial z^2}(\tau, z) = 0, \text{ where } g(\tau, z, M) = \frac{1}{2} z^2 |\varrho_{M-\tau} - b(M - \tau, T)|^2. \quad (549)$$

Carrying the last change into (547) we obtain:

$$\left\{ \begin{array}{ll} \left(\frac{\partial u}{\partial \tau} - g(\tau, z, M) \frac{\partial^2 u}{\partial z^2} \right) \times \\ \quad \times (u(\tau, z) - \Phi(z)) (\Psi(z) - u(\tau, z)) = 0 & 0 < \tau < M, 0 < z \\ g(\tau, z, M) = \frac{1}{2} z^2 \|\varrho_{M-\tau} - b(M - \tau, T)\|^2 & \\ \Psi(z) \geq u(t, z) \geq \Phi(z) & 0 < t < M, 0 \leq z \\ u(0, z) = \Phi(z) & 0 \leq z \\ u(\tau, 0) = 0 & 0 \leq \tau \leq M \end{array} \right. \quad (550)$$

Finally, if the cancelation payoff is not considered in the solution of (547) (or which is equivalent, if \mathbf{p} is made equal to infinity) we will obtain the arbitrage price of the corresponding American version of the contract.

5.7 *Finite difference discretization of the partial differential equation for an American game call on a forward price*

Here we will consider a finite difference approximation to (512), and so we will “solve” the problem of finding an approximate solution to equation (512) in a certain region of the $z\tau$ -plane in which a regular mesh (equally spaced grid) will be defined.

Intrinsic to all numerical computations is the restriction to work on finite regions of space, in which discrete approximations are computed. Such a restriction enters in direct conflict with our particular problem since our equation is given in an unbounded domain. To cope with this difficulty we need to introduce smart changes in the side conditions of our problem. For example, if we are going to look for a solution in the rectangle $[\tau_{\min}, \tau_{\max}] \times [z_{\min} = 0, z_{\max}]$ we will like z_{\max} to be “big enough”, that is $z_{\max} > K$, and big enough as to ensure that $u(\tau, z_{\max}) \sim z_{\max} - K$.

On the other hand, since we are interested in obtaining the initial price of the option it is clear that $\tau_{\max} = M$, on the other hand, $\tau_{\min} = 0$ corresponds to maturity time.

Let δz and $\delta \tau$ (both assumed positive) be the step sizes in directions z and τ respectively. We will call N_τ the number of time subdivisions and N_z the number of spatial subdivisions. That way we will have

$$\begin{aligned} z_i &= i\delta z, & i &\in \mathbb{N}_{N_z}^*, & z_0 &= 0, & z_{N_z} &= z_{\max}, & \tau_0 &= 0, & \tau_{N_\tau} &= M \\ \tau_j &= j\delta \tau, & j &\in \mathbb{N}_{N_\tau}^*. \end{aligned} \quad (551)$$

Similarly, if $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of two variables, we will use the following notation

$$\left\{ \begin{aligned} u_n^m &= u_{n,m} = u(\tau_m, z_n) \\ u_{n^{(i)}}^m &= u_{n^{(i)},m} = \frac{\partial^i u}{\partial z^i} \Big|_{(\tau_m, z_n)} \\ u_n^{m^{(j)}} &= u_{n,m^{(j)}} = \frac{\partial^j u}{\partial \tau^j} \Big|_{(\tau_m, z_n)} \\ u_{n^{(i)}}^{m^{(j)}} &= u_{n^{(i)},m^{(j)}} = \frac{\partial^{(i+j)} u}{\partial z^i \partial \tau^j} \Big|_{(\tau_m, z_n)} \end{aligned} \right. \quad (552)$$

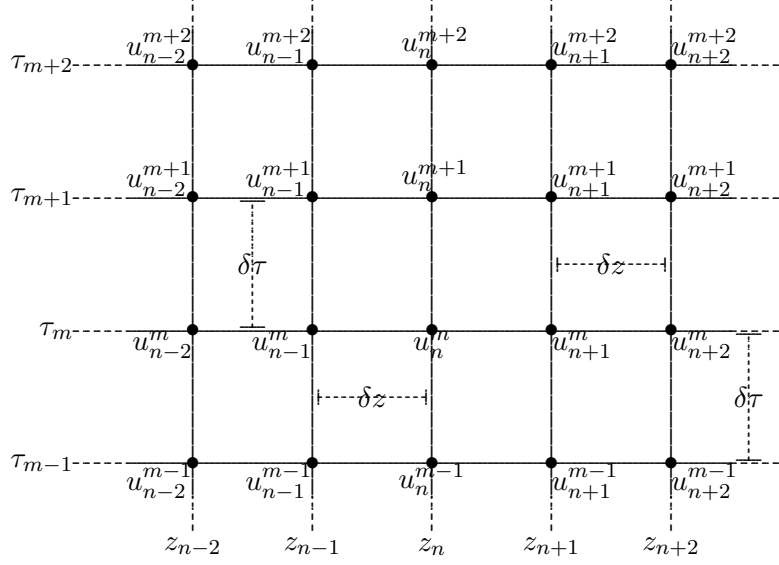


Figure 46: Regular mesh.

Also, we will define α , the *mesh constant*¹⁸ as

$$\alpha = \frac{\delta \tau}{(\delta z)^2} \quad (553)$$

We will approximate u with u , solution to a finite elements problem, where $u(\tau, z) \simeq u(\tau, z)$, and (τ, z) is a point in the mesh. In principle, we are to deal with a parabolic equation which is resemblant to the Heat equation. The problem in our case is that the coefficient of the second order (spatial) derivative is not a constant, but a function of time and the spatial variable. Thus, numerical procedures averaging over different schemes to compute the derivatives (Crank–Nicholson for example) may not have the expected effect. Instead, we will consider a fully implicit method, where

$$\begin{aligned} \frac{\partial u}{\partial \tau} &\simeq \frac{u_n^{m+1} - u_n^m}{\delta \tau} \\ \frac{\partial^2 u}{\partial z^2} &\simeq \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta z)^2} \end{aligned} \quad (554)$$

thus (549) will correspond to the problem

$$\frac{u_n^{m+1} - u_n^m}{\delta \tau} = g(\tau_{m+1}, z_n, M) \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta z)^2}$$

¹⁸This definition makes sense once we consider equation (543) in which first order terms in time are coupled with second order spatial terms. Since $\delta \tau$ is assumed positive, it is clear that α is also a real positive constant (once $\delta \tau$ and δz have been selected).

which is equivalent to

$$-\alpha g(\tau_{m+1}, z_n, M)u_{n+1}^{m+1} + (1 + 2\alpha g(\tau_{m+1}, z_n, M))u_n^{m+1} - \alpha g(\tau_{m+1}, z_n, M)u_{n-1}^{m+1} = u_n^m \quad (555)$$

it is easy to see that (555) is of orders $o(\delta\tau)$ and $o((\delta z)^2)$ and that the method is stable.

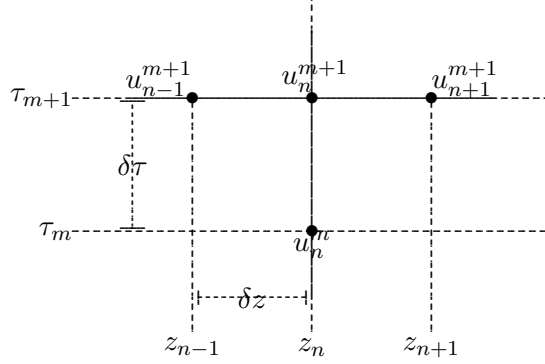


Figure 47: Four points involved in the Implicit method.

Proposition 5.7.1. *The method (555) is stable for any selection of the mesh constant.*

Proof. We apply von Neumann's stability analysis. Assume $\epsilon_n^m = \xi^m e^{ikn\delta x}$. Then $\epsilon_n^{m+1} = \xi \epsilon_n^m$ and $\epsilon_{n\pm 1}^m = e^{\pm ik\delta x} \epsilon_n^m$. Thus, from (555) we obtain

$$\begin{aligned} \epsilon_n^{m+1} - \epsilon_n^m &= \alpha g(\tau_{m+1}, x_n, M)(\epsilon_{n+1}^{m+1} - 2\epsilon_n^{m+1} + \epsilon_{n-1}^{m+1}) \\ \implies (\xi - 1)\epsilon_n^m &= \xi \alpha g(\tau_{m+1}, x_n, M)(e^{ik\delta x} - 2 + e^{-ik\delta x})\epsilon_n^m \\ \implies \xi - 1 &= \xi \alpha g(\tau_{m+1}, x_n, M)(e^{ik\delta x/2} - e^{-ik\delta x/2})^2 \\ \implies \xi - 1 &= -4\xi \alpha g(\tau_{m+1}, x_n, M) \sin^2\left(k \frac{\delta x}{2}\right) \\ \implies \xi &= \frac{1}{1 + 4\alpha g(\tau_{m+1}, x_n, M) \sin^2\left(k \frac{\delta x}{2}\right)} \end{aligned} \quad (556)$$

since $\alpha \geq 0$ and $g(\tau_{m+1}, x_n, M) \geq 0$, $\forall m, \forall n$, we know $1 + 4\alpha g(\tau_{m+1}, x_n, M) \sin^2\left(k \frac{\delta x}{2}\right) \geq 1$ $\forall \alpha, \forall m, \forall n, \forall \delta x$. Therefore $|\xi| \leq 1$ and our method is stable for any choice of the mesh constant. \square

As we mentioned before, we will solve our numerical problem in a subregion, $[0, M] \times [0, z_{\max}]$ of the τz space, where a mesh of dimensions N_τ and N_z and mesh constant α is

defined¹⁹.

5.8 Numerical Solutions

Equation (555) is equivalent to several tridiagonal, diagonally dominant linear systems that can be solved iteratively by means of a variant to the well known PSOR (projected successive over-relaxation) method.

Explicitly (512) can be discretized as follows:

$$\begin{cases} \mathbf{A}_{m+1} \vec{\mathbf{u}}_{m+1} = \vec{\mathbf{b}}_m ; & \vec{\mathbf{u}}_{m+1} \geq \vec{\mathbf{c}} ; & \vec{\mathbf{u}}_{m+1} \leq \vec{\mathbf{c}} + \mathbf{p} \vec{\mathbf{1}} \\ (\mathbf{A}_{m+1} \vec{\mathbf{u}}_{m+1} - \vec{\mathbf{b}}_m) \square (\vec{\mathbf{u}}_{m+1} - \vec{\mathbf{c}}) \square (\vec{\mathbf{c}} + \mathbf{p} \vec{\mathbf{1}} - \vec{\mathbf{u}}_{m+1}) = \vec{\mathbf{0}} \\ \vec{\mathbf{u}}_0 = \vec{\mathbf{c}} \\ u_n^m = c_n & 0 \leq m \leq N_\tau; \quad n \in \{0, N_z\} \end{cases} \quad (557)$$

Where, $\vec{\mathbf{u}}_{m+1} = (u_n^{m+1})_{0 < n < N_z}$, $\vec{\mathbf{b}}_m = (b_n^m)_{0 < n < N_z} = \vec{\mathbf{u}}_m$, $\vec{\mathbf{c}} = (c_n)_{0 < n < N_z} = (z_n - \mathcal{K})^+$, and $\mathbf{A}_m = \text{tridiag}(-\alpha g(\tau_m, x_n, M), 1 + 2\alpha g(\tau_m, x_n, M), -\alpha g(\tau_m, x_n, M); 0 < n < N_z)$. The \mathbf{A}_m are N_τ diagonally dominant tri-diagonal $(N_z - 1) \times (N_z - 1)$ symmetric matrices. $\vec{\mathbf{a}} \geq \vec{\mathbf{b}}$ is defined as $a_n \geq b_n \quad \forall n \in I$ and $\vec{\mathbf{a}} \square \vec{\mathbf{b}} \square \vec{\mathbf{c}} = (a_n \times b_n \times c_n)_{n \in I}$ for $\vec{\mathbf{a}} = (a_n)_{n \in I}$, $\vec{\mathbf{b}} = (b_n)_{n \in I}$, $\vec{\mathbf{c}} = (c_n)_{n \in I}$, I some index set. As before $\vec{\mathbf{1}}$ represents a vector of ones of the appropriate dimensionality. $\vec{\mathbf{0}} = 0\vec{\mathbf{1}}$ represents a vector of zeroes of the appropriate dimensionality.

$\vec{\mathbf{u}}_m$ contains our approximations to the time τ_m price of the American Game Call described in §5.6.

As mentioned before, we implement (557) using a variant of the PSOR method. If $u_{n,k}^{m+1}$ is used to represent the k^{th} iteration in our computation of u_n^{m+1} , and $v_{n,k}^{m+1}$ is used for

¹⁹That is, the intervals $[0, M]$ and $[0, z_{\max}]$ are subdivided into subintervals of lengths $\delta z = \frac{z_{\max}}{N_z}$ and $\delta \tau = \frac{M}{N_\tau}$ respectively.

temporary storage, $m \in \mathbb{N}_{N_z-1}$:

$$\begin{aligned}
v_{1,k}^{m+1} &= \frac{1}{1 + 2\alpha g_1^{m+1}} \left\{ b_1^m + \alpha g_1^{m+1} u_{2,k-1}^{m+1} \right\} \\
v_{n,k}^{m+1} &= \frac{1}{1 + 2\alpha g_n^{m+1}} \left\{ b_n^m + \alpha g_n^{m+1} (u_{n+1,k-1}^{m+1} + u_{n-1,k}^{m+1}) \right\}; \quad 1 < n < N_z - 1 \\
v_{N_z-1,k}^{m+1} &= \frac{1}{1 + 2\alpha g_{N_z-1}^{m+1}} \left\{ b_{N_z-1}^m + \alpha g_{N_z-1}^{m+1} u_{N_z-2,k}^{m+1} \right\} \\
u_{n,k}^{m+1} &= u_{n,k-1}^{m+1} + \theta \{v_{n,k}^{m+1} - u_{n,k-1}^{m+1}\}; \quad -1 \leq n \leq N_z - 1 \\
&\quad \theta \in [0, 2]
\end{aligned}$$

to account for the double obstacle in our computations, our rendition of the PSOR method must compare, at every step of the iterative process²⁰, the approximation to u_n^{m+1} with $c_n + \mathbf{p}$ and c_n :

$$\begin{aligned}
temp &:= b_i + \alpha * g_n * (u_{i-1}^m + u_{i+1}^m) / (1 + 2 * \alpha * g_n) \\
temp &:= \min(c_i + \mathbf{p}, u_i^m + \theta * (temp - u_i^m)) \\
temp &:= \max(c_i, temp)
\end{aligned} \tag{558}$$

In our computations we assumed that ϱ_t was a non null constant²¹, $\varrho_t = \varrho$, that the stock price process is uncorrelated to the zcb's price process and that the interest rate followed a Hull–White model as in Chapter 4. That way bond volatility, $b(t, T)$ is given by (295). As we saw in Chapter 4, for a Vasicek or extended Vasicek model, $b(t, T)$ is given by expressions (295) and (350), that is by an expression of the form

$$b(t, T) = -\frac{\gamma(t)}{a_2} \left(1 - e^{-a_2(T-t)} \right). \tag{559}$$

thus, we will have

$$g(\tau, z, M) = \frac{1}{2} z^2 \left(\varrho^2 + \left(\frac{\gamma(M-\tau)}{a_2} \left(1 - e^{-a_2(T-M+\tau)} \right) \right)^2 \right) \tag{560}$$

and in particular

$$c(\tau) = \|\vec{\varrho}_{M-\tau} - \vec{b}(M-\tau, T)\|^2 = \varrho^2 + \left(\frac{\gamma(M-\tau)}{a_2} \left(1 - e^{-a_2(T-M+\tau)} \right) \right)^2 \tag{561}$$

²⁰These steps rely in our assumption that both writer and holder are rational players.

²¹In this section we will abuse a little our notation and take both ϱ_t and $b(t, T)$ not as vector processes but also as the corresponding non null components of such processes.

is a deterministic, integrable function of τ that satisfies our assumptions of boundedness, (546). Recall the meaning of the different parameters that appear in equation (561); ϱ is the stock volatility which we assume is given, a_2 is the speed of mean reversion, and $\gamma(\cdot)$ is the volatility parameter of the Hull–White model we use. M is the option’s maturity and T the zcb maturity (which we also assume given). As commented in Chapter 4, and depending on the variant of the Hull–White model one chooses to use in the valuation, parameters a_2 and $\gamma(\cdot)$ can be obtained through a calibration process. For example, if a Vasicek model is used, $\gamma(\cdot)$ will be also a number and both, a_2 and $\gamma(\cdot) = a_3$ can be obtained after a calibration process as that shown in section §4.2, subsection §4.2.1, see for example (340). If instead a model of shifted rates as those discussed in §4.5 is used, $\gamma(\cdot)$ will represent a piecewise constant as that shown in Figure 34.

We conducted several numerical experiments using made up parameter values and values taken from our calibrations in Chapter 4. The following figures exemplify our results.

Figure 48 shows time $t = 0$ u curves for different penalty parameters, as expected, see Proposition 5.1.1, u increases as \mathbf{p} increases. In our computations, \mathbf{p} was given the values of 6.00, 4.00, 2.00, 1.50, 1.00, 0.90, 0.80, 0.75, 0.50, 0.25, 0.10, $\mathcal{K} = 50$, $M = 1$, $T = 2$, $\gamma(t) = 0.2$, $a_2 = 0.1$, $\varrho = 0.25$, $z_{\max} = 120$, $N_z = 250$, and $N_\tau = 80$.

Figure 49 shows a zoom of one of such curves, $\mathbf{p} = 2$, we also show the cancellation payoff for that penalization parameter and the execution payoff.

Next, we considered different zcb maturities $T = 1.00, 2.00, 3.00, 4.00, 5.00, 7.00, 10.0$ and computed the corresponding initial curves for two different penalization parameters $\mathbf{p} = 6$ and $\mathbf{p} = 4$, preserving all other parameters unchanged. Our results are shown in Figure 50.

Using the same parameters, we also computed the initial values of prices per unit of zcb price for the corresponding European and American versions of the contracts, which we compare against our previous computations in Figure 51 and Figure 52. Recall from Proposition 5.1.2 that the price of the corresponding American version of the contract is always larger than the price of the Game contract.

Similar numerical experiments were performed using volatility parameters extracted from

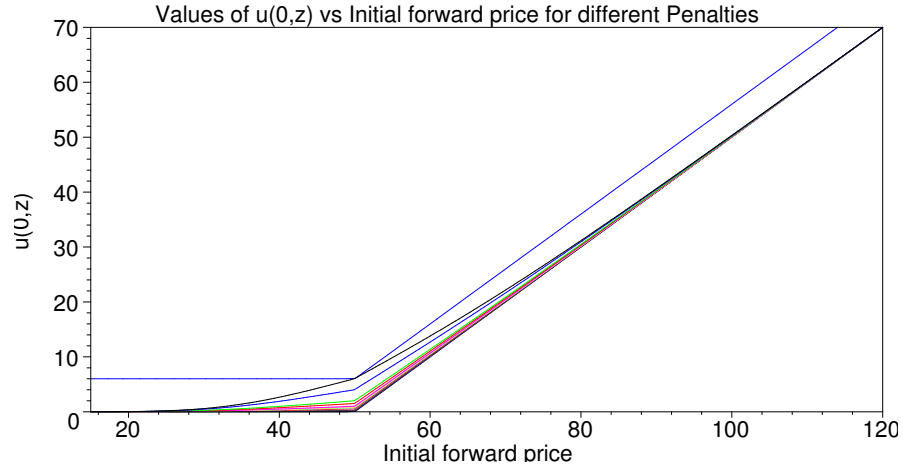


Figure 48: This plot shows $u(z,0)$ for different values of p , all other parameters were left unchanged. As reference, the cancellation payoff for a penalization parameter $p = 6.00$ and the execution payoff are also shown. In this plot, the upper curve (just beneath the cancelation payoff) corresponds to a penalization parameter of 6.00, for a given initial forward price z , the corresponding value of $u(z,0)$ decreases as p decreases.

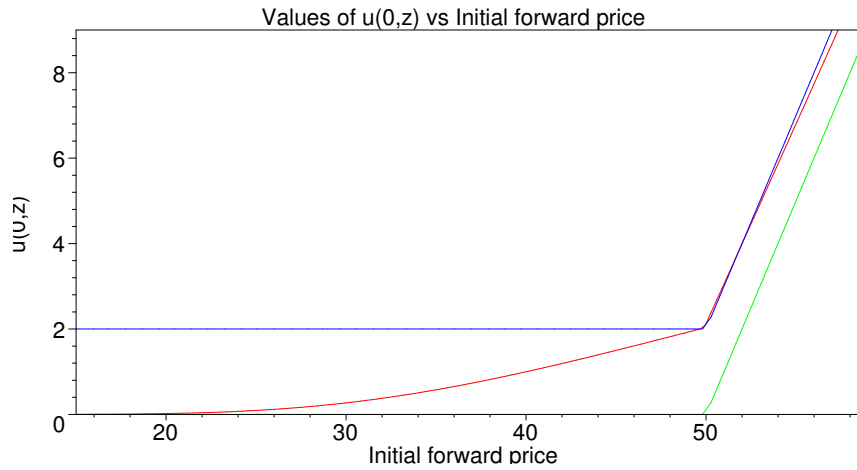


Figure 49: This plot shows $u(z,0)$ for $p = 2$, all other parameters were left unchanged. The cancelation payoff and the execution payoff are also shown.

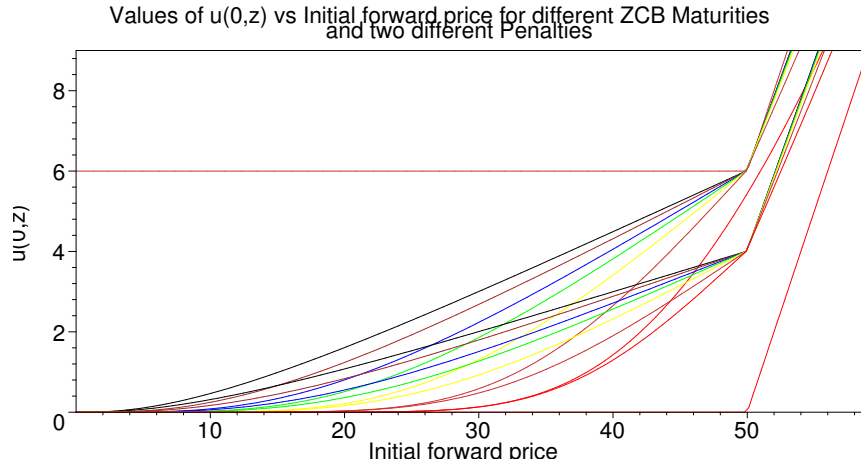


Figure 50: $u(z, 0)$ was computed for $p = 6$ and $p = 4$, and different zcb maturities. All other parameters were left unchanged. The cancellation payoff (for $p = 6$) and the execution payoff are also shown. The “upper” set of curves corresponds to options with a penalization p of 6, the “lower” set of curves corresponds to $p = 4$. Within each set of curves we see that the initial forward price of the option increases as the maturity of the underlying zcb increases (the red curves correspond to a zcb of 1 year maturity, the black curves to a zcb of 10.0 years maturity, all options are of maturity 1 year)

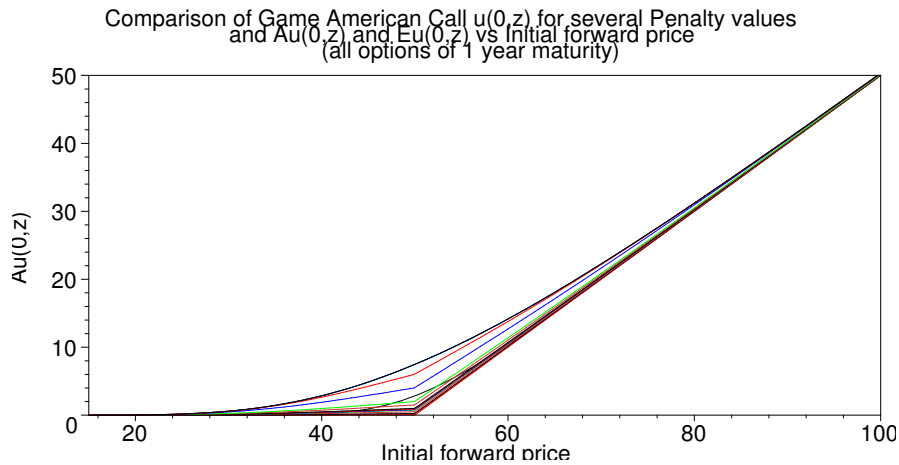


Figure 51: Comparison between the initial prices per unit of zcb value of American, American Game and European Calls on a Forward price. Initial American Game prices (per unit of zcb value) are shown for several penalization parameters. As expected, the initial price of the corresponding American Call contract (upper curve) dominates over the initial prices of all other contracts (European Call contract included, this is the black curve that seems to cross through several of the Game American curves).

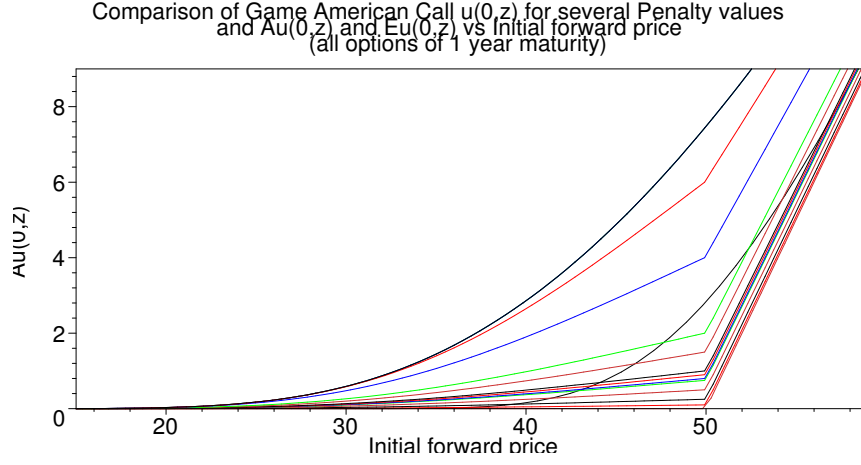


Figure 52: A detail of the same results presented in Figure 51 is shown here.

our sample calibrations (see Chapter 4).

Figure 53 shows the initial forward prices of Game American Calls for different penalties (all options of maturity 1, on a 2 year forward) under the Vasicek model of interest rates (see §4.2.1), in this case we use interest rate parameters from (342), namely speed of mean reversion $a_2 \sim 0.054008$, and interest rate volatility $a_3 \sim 0.032696$. All other parameters involved in our computations are as before: \mathbf{p} was given the values of 6.00, 4.00, 2.00, 1.50, 1.00, 0.90, 0.80, 0.75, 0.50, 0.25, 0.10, $\mathcal{K} = 50$, $M = 1$, $T = 2$, $\varrho = 0.25$, $z_{\max} = 120$, $N_z = 250$, and $N_\tau = 80$.

Figure 54 shows similar computations assuming an extended Vasicek model of interest rates (see §4.2.2 where we considered the case of an extended Vasicek model with polynomial mean reversion parameter, (357) lists the approximate values of the model parameters found by a least squares regression). From (357), we have $a_2 \sim 0.17908$, and (interest rate volatility) $a_3 \sim 3.9993e-12$. As before, \mathbf{p} was given the values of 6.00, 4.00, 2.00, 1.50, 1.00, 0.90, 0.80, 0.75, 0.50, 0.25, 0.10, $\mathcal{K} = 50$, $M = 1$, $T = 2$, $\varrho = 0.25$, $z_{\max} = 120$, $N_z = 250$, and $N_\tau = 80$.

For our last numerical experiment, see Figure 58, Figure 59, and Figure 60, we used the results of our calibration to initial spot volatilities using a shifted rate model with piecewise constant volatility parameter (see Chapter 4, §4.5). Figure 35 shows a graph of

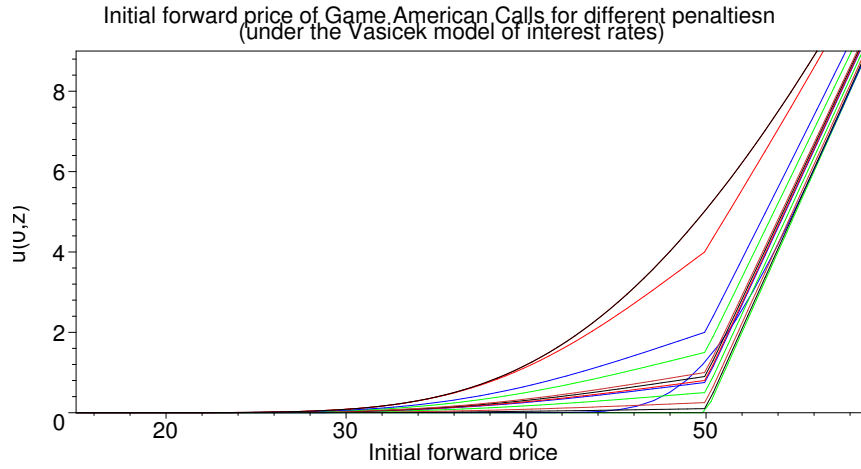


Figure 53: Initial American Game prices per unit of zcb value for different maturities were computed under the assumption of a Vasicek model of interest rates using parameters found in our calibrations (see Chapter 4, §4.2.1 and in particular (342)), initial American Call and initial European Call prices per unit of zcb are also shown. As before, the initial forward price of the American Call dominates over the other initial forward prices, also we observe that initial forward price of the American Game Calls increases as the penalization increases.

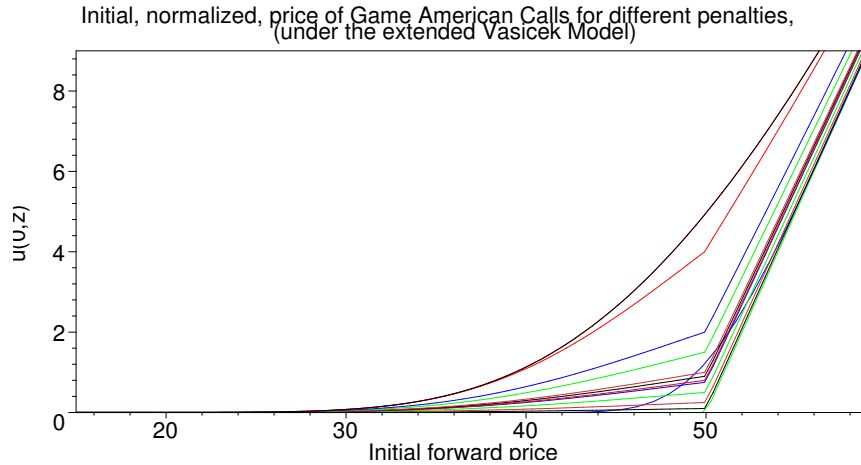


Figure 54: Initial American Game prices per unit of zcb value for different maturities were computed under the assumption of an extended Vasicek model of interest rates using parameters found in our calibrations (see Chapter 4, §4.2.2 and in particular (357)), initial American Call and initial European Call prices per unit of zcb are also shown. As before, the initial forward price of the American Call dominates over the other initial forward prices, also we observe that initial forward price of the American Game Calls increases as the penalization increases.

such a piecewise constant interest rate volatility parameter (corresponding to the initial data shown in Table 6 and under the assumption that the speed of mean reversion parameter is $a_2 = 10$). As before, we used $\mathbf{p} = 6.00, 4.00, 2.00, 1.50, 1.00, 0.90, 0.80, 0.75, 0.50, 0.25, 0.10$, $\mathcal{K} = 50$, $M = 1$, $T = 2$, $\varrho = 0.25$, $z_{\max} = 120$, $N_z = 250$, and $N_\tau = 80$. Figure 55 shows the corresponding forward volatility curve for a zcb of maturity 10 years.

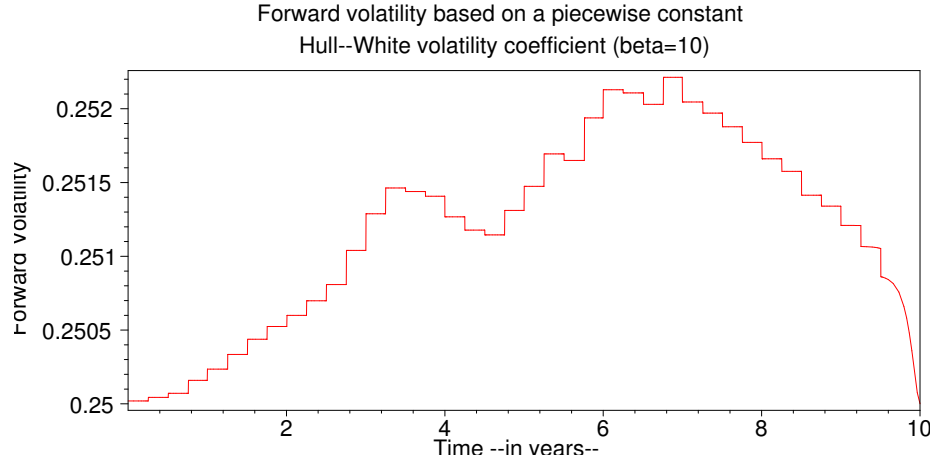


Figure 55: Piecewise constant Forward volatility curve. We assume a stock with a constant volatility of $\varrho = 0.25$ uncorrelated to a zcb of maturity 10.0 under a model of interest rates as shown in §4.5 which we calibrated to initial volatility data, Table 6.

Figure 56 show a plot of the diffusion coefficient for a zcb of maturity 10.0 under the shifted rate model of §4.5 that we calibrated to initial flat volatility data, the profile of such diffusion coefficient, as can be seen in Figure 57, is determined by the forward volatility (compare with Figure 55).

Figure 58, Figure 59, and Figure 60 show plots of the results of our last numerical experiment.

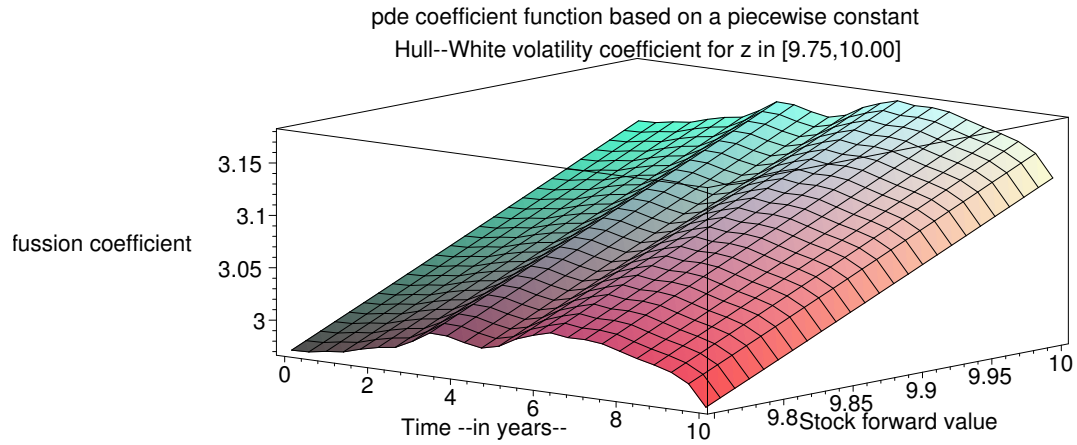


Figure 56: Diffusion coefficient under a model of shifted rates. We assumed that stock and zcb are uncorrelated, and that the corresponding stock volatility is constant and equal to $\varrho = 0.25$. The surface shown here correspond to a zcb of maturity 10.0 years. As mentioned before we also assumed that interest rates follow a Hull–White model like (411).

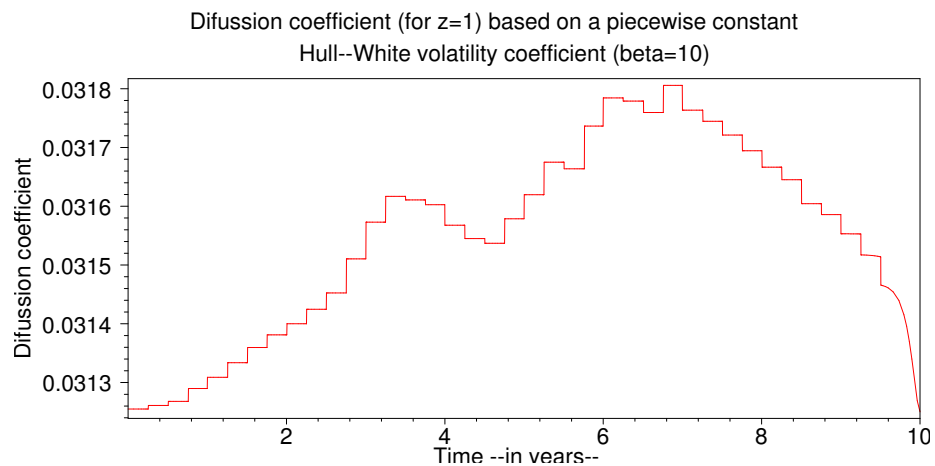


Figure 57: Profile of the diffusion coefficient under a model of shifted rates.

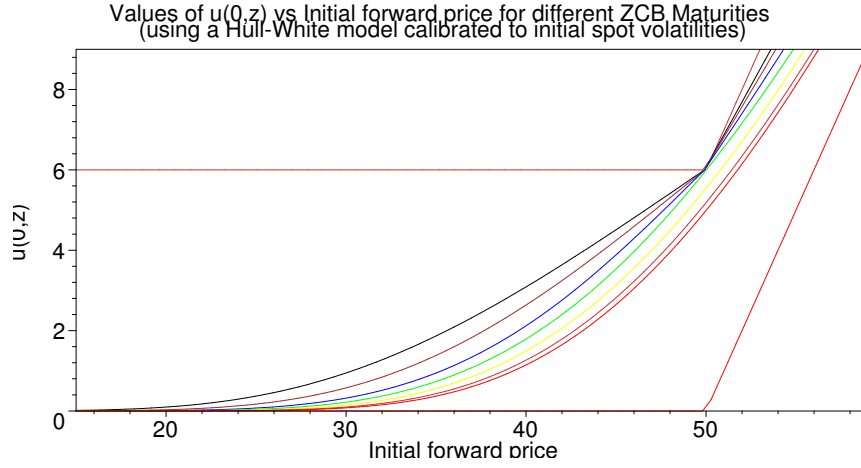


Figure 58: We computed $u(z, 0)$ for $p = 6$ and different zcb maturities under the assumption of a shifted rates model, as that shown in §4.5, calibrated to initial flat volatility. As a result we obtained a piecewise constant Forward volatility curve (see Figure 55) that determined the diffusion parameter of our pde. We assume a stock with a constant volatility of $\varrho = 0.25$ uncorrelated to our zcb's. All other parameters were left unchanged. As before, we see that the initial forward price of the option increases as the maturity of the underlying zcb increases (the red curves correspond to a zcb of 1 year maturity, the black curves to a zcb of 10.0 years maturity, all options are of maturity 1 year)

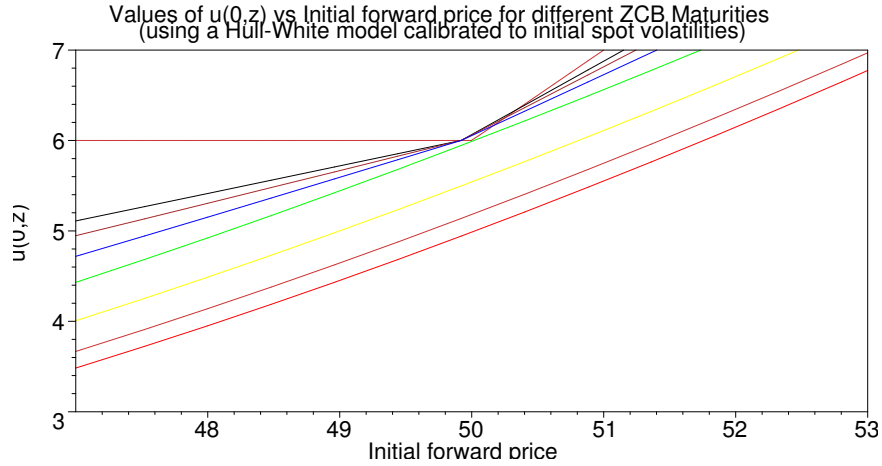


Figure 59: Initial forward price of Game American calls under a Hull–White model calibrated to initial flat volatility using a piecewise constant volatility parameter, detail of Figure 58

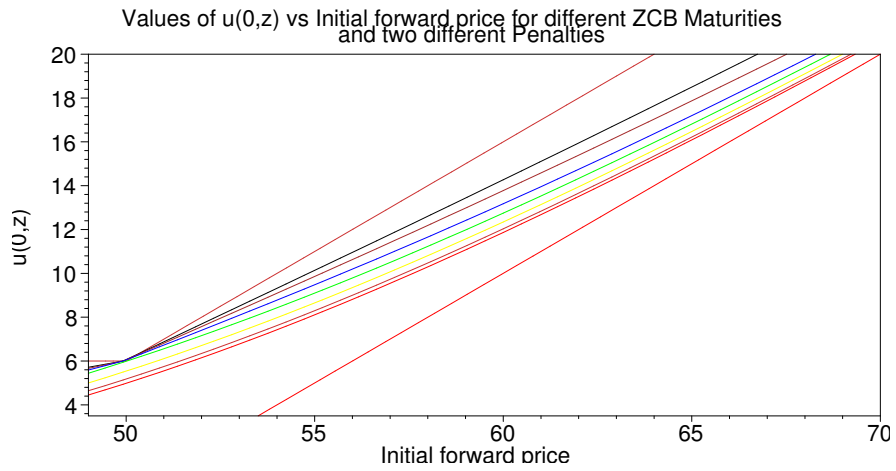


Figure 60: Initial forward price of Game American calls under a Hull–White model calibrated to initial flat volatility using a piecewise constant volatility parameter, detail of Figure 58

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VITA

Luis Gustavo Hernández Ureña came to life in the tropical country of Costa Rica, in the city of San José. He lived with his family in the small town of San Rafael Arriba de Desamparados and attended the first three years of his primary school at the local public school, named Escuela Manuel Ortuño Boutin. Afterwards he was transferred to the Conservatorio de Castella where he finished his primary school and attended secondary school. There he learned to play the clarinet and participated in many other forms of artistic expression. His education continued at the Universidad de Costa Rica where he studied Mathematics and Physics. He earned the title of Licenciado en Matemática at the Universidad de Costa Rica working with Dr. Ricardo Estrada Navas. Between the time period in which he earned his Bachelors of Mathematics and the moment in which he came to the Georgia Institute of Technology, he worked as an assistant professor at the Universidad de Costa Rica. Luis came to the United States on September 25th, 1996 to enroll in the Ph.D. program in Mathematics at the Georgia Institute of Technology. While pursuing his studies in Mathematics, Luis obtained a Masters in Quantitative and Computational Finance and is one of the first three graduates from that program. At Georgia Tech he worked on Financial Mathematics under the direction of Dr. Robert P. Kertz. He plans to graduate in May 2005, and to return to Costa Rica after his defense to continue his work at the Universidad de Costa Rica.