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TWO-DIMENSIONAL ESTIMATION FROM A

RESTRICTED SAMPLING LATTICE

A THESIS

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RESTRICTED SAMPLING LATTICE

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SUMMARY

This dissertation is concerned with the estimation of a process signal over a finite two-dimensional plane from data sampled at discrete points along a prescribed scan path in the plane. One coordinate of the plane is referred to as the spatial coordinate, while the other coordinate is referred to as the temporal coordinate. The scan path is such that the spatial coordinate is a single-valued function of the temporal coordinate. The process under consideration belongs to a class of nonstationary processes in which the signal is the sum of a Gaussian stationary component of unknown statistics and a nonrandom component that is a function of the spatial coordinate only. The estimates are linear combinations of the measured signals with the weighting factors being chosen to minimize the expected value of the square of the estimation error.

Efficient two-dimensional estimation techniques require a knowledge of the autocorrelation function of the process, which is not assumed to be known a priori in this case. Due to the nature of the scan path, the autocorrelation function can be estimated only in a restricted sector of the two-dimensional plane. Since this restriction poses a difficult problem in two-dimensional estimation, simpler one-dimensional estimation techniques have been popular in the past. A major contribution of this work lies in solving this problem. The extrapolation and refinement of the

autocorrelation function is achieved by fitting an appropriate form of the autocorrelation function to the estimated autocorrelation function in the restricted sector. In some cases the form is known for a particular class of stochastic processes. In other cases the form has to be approximated by an exponential-cosine form.

The approach used is to first estimate the autocorrelation function in the restricted sector from the scan path data. To the estimated autocorrelation function is fitted the appropriate form by the weighted least-squares technique. The fitted auto-correlation function is then used in the estimation of the signal. A comparison of the resulting expected value of the square of the estimation error is made with the one obtained by using the real autocorrelation function. Also, a comparison is made with other existing methods. A sensitivity analysis is performed to determine the allowable flexibility in the parameters of the functional form.

The results are demonstrated by using two simulated processes and a real sheet paper process. The two-dimensional stochastic processes having an exponential-cosine autocorrelation function are simulated by passing discrete white noise through a synthesizing filter. The advantage of working with simulated processes is that the real autocorrelation function is known a priori. The developed method is also applied to the scan data from a sheet paper process and the results are used in comparing the developed method with existing ones.

CHAPTER I

INTRODUCTION

The research reported in this thesis began as an attempt to improve the on-line estimation of the basis weight of Kraft paper. The author and his advisor visited the Westvaco paper mill operation in Charleston, South Carolina, as guests of Dr. D. B. Brewster, then Director of Process Control. In a conference, Dr. D. B. Brewster indicated the poor control of the basis weight obtained by using estimation techniques in the temporal coordinate of the sheet paper and suggested the existence of correlation in the temporal and spatial coordinates of the sheet.

After reviewing existing literature in multidimensional estimation, algorithms for basis weight estimation and identification of random process statistics, it was decided that a significant contribution to the state of the art could be made through examination of this problem. A general mathematical problem, applicable to most sheet processes, was thus formulated from the basis weight problem.

Statement of the Problem

Consider a class of two-dimensional nonstationary random processes $q(x, t)$ in which the signal is the sum of a Gaussian stationary component $s(x, t)$ and a nonrandom component $p(x)$. The statistics of the stationary part and the form of the nonrandom component are not given a priori. From the data sampled at regular

intervals in the past up to time t_p along a prescribed scan path, it is desired to determine a discrete point estimate $\hat{q}(x_p + \beta, t_p + \eta)$ such that the expected value of the squared error

$$\epsilon = E\left\{(\hat{q} - q)^2\right\} \qquad (1.1)$$

is minimized. The upper and lower bounds on $(x_p + \beta)$ and $(t_p + \eta)$ are such as to allow for smoothing and prediction in a reasonable subset of the total space. In addition, the random process $s(x, t)$ is ergodic, and the limiting conditions given in Table 1 hold. The form of the scan path is

$$x = F(t) \qquad (1.2)$$

where x is a real, single-valued function of t and the sampling period on the scan path is chosen so as to eliminate aliasing.

Since the nonrandom component can be recovered by exponential smoothing in the t coordinate, the problem essentially reduces to estimation of the stationary part. Optimal linear estimators require the knowledge of the autocorrelation function of the random process. In the process described above, only an estimate of the autocorrelation function is available in the restricted regions related to the scan path. Hence, the major portion of this work is devoted to extrapolation and refinement of the estimated autocorrelation function.

Review of Literature

Past attempts to estimate the basis weight of sheet paper

Table 1. Limiting Conditions on the Autocovariance

Δx	Δt	Autocovariance $C_{ss}(\Delta x, \Delta t)$
0	0	σ^2
∞	∞	0
0	∞	0
∞	0	0
Δx	0	$C_{ss}(\Delta x)$
Δx	∞	0
0	Δt	$C_{ss}(\Delta t)$
∞	Δt	0

$$C_{ss}(\infty) = 0 \quad \text{and} \quad C_{ss}(0) = \sigma^2$$

have largely ignored optimal two-dimensional estimation theory
and the existence of correlation between the spatial and temporal
coordinates. Examples of such attempts can be found in the works
of Brewster (1),[*] Astrom (2), Dahlin and Brewster (3), and Wingrove,
Madeley and Shabi (4). In reference (1) the spatial component of
the signal is arrived at by exponential smoothing in the temporal
coordinate, while the temporal component is estimated by filter-
ing out the high frequencies in the temporal direction. The
algorithm of reference (1) is presented in Chapter V. A logical

[*]Numbers in parentheses not following "Equation" refer to
items in the Bibliography.

improvement of this algorithm, assuming no correlation in the
spatial and temporal coordinates, is the estimation of the
temporal coordinate using one-dimensional linear optimum estima-
tion theory. References (2) and (3) are in this line of think-
ing, excepting they tend to be more oriented towards fitting a
state model to the process in the temporal coordinate. Tuning
the parameters in the state model requires input-output informa-
tion. Unfortunately, only very few inputs contributing to the
output signal can be monitored and, hence, the remaining inputs
go undetected. These undetected inputs can cause considerable
deviation of the actual output signal from the state model pre-
dicted output signal. Reference (2) is primarily concerned with
the assessment of the spatial basis weight profile and this asses-
ment is used for the purpose of deciding whether or not the paper
will run satisfactorily during operations on or subsequent to the
paper machine (coating and reeling). It has very little to con-
tribute to the problem being considered in this thesis since the
method is empirical and no definite design criterion is used. The
method is mentioned here only to illustrate the variety of work
done in the field.

All the methods discussed so far do not consider the exis-
tence of correlation between the spatial and temporal coordinates.
Dr. D. B. Brewster's experience indicated that correlation does
exist. The same conclusion can be drawn by considering the sheet
paper as a two-dimensional random process. Even though the exis-
tence of this correlation was known to the investigators in the

past, the way to extract the correlation over the whole two-dimensional plane from the scan path data was not known. This presents an obstacle in the use of optimal multidimensional estimation theory.

Optimal multidimensional estimation theory requires the multidimensional autocorrelation function to be given a priori. References on this topic are Peterson and Middleton (5), Blum (6) and Repjar (7). Reference (5) deals with optimal estimation of multidimensional stationary random processes, whose autocorrelation function is given a priori. With a slight modification described in Chapter IV, their method can be made to handle the nonstationarity under consideration in this thesis. Reference (6) is concerned with a class of one-dimensional nonstationary processes similar to the one described in the problem statement. With appropriate modifications, it can be extended to handle the two-dimensional case as shown in Chapter IV. It also requires prior knowledge of the autocorrelation function. However, not all multidimensional estimation methods require the autocorrelation function, as shown by Reference (7). It deals with two-dimensional estimation which is not optimal and at the same time does not require any knowledge of the autocorrelation function. Its application is restricted to pattern recognition since it requires sampled data on the whole two-dimensional plane and cannot be used with data on a restricted sampling lattice.

The problem of obtaining the two-dimensional autocorrelation function from a restricted sampling lattice has not been treated

in literature. The commonly known works in the estimation of the one-dimensional autocorrelation function are by Parzen (8), Blackman and Tukey (9), Balchen and Blandhol (10), and Uskov and Orlov (11). References (10) and (11) are primarily concerned with the error involved in the estimation of the autocorrelation function and designing the experimental parameters so as to reduce this error. References (8) and (9) develop many different ways for smoothing an estimate of the autocorrelation function by passing it through various lag windows. The single most important clue to the extrapolation and refinement of the estimated autocorrelation function came from Bendat (12), who is his book showed a number of one-dimensional physical processes that obey the exponential-cosine form of the autocorrelation function. He suggests it as a means of refining the estimated autocorrelation function. This concept has been developed in the thesis and used not only in the refinement but also in the extrapolation of the estimated autocorrelation function.

Two methods very relevant to the above concept and used as solution tools in this thesis are the Blackman and Tukey's power spectrum analyzer and Fletcher and Powell's function minimization technique. Blackman and Tukey (9) describes an indirect technique of obtaining the power spectrum from finite length discrete-time data. The term "indirect" refers to computing the power spectrum from the autocorrelation function instead of directly from the discrete-time data as considered by Welch (13). The significant advantage of this method is its computational ease. The autocor-

relation function is first obtained from the discrete-time data.
Then employing the standard definition of power spectrum as the
Fourier transform of the autocorrelation function, the raw power
spectrum is obtained at discrete points in the frequency plane.
This estimate of the power spectrum is then refined by convolving
the raw estimates with different sets of weights to obtain either
the Hamming or Hanning power spectrum. The power spectrum analyzer
has been used in this thesis to identify the frequencies present
in the autocorrelation function.

In Fletcher and Powell's paper (14) a powerful iterative
descent method for finding a local minimum of a function of sev-
eral variables is described. A number of theorems are proved to
show that the method always converges and that it converges rapidly.
The authors mention in their paper that the method has been used
successfully to solve a system of one hundred non-linear simul-
taneous equations. The method falls under the classification of
conjugate gradient techniques and requires the analytical form of
the gradients. This class of techniques has the property of quad-
ratic convergence in that the minimum of the quadratic objective
function is found within some finite number of iterations. The
set of directions chosen to ensure that the optimum of the quad-
ratic function is found in a finite number of iterations are known
as conjugate directions. Fletcher and Powell have shown a way of
choosing these conjugate directions and the distances to move in
these directions.

They assume that the function obeys the standard quadratic

form in n-dimensions. The analytical solution of the minimum is given by the Newton Raphson method, if the Hessian matrix is available. In their method the inverse of the Hessian matrix is approximated initially by a unit matrix so that the first direction is down the line of steepest descent. This approximation is subsequently improved until at the minimum, it converges to the true inverse of the Hessian matrix. The improvement is achieved by noting that the current gradient vector is orthogonal to the past incremental vector. For obtaining the minimum along a line they suggest a procedure which uses cubic interpolation and is based on that given by Davidon (15).

The computer subroutine for function minimization using the Fletcher and Powell algorithm is available from IBM Corporation. Dr. D. E. Fyffe of Georgia Tech's Industrial Engineering Department, made this subroutine available to the author for use in this thesis and it is included in the Appendix.

Properties of a Two-dimensional Stochastic Process

A stationary process of order two is defined as a stochastic process $s(x,t)$ whose first and second order density functions are not affected by a shift in the x and t origins, i.e. the first order density function

$$f(s;x,t) = f(s;x + \xi, t+\lambda) \tag{1.3}$$

and the second order density function

$$f(s_1,s_2;x_1,x_2,t_1,t_2) = f(s_1,s_2;x_1 + \xi,x_2+\xi,t_1+\lambda, t_2+\lambda) \tag{1.4}$$

This, in turn, means that

$$f(s;x,t) = \text{Constant} \qquad (1.5)$$

and

$$f(s_1,s_2;x_1,x_2,t_1,t_2) = f(s_1,s_2; \Delta x, \Delta t) \qquad (1.6)$$

where

$$\Delta x = x_1 - x_2 \quad \text{and} \quad \Delta t = t_1 - t_2$$

In terms of the statistics of the process, the mean is a constant and the autocorrelation function

$$R_{ss}(x_1,x_2,t_1,t_2) = E\left\{s(x_1,t_1)s(x_2,t_2)\right\} \qquad (1.7)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_1 s_2 f(s_1,s_2; x_1,x_2,t_1,t_2)ds_1 ds_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_1 s_2 f(s_1,s_2; \Delta x, \Delta t)ds_1 ds_2$$

$$= E\left\{s(x,t)s(x + \Delta x, t + \Delta t)\right\}$$

$$= R_{ss}(\Delta x, \Delta t)$$

A process $s(x,t)$ is wide-sense stationary if its expected value is a constant and its autocorrelation function obeys Equation (1.7). It does not, however, imply stationarity of order two. If Δx and Δt are assumed positive in Equation (1.7), then from the definition of autocorrelation function it follows that

$$R_{ss}(\Delta x, \ \Delta t) = R_{ss}(-\Delta x, \ -\Delta t) \tag{1.8}$$

and equivalently

$$R_{ss}(-\Delta x, \ \Delta t) = R_{ss}(\Delta x, \ -\Delta t) \tag{1.9}$$

In special cases where

$$R_{ss}(\Delta x, \ \Delta t) = R_{ss}(-\Delta x, \ \Delta t) \tag{1.10}$$

a smaller sampling lattice will suffice to make a crude estimate of the autocorrelation function. Therefore, it is always recommended that a real process be tested to see if this feature exists.

The power spectrum $S_{ss}(\omega_x, \ \omega_t)$ is the double Fourier transform of the autocorrelation function, i.e.,

$$S_{ss}(\omega_x, \ \omega_t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ss}(\Delta x, \ \Delta t) e^{-j\omega_x \Delta x - j\omega_t \Delta t} d\Delta x d\Delta t \tag{1.11}$$

When Equation (1.10) holds, the power spectrum $S_{ss}(\omega_x, \ \omega_t)$ is real. The reason being that the autocorrelation function is even about Δx and Δt, and this causes the odd terms, which includes all the imaginary terms, to vanish upon integration from $-\infty$ to $+\infty$. This power spectrum has the property of always being positive. The indirect proof of this statement is given in reference (16) for a one-dimensional process. It can be extended to the two-dimensional case by simply replacing the one-dimensional quantities by their two-dimensional counterparts. When Equation (1.10) does not hold, the power spectrum becomes a complex value (real and imaginary parts).

Power spectrum is very useful when performing a linear filter synthesis. This is because it does not involve the difficult to solve convolution formulas associated with using autocorrelation functions in the synthesis. The important result to remember is

$$S_{zz}(\omega_x, \omega_t) = S_{ss}(\omega_x, \omega_t) |G(j\omega_x, j\omega_t)|^2 \tag{1.12}$$

where $S_{zz}(\omega_x, \omega_t)$ and $S_{ss}(\omega_x, \omega_t)$ are the power spectrum of the output and input signals and $G(j\omega_x, j\omega_t)$ is the transfer function of the linear filter.

A normal process $s(x, t)$ is one in which the random variables

$$s(x_1, t_1), \; s(x_2, t_2), \; s(x_3, t_3), \ldots, \; s(x_n, t_n)$$

are jointly normal for any n and the nth-order density function is completely determined in terms of its expected value and auto-correlation function (16). Now if the process is normal, wide-sense stationary and has zero mean, the first order density function becomes

$$f(s;x,t) = \frac{1}{\sqrt{2\pi R(0,0)}} \; e^{-s^2/2R(0,0)} \tag{1.13}$$

$$= f(s;x + \xi, \; t + \lambda)$$

and

$$f(s_1,s_2;x_1,x_2,t_1,t_2) = \frac{1}{2\pi \sqrt{R^2(0,0) - R^2(\Delta x, \Delta t)}} \; e^{-\frac{R(0,0)s_1^2 - 2R(\Delta x, \Delta t) + R(0,0)s_2^2}{2[R^2(0,0) - R^2(\Delta x, \Delta t)]}} \tag{1.14}$$

$$= f(s_1, \; s_2; \; \Delta x, \; \Delta t)$$

These two equations are equivalent to Equations (1.3) and (1.6),

and, therefore, the process is stationary of order two. The same results hold when the process mean is not zero. However from the above definition of a normal process, the statistics are uniquely determined in terms of the expected value and autocorrelation. Hence the process is stationary of order n where n tends to infinity, or strictly stationary.

The normal property is useful since it ensures that a linear estimator is the best estimator (16). It can however be relaxed in a practical situation with the penalty that the linear estimator is not necessarily the best estimator.

Ergodicity is perhaps one of the most desirable properties in any stationary random process. It enables the determination of the mean and autocorrelation function without having to calculate the ensemble averages. Ergodic theory states (16) that "s(t) is ergodic in the most general form if (with probability one) all its statistics can be determined from a single function s(t, ζ) of the process." According to Grenander and Rosenblatt (17) "If s(t) is a normal process one can show that a necessary and sufficient condition for it to be ergodic is that the spectrum be continuous." Since many real processes are approximately normal and have a continuous power spectrum, they are ergodic. Hence the mean and autocorrelation function can be estimated from

$$E\{s\} = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{1}{mn} \int_0^m \int_0^n s(x,t)\, dx\, dt \qquad (1.15)$$

and

$$R_{ss}(\Delta x, \Delta t) = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{1}{mn} \int_0^m \int_0^n s(x,t)s(x+\Delta x, t+\Delta t) \; dxdt \qquad (1.16)$$

If only discrete values of the signal are available at spacings of X and T in the x and t directions, then the mean and auto-correlation function are estimated by

$$E\{s\} = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{1}{mn} \sum_{j=0}^{m} \sum_{i=0}^{n} s(iX, jT) \qquad (1.17)$$

and

$$R_{ss}(kX, rT) = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{1}{mn} \sum_{j=0}^{m} \sum_{i=0}^{n} s(iX, jT)s(iX + kX, jT + rT) \qquad (1.18)$$

In filter synthesis, a process known as white noise is fre-quently used. Two types of white noises have been considered in this thesis; namely, discrete Gaussian white noise and dense Gaussian white noise. Discrete Gaussian white noise n(iX, jT) is defined as a process made up of a two-dimensional array of uncor-related random variables belonging to a Gaussian distribution. Its autocorrelation function is given by

$$R_{nn}(iX, jT) = \sigma^2 \quad \text{for } i = j = 0 \qquad (1.19)$$
$$0 \quad \text{elsewhere}$$

where σ^2 is the variance of the random variables and X and T are the array spacings in the x and t directions.

Dense Gaussian white noise $z(x,t)$ is defined as a process made up of Gaussian uncorrelated random variables densely packed over the two-dimensional plane. Its autocorrelation function is given by

$$R_{zz}(\Delta x, \Delta t) = \sigma^2 \delta(\Delta x, \Delta t) \qquad (1.20)$$

Consequently, the power spectrum is flat with a value of σ^2.

Approach to the Problem

A major portion of this thesis is concerned with the estimation of the stationary component of the nonstationary process as mentioned in the Statement of the Problem.

The stationary part was modeled as Gaussian and ergodic since these properties are very practical assumptions with respect to sheet processes. The central limit theorem (16) states that under fairly general conditions the sum of n independent random variables tends to the Gaussian distribution as n tends to infinity. The criterion to minimize the expected value of the square of the estimation error under the above assumptions leads to a consistent and efficient estimator (18). This criterion was chosen as compared to other consistent and efficient estimators since it does not require the probability density function of the process and is mathematically easy to solve.

For the Gaussian stationary process, the linear estimator is the best estimator when the above criterion is used (16). Optimal linear estimators require the autocorrelation function of the

process to be specified a priori. Since estimation of the total autocorrelation function is not possible from the scan path data, the functional form of the autocorrelation function is used to advantage. If this form is known a priori, then the parameters in the form are estimated by either the maximum likelihood or the least-squares method. If the form is not known, the approach taken is to fit an exponential-cosine form to the autocorrelation function estimated from the scan path data. The choice of the exponential-cosine form stems from the observations of one-dimensional random processes by Bendat (12). According to his observations, most real random process with continuous power spectrum have an approximately exponential-cosine autocorrelation function. The least-squares method will be used to estimate the parameters in the functional form because it is mathematically easy to use.

The effectiveness of the proposed method is determined by comparing the mean square estimation error of the proposed method versus the error obtained by using the theoretical autocorrelation function. A simulated process will be used for this purpose. Comparison with other existing methods will also be made to see if significant improvements in the estimation of the signal are obtainable by using the proposed method. A real sheet paper process will be used for this purpose.

Outline of the Thesis

In this chapter the problem has been defined, relevant literature has been reviewed and the proposed method of attack has been stated. In Chapter II, two stationary random processes with known

autocorrelation functions will be simulated. The first process will be used as an analysis problem in this thesis. The second process will be used only in Chapter II to show that the frequency decomposition technique works when more than one set of frequencies are present in the spatial and temporal coordinates. In Chapter III, a technique to estimate, refine and extrapolate the autocorrelation function from the scan path data will be devised. In Chapter IV, optimal estimation will be performed using the fitted autocorrelation function, the theoretical autocorrelation function and the one-dimensional autocorrelation function. Also a sensitivity analysis will be performed for the parameters introduced in Chapter III. In Chapter V, the techniques developed in the previous chapters will be applied to a paper mill process and the advantages of the developed scheme highlighted. Chapter VI will present the conclusions and recommendations of this thesis.

CHAPTER II

FILTER SYNTHESIS FOR GENERATING A TWO-DIMENSIONAL
RANDOM PROCESS FROM DISCRETE WHITE NOISE

In this chapter a technique for synthesizing two-dimensional filters is devised. It is desired to design a filter which when excited by discrete Gaussian white noise, generates a random process with the chosen exponential-cosine autocorrelation function. The approach used is analogous to synthesizing a continuous one-dimensional filter with several modifications. These modifications are necessary because the two-dimensional continuous approach requires dense white noise as input to the filter. To use discrete white noise instead of dense white noise, the convolution integral in the continuous case is separated into smaller integrals. It is difficult to proceed beyond this point without approximating the filter impulse response as constants in the small intervals of integration. After making this approximation and moving the constants outside the integration signs, the remaining integrals are shown to be discrete Gaussian white noise. Also, in the two-dimensional case the filter synthesis procedure is hampered by the nonseparable property of the power spectrum. This problem is resolved by introducing a multiplicity of independent discrete white noise passed through separate filters and summed.

Background Information

In the generation of one-dimensional random processes, dense white noise is passed through an appropriate filter yielding a process with the desired autocorrelation function. The resulting power spectrum is given by

$$S_{ss}(\omega) = |G(j\omega)|^2 \times 1 \tag{2.1}$$

where $S_{ss}(\omega)$ is the power spectrum of the desired process and $G(j\omega)$ is the transfer function of the filter. It is known, however, that if $S_{ss}(\omega)$ is given, then $G(j\omega)$ is not uniquely determined, but is given by

$$G(j\omega) = \sqrt{S_{ss}(\omega)} \ e^{j\varphi(\omega)} \tag{2.2}$$

where $\varphi(\omega)$ is arbitrary (19). The inverse of $G(j\omega)$ is causal, if the $S_{ss}(\omega)$ satisfies the Paley-Wiener condition (19). In the generation of a two-dimensional random process, the problems are further compounded by the complicated form of the desired power spectrum.

Before proceeding to the synthesis problem, a fundamental concept is reviewed. Consider the process shown in Figure 1. Independently generated dense white noises z_1, z_2, ..., z_n are passed through separate filters. If the output signal $s(t)$ is given by

$$s(t) = s_1(t) + s_2(t) + s_3(t) \ldots s_n(t) \tag{2.3}$$
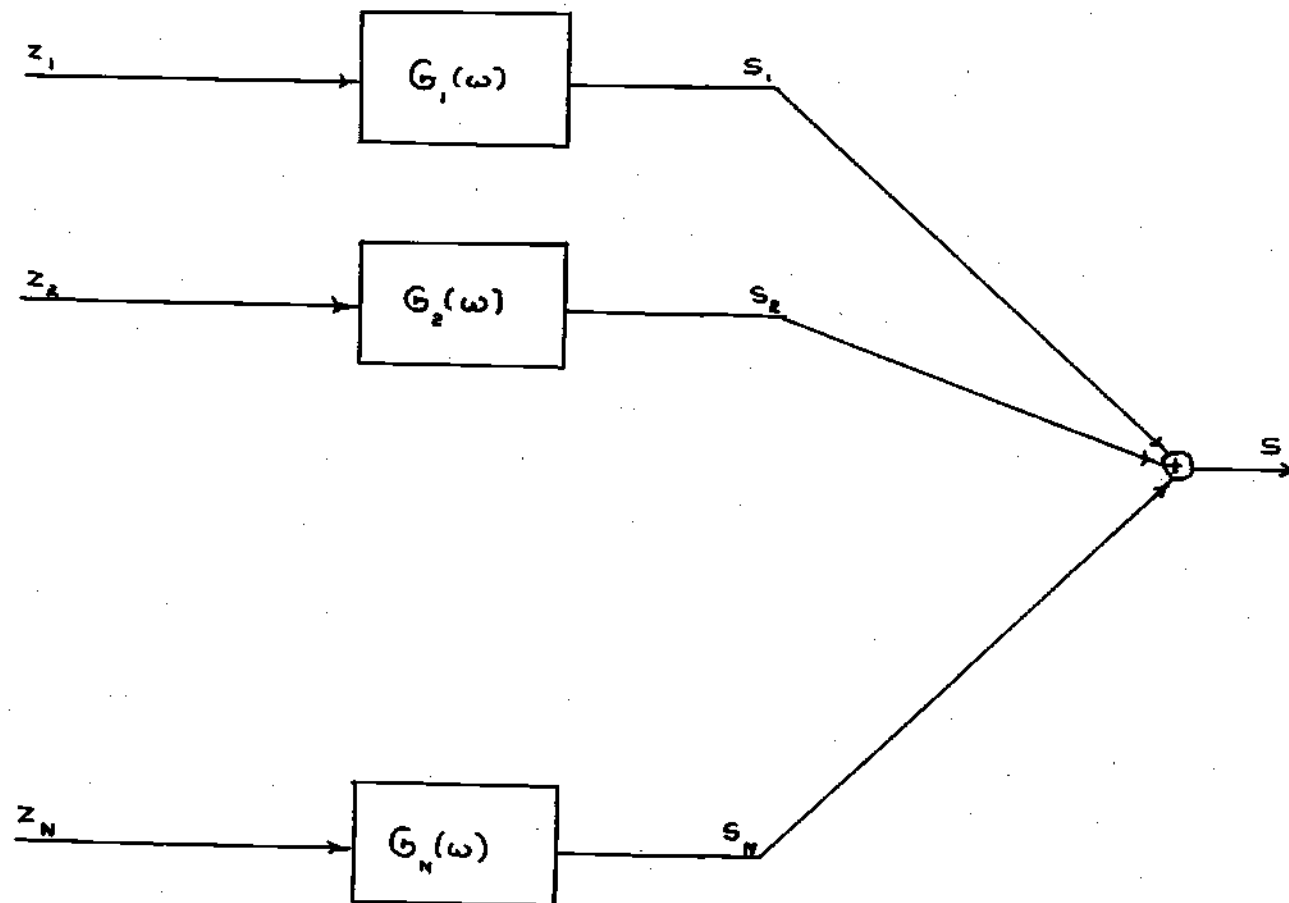
Figure 1. Scheme for Generating a Random Process.

then it follows that the output autocorrelation function $R_{ss}(\Delta t)$ is given by

$$R_{ss}(\Delta t) = R_{s_1 s_1}(\Delta t) + R_{s_2 s_2}(\Delta t) + R_{s_3 s_3}(\Delta t) \ldots + R_{s_n s_n}(\Delta t)$$

$$(2.4)$$

and the output spectrum $S_{ss}(\omega)$ is given by

$$S_{ss}(\omega) = S_{s_1 s_1}(\omega) + S_{s_2 s_2}(\omega) + S_{s_3 s_3}(\omega) \ldots + S_{s_n s_n}(\omega)$$

$$(2.5)$$

Synthesis Procedure

Consider a class of processes described by the autocorrelation function

$$R_{ss}(\Delta x, \Delta t) = \sum_i D_i e^{-A_i |\Delta x| - B_i |\Delta t|} \cos \omega_{xi} \Delta x \cos \omega_{ti} \Delta t \quad (2.6)$$

It is desired to simulate this process rather than the more common

$$R_{ss}(\Delta x, \Delta t) = \sum_i D_i e^{-\sqrt{A_i \Delta x^2 + B_i \Delta t^2}} \cos \omega_{xi} \Delta x \cos \omega_{ti} \Delta t \quad (2.7)$$

because of the difficulty in handling integrals of the type

$$\int \int e^{-\sqrt{A_i \Delta x^2 + B_i \Delta t^2}} e^{-j \omega_x \Delta x - j \omega_t \Delta t} \, d\Delta x \, d\Delta t$$

These forms will be discussed in the next chapter, but for the present note that A_i, B_i, and D_i are constants, and ω_{xi} and ω_{ti}

represent frequencies in the x and t directions. Its power spectrum is given by

$$S_{ss}(\omega_x, \omega_t) = \sum_i 4D_i A_i B_i \left[\frac{\omega_x^2 + \omega_{xi}^2 + A_i^2}{(A_i^2 + \omega_{xi}^2 - \omega_x^2)^2 + 4A_i^2\omega_x^2} \right] \cdot \qquad (2.8)$$

$$\left[\frac{\omega_t^2 + \omega_{ti}^2 + B_i^2}{(B_i^2 + \omega_{ti}^2 - \omega_t^2)^2 + 4B_i^2\omega_t^2} \right]$$

which can be rewritten as

$$S_{ss}(\omega_x, \omega_t) = \sum_i 4D_i A_i B_i \frac{(j\omega_x + \sqrt{A_i^2 + \omega_{xi}^2})(-j\omega_x + \sqrt{A_i^2 + \omega_{xi}^2})}{[(A_i + j\omega_x)^2 + \omega_{xi}^2][(A_i - j\omega_x)^2 + \omega_{xi}^2]} \qquad (2.9)$$

$$\frac{(j\omega_t + \sqrt{B_i^2 + \omega_{ti}^2})(-j\omega_t + \sqrt{B_i^2 + \omega_{ti}^2})}{[(B_i + j\omega_t)^2 + \omega_{ti}^2][(B_i - j\omega_t)^2 + \omega_{ti}^2]}$$

$S_{ss}(\omega_x, \omega_t)$ in Equation (2.9) is not separable in the form $G(j\omega_x, j\omega_t)G(-j\omega_x, -j\omega_t)$ because of the summation sign. However, if each term under the summation sign is the result of a separate filter excited by independent white noise and then summed, the nonseparable problem of Equation (2.9) does not arise. The power spectrum can be written as

$$S_{ss}(\omega_x, \omega_t) = \sum_i G_i(j\omega_x, j\omega_t)G_i(-j\omega_x, -j\omega_t) \qquad (2.10)$$

in which the various $G_i(j\omega_x, j\omega_t)$ represent the transfer functions of the different filters excited by independent white noise. Hence, the transfer function of a particular filter is given by

$$G_i(j\omega_x, j\omega_t) = \sqrt{4A_i B_i D_i} \; \frac{(j\omega_x + \sqrt{A_i^2 + \omega_{xi}^2})(j\omega_t + \sqrt{B_i^2 + \omega_{ti}^2})}{[(A_i + j\omega_x)^2 + \omega_{xi}^2][(B_i + j\omega_t)^2 + \omega_{ti}^2]} \qquad (2.11)$$

and the causal impulse response is given by

$$W(\Delta x, \Delta t) = \sqrt{4A_i B_i D_i}\left[e^{-A_i \Delta x} \cos \omega_{xi} \Delta x + \frac{(\sqrt{A_i^2 + \omega_{xi}^2} - A_i)}{\omega_{xi}} \cdot \right. \qquad (2.12)$$
$$\left. e^{-A_i \Delta x} \sin \omega_{xi} \Delta x \right]$$
$$\left[e^{-B_i \Delta t} \cos \omega_{ti} \Delta t + \frac{(\sqrt{B_i^2 + \omega_{ti}^2} - B_i)}{\omega_{ti}} e^{-B_i \Delta t} \sin \omega_{ti} \Delta t \right]$$

Independent dense white noises are convolved with the different impulse responses and then summed to generate the desired random process. The convolution equation from linear continuous filtering theory is given by

$$s(x,t) = \int_0^\infty \int_0^\infty W(\chi, \tau) \, z \, (x - \chi, t - \tau) d\chi \, d\tau \qquad (2.13)$$

where $z(x,t)$ is the dense Gaussian white noise and $W(x,t)$ is the impulse response of the continuous filter given by Equation (2.12). Equation (2.13) can be separated in the form

$$s(x,t) = \int_0^T \int_0^X W(\chi,\tau) z(x-\chi, t-\tau) d\chi\, d\tau + \int_T^{2T} \int_0^X (\quad) d\chi\, d\tau \quad (2.14)$$

$$+ \int_0^T \int_X^{2X} (\quad) d\chi\, d\tau + \int_T^{2T} \int_X^{2X} (\quad) d\chi\, d\tau + \ldots$$

Since the computer generated white noise is discrete, Equation (2.14) cannot be used without modifications. The impulse responses under the integral signs in Equation (2.14) are approximated by constants as shown in Figure 2. The value of the constant is chosen to be the value of the impulse response at the coordinates specified by the centroid of the area bounded by the limits of the integral.

Hence, Equation (2.14) results in

$$s(x,t) = W(\tfrac{X}{2}, \tfrac{T}{2}) \int_0^T \int_0^X z(x-\chi, t-\tau) d\chi\, d\tau + W(\tfrac{X}{2}, \tfrac{3T}{2}) \int_T^{2T} \int_0^X \cdot \quad (2.15)$$

$$z(x-\chi, t-\tau) d\chi\, d\tau$$

$$+ W(\tfrac{3X}{2}, \tfrac{T}{2}) \int_0^T \int_X^{2X} z(x-\chi, t-\tau) d\chi\, d\tau + W(\tfrac{3X}{2}, \tfrac{3T}{2}) \int_T^{2T} \int_X^{2X} \cdot$$

$$z(x-\chi, t-\tau) d\chi\, d\tau + \ldots$$

The double integrals in Equation (2.15) are uncorrelated random variables and are denoted by $n_{x,t}(i,j)$. Since the white noise $z(x,t)$ is Gaussian, its integral which represents a summation of Gaussian distributed random variables, is also Gaussian. The expected value of $n_{x,t}(i,j)$ is

$$E\left\{n_{x,t}(i,j)\right\} = E\left\{\int_{jT}^{(j+1)T} \int_{iX}^{(i+1)X} z(x-\chi, t-\tau) d\chi\, d\tau\right\} \quad (2.16)$$

$$= XTE\left\{z\right\}$$

Impulse Response of the Filter
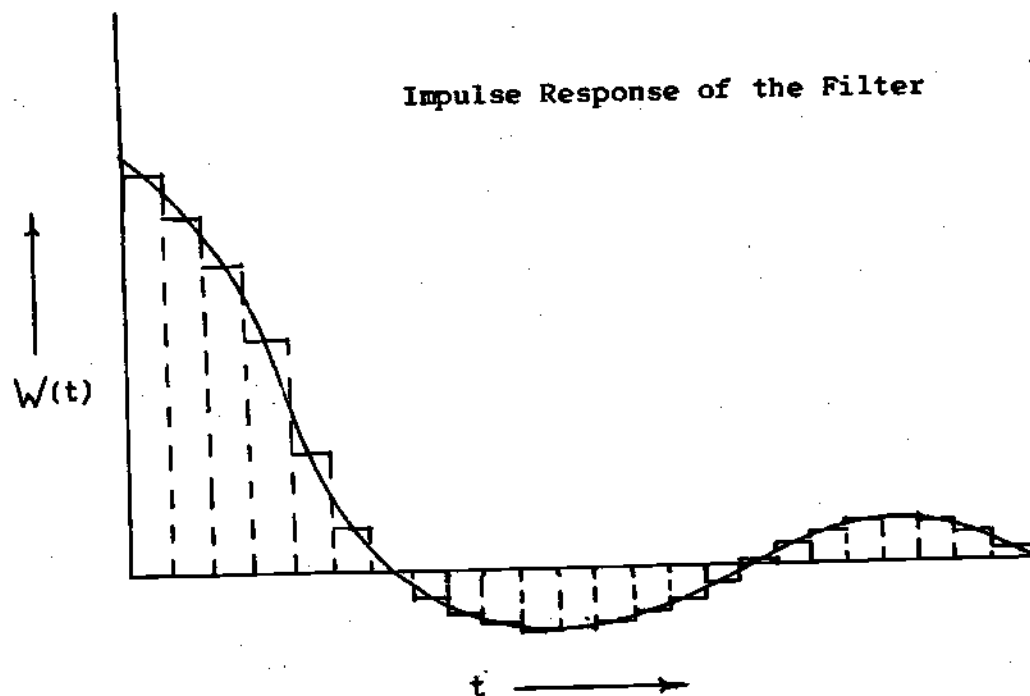
Figure 2. Digitalization of the Analog Filter.

and the variance is

$$\sigma_n^2 = E\left\{ \left[\iint z(x-\chi, t-\tau) d\chi d\tau \right]^2 \right\} \qquad (2.17)$$

Since $z(x,t)$ is white noise, therefore, Equation (2.17) becomes

$$\sigma_n^2 = \int_{jT}^{(j+1)T} \int_{iX}^{(i+1)X} \sigma_z^2 \, d\chi d\tau \qquad (2.18)$$

$$= XT \, \sigma_z^2$$

If $z(x,t)$ has zero mean and unit variance as selected for the two simulations in this chapter, then from Equations (2.17) and (2.18) $E\left\{ n_{x,t}(i,j) \right\}$ becomes zero and σ_n^2 becomes XT.

In both the simulated processes, X and T are unity and hence, σ_n^2 becomes unity. The discrete white noise sequence generated on the computer with mean zero and unit variance can now be used in place of the double integrals in Equation (2.15) which becomes

$$s(x,t) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} W(i + \tfrac{1}{2}, \ j + \tfrac{1}{2}) n_{x,t}(i,j) \qquad (2.19)$$

If once the random sequence $n_{x,t}(i,j)$ is chosen for a particular x and t, then the random sequence for other x and t is correlated to the first sequence. However, moving x and t by unity from the x and t that established the first random sequence, one is able to reuse the same sequence. In order to express this mathematically, define

$$n^*(i,\ j) = \int_{jT}^{(j+1)T} \int_{iX}^{(i+1)X} z(x,t)\,dxdt \qquad (2.20)$$

then Equation (2.19) reduces to

$$s(x^*,t^*) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} W(i+\tfrac{1}{2},\ i+\tfrac{1}{2})n^*(x^*-i,t^*-j) \qquad (2.21)$$

where x^* and t^* represent discrete integer values of x and t, and i and j are integers.

Two types of errors can arise when using this technique of digitalization. The first error is the approximation of

$$\int_{j}^{j+1} \int_{i}^{i+1} W(x,t)\,dxdt$$

by a constant $W(i+\tfrac{1}{2},\ j+\tfrac{1}{2})$ which does not even represent the mean value of the weighting function in the interval. The reason for choosing $W(i+\tfrac{1}{2},\ j+\tfrac{1}{2})$ is to avoid the excessive computation which arises if the mean value is to be found for each interval. The second source of error is the truncation of the summation of Equation (2.21). In most practical cases, truncation is essential due to limited computer memory and computer time. The effect of the two types of errors on the variance of the output signal will be studied in the next paragraph.

Let σ_s^2 and σ_n^2 represent the variances of the output and input signals. Express Equation (2.21) by

$$s(x^*,t^*) = \sum_{j} \sum_{i} a_{ij}n(x_i,t_j) \qquad (2.22)$$

Taking the expected value of the signal squared, results in

$$E\left\{s^2(x^*,t^*)\right\} = E\left\{\left[\sum_i \sum_j a_{ij}n(x_i,t_j)\right]^2\right\} \qquad (2.23)$$

$$= \sum_i \sum_j E\left\{a_{ij}^2 n^2(x_i,t_j)\right\}$$

$$= \sum_i \sum_j a_{ij}^2 \sigma_n^2 = \sigma_n^2 \sum_i \sum_j a_{ij}^2$$

since $n(x,t)$ is white noise. Now if $\sigma_n^2 = 1$ then from Equation (2.23) it follows that

$$\sum_i \sum_j a_{ij}^2 = \sigma_s^2 \qquad (2.24)$$

The truncation error causes

$$\sum_i \sum_j a_{ij}^2 < \sigma_s^2 \qquad (2.25)$$

and the approximation of the analog filter of Equation (2.12) by a histogram form causes

$$\sum_i \sum_j a_{ij}^2 > \sigma_s^2 \qquad (2.26)$$

Equation (2.26) is a result of the shape of the weighting function of Equation (2.12)

Results

Two random processes are simulated. The first one employs a shaper with the impulse response

$$W(\Delta x, \Delta t) = .212(e^{-.075\Delta x}\cos\cdot 3\Delta x + \frac{.235}{.3}e^{-.075\Delta x}\sin\cdot 3\Delta x) \quad (2.27)$$

$$(e^{-.075\Delta t}\cos\cdot 3\Delta t + \frac{.235}{.3}e^{-.075\Delta t}\sin\cdot 3\Delta t)$$

which when excited by discrete Gaussian white noise over a rectangular lattice, shown in Figure 3, yields a process with the autocorrelation function

$$R_{ss}(\Delta x, \Delta t) = 2e^{-.075\,|\Delta x|-.075\,|\Delta t|}\cos\cdot 3\Delta x \cos\cdot 3\Delta t \quad (2.28)$$

This process is used in the extrapolation of the autocorrelation function, signal estimation, comparison of the expected value of the squared error with the theoretical value, sensitivity analysis of the parameters, and comparison of estimation errors of the developed method with the other existing methods. The second random process employs two shapers with the impulse responses

$$W_1(\Delta x, \Delta t) = .15(e^{-.075\Delta x}\cos\cdot 165\Delta x \quad (2.29)$$

$$+ \frac{.106}{.165}e^{-.075\Delta x}\sin\cdot 165\Delta x)$$

$$(e^{-.075\Delta t}\cos\cdot 165\Delta t + \frac{.106}{.165}e^{-.075\Delta t}\sin\cdot 165\Delta t)$$

and
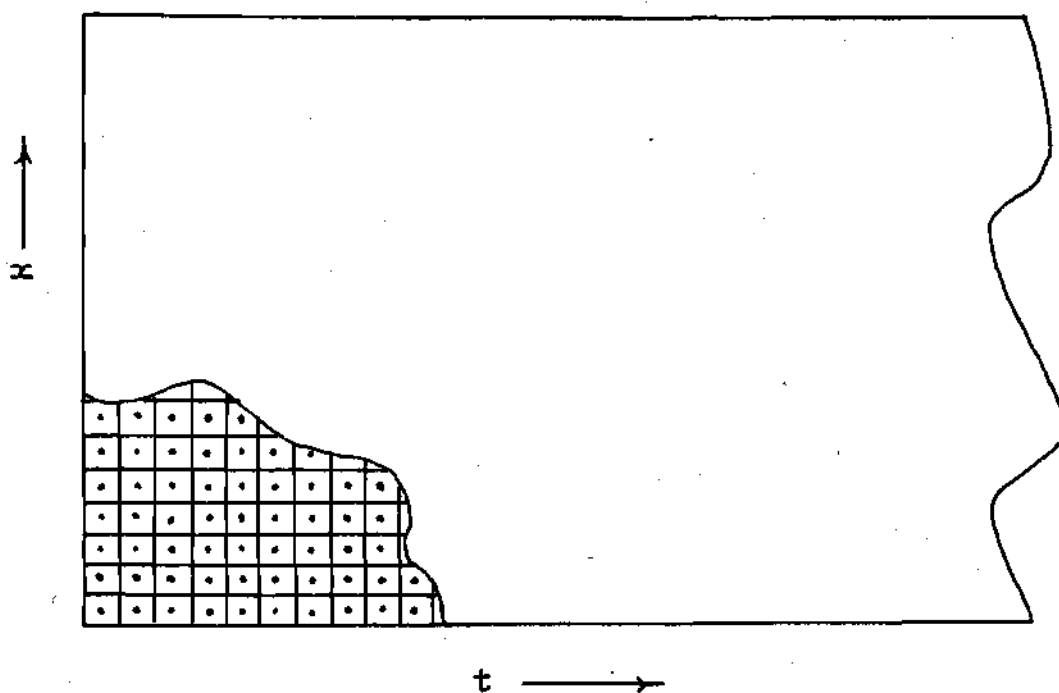
Figure 3.   Rectangular Lattice.

$$W_2(\Delta x, \Delta t) = .212(e^{-.075\Delta x}\cos\cdot3\Delta x + \frac{.235}{.3}e^{-.075\Delta x}\sin\cdot3\Delta x)$$

$$(2.30)$$

$$(e^{-.075\Delta t}\cos\cdot3\Delta t + \frac{.235}{.3}e^{-.075\Delta t}\sin\cdot3\Delta t)$$

which when excited by two independent discrete white noises, yield a process with the autocorrelation function

$$R_{ss}(\Delta x, \Delta t) = e^{-.075|\Delta x|-.075|\Delta t|}[\cos\cdot165\Delta x\cos\cdot165\Delta t \qquad (2.31)$$
$$+ 2\cos\cdot3\Delta x\cos\cdot3\Delta t]$$

The second process is used chiefly to demonstrate the ability of the power spectrum analysis to handle cases in which more than one pair of frequencies are present. It is used only in Chapter III.

The process lattice in both cases extends over 40 points in the x direction and 1200 points in the t direction. The white noise lattice extends over 79 points in the x direction and 1,240 points in the t direction. Since a 40 by 40 causal weighting matrix is used, the additional points help the filter operate in the steady state region. The two-dimensional random processes are generated on 30 scans having 40 points in each straight line scan. These scans have alternately +45° and -45° angles. The gap between each scan constitutes one unit in the t direction. The computer program used in the generation of the process is devised so as to remain in the 60K memory limit of UNIVAC 1108. It stores 97,960 values of the normal white noise sequence (zero mean and unit variance) on a tape, calls for only 3,160 random numbers at a time, and generates the random process on one scan at a time. At any time it only has 6,241 values of the random numbers in its memory. The program is included in the Appendix.

The results of the first simulation are shown in Figure 4.
The ordinate represents the value of the generated process, and
the abscissa represents 80 points on two consecutive scans.
The results of the second simulation are shown in Figures 5 and
6. The ordinate represents the values of the generated process,
and the abscissa represents 1,200 points on thirty consecutive
scans. Although the computer generated plots look continuous,
in reality they represent discrete values of the process on
1,200 abscissa points.

Discussion of the Results and Conclusions

The results of the simulation appear to be good. In
Chapter III 99.76 percent confidence bounds will be set around
the estimated autocorrelation function obtained by using this
data, and it will be shown that the desired autocorrelation func-
tion falls well within these bounds. The effect of truncation
and digitalization errors appear to be negligible since for the
first process equation (2.24) becomes

$$\sum_{i=1}^{40} \sum_{j=1}^{40} a_{ij}^2 = 2.028 \tag{2.32}$$

as compared to the theoretical summation

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 = 2.000 \tag{2.33}$$

In this chapter two random processes have been generated
which are Gaussian, stationary and possess the exponential-cosine

autocorrelation function. The simulation on the scan path constitutes the values of the signal sampled at regular intervals on the scan path as described in the "Statement of the Problem." This scan path data will be employed in Chapter III for extrapolation and refinement of the autocorrelation function.
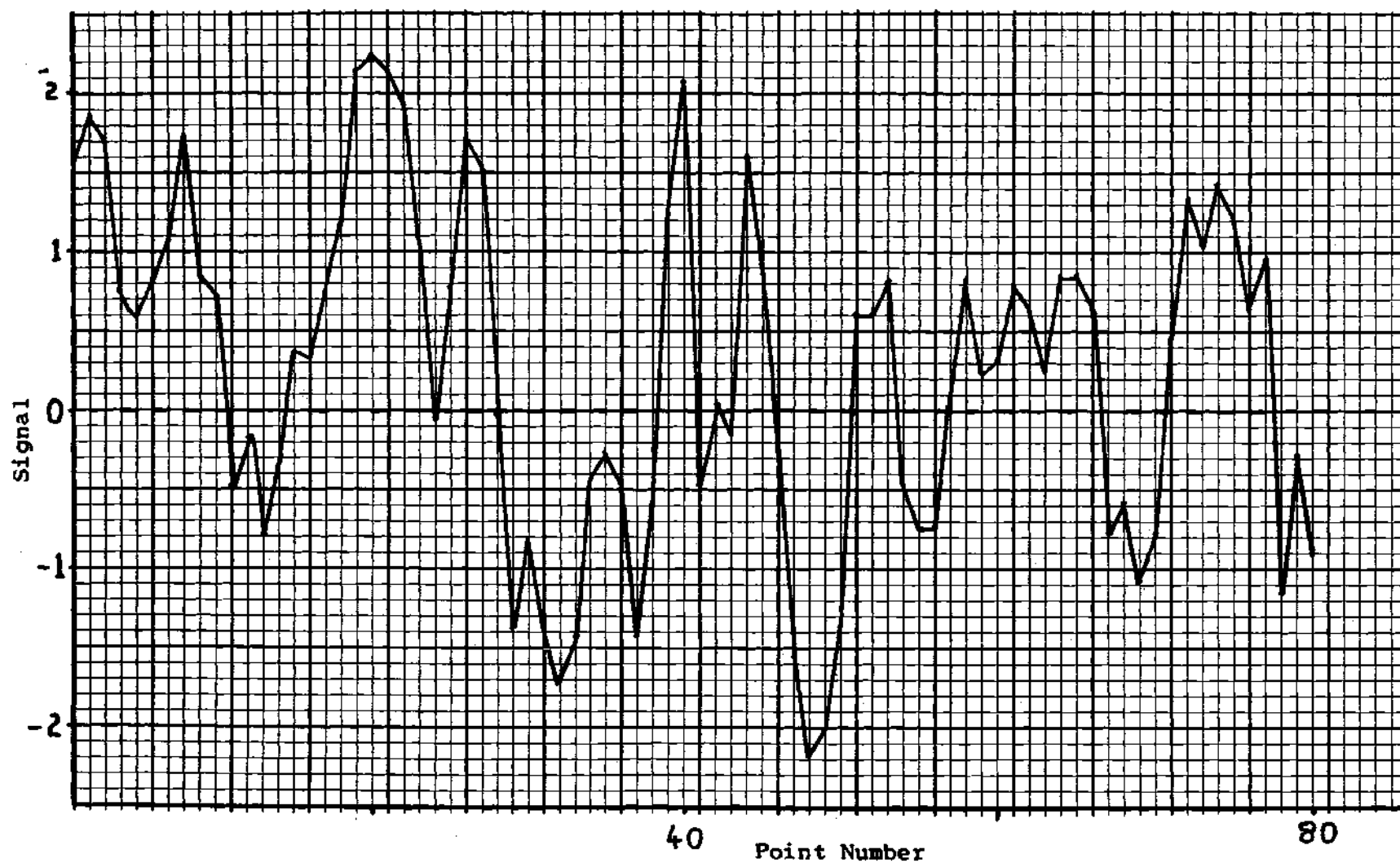
Figure 4. Two Consecutive Scan Display of Process 1.

D = 2, A = .075, B = .075 ω_x = .3, ω_t = .3

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$D_1 = 1$

$A_1 = .075$

$B_1 = .075$

$\omega_{xi} = .165$

$\omega_{ti} = .165$

Point Number * 10^1
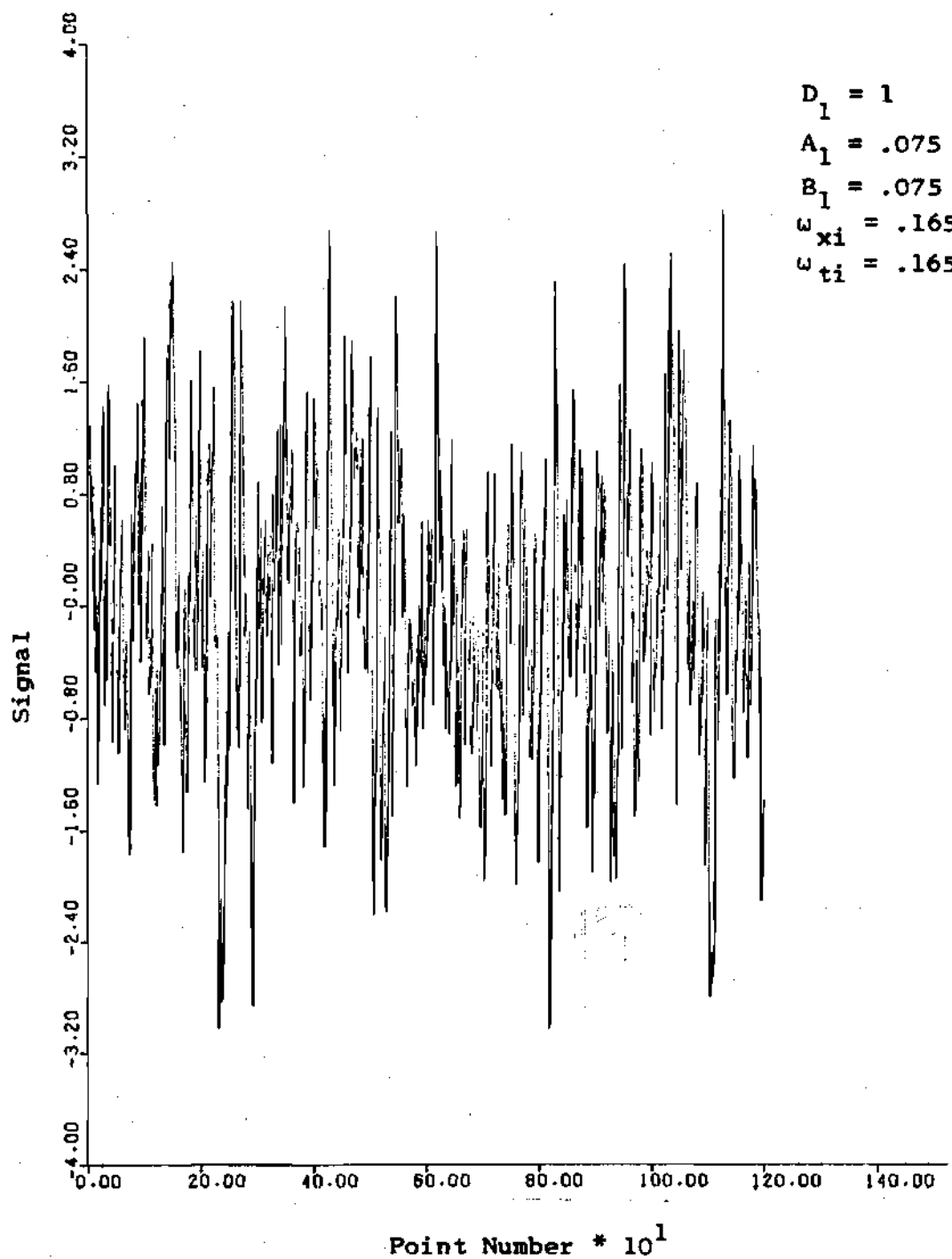
Figure 5.   Thirty Consecutive Scan Display of Process 2
            Using Shaper Given by Equation (2.29).

$D_2 = 2$
$A_2 = .075$
$B_2 = .075$
$\omega_{x2} = .3$
$\omega_{t2} = .3$

Point Number $* \ 10^1$
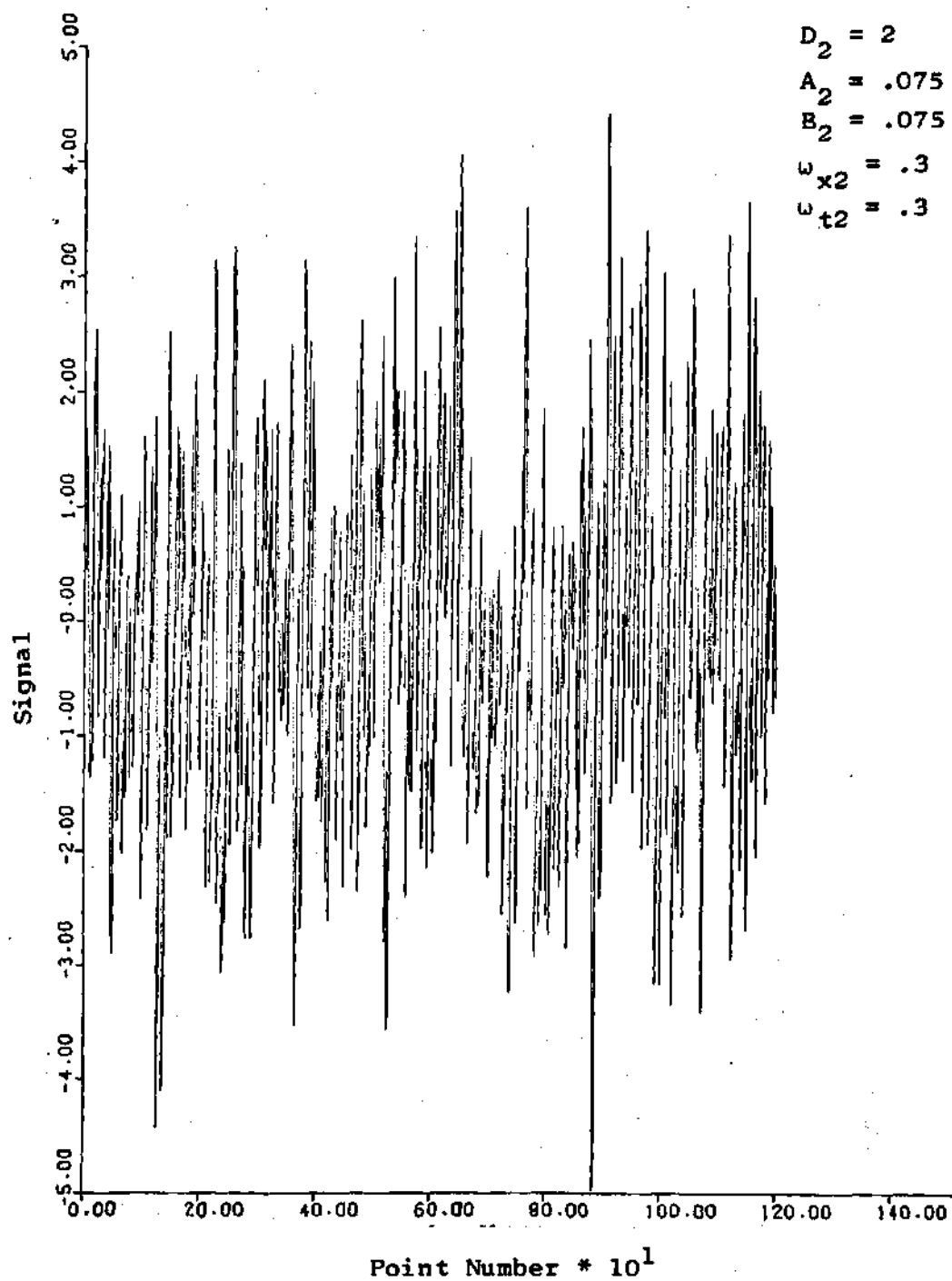
Figure 6. Thirty Consecutive Scan Display of Process 2
Using Shaper Given by Equation (2.30).

CHAPTER III

FITTING AN EXPONENTIAL-COSINE FORM TO

THE AUTOCORRELATION FUNCTION

To use the finite memory estimation procedure of D. P.
Peterson and D. Middleton (5), an accurate estimate of the auto-
correlation function is needed. As mentioned in the problem
statement, signal information is only available on the scan
path. This permits the estimation of the autocorrelation func-
tion in a limited sector of the $(\Delta x, \Delta t)$ plane. Due to the
finite nature of the sampling lattice, these estimates are
quite crude and will be referred henceforth as the crude auto-
correlation function. The procedure developed in this chapter
permits the extrapolation and refinement of the crude autocor-
relation function by fitting it into a closed functional form.
There are situations in which the form is known a priori, but
in most practical cases the form is not known. Hence, the
first step is to determine the nature of the closed functional
form that is applicable to most two dimensional random processes.
Once this form is established, it is fitted to the crude auto-
correlation function by a weighted least-squares technique. The
fitting procedure is not straightforward because of the possi-
bility of obtaining the local minimum of the cost function
instead of the global minimum. The scheme shown in Figure 7 is
devised to make the fitting procedure converge to its global

Spectrum
Analysis

Data

Crude
Autocorre-
lation
Function
in Limited
Sector

Fitting
Exp.-Cosine
Form to
Crude
Autocorre-
lation
Function
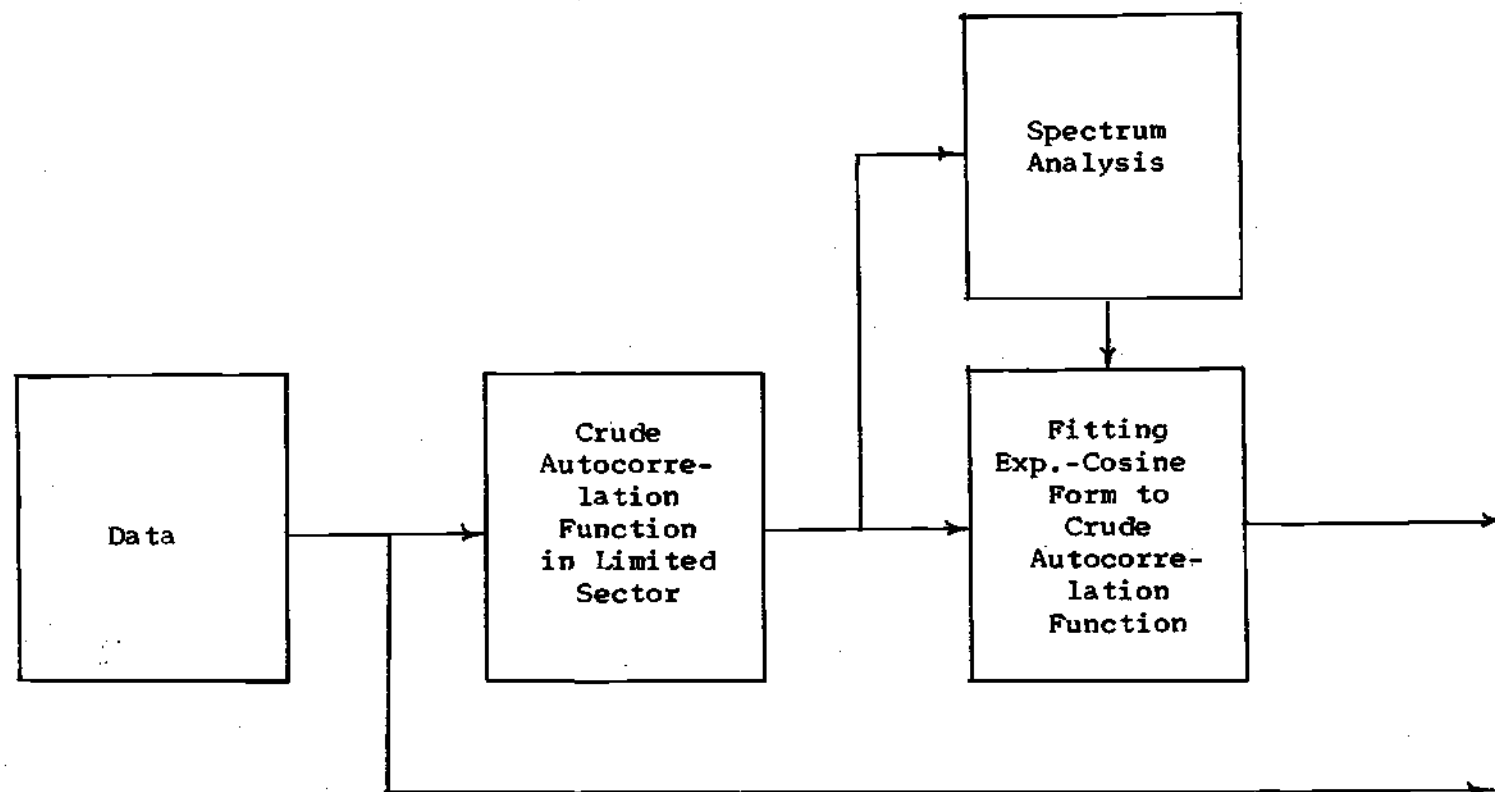
Figure 7.   Scheme for Extrapolation and Refinement of
the Crude Autocorrelation Function.

minimum by estimating the initial values of the critical parameters by another technique. The frequency terms in the closed form are critical parameters and their estimates are obtained by a frequency decomposition technique. A 99.76 percent confidence bound is set around the crude autocorrelation function and is used as a check on the fitted autocorrelation function.

Closed Functional Form

Consider a process with a continuous real power spectrum. Assume this power spectrum is expressed in the form

$$\sum_i S_{ss}^{(i)}(\omega_x - \omega_{xi}, \ \omega_t - \omega_{ti})$$

as shown in Figure 8.

If the inverse transform of $S_{ss}^{(i)}(\omega_x, \ \omega_t)$ is given by $R_{ss}^{(i)}(\Delta x, \ \Delta t)$, then

$$S_{ss}(\omega_x, \ \omega_t) = \sum_i S_{ss}^{(i)}(\omega_x - \omega_{xi}, \ \omega_t - \omega_{ti}) \qquad (3.1)$$

$$= \sum_i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ss}^{(i)}(\Delta x, \ \Delta t) e^{-j(\omega_x - j\omega_{xi})\Delta x - j(\omega_j - \omega_{ti})\Delta t} \ d\Delta x \ d\Delta t$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_i R_{ss}^{(i)}(\Delta x, \ \Delta t) e^{j\omega_{xi} + j\omega_{ti}} \right] e^{-j\omega_x \Delta x - j\omega_t \Delta t} \ d\Delta x \ d\Delta t$$

Hence the inverse transform of $S_{ss}(\omega_x, \ \omega_t)$ is given by

$$S_{ss}(\omega_x, \omega_t)$$

$$\equiv \quad S_{ss}^{(1)}(\omega_x - \omega_{x1}, \omega_t - \omega_{t1}) \quad + \quad S_{ss}^{(2)}(\omega_x - \omega_{x2}, \omega_t - \omega_{t2}) \quad + \quad S_{ss}^{(3)}(\omega_x - \omega_{x3}, \omega_t - \omega_{t3})$$
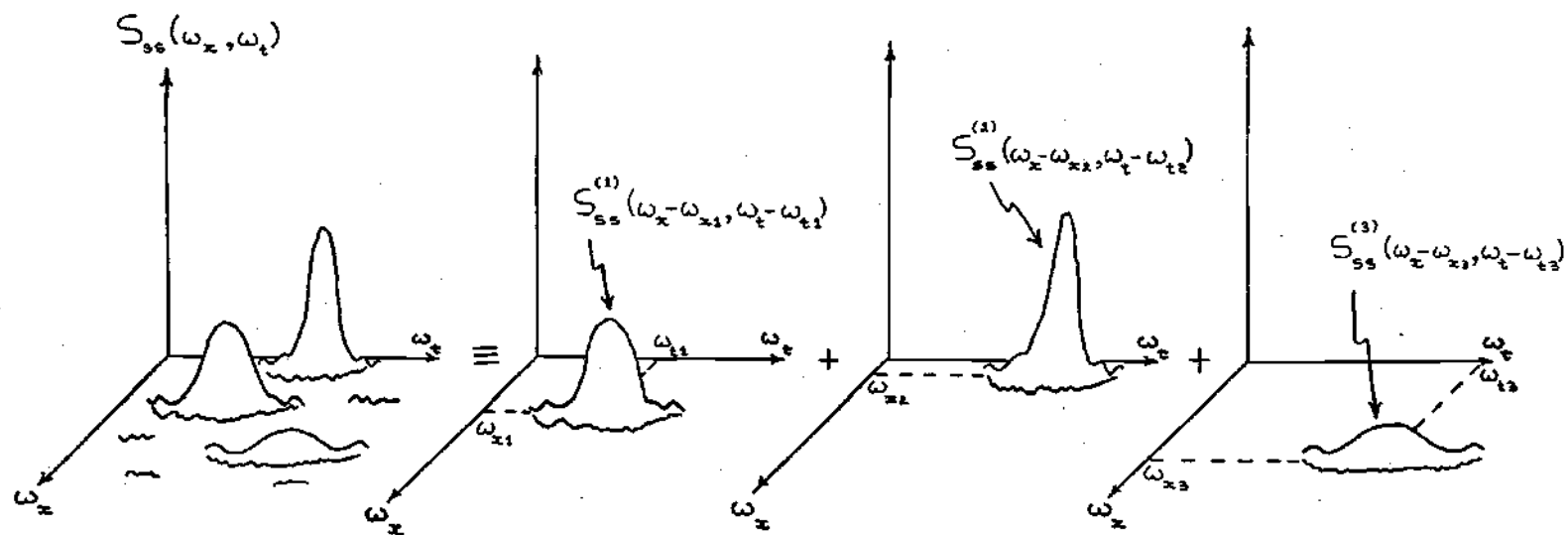
Figure 8.   Splitting of the Power Spectrum.

$$\sum_i R_{ss}^{(i)}(\Delta x, \Delta t)e^{j\omega_{xi} + j\omega_{ti}}$$

which is the autocorrelation function of the process. Separating the real part from the imaginary part and cancelling out odd terms due to symmetry in the four quadrants, one has

$$R_{ss}(\Delta x, \Delta t) = \sum_i R_{ss}^{(i)}(\Delta x, \Delta t)\cos \omega_{xi} \Delta x \cos \omega_{ti} \Delta t \quad (3.2)$$

In many one-dimensional random processes (20), $R_{ss}^{(i)}(\Delta t)$ obeys an equation of the form

$$R_{ss}^{(i)}(\Delta t) = D_i e^{-A|\Delta t|} \quad (3.3)$$

It is assumed that two-dimensional random processes obey a similar equation of the form

$$R_{ss}^{(i)}(\Delta x, \Delta t) = D_i e^{-\sqrt{A_i \Delta x^2 + B_i \Delta t^2}} \quad (3.4)$$

which represents exponential decay with elliptical contours. Hence, the closed form becomes

$$R_{ss}(\Delta x, \Delta t) = \sum_i D_i e^{-\sqrt{A_i \Delta x^2 + B_i \Delta t^2}} \cos \omega_{xi} \Delta x \cos \omega_{ti} \Delta t \quad (3.5)$$

It is emphasized again, that the form of Equation (3.5) should

only be used when no priori knowledge is available for the true form.

Weighted Least-Squares Fitting

The next step in the procedure is to fit the form to the crude autocorrelation function. A number of techniques exist in parameter estimation, the most notable being the maximum likelihood method and the weighted least-squares method. It has been shown in (18) that in cases of linear models both the methods yield the same solution subject to Gaussian probability distribution of the dependent variable. The maximum likelihood method is not chosen, since dealing with likelihood functions and their maximization can become difficult for nonlinear models. The weighted least-squares method is chosen for its ease of handling.

Let $\hat{R}_k$ represent the crude autocorrelation function at point k. Then, by the method of weighted least-squares, the cost function

$$F = \sum_{k=0}^{N} \gamma_k (\hat{R}_k - \sum_i D_i e^{-\sqrt{A_i \Delta x_k^2 + B_i \Delta t_k^2}} \cos \omega_{xi} \Delta x_k \cos \omega_{ti} \Delta t_k)^2$$

(3.6)

is minimized with respect to the parameters A_i, B_i, D_i, ω_{xi}, and ω_{ti}. The choice of the weighting terms γ_k depend on the accuracy with which $\hat{R}_k$ can be estimated. In case of the first simulated process, where $\hat{R}_k$ is estimated only on the straight line scan path, the choice of γ_k was

$$\gamma_k = \begin{cases} (1 - \frac{k}{N}) & \text{for } k \leq N \\ 0 & \text{for } k > N \end{cases} \tag{3.7}$$

where increasing values of k represent increasing values of Δx_k and Δt_k. Minimizing the cost function of Equation (3.6) with analytical or numerical function minimization techniques normally leads to a local minimum instead of a global minimum. Some computer plots of this cost function are shown in Figures 9 and 10. Invariably, it is the frequency variables that prevent convergence to the global minimum. To minimize this problem, a technique described in the next section was devised to obtain an estimate of the frequencies present and these estimates were used as initial values of the frequency parameters in the function minimization methods.

Beveridge and Schechter (20) and Fletcher (21) give an excellent evaluation of both analytical and numerical techniques to be used in function minimization. Fletcher and Powell's algorithm (14) is used in this thesis since it is recommended by Fletcher (21) in cases where the first derivatives of F are available in the analytic form and the problem is medium sized and unconstrained. The convergence of this algorithm is superior to most other algorithms and the rate of convergence is also good. The algorithm is described briefly in the section entitled "Review of Literature."

Power Spectrum Analysis on the Scan Path

A spectrum analysis operation indicated in Figure 7 serves
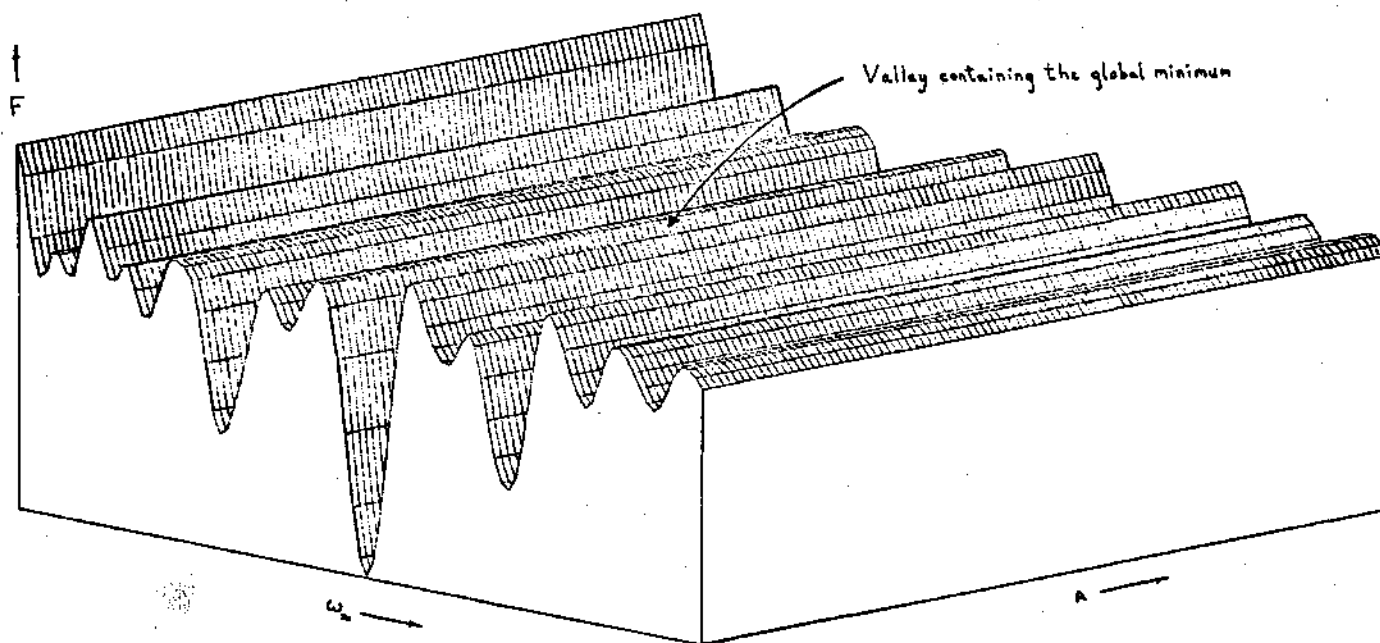
Figure 9. Cost as a Function of ω_x and A.
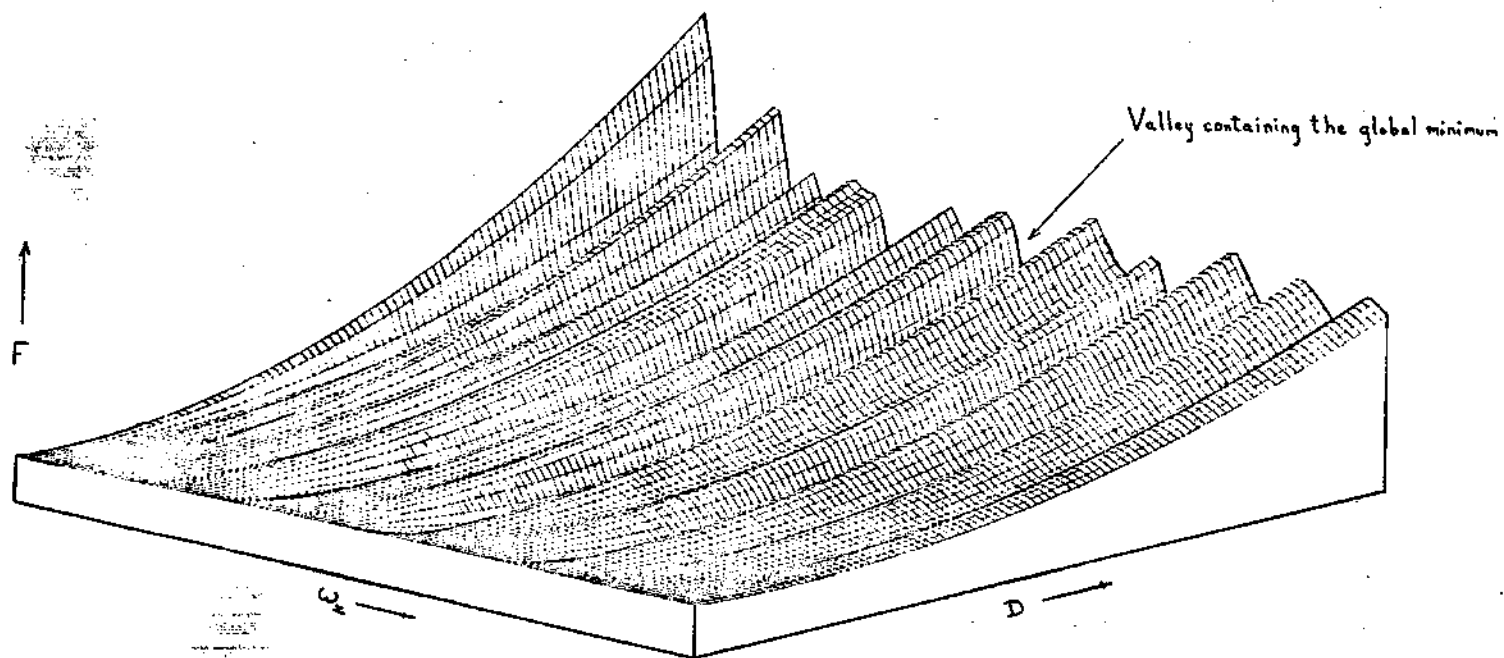
Figure 10.   Cost as a Function of ω_x and D.

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the function of furnishing the initial estimates of the frequencies ω_{xi} and ω_{ti} present in the process. If these frequencies are extracted to great accuracy, it may not even be necessary to include them in the functional fitting block as parameters, which would simplify the functional fitting procedure considerably.

Frequency Decomposition Technique

For the general case, consider the scan pattern to be periodic as shown in Figure 11 and described by the equation

$$x = f(t) \tag{3.8}$$

Let (x_o, t_{oi}), the starting point of each period in the x and t directions be represented by the vector y_i. Then, about any reference y_i in time

$$\Delta x \big|_{y_i} = F(\Delta t \big|_{y_i}) \tag{3.9}$$

where

$$\Delta t \big|_{y_i} = t - f^{-1}(y_i) \tag{3.10}$$

The superscript -1 stands for the inverse of a function. Now if the exponential decaying terms of the autocorrelation function are neglected for convenience

$$R(\Delta x, \Delta t) = \sum_i D_i \cos \omega_{xi} \Delta x \cos \omega_{ti} \Delta t \tag{3.11}$$
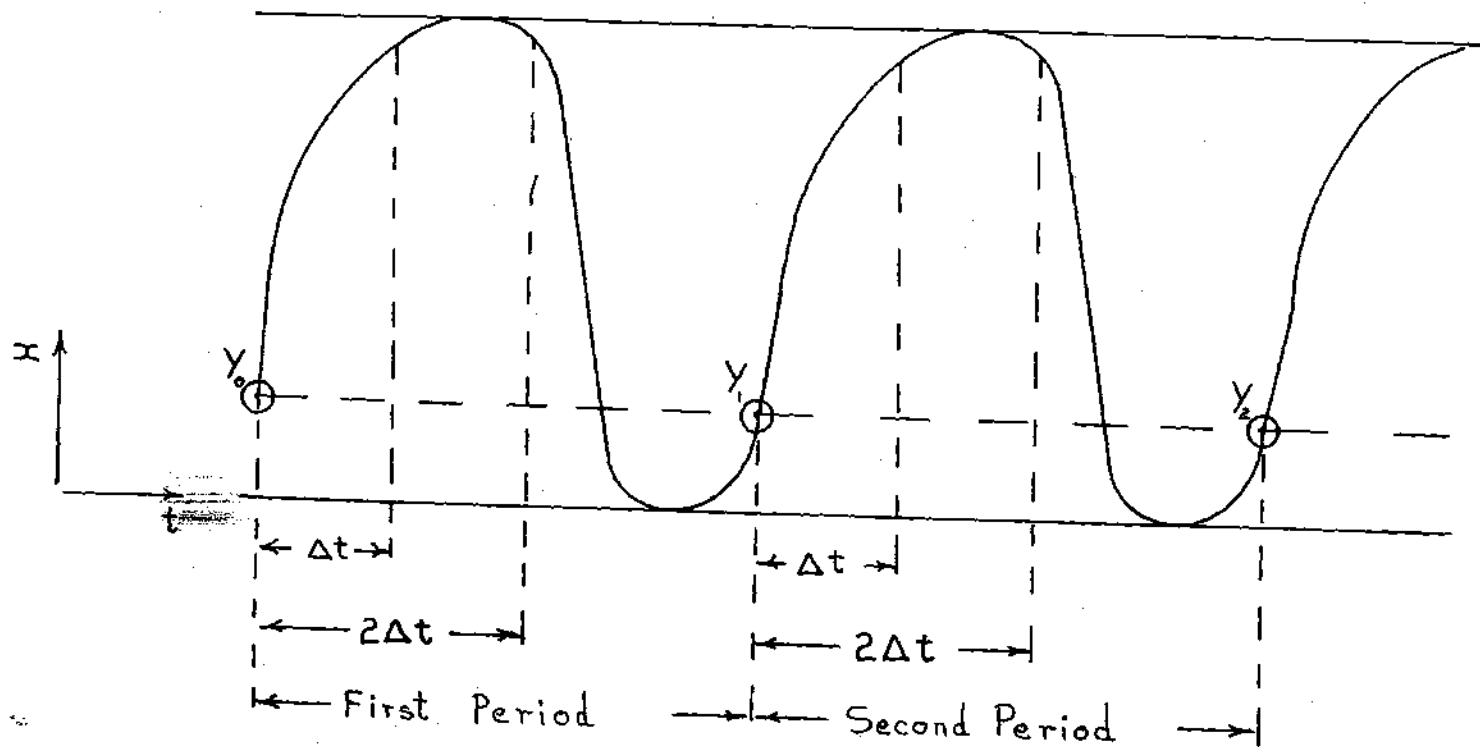
Figure 13. Periodic Scan Path.

$$= \sum_i \frac{D_i}{2} \left[\cos\{\omega_{xi}\Delta x + \omega_{ti}\Delta t\} + \cos\{\omega_{xi}\Delta x - \omega_{ti}\Delta t\} \right]$$

$$= \sum_i \frac{D_i}{2} \left[\cos\{\omega_{xi}F(\Delta t|_{y_i}) + \omega_{ti}\Delta t\} + \cos\{\omega_{xi}F(\Delta t|_{y_i}) - \omega_{ti}\Delta t\} \right]$$

$$= \sum_i \frac{D_i}{2} \left[\cos \Delta t|_{y_i}\{\omega_{xi}Q(\Delta t|_{y_i}) + \omega_{ti}\} + \cos \Delta t|_{y_i}\{\omega_{xi}Q(\Delta t|_{y_i}) - \omega_{ti}\} \right]$$

where

$$F(\Delta t) = \Delta t Q(\Delta t) \qquad (3.12)$$

By conducting an indirect power spectrum analysis over all $\Delta t|_{y_i}$ on the scan path, one is able to determine the frequencies ω_{si} on which the power peaks are concentrated. This power spectrum analysis can be performed by gathering the autocorrelation functions for all discrete $\Delta t|_{y_i}$ and then taking its Fourier Transform according to the procedure described by Blackman and Tukey (9). Since $\Delta t|_{y_i}$ is related to ω_s by

$$\Delta t|_{y_i} = \frac{2\pi}{\omega_s} \qquad (3.13)$$

the equations

$$\omega_{xi}Q(\frac{2\pi}{\omega_{si1}}) + \omega_{ti} = \omega_{si1} \qquad (3.14)$$

and

$$\omega_{xi}Q(\frac{2\pi}{\omega_{si2}}) - \omega_{ti} = \omega_{si2} \qquad (3.15)$$

can be solved for ω_{xi} and ω_{ti}. Grouping of the correct ω_{si1} and ω_{si2} may present a problem, which will be discussed further in the following simplified case.

Consider a simplified version of the general form where the scan path is a straight line scan as shown in Figure 12. In this case

$$\Delta x = f(\Delta t) = K\Delta t \qquad (3.16)$$

where K is constant. Therefore Equation (3.11) becomes

$$R(\Delta x, \Delta t) = \sum_i \frac{D_i}{2} \left[\cos(\Delta l \cos \theta)\left\{K\omega_{xi}+\omega_{ti}\right\} + \cos(\Delta l \cos\theta)\left\{K\omega_{xi}-\omega_{ti}\right\}\right]$$

$$(3.17)$$

where

$$\Delta t = \Delta l \cos \theta \qquad (3.18)$$

Conducting a spectrum analysis along just one straight line will obtain the values of ω_s. Again, solving the equations

$$K\omega_{xi} + \omega_{ti} = \omega_{si1}/\cos \theta \qquad (3.19)$$

and

$$K\omega_{xi} - \omega_{ti} = \omega_{si2}/\cos \theta \qquad (3.20)$$

one is able to obtain the values of ω_{xi} and ω_{ti}.

The pairing of ω_{si1} and ω_{si2} can be done easily if the power spectrum intensities can be distinguished readily as
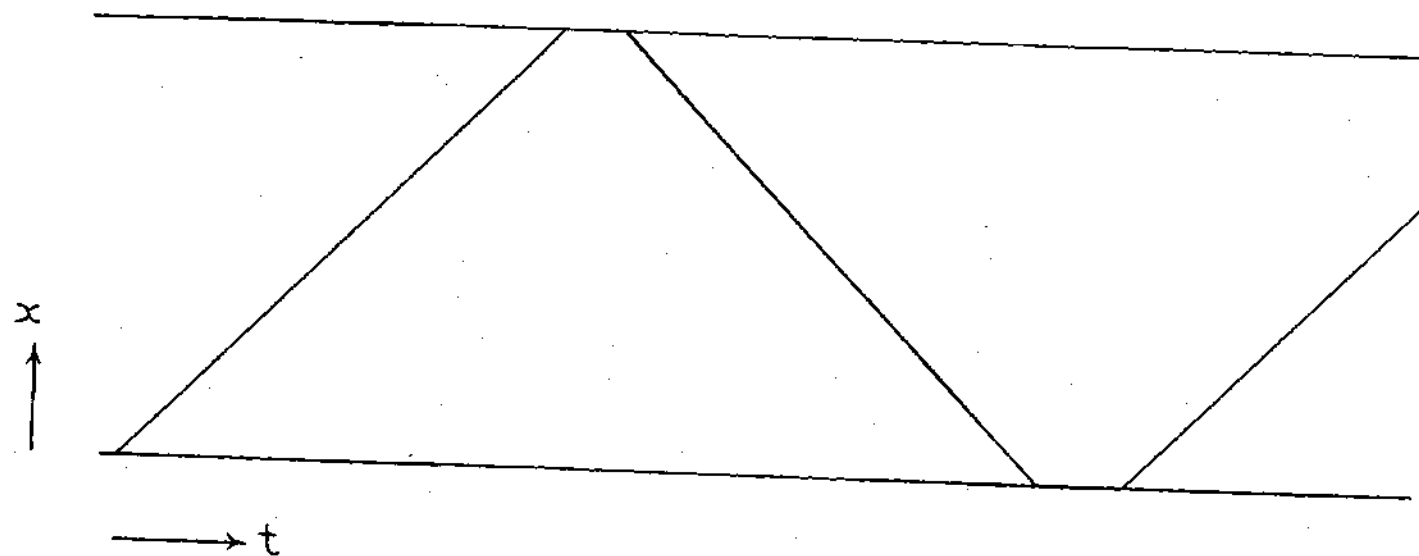
Figure 12. Straight Line Scan Path.

shown in Figure 13. This identification follows from Equation (3.17) which indicates that the power over any paired frequencies must be the same. If these intensities are not distinguishable then it is best to obtain the values of ω_{xi} and ω_{ti} by allowing all possible combinations and using these initial values in the functional fitting process of Fletcher and Powell. The correct combination must yield the lowest cost function. Also, if one suspects that wave lengths larger than half the scan wave length are present in the process, then it is necessary to extend this simplified version to the more general version described previously.

Confidence Bounds Around the Crude Autocorrelation Function

This section shows a practical way of placing 99.76 percent or (3σ) confidence bounds around the experimentally obtained autocorrelation function. The method uses Fisher's Z, a variation of which is asymptotically normally distributed with zero mean and unit variance. This method has an advantage over other methods since its asymptotic distribution function tends toward normality faster.

Past attempts to place confidence bounds around autocorrelation functions have been centered around assuming the distribution of the autocorrelation function $R(\tau)$ (for particular τ) as Gaussian. Examples of this can be found in (10) and (22). In both references, the authors state that it is very difficult to obtain the real distribution of the autocorrelation function and in cases in which the number of samples are large, a Gaussian
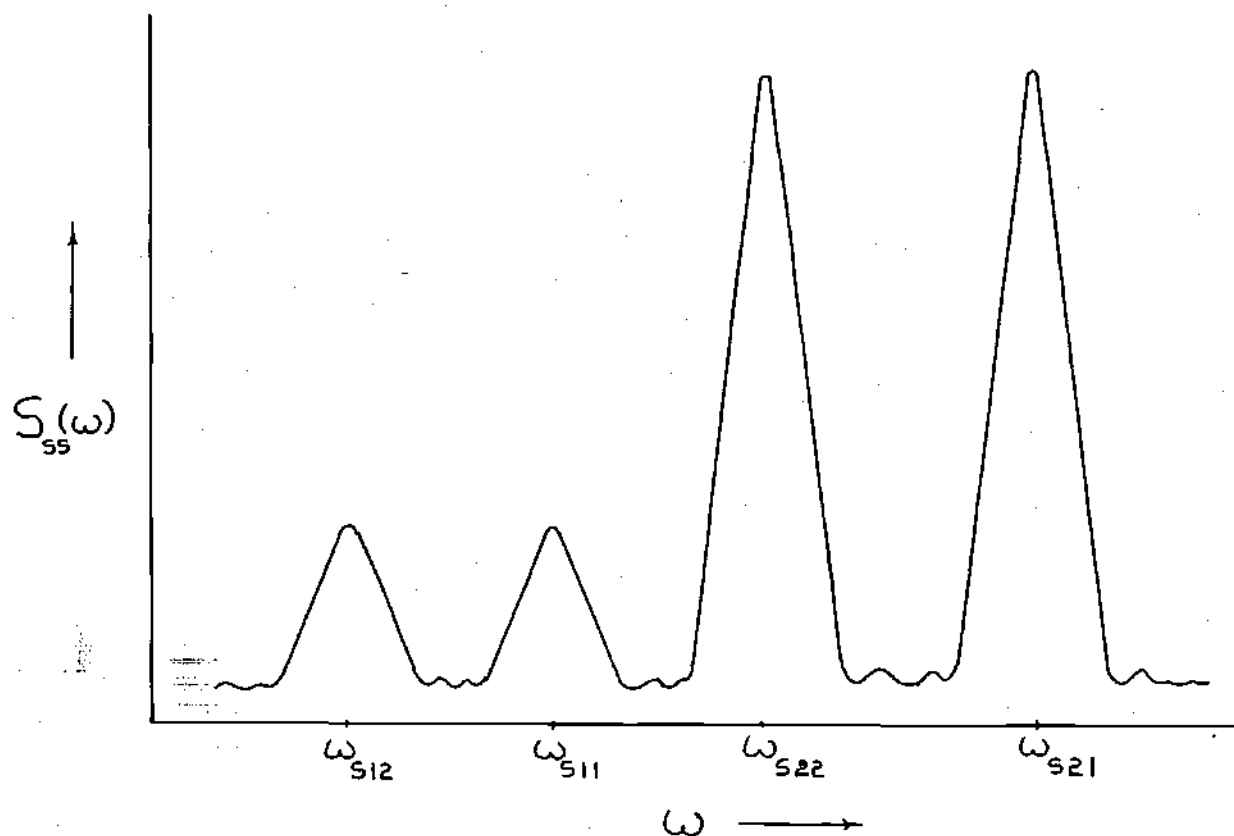
Figure 13.   Identification of Frequencies to be Paired.

approximation is valid. Under these circumstances, if the variance σ^2 of $R(\tau)$ and sample size n are known, then for 99.76 percent confidence the following probability statement is used:

$$Pr\left\{-3 < \frac{(\hat{R}(\tau) - R(\tau))\sqrt{n}}{\sigma} < +3\right\} = .9976 \qquad (3.21)$$

which results in

$$Pr\left\{\hat{R}(\tau) - 3\frac{\sigma}{\sqrt{n}} < R(\tau) < \hat{R}(\tau) + 3\frac{\sigma}{\sqrt{n}}\right\} = .9976 \qquad (3.22)$$

where $\hat{R}(\tau)$ is the experimentally obtained estimate of $R(\tau)$. Dr. J. J. Goode of Georgia Institute of Technology recommends the use of the so-called Fisher's Z, which has the important property of approaching normality faster than any other statistic of $R(\tau)$. In addition to this important property, the method has the advantage of not requiring any prior knowledge about σ^2.

Anderson in his book (23) has a section on "The Asymptotic Distribution of a Sample Correlation Coefficient and Fisher's Z." His approach uses the correlation coefficient $\hat{r}$, which in the present case is

$$\hat{r} = \hat{R}(\tau)/\hat{R}(0) \qquad (3.23)$$

The so-called Fisher's Z is

$$\hat{Z} = \frac{1}{2} \log_e \frac{1 + \hat{r}}{1 - \hat{r}} \qquad (3.24)$$

If $\hat{r}$ and $\hat{Z}$ are the estimates of r and Z, then according to Anderson, the statistic $\sqrt{n}\,(Z - \hat{Z})$ is asymptotically normally

distributed with zero mean and unit variance for n > 24. This results in the probability statement

$$\Pr\left\{-3 < \sqrt{(n-2)} \ (Z-\hat{Z}) < +3\right\} = .9976 \qquad (3.25)$$

the details of which are explained in (23). Thus, the confidence region for n > 24 is

$$\tanh(\hat{Z} - 3/\sqrt{n-2}) < \frac{R(\tau)}{R(0)} < \tanh(\hat{Z} + 3/\sqrt{n-2}) \qquad (3.26)$$

For n < 24, David's tables are used as given in the handbook by Beyer (24) under the title "Confidence Limits for the Population Correlation Coefficient." This case is treated in (23) and will not be considered further due to the large confidence bounds for small n.

Results

The scan path data for the first simulated process is analyzed by a computer program, and the values of the crude autocorrelation function are obtained as shown in Table 2. Also, a computer plot of the crude autocorrelation function on the scan path is shown in Figure 14. A subroutine of the same program places 99.76 percent confidence bounds around this crude autocorrelation function and the results are shown in Table 3 and Figure 15. The purpose of this confidence band is to bound the deviation of the fitted autocorrelation function from the crude autocorrelation function. Figure 16 shows the three dimensional plot of the desired autocorrelation function of the simulated

Table 2. Crude Autocorrelation Function
on Scan Path

| Δx	Δt	Crude $R_{ss}(\Delta x, \Delta t)$
.0000	.0000	1.9201
1.0000	1.0000	1.5319
2.0000	2.0000	1.0589
3.0000	3.0000	.5998
4.0000	4.0000	.2638
5.0000	5.0000	.0888
6.0000	6.0000	.0879
7.0000	7.0000	.2021
8.0000	8.0000	.3823
9.0000	9.0000	.4976
10.0000	10.0000	.5390
11.0000	11.0000	.5355
12.0000	12.0000	.4443
13.0000	13.0000	.3062
14.0000	14.0000	.1345
15.0000	15.0000	-.0288
16.0000	16.0000	-.1196
17.0000	17.0000	-.0935
18.0000	18.0000	-.0348
19.0000	19.0000	-.0114
20.0000	20.0000	.0683
21.0000	21.0000	.1296
22.0000	22.0000	.1588
23.0000	23.0000	.1767
24.0000	24.0000	.1901
25.0000	25.0000	.1557
26.0000	26.0000	.1171
27.0000	27.0000	.0588
28.0000	28.0000	-.0265
29.0000	29.0000	-.0889
30.0000	30.0000	-.1132
31.0000	31.0000	-.0808
32.0000	32.0000	-.0177
33.0000	33.0000	-.1159
34.0000	34.0000	.1865
35.0000	35.0000	.2478
36.0000	36.0000	.2723
37.0000	37.0000	.2830
38.0000	38.0000	.2921
39.0000	39.0000	.3221
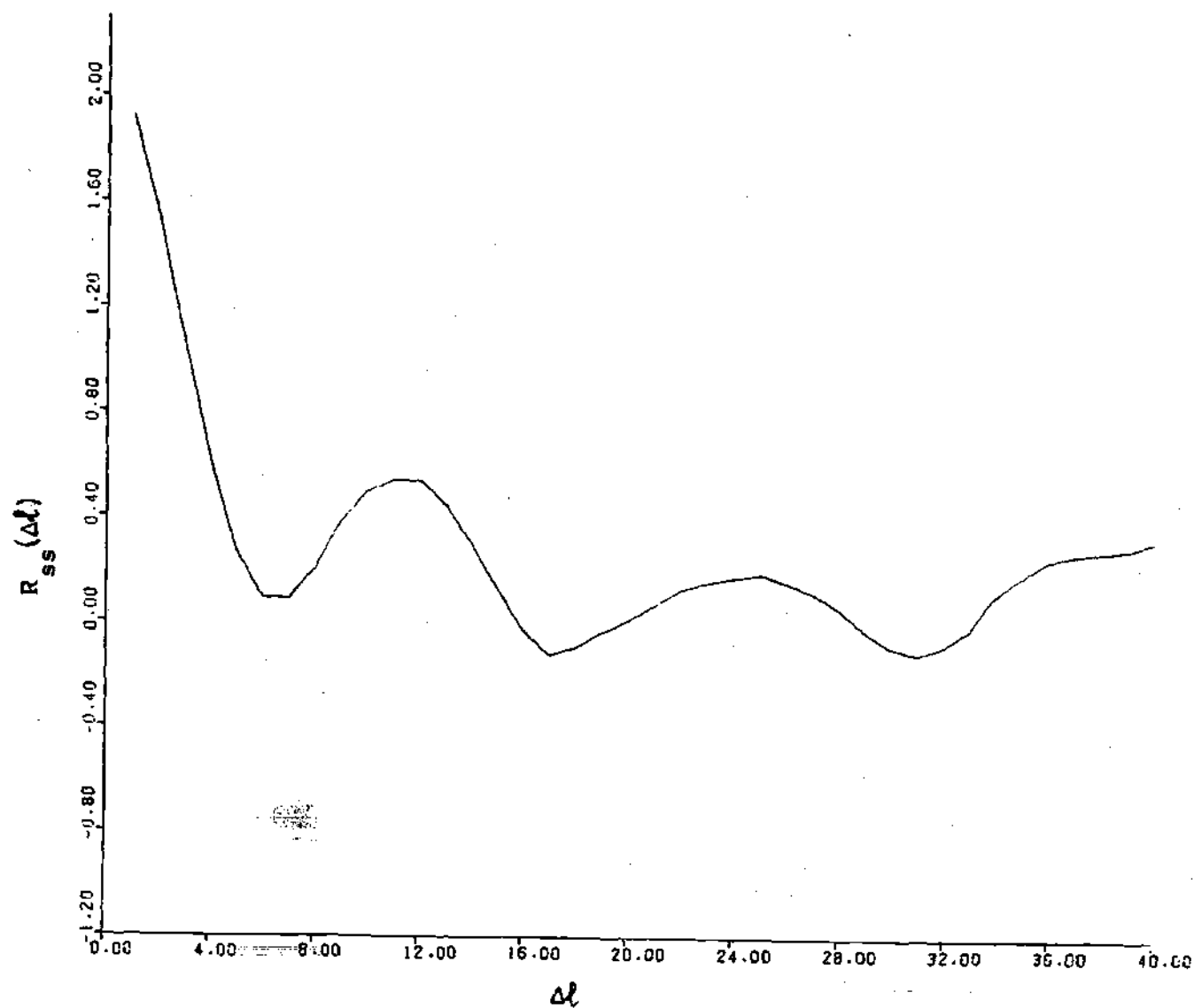
Figure 14. Crude Autocorrelation Function on the Scan Path.

Table 3. Confidence Bounds of 99.76 Percent Around
the Crude Autocorrelation Function.

Lower Bound	Crude Autocorrelation Function	Upper Bound	Sample Size	Total Bound
1.46505018	1.53193997	1.58995549	1131.00000000	.12490532
.93206641	1.05890194	1.17371573	1102.00000000	.24164931
.43673472	.59978456	.75378090	1073.00000000	.31704618
.08693592	.26376214	.43614284	1044.00000000	.34920692
-.09203517	.08881197	.26809359	1015.00000000	.36012876
-.09554544	.08794269	.26983513	986.00000000	.36538057
.01642429	.20206959	.38397152	957.00000000	.36754723
.19748273	.38230004	.56002388	928.00000000	.36254116
.31400017	.49756379	.67186905	899.00000000	.35786889
.35429088	.53903892	.71355214	870.00000000	.35926126
.34734263	.53550146	.71313174	841.00000000	.36578912
.24867198	.44427593	.63059891	812.00000000	.38192693
.10263906	.30621686	.50296713	783.00000000	.40032808
-.07529238	.13450144	.34111679	754.00000000	.41640917
-.24169482	-.02875719	.18489037	725.00000000	.42658519
-.33490502	-.11955545	.09885662	696.00000000	.43376164
-.31407802	-.09348002	.12961979	667.00000000	.44369781
-.26161317	-.03483954	.19291061	638.00000000	.45452377
-.22142622	.01138372	.24385934	609.00000000	.46528555
-.17078850	.06832306	.30533139	580.00000000	.47611989
-.11592789	.12957329	.37089099	551.00000000	.48681888
-.09338210	.15878248	.40555001	522.00000000	.49893212
-.08271195	.17669000	.42974603	493.00000000	.51245797
-.07725686	.19006595	.45014897	464.00000000	.52740583
-.12060931	.15570685	.42567993	435.00000000	.54628924
-.16887551	.11712237	.39799714	406.00000000	.56687265
-.23746530	.05875015	.35219262	377.00000000	.58965792
-.33276956	-.02649046	.28114299	348.00000000	.61391255
-.40627377	-.08891622	.23338583	319.00000000	.63965961
-.44452581	-.11317883	.22507371	290.00000000	.66959952
-.43124995	-.08075983	.27520611	261.00000000	.70645606
-.39190006	-.01766524	.35791673	232.00000000	.74981679
-.28807804	.11588328	.50980415	203.00000000	.79788219
-.25071397	.18646841	.60496660	174.00000000	.85568058
-.23140337	.24776986	.69748729	145.00000000	.92889065
-.26364613	.27232229	.76821920	116.00000000	1.03186531
-.33624469	.28295992	.84732979	87.00000000	1.18357447
-.46588112	.29209390	.96707698	58.00000000	1.43295810
-.74265748	.32206067	1.21572350	29.00000000	1.95838097
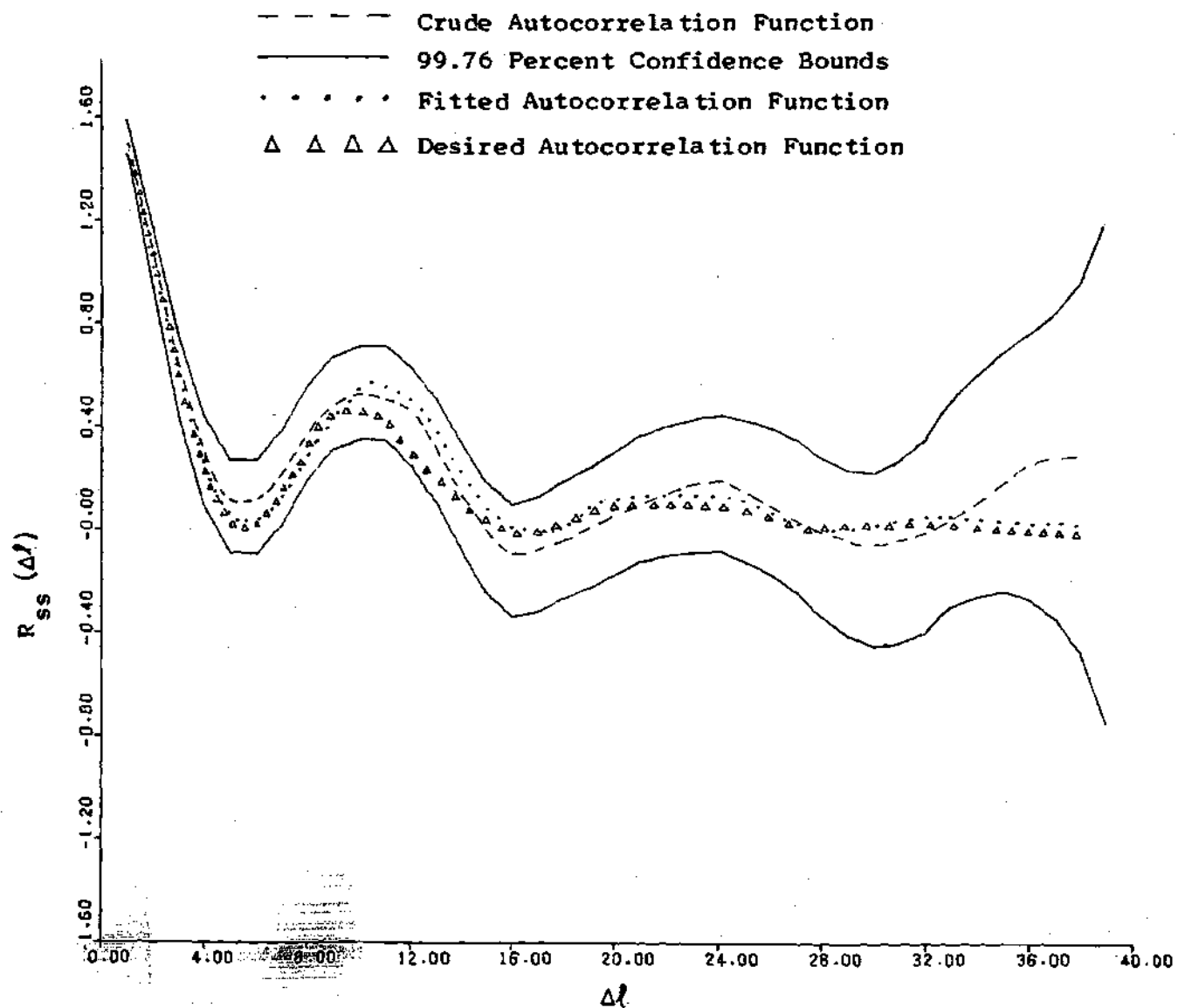
Figure 15. Confidence Bounds of 99.76 Percent Around the Crude Autocorrelation
Function along with the Fitted Autocorrelation Function and the
Desired Autocorrelation Function.
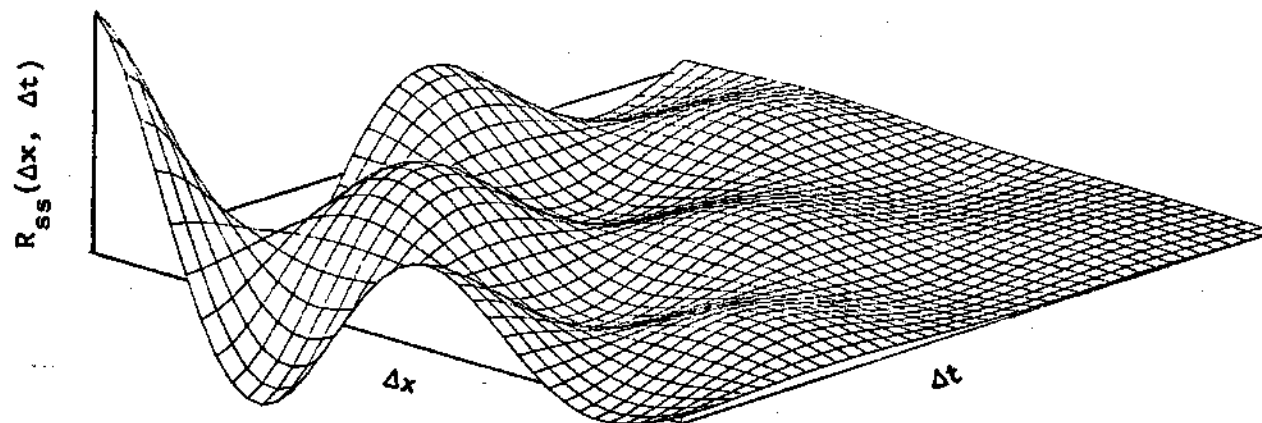
Figure 16.   Three-Dimensional Plot of the Desired
             Autocorrelation Function.

process, while Table 4 and Figure 17 show the same autocorrelation function but on the scan path only.

The indirect power spectrum analysis is then performed on the crude autocorrelation function using appropriate lag filters such as Bartlett, Hamming and Hanning windows (9) and the results obtained are shown in Tables 5 and 6 and Figures 18 and 19. The two predominant peaks occur at ω_{s1} = .0 and ω_{s2} = .39. Applying the analysis described in an earlier section

$$\omega_{x1} + \omega_{t1} = .39/\cos 45$$

and

$$\omega_{x1} - \omega_{t1} = .0/\cos 45$$

which yields ω_{x1} = .28 and ω_{t1} = .28. These compare well with the original ω_{x1} = .3 and ω_{t1} = .3, especially since the least count of the angular frequency is 0.05552 in Table 5. The value of the least count is fixed from the expression for the raw power spectrum (9)

$$S_{ss}(\frac{2\pi j}{2m\Delta\tau}) = \Delta\tau[R_{ss}(0) + 2\sum_{r\Delta\tau=1}^{n-1} R_{ss}(r\Delta\tau)\cos(\frac{rj\pi}{m}) + R_{ss}(m\Delta\tau)\cos(j\pi)]$$

$$\text{for } j = 0,1,2,\ldots,m \qquad (3.27)$$

where m is equal to the number of autocorrelation function values available and $\Delta\tau$ is the spacing interval in time. In the present case m = 40 and $\Delta\tau = \sqrt{2}$ and, therefore, the least count is .05552

Table 4. Desired Autocorrelation Function on Scan Path

Δx	Δt	Desired R_{ss} (Δx, Δt)
.0000	.0000	2.00000
1.0000	1.0000	1.57108
2.0000	2.0000	1.00926
3.0000	3.0000	.49276
4.0000	4.0000	.14412
5.0000	5.0000	.00473
6.0000	6.0000	.04197
7.0000	7.0000	.17838
8.0000	8.0000	.32755
9.0000	9.0000	.42378
10.0000	10.0000	.43737
11.0000	11.0000	.37454
12.0000	12.0000	.26586
13.0000	13.0000	.14995
14.0000	14.0000	.05887
15.0000	15.0000	.00937
16.0000	16.0000	.00139
17.0000	17.0000	.02231
18.0000	18.0000	.05415
19.0000	19.0000	.08061
20.0000	20.0000	.09180
21.0000	21.0000	.08568
22.0000	22.0000	.06661
23.0000	23.0000	.04225
24.0000	24.0000	.02022
25.0000	25.0000	.00565
26.0000	26.0000	.00012
27.0000	27.0000	.00207
28.0000	28.0000	.00809
29.0000	29.0000	.01447
30.0000	30.0000	.01844
31.0000	31.0000	.01883
32.0000	32.0000	.01596
33.0000	33.0000	.01120
34.0000	34.0000	.00622
35.0000	35.0000	.00237
36.0000	36.0000	.00034
37.0000	37.0000	.00008
38.0000	38.0000	.00104
39.0000	39.0000	.00242

Table 5. Raw Power Spectrum and Power Spectrum
Using Bartlett Window

Raw Spectrum	Bartlett Smoothed Spectrum	Plot Abs- cissa	Frequency	Angular Fre- quency
27.3013	19.1250	1	.00000	.00000
14.1386	15.6754	2	.00884	.05552
13.5794	10.9201	3	.01768	.11103
4.8372	7.4121	4	.02652	.16655
6.4526	5.2429	5	.03536	.22207
1.4038	4.4618	6	.04420	.27758
8.2260	7.6669	7	.05304	.33310
14.5298	10.4950	8	.06188	.38861
7.0040	7.9864	9	.07072	.44413
7.0644	5.6755	10	.07956	.49965
2.1334	3.3852	11	.08840	.55516
2.1179	2.2308	12	.09724	.61068
1.9550	2.1009	13	.10608	.66620
1.6928	1.6099	14	.11492	.72171
.7425	1.1019	15	.12376	.77723
.6786	.7208	16	.13260	.83274
.1490	.5934	17	.14144	.88826
.8998	.7607	18	.15028	.94378
.7096	.8095	19	.15912	.99929
.7408	.7105	20	.16796	1.05481
.4432	.5354	21	.17680	1.11033
.2940	.4764	22	.18564	1.16584
.6305	.4849	23	.19448	1.22136
.1823	.4554	24	.20332	1.27687
.6593	.3561	25	.21216	1.33239
-.3350	.2232	26	.22100	1.38791
.7005	.3145	27	.22984	1.44342
-.0256	.3991	28	.23868	1.49894
.8052	.3627	29	.24752	1.55446
-.2224	.3492	30	.25636	1.60997
.9210	.3340	31	.26521	1.66549
-.4070	.2884	32	.27405	1.72100
.9185	.2469	33	.28289	1.77652
-.6034	.2296	34	.29173	1.83204
1.0773	.2507	35	.30057	1.88755
-.7424	.2122	36	.30941	1.94307
1.1708	.2928	37	.31825	1.99859
-.5804	.2932	38	.32709	2.05410
1.0885	.2892	39	.33593	2.10962
-.5348	.2863	40	.34477	2.16513

Table 6.   Hamming and Hanning Smoothed
Power Spectra

Plot Abscissa	Hamming Smoothed Spectrum	Hamming Smoothed Spectrum	Raw Spectrum
2	17.28947	17.03740	14.13860
3	11.53364	11.69730	13.57939
4	7.42658	7.21943	4.83717
5	4.78654	4.91982	6.45260
6	4.37153	4.13411	1.40378
7	8.09638	8.10675	8.22595
8	11.07241	11.34901	14.52985
9	8.90057	8.74884	7.00400
10	5.81656	5.91639	7.06443
11	3.36228	3.26397	2.13338
12	2.08107	2.08402	2.11793
13	1.93020	1.93219	1.95503
14	1.52080	1.53456	1.69281
15	.96413	.94640	.74253
16	.56220	.57151	.67863
17	.46911	.44350	.14900
18	.66454	.68336	.89980
19	.76492	.76049	.70956
20	.65857	.66515	.74077
21	.48028	.47731	.44320
22	.41540	.40568	.29396
23	.43431	.45000	.63047
24	.41361	.39511	.18235
25	.29149	.32091	.65929
26	.17246	.13187	-.33497
27	.26011	.29534	.70050
28	.36364	.33250	-.02558
29	.34062	.37779	.80521
30	.32037	.27695	-.22237
31	.30316	.35259	.92099
32	.25639	.20332	-.40697
33	.20667	.26361	.91850
34	.19726	.13321	-.60336
35	.20218	.27219	1.07725
36	.19080	.11614	-.74241
37	.25468	.32797	1.17077
38	.27461	.20621	-.58040
39	.26543	.33128	1.08848
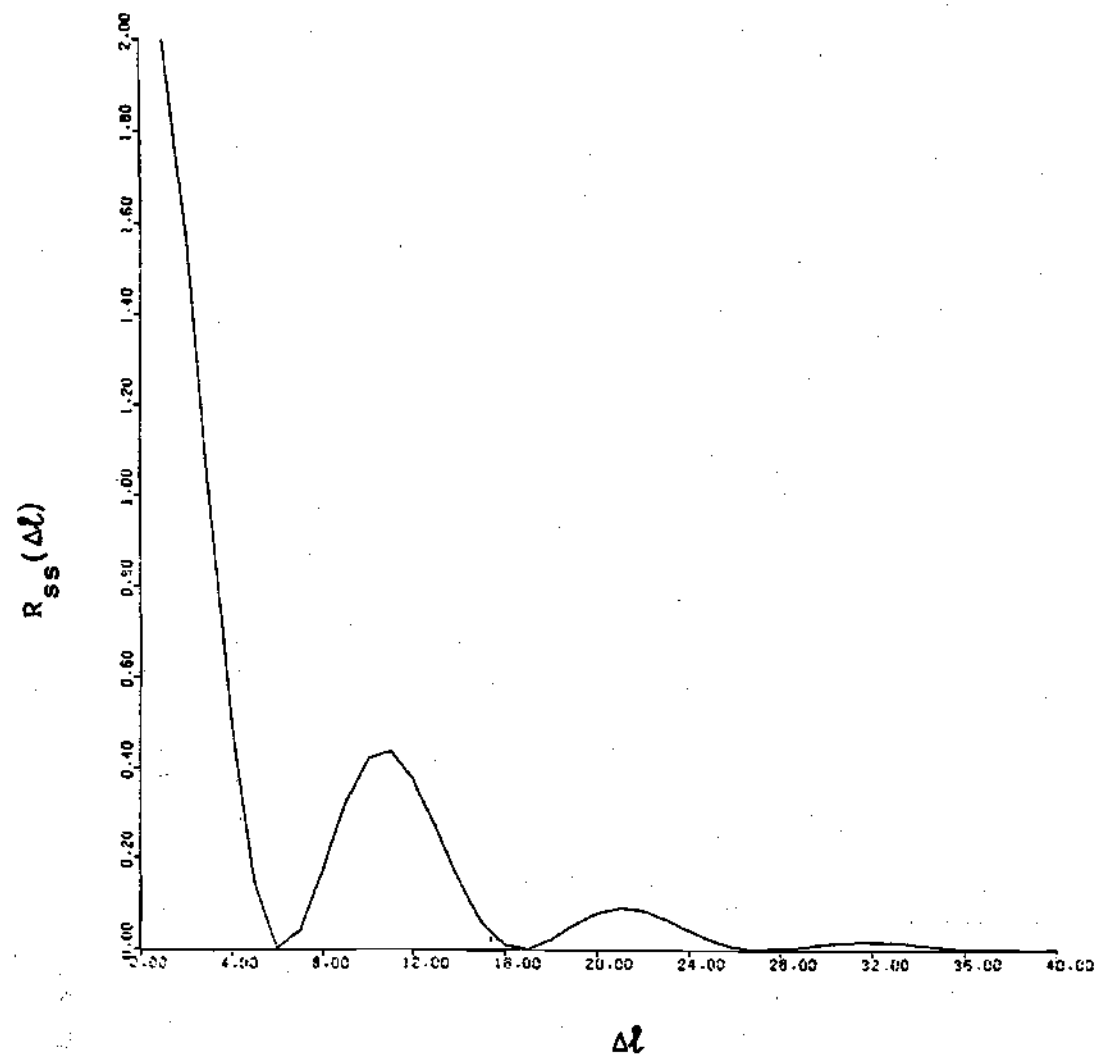
Figure 17. Desired Autocorrelation Function on the Scan Path.
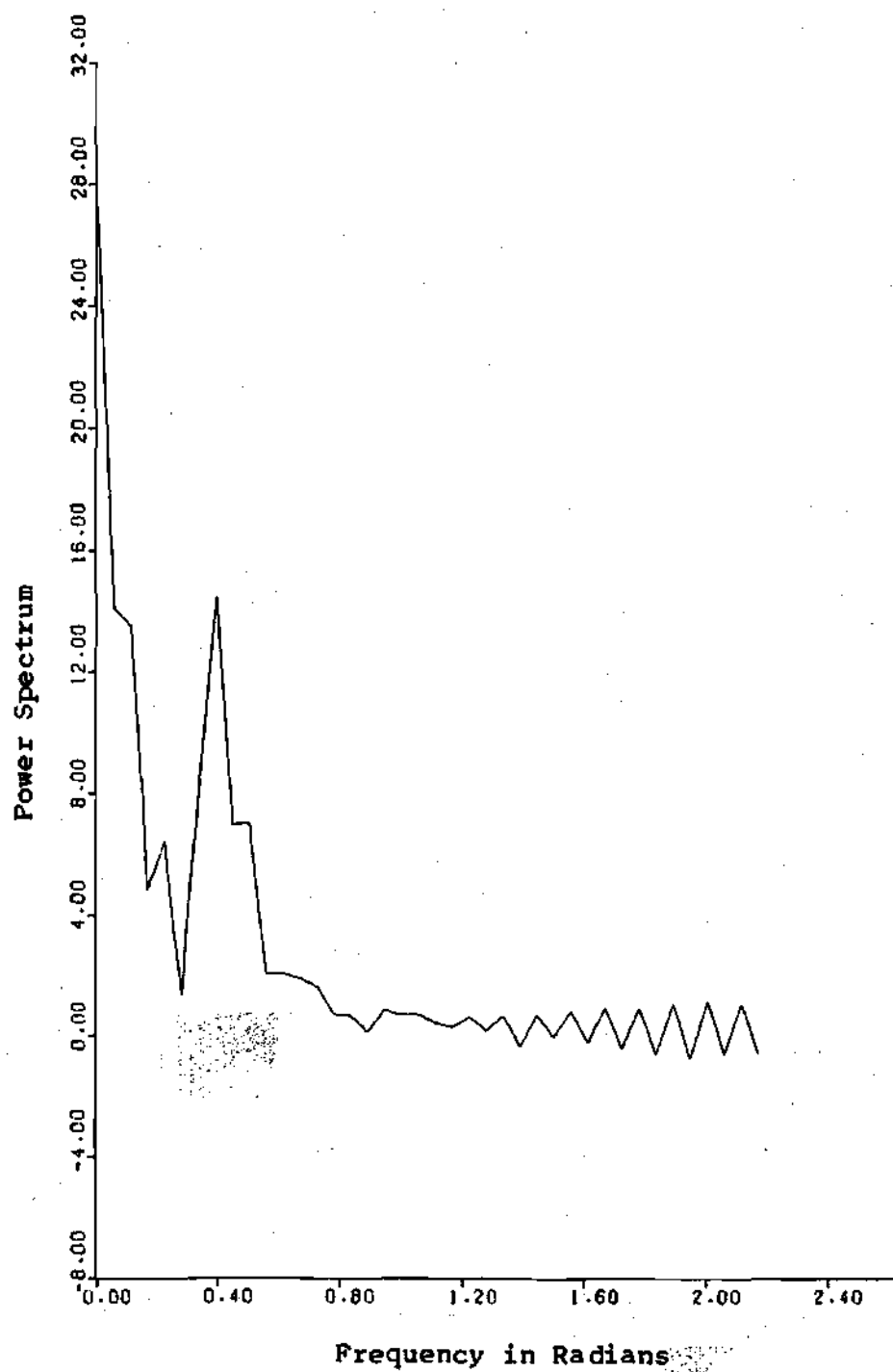
Figure 18.   Raw Power Spectrum from Scan Path
Data of Process 1.
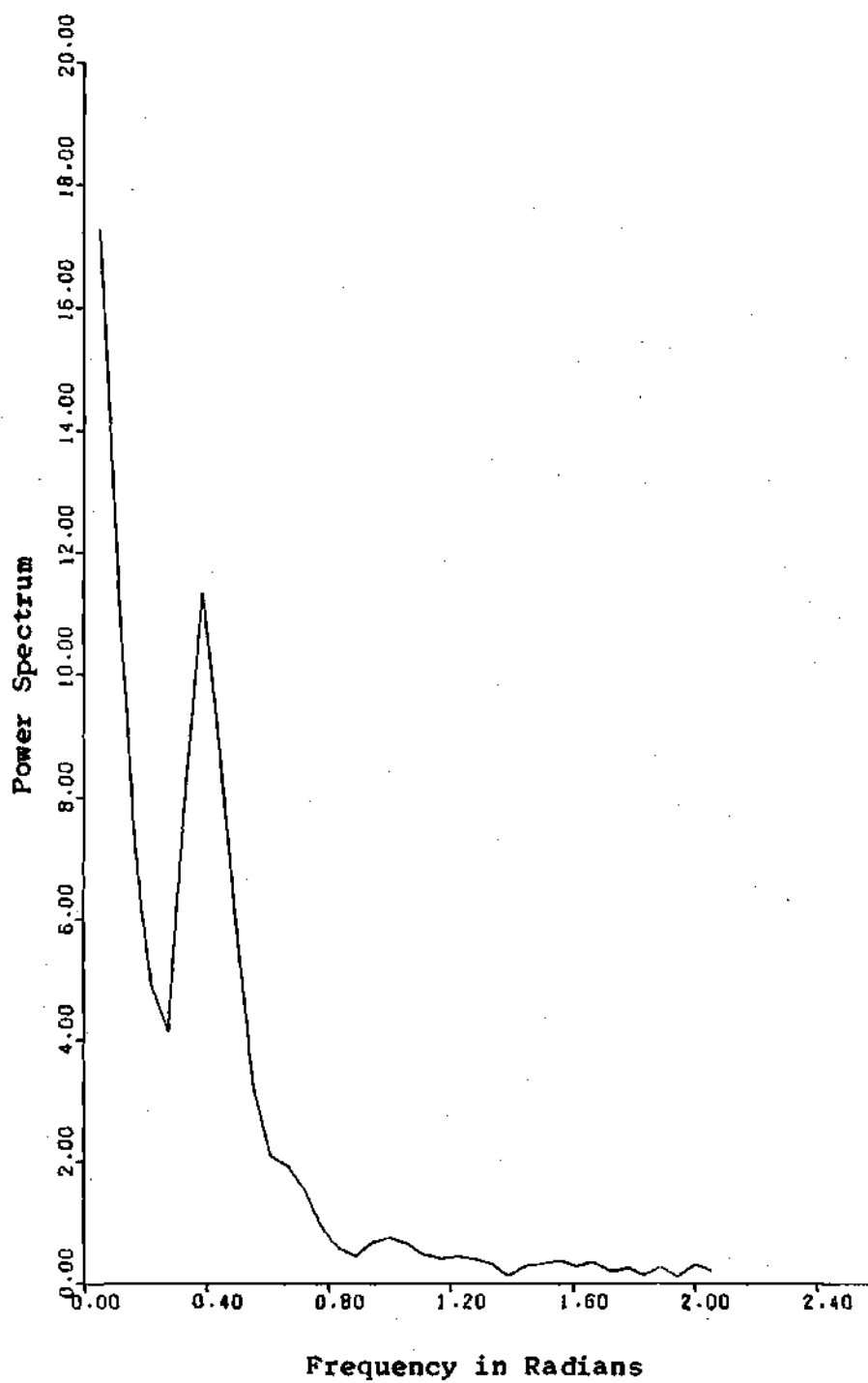
Figure 19. Hamming Smoothed Power Spectrum from
Scan Path Data of Process 1.

for the angular frequency.

These frequencies are used as initial values in the functional fitting program of Fletcher and Powell along with several arbitrary initial values for the other parameters. The minimum cost of .444 is obtained in 25 iterations yielding the final parameter values as $A_1 = .054$, $B_1 = .054$, $D_1 = 1.87$, $\omega_{x1} = .277$ and $\omega_{t1} = .277$. The values of the fitted autocorrelation function fall well within the 99.76 percent confidence band limits of the crude autocorrelation function.

The power spectrum analysis of the second simulated process is performed in the same manner as the first, and the result is illustrated in Figures 20 and 21. Three distinct peaks are identified at ω_s equal to .0555, .2775 and .444, one of which is approximately double the size of the other two. This suggests that ω_s equal to .0555 is contained in both the frequency pairs. Setting

$$\omega_{x1} + \omega_{t1} = .2775/\cos 45$$

$$\omega_{x1} - \omega_{t1} = .0555/\cos 45$$

and

$$\omega_{x2} + \omega_{t2} = .444/\cos 45$$

$$\omega_{x2} - \omega_{t2} = .0555/\cos 45$$

yields $\omega_{x1} = .235$, $\omega_{t1} = .157$, $\omega_{x2} = .35$ and $\omega_{t2} = .275$. The frequencies contained in the simulated process were $\omega_{x1} = .165$,
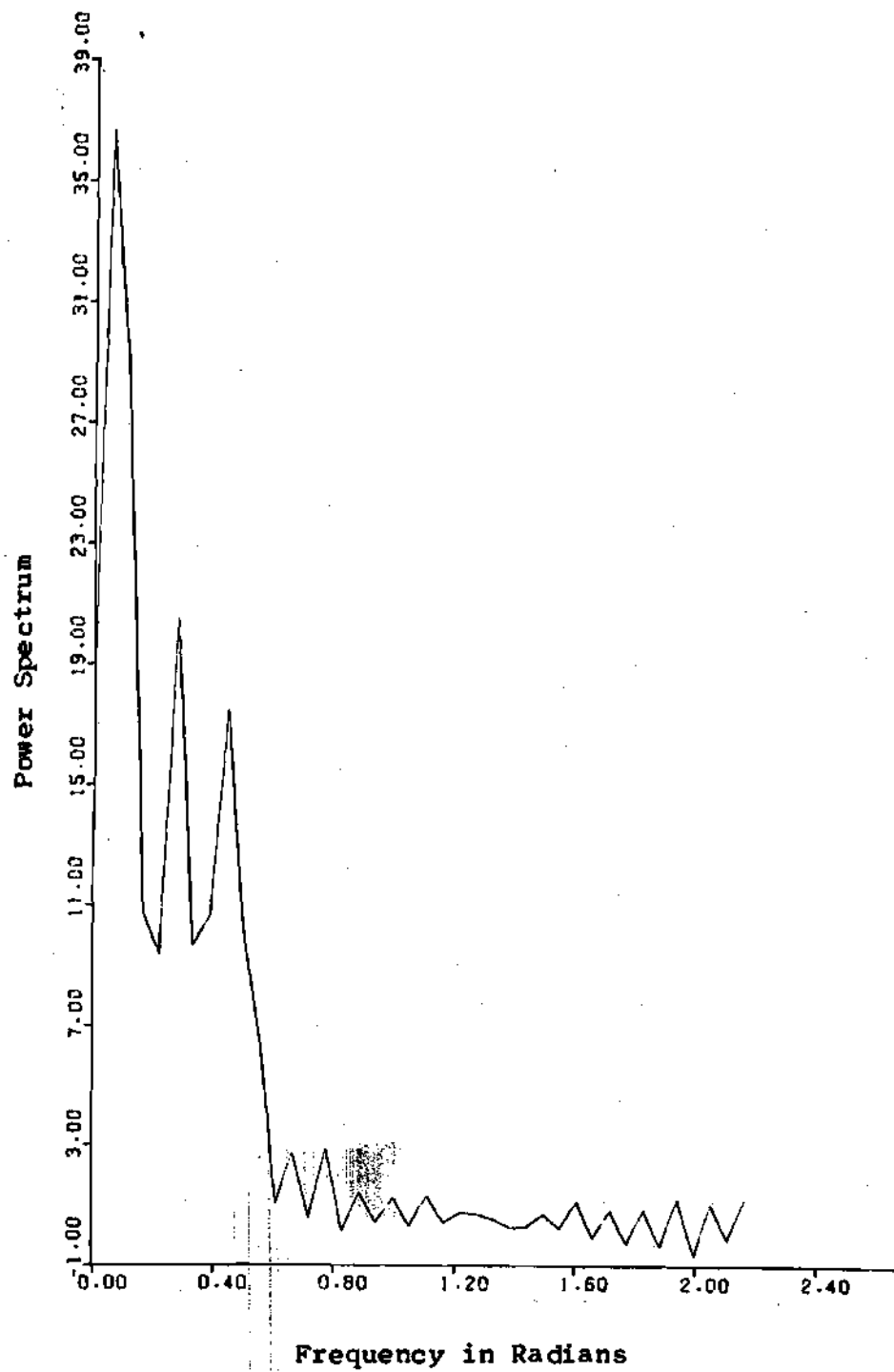
Figure 20. Raw Power Spectrum from Scan Path Data of Process 2.
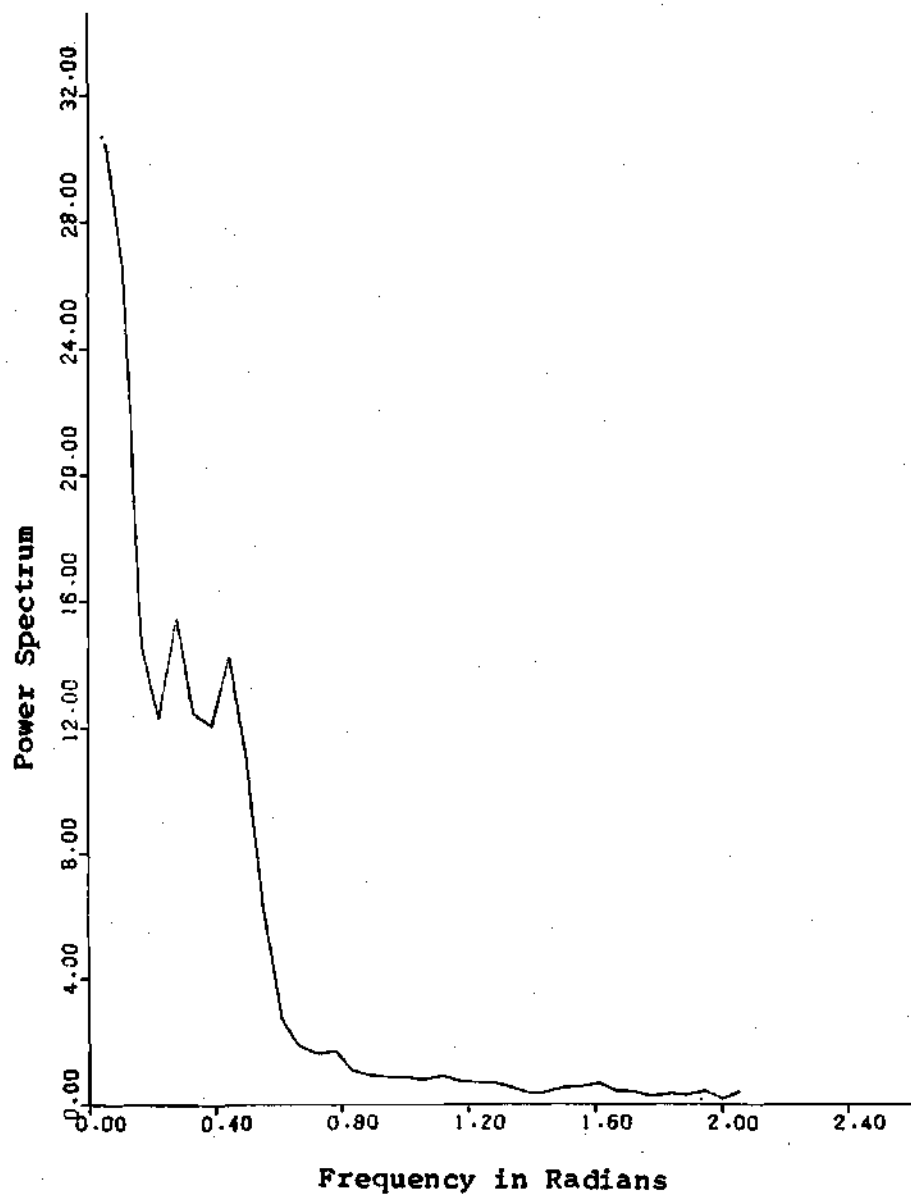
Figure 21. Hamming Smoothed Power Spectrum from
Scan Path Data of Process 2.

$\omega_{t1} = .165$, $\omega_{x2} = .3$ and $\omega_{t2} = .3$. The results obtained are
sufficiently close to enable the fitting process to converge
to its global minimum. Note that if ω_{s11} and ω_{s21} were 0.00,
instead of .0555, then $\omega_{x1} \approx .196$, $\omega_{t1} = .196$, $\omega_{x2} = .315$ and
$\omega_{t2} = .315$.

Discussion of the Results and Conclusions

In Chapter II two random processes were generated; however,
at that stage no tools were available to check whether the
processes had the desired autocorrelation function. In this
chapter it has been shown with the help of Figure 15 that the
desired autocorrelation function falls within the 99.76 percent
confidence bounds of the crude autocorrelation function. This
result offers experimental backing to the filter synthesizing
theory of Chapter II. However, this does not mean that there is
no room for improvements. The approximation

$$\int_j^{j+1} \int_i^{i+1} W(x,t)\,dx\,dt = W(i + \tfrac{1}{2},\ j + \tfrac{1}{2}) \qquad (3.28)$$

is made in Equation (2.15) to avoid the excessive computations
which arise if the mean value of each interval is to be found.
Perhaps there are better assumptions than the one made above.
Also, in Figure 15, the fluctuations of the crude autocorrela-
tion function around the desired autocorrelation function in-
crease as Δl increases. This was expected since the number of
samples available to make an estimate decrease as Δl increases.

The frequency decomposition technique performed remarkably

well, especially since it extracted the frequencies present in process 1 within 7.0 percent of the original values. Also, the identification of the frequency pairs was relatively easy, as shown in Figures 20 and 21 for process 2.

The results show that the fitted autocorrelation function has parameter values that do not exactly match the original values. This is to be expected since the fitting process is being performed on the crude autocorrelation function and not the real autocorrelation function. One of the two criteria used in judging the merits of the fitting process is that the fitted autocorrelation must fall within the 99.76 percent confidence band around the crude autocorrelation function. This is satisfied as shown in Figure 15. The other criterion is its effectiveness in estimation, which will be discussed in the next chapter.

CHAPTER IV

TWO-DIMENSIONAL ESTIMATION AND AN EVALUATION OF

THE EXPECTED VALUE OF THE SQUARED ERROR

In the last chapter a method to determine the closed
functional form of the autocorrelation function has been des-
cribed. This autocorrelation function is to be used in the
estimation of the signal. Fortunately, signal estimation given
the statistics of the process has been investigated by many
researchers since the time of Wiener (25) and the methods suited
for use in this thesis have already been developed. Peterson
and Middleton (5) have investigated the case of stationary multi-
dimensional estimation, while M. Blum (6) has obtained the opti-
mum linear estimator for a one-dimensional nonstationary random
process, the nonstationarity being similar to the one under con-
sideration in this thesis. With a slight modification, both of
these methods could be adapted to the estimation of the two-
dimensional non-stationary signal. This chapter is primarily
designed to show these modifications and to develop a way of
checking the effectiveness of the extrapolation and smoothing
technique of Chapter III.

Modification to the Peterson and Middleton Estimator

The Peterson and Middleton estimator is of the form

$$\hat{s}(y) = \sum_{k=1}^{M} s(Y_k)w(Y - Y_k) \tag{4.1}$$

where the weighting term $w(Y - Y_k)$ must satisfy

$$R_{ss}(Y - Y_r) = \sum_{k=1}^{M} R_{ss}(Y_r - Y_k)w(Y - Y_k) \quad r=1,2,\ldots,M \tag{4.2}$$

Equations (4.1) and (4.2) are subject to the condition

$$M \leq N \tag{4.3}$$

where N represents all the points in the sampling lattice.
The vector Y is described by

$$Y = [x,t] \tag{4.4}$$

To use linear algebra computation techniques for solving simultaneous equations, Equation (4.2) is transformed to its matrix form

$$\underset{(M \times 1)}{W} = \underset{(M \times M)}{V^{-1}} \quad \underset{(M \times 1)}{U} \tag{4.5}$$

where W is a column vector with elements $w(Y-Y_k)$, V is a square matrix with elements $R_{ss}(Y_r-Y_k)$ and U is a column vector with elements $R_{ss}(Y-Y_r)$. Normally Equation (4.1) would be solved on-line, whereas Equation (4.5) would be solved off-line. For cases in which M and N are the same, the estimation of K points on any cross-section in the x direction requires

K(M)	on-line multiplications
K(M)	on-line additions
K(M^2)L	off-line multiplications
K(M^2)L	off-line additions
K(L)	off-line inversions of a (MxM) matrix

where L denotes the total number of points scanned on the forward and reverse stroke of the scanning gauge. The on-line storage requirement is for M+(MLK) values.

As can be seen from the above numbers, the choice of M can result in a considerable savings in computation time. The criterion for the choice of M is a term known as the Figure of Merit, which determines the change in ε obtained by removing the rth point in the sampling lattice. The Figure of Merit is described by

$$\varepsilon_{min}^{(N-1)} - \varepsilon_{min}^{(N)} = \frac{[w(Y-Y_r)]^2 |V|}{|V_{rr}|} \qquad (4.6)$$

where the rth point is the one being tested for possible elimination, $|V|$ is the determinant of the N by N matrix of V and $|V_{rr}|$ is the determinant of the N-1 by N-1 matrix of V after the removal of the rth row and column. If the figure of merit is below a certain threshold value, the rth point is discarded. In this way the M points to be retained are determined.

To handle the nonstationarity of this problem, consider the scheme shown in Figure 22. The nonrandom part p(x) is estimated by performing exponential smoothing on the data for discrete values of x in the machine direction t. Mathematically, this is represented by
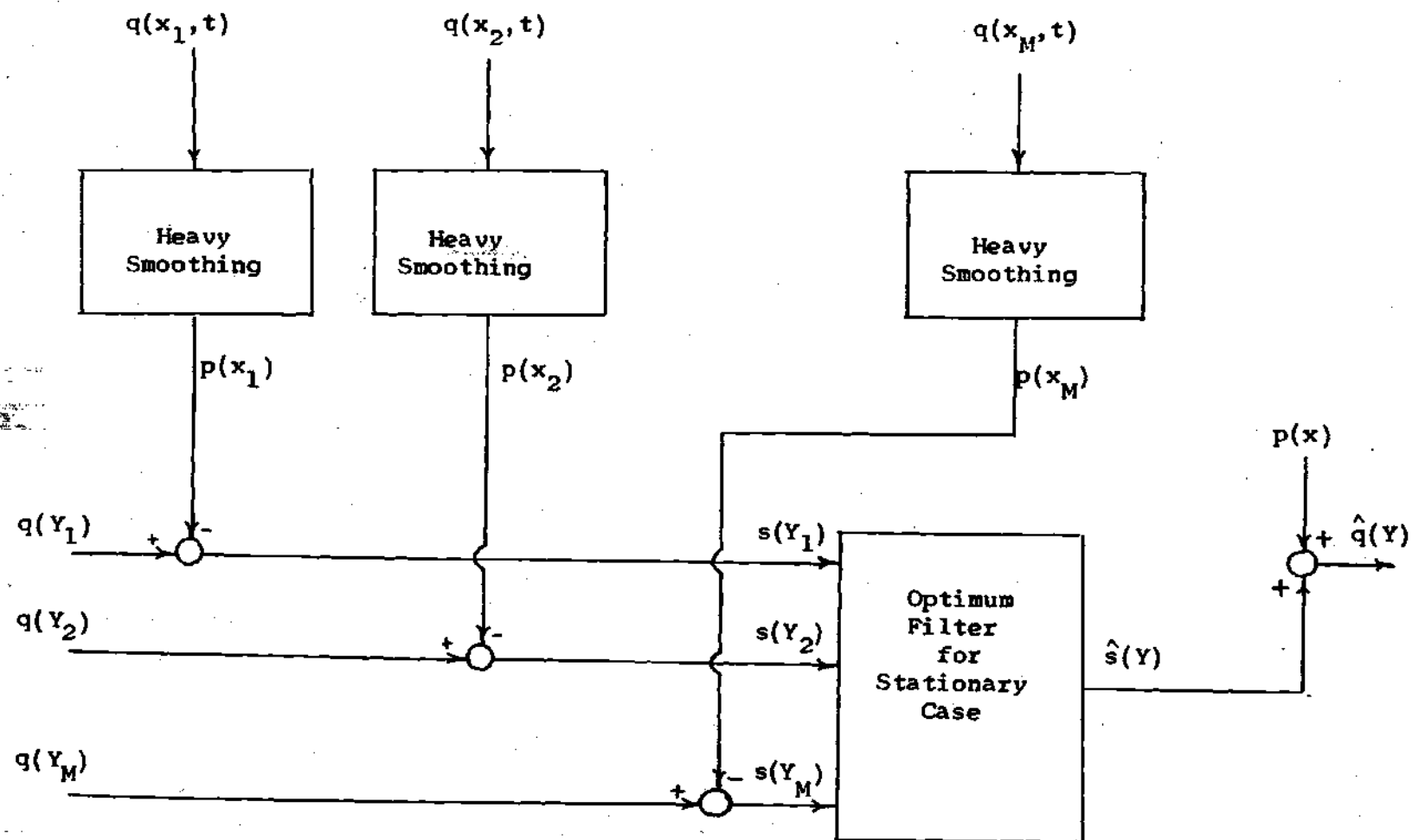
Figure 22. Scheme to Handle the Nonstationarity with Peterson and Middleton's Estimator.

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$$P_n(x_k) = \alpha P_{n-1}(x_k) + (1-\alpha)q(x_k, t_n) \qquad (4.7)$$

where $P_n(x_k)$ is the estimate of $p(x)$ using the last estimate $P_{n-1}(x_k)$ and the new incoming data $q(x_k, t_n)$, and α is the smoothing coefficient. The coefficient α determines the time constant of the filter and its choice is dependent upon the characteristics of the process under consideration. A discussion of this topic can be found in the book by Brown (26). The estimated value of $p(x)$ is subtracted from the signal to give the stationary part $s(x,t)$. Peterson and Middleton's estimator is used to give the best estimate of the stationary part, which is combined with the nonrandom estimate to give a suboptimal estimate of the signal.

Note that although the off-line computation remains nearly the same as in the stationary case, the on-line computations have increased by two multiplications and K+1 additions, and the storage space has increased by K values.

Modification to the Blum's Estimator

Blum (6) has developed a way of finding the general linear operator of a one-dimensional signal such that the mean square error of prediction is a minimum. The output of his filter is

$$\hat{q}(t) = \sum_{k=1}^{M} q(t_k)w(t-t_k) \qquad (4.8)$$

His signal q(t) is made up of a stationary part s(t) and a non-random part

$$p(t) = \sum_{k=1}^{D} a_k p_k(t) \qquad (4.9)$$

where $p_k(t)$ are known a priori, but the parameters a_k need not be known. He achieves his objective by setting the expected value of the estimation error equal to zero which yields a set of restraint equations. The expected value of the squared error is then minimized subject to these restraint equations and the resulting weighting matrix is of the form

$$W = V^{-1}U + V^{-1}P^T(PV^{-1}P^T)^{-1}Q - V^{-1}P^T(PV^1P^T)^{-1}PV^{-1}U \qquad (4.10)$$

where W, V and U retain the same definitions as in the Peterson and Middleton method,

$$Q = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdot \\ \cdot \\ \cdot \\ p_D(t) \end{bmatrix} \qquad (4.11)$$

and

$$P = \begin{bmatrix} p_1(t_m) & p_1(t_{m-1}) & \cdots & p_1(t_1) \\ p_2(t_m) & p_2(t_{m-1}) & \cdots & p_2(t_1) \\ \vdots & & & \\ p_D(t_m) & p_D(t_{m-1}) & \cdots & p_D(t_1) \end{bmatrix} \qquad (4.12)$$

Equation (4.10) holds for all non-singular $(PV^{-1}P^T)$ and V matrices. For $(PV^{-1}P^T)$ to be non-singular (i.e., determinant $(PV^{-1}P^T) \neq 0$), the value of D must be less than or equal to M.

The same relations hold true in the two-dimensional case where, instead of the signal just being a function of t, it becomes a function of a vector Y consisting of x and t. The form of p(x) can be modeled by a truncated Fourier series

$$p(x) = \sum_{i=1}^{\frac{D-1}{2}} (a_i \cos \frac{2\pi i}{X} x + b_i \sin \frac{2\pi i}{X} x) + a_o \quad (4.13)$$

where X is the total width of the sheet, (D-1) is the total number of points sampled in the x direction and a_o constitutes the steady part of the signal. The highest angular frequency is chosen as (D-1)/2X since the frequencies higher than this will not be identified at the fixed sampling rate of D points per scan. The lowest angular frequency is 1/X, which accounts for the periodic wave length X used in a Fourier series expansion.

Because of the tremendous computation and storage needed in this method, its practical application is limited. It requires twelve additional steps beyond the Peterson and Middleton algorithm. These include one function generation, one Q matrix generation, one matrix transformation, six matrix multiplications, one matrix inverse and two matrix additions. In terms of a 40 by 40 matrix or larger, this results in an excessive amount of computer time.

Effectiveness of the Method Proposed
in Chapter III

The effectiveness of the method proposed in Chapter III can be judged by the performance of the estimator which uses the results of the proposed method. It is possible to do this testing with the stationary simulated process since its real autocorrelation function is known.

The criterion for effectiveness testing is the expected value of the squared error ε. If the linear estimator is denoted by

$$\hat{s}(Y) = \sum_{k=1}^{M} s(Y_k) w(Y, Y_k) \tag{4.14}$$

then the expected value of the squared error is given by

$$\varepsilon = E\left\{ [\hat{s}(Y) - s(Y)]^2 \right\} \tag{4.15}$$

$$= E\left\{ [s(Y)]^2 \right\} - 2\sum_{k=1}^{M} w(Y, Y_k) E\left\{ s(Y) s(Y_k) \right\}$$

$$+ \sum_{i=1}^{M} \sum_{k=1}^{M} w(Y, Y_k) w(Y, Y_i) E\left\{ s(Y_k) s(Y_i) \right\}$$

$$= R_{ss}(0) - 2\sum_{k=1}^{M} w(Y, Y_k) R_{ss}(Y, Y_k)$$

$$+ \sum_{i=1}^{M} \sum_{k=1}^{M} w(Y, Y_k) w(Y, Y_i) R_{ss}(Y_k, Y_i)$$

In the theoretical case, according to Peterson and Middleton,

the weighting terms $w(Y, Y_k)$ are obtained from the equations

$$R_{ss}(Y, Y_r) = \sum_{k=1}^{M} R_{ss}(Y_r, Y_k)w(Y, Y_k) \qquad (4.16)$$

for $r = 1,2,\ldots,M$. However, in a practical case the weighting terms are obtained from

$$\hat{R}_{ss}(Y, Y_r) = \sum_{k=1}^{M} \hat{R}_{ss}(Y_r, Y_k)\hat{w}(Y, Y_k) \qquad (4.17)$$

for $r = 1,2,\ldots,M$, where $\hat{R}_{ss}(Y, Y_r)$ represents the fitted autocorrelation function instead of the real autocorrelation function $R_{ss}(Y, Y_r)$. Hence the theoretical ϵ is given by

$$\epsilon_{Theo} = R_{ss}(0) - 2\sum_{k=1}^{M} w(Y,Y_k)R_{ss}(Y,Y_k) + \sum_{i=1}^{M}\sum_{k=1}^{M} w(Y, Y_k) \cdot \qquad (4.18)$$

$$w(Y, Y_i)R_{ss}(Y_k, Y_i)$$

whereas the ϵ obtained by using the fitted autocorrelation function is given by

$$\epsilon_{Actual} = R_{ss}(0) - 2\sum_{k=1}^{M} \hat{w}(Y,Y_k)R_{ss}(Y,Y_k) + \sum_{i=1}^{M}\sum_{k=1}^{M} \hat{w}(Y,Y_k) \cdot \qquad (4.19)$$

$$\hat{w}(Y,Y_i)R_{ss}(Y_k,Y_i)$$

Results

For the first simulated process the theoretical and actual ϵ are determined for three prediction points. Their position is shown in Figure 23. A comparison is made with the scheme using a one-dimensional autocorrelation function in the t direction even though the stationary process is two dimensional. This comparison is interesting because, in most sheet
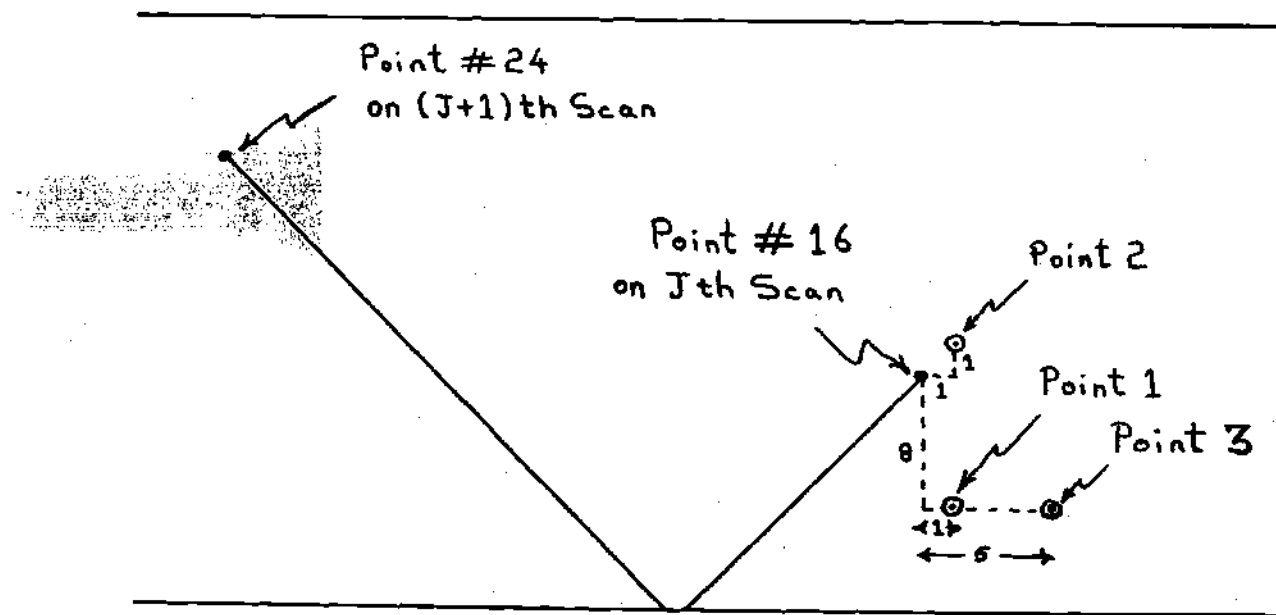
Figure 23.   Position of the Three Prediction Points.

processes, one-dimensional estimation is still very popular.
The results are shown in Table 7 and the advantages of using
two-dimensional estimation are highlighted. It is interesting
to note that the maximum ϵ for the theoretical case is 2.

<div align="center">Table 7. Comparison of ϵ for the Three
Prediction Points</div>

Estimation Technique Using	ϵ at Points		
	1	2	3
Real $R_{ss}(\Delta x, \Delta t)$	1.2992	.6735	1.7089
Fitted $R_{ss}(\Delta x, \Delta t)$	1.527	.698	1.94
1-Dimensional $R_{ss}(\Delta x, \Delta t)$	5.0200	.7800	2.4080
Mean as the Best Estimate	2.0000	2.0000	2.0000
General Form of Equation (2.7)	2.53	.722	2.56

Discussion of the Results and Conclusions

From the results of Table 7 two conclusions can be drawn.
The first conclusion is that the two-dimensional estimator is
superior to the one-dimensional estimator when the process is
two-dimensional. The percentage deviations of ϵ from the theo-
retical best values are 17.5, 3.7, and 12.4 percents for the
two-dimensional fitted autocorrelation as compared to 286.0,
16.5, and 41.0 percents for the one-dimensional autocorrelation
function. This result was anticipated from the beginning of
this research. Even the choice of the mean as the best estimate
gives better results than the one-dimensional case for two of

the three points. The second conclusion is that the method
devised in Chapter III to extrapolate and refine the autocorrelation function works more efficiently for estimating points
closer to the sampling lattice than distant ones. The reason
for the second conclusion is that the weighted least-squares
method of Chapter III lays more weight on the crude $R_{ss}(\Delta x, \Delta t)$
when Δx and Δt are small and, hence, the autocorrelation function fit is poor in regions where Δx and Δt are large. This
fact can be evidenced in Figure 15 where the fitted autocorrelation function follows the crude autocorrelation function
closely in regions of small $\Delta \ell$ and exhibits poor fit for large
$\Delta \ell$. Also note that for $\Delta \ell$ between 8 and 16 the fitted autocorrelation function deviates considerably from the desired autocorrelation function, thus, causing the percentage deviation of
ϵ from the theoretical value to increase.

In the simulated processes, points whose Δx and Δt separations are greater than 40 units are not correlated because
the weighting matrix was truncated. If estimation is attempted
for points whose separation from the sampling lattice is such
as to cause most sampling lattice points to fall outside the 40
by 40 range, the results obtained by using the fitted autocorrelation function will be in large error. In these situations
it is best to use the mean as the best estimate. A criterion
for judging the range of the estimator is that ϵ for a particular point must not exceed the variance of the stationary process.

Empirical Sensitivity of the Parameters
in the Functional Form

In this section the effect of variation in the parameters
of the closed functional form (of the autocorrelation function)
on the expected value of the squared error will be studied.
This study is important because from time to time the process
characteristics change, e.g., drift in frequency or change in
the decay constant of the autocorrelation function, and it is
essential to know how the estimator will perform under these
new circumstances.

For this investigation the correct autocorrelation func-
tion form as given by Equation (2.6) for process 1 is considered.
The values of the parameters in Equation (2.6) are $A = .075$,
$B = .075$, $D = 2$, $\omega_x = 0.3$ and $\omega_t = 0.3$. The sensitivity of
the parameters is performed empirically about point 1 of Figure
23. One parameter is varied at a time keeping the remaining
parameters constant at their true value, and its effect on ε is
noted. The results are shown in Table 8. One hundred fifty
percent variation in each direction around the true value of
ω_x and ω_t is the maximum permissible limit, since at that limit
ε is greater than the variance of the process. In the cases of
A and B, 33.3 percent variation in the negative direction and
300 percent variation in the positive direction is the maximum
permissible limit. As expected, variation in D did not produce
any change in ε. When empirical sensitivity analysis is per-
formed about point 2 in Figure 23, the percentage maximum per-
missible limits substantially increase. This fact is illustrated

Table 8. Sensitivity of ε on the Parameters

Parameter Being Varied		About Point 1
ω_x	.1	1.870
	.2	1.429
	.3	1.299
	.4	1.409
	.5	1.671
	1.0	2.167
A	.01	7.624
	.03	5.600
	.06	1.376
and	.075	1.299
	.1	1.340
B	.15	1.478
	.25	1.729
D	.5	1.299
	2.0	1.299
	6.0	1.299

Results for ω_t are the same as ω_x

Fixed values are kept at ω_x = .3, ω_t = .3, D = 2, A = .075 and B = .075

in Figure 24, and suggests that further the separation of the prediction point and the sampling lattice, less flexibility is available in the parameters.

Sensitivity of the functional form itself is analyzed by fitting the form of Equation (2.7) to the first simulated process having the form of Equation (2.6). The results are shown in
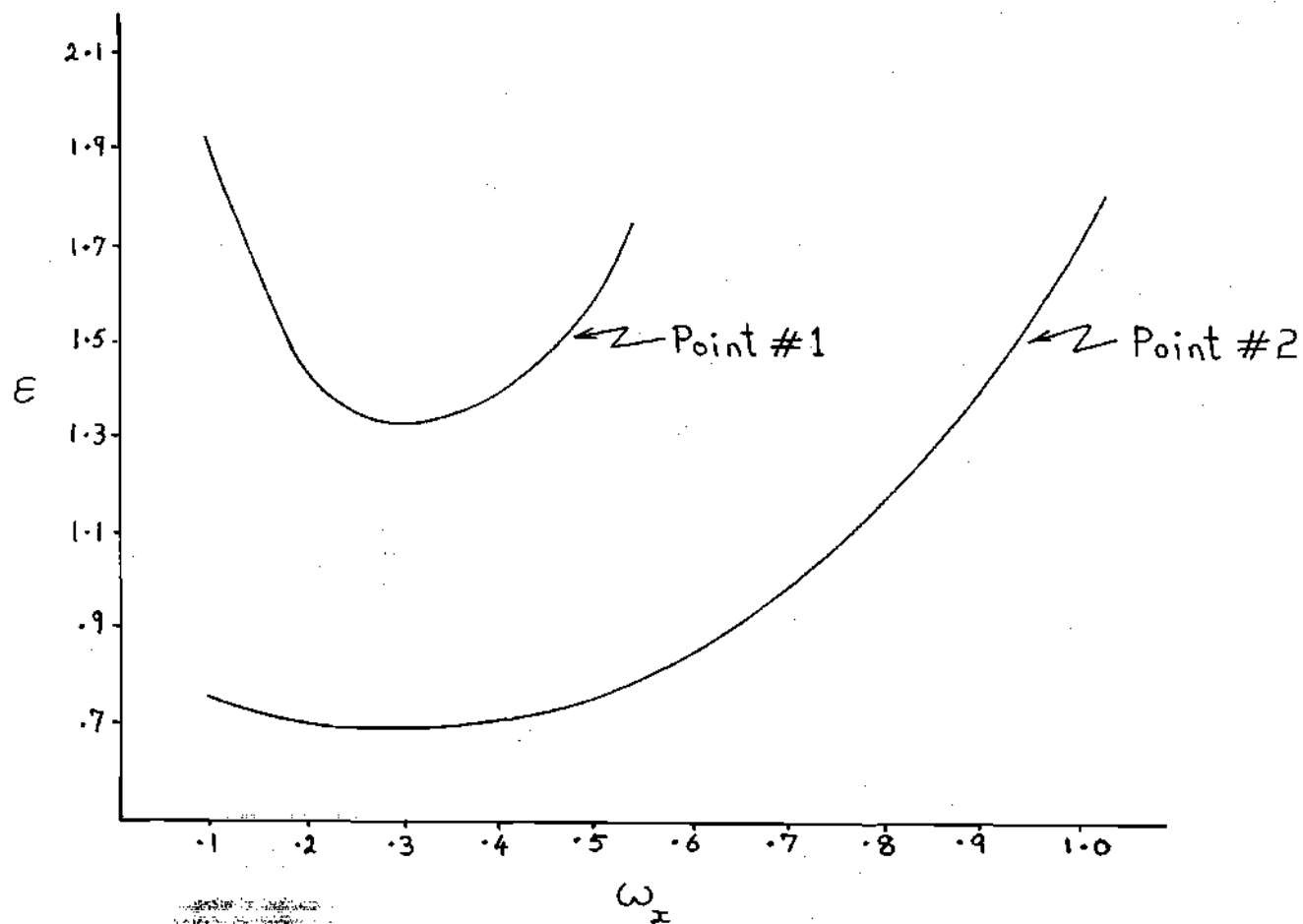
Figure 24.   Sensitivity of ϵ on ω_x for Different Prediction Points.

Table 7 for the same three prediction points of Figure 23. Although the percentage variation in ε from the correct form varies for each point, its performance is still acceptable for points close to the sampling lattice. This suggests that in cases where the correct functional form of the autocorrelation function is not known, a good approximation is the form of Equation (3.5).

CHAPTER V

APPLICATION OF THE METHOD TO A PAPER MILL PROCESS

AND A COMPARISON WITH SOME EXISTING METHODS

A schematic view of the paper making process is shown
in Figure 25. The homogeneous mixture of cellulose fiber and
water is forced out of the headbox slot and is carried by a
bronze screen to a complex of presses, dryers and calendars,
the output of which is wound on a reel. A traversing Beta
Gauge senses the instantaneous basis weight (essentially mass
per unit area) of the paper in the region between the calendars
and the winder. A typical paper sheet is 20 feet wide and
travels at the rate of 2000 feet per minute. Relative to this
high speed, the Beta Gauge moves at a slow pace of 20 feet per
minute which generates an angle of about 40 minutes between the
gauge path and edge of paper. The instantaneous output signal
from the Beta Gauge is the basis weight signal, the undesired
sensor noise being substantially removed by a built-in filter.
The randomness of the basis weight is due to the slot profile,
pulsating flow caused by pumps, vibration of the machine members
and fluctuation of the fiber to water ratio in the headbox among
other things.

Field data was collected by a Southeastern paper mill.
Since the machine is a production unit, the author had very
little control over the type of data that was collected. The

Figure 25. Panorama of Papermaking Today

data received is only good for prediction on the scan path since
no verification means are available for points outside the scan.
The data was collected as the scanning gauge was traversing
from the front to the back of the machine. The first observa-
tion was collected at the same start position and each observa-
tion thereafter, for a total of 79, was collected on a one second
interval. The data was punched into cards with 10 successive
cards constituting one set or one scan. The first value in card
one was the time at which the first observation for the scan
was obtained. Machine direction data was also collected on a
one second interval with the scanner in a stationary position
towards the middle of the web. The speed of the paper machine
was 2201 feet per minute and the sensor speed was 2.25 inches
per second. The average basis weight was 38 lbs. and moisture
was in the range of 4.5 - 5 percent.

The persistent cross-directional profile is first sepa-
rated from the data and the remainder is verified for wide
sense stationarity. The crude autocorrelation function is then
extracted and 95 percent confidence bounds placed around it.
This crude autocorrelation function as shown in Figure 26 is
passed through a Bartlett lag window to yield the refined
autocorrelation function. Extrapolation of the autocorrelation
function is not performed since verification of the results is
not possible outside the scan path. The refined autocorrelation
function is then used in one point ahead estimation of four
points on the scan path and the signal is reconstructed by adding
the persistent cross-directional component. One point ahead
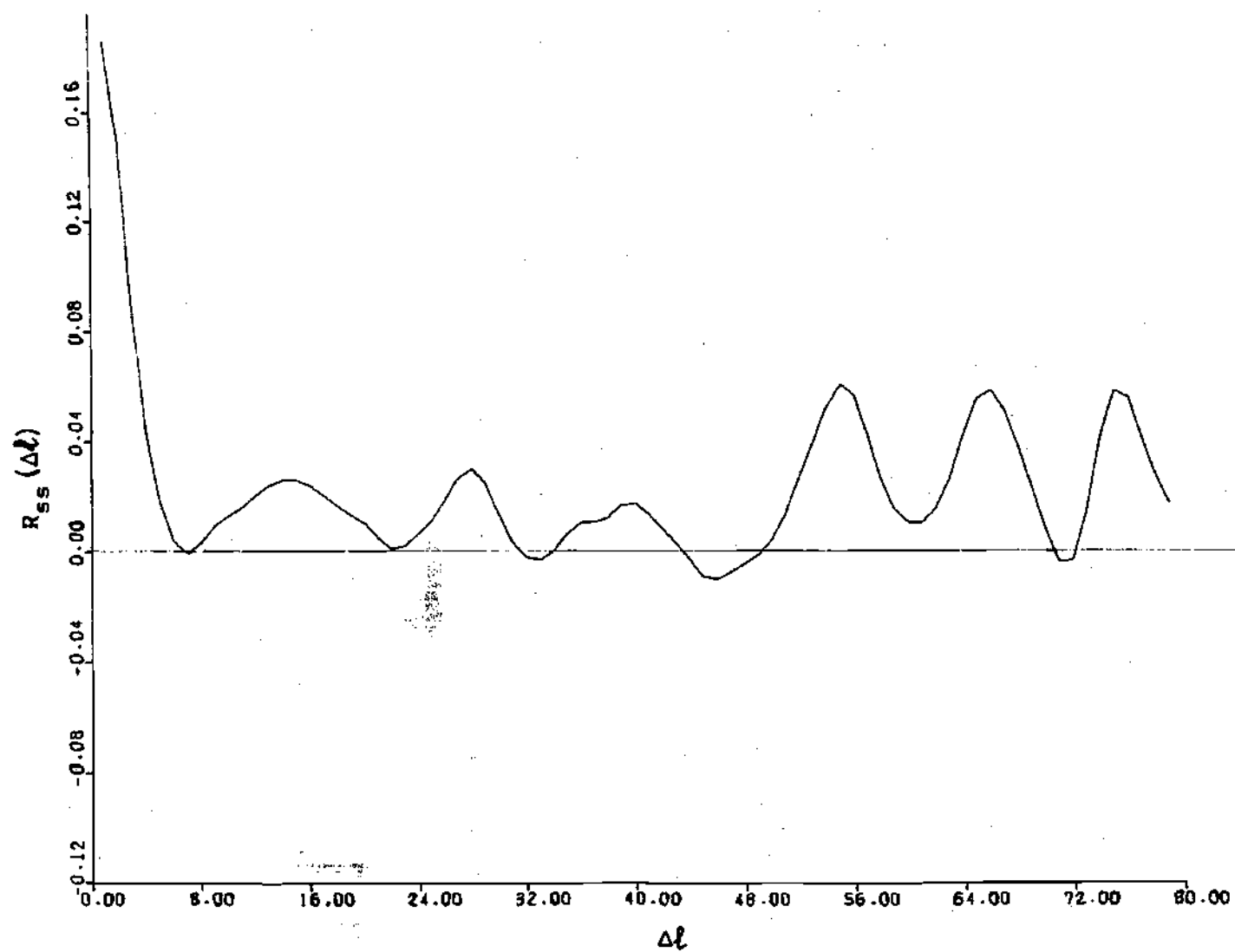
Figure 26. Crude Autocorrelation Function of the Basis Weight.

estimation means predicting one unit in the future on the scan path. Variance of the reconstructed signal is also determined over eighteen scans. This variance is shown in Table 9, in which a comparison is made with the variances for the same points obtained by using three other methods, some of which are being used in the paper mills.

The three other methods are: Brewster's method, one-dimensional estimator with Bartlett filter and two-dimensional estimator without Bartlett filter. Their algorithms are presented below along with the algorithm for the variance about zero mean.

Brewster's Method (1)

Consider q_{ij} as representing the signal on the jth scan and i^{th} point on that scan, then

Table 9. Comparison of Experimental ε for Various
Methods used in Basis Weight Estimation

| Methods | One Point Ahead Prediction Variance at Points | | | |
	1	25	50	79
Brewster's Method	5.339	.874	3.119	3.416
1-Dimensional with Bartlett Filter	3.559	2.776	1.888	1.629
2-Dimensional without Bartlett Filter	3.823	1.484	10.167	2.039
2-Dimensional with Bartlett Filter	2.797	.905	.8875	1.772
Variance about Zero Mean	3.842	3.574	2.528	3.510

$$\overline{q}_j = \frac{1}{N} \sum_{i=1}^{N} q_{ij} \tag{5.1}$$

$$\hat{c}_{ij} = \alpha(q_{ij} - \overline{q}_j) - (1 - \alpha)\hat{c}_{i,j-1} \tag{5.2}$$

$$\hat{m}_{ij} = q_{ij} - \hat{c}_{i,j-1} \tag{5.3}$$

$$m^*_{ij} = \zeta \hat{m}_{ij} + (1 - \zeta)\hat{m}_{i-1,j} \tag{5.4}$$

$$\hat{q}_{i+1,j} = m^*_{ij} + \hat{c}_{i+1,j} \tag{5.5}$$

where $\hat{c}_{ij}$ represents the cross-directional profile, $\hat{m}_{ij}$ represents the machine-direction deviation, m^*_{ij} represents the filtered $\hat{m}_{ij}$ to be used in control and $\hat{q}_{i+1,j}$ represents the one point ahead prediction. The value of α is chosen as .3 and the value of ζ as .95.

One-Dimensional Estimator with Bartlett Filter

The machine-direction data is used to obtain the crude one-dimensional autocorrelation function. This crude autocorrelation function is passed through a Bartlett lag window (9) to give the refined one-dimensional autocorrelation function. The refined autocorrelation function is used along with the scan path data for estimation. The rest of the algorithm is the same as in Chapters III and IV.

Two-Dimensional Estimator without Bartlett Filter

In this method, instead of using the refined two-dimensional autocorrelation function, the crude one is used.

Variance about Zero Mean

In this method, the mean of the process or the persistent cross-direction profile is considered as the best estimate of the signal. The algorithm consists of

$$\bar{q}_i = \frac{1}{N} \sum_{j=1}^{N} q_{ij} \tag{5.6}$$

$$s_{ij} = q_{ij} - \bar{q}_i \tag{5.7}$$

$$\sigma_i^2 = (s_{ij} - \bar{q}_i)^2 \tag{5.8}$$

where q_{ij} is the signal on the ith point of the jth scan and σ^2 is the variance for the ith strip.

Discussion of the Results and Conclusions

In this chapter the approach developed in this thesis is applied to a real sheet paper process. Extrapolation was not performed because the available data could not be used for verification of the extrapolation results. Instead, only the refined autocorrelation function was used in one point ahead prediction on the scan path. The modified Peterson and Middleton's estimation technique of Chapter IV, is used because of its computational advantage over Blum's technique. It is evident from Table 9 that the proposed method is superior to the other methods. The reason for the percentage differences not remaining constant is because only eighteen values were used in obtaining the variance and the variance did not have a chance to reach a steady

value. The choice of the number eighteen was governed abso-lutely by the data base furnished in which only eighteen scans were consecutive. A comparison with the variance of the process shows that the other methods are quite marginal and that in many cases it would have been better to choose the mean as the best estimate.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In this thesis a solution is obtained for the difficult problem of two-dimensional estimation from a restricted sampling lattice. Attention is restricted to a class of two-dimensional stochastic processes which is a sum of a Gaussian stationary process and a nonrandom process that is a function of the spatial coordinate only. The concept of using the functional form of the autocorrelation function for attacking the problem of optimal estimation from a restricted sampling lattice is a significant contribution to the state of the art. When the functional form is not known a priori, the exponential - cosine form proves to be a good approximation. The estimates obtained with the proposed approach have smaller mean square estimation errors than the ones obtained by using existing methods, and in this respect, the proposed approach has a distinct advantage over existing schemes.

In Chapter I the problem is stated and the approach to be followed is discussed. The properties of two-dimensional stationary processes are presented from a practical view point. All the techniques developed in this thesis make use of these properties and this in itself, should suggest the importance of these properties.

In Chapter II the stationary part of the process described
in the "Statement of the Problem" is generated on a straight line
scan path. A technique to simulate Gaussian stationary processes
with the desired exponential-cosine autocorrelation function
from discrete white noise has been developed. The results of
the simulation appear to be good. The desired autocorrelation
function lies within the 99.76 percent confidence bounds of the
crude autocorrelation function. Also, the truncation and digi-
talization errors appear to be negligible. The method is not
limited to exponential-cosine forms but will synthesize any
process where the synthesizing filter impulse response is known.

In Chapter III the crude autocorrelation function is
successfully extrapolated and refined by taking advantage of
the functional form of the autocorrelation function. The results
of the first simulated process show that the fitted autocorre-
lation function lies well within the 99.76 percent confidence
bounds of the crude autocorrelation function. Also, the power
spectrum analysis is able to extract the initial values of the
frequencies present within 7.0 percent of the original values.

The real test of the procedure of Chapter III is presented
in Chapter IV where the estimation error, or the expected value
of the squared error using the fitted autocorrelation function,
is compared with the one using the real autocorrelation func-
tion. For the three estimation points chosen the percentage
differences of the expected values of the squared errors were
17.5, 3.7, and 12.4 percents, the larger percentages arising as

one goes further from the sampling lattice. However, all of them are well within the maximum permissible limit, beyond which the variance of the process is exceeded. The comparison in Table 7 shows the superiority of the proposed method over existing methods. In the same chapter, the empirical sensitivity of the parameters in the functional form of the autocorrelation function is studied. From the results it is clear that a certain margin is available in which the expected value of the squared error is still below the permissible limit, for drift in the parameter values. This margin is a function of the distance of the estimation point from the sampling lattice and the parameter itself. Also, a very limited empirical sensitivity analysis of the form of the autocorrelation function is performed and the results show that the exponential-cosine form is indeed very promising in cases where the form is not given a priori,

In Chapter V the approach developed in this thesis is applied to a real sheet paper process. Extrapolation was not performed because the available data could not be used for verification of the extrapolation results. Instead, only the refined autocorrelation function was used in one point ahead prediction on the scan path. It is evident from the results shown in Table 9, that the estimation error is least for the proposed method and, hence, the proposed method is superior to the existing methods.

Recommendations

The author is of the opinion that the work reported in this thesis is only the beginning of many works to appear in the field of estimation from restricted sampling lattices. More and more scientists and engineers sense the need for techniques which will enable them to take advantage of the sophisticated optimal estimation theories. These techniques will act as an interface between the data available in practice and the requirements of optimal estimation theory. In this thesis, the process was Gaussian and stationary. There will be situations when either one or both these conditions need to be relaxed. When the Gaussian condition is released, nonlinear estimation theory will be used. When stationarity is relaxed, optimal estimation will be performed using the process dynamics. The point being emphasized is that the requirements of optimal estimation change for different class of stochastic processes and, hence, new interface tools have to be devised.

Certain topics discussed below, have not been given sufficient consideration in this thesis due to lack of time and could be developed in the future. The sensitivity of the scan path form on the extrapolation and refinement of the autocorrelation function would be an interesting study to undertake. This topic is of special interest to paper manufacturers since their scan angle is only $0^{o} - 4'$ and the author is of the opinion that this angle is very poor for extrapolation purposes. The reason is that the contribution of ω_{xi} in Equations (3.19) and (3.20)

becomes negligible and, hence, the difference between ω_{si1} and ω_{si2} becomes very small. The indirect power spectrum technique is not able to distinguish between ω_{si1} and ω_{si2}.

In the fitting process of Chapter III, the weighting term γ_k was given the form of Equation (3.7). This form of γ_k resembles the Bartlett lag window for recovery of the process statistics. The author is aware of the Hamming, Hanning and half a dozen more lag windows which can be used. It is conceivable that there exists an optimal lag window for a particular class of stochastic processes. An understanding of optimal lag windows and their performance in estimation of a signal would be a valuable contribution to the state of the art.

Another topic that needs further study is the exponential-cosine form of Equation (3.5). In this thesis the choice of Equation (3.5) stems from Bendat's observations that most one-dimensional random processes in practice have the exponential-cosine form for the autocorrelation function. It would be desirable to study a whole class of two-dimensional random processes to verify Equation (3.5).

APPENDIX

APPENDIX

COMPUTER PROGRAMS

The useful computer programs and subroutines which have been used in this thesis, are included here.

Generation of 2-D Random Process from White Noise

This program uses discrete Gaussian white noise to generate a two-dimensional random process with an exponential-cosine autocorrelation function. This program calls for subroutine Shaper which is also included.

Subroutine SPECT

This subroutine is used for generating the raw, Hamming and Hanning power spectra by the indirect technique of Blackman and Tukey (9).

Subroutine BAND

This subroutine is used to place 99.76 percent confidence bounds around the crude autocorrelation function by using the Fisher's Z statistic (23).

Subroutine DIPLOT

This subroutine plots YARRAY versus XARRAY, where XARRAY and YARRAY could be any two variables.

Subroutine PLOT3D

This subroutine plots the function FCN2D (I, J) on a IMAX by JMAX array. XSIZE and YSIZE represent the maximum horizontal

and vertical length of the base on which FCN2D(I, J) is plotted. HEIGHT represents a scaling factor which is multiplied with FCN2D(I, J). The result of multiplication with the largest value of FCN2D(I, J) must not exceed (10" - YSIZE), where ten inches represents the width of the plotting paper. This program calls for subroutine PLTT which is also included.

Subroutine FMFP

This subroutine is used to find the local minimum of a function of several variables by the method of Fletcher and Powell (14).

Generation of 2-D Random Process from White Noise

```
C     GENERATION OF 2-D RANDOM PROCESS FROM WHITE NOISE.
C     INPUT  - NS=NO OF SCANS, N=NO OF POINTS IN T DIRECTION,M= NO OF
C     POINTS IN X DIRECTION, NN= NO OF POINTS IN T DIRECTION PLUS
C     WEIGHTING MATRIX ARRAY IN T DIRECTION, MM=NO OF POINTS IN X DIR.
C     PLUS WEIGHTING MATRIX ARRAY IN X DIRECTION. NW=WEIGHTING MATRIX
C     GRID IN T DIR., MW=WEIGHTING MATRIX GRID IN X DIR., NNN=WHITE
C     NOISE ARRAY.            OUTPUT... TAPE LABELED DATA,PLOT OF
C     DATA ON SCAN.           INPUT... INPUT TAPE LABELED NOISE AND
C     INPUT TAPE LABELED WHITE, NBLO=BLOCK LENGTH ON TAPE WHITE AND NOISE.
C     CONSISTING OF 31 BLOCKS,NNN=NBLO*2.
C     NSET=NO OF FREQUENCY SETS
      PARAMETER NSET=2
      PARAMETER MM=79
      PARAMETER NN=79
      PARAMETER M=40
      PARAMETER N=40
      PARAMETER NW=40
      PARAMETER MW=40
      PARAMETER NS=30
      PARAMETER NBLO=3160
      PARAMETER NNN=6320
      DIMENSION W(MW,NW),S(MM,NN),      Z(NNN),Y(NS,M),PL(1202),
     1CLOCK(1202)
      DIMENSION SUM(NS,M)
      JER=1
   16 CONTINUE
      K=1
      IER=-1
      DO 30 I=1,2
      L1=NBLO*(I-1)+1
      L2=L1+(NBLO-1)
      IF (JER.EQ.1)READ (3)(Z(J),J=L1,L2)
   30 IF (JER.EQ.2) READ(4)(Z(J),J=L1,L2)
      DO 1 I=1,MM
      DO 2 J=1,NN
      S(I,J)=Z(K)
    2 K=K+1
    1 CONTINUE
      CALL SHAPER(W,NW,MW)
      K=1
      IO=1
      X=1.
      T=1.
    9 IX=X
      IT=T
      Y(IO,IX)=0.
      DO 3 I=1,MW
      DO 4 J=1,NW
    4 Y(IO,IX)=Y(IO,IX)+S(IX+I-1,IT+J-1)*W(I,J)
    3 CONTINUE
      SUM(IO,IX)=SUM(IO,IX)+Y(IO,IX)
      PL(K)=Y(IO,IX)
      CLOCK(K)=K
      K=K+1
    6 IF(IER)7,7,8
    7 X=X+1
      T=T+1
      IF(X-M)9,9,10
   10 IER=+1
      X=M
      T=1.
      GO TO 21
    8 X=X-1
      T=T+1
      IF(X)11,11,9
   11 IER=-1
      X=1.
      T=1.
   21 NAS=N-1
      IF(IO.EQ.NS)GO TO 5
      IO=IO+1
      DO 22 I=1,MM
```

```
      DO 23 J=1,NAS
   23 S(I,J)=S(I,J+N)
   22 CONTINUE
      IF (JER.EQ.1)READ (3)(Z(J),J=1,NBLO)
      IF (JER.EQ.2) READ(4)(Z(J),J=1,NBLO)

      KK=1
      DO 18 I=1,MM
      DO 19 J=N,NN
      S(I,J)=Z(KK)
   19 KK=KK+1
   18 CONTINUE
      GO TO 9
    5 CALL D1PLOT(CLOCK,PL,1202)
      WRITE (6,31)
   31 FORMAT (2X,'2-D RANDOM PROCESS')
      WRITE (6,20)((Y(I,J),J=1,M),I=1,NS)
      JER=JER+1
      IF(JER-NSET)16,16,17
   17 WRITE (2)((SUM(I,J),J=1,M),I=1,NS)
   20 FORMAT (7F10.5)
      END
      SUBROUTINE SHAPER(W,NW,MW)
C     INPUT....VALUES OF PARAMETERS TO BE USED IN THE FUNCTIONAL FORM.
      DIMENSION W(MW,NW)
      SUM=0.
      READ(5,20,END=70)A,D,WX,WT,D1,D2
C     .075         .15         .165        .165        .642        .642
C     .075         .212        .3          .3          .785        .785
      WRITE(6,30)
   30 FORMAT (2X,'WEIGHTING MATRIX')
      DO 1 IX=1,MW
      DO 2 IT=1,NW
      F=1.
      T=IT-1
      X=IX-1
      X=X+.5
      T=T+.5
      EX=EXP(-A*X)
      ET=EXP(-A*T)
      CX=EX*COS(WX*X)
      SX=D1*EX*SIN(WX*X)
      CT=ET*COS(WT*T)
      ST=D2*ET*SIN(WT*T)

      W(IX,IT)=D*(CX+SX)*(CT+ST)*F
      SUM=SUM+W(IX,IT)**2
    2 CONTINUE
    1 CONTINUE
      WRITE(6,20)((W(I,J),J=1,NW),I=1,MW)
      WRITE (6,20)SUM
   20 FORMAT (7F10.5)
      RETURN
   70 STOP
      END
```

Subroutine SPECT

```
      SUBROUTINE SPECT(C,DT,KG)
C     C=AUTOCOVARIANCE VECTOR OF DIMENSION KG,DT=LENGTH OF SMALLEST
C     ELEMENT, KG=DIMENSION OF C VECTOR
C     OBTAIN POWER SPECTRUM 1-D
      DIMENSION C(1000),V(1000),HM(1000),HN(1000),U(1000),OMEGA(1000)
      DIMENSION T(1000),Q(1000)
      WRITE (6,30)
   30 FORMAT (2X,'SPECTRUM,TRIANGULAR LAG SPECT,POINTS, FREQ, OMEGA')
      K=KG
      Z=K
C     TRIANGULAR LAG WINDOW
      DO 4 I=1,K
      Y=I
    4 T(I)=C(I)*(1.-Y/Z)
      DO 2 I=1,K
      SUM=0.0
      SUP=0.
      M=K-1
```

```
      DO 1 J=2,M
      B=I-1
      A=J-1
      SUP=SUP+T(J)*COS(A*B*3.14/Z)
    1 SUM=SUM+C(J)*COS(A*B*3.14/Z)
      V(I)=DT*(C(1)+2.*SUM+C(K) *COS(B*3.14))
      Q(I)=DT*(T(1)+2.*SUP+T(K) *COS(B*3.14))
      FREQ=B/(2.*Z *DT)
      OMEGA(I)=2.*3.14*FREQ
    2 WRITE(6,21)V(I),Q(I),I,FREQ,OMEGA(I)
   21 FORMAT (F10.4,F10.4,I10,F10.5,F10.5)
   20 FORMAT (3F10.4)
   22 FORMAT (I10,F10.5,F10.5,F10.5)
   23 FORMAT (I10,F10.5,F10.5)
      MO=KG+2
      CALL D1PLOT(OMEGA,V,MO)
C     HAMMING AND HANNING ESTIMATES
      WRITE (6,31)
   31 FORMAT (2X,'POINT,HANNING, HAMMING,RAW SPECTRUM')
      M=K-1
      DO 3 I=2,M
      HN(I)=.25*V(I-1)+.5*V(I)+.25*V(I+1)
      HM(I)=.23*V(I-1)+.54*V(I)+.23*V(I+1)
    3 WRITE (6,22) I,HN(I),HM(I),V(I)
      MO=KG+2-2
      CALL D1PLOT(OMEGA,HM,MO)
C     HIGHER ORDER SMOOTHING
      M2=K-2
      DO 5 I=3,M2
      U(I)=.16*HN(I-1)+.68*HN(I)+.16*HN(I+1)
    5 WRITE (6,23) I,U(I),V(I)
      END
```

Subroutine BAND

```
      SUBROUTINE BAND(C,KK,SAMP,CZERO)
C     SUBROUTINE FOR PLACING99.7PERCENT CONFIDENCE BOUNDS
C     INPUT  C=AUTOCOVARIANCE,KK=DIMENSION OF C,SAMP=SAMPLE SIZE
C     M2=(KK*2-2)+2 , CZERO= VARIANCE
      PARAMETER M2=80
      DIMENSION CLOCK(M2),Y(M2)
      DIMENSION C(KK),SAMP(5000)
      WRITE (6,30)
   30 FORMAT (2X,'LOWER, AUTOCORRELATION, UPPER, SAMPLE SIZE,DIFF')
      DO 4 K=2,KK
      CLOCK(K-1)=K-1
      CLOCK(KK+K-2)=K-1
      R=C(K)/CZERO
      Z=(ALOG((1.+R)/(1.-R)))/2.
      COWER=CZERO*TANH(Z-3.00/SQRT(SAMP(K)-2.))
      UPPER=CZERO*TANH(Z+3.00/SQRT(SAMP(K)-2.))
      Y(K-1)=UPPER
      Y(KK+K-2)=COWER
      DIFF=ABS(UPPER-COWER)
      WRITE (4)DIFF
    4 WRITE (6,20)COWER,C(K),UPPER,SAMP(K),DIFF
      CALL D1PLOT(CLOCK,Y,M2)
   20 FORMAT (5F14.8)
      END
```

Subroutine BAND

```
      SUBROUTINE D1PLOT(XARRAY,YARRAY,MO)
C     INPUT - MO=NO+2=NUMBER OF DATA POINTS PLUS 2.  PLOT TAPE IS ALWAYS
C     LABELED PLOT AND IS NUMBERED 9)
C     OUTPUT - PLOT WILL APPEAR OVER A 10*10 INCH GRID.
      DIMENSION IBUF(1000)
      DIMENSION XARRAY(MO),YARRAY(MO)
      NO=MO-2
      CALL PLOTS(IBUF,1000,9)
      CALL PLOT (.0,-.5,3)
```

```
      CALL SCALE (XARRAY,10.,NO,1)
      CALL SCALE (YARRAY,10.,NO,1)
      KO=NO+1
      CALL AXIS (0.,0.,10HX-ABSCISSA,-10,10.,.0,XARRAY(KO),XARRAY(MO))
      CALL AXIS (0.,0.,10HY-ABSCISSA,10,10.,90.,YARRAY(KO),YARRAY(MO))
      CALL LINE (XARRAY,YARRAY,NO,1,0,INTEG)
      CALL PLOT (12.,0.,-3)
      CALL PLOT (.0,.0,999)
      END
```

Subroutine PLOT3D

```
      SUBROUTINE PLOT3D(XSIZE,YSIZE,HEIGHT,FCN2D,IMAX,JMAX)
      DIMENSION FCN2D(IMAX,JMAX)
      DIMENSION HID(600)
      COMMON IBUF(10000)
      REAL LASTX,LASTY,LASTH,LASTHM
      XPAGE=0.0
      YPAGE=0.0
      LASTHM=0.0
      NIJ=IMAX+JMAX
      RI=IMAX-1.0
      RJ=JMAX-1.0
      CALL PLTT(0.0,-12.5,-3)
      CALL PLTT(0.0,2.0,-3)
      DO 1 I=1,NIJ
    1 HID(I)=-0.5
      DO 7 J=1,JMAX
      AJ=J-1.0
      DO 7 I=1,IMAX
      AI=I-1.0
      LASTX=XPAGE
      XPAGE=(AJ+AI)*XSIZE/(RI+RJ)
      LASTY=YPAGE
      YPAGE=(AJ*RI/RJ-AI*RJ/RI+RJ)*YSIZE/(RJ+RI)+HEIGHT*FCN2D(I,J)
      LASTH=LASTHM
      LASTHM=HID(I+J)
      IF(YPAGE-HID(I+J)) 5,5,2
    2 IF(I.NE.1) GO TO 3
      CALL PLTT(XPAGE,YPAGE,3)
      IPEN=2
      GO TO 4
    3 CALL PLTT(XPAGE,YPAGE,IPEN)
      IPEN=2
    4 HID(I+J)=YPAGE
      GO TO 7
    5 IF(I.EQ.1) IPEN=3
      IF(IPEN.EQ.3) GO TO 6
      X1N=LASTX*HID(I+J)-LASTH*XPAGE-LASTX*YPAGE+XPAGE*LASTY
      X1D=HID(I+J)-LASTH-YPAGE+LASTY
      X1=X1N/X1D
      Y1=(X1*(HID(I+J)-LASTH)+LASTH*XPAGE-LASTX*HID(I+J))/(XPAGE-LASTX)
      CALL PLTT(X1,Y1,2)
      IPEN=3
    6 CALL PLTT(XPAGE,YPAGE,IPEN)
    7 CONTINUE
      DO 8 I=1,NIJ
    8 HID(I)=-0.5
      DO 16 I=IMAX,1,-1
      AI=I-1.0
      DO 16 J=1,JMAX
      AJ=J-1
      LASTX=XPAGE
      XPAGE=(AJ+AI)*XSIZE/(RI+RJ)
      LASTY=YPAGE
      YPAGE=(AJ*RI/RJ-AI*RJ/RI+RJ)*YSIZE/(RJ+RI)+HEIGHT*FCN2D(I,J)
      LASTH=LASTHM
      LASTHM=HID(I+J)
      IF(YPAGE-HID(I+J)) 13,14,9
    9 IF(J.NE.1) GO TO 10
      CALL PLTT(XPAGE,YPAGE,3)
      IPEN=2
      GO TO 12
   10 IF(IPEN.EQ.2) GO TO 11
      X1N=LASTX*YPAGE-LASTY*XPAGE-LASTX*HID(I+J)+XPAGE*HID(I+J-1)
```

```
            X1D=YPAGE-LASTY-HID(I+J)+HID(I+J-1)
            X1=X1N/X1D
            Y1=(X1*(YPAGE-LASTY)+LASTY*XPAGE-LASTX*YPAGE)/(XPAGE-LASTX)
            CALL PLTT(X1,Y1,3)
            IPEN=2
   11 CALL PLTT(XPAGE,YPAGE,IPEN)
   12 HID(I+J)=YPAGE
            GO TO 16
   13 IPEN=3
            GO TO 15
   14 IF(J.EQ.1) IPEN=3
   15 CALL PLTT(XPAGE,YPAGE,IPEN)
   16 CONTINUE
            CALL PLTT(XSIZE+4.0,-1.0,-3)
            RETURN
            END
            SUBROUTINE PLTT(X,Y,IPEN)
            COMMON IBUF(10000)
            XLAST=XN
            YLAST=YN
            ILAST=IN
            XN=X
            YN=Y
            IN=IPEN
            IF(IPEN.EQ.2.AND.ILAST.EQ.2) CALL PLOT(X,Y,IPEN)
            IF(IPEN.EQ.2.AND.ILAST.EQ.3) CALL PLOT(XLAST,YLAST,ILAST)
            IF(IPEN.EQ.2.AND.ILAST.EQ.3) CALL PLOT(X,Y,IPEN)
            IF(IPEN.NE.2.AND.IPEN.NE.3) CALL PLOT(X,Y,IPEN)
            RETURN
            END
```

Subroutine FMFP

```
C          SUBROUTINE FMFP
C
C          PURPOSE
C              TO FIND A LOCAL MINIMUM OF A FUNCTION OF SEVERAL VARIABLES
C              BY THE METHOD OF FLETCHER AND POWELL
C
C          USAGE
C              CALL FMFP(FUNCT,N,X,F,G,EST,EPS,LIMIT,IER,H)
C
C          DESCRIPTION OF PARAMETERS
C              FUNCT  - USER-WRITTEN SUBROUTINE CONCERNING THE FUNCTION TO
C                       BE MINIMIZED. IT MUST BE OF THE FORM
C                       SUBROUTINE FUNCT(N,ARG,VAL,GRAD)
C                       AND MUST SERVE THE FOLLOWING PURPOSE
C                       FOR EACH N-DIMENSIONAL ARGUMENT VECTOR ARG,
C                       FUNCTION VALUE AND GRADIENT VECTOR MUST BE COMPUTED
C                       AND, ON RETURN, STORED IN VAL AND GRAD RESPECTIVELY.
C              N      - NUMBER OF VARIABLES
C              X      - VECTOR OF DIMENSION N CONTAINING THE INITIAL
C                       ARGUMENT WHERE THE ITERATION STARTS. ON RETURN,
C                       X HOLDS THE ARGUMENT CORRESPONDING TO THE
C                       COMPUTED MINIMUM FUNCTION VALUE
C              F      - SINGLE VARIABLE CONTAINING THE MINIMUM FUNCTION
C                       VALUE ON RETURN, I.E. F=F(X).
C              G      - VECTOR OF DIMENSION N CONTAINING THE GRADIENT
C                       VECTOR CORRESPONDING TO THE MINIMUM ON RETURN,
C                       I.E. G=G(X).
C              EST    - IS AN ESTIMATE OF THE MINIMUM FUNCTION VALUE.
C              EPS    - TESTVALUE REPRESENTING THE EXPECTED ABSOLUTE ERROR.
C                       A REASONABLE CHOICE IS 10**(-6), I.E.
C                       SOMEWHAT GREATER THAN 10**(-D), WHERE D IS THE
C                       NUMBER OF SIGNIFICANT DIGITS IN FLOATING POINT
C                       REPRESENTATION.
C              LIMIT  - MAXIMUM NUMBER OF ITERATIONS.
C              IER    - ERROR PARAMETER
C                       IER = 0 MEANS CONVERGENCE WAS OBTAINED
C                       IER = 1 MEANS NO CONVERGENCE IN LIMIT ITERATIONS
C                       IER =-1 MEANS ERRORS IN GRADIENT CALCULATION
C                       IER = 2 MEANS LINEAR SEARCH TECHNIQUE INDICATES
C                       IT IS LIKELY THAT THERE EXISTS NO MINIMUM.
C              H      - WORKING STORAGE OF DIMENSION N*(N+7)/2.
```

```
C      REMARKS
C         I) THE SUBROUTINE NAME REPLACING THE DUMMY ARGUMENT  FUNCT
C            MUST BE DECLARED AS EXTERNAL IN THE CALLING PROGRAM.
C        II) IER IS SET TO 2 IF , STEPPING IN ONE OF THE COMPUTED
C            DIRECTIONS, THE FUNCTION WILL NEVER INCREASE WITHIN
C            A TOLERABLE RANGE OF ARGUMENT.
C            IER = 2 MAY OCCUR ALSO IF THE INTERVAL WHERE F
C            INCREASES IS SMALL AND THE INITIAL ARGUMENT WAS
C            RELATIVELY FAR AWAY FROM THE MINIMUM SUCH THAT THE
C            MINIMUM WAS OVERLEAPED. THIS IS DUE TO THE SEARCH
C            TECHNIQUE WHICH DOUBLES THE STEPSIZE UNTIL A POINT
C            IS FOUND WHERE THE FUNCTION INCREASES.
C
C      SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C         FUNCT
C
C      METHOD
C         THE METHOD IS DESCRIBED IN THE FOLLOWING ARTICLE
C         R. FLETCHER AND M.J.D. POWELL, A RAPID DESCENT METHOD FOR
C         MINIMIZATION,
C         COMPUTER JOURNAL VOL.6,ISS. 2, 1963, PP.163-168.
C
C      ...................................................................
C
       SUBROUTINE FMFP(FUNCT,N,X,F,G,EST,EPS,LIMIT,IER,H)
C
C      DIMENSIONED DUMMY VARIABLES
       DIMENSION H(1000),X(1000),G(1000)
C
C      COMPUTE FUNCTION VALUE AND GRADIENT VECTOR FOR INITIAL ARGUMENT
       CALL FUNCT(N,X,F,G)
C
C      RESET ITERATION COUNTER AND GENERATE IDENTITY MATRIX
       IER=0
       KOUNT=0
       N2=N+N
       N3=N2+N
       N31=N3+1
     1 K=N31
       DO 4 J=1,N
       H(K)=1.
       NJ=N-J
       IF(NJ)5,5,2
     2 DO 3 L=1,NJ
       KL=K+L
     3 H(KL)=0.
     4 K=KL+1
C
C      START ITERATION LOOP
     5 KOUNT=KOUNT +1
C
C      SAVE FUNCTION VALUE, ARGUMENT VECTOR AND GRADIENT VECTOR
       OLDF=F
       DO 9 J=1,N
       K=N+J
       H(K)=G(J)
       K=K+N
       H(K)=X(J)
C
C      DETERMINE DIRECTION VECTOR H
       K=J+N3
       T=0.
       DO 8 L=1,N
       T=T-G(L)*H(K)
       IF(L-J)6,7,7
     6 K=K+N-L
       GO TO 8
     7 K=K+1
     8 CONTINUE
     9 H(J)=T
C
C      CHECK WHETHER FUNCTION WILL DECREASE STEPPING ALONG H.
       DY=0.
       HNRM=0.
       GNRM=0.
C
C      CALCULATE DIRECTIONAL DERIVATIVE AND TESTVALUES FOR DIRECTION
C      VECTOR H AND GRADIENT VECTOR G.
       DO 10 J=1,N
```

```
      HNRM=HNRM+ABS(H(J))
      GNRM=GNRM+ABS(G(J))
   10 DY=DY+H(J)*G(J)
C
C        REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF DIRECTIONAL
C        DERIVATIVE APPEARS TO BE POSITIVE OR ZERO.
      IF(DY)11,51,51
C
C        REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF DIRECTION
C        VECTOR H IS SMALL COMPARED TO GRADIENT VECTOR G.
   11 IF(HNRM/GNRM-EPS)51,51,12
C
C        SEARCH MINIMUM ALONG DIRECTION H
C
C        SEARCH ALONG H FOR POSITIVE DIRECTIONAL DERIVATIVE
   12 FY=F
      ALFA=2.*(EST-F)/DY
      AMBDA=1.
C
C        USE ESTIMATE FOR STEPSIZE ONLY IF IT IS POSITIVE AND LESS THAN
C        1. OTHERWISE TAKE 1. AS STEPSIZE
      IF(ALFA)15,15,13
   13 IF(ALFA-AMBDA)14,15,15
   14 AMBDA=ALFA
   15 ALFA=0.
C
C        SAVE FUNCTION AND DERIVATIVE VALUES FOR OLD ARGUMENT
   16 FX=FY
      DX=DY
C
C        STEP ARGUMENT ALONG H
      DO 17 I=1,N
   17 X(I)=X(I)+AMBDA*H(I)
C
C        COMPUTE FUNCTION VALUE AND GRADIENT FOR NEW ARGUMENT
      CALL FUNCT(N,X,F,G)
      FY=F
C
C        COMPUTE DIRECTIONAL DERIVATIVE DY FOR NEW ARGUMENT. TERMINATE
C        SEARCH, IF DY IS POSITIVE. IF DY IS ZERO THE MINIMUM IS FOUND
      DY=0.
      DO 18 I=1,N
   18 DY=DY+G(I)*H(I)
      IF(DY)19,36,22
C
C        TERMINATE SEARCH ALSO IF THE FUNCTION VALUE INDICATES THAT
C        A MINIMUM HAS BEEN PASSED
   19 IF(FY-FX)20,22,22
C
C        REPEAT SEARCH AND DOUBLE STEPSIZE FOR FURTHER SEARCHES
   20 AMBDA=AMBDA+ALFA
      ALFA=AMBDA
C        END OF SEARCH LOOP
C
C        TERMINATE IF THE CHANGE IN ARGUMENT GETS VERY LARGE
      IF(HNRM*AMBDA-1.E10)16,16,21
C
C        LINEAR SEARCH TECHNIQUE INDICATES THAT NO MINIMUM EXISTS
   21 IER=2
      RETURN
C
C        INTERPOLATE CUBICALLY IN THE INTERVAL DEFINED BY THE SEARCH
C        ABOVE AND COMPUTE THE ARGUMENT X FOR WHICH THE INTERPOLATION
C        POLYNOMIAL IS MINIMIZED
   22 T=0.
   23 IF(AMBDA)24,36,24
   24 Z=3.*(FX-FY)/AMBDA+DX+DY
      ALFA=AMAX1(ABS(Z),ABS(DX),ABS(DY))
      DALFA=Z/ALFA
      DALFA=DALFA*DALFA-DX/ALFA*DY/ALFA
      IF(DALFA)51,25,25
   25 W=ALFA*SQRT(DALFA)
      ALFA=(DY+W-Z)*AMBDA/(DY+2.*W-DX)
      DO 26 I=1,N
   26 X(I)=X(I)+(T-ALFA)*H(I)
C
C        TERMINATE, IF THE VALUE OF THE ACTUAL FUNCTION AT X IS LESS
C        THAN THE FUNCTION VALUES AT THE INTERVAL ENDS. OTHERWISE REDUCE
C        THE INTERVAL BY CHOOSING ONE END-POINT EQUAL TO X AND REPEAT
```

```
C          THE INTERPOLATION.  WHICH END-POINT IS CHOOSEN DEPENDS ON THE
C          VALUE OF THE FUNCTION AND ITS GRADIENT AT X
C
       CALL FUNCT(N,X,F,G)
       IF(F-FX)27,27,28
    27 IF(F-FY)36,36,28
    28 DALFA=0.
       DO 29 I=1,N
    29 DALFA=DALFA+G(I)*H(I)
       IF(DALFA)30,33,33
    30 IF(F-FX)32,31,33
    31 IF(DX-DALFA)32,36,32
    32 FX=F
       DX=DALFA
       T=ALFA
       AMBDA=ALFA
       GO TO 23
    33 IF(FY-F)35,34,35
    34 IF(DY-DALFA)35,36,35
    35 FY=F
       DY=DALFA
       AMBDA=AMBDA-ALFA
       GO TO 22
C
C          COMPUTE DIFFERENCE VECTORS OF ARGUMENT AND GRADIENT FROM
C          TWO CONSECUTIVE ITERATIONS
    36 DO 37 J=1,N
       K=N+J
       H(K)=G(J)-H(K)
       K=N+K
    37 H(K)=X(J)-H(K)
C
C          TERMINATE, IF FUNCTION HAS NOT DECREASED DURING LAST ITERATION
       IF(OLDF-F+EPS)51,38,38
C
C          TEST LENGTH OF ARGUMENT DIFFERENCE VECTOR AND DIRECTION VECTOR
C          IF AT LEAST N ITERATIONS HAVE BEEN EXECUTED, TERMINATE, IF
C          BOTH ARE LESS THAN  EPS
    38 IER=0
       IF(KOUNT-N)42,39,39
    39 T=0.
       Z=0.
       DO 40 J=1,N
       K=N+J
       W=H(K)
       K=K+N
       T=T+ABS(H(K))
    40 Z=Z+W*H(K)
       IF(HNRM-EPS)41,41,42
    41 IF(T-EPS)56,56,42
C
C          TERMINATE, IF NUMBER OF ITERATIONS WOULD EXCEED  LIMIT
    42 IF(KOUNT-LIMIT)43,50,50
C
C          PREPARE UPDATING OF MATRIX H
    43 ALFA=0.
       DO 47 J=1,N
       K=J+N3
       W=0.
       DO 46 L=1,N
       KL=N+L
       W=W+H(KL)*H(K)
       IF(L-J)44,45,45
    44 K=K+N-L
       GO TO 46
    45 K=K+1
    46 CONTINUE
       K=N+J
       ALFA=ALFA+W*H(K)
    47 H(J)=W
C
C          REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF RESULTS
C          ARE NOT SATISFACTORY
       IF(Z*ALFA)48,1,48
C
C          UPDATE MATRIX H
    48 K=N31
       DO 49 L=1,N
       KL=N2+L
       DO 49 J=L,N
```

```
      NJ=N2+J
      H(K)=H(K)+H(KL)*H(NJ)/2-H(L)*H(J)/ALFA
   49 K=K+1
      GO TO 5
C         END OF ITERATION LOOP
C
C         NO CONVERGENCE AFTER  LIMIT  ITERATIONS
   50 IER=1
      RETURN
C
C         RESTORE OLD VALUES OF FUNCTION AND ARGUMENTS
   51 DO 52 J=1,N
      K=N2+J
   52 X(J)=H(K)
      CALL FUNCT(N,X,F,G)
C
C         REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF DERIVATIVE
C         FAILS TO BE SUFFICIENTLY SMALL
      IF(GNRM-EPS)55,55,53
C
C         TEST FOR REPEATED FAILURE OF ITERATION
   53 IF(IER)56,54,54
   54 IER=-1
      GO TO 1
   55 IER=0
   56 RETURN
      END
```

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VITA

Dady Jal Patel, son of Jal and Bacha Patel, was born in Bombay, India, on December 5, 1943. He attended St. Mary's High School, St. Xavier's College of Arts and Sciences and Walchand College of Engineering from which he successively earned a Secondary School Certificate, Inter-Science Diploma and Bachelors in Mechanical and Electrical Engineering. In September 1964 he came to the U.S.A. to do his Masters at the University of Missouri at Rolla. Upon completion, he served with Beaunit Fibers in Tennessee, as design and projects engineer. In September 1967 he entered Georgia Tech to pursue studies in control systems leading toward a Doctorate.

During his stay at Georgia Tech he served two years as a Senator to the Graduate Student Senate, one year as Vice President of the International Students Organization and one year as chairman of the Senate's International Students Committee. For his services he was awarded the Outstanding International Student Award for 1970. He has been a recipient of eight scholarships from India and a teaching assistantship from Georgia Tech.

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