

MATCHING PROBLEMS IN HYPERGRAPHS

A Dissertation
Presented to
The Academic Faculty

By

Xiaofan Yuan

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
College of Sciences
School of Mathematics

Georgia Institute of Technology

August 2022

© Xiaofan Yuan 2022

MATCHING PROBLEMS IN HYPERGRAPHS

Thesis committee:

Dr. Anton Bernshteyn
School of Mathematics
Georgia Institute of Technology

Dr. Josephine Yu
School of Mathematics
Georgia Institute of Technology

Dr. Hao Huang
Department of Mathematics
National University of Singapore

Dr. Xingxing Yu
School of Mathematics
Georgia Institute of Technology

Dr. Santosh Vempala
College of Computing
Georgia Institute of Technology

Date approved: June 9, 2022

ACKNOWLEDGMENTS

I would like to first express my sincere gratitude to my advisor Xingxing Yu for his guidance and support in my Ph.D. study. I am really grateful for his patience and energy during our collaborations, as well as for his kindness and help in my daily life challenges as a new foreigner here. I would like to thank Professor Hongliang Lu for introducing me to my dissertation topic and many interesting techniques. I am sincerely grateful for his valuable advising and many fruitful discussions as my unofficial mentor.

I would like to thank my coauthors Qiqin Xie and Shijie Xie for introducing me to new ideas in mathematics and giving me helpful advice. I am grateful to my committee, Professors Anton Bernshteyn, Hao Huang, Santosh Vempala, Josephine Yu, and Xingxing Yu, for taking the time to review this thesis.

I would like to thank all the faculty and staff in the School of Mathematics at Georgia Tech. They are always there and always willing to offer help whenever needed. In particular, thanks to Professors Prasad Tetali, Robin Thomas, and Lutz Warnke for serving on my committee for proposal and teaching me many things in combinatorics. Special thanks to Professors Wing Suet Li and Yao Yao for their support and encouragement, which helped me get through the toughest time during my PhD journey.

I would like to thank Professor Jie Ma for introducing me to mathematical research when I was an undergraduate student at USTC, as well as providing professional advice and always encouraging me along the years. I am grateful to Professors Hao Huang, Wing Suet Li, Jie Ma, Lutz Warnke, and Xingxing Yu for their advice and help during my application for postdoc.

I owe special thanks to the other students in my cohort and all the good friends I made here. Thanks to Cyrus Hettle, Yingjie Qian, Jad Salem, Reuben Tate, and Youngho Yoo for willingness to work together on our ACO coursework; and thanks to other young researchers in combinatorics: He Guo, Jie Han, Guangming Jing, Xiaonan Liu, Xizhi Liu,

Mihalis Sarantis, Songling Shan, Zhiyu Wang, Fan Wei, and Dantong Zhu for all the enjoyable discussions, which have also taught me a lot. Thanks to my friends for all the good days we spent together in Atlanta, especially to Renyi Chen, Tongzhou Chen, Zhehui Chen, Jaihui Cheng, Christina Giannitsi, Marc Härkönen, Tong Jin, Changong Li, Kunyang Li, Ruilin Li, Shasha Liao, Longmei Shu, Jieun Seong, Haodong Sun, Hao Wu, Jiaqi Yang, Yian Yao, Weiwei Zhang, Hongyi Zhou, Xingyu Zhu.

Finally, I would like to express my deepest gratitude to my parents for their unwavering love and consistent support. I would not be here without them.

TABLE OF CONTENTS

Acknowledgments	iii
Summary	viii
Chapter 1: Introduction and Background	1
1.1 Background on Matching Problems	1
1.1.1 Matchings in Graphs	1
1.1.2 Matchings in Hypergraphs	1
1.2 Degree Conditions	3
1.2.1 Perfect Matchings	3
1.2.2 Large Matchings	4
1.3 Rainbow Matchings	5
1.3.1 Degree Version	6
1.4 Remarks	6
Chapter 2: Ideas and Techniques	7
2.1 Large Matchings	7
2.1.1 Tightness	7
2.1.2 Proof Ideas	7
2.2 Rainbow Matchings in 3-Graphs	9

2.2.1	Tightness	9
2.2.2	Proof Ideas	9
2.3	Organization	10
Chapter 3: Small Matchings		12
3.1	Rainbow Matchings	12
3.2	Matchings in k -Graphs	13
Chapter 4: Extremal Cases		14
4.1	Close to Extremal Configuration at Each Vertex	14
4.1.1	k -Graphs Close to $H_k^{k-l}(U, W)$ at Each Vertex	14
4.1.2	$(1, 3)$ -Partite 4-Graphs Close to $H_{1,3}(n, n/3)$ at Each Vertex	17
4.2	$(1, 3)$ -Partite 4-Graphs Close to $H_{1,3}(n, n/3)$	18
4.3	k -Graphs Close to $H_k^{k-l}(U, W)$	21
Chapter 5: Absorbing Lemmas		30
5.1	Absorbing Matchings in $(1, 3)$ -Partite 4-Graphs	30
5.2	Absorbing Matchings in k -Graphs	33
Chapter 6: Non Extremal Cases and Independent Sets		38
6.1	Independent Sets in $(1, 3)$ -Partite 4-Graphs	39
6.2	Independent Sets in k -Graphs	43
Chapter 7: Perfect Fractional Matchings in k-Graphs		47
7.1	Shadows and Stable Families	47

7.2	Perfect Fractional Matchings	54
Chapter 8: Perfect Fractional Matchings in $(1, 3)$-Partite 4-Graphs		58
8.1	Stable Graphs	58
8.2	Perfect Fractional Matchings	60
Chapter 9: Almost Perfect Matchings		63
9.1	Almost Perfect Matchings in k -Graphs	63
9.2	Balancing in $(1, 3)$ -Partite 4-Graphs	68
9.3	Second Round of Sampling in $(1, 3)$ -Partite 4-Graphs	72
Chapter 10: Conclusions and Remarks		76
10.1	Proof of Theorem 1.3	76
10.2	Proof of Theorem 2.8	77
10.3	Remarks	77
References		79

SUMMARY

Kühn, Osthus, and Treglown [31] and, independently, Khan [28] proved that if H is a 3-uniform hypergraph with n vertices, where $n \in 3\mathbb{Z}$ and large, and $\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$, then H contains a perfect matching.

We [34] show that for $n \in 3\mathbb{Z}$ sufficiently large, if $F_1, \dots, F_{n/3}$ are 3-uniform hypergraphs with a common vertex set and $\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ for $i \in [n/3]$, then $\{F_1, \dots, F_{n/3}\}$ admits a rainbow matching, i.e., a matching consisting of one edge from each F_i . This is done by converting the rainbow matching problem to a perfect matching problem in a special class of uniform hypergraphs.

We [33] also prove that, for any integers k, l with $k \geq 3$ and $k/2 < l \leq k-1$, there exists a positive real μ such that, for all sufficiently large integers m, n satisfying

$$\frac{n}{k} - \mu n \leq m \leq \frac{n}{k} - 1 - \left(1 - \frac{l}{k}\right) \left\lceil \frac{k-l}{2l-k} \right\rceil,$$

if H is a k -uniform hypergraph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$, then H has a matching of size $m+1$. This improves upon an earlier result of Hàn, Person, and Schacht [22] for the range $k/2 < l \leq k-1$. In many cases, our result gives tight bound on $\delta_l(H)$ for near perfect matchings (e.g., when $l \geq 2k/3$, $n \equiv r \pmod{k}$, $0 \leq r < k$, and $r+l \geq k$, we can take $m = \lceil n/k \rceil - 2$).

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Background on Matching Problems

1.1.1 Matchings in Graphs

There is a long history on problems concerning the maximum number of disjoint edges in graphs. We have the following terminology for this well studied problem.

Definition 1.1. A set M of disjoint edges in a graph $G = (V, E)$ is called a *matching*. A matching is called *perfect* if it covers all vertices in the graph.

Dating back to 1935, Hall [21] proved a necessary and sufficient condition for finding a matching that covers at least one side of a bipartite graph. In 1947, Tutte [43] showed a necessary and sufficient condition for the existence of a perfect matching in a general graph. The problem of finding maximum matchings in graphs can be solved in polynomial time (see, for example, Ford-Fulkerson algorithm [12] for bipartite graph, and Blossom algorithm [9] for general graphs).

1.1.2 Matchings in Hypergraphs

A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. We write $v(H) := |V(H)|$, $e(H) := |E(H)|$ and often identify $E(H)$ with H . For any positive integer k and any set S , let $[k] := \{1, \dots, k\}$ and $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. For any positive integer k , a hypergraph H is *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and a *k-uniform* hypergraph is also called a *k-graph*.

Let H be a hypergraph. For $S \subseteq V(H)$, we use $H - S$ to denote the hypergraph obtained from H by deleting S and all edges of H with a vertex in S , and we use $H[S]$ to denote the hypergraph with vertex set S and edge set $\{e \in E(H) : e \subseteq S\}$. For $S \subseteq R \subseteq$

$V(H)$, let $N_{H-R}(S) = \{T \subseteq V(H) \setminus R : S \cup T \in E(H)\}$, and let $N_H(S) := N_{H-S}(S)$. For any positive integer n , let $[n] := \{1, \dots, n\}$.

A *matching* in a hypergraph H is a set of pairwise disjoint edges in H . If M is a matching in H , we write $V(M) := \bigcup_{e \in M} e$. The size of a largest matching in H is denoted by $\nu(H)$, known as the *matching number* of H . A matching in H is *perfect* if it covers all vertices of H . A matching is *nearly perfect* in H if it covers all but a constant number of vertices. Moreover, a matching in a k -graph is *near perfect* if it covers all but at most k vertices.

The problem for finding maximum matchings in hypergraphs is NP-hard, even for 3-graphs [25]. It is of interest to find good sufficient conditions that guarantee large matchings.

Erdős [11] proposed the following conjecture in 1965.

Conjecture 1.2 (Erdős [11]). *For positive integers k, n, t , if H is a k -graph on n vertices and $\nu(H) < t$, then*

$$e(H) \leq \max \left\{ \binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k} \right\}.$$

This bound is tight because of the complete k -graph on $kt-1$ vertices and the k -graph on n vertices in which every edge intersects a fixed set of $t-1$ vertices. In the same paper Erdős proved this conjecture to be true for $n > n_0(k, s)$. Later in 1976, Bollobás, Daykin and Erdős [8] proved this conjecture for $n > 2k^3s$. For recent progress on this conjecture, see [3, 4, 14, 15, 18, 24, 35]. In particular, Huang, Loh, and Sudakov [24] proved this conjecture for $n > 3k^2s$; Frankl [14] proved that if $n \geq (2t-1)k - (t-1)$ and $\nu(H) < t$ then $e(H) \leq \binom{n}{k} - \binom{n-t+1}{k}$; and this result was further improved by Frankl and Kupavskii [17]. In the case of $k = 3$, Frankl, Rödl, and Ruciński [19] proved this conjecture for $n \geq 4s$; Łuczak and Mieczkowska [35] proved this conjecture for $s > s_0$; and Frankl [15] proved this conjecture to be true for any 3-graph.

1.2 Degree Conditions

There has been extensive study on degree conditions for large matchings in uniform hypergraphs. Let H be a hypergraph and $T \subseteq V(H)$. The *degree* of T in H , denoted by $d_H(T)$, is the number of edges in H containing T . For any integer $l \geq 0$, let $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ denote the minimum l -degree of H . Note that $\delta_0(H)$ is the number of edges in H , and $\delta_1(H)$ is often called the minimum *vertex degree* of H . When H is a k -graph for some positive integer k , $\delta_{k-1}(H)$ is known as the minimum *codegree* of H .

For $u \in V(H)$, let $N_H(u) := \{e : e \subseteq V(H) \setminus \{u\} \text{ and } e \cup \{u\} \in E(H)\}$. When there is no confusion, we also view $N_H(u)$ as a hypergraph with vertex set $V(H) \setminus \{u\}$ and edge set $N_H(u)$.

For integers n, k, d satisfying $0 \leq d \leq k - 1$ and $n \in k\mathbb{Z}$, let $m_d(k, n)$ denote the minimum integer m such that every k -graph H on n vertices with $\delta(H) \geq m$ has a perfect matching.

Bollobás, Daykin, and Erdős [8] considered minimum vertex degree conditions for matchings in k -graphs. They proved that if H is a k -graph of order $n \geq 2k^2(m + 2)$ and $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then $\nu(H) \geq m$.

1.2.1 Perfect Matchings

For 3-graphs, Kühn, Osthus, and Treglown [31] and, independently, Khan [28] proved the following stronger result: There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geq n_0$, $m \leq n/3$, and $\delta_1(H) > \binom{n-1}{2} - \binom{n-m}{2}$, then $\nu(H) \geq m$.

In [29], Kühn and Osthus proved that there exists $n_0 \in \mathbb{N}$ such that if H is a k -graph of order $n \geq n_0$ and $\delta_{k-1}(H) \geq n/2 + 3k^2\sqrt{n \log n}$, then H has a perfect matching. Rödl, Ruciński, and Szemerédi [39] determined the minimum codegree threshold for the existence of a perfect matching in a k -graph. Pikhurko [37] showed if $l \geq k/2$ and H is a k -graph whose order n is divisible by k then H has a perfect matching provided that $\delta_l(H) \geq$

$(1/2 + o(1))\binom{n}{k-l}$. Treglown and Zhao [42] determined the exact l -degree threshold for perfect matching when $k/2 \leq l \leq k-1$, where they also determined the extremal families.

Hàn, Person, and Schacht [22] considered the minimum l -degree condition for perfect matchings in the range $1 \leq l \leq k/2$. In particular, they showed that if H is a 3-graph and $\delta_1(H) > (1 + o(1))\frac{5}{9}\binom{|V(H)|}{2}$ then H has a perfect matching.

1.2.2 Large Matchings

For near perfect matchings, Han [23] proved a conjecture of Rödl, Ruciński, and Szemerédi [39] that, for $n \not\equiv 0 \pmod{k}$, the co-degree threshold for the existence of a near perfect matching in a k -graph H is $\lfloor n/k \rfloor$. This is much smaller than the co-degree threshold (roughly $n/2$) obtained by Rödl, Ruciński, and Szemerédi [39] for perfect matchings.

For nearly perfect matchings, Hàn, Person, and Schacht [22] proved the following result: For any integers $k > l > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ with $n \in k\mathbb{Z}$ and for every n -vertex k -graph H with

$$\delta_l(H) \geq \frac{k-l}{k} \binom{n}{k-l} + k^{k+1} (\ln n)^{1/2} n^{k-l-1/2},$$

H contains a matching covering all but $(l-1)k$ vertices.

Our Result on Large Matchings

We [33] improve this bound for the range $k/2 < l \leq k-1$, by providing an exact l -degree threshold for the existence of a matching covering all but at most $(k-l)\lceil(k-l)/(2l-k)\rceil + k-1$ vertices.

Theorem 1.3 (Lu, Yu, and Yuan [33], 2021). *For any integers k, l satisfying $k \geq 3$ and $k/2 < l \leq k-1$, there exists a positive real μ such that, for all integers m, n satisfying*

$$\frac{n}{k} - \mu n \leq m \leq \frac{n}{k} - 1 - \left(1 - \frac{l}{k}\right) \left\lceil \frac{k-l}{2l-k} \right\rceil \quad (1.1)$$

and n sufficiently large, if H is a k -graph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$ then $\nu(H) \geq m + 1$.

When $l \geq 2k/3$, we have $(k-l)/(2l-k) \leq 1$. Moreover, if $n \equiv r \pmod{k}$, $0 \leq r < k$, and $r + l \geq k$ then Theorem 1.3 with $m = \lceil n/k \rceil - 2$ implies that H has a matching covering all but at most k vertices. In general, if the interval $[n/k - 2, n/k - 1 - (1 - l/k) \lceil (k-l)/(2l-k) \rceil]$ contains an integer, then by letting m be that integer, the conditions of Theorem 1.3 imply that H has a near perfect matching.

The bound on $\delta_l(H)$ in Theorem 1.3 is best possible because of the classical space barrier: Consider a k -graph in which edges are all the k -subsets intersecting a specific set of size m , then this k -graph satisfies the degree condition, but its matching number is at most m .

1.3 Rainbow Matchings

There are attempts to extend the above conjecture of Erdős to families of hypergraphs. Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family of hypergraphs. A set of pairwise disjoint edges, one from each F_i , is called a *rainbow matching* for \mathcal{F} . (In this situation, we also say that \mathcal{F} or $\{F_1, \dots, F_t\}$ *admits* a rainbow matching.) Aharoni and Howard [1] made the following conjecture, which first appeared in Huang, Loh, and Sudakov [24]:

Conjecture 1.4. *Let t be a positive integer and $\mathcal{F} = \{F_1, \dots, F_t\}$ such that, for $i \in [t]$, $F_i \subseteq \binom{[n]}{k}$ and $e(F_i) > \max \left\{ \binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k} \right\}$; then \mathcal{F} admits a rainbow matching.*

Huang, Loh, and Sudakov [24] showed that this conjecture holds when $n > 3k^2t$. When $k = 2$, this conjecture is true as a direct consequence of an earlier result of Akiyama and Frankl [2]. Recently, Keller and Lifshitz [27] showed that this conjecture holds when $n \geq f(t)k$ for some large constant $f(t)$ which only depends on t ; Frankl and Kupavskii [16] improved the bound to $n \geq 12tk \log(e^2t)$; and Lu, Wang, and Yu [32] improved the bound

to $n \geq 2kt$ and n is sufficiently large. Gao, Lu, Ma, and Yu [20] confirmed Conjecture 1.4 for $k = 3$ and sufficiently large n that does not depend on t .

1.3.1 Degree Version

We [34] prove a degree version of the above conjecture for rainbow matchings, which extends the results of Kühn, Osthus, and Treglown [31] and, independently, of Khan [28] for 3-graphs to families of 3-graphs. Here we use \mathbb{Z} to denote the set of all integers, and $3\mathbb{Z}$ is the set of integers divisible by 3.

Theorem 1.5 (Lu, Yu, and Yuan [34], 2021). *Let $n \in 3\mathbb{Z}$ be positive and sufficiently large and let $\mathcal{F} = \{F_1, \dots, F_{n/3}\}$ be a family of n -vertex 3-graphs such that $V(F_i) = V(F_1)$ for $i \in [n/3]$. If $\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ for $i \in [n/3]$, then \mathcal{F} admits a rainbow matching.*

The bound on $\delta_1(F_i)$ in Theorem 1.5 is sharp because of a classical space obstruction, which will be described in details in the next chapter.

1.4 Remarks

This thesis is based on joint work with Hongliang Lu and Xingxing Yu [33], [34].

CHAPTER 2

IDEAS AND TECHNIQUES

2.1 Large Matchings

In this section, we discuss the tightness and proof ideas of Theorem 1.3.

2.1.1 Tightness

To see that the bound on $\delta_l(H)$ in Theorem 1.3 is best possible, we define the following graphs, which is often call a ‘space barrier’ in the literature.

Definition 2.1. Let $H_k^l(U, W)$ denote the k -graph such that U, W is a partition of $V(H_k^l(U, W))$ and the edges of $H_k^l(U, W)$ are precisely those k -subsets of $V(H_k^l(U, W))$ intersecting W at least once.

For integers k, l, n with $k \geq 2$ and $0 < l < k$ and for large n , $\delta_l(H_k^l(U, W)) = \binom{n-l}{k-l} - \binom{(n-l)-|W|}{k-l}$ and the matching number of $H_k^l(U, W)$ is $|W|$. Thus, the bound on $\delta_l(H)$ in Theorem 1.3 is best possible (by letting $|W| = m$).

2.1.2 Proof Ideas

To prove Theorem 1.3, we need to refine the definition of $H_k^l(U, W)$ to $H_k^s(U, W)$ for all $s \in [k]$.

Definition 2.2. For each $s \in [k]$, let $H_k^s(U, W)$ be a k -graph, where U, W is a partition of the vertex set, and the edges of $H_k^s(U, W)$ are precisely those k -subsets of $V(H_k^s(U, W))$ intersecting W at least once and at most s times.

We also need the following definition describing the ‘closeness’ of two k -graphs.

Definition 2.3. Given two k -graphs H_1, H_2 and a real number $\varepsilon > 0$, we say that H_2 is ε -close to H_1 if $V(H_1) = V(H_2)$ and $|E(H_1) \setminus E(H_2)| \leq \varepsilon |V(H_1)|^k$.

Roughly speaking, most edges of H_1 are also edges of H_2 . Our proof of Theorem 1.3 consists of two parts by considering whether or not H is “close” to $H_k^{k-l}(U, W)$, which is similar to arguments in [39]. In the next two paragraphs, we give an outline for each case.

We first consider the case when $V(H)$ has a partition U, W with $|W| = m$ such that H is close to $H_k^{k-l}(U, W)$. If every vertex of H is “good” (to be made precise later) with respect to $H_k^{k-l}(U, W)$ then we find the desired matching by a greedy argument. Otherwise, we find the desired matching in two steps by first finding a matching M' such that every vertex in $H - V(M')$ is good, thereby reducing the problem to the previous case.

The other case is when H is not close to $H_k^{k-l}(U, W)$ for any partition $V(H)$ into U, W with $|W| = m$. We will see that such H does not have any sparse subset of very large size. To deal with this case, we will use the following approach of Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [3]:

- Find a small absorbing matching M_a in H ,
- find random subgraphs of $H - V(M_a)$ with perfect fractional matchings (see Chapter 7 for definition),
- use those random subgraphs and a theorem of Frankl and Rödl to find an almost perfect matching M' in $H - V(M_a)$ (see Lemma 9.4), and
- use the matching M_a to absorb the remaining vertices in $V(H) \setminus (V(M_a) \cup V(M'))$.

To find a perfect fractional matching in certain random subgraphs of $H - V(M_a)$ we need to prove a stability version of a result of Frankl [14] on the Erdős matching conjecture [11], which might be of independent interest. We also need to use the hypergraph container result of Balogh, Morris, and Samotij [6] (proved independently by Saxton and Thomason [41]) to bound the independence number of random subgraphs of H .

2.2 Rainbow Matchings in 3-Graphs

In this section, we discuss the tightness and proof ideas of Theorem 1.5.

2.2.1 Tightness

The bound on $\delta_1(F_i)$ in Theorem 1.5 is sharp. To see this, we consider the following 3-graph, which is a special case of $H_k^{k-l}(U, W)$ when $k = 3$ and $l = 1$.

Definition 2.4. Let $m \leq n/3$ and let $H(n, m)$ denote a 3-graph that is isomorphic to the 3-graph with vertex set $[n]$ and edge set

$$\left\{ e \in \binom{[n]}{3} : e \not\subseteq [m] \text{ and } e \cap [m] \neq \emptyset \right\}.$$

Note that for $n \in 3\mathbb{Z}$, $\delta_1(H(n, n/3 - 1)) = \binom{n-1}{2} - \binom{2n/3}{2}$ and $H(n, n/3 - 1)$ has no perfect matching. Hence, the family of $n/3$ copies of $H(n, n/3 - 1)$ admits no rainbow matching.

2.2.2 Proof Ideas

To prove Theorem 1.5, we convert this rainbow matching problem to a perfect matching problem for a special class of hypergraphs.

Definition 2.5. For any integer $k \geq 2$, a k -graph H is $(1, k - 1)$ -*partite* if there exists a partition of $V(H)$ into sets V_1, V_2 (called *partition classes*) such that for any $e \in E(H)$, $|e \cap V_1| = 1$ and $|e \cap V_2| = k - 1$.

Definition 2.6. A $(1, k - 1)$ -partite k -graph with partition classes V_1, V_2 is *balanced* if $(k - 1)|V_1| = |V_2|$.

Definition 2.7. Let $n \in 3\mathbb{Z}$, let P and Q be disjoint sets such that $|P| = n$ and $|Q| = n/3$, and let $Q = \{v_1, \dots, v_{n/3}\}$. Let $\mathcal{F} = \{F_1, \dots, F_{n/3}\}$ be a family of 3-graphs on the same

vertex set P . We use $H_{1,3}(\mathcal{F})$ to represent the balanced $(1, 3)$ -partite 4-graph with partition classes Q, P and edge set $\bigcup_{i=1}^{n/3} E_i$, where $E_i = \{e \cup \{v_i\} : e \in E(F_i)\}$ for $i \in [n/3]$. If $E(F_i) = E(H(n, n/3))$ and $V(F_i) = V(H(n, n/3))$ for all $i \in [n/3]$, then we write $H_{1,3}(n, n/3)$ for $H_{1,3}(\mathcal{F})$.

The following observations will be useful:

- (i) $E(F_i)$ is the neighborhood of v_i in $H_{1,3}(\mathcal{F})$ for $i \in [n/3]$, and \mathcal{F} admits a rainbow matching if, and only if, $H_{1,3}(\mathcal{F})$ has a perfect matching.
- (ii) $e(F_i) \geq \frac{n}{3} \delta_1(F_i)$ for all $i \in [n/3]$, and $d_{H_{1,3}(\mathcal{F})}(v) \geq \sum_{i=1}^{n/3} \delta_1(F_i)$ for $v \in P$.
- (iii) $d_{H_{1,3}(\mathcal{F})}(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$, provided $\delta_1(F_i) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for $i \in [n/3]$.
- (iv) $\delta_1(H_{1,3}(\mathcal{F})) \geq \frac{n}{3} \left(\binom{n-1}{2} - \binom{2n/3}{2} + 1 \right)$, provided $d_{H_{1,3}(\mathcal{F})}(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$.

By observations (i) and (iii), Theorem 1.5 follows from the following result.

Theorem 2.8. *Let $n \in 3\mathbb{Z}$ be positive and sufficiently large, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $|P| = n$ and $|Q| = n/3$. Suppose $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$. Then H has a perfect matching.*

To prove Theorem 2.8, we take the usual approach by considering whether or not H is close to some $H_{1,3}(n, n/3)$ on the same vertex set of H .

2.3 Organization

Given $\varepsilon > 0$ and two k -graphs H_1, H_2 with $V(H_1) = V(H_2)$, we say that H_2 is ε -close to H_1 if $|E(H_1) \setminus E(H_2)| < \varepsilon |V(H_1)|^k$.

In the next two chapters, we prove Theorem 1.3 for k -graphs H such that $V(H)$ has a partition U, W with $|W| = m$ and H is ε -close to $H_k^{k-l}(U, W)$. We also prove Theorem 2.8

when H is close to some $H_{1,3}(n, n/3)$ using the structure of $H_{1,3}(n, n/3)$ to find a perfect matching in H greedily. We call this the extremal case, because the structures of H are close to the extremal configurations.

In Chapter 5, we prove two absorbing lemmas, using a standard second moment method. In Chapter 6, we relate the property of “close to extremal configurations” to the existence of certain large independent sets, which allows us to consider small random induced subgraphs that inherit this property. In those small random induced subgraphs, we seek for perfect fractional matchings, which will be taken care of in Chapters 7 and 8. Then we take a second round of random sampling to find almost perfect matchings in Chapter 9.

In the last chapter, we conclude our proofs of Theorem 1.3 and Theorem 2.8, and make some related remarks.

CHAPTER 3

SMALL MATCHINGS

The goal for this chapter and the next chapter is to prove that Theorem 1.3 and Theorem 2.8 hold for the case when the hypergraph H is close to the extremal construction, that is, when $V(H)$ has a partition U, W with $|W| = m$ such that H is close to $H_k^{k-l}(U, W)$ in Theorem 1.3, or when H is close to some $H_{1,3}(n, n/3)$ on $V(H)$ in Theorem 2.8. In fact, in this case, the assertion of Theorem 1.3 holds for all $m \leq n/k - 1$.

In this chapter, we prove a result on rainbow matchings for a small family of hypergraphs, and as a direct corollary, the assertion of Theorem 1.3 holds for $m \leq n/(2k^4)$. These lemmas will serve as induction bases for our proofs.

3.1 Rainbow Matchings

Lemma 3.1. *Let n, t, k be positive integers such that $n > 2k^4t$. Let $F_i, i \in [t]$, be n -vertex k -graphs with a common vertex set. If $\delta_1(F_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1}$ for $i \in [t]$ then $\{F_1, \dots, F_t\}$ admits a rainbow matching.*

Proof. We apply induction on t . Note that the assertion is trivial when $t = 1$. So assume $t > 1$ and the assertion holds for $t - 1$. Then, since $\delta_1(F_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1} > \binom{n-1}{k-1} - \binom{n-(t-1)}{k-1}$, $\{F_1, \dots, F_{t-1}\}$ admits a rainbow matching, say M .

Suppose for a contradiction that $\{F_1, \dots, F_t\}$ does not admit a rainbow matching. Then every edge of F_t must intersect M . So there exists $v \in V(M)$ such that $d_{F_t}(v) > e(F_t)/(kt)$. Note that

$$\begin{aligned} \delta_1(F_t) &> \binom{n-1}{k-1} - \binom{n-t}{k-1} > \binom{n-1}{k-1} \left(1 - \left(1 - \frac{t-1}{n-1}\right)^{k-1}\right) \\ &> \frac{t(k-1)}{2(n-1)} \binom{n-1}{k-1} \end{aligned}$$

since $n > 2k^4t$. So we have

$$d_{F_t}(v) > \frac{\delta_1(F_t)n/k}{kt} > \frac{t(k-1)n}{2(n-1)k^2t} \binom{n-1}{k-1} > \frac{1}{2k^2} \binom{n-1}{k-1}.$$

Let $F'_i = F_i - v$ for $i \in [t-1]$. Since

$$\delta_1(F'_i) \geq \delta_1(F_i) - \binom{n-2}{k-2} > \binom{n-1}{k-1} - \binom{n-t}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1} - \binom{n-t}{k-1},$$

it follows from induction hypothesis that $\{F'_1, \dots, F'_{t-1}\}$ admits a rainbow matching, say M' .

Note that the number of edges in F_t containing v and intersecting M' is at most

$$k(t-1) \binom{n-2}{k-2} < \frac{1}{2k^2} \binom{n-1}{k-1} < d_{F_t}(v),$$

as $n \geq 2k^4t$. Hence, v is contained in some edge of $F_t - V(M')$, say e . Now $M' \cup \{e\}$ is a rainbow matching for $\{F_1, \dots, F_t\}$, a contradiction. \blacksquare

3.2 Matchings in k -Graphs

In the above lemma, if we consider the case when F_1, \dots, F_t are identical graphs, then the result converts back to a result of the original matching problem in a hypergraph. More specifically, we have the following result.

Lemma 3.2. *Let n, m, k, l be positive integers such that $k \geq 3$, $m \leq n/(2k^4)$, and $l \in [k-1]$. Let H be a k -graph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$. Then $\nu(H) \geq m+1$.*

CHAPTER 4

EXTREMAL CASES

In this chapter, we use the structure of $H_k^{k-l}(U, W)$ (or $H_{1,3}(n, n/3)$, respectively) to construct the desired matching in H when H is close to $H_k^{k-l}(U, W)$ (or $H_{1,3}(n, n/3)$, respectively). We first deal with the case when all vertices of H are “good”, and then deal with those “bad” vertices. The “good” vertices and “bad” vertices are defined precisely in the following sections.

4.1 Close to Extremal Configuration at Each Vertex

In this section, we prove Theorem 1.3 for the case where, for each vertex $v \in V(H)$, only a small number of edges of $H_k^{k-l}(U, W)$ containing v do not belong to H ; and we prove Theorem 2.8 for the case when, for every vertex v , most of the edges of $H_{1,3}(n, n/3)$ containing v also lie in H .

4.1.1 k -Graphs Close to $H_k^{k-l}(U, W)$ at Each Vertex

Let H be a k -graph and let U, W be a partition of $V(H)$ and let $n = |U| + |W|$.

Definition 4.1. Given real number α with $0 < \alpha < 1$, a vertex $v \in V(H)$ is called α -good with respect to $H_k^{k-l}(U, W)$, if

$$\left| N_{H_k^{k-l}(U, W)}(v) \setminus N_H(v) \right| \leq \alpha n^{k-1};$$

and, otherwise, v is called α -bad.

Roughly speaking, a vertex x is good if the neighborhood of x in H contains most of the neighborhood of x in $H_k^{k-l}(U, W)$.

This notion quantifies the closeness of H to $H_k^{k-l}(U, W)$ at a vertex. Clearly, if H is ε -close to $H_k^{k-l}(U, W)$, then the number of α -bad vertices in H is at most $k\varepsilon n/\alpha$, as, otherwise,

$$\begin{aligned} |E(H_k^{k-l}(U, W)) \setminus E(H)| &\geq \frac{1}{k} \sum_{v \in V(H)} |N_{H_k^{k-l}(U, W)}(v) \setminus N_H(v)| \\ &\geq \frac{1}{k} (k\varepsilon n/\alpha)(\alpha n^{k-1}) = \varepsilon n^k, \end{aligned}$$

a contradiction. Note that in the statement of the lemma below we use $m \geq n/(2k^5)$ rather than $m \geq n/(2k^4)$ as opposed to Lemma 3.2. The reason is for its application in the proof of Lemma 4.6, where we use it to deal with a graph obtained by deleting all bad vertices and some neighbors.

Lemma 4.2. *Let k, l, m, n be integers and α be a positive real, such that $k \geq 3$, $l \in [k-1]$, $\alpha < (8^{k-1}k^{5(k-1)}k!)^{-1}$, $n \geq 8k^6$, and $n/(2k^5) \leq m \leq n/k$. Suppose that H is a k -graph on n vertices and U, W is a partition of $V(H)$ with $|W| = m$ such that every vertex of H is α -good with respect to $H_k^{k-l}(U, W)$. Then $\nu(H) \geq m$.*

Proof. We find a matching of size m in H using those edges that intersect W just once. Let M be a maximum matching in H such that $|e \cap W| = 1$ for each $e \in M$, and let $t = |M|$. We may assume $t < m$; or else M is the desired matching. So $W \setminus V(M) \neq \emptyset$. By the maximality of M , $N_H(x) \cap \binom{U \setminus V(M)}{k-1} = \emptyset$ for all $x \in W \setminus V(M)$.

We claim that $t \geq m/2$. For, suppose $t < m/2$. Since $m \leq n/k$, $t < n/(2k)$; so $|V(H) \setminus V(M)| = n - tk > n - n/2 = n/2$. Hence,

$$|U \setminus V(M)| > |V(H) \setminus V(M)| - |W| \geq n/2 - n/k \geq n/6.$$

Thus, for any $x \in W \setminus V(M)$,

$$|N_{H_k^{k-l}(U, W)}(x) \setminus N_H(x)| \geq \left| \binom{|U \setminus V(M)|}{k-1} \right| > \binom{n/6}{k-1} > \alpha n^{k-1},$$

contradicting the assumption that every vertex in H is α -good.

Since $t < m \leq n/k$ and $|e \cap W| = 1$ for each $e \in M$, there exists a k -set $S = \{u_1, \dots, u_k\} \subseteq V(H) \setminus V(M)$ such that $u_k \in W$ and $S \setminus \{u_k\} \subseteq U$. Since $m \geq n/(2k^5) > 2k$, we have $t \geq m/2 > k$. We will consider the k -subsets of vertices in M and a fixed S , and seek for a contradiction to the assumption that u_1, \dots, u_k are good.

Arbitrarily choose $k - 1$ pairwise distinct edges e_1, \dots, e_{k-1} from M and write $e_i := \{v_{i,1}, v_{i,2}, \dots, v_{i,k}\}$ such that $v_{i,k} \in W$, and $v_{i,j} \in U$ for $j \in [k - 1]$. For convenience, let $v_{k,j} := u_j$ for $j \in [k]$. For $i \in [k]$, define $f_i := \{v_{1,1+i}, v_{2,2+i}, \dots, v_{k-1,(k-1)+i}, v_{k,k+i}\}$, where the addition in the subscripts is modulo k (except that we write k for 0). Then $f_i \notin E(H)$ for some $i \in [k]$ as, otherwise, $(M \setminus \{e_i : i \in [k - 1]\}) \cup \{f_i : i \in [k]\}$ is a matching in H that contradicts the maximality of M .

Note that for different choices of $e_1, \dots, e_{k-1} \in M$ and $e'_1, \dots, e'_{k-1} \in M$, the corresponding sets $\{f_1, \dots, f_k\}$ and $\{f'_1, \dots, f'_k\}$ constructed in the above paragraph are disjoint, that is, $f_i \neq f'_j$ for $i \neq j$ and $i, j \in [k]$. Since there are $\binom{t}{k-1}$ choices of e_1, \dots, e_{k-1} from M , and each provides at least one edge in $H_k^{k-l}(U, W)$ but not in H , we have

$$\begin{aligned}
& \sum_{i=1}^k \left| N_{H_k^{k-l}(U, W)}(u_i) \setminus N_H(u_i) \right| \\
& \geq \binom{t}{k-1} \\
& > \frac{(t - (k-1) + 1)^{k-1}}{(k-1)!} \\
& > \frac{(n/(4k^5) - (k-1))^{k-1}}{(k-1)!} \quad (\text{since } t \geq m/2 > n/(4k^5)) \\
& > \frac{(n/(8k^5))^{k-1}}{(k-1)!} \quad (\text{since } n \geq 8k^6) \\
& = (8^{k-1} k^{5(k-1)} k!)^{-1} k n^{k-1} \\
& > \alpha k n^{k-1} \quad (\text{since } \alpha < (8^{k-1} k^{5(k-1)} k!)^{-1}).
\end{aligned}$$

Thus there exists $u_j \in S$ such that

$$\left| N_{H_k^{k-l}(U,W)}(u_j) \setminus N_H(u_j) \right| > \alpha n^{k-1},$$

contradicting the assumption that every vertex in H is α -good. ■

4.1.2 $(1, 3)$ -Partite 4-Graphs Close to $H_{1,3}(n, n/3)$ at Each Vertex

Next we prove Theorem 2.8 for the case when the $(1, 3)$ -partite 4-graphs is close to $H_{1,3}(n, n/3)$ everywhere, using a similar idea to the proof of Lemma 4.2.

Definition 4.3. Given $\alpha > 0$, $H_{1,3}(n, n/3)$ defined in Definition 2.7, and a $(1, 3)$ -partite 4-graph H with $V(H) = V(H_{1,3}(n, n/3))$, we say that a vertex $v \in V(H)$ is α -good with respect to $H_{1,3}(n, n/3)$ if $|N_{H_{1,3}(n, n/3)}(v) \setminus N_H(v)| \leq \alpha n^3$. Otherwise we say that v is α -bad with respect to $H_{1,3}(n, n/3)$.

Lemma 4.4. *Let n be sufficiently large positive integer and H be a balanced $(1, 3)$ -partite 4-graph on $4n/3$ vertices, and let α be a positive constant less than 2^{-12} . If all vertices of H are α -good with respect to some $H_{1,3}(n, n/3)$ on $V(H)$, then H has a perfect matching.*

Proof. Let Q, P be the partition classes of H , and let $U \cup W$ be partition classes of $H(n, n/3)$ (as in Definition 2.4), such that

$$|Q| = |W| = n/3, \quad |U| = 2n/3, \quad \text{and } V(H(n, n/3)) = P.$$

Let M be a matching in H that only uses edges consisting of two vertices from U and one vertex from each of Q and W , and choose such M that $|M|$ is maximum. Let $Q' := Q \setminus V(M)$, $U' = U \setminus V(M)$, and $W' = W \setminus V(M)$. Then $|U'|/2 = |W'| = |Q'|$.

Note that $|M| \geq n/4$. For, otherwise, $|U'|/2 = |W'| = |Q'| = n/3 - |M| > n/12$.

Then, by the maximality of M , we have, for any $u \in U'$,

$$|N_{H_{1,3}(n, n/3)}(u) \setminus N_H(u)| \geq |Q'| |W'| (|U'| - 1) > n^3/12^3 > \alpha n^3,$$

a contradiction.

Now suppose M is not a perfect matching in H . Then Q', U', W' are all non-empty. Let $v \in Q'$, $u_1, u_2 \in U'$ be distinct, and $w \in W'$.

Let $\{e_1, e_2, e_3\}$ be an arbitrary set of three pairwise distinct edges from M . By the maximality of M , no matching of size 4 in H is contained in $e_1 \cup e_2 \cup e_3 \cup \{v, w, u_1, u_2\}$ and uses only edges with two vertices from U and one vertex from each of Q and W . Hence, there exists $S \in E(H_{1,3}(n, n/3)) \setminus E(H)$ such that $S \subseteq e_1 \cup e_2 \cup e_3 \cup \{v, w, u_1, u_2\}$, $|S \cap e_i| = 1$ for $i \in [3]$, $|S \cap \{v, w, u_1, u_2\}| = 1$, and S has two vertices from U and one vertex from each of Q and W .

Note that there are $\binom{m}{3}$ choices for $\{e_1, e_2, e_3\}$, which result in distinct choices for S . So the number of edges in $E(H_{1,3}(n, n/3)) \setminus E(H)$ containing exactly one vertex from $\{v, w, u_1, u_2\}$ is at least

$$\binom{m}{3} \geq \binom{n/4}{3} > n^3/(2^{10}).$$

This implies that for some $u \in \{v, w, x_1, x_2\}$,

$$|N_{H_{1,3}(n, n/3)}(u) \setminus N_H(u)| > n^3/(2^{12}) > \alpha n^3,$$

a contradiction. ■

4.2 (1, 3)-Partite 4-Graphs Close to $H_{1,3}(n, n/3)$

Now we are ready to complete the proof of Theorem 2.8 in the case when H is close to some $H_{1,3}(n, n/3)$. We use $\alpha \ll \beta$ to mean that α is sufficiently smaller than β .

Lemma 4.5. *Let $n \in 3\mathbb{Z}$ be positive and $\varepsilon > 0$ be sufficiently small, and let H be a*

balanced $(1, 3)$ -partite 4-graph with partition classes Q, P and $3|Q| = |P| = n$. Suppose H is ε -close to some $H_{1,3}(n, n/3)$ with $P = V(H(n, n/3))$. If $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$, then H has a perfect matching.

Proof. Let U, W denote the partition of $P = V(H(n, n/3))$ such that $|W| = |U|/2 = n/3$ (as in Definition 2.4 when $m = n/3$, $W = [m]$ and $U = V \setminus W$). Note that $|Q| = n/3$. Let B denote the set of $\sqrt{\varepsilon}$ -bad vertices of H with respect to $H_{1,3}(n, n/3)$. Since H is ε -close to $H_{1,3}(n, n/3)$, we have $|B| \leq 4\sqrt{\varepsilon}n$. Let $Q \cap B = \{v_1, \dots, v_q\}$ and $Q = \{v_1, \dots, v_{n/3}\}$, and let $W' \subseteq W \setminus B$ such that $|W'| = n/3 - (q + |W \cap B|) \geq n/3 - 4\sqrt{\varepsilon}n$.

First, we find a matching M'_0 in $H - W'$ covering $Q \cap B$. For this, let F_i be the subgraph of $N_H(v_i)$ induced by $N_H(v_i) - W'$ for $i \in [n/3]$. Note that, for $i \in [n/3]$, $\delta_1(N_H(v_i)) = \min\{d_H(\{u, v_i\}) : u \in P\} \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$. Hence,

$$\begin{aligned} \delta_1(F_i) &\geq \delta_1(N_H(v_i)) - \left(\binom{n-1}{2} - \binom{n-|W'| - 1}{2} \right) \\ &> \binom{n-|W'| - 1}{2} - \binom{2n/3}{2} \\ &= \binom{n-|W'| - 1}{2} - \binom{n-|W'| - (q + |W \cap B|)}{2}. \end{aligned}$$

Since $|B| \leq 4\sqrt{\varepsilon}n$, $q + |W \cap B| = n/3 - |W'| < (n - |W'|)/(2 \cdot 3^4)$. Hence by Lemma 3.1, $\{F_1, \dots, F_{q+|W \cap B|}\}$ admits a rainbow matching, say M_0 . Let $M_0 = \{e_i \in E(F_i) : i \in [q + |W \cap B|]\}$, and let $M'_0 = \{e_i \cup \{v_i\} : i \in [q + |W \cap B|]\}$. Then M'_0 is a matching in H and $Q \cap B \subseteq V(M'_0)$.

Next, we find a matching in $H_1 := H - V(M'_0)$ covering $B \setminus V(M'_0)$, in two steps. Since ε is very small, we can choose η such that $\sqrt{\varepsilon} \ll \eta \ll 1$. We divide $B \setminus V(M'_0)$ to two disjoint sets B_1, B_2 such that, for each $x \in B \setminus V(M'_0)$, $x \in B_1$ if, and only if, H_1 has at least ηn^3 edges each of which contains x and exactly one vertex in W' .

We greedily pick a matching M_1 in H_1 such that $B_1 \subseteq V(M_1)$ and every edge of M_1 contains at least one vertex from B_1 and exactly one vertex from W' . This can be done since each time we pick an edge e for a vertex $x \in B_1$, we have at least ηn^3 choices and at

most $4(4\sqrt{\varepsilon}n)n^2$ ($\ll \eta n^3$ as $\sqrt{\varepsilon} \ll \eta$) of which intersect a previously chosen edge.

Now we find a matching M_2 in $H_2 := H_1 - V(M_1)$ such that $B_2 \subseteq V(M_2)$. Note that

$$\delta_1(H_2) \geq \delta_1(H) - 4|M'_0 \cup M_1|n^2 \geq \frac{n}{3} \left(\binom{n-1}{2} - \binom{2n/3}{2} + 1 \right) - 16\sqrt{\varepsilon}n^3.$$

Hence, for any $x \in B_2$, the number of edges containing x and disjoint from W' is at least

$$\delta_1(H_2) - \eta n^3 - |Q| \binom{|W'|}{2} > \eta n^3,$$

as $\sqrt{\varepsilon} \ll \eta \ll 1$ and $n/3 = |Q| \geq |W'|$. Thus, since $\sqrt{\varepsilon} \ll \eta$, we greedily find a matching M_2 in $H_1 - V(M_1)$ such that $B_2 \subseteq V(M_2)$, M_2 is disjoint from W' , and every edge of M_2 contains at least one vertex from B_2 .

Thus, $M_1 \cup M_2$ gives the desired matching in $H_1 := H - V(M'_0)$ covering $B \setminus V(M'_0)$. Note that $|M'_0 \cup M_1 \cup M_2| \leq (q + |W \cap B|) + |B_1| + |B_2| \leq 2|B| \leq 8\sqrt{\varepsilon}n$. Also note that each vertex of $H - V(M'_0 \cup M_1 \cup M_2)$ is $\sqrt{\varepsilon}$ -good in H (with respect to $H_{1,3}(n, n/3)$). Thus, for every vertex $u \in U \setminus V(M'_0 \cup M_1 \cup M_2)$, the number of edges of $H - V(M'_0 \cup M_1 \cup M_2)$ containing u and exactly two vertices of $W \setminus V(M'_0 \cup M_1 \cup M_2)$ is at least

$$\frac{n}{3} \binom{n/3}{2} - \sqrt{\varepsilon}n^3 - 4|M'_0 \cup M_1 \cup M_2|n^2 > \eta n^3,$$

as $\sqrt{\varepsilon} \ll \eta \ll 1$. Hence, we may greedily find a matching M'_2 in $H - V(M'_0 \cup M_1 \cup M_2)$ such that $|M'_2| = |M_2|$ and every edge of M'_2 contains exactly two vertices of W' .

Let $M = M'_0 \cup M_1 \cup M_2 \cup M'_2$ and $m = |M|$. Then $m \leq 8\sqrt{\varepsilon}n$. Let $H_3 = H - V(M)$. Let $H_{1,3}(n - 3m, n/3 - m)$ be obtained from $H_{1,3}(n, n/3)$ by removing $V(M)$. Then, for

any $x \in V(H_3)$,

$$\begin{aligned}
& |N_{H_{1,3}(n-3m, n/3-m)}(x) \setminus N_{H_3}(x)| \\
& \leq |N_{H_{1,3}(n, n/3)}(x) \setminus N_H(x)| \\
& \leq \sqrt{\varepsilon} n^3 \\
& \leq 2\sqrt{\varepsilon} (n-3m)^3.
\end{aligned}$$

Thus, every vertex of H_3 is $2\sqrt{\varepsilon}$ -good with respect to $H_{1,3}(n-3m, n/3-m)$. By Lemma 4.4, H_3 contains a perfect matching, say M_3 . Now $M_3 \cup M$ is a perfect matching in H . ■

4.3 k -Graphs Close to $H_k^{k-l}(U, W)$

In this section we prove Theorem 1.3 for the case when $m > n/(2k^4)$ and H is ε -close to $H_k^{k-l}(U, W)$. The idea is similar to the proof of Lemma 4.5 while the argument is a bit more complicated. We first find two matchings (in two steps and using Lemma 3.2) that cover all $\sqrt{\varepsilon}$ -bad vertices. We then apply Lemma 4.2 to the hypergraph obtained from H by deleting these two matchings.

Lemma 4.6. *Let k, l, m, n be integers and let $0 < \varepsilon < (8^{k-1} k^{5(k-1)} k!)^{-3}$, such that $k \geq 3$, $l \in [k-1]$, $n \geq 8k^6/(1-5k^2\sqrt{\varepsilon})$, and $n/(2k^4) < m \leq n/k - 1$. Suppose H is a k -graph on n vertices and U, W is a partition of $V(H)$ with $|W| = m$, such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l}$ and H is ε -close to $H_k^{k-l}(U, W)$. Then $\nu(H) \geq m+1$ when $m < n/k - 1$ or $l \leq k-2$, and $\nu(H) \geq m$ when $l = k-1$ and $m = n/k - 1$.*

Proof. Since H is ε -close to $H_k^{k-l}(U, W)$, all but at most $k\sqrt{\varepsilon}n$ vertices of H are $\sqrt{\varepsilon}$ -good with respect to $H_k^{k-l}(U, W)$. Let U^{bad} and W^{bad} denote the set of $\sqrt{\varepsilon}$ -bad vertices in U and W , respectively. So $|U^{bad}| + |W^{bad}| \leq k\sqrt{\varepsilon}n$. Let $c := |W^{bad}|$, $V_1 := U \cup W^{bad}$, and $W_1 := W \setminus W^{bad}$. Note that possibly $c = 0$. We deal with vertices in W_1 later since at those vertices H and $H_k^{k-l}(U, W)$ are “close”. We claim that

(1) $H[V_1]$ has a matching M_1 of size $c + 1$.

To see this, let s be the maximum number of edges in H intersecting W_1 and containing a fixed l -set in V_1 . Then $s \leq \binom{n-l}{k-l} - \binom{n-l-(m-c)}{k-l}$ and $\delta_l(H[V_1]) \geq \delta_l(H) - s$. Hence,

$$\delta_l(H[V_1]) \geq \delta_l(H) - s > \binom{(n-m+c)-l}{k-l} - \binom{(n-m+c)-l-c}{k-l}.$$

Since $n/(2k^4) < m < n/k \leq n/3$, we have $n - m + c > 2m + c > n/k^4 + c$. Thus, since $c \leq k\sqrt{\varepsilon}n$, $n - m + c > 2k^4c$ by the choice of ε . So by Lemma 3.2, $H[V_1]$ contains a matching of size $c + 1$. This completes the proof of (1). \square

Let $H_1 := H - V(M_1)$. Next, we cover $U^{bad} \cup W^{bad}$ with two matchings in H_1 , using edges intersecting W_1 at most once. First note that, for each l -set $S \subseteq V_1 \setminus V(M_1)$, H_1 has lots of edges containing S and intersecting W_1 just once, or H_1 has lots of edges containing S and contained in $V_1 \setminus V(M_1)$. More precisely, we show that

(2) for any real number β with $2k^2\sqrt{\varepsilon} < \beta < (2k)^{-(k-l+3)}/2 - k^2\sqrt{\varepsilon}$ (which exists as $\varepsilon < (2k)^{-2k-11}$ and $k \geq 3$) and for any $S \in \binom{V_1 \setminus V(M_1)}{l}$, we have

$$|\{T \in N_{H_1}(S) : |T \cap W_1| = 1\}| \geq \beta n^{k-l}, \quad \text{or}$$

$$|\{T \in N_{H_1}(S) : T \subseteq V_1 \setminus V(M_1)\}| \geq \beta n^{k-l}.$$

To prove (2), let $S \in \binom{V_1 \setminus V(M_1)}{l}$ and assume $|\{T \in N_{H_1}(S) : |T \cap W_1| = 1\}| < \beta n^{k-l}$.

Since

$$|\{T \in N_{H_1}(S) : |T \cap W_1| \geq 2\}| \leq \sum_{i=2}^{k-l} \binom{m}{i} \binom{n-l-m}{k-l-i}$$

and

$$|\{T \in N_H(S) : |T \cap V(M_1)| \geq 1\}| \leq k(c+1)n^{k-l-1} < 2k^2\sqrt{\varepsilon}n^{k-l},$$

we have

$$\begin{aligned}
& |\{T \in N_{H_1}(S) : T \subseteq V_1 \setminus V(M_1)\}| \\
& > \delta_l(H) - |\{T \in N_{H_1}(S) : |T \cap W_1| \geq 2\}| \\
& \quad - |\{T \in N_{H_1}(S) : |T \cap W_1| = 1\}| - 2k^2\sqrt{\varepsilon}n^{k-l} \\
& > \left(\binom{n-l}{k-l} - \binom{n-l-m}{k-l} \right) - \sum_{i=2}^{k-l} \binom{m}{i} \binom{n-l-m}{k-l-i} - \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} \\
& = m \binom{n-l-m}{k-l-1} - 2k^2\sqrt{\varepsilon}n^{k-l} - \beta n^{k-l} \\
& > n^{k-l}/(2k)^{k-l+3} - 2k^2\sqrt{\varepsilon}n^{k-l} - \beta n^{k-l} \quad (\text{since } n/(2k^4) \leq m < n/k \text{ and } n \geq 8k^6) \\
& \geq \beta n^{k-l} \quad (\text{by the choice of } \beta).
\end{aligned}$$

This completes the proof of (2). \square

To find matchings in H_1 covering $(U^{bad} \cup W^{bad}) \setminus V(M_1)$, we fix a set $B \subseteq V_1 \setminus V(M_1)$ such that $|B| \equiv 0 \pmod{l}$, $(U^{bad} \cup W^{bad}) \setminus V(M_1) \subseteq B$, and $|B \setminus (U^{bad} \cup W^{bad})| < l$. For convenience, let $q = |B|/l$. Then

$$q \leq k\sqrt{\varepsilon}n.$$

We partition B into q disjoint l -sets B_1, \dots, B_q . By (2), we may assume that, for some $q_1 \in [q] \cup \{0\}$, $|\{T \in N_{H_1}(B_i) : |T \cap W_1| = 1\}| \geq \beta n^{k-l}$ for $1 \leq i \leq q_1$ and $|\{T \in N_{H_1}(B_j) : T \subseteq V_1 \setminus V(M_1)\}| \geq \beta n^{k-l}$ for $q_1 < j \leq q$. We claim that

(3) there exist disjoint matchings M_{21} and M_{22} in H_1 such that

- $|M_{21}| + |M_{22}| \leq k\sqrt{\varepsilon}n$,
- M_{21} covers $\bigcup_{i=1}^{q_1} B_i$ and each edge in M_{21} intersects W_1 just once, and
- M_{22} covers $\bigcup_{i=q_1+1}^q B_i$ and each edge in M_{22} is disjoint from W_1 .

First, we find the matching M_{21} covering $\bigcup_{i=1}^{q_1} B_i$ (which is empty if $q_1 = 0$). Suppose for

some $0 \leq h < q_1$ we have chosen pairwise disjoint edges e_1, \dots, e_h of $H_1 = H - V(M_1)$ (which is empty when $h = 0$), such that, for $i \in [h]$, we have $|e_i \cap W_1| = 1$ and $B_i \subseteq e_i$. Since $|\{T \in N_{H_1}(B_{h+1}) : |T \cap W_1| = 1\}| \geq \beta n^{k-l}$ and $h \leq q_1 - 1 \leq k\sqrt{\varepsilon}n - 1$, the number of edges of H disjoint from $V(M_1) \cup \left(\bigcup_{i=1}^h e_i\right)$ but containing B_{h+1} and exactly one vertex from W_1 is at least

$$\beta n^{k-l} - k|M_1|n^{k-l-1} - (hk)n^{k-l-1} \geq \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} > 0.$$

Thus, there is an edge e_{h+1} of H_1 such that $|e_{h+1} \cap W_1| = 1$, $B_{h+1} \subseteq e_{h+1}$, and $e_{h+1} \cap \left(\bigcup_{j=1}^h e_j\right) = \emptyset$. Since $q_1 \leq q \leq k\sqrt{\varepsilon}n$, we may continue this process till $h = q_1 - 1$. Now $M_{21} = \{e_1, \dots, e_{q_1}\}$ is the desired matching that covers $\bigcup_{i=1}^{q_1} B_i$.

Next, we find the matching $M_{22} = \{e_j : q_1 < j \leq q\}$, such that for $q_1 < j \leq q$, $B_j \subseteq e_j$ and $e_j \subseteq V_1 \setminus \left(V(M_1) \cup \left(\bigcup_{s=1}^{j-1} e_s\right)\right)$. Suppose that we have chosen $e_1, \dots, e_{q_1}, \dots, e_s$ for some s with $q_1 \leq s < q$ (which is empty if $q_1 = q$). Since $|\{T \in N_{H_1}(B_{s+1}) : T \subseteq V_1 \setminus V(M_1)\}| \geq \beta n^{k-l}$ and $s \leq q - 1 \leq k\sqrt{\varepsilon}n - 1$, the number of edges in H disjoint from $V(M_1) \cup \left(\bigcup_{i=1}^s e_i\right) \cup W_1$ but containing B_{s+1} is at least

$$\beta n^{k-l} - k|M_1|n^{k-l-1} - (sk)n^{k-l-1} \geq \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} > 0.$$

So there exists an edge e_{s+1} of H_1 such that $B_{s+1} \subseteq e_{s+1}$ and $e_{s+1} \cap \left(\bigcup_{i=1}^s e_i\right) = \emptyset$. Since $q \leq k\sqrt{\varepsilon}n$, we may continue this process till $s = q - 1$. Now $M_{22} = \{e_{q_1+1}, \dots, e_q\}$ gives the desired matching that covers $\bigcup_{i=q_1+1}^q B_i$. This completes the proof of (3). \square

Now, every vertex in $V(H) \setminus V(M_1 \cup M_{21} \cup M_{22})$ (as a vertex of H) is $\sqrt{\varepsilon}$ -good with respect to $H_k^{k-l}(U, W)$. In order to apply Lemma 4.2, we find a matching M_{23} in $H_1 - V(M_{21} \cup M_{22})$ such that every vertex of $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ is $\varepsilon^{1/3}$ -good with respect to $H_k^{k-l}(U^*, W^*)$, where $U^* = U \cap V(H_2)$ and $W^* = W \cap V(H_2)$, $|U^*| + |W^*| \geq 8k^6$, and $(|U^*| + |W^*|)/(2k^4) < |W^*| \leq (|U^*| + |W^*|)/k$. So we need to prove (4) and (5) below.

(4) There exists a matching M_{23} in $H_1 - V(M_{21} \cup M_{22})$ with $|M_{23}| < k\sqrt{\varepsilon}n$ and satisfying the following property: If we let $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$, $U' = U \cap V(H_2)$, $W' = W \cap V(H_2)$, then, for some $r \in \{0, 1\}$ with $r = 0$ for $l \leq k - 2$, we have

- $|W'| - r = m - c - |M_{21}| - |M_{22}| - |M_{23}|$, $|U'| + |W'| - r \geq 8k^6$, and $(|U'| + |W'| - r)/(2k^5) < |W'| - r$,
- $|W'| - r \leq (|U'| + |W'| - r)/k$ when $l \leq k - 2$ or $m < n/k - 1$, and
- $|W'| - r \leq (|U'| + |W'|)/k$ when $l = k - 1$ and $m = n/k - 1$.

We prove (4) by considering two cases. Note $|M_1 \cup M_{21} \cup M_{22}| = (c + 1) + q \leq 3k\sqrt{\varepsilon}n$ as $c, q \leq k\sqrt{\varepsilon}n$.

Case 1. $l \leq k - 2$.

In this case, we construct the matching M_{23} as follows. Suppose for some $1 \leq t \leq q - q_1$, we found vertices x_1, \dots, x_{t-1} in $U \setminus V(M_1 \cup M_{21} \cup M_{22})$ and edges f_1, \dots, f_{t-1} in $H_1 - V(M_{21} \cup M_{22})$ such that, for $i \in [t - 1]$, we have $x_i \in f_i$, $|f_i \cap W_1| = 2$, and $f_i \cap \left(\bigcup_{j=1}^{i-1} f_j\right) = \emptyset$. (When $t = 1$, these sequences are empty.) Let $x_t \in U \setminus V(M_1 \cup M_{21} \cup M_{22}) \setminus \left(\bigcup_{i=1}^{t-1} f_i\right)$. Since x_t is $\sqrt{\varepsilon}$ -good with respect to $H_k^{k-l}(U, W)$, the number of edges of $H_1 - V(M_{21} \cup M_{22}) - \left(\bigcup_{i=1}^{t-1} f_i\right)$ containing x_t and exactly two vertices in W_1 is at least

$$\binom{m - c - 2(t - 1)}{2} \binom{n - m - 1}{k - 3} - \sqrt{\varepsilon}n^{k-1} - (3k\sqrt{\varepsilon}n)n^{k-2} - (kt)n^{k-2} > 0,$$

as $n/(2k^4) < m$, $c < k\sqrt{\varepsilon}n$, $t < k\sqrt{\varepsilon}n$, and $\varepsilon < (8^{k-1}k^{5(k-1)}k!)^{-3}$. So there exists an edge f_t in $H_1 - V(M_{21} \cup M_{22}) - \left(\bigcup_{i=1}^{t-1} f_i\right)$ such that $x_t \in f_t$ and $|f_t \cap W_1| = 2$. This process works as long as $t \leq q - q_1$. Thus, we have a matching $M_{23} = \{f_j : j \in [q - q_1]\}$ such that, for $j \in [q - q_1]$, $f_j \subseteq V(H_1) \setminus \left(V(M_{21} \cup M_{22}) \cup \left(\bigcup_{i=1}^{j-1} f_i\right)\right)$ and $|f_j \cap W_1| = 2$.

Let $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ and let $U' = U \cap V(H_2)$ and $W' = W \cap V(H_2)$.

Note that $|M_{23}| = |M_{22}|$, and note that

$$\begin{aligned} |W'| &= |W| - c - |M_{21}| - 2|M_{23}| = |W| - c - |M_{21}| - |M_{22}| - |M_{23}|, \text{ and} \\ |U'| &= |U| - (k(c+1) - c) - (k-1)|M_{21}| - k|M_{22}| - (k-2)|M_{23}| \\ &= |U| - (k-1)(c+1 + |M_{21}| + |M_{22}| + |M_{23}|) - 1. \end{aligned}$$

Hence, we have

$$|U'| + |W'| = |U| + |W| - k(c+1) - k|M_{21}| - k|M_{22}| - k|M_{23}|.$$

Thus, $|U'| + |W'| \geq n - 5k^2\sqrt{\varepsilon}n \geq 8k^6$ and, since $m \leq n/k - 1$,

$$\begin{aligned} (|U'| + |W'|)/k &= (|U| + |W|)/k - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &\geq (|W| + 1) - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &= |W'|. \end{aligned}$$

Moreover, since $|W| > n/(2k^4)$ and $|W| \geq |W'| \geq |W| - 3k\sqrt{\varepsilon}n$, we have

$$\begin{aligned} &(|U'| + |W'|) - 2k^5|W'| \\ &= |U| + |W| - k(c+1) - k|M_{21}| - k|M_{22}| - k|M_{23}| - 2k^5|W'| \\ &< |U| + |W| - 2k^5|W'| \\ &< 2k^4|W| - 2k^5|W'| \\ &< 0 \quad (\text{since } n \text{ is large and } \varepsilon \text{ is small}) \end{aligned}$$

Case 2. $l = k - 1$.

Arbitrarily choose $q - q_1$ pairwise disjoint $(k-1)$ -sets in $V(H) \setminus V(M_1 \cup M_{21} \cup M_{22})$, each containing exactly two vertices in W_1 . Note that this can be done, because $|W_1| =$

$m - c \geq n/(2k^4) - k\sqrt{\varepsilon}n > 2q$. Since $\delta_{k-1}(H) > m \geq n/(2k^4) > 5k^2\sqrt{\varepsilon}n \geq k((c + 1) + 3q)$, we can extend these $q - q_1$ sets to $q - q_1$ pairwise disjoint edges f_1, \dots, f_{q-q_1} in $H - V(M_1 \cup M_{21} \cup M_{22})$.

Clearly, each f_i contains either two or three vertices from W_1 . Thus, there exists some integer p with $0 \leq p \leq q - q_1$ such that $q - q_1 + p - 1 \leq |W_1 \cap (\bigcup_{i=1}^p f_i)| \leq q - q_1 + p$. Let $M_{23} = \{f_1, \dots, f_p\}$, $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$, and $U' = U \cap V(H_2)$ and $W' = W \cap V(H_2)$.

Note that $|W_1 \cap V(M_{23})| = |M_{22}| + |M_{23}| - r$ for some $r \in \{0, 1\}$. Hence,

$$|W'| = |W| - c - |M_{21}| - |W_1 \cap V(M_{23})| = |W| - c - |M_{21}| - |M_{22}| - |M_{23}| + r$$

and

$$|U'| = |U| - (k(c + 1) - c) - (k - 1)|M_{21}| - k|M_{22}| - (k|M_{23}| - |W_1 \cap V(M_{23})|).$$

Therefore,

$$|U'| + |W'| - r = (|U| + |W| - r) - k(c + 1) - k|M_{21}| - k|M_{22}| - k|M_{23}|.$$

It is easy to see that the same calculations in Case 1 also allow us to conclude that $|U'| + |W'| - r \geq 8k^6$ and $(|U'| + |W'| - r) - 2k^5(|W'| - r) < 0$. Moreover, if $r = 0$ then the same argument in Case 1 shows that $|W'| \leq (|U'| + |W'|)/k$. So we may assume $r = 1$.

First, suppose $m < n/k - 1$. Then $(|U| + |W|)/k \geq |W| + 1 + 1/k$; so

$$\begin{aligned} (|U'| + |W'| - 1)/k &= (|U| + |W| - 1)/k - (c + 1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &\geq (|W| + 1 + 1/k) - 1/k - (c + 1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &= |W'| - 1. \end{aligned}$$

Now suppose $m = n/k - 1$ (so $n \in k\mathbb{Z}$). Then

$$\begin{aligned}
(|U'| + |W'|)/k &= (|U| + |W|)/k - (c + 1) - |M_{21}| - |M_{22}| - |M_{23}| \\
&\geq (|W| + 1) - (c + 1) - |M_{21}| - |M_{22}| - |M_{23}| \\
&\geq |W'| - 1.
\end{aligned}$$

So $|W'| - r \leq (|U'| + |W'|)/k$, completing the proof of (4). \square

We now define $W^* \subseteq W'$ and $U^* = V(H) \setminus W^*$ as follows: If $r = 0$ let $W^* = W'$. If $r = 1$ and $n \notin k\mathbb{Z}$ or $m < n/k - 1$ then choose some $w \in W'$ and let $W^* = W' \setminus \{w\}$. If $r = 1$, $n \in k\mathbb{Z}$, and $m = n/k - 1$ then choose $w_1, w_2 \in W'$ and let $W^* = W' \setminus \{w_1, w_2\}$.

(5) Every vertex of $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ is $\varepsilon^{1/3}$ -good with respect to $H_k^{k-l}(U^*, W^*)$.

To prove (5), we note that $k|M_1 \cup M_{21} \cup M_{22} \cup M_{23}| + 2 \leq k((c + 1) + 3q) + 2 \leq 5k^2\sqrt{\varepsilon}n$. For each $x \in V(H_2)$, since x is $\sqrt{\varepsilon}$ -good with respect to $H_k^{k-l}(U, W)$, we have

$$|N_{H_k^{k-l}(U, W)}(x) \setminus N_H(x)| \leq \sqrt{\varepsilon}n^{k-1}.$$

Thus,

$$\begin{aligned}
&\left| N_{H_k^{k-l}(U^*, W^*)}(x) \setminus N_{H_2}(x) \right| \\
&\leq \left| N_{H_k^{k-l}(U, W)}(x) \setminus N_H(x) \right| + (k|M_1 \cup M_{21} \cup M_{22} \cup M_{23}| + 2)n^{k-2} \\
&\leq \sqrt{\varepsilon}n^{k-1} + 5k^2\sqrt{\varepsilon}n^{k-1} \\
&< \varepsilon^{1/3}n^{k-1}.
\end{aligned}$$

This completes the proof of (5). \square

Hence, by (4) and (5), it follows from Lemma 4.2 that there is a matching M_3 in H_2 of size $|W^*|$. Let $M := M_1 \cup M_{21} \cup M_{22} \cup M_{23} \cup M_3$. Then M is a matching in H . By (4),

$|W'| - r = m - (|M_1| - 1) - |M_{21}| - |M_{22}| - |M_{23}|$. If $l = k - 1$ and $m = n/k - 1$, then $|W^*| \geq |W'| - r - 1$; so

$$|M| \geq (|W'| - r - 1) + |M_1| + |M_{21}| + |M_{22}| + |M_{23}| = n/k - 1.$$

Otherwise, $|W^*| = |W'| - r$ and

$$|M| = (|W'| - r) + |M_1| + |M_{21}| + |M_{22}| + |M_{23}| = m + 1.$$

■

CHAPTER 5

ABSORBING LEMMAS

A typical approach to finding large matchings in a dense k -graph H is to find a small matching M_a in H that can be used to ‘absorb’ small sets of vertices. More precisely, we look for a small matching M_a such that for each small subset $S \subseteq V(H) \setminus V(M_a)$, $H[V(M_a) \cup S]$ has a large matching. Such a matching M_a is known as an *absorbing matching* (or *absorber*), often found by applying the second moment method. This approach was initiated by Rödl, Ruciński, and Szemerédi [40].

Let $Bi(n, p)$ be the binomial distribution with parameters n and p . The following lemma on Chernoff bound can be found in Alon and Spencer [5] (page 313, also see [36]).

Lemma 5.1 (Chernoff). *Suppose X_1, \dots, X_n are independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then, for any $0 < \delta \leq 1$,*

$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu / 3} \text{ and } \mathbb{P}[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu / 2}.$$

In particular, when $X \sim Bi(n, p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}(|X - np| \geq \lambda) \leq e^{-\Omega(\lambda^2 / np)}.$$

In the following two sections, we prove the existence of desired absorbing matchings to help us prove Theorem 2.8 and Theorem 1.3, respectively.

5.1 Absorbing Matchings in $(1, 3)$ -Partite 4-Graphs

Lemma 5.2. *Let $n \in 3\mathbb{Z}$ be large enough and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P|$ and $\delta_1(H) \geq (n/3) \left(\binom{n-1}{2} - \binom{2n/3}{2} + 1 \right)$.*

Let ρ, ρ' be constants such that $0 < \rho' \ll \rho \ll 1$. Then H has a matching M' such that $|M'| \leq \rho n$ and, for any subset $S \subseteq V(H) \setminus V(M')$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap P|$, $H[S \cup V(M')]$ has a perfect matching.

Proof. Our proof follows along the same lines as in [40].

We call a balanced 12-element set $A \subseteq V(H)$ an *absorbing set* for a balanced 4-element set $T \subseteq V(H)$ if $H[A]$ has a matching of size 3 and $H[A \cup T]$ has a matching of size 4. Denote by $\mathcal{L}(T)$ the collection of all absorbing sets for T . Then

(1) for every balanced $T \in \binom{V(H)}{4}$, $|\mathcal{L}(T)| > 10^{-8} n^{12}/12!$.

Let $T = \{u_0, u_1, u_2, u_3\} \in \binom{V(H)}{4}$ be balanced, with $u_0 \in Q$ and $u_1, u_2, u_3 \in P$. We form an absorbing set for T by choosing four pairwise disjoint 3-sets U_0, U_1, U_2, U_3 in order.

First, we choose a 3-set $U_0 \subseteq P \setminus T$ such that $U_0 \cup \{u_0\} \in E(H)$. The number of choices for U_0 is at least

$$d_H(u_0) - 3 \binom{n-3}{2} > \delta_1(H) - 3 \binom{n-1}{2} > \frac{n}{9} \binom{n-1}{2}.$$

Now fix a choice of U_0 , and let $U_0 = \{w_1, w_2, w_3\}$. Note that, for each $x \in P$, $N_H(x)$ is a subset of $\{\{x_0, x_1, x_2\} : x_0 \in Q, x_1, x_2 \in P\}$. Hence, $|N_H(u_i) \cup N_H(w_i)| \leq \frac{n}{3} \binom{n}{2}$. Thus, for $i \in [3]$,

$$|N_H(u_i) \cap N_H(w_i)| \geq \frac{2n}{3} \left(\binom{n-1}{2} - \binom{2n/3}{2} + 1 \right) - \frac{n}{3} \binom{n}{2} \geq \frac{n}{30} \binom{n-1}{2}.$$

For $i \in [3]$, we choose 3-sets U_i from $(V(H) \setminus T) \setminus \bigcup_{j=0}^{i-1} U_j$ such that $U_i \cup \{u_i\}$ and $U_i \cup \{w_i\}$ are both edges of H . For each choice of U_j , $0 \leq j \leq i-1$, the number of choices for U_i is at least

$$|N_H(u_i) \cap N_H(w_i)| - 13(n/3)n \geq \frac{n}{30} \binom{n-1}{2} - 13n^2/3 > \frac{n}{50} \binom{n-1}{2}.$$

Let $A = \bigcup_{i=0}^3 U_i$. Then $\{U_i \cup \{w_i\} : i \in [3]\}$ is a matching in $H[A]$, and $\{U_i \cup \{u_i\} : i \in [3] \cup \{0\}\}$ is a matching in $H[A \cup T]$. Thus A is an absorbing set for T . Since there are more than $10^{-8}n^{12}$ choices of (U_0, U_1, U_2, U_3) , there are more than $10^{-8}n^{12}/12!$ absorbing sets for T . \square

Now, form a family \mathcal{F} of subsets of $V(H)$ by selecting each of the $\binom{n/3}{3}\binom{n}{9}$ possible balanced 12-sets independently with probability

$$p = \frac{\rho n}{2\binom{n/3}{3}\binom{n}{9}}.$$

Then, it follows from Lemma 5.1 that, with probability $1 - o(1)$ (as $n \rightarrow \infty$),

$$(2) \quad |\mathcal{F}| \leq \rho n, \text{ and}$$

$$(3) \quad |\mathcal{L}(T) \cap \mathcal{F}| \geq p|\mathcal{L}(T)|/2 \geq 10^{-10}\rho n \text{ for all balanced } T \in \binom{V(H)}{4}.$$

Furthermore, the expected number of intersecting pairs of sets in \mathcal{F} is at most

$$\binom{n/3}{3}\binom{n}{9} \left[3\binom{n/3-1}{2}\binom{n}{9} + 9\binom{n-1}{8}\binom{n/3}{3} \right] p^2 < \rho^{1.5}n.$$

Thus, using Markov's inequality, we derive that, with probability at least $1/2$,

$$(4) \quad \mathcal{F} \text{ contains at most } 2\rho^{1.5}n \text{ intersecting pairs.}$$

Hence, with positive probability, \mathcal{F} satisfies (2), (3), and (4). Let \mathcal{F}' be obtained from \mathcal{F} by removing one set from each intersecting pair and deleting all non-absorbing sets. Then \mathcal{F}' consists of pairwise disjoint absorbing sets, such that, for each $T \in \binom{V(H)}{4}$,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq 10^{-10}\rho n/2.$$

Since \mathcal{F}' consists only of pairwise disjoint absorbing sets, $H[V(\mathcal{F}')] has a perfect matching, say M' . Then $|M'| \leq \rho n$. To complete the proof, take an arbitrary $S \subseteq$$

$V(H) \setminus V(M')$ with $|S| \leq \rho'n$ and $3|S \cap Q| = |S \cap P|$, where $\rho' \leq 10^{-10}\rho/2$. Note that S can be partitioned into t balanced 4-sets, say T_1, \dots, T_t , for some $t \leq \rho'n/4 < 10^{-10}\rho n/2$. We can greedily choose distinct absorbing sets $A_i \in \mathcal{F}'$ in order for $i = 1, \dots, t$, such that $H[A_i \cup T_i]$ has a perfect matching. Hence, $H[S \cup V(M')]$ has a perfect matching as required. \blacksquare

5.2 Absorbing Matchings in k -Graphs

In this section, we prove the following lemma for absorbing matchings in k -graphs with large l -degree for $k/2 < l \leq k - 1$. We are able to do this partly due to the existence of positive integers a, h satisfying $h \leq l$, $a \leq k - l$, and $al \geq a(k - l) + (k - h)$. (One can check that $a = k - l$ and $h = l$ satisfies this requirement.)

We will frequently use the following fact: For integers $0 \leq l' < l \leq k - 1$ and any k -graph H , if $\delta_l(H) \geq c \binom{n-l}{k-l}$ for some $0 \leq c \leq 1$, then $\delta_{l'}(H) \geq c \binom{n-l'}{l-l'} \binom{n-l}{k-l} / \binom{k-l'}{l-l'} \geq c \binom{n-l'}{k-l'}$, which can be proved by a standard double-counting.

Lemma 5.3. *Let k, l be integers with $k \geq 3$ and $k/2 < l \leq k - 1$, and let $c > 0$ be a constant with $c < 1/k!$. Then there exist $\rho > 0$ and $c' > 0$ with $0 < \rho \ll c' \ll c$, such that the following holds for all sufficiently large integers n :*

Let a, h be positive integers satisfying $h \leq l$, $a \leq k - l$, and $al \geq a(k - l) + (k - h)$. Let H be a k -graph on n vertices with $\delta_l(H) \geq c \binom{n-l}{k-l}$. Then there exists a matching M in H such that

- $|M| \leq 2k\rho n$ and
- *for any subset $S \subseteq V(H)$ with $|S| \leq c'\rho n$, $H[V(M) \cup S]$ has a matching covering all but at most $al + h - 1$ vertices.*

Proof. For $R \in \binom{V(H)}{al+h}$ and $Q \in \binom{V(H)}{ak}$, we say that Q is R -absorbing if $\nu(H[Q \cup R]) \geq a + 1$ and Q is the vertex set of a matching in H . (In particular, this requires $al + h \geq k$,

which is guaranteed by assumption.) Let $\mathcal{L}(R)$ denote the collection of all R -absorbing sets in H . We claim that

(1) there exists $c' = c'(c, k) > 0$ such that $|\mathcal{L}(R)| \geq c'n^{ak}$ for every $R \in \binom{V(H)}{al+h}$.

To prove (1), let $R \in \binom{V(H)}{al+h}$. We wish to extend R to a matching of size $a+1$ by adding a set of size $(a+1)k - (al+h) = a(k-l) + (k-h)$. Partition R into $a+1$ pairwise disjoint subsets R_1, \dots, R_{a+1} , with $|R_{a+1}| = h$ and $|R_i| = l$ for $i \in [a]$. Next we choose $(k-l)$ -sets T_s for $s \in [a]$ and a $(k-h)$ -set T_{a+1} such that $\{R_s \cup T_s : s \in [a+1]\}$ form a matching in H .

For $j \in [a]$, since $d_H(R_j) \geq \delta_l(H) \geq c \binom{n-l}{k-l}$, we have, for large n ,

$$\left| N_{(H-R) - \bigcup_{s=1}^{j-1} T_s}(R_{j+1}) \right| \geq c \binom{n-l}{k-l} - ((al+h) + (k-l)j) \binom{(n-l)-1}{(k-l)-1} > \frac{c}{2} \binom{n-l}{k-l};$$

thus, we have more than $\frac{c}{2} \binom{n-l}{k-l}$ choices for each T_j with $j \in [a]$. Similarly, since $d_H(R_{a+1}) \geq c \binom{n-h}{k-h}$ as $h \leq l$, we have

$$\left| N_{(H-R) - \bigcup_{s=1}^a T_s}(R_{a+1}) \right| \geq c \binom{n-h}{k-h} - ((al+h) + (k-l)a) \binom{(n-h)-1}{(k-h)-1} > \frac{c}{2} \binom{n-h}{k-h};$$

hence, we have more than $\frac{c}{2} \binom{n-h}{k-h}$ choices for T_{a+1} .

Fix an arbitrary choice of $T_i \in N_{(H-R) - \bigcup_{s=1}^{i-1} T_s}(R_i)$, $i \in [a+1]$, such that $\{R_s \cup T_s : s \in [a+1]\}$ form a matching in H . Let $T = \bigcup_{i=1}^{a+1} T_i$. Next, we form an R -absorbing set Q by extending the set T to a matching of size a . We partition T into subsets T'_1, \dots, T'_a such that $1 \leq |T'_i| \leq l$ for $i \in [a]$. Such a partition exists since $|T| = a(k-l) + (k-h) \leq al$. Similar to the arguments in the previous paragraph, we can show that there exist $P_i \in N_{(H-(R \cup T)) - \bigcup_{s=1}^{i-1} P_s}(T'_i)$ for $i \in [a]$, such that

$$\left| N_{(H-(R \cup T)) - \bigcup_{s=1}^{i-1} P_s}(T'_i) \right| > \frac{c}{2} \binom{n-|T'_i|}{k-|T'_i|}.$$

This means that there are more than $\frac{c}{2} \binom{n-|T'_i|}{k-|T'_i|}$ choices for each P_i with $i \in [a]$. Let $Q =$

$T \cup (\bigcup_{i=1}^a P_i)$. Then Q is the vertex set of a matching of size a in H . Hence Q is an R -absorbing set.

Note that each such ak -set Q can be produced at most $(ak)!$ times by the above process, and recall that $\sum_{i=1}^a |T'_i| = a(k-l) + (k-h)$. Hence, for large n (compared with k), we have

$$\begin{aligned} |\mathcal{L}(R)| &> ((ak)!)^{-1} \left(\frac{c}{2} \binom{n-l}{k-l} \right)^a \left(\frac{c}{2} \binom{n-h}{k-h} \right) \prod_{i=1}^a \left(\frac{c}{2} \binom{n-|T'_i|}{k-|T'_i|} \right) \\ &> (2(ak)!)^{-1} \left(\frac{c}{2} \right)^{2a+1} \left(\frac{n^{a(k-l)}}{((k-l)!)^a} \right) \left(\frac{n^{k-h}}{(k-h)!} \right) \left(\frac{n^{ak-(a(k-l)+(k-h))}}{(ak-(a(k-l)+(k-h)))!} \right) \\ &> c' n^{ak}, \end{aligned}$$

by choosing $c' < (2(ak)!)^{-1} (c/2)^{2a+1} (((k-l)!)^a (k-h)! (ak-(a(k-l)+(k-h))))^{-1}$. This completes the proof of (1). \square

Choose $\rho < c'/(2a^2k^2)$. We form a family $\mathcal{F} \subseteq \binom{V(H)}{ak}$ by choosing each member of $\binom{V(H)}{ak}$ independently at random with probability

$$p = \frac{\rho n}{\binom{n}{ak}}.$$

Then

(2) with probability $1/2 - o(1)$, all of the following hold:

(2a) $|\mathcal{F}| \leq 2\rho n$,

(2b) $|\mathcal{L}(R) \cap \mathcal{F}| \geq 2c'\rho n$ for all $(al+h)$ -sets R , and

(2c) \mathcal{F} contains less than $c'\rho n$ intersecting pairs.

Clearly, $\mathbb{E}(|\mathcal{F}|) = \rho n$ and, by (1), $\mathbb{E}(|\mathcal{L}(R) \cap \mathcal{F}|) > c'n^{ak}p > 4c'\rho n$ (as $a \geq 1$ and $k \geq 3$).

So by Lemma 5.1, with probability $1 - o(1)$,

$$|\mathcal{F}| \leq 2\rho n,$$

and, for each fixed $(al + h)$ -set R , with probability at least $1 - e^{-\Omega(\rho n)}$, \mathcal{F} satisfies

$$|\mathcal{L}(R) \cap \mathcal{F}| \geq 2c'\rho n.$$

Hence given n sufficiently large, it follows from union bound that, with probability $1 - o(1)$, (2a) and (2b) hold.

Furthermore, the expected number of intersecting pairs in \mathcal{F} is at most

$$\binom{n}{ak} \binom{ak}{1} \binom{n-1}{ak-1} p^2 = a^2 k^2 \rho^2 n < c'\rho n/2.$$

Thus, using Markov's inequality, we derive that with probability at least $1/2$, \mathcal{F} contains less than $c'\rho n$ intersecting pairs of ak -sets. Hence, by union bound, (2a), (2b), (2c) hold with probability $1/2 - o(1)$, completing the proof of (2). \square

Let \mathcal{F}' denote the family obtained from \mathcal{F} by deleting one ak -set from each intersecting pair of sets in \mathcal{F} and removing all ak -sets that are not the vertex set of a matching in H . (Note that the latter are not in $\mathcal{L}(R)$ for any $(al + h)$ -set R .) Then \mathcal{F}' consists of pairwise disjoint vertex sets of matchings of size a in H . Moreover, for all $(al + h)$ -sets R ,

$$|\mathcal{L}(R) \cap \mathcal{F}'| \geq 2c'\rho n - c'\rho n \geq c'\rho n.$$

For each $F \in \mathcal{F}'$, let M_F be a matching in H with $V(M_F) = F$. Then $M = \bigcup_{F \in \mathcal{F}'} M_F$ is a perfect matching in $H[V(\mathcal{F}')]$, and $|M| \leq a|\mathcal{F}| \leq k|\mathcal{F}| \leq 2k\rho n$. It remains to show that M absorbs small sets.

Let S be an arbitrary subset of $V(H) \setminus V(M)$ with $|S| \leq c'\rho n$. We use M to absorb $(al + h)$ -sets iteratively, starting with an arbitrary $(al + h)$ -subset of S . Let $S_0 := S$ and let $R_0 \subseteq S_0$ with $|R_0| = al + h$. Since $|\mathcal{L}(R_0) \cap \mathcal{F}'| \geq c'\rho n$, we can find $Q_0 \in \mathcal{F}'$ such that $H[R_0 \cup Q_0]$ has a matching M_0 with $|M_0| = a + 1$. Let $t \geq 0$ be the maximal integer such that there exist

- sets S_0, \dots, S_t with $|S_i| \geq al + h$ for $i \in [t] \cup \{0\}$,
- $(al + h)$ -sets R_0, \dots, R_t with $R_i \subseteq S_i$ for $i \in [t] \cup \{0\}$,
- pairwise disjoint sets $Q_0, \dots, Q_t \in F'$ with Q_i being R_i -absorbing for $i \in [t] \cup \{0\}$,
- pairwise disjoint $(a + 1)$ -matchings M_0, \dots, M_t , with M_i in $H[R_i \cup Q_i]$ for $i \in [t] \cup \{0\}$,

satisfying the property that $S_i = (S_{i-1} \cup Q_{i-1}) \setminus V(M_{i-1})$ for $i \in [t]$

Then $|S_i| = |S_{i-1}| - k$ for $i \in [t]$. Let $S_{t+1} = (S_t \cup Q_t) \setminus V(M_t)$. If $|S_{t+1}| < al + h$ then M is the desired matching. So assume $|S_{t+1}| \geq al + h$ and let R_{t+1} be an $(al + h)$ -subset of S_{t+1} . Since $|\mathcal{L}(R_{t+1}) \cap \mathcal{F}'| \geq c'\rho n$ and $t + 1 \leq |S|/k + 1 \leq c'\rho n - 1$, there exists $Q_{t+1} \in \mathcal{F}' \setminus \{Q_0, \dots, Q_t\}$ such that $H[R_{t+1} \cup Q_{t+1}]$ has a matching M_{t+1} with $|M_{t+1}| = a + 1$. This contradicts the maximality of t . ■

CHAPTER 6

NON EXTREMAL CASES AND INDEPENDENT SETS

In this section we point out the relationship between ‘close to extremal constructions’ and ‘having certain independent sets’ in locally dense uniform hypergraphs, and use that to prove properties from ‘non-closeness’ for certain induced subgraphs, which is an important step in our solution to the non extremal case.

Definition 6.1. Any subset $I \subseteq V(H)$ that contains no edge of H is called an *independent set*. We use $\alpha(H)$ to denote the size of a largest independent set in the hypergraph H .

When H is not close to the extremal constructions, we will show that the random induced subgraphs obtained by independently sampling each vertex not in the absorber with a certain probability do not have independent sets larger than a certain size (see Lemma 6.6 and Lemma 6.8) with high probability. Then we use this property of a random induced subgraph H' , as well as some degree conditions inherited from H , to show that H' has a perfect fractional matching (see Lemma 8.3 and Lemma 7.4).

To obtain this property, we use the hypergraph container method developed by Balogh, Morris, and Samotij [6] and, independently, by Saxton and Thomason [41].

Definition 6.2. A family \mathcal{F} of subsets of a set V is said to be *increasing* if, for any $A \in \mathcal{F}$ and $B \subseteq V$, $A \subseteq B$ implies $B \in \mathcal{F}$.

We use $\Delta_l(H)$ to denote the maximum l -degree of H , and $\mathcal{I}(H)$ to denote the collection of all independent sets in H .

Definition 6.3. Let $\varepsilon > 0$ and let \mathcal{F} be a family of subsets of $V(H)$. We say that H is $(\mathcal{F}, \varepsilon)$ -dense if $e(H[A]) \geq \varepsilon e(H)$ for every $A \in \mathcal{F}$. We use $\overline{\mathcal{F}}$ to denote the family consisting of subsets of $V(H)$ not in \mathcal{F} .

We will use the following container result from [6].

Theorem 6.4 (Balogh, Morris, and Samotij, 2015). *For every $k \in \mathbb{N}$ and all positive c and ε , there exists a positive constant C such that the following holds. Let H be a k -graph and let \mathcal{F} be an increasing family of subsets of $V(H)$ such that $|A| \geq \varepsilon v(H)$ for all $A \in \mathcal{F}$. Suppose that H is $(\mathcal{F}, \varepsilon)$ -dense and $p \in (0, 1)$ is such that, for every $l \in [k]$,*

$$\Delta_l(H) \leq cp^{l-1} \frac{e(H)}{v(H)}.$$

Then there exist a family $\mathcal{S} \subseteq \binom{V(H)}{\leq Cpv(H)}$ and functions $f : \mathcal{S} \rightarrow \overline{\mathcal{F}}$ and $g : \mathcal{I}(H) \rightarrow \mathcal{S}$ such that, for every $I \in \mathcal{I}(H)$,

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$

In each of the following two sections, we first prove the hypergraph H in question has a family of sets with the desired density, and then show H has no large independent sets.

6.1 Independent Sets in $(1, 3)$ -Partite 4-Graphs

In order to apply Theorem 6.4 we need a family \mathcal{F} of subsets of $V(H)$ so that H is $(\mathcal{F}, \varepsilon)$ -dense, which is possible when H is not close to any $H_{1,3}(n, n/3)$.

Lemma 6.5. *Let ρ, ε be reals such that $0 < \rho \leq \varepsilon/4 \ll 1$, let $n \in 3\mathbb{Z}$ be large, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$ and $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for any $v \in Q$ and $u \in P$. If H is not ε -close to any $H_{1,3}(n, n/3)$ with $V(H_{1,3}(n, n/3)) = P \cup Q$, then H is $(\mathcal{F}, \varepsilon/6)$ -dense, where $\mathcal{F} = \{A \subseteq V(H) : |A \cap Q| \geq (1/3 - \varepsilon/8)n \text{ and } |A \cap P| \geq (2/3 - \varepsilon/8)n\}$.*

Proof. Suppose to the contrary that there exists $A \subseteq V(H)$ such that $|A \cap Q| \geq (1/3 - \varepsilon/8)n$, $|A \cap P| \geq (2/3 - \varepsilon/8)n$, and $e(H[A]) \leq \varepsilon e(H)/6$. Choose such A that $|P \setminus A| \geq n/3$ and let $W \subseteq P \setminus A$ such that $|W| = n/3$. Let $A_1 = A \cap P$ and $A_2 = A \cap Q$, and let

$B_1 = (P \setminus W) \setminus A_1$, $B_2 = Q \setminus A_2$, and $B = B_1 \cup B_2$. Then $|A_1| \leq 2n/3$ and, by the choice of A , $|B_1| \leq \varepsilon n/8$ and $|B_2| \leq \varepsilon n/8$.

Let $U = P \setminus W = A_1 \cup B_1$ and let H_0 denote the $H_{1,3}(n, n/3)$ with partition classes Q, U, W . We derive a contradiction by showing that $|E(H_0) \setminus E(H)| < \varepsilon n^4$. By the definition of $H(n, n/3)$, each $f \in E(H_0) \setminus E(H)$ intersects U . So

$$|E(H_0) \setminus E(H)| \leq |\{f \in E(H_0) : f \cap B_1 \neq \emptyset\}| + |\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}|.$$

Since $|B_1| \leq \varepsilon n/8$, we have $|\{f \in E(H_0) : f \cap B_1 \neq \emptyset\}| \leq |B_1||Q||P|^2/2 \leq \varepsilon n^4/48$. To bound $|\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}|$, we note that, for each fixed $u \in A_1$,

$$|\{f \in E(H) : u \in f, f \cap B \neq \emptyset\}| \leq |B_1||P||Q| + |B_2||P|^2/2 < \varepsilon n^3/8,$$

and that, for each $f \in E(H)$ with $u \in f$, we have $f \cap B \neq \emptyset$, or $f \subseteq A$, or $f \in E(H_0)$. So for any $u \in A_1$,

$$\begin{aligned} & |\{f \in E(H) : u \in f, f \in E(H_0)\}| \\ & \geq d_H(u) - |\{f \in E(H) : u \in f, f \cap B \neq \emptyset\}| - |\{f \in E(H) : u \in f, f \subseteq A\}| \\ & \geq d_H(u) - \varepsilon n^3/8 - d_{H[A]}(u). \end{aligned}$$

Hence,

$$\begin{aligned} & |\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}| \\ & \leq \sum_{u \in A_1} |\{f \in E(H_0) \setminus E(H) : u \in f\}| \\ & \leq \sum_{u \in A_1} (d_{H_0}(u) - |\{f \in E(H) : u \in f, f \in E(H_0)\}|) \\ & \leq \sum_{u \in A_1} (d_{H_0}(u) - d_H(u) + \varepsilon n^3/8 + d_{H[A]}(u)). \end{aligned}$$

Since for $u \in A_1$, $d_{H_0}(u) = \frac{n}{3} \left(\binom{n-1}{2} - \binom{2n/3-1}{2} \right)$ and $d_H(u) = \sum_{v \in Q} d_H(\{u, v\}) \geq \frac{n}{3} \left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2 \right)$, we have $d_{H_0}(u) - d_H(u) \leq \rho n^3/3$ (for large n). Hence,

$$\begin{aligned}
|E(H_0) \setminus E(H)| &\leq \varepsilon n^4/48 + |A_1| (\rho/3 + 3\varepsilon/8) n^3 + \sum_{u \in A_1} d_{H[A]}(u) \\
&\leq (\varepsilon/48 + 4\rho/9 + \varepsilon/4) n^4 + 3e(H[A]) \quad (\text{since } |A_1| \leq 2n/3) \\
&\leq (1/48 + 1/9 + 1/4) \varepsilon n^4 + 3\varepsilon n^4/6 \quad (\text{since } e(H[A]) \leq \varepsilon e(H)/6) \\
&< \varepsilon n^4,
\end{aligned}$$

a contradiction. ■

We now use Theorem 6.4 to control the independence number of a random subgraph.

Lemma 6.6. *Let $c, \varepsilon', \alpha_1, \alpha_2$ be positive reals, let $\gamma > 0$ with $\gamma \ll \min\{\alpha_1, \alpha_2\}$, let k, n be positive integers with $n \in 3\mathbb{Z}$, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$, $e(H) \geq cn^4$, and $e(H[F]) \geq \varepsilon' e(H)$ for all $F \subseteq V(H)$ with $|F \cap P| \geq \alpha_1 n$ and $|F \cap Q| \geq \alpha_2 n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then, with probability at least $1 - n^{O(1)} e^{-\Omega(n^{0.1})}$, every independent set J in $H[R]$ satisfies $|J \cap P| \leq (\alpha_1 + \gamma + o(1))n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \gamma + o(1))n^{0.1}$.*

Proof. Define $\mathcal{F} := \{A \subseteq V(H) : e(H[A]) \geq \varepsilon' e(H) \text{ and } |A| \geq \varepsilon' n\}$. Then \mathcal{F} is an increasing family, and H is $(\mathcal{F}, \varepsilon')$ -dense. Let $p = n^{-1}$ and $v(H) = 4n/3$. Then, for $l \in [4]$,

$$\Delta_l(H) \leq \binom{4n/3}{4-l} \leq (4n/3)^{4-l} \leq (4/3)^{4-l} c^{-1} n^{-l} e(H) = (4/3)^{4-l+1} c^{-1} p^{l-1} \frac{e(H)}{v(H)}.$$

Thus by Lemma 6.4, there exist constant C , family $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, and function $f : \mathcal{S} \rightarrow \overline{\mathcal{F}}$, such that every independent set in H is contained in some $T \in \mathcal{T} := \{F \cup S : F \in f(\mathcal{S}), S \in \mathcal{S}\}$.

Since $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, $|\mathcal{S}| \leq C(4n/3)^C$ and, hence,

$$|\mathcal{T}| = |\mathcal{S}|f(\mathcal{S}) \leq |\mathcal{S}|^2 \leq C^2(4n/3)^{2C}.$$

Since for $T \in \mathcal{T}$ it is possible that $|T \cap P| < \alpha_1 n + C$ or $|T \cap Q| < \alpha_2 n + C$, we need to make the sets in \mathcal{T} slightly larger in order to apply Lemma 5.1. For each $T \in \mathcal{T}$, let T' be a set obtained from T by adding vertices such that $|T' \cap P| = \max\{|T \cap P|, \lceil \alpha_1 n + C \rceil\}$ and $|T' \cap Q| = \max\{|T \cap Q|, \lceil \alpha_2 n + C \rceil\}$. (We choose one such T' for each T .) Let $\mathcal{T}' := \{T' : T \in \mathcal{T}\}$. Then

$$|\mathcal{T}'| \leq |\mathcal{T}| \leq C^2(4n/3)^{2C}.$$

Note that for each fixed $T' \in \mathcal{T}'$, we have $|R \cap T' \cap P| \sim \text{Bi}(|T' \cap P|, n^{-0.9})$ and $|R \cap T' \cap Q| \sim \text{Bi}(|T' \cap Q|, n^{-0.9})$. Hence, $\mathbb{E}(|R \cap T' \cap P|) = n^{-0.9}|T' \cap P|$ and $\mathbb{E}(|R \cap T' \cap Q|) = n^{-0.9}|T' \cap Q|$. Applying Lemma 5.1 to $|R \cap T' \cap P|$ and $|R \cap T' \cap Q|$ by taking $\lambda = \gamma n^{0.1}$, we have,

$$\begin{aligned} \mathbb{P}(|R \cap T' \cap P| - n^{-0.9}|T' \cap P| \geq \lambda) &\leq e^{-\Omega(\lambda^2/(n^{-0.9}|T' \cap P|))} \leq e^{-\Omega(n^{0.1})}, \text{ and} \\ \mathbb{P}(|R \cap T' \cap Q| - n^{-0.9}|T' \cap Q| \geq \lambda) &\leq e^{-\Omega(\lambda^2/(n^{-0.9}|T' \cap Q|))} \leq e^{-\Omega(n^{0.1})}. \end{aligned}$$

So with probability at most $2e^{-\Omega(n^{0.1})}$, $|R \cap T' \cap P| \geq n^{-0.9}|T' \cap P| + \lambda \geq (\alpha_1 + \gamma + C/n)n^{0.1}$ and $|R \cap T' \cap Q| \geq n^{-0.9}|T' \cap Q| + \lambda \geq (\alpha_2 + \gamma + C/n)n^{0.1}$.

Therefore, with probability at most $2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, there exists some $T' \in \mathcal{T}'$ such that $|R \cap T' \cap P| \geq (\alpha_1 + \gamma + C/n)n^{0.1}$ and $|R \cap T' \cap Q| \geq (\alpha_2 + \gamma + C/n)n^{0.1}$. Hence, with probability at least $1 - 2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, $|R \cap T' \cap P| < (\alpha_1 + \gamma + C/n)n^{0.1}$ or $|R \cap T' \cap Q| < (\alpha_2 + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$.

Now let J be an independent set in $H[R]$. Then J is also an independent set in H ; so there exist $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $J \subseteq T \subseteq T'$. Thus $J \subseteq R \cap T'$; so

$|J \cap P| \leq |R \cap T' \cap P|$ and $|J \cap Q| \leq |R \cap T' \cap Q|$. Hence, with probability at least $1 - 2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, for all independent set J in $H[R]$, $|J \cap P| \leq (\alpha_1 + \gamma + C/n)n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \gamma + C/n)n^{0.1}$. \blacksquare

6.2 Independent Sets in k -Graphs

In this section we prepare ourselves to deal with hypergraphs not close to $H_k^{k-l}(U, W)$.

The following lemma says that, if an n -vertex k -graph H is not ε -close to $H_k^{k-l}(U, W)$ and $\delta_l(H) \geq \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$ then H is $(\mathcal{F}, \varepsilon')$ -dense.

Lemma 6.7. *Let k, l be integers with $k \geq 2$ and $l \in [k-1]$. Let $0 < \varepsilon \ll 1$, $\rho' \leq \varepsilon/8$, and $0 < \mu \leq \varepsilon/40$. Let m, n be sufficiently large integers such that $n/k - \mu n \leq m \leq n/k$. Suppose H is a k -graph with order n such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$, and H is not ε -close to $H_k^{k-l}(U, W)$ for any partition of $V(H)$ into U, W with $|W| = m$. Then H is $(\mathcal{F}, \varepsilon/(2k!))$ -dense, where $\mathcal{F} = \{A \subseteq V(H) : |A| \geq (1 - 1/k - \varepsilon/4)n\}$.*

Proof. Suppose to the contrary that there exists $A \subseteq V(H)$ such that $|A| \geq (1 - 1/k - \varepsilon/4)n$ and $e(H[A]) \leq \varepsilon e(H)/(2k!)$. By removing vertices if necessary, we may choose A such that $|V(H) \setminus A| \geq m$ (as $m \leq n/k$). Let $W \subseteq V(H) \setminus A$ such that $|W| = m$. For convenience, let $B = V(H) \setminus (W \cup A)$. Then

$$|B| \leq n - m - (1 - 1/k - \varepsilon/4)n \leq \varepsilon n/4 + n/k - (1/k - \mu)n \leq 11\varepsilon n/40.$$

Let $U = V(H) \setminus W$ and $H_0 = H_k^{k-l}(U, W)$. We derive a contradiction by showing that $|E(H_0) \setminus E(H)| < \varepsilon n^k$.

Note that, for each $f \in E(H_0) \setminus E(H)$, we have $1 \leq |f \cap W| \leq k-l$ (by definition of H_0); so $|f \cap B| > 0$ or $|f \cap A| \geq l$. Thus

$$|E(H_0) \setminus E(H)| \leq |\{f \in E(H_0) : |f \cap B| > 0\}| + |\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}|.$$

It is easy to see that

$$|\{f \in E(H_0) : |f \cap B| > 0\}| \leq |B||W|n^{k-2} \leq (11\varepsilon n/40)(n/k)n^{k-2} = \frac{11\varepsilon}{40k}n^k.$$

Next, we bound $|\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}|$. Fix an arbitrary l -set $S \subseteq A$.

Note that

$$|\{f \in E(H) : S \subseteq f \text{ and } f \cap B \neq \emptyset\}| \leq |B|n^{k-l-1} \leq \frac{11\varepsilon}{40}n^{k-l}.$$

For any $f \in E(H)$ and $S \subseteq f$, we have $f \cap B \neq \emptyset$, or $f \subseteq A$, or $f \in E(H_0)$. So

$$\begin{aligned} & |\{f \in E(H) : S \subseteq f \text{ and } f \in E(H_0)\}| \\ & \geq d_H(S) - |\{f \in E(H) : S \subseteq f \text{ and } f \cap B \neq \emptyset\}| - |\{f \in E(H) : S \subseteq f \text{ and } f \subseteq A\}| \\ & \geq d_H(S) - \frac{11\varepsilon}{40}n^{k-l} - d_{H[A]}(S). \end{aligned}$$

Hence,

$$\begin{aligned} & |\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}| \\ & \leq \sum_{S \in \binom{A}{l}} |\{f \in E(H_0) \setminus E(H) : S \subseteq f\}| \\ & \leq \sum_{S \in \binom{A}{l}} (d_{H_0}(S) - |\{f \in E(H) : f \in E(H_0) \text{ and } S \subseteq f\}|) \\ & \leq \sum_{S \in \binom{A}{l}} \left(d_{H_0}(S) - d_H(S) + \frac{11\varepsilon}{40}n^{k-l} + d_{H[A]}(S) \right). \end{aligned}$$

Note that for $S \in \binom{A}{l}$, $d_{H_0}(S) = \binom{n-l}{k-l} - \binom{n-l-m}{k-l}$; so $d_{H_0}(S) - d_H(S) < \rho'n^{k-l}$ by the

assumption on $\delta_l(H)$. Hence,

$$\begin{aligned}
|E(H_0) \setminus E(H)| &< \frac{11\varepsilon}{40k} n^k + \binom{|A|}{l} \left(\rho' + \frac{11\varepsilon}{40} \right) n^{k-l} + \sum_{S \in \binom{A}{l}} d_{H[A]}(S) \\
&\leq \left(\frac{11\varepsilon}{40k} + \rho' + \frac{11\varepsilon}{40} \right) n^k + \binom{k}{l} e(H[A]) \\
&\leq \left(\frac{11}{120} + \frac{1}{8} + \frac{11}{40} \right) \varepsilon n^k + \binom{k}{l} \frac{\varepsilon n^k}{2k!} \quad (\text{since } k \geq 3 \text{ and } \rho' \leq \varepsilon/8) \\
&< \varepsilon n^k,
\end{aligned}$$

a contradiction. ■

We now use Lemma 6.4 to show that one can control, with high probability, the independence number of a subgraph of a k -graph induced by a random subset of vertices.

Lemma 6.8. *Let c, ε', α be positive reals and let k, n be positive integers. Let H be an n -vertex k -graph such that $e(H) \geq cn^k$ and $e(H[S]) \geq \varepsilon' e(H)$ for all $S \subseteq V(H)$ with $|S| \geq \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H independently and uniformly at random with probability $n^{-0.9}$. Then, for any positive $\gamma \ll \alpha$, the independence number of $H[R]$ is at most $(\alpha + \gamma + o(1))n^{0.1}$, with probability at least $1 - n^{O(1)}e^{-\Omega(n^{0.1})}$.*

Proof. Define $\mathcal{F} := \{A \subseteq V(H) : e(H[A]) \geq \varepsilon' e(H) \text{ and } |A| \geq \varepsilon' n\}$. Then \mathcal{F} is an increasing family, and H is $(\mathcal{F}, \varepsilon')$ -dense. Let $p = n^{-1}$ and $v(H) = n$. Then

$$\Delta_l(H) \leq \binom{n}{k-l} \leq n^{k-l} \leq c^{-1} n^{-l} e(H) = c^{-1} p^{l-1} \frac{e(H)}{v(H)}.$$

Thus by Lemma 6.4, there exist a constant C (depending only on ε' and c), a family $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, a function $f : \mathcal{S} \rightarrow \overline{\mathcal{F}}$, and a family $\mathcal{T} := \{F \cup S : F \in f(\mathcal{S}), S \in \mathcal{S}\}$, such that every independent set in H is contained in some $T \in \mathcal{T}$. Since $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, $|\mathcal{S}| \leq Cn^C$ and, hence,

$$|\mathcal{T}| = |\mathcal{S}| |f(\mathcal{S})| \leq |\mathcal{S}|^2 \leq C^2 n^{2C}.$$

We claim that $|T| < \alpha n + C$ for all $T \in \mathcal{T}$. To see this, let $T = F \cup S$ for some $F \in f(\mathcal{S})$ and $S \in \mathcal{S}$. By definition, $F \in \overline{\mathcal{F}}$ and hence, $e(H[F]) < \varepsilon' e(H)$. Since $e(H[S]) \geq \varepsilon' e(H)$ for any $S \subseteq V(H)$ with $|S| \geq \alpha n$, we have $|F| < \alpha n$. Therefore, $|T| \leq |F| + |S| < \alpha n + C$.

We wish to apply Lemma 5.1 and, hence, we need to make sets in \mathcal{T} slightly larger. Take an arbitrary map $h : \mathcal{T} \rightarrow \binom{V(H)}{[\alpha n + C]}$ such that $T \subseteq h(T)$ for all $T \in \mathcal{T}$, and let $\mathcal{T}' = h(\mathcal{T})$. Then

$$|\mathcal{T}'| \leq |\mathcal{T}| \leq |\mathcal{S}|^2 \leq C^2 n^{2C}.$$

Note that for each fixed $T' \in \mathcal{T}'$, we have $|R \cap T'| \sim \text{Bi}(|T'|, n^{-0.9})$ and $\mathbb{E}(|R \cap T'|) = n^{-0.9}|T'| = \lfloor \alpha n + C \rfloor n^{-0.9}$. We apply Lemma 5.1 to $|R \cap T'|$ by taking $\lambda = \gamma n^{0.1}$, where γ is fixed and $\gamma \ll \alpha$. Then

$$\mathbb{P}(|R \cap T'| - n^{-0.9}|T'| \geq \lambda) \leq e^{-\Omega(\lambda^2/(n^{-0.9}|T'|))} = e^{-\Omega(n^{0.1})}.$$

So with probability at most $e^{-\Omega(n^{0.1})}$, we have $|R \cap T'| \geq n^{-0.9}|T'| + \lambda$. Hence, $|R \cap T'| \geq (\alpha + \gamma + C/n)n^{0.1}$ with probability at most $e^{-\Omega(n^{0.1})}$.

Therefore, with probability at most $C^2 n^{2C} e^{-\Omega(n^{0.1})}$ (from union bound), there exists some $T' \in \mathcal{T}'$ such that $|R \cap T'| \geq (\alpha + \gamma + C/n)n^{0.1}$. Hence, with probability at least $1 - C^2 n^{2C} e^{-\Omega(n^{0.1})}$, $|R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$.

It remains to show that, conditioning on $|R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$, $|J| \leq (\alpha + \gamma + C/n)n^{0.1}$ for every independent set J in $H[R]$. Since such J is also an independent set in H , there exist $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $J \subseteq T \subseteq T'$. Thus $J \subseteq R \cap T'$ and $|J| \leq |R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$.

Hence, $\alpha(H[R]) \leq (\alpha + \gamma + C/n)n^{0.1}$, with probability at least $1 - C^2 n^{2C} e^{-\Omega(n^{0.1})}$. ■

CHAPTER 7

PERFECT FRACTIONAL MATCHINGS IN K -GRAPHS

Definition 7.1. A *fractional matching* in a k -graph H is a function $w : E(H) \rightarrow [0, 1]$ such that for any $v \in V(H)$, $\sum_{\{e \in E(H) : v \in e\}} w(e) \leq 1$. A fractional matching is called *perfect* if $\sum_{e \in E(H)} w(e) = |V(H)|/k$.

In this chapter, we show that for any reals $0 < \rho \ll \varepsilon$, if an n -vertex k -graph H has $\alpha(H) \leq (1 - 1/k - \varepsilon/5)n$ and $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho n^{k-l}$, then H admits a perfect fractional matching. Note that the term $-\rho n^{k-l}$ is from removing an absorbing matching in the original graph, and the deviation from random sampling.

7.1 Shadows and Stable Families

We need to consider matchings in the “link” graph of an l -set in a k -graph, which is a $(k - l)$ -graph. This is related to the following well known conjecture of Erdős [11] on matchings in uniform hyperegraphs: If F is a k -graph on n vertices and $\nu(F) = s$, then $e(F) \leq \max \left\{ \binom{n}{k} - \binom{n-s}{k}, \binom{ks+1}{k} \right\}$. Frankl [14] proved that if $n \geq (2s + 1)k - s$ then $e(F) \leq \binom{n}{k} - \binom{n-s}{k}$, with $H_k^k(U, W)$ (where $|W| = s$ and $|U| = n - s$) as extremal graphs. Very recently, Frankl and Kupavskii [17] further improved the lower bound to $n \geq (5k/3 - 2/3)s$ for large s .

Ellis, Keller, and Lifshitz [10] recently proved the following stability version of Frankl’s result, which we state as follows using our notation: For any $s \in \mathbb{N}$, $\eta > 0$, and $\varepsilon > 0$, there exists $\delta = \delta(s, \eta, \varepsilon) > 0$ such that the following holds. Let $n, k \in \mathbb{N}$ with $k \leq (\frac{1}{2s+1} - \eta)n$. Suppose $H \subseteq \binom{[n]}{k}$ with $\nu(H) \leq s$ and $e(H) \geq \binom{n}{k} - \binom{n-s}{k} - \delta \binom{n-s}{k-1}$. Then there exists $W \in \binom{[n]}{s}$ such that $|E(H) \setminus E(H_k^k(U, W))| < \varepsilon \binom{n-s}{k}$.

The lower bound on $e(H)$ in the above result of Ellis, Keller, and Lifshitz is too large

for our purpose. Using LP duality we only need to consider “stable” hypergraphs and for such hypergraphs we can improve the bound on $e(H)$ to $\binom{n}{k} - \binom{n-s}{k} - \xi n^k$.

For subsets $e = \{u_1, \dots, u_k\}, f = \{v_1, \dots, v_k\} \subseteq [n]$ with $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for $i \in [k-1]$, we write $e \leq f$ if $u_i \leq v_i$ for all $i \in [k]$. A hypergraph $H \subseteq \binom{[n]}{k}$ is said to be *stable* if, for $e, f \in \binom{[n]}{k}$ with $e \leq f$, $f \in E(H)$ implies $e \in E(H)$. Our proof of a stability version of Frankl’s theorem for stable hypergraphs uses the same ideas as in [14]. The following result from [14] is an extension of Katona’s Intersection Shadow Theorem [26].

Lemma 7.2. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) = s$. Then $s|\partial\mathcal{F}| \geq |\mathcal{F}|$, where $\partial\mathcal{F}$ is the shadow of \mathcal{F} , defined by*

$$\partial\mathcal{F} := \left\{ G \in \binom{[n]}{k-1} : G \subseteq F \text{ for some } F \in \mathcal{F} \right\}.$$

We can now state and prove the following stability version of Frankl’s result on matchings for stable hypergraphs. Note that we allow $k = 1$.

Lemma 7.3. *Let k be a positive integer, and let c and ξ be constants such that $0 < c < 1/(2k)$ and $0 < \xi \leq (1 + 18(k-1)!/c)^{-2}$. Let n, m be positive integers such that n is sufficiently large and $cn \leq m \leq n/(2k)$. Let H be a k -graph with vertex set $[n]$ such that H is stable and $\nu(H) \leq m$. If $e(H) > \binom{n}{k} - \binom{n-m}{k} - \xi n^k$, then H is $\sqrt{\xi}$ -close to $H_k^k([n] \setminus [m], [m])$.*

Proof. Suppose $e(H) > \binom{n}{k} - \binom{n-m}{k} - \xi n^k$. When $k = 1$, each edge of H consists of a single vertex. In this case, since $e(H) > m - \xi n \geq m - \sqrt{\xi}n$ and because H is stable and $e(H) = \nu(H) \leq m$, we have that H is $\sqrt{\xi}$ -close to $H_1^1([n] \setminus [m], [m])$.

Thus, we may assume $k \geq 2$. To show that H is close to $H_k^k([n] \setminus [m], [m])$, we bound $e(H - [m])$ (as edges in $H - [m]$ are not in $H_k^k([n] \setminus [m], [m])$). Since H is stable, the vertex $m+1$ has the maximum degree in $H - [m]$. So

$$e(H - [m]) \leq \frac{(n-m)}{k} |\{e \in E(H - [m]) : m+1 \in e\}|.$$

Hence, our objective is to bound the size of $\mathcal{F}(\{m+1\}) := \{e \in E(H - [m]) : m+1 \in e\}$.

Let

$$\sigma = \frac{2\xi(k-1)!}{c}.$$

First, we may assume that

$$(1) \quad |\mathcal{F}(\{m+1\})| \geq 9k\sigma n^{k-1}.$$

For, suppose $|\mathcal{F}(\{m+1\})| < 9k\sigma n^{k-1}$. Then

$$e(H - [m]) \leq \frac{(n-m)}{k} |\mathcal{F}(\{m+1\})| < 9\sigma n^k.$$

Thus

$$\begin{aligned} & |E(H_k^k([n] \setminus [m], [m])) \setminus E(H)| \\ &= e(H_k^k([n] \setminus [m], [m])) - (e(H) - e(H - [m])) \\ &< \left(\binom{n}{k} - \binom{n-m}{k} \right) - \left(\binom{n}{k} - \binom{n-m}{k} - \xi n^k - 9\sigma n^k \right) \\ &= \xi n^k + 9 \cdot \frac{2\xi(k-1)!}{c} n^k \\ &\leq \sqrt{\xi} n^k, \end{aligned}$$

as $\xi \leq (1 + 18(k-1)!/c)^{-2}$. That is, H is $\sqrt{\xi}$ -close to $H_k^k([n] \setminus [m], [m])$, and the assertion of the lemma holds. So we may assume that (1) holds. \square

To proceed further, we extend the notation $\mathcal{F}(\{m+1\})$ to all $Q \subseteq [m+1]$, by letting

$$\mathcal{F}(Q) = \{e \in E(H) : e \cap [m+1] = Q\}.$$

Note that $|\mathcal{F}(Q)| \leq \binom{n-(m+1)}{k-|Q|} = \binom{n-m-1}{k-|Q|}$. Also note that, since H is stable, $|\mathcal{F}(\{m+$

$1\})| \geq |\partial\mathcal{F}(\emptyset)|$. So Lemma 7.2 gives

$$m|\mathcal{F}(\{m+1\})| \geq m|\partial\mathcal{F}(\emptyset)| \geq |\mathcal{F}(\emptyset)|.$$

We claim that

$$(2) \quad (\sum_{i=1}^{m+1} |\mathcal{F}(\{i\})|) + m|\mathcal{F}(\{m+1\})| > m \binom{n-m}{k-1} (1 - \sigma).$$

To prove (2), it suffices to show $|\mathcal{F}(\emptyset)| + \sum_{i=1}^{m+1} |\mathcal{F}(\{i\})| > m \binom{n-m}{k-1} (1 - \sigma)$. Note that

$$\sum_{Q \subseteq [m+1], |Q| \geq 2} |\mathcal{F}(Q)| \leq \sum_{i=2}^k \binom{m+1}{i} \binom{n-(m+1)}{k-i}$$

and

$$\begin{aligned} \binom{n}{k} &= \binom{n-(m+1)}{k} + (m+1) \binom{n-(m+1)}{k-1} + \sum_{i=2}^k \binom{m+1}{i} \binom{n-(m+1)}{k-i} \\ &= \binom{n-m}{k} + m \binom{n-(m+1)}{k-1} + \sum_{i=2}^k \binom{m+1}{i} \binom{n-(m+1)}{k-i}. \end{aligned}$$

Thus,

$$\begin{aligned} & |\mathcal{F}(\emptyset)| + \sum_{i=1}^{m+1} |\mathcal{F}(\{i\})| \\ &= e(H) - \sum_{Q \subseteq [m+1], |Q| \geq 2} |\mathcal{F}(Q)| \\ &> \binom{n}{k} - \binom{n-m}{k} - \xi n^k - \sum_{i=2}^k \binom{m+1}{i} \binom{n-(m+1)}{k-i} \\ &= m \binom{n-(m+1)}{k-1} - \xi n^k \\ &> m \binom{n-m}{k-1} (1 - \sigma) \quad (\text{since } cn \leq m \leq n/(2k) \text{ and } n \text{ large}). \end{aligned}$$

This proves (2). \square

Let $t = \lceil (2 + 1/k)m \rceil$. Since $n \geq 2km$ and $m \geq cn$ (where n is sufficiently large),

$$n - (m + 1) \geq 2km - (m + 1) = (2 + 1/(k - 1))m(k - 1) - 1 > t(k - 1).$$

Let $\mathcal{M} = \{f_1, \dots, f_t\}$ be t pairwise disjoint $(k-1)$ -subsets of $[n] \setminus [m+1]$ chosen uniformly at random. Let $\mathcal{F}_i := \{e \setminus \{i\} : e \in \mathcal{F}(\{i\})\}$ for $i \in [m+1]$. Then $\mathcal{F}_{m+1} \subseteq \mathcal{F}_m \subseteq \dots \subseteq \mathcal{F}_1$ (since H is stable) and, for each fixed pair i, j ,

$$\mathbb{P}(f_j \in \mathcal{F}_i) = \frac{|\mathcal{F}_i|}{\binom{n-(m+1)}{k-1}}.$$

Let

$$x_i = \begin{cases} 1, & f_i \in \mathcal{F}_{m+1}, \\ 0, & f_i \notin \mathcal{F}_{m+1}, \end{cases}$$

and let $p = \mathbb{P}(x_i = 1)$ (which is the same for all $i \in [t]$). Now $|\mathcal{F}_{m+1}| = p \binom{n-(m+1)}{k-1}$. So by (1), we have

$$(3) \quad p > 9k\sigma.$$

We claim that

$$(4) \quad \text{for } 1 \leq i < j \leq t, \mathbb{P}(x_i x_j = 1) \leq \left(1 + \frac{1}{4k}\right) p^2.$$

This is because

$$\begin{aligned} \mathbb{P}(x_i x_j = 1) &= \mathbb{P}(x_j = 1 | x_i = 1) \mathbb{P}(x_i = 1) \\ &\leq \frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)-(k-1)}{k-1}} \frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)}{k-1}} \\ &= \frac{\binom{n-(m+1)}{k-1}}{\binom{n-(m+1)-(k-1)}{k-1}} \cdot p^2 \\ &\leq \left(1 + \frac{1}{4k}\right) p^2, \end{aligned}$$

as $n - (m + 1) \geq (1 - 1/(2k))n - 1$ and n is large. This completes the proof of (4). \square

Define a bipartite graph G with partition sets \mathcal{M} and $\{\mathcal{F}_1, \dots, \mathcal{F}_{m+1}\}$, where $f_j \in \mathcal{M}$ is adjacent to \mathcal{F}_i if, and only if, $f_j \in \mathcal{F}_i$. Note that a matching of size $m + 1$ in G gives rise to a matching of size $m + 1$ in H . Thus, $\nu(G) \leq m$. So by a theorem of König, G has a vertex cover of size m , say T . Let $x = |T \cap \mathcal{M}|$; then $|T \cap \{\mathcal{F}_1, \dots, \mathcal{F}_{m+1}\}| = m - x$. Since $\mathcal{F}_{m+1} \subseteq \mathcal{F}_m \subseteq \dots \subseteq \mathcal{F}_1$, $d_G(f_j) = m + 1$ for $f_j \in \mathcal{F}_{m+1}$; so $f_j \in T$ for all $f_j \in \mathcal{F}_{m+1}$. Hence $0 \leq b \leq x \leq m$, where $b := |\mathcal{M} \cap \mathcal{F}_{m+1}| = \sum_{i=1}^t x_i$. So $pt = \mathbb{E}(b) \leq m \leq t/(2 + 1/k)$. This implies

$$(5) \quad p \leq 1/(2 + 1/k) < 1/2.$$

Moreover,

$$\sum_{i=1}^{m+1} |\mathcal{M} \cap \mathcal{F}_i| = e(G) \leq t(m - x) + x((m + 1) - (m - x)) = x^2 - (t - 1)x + mt.$$

Thus, letting $h(x, b) := x^2 - (t - 1)x + mt + mb$, we have

$$\begin{aligned} \mathbb{E}(h(x, b)) &\geq \mathbb{E} \left(m|\mathcal{M} \cap \mathcal{F}_{m+1}| + \sum_{i=1}^{m+1} |\mathcal{M} \cap \mathcal{F}_i| \right) \\ &= mt \frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)}{k-1}} + \sum_{i=1}^{m+1} t \frac{|\mathcal{F}_i|}{\binom{n-(m+1)}{k-1}} \\ &= \frac{t}{\binom{n-(m+1)}{k-1}} \left(m|\mathcal{F}(\{m+1\})| + \sum_{i=1}^{m+1} |\mathcal{F}(\{i\})| \right) \\ &> mt(1 - \sigma) \quad (\text{by (2)}). \end{aligned}$$

Next we obtain an upper bound on $\mathbb{E}(h(x, b))$. Using the convexity of $h(x, b)$ (as a function of x over the interval $[b, m]$) and the fact that $h(b, b) - h(m, b) = (t - 1 - m - b)(m - b) \geq 0$, we have

$$h(x, b) \leq \max\{h(b, b), h(m, b)\} = h(b, b) = b^2 - (t - 1)b + mt + mb.$$

Thus,

$$\begin{aligned}
\mathbb{E}(h(x, b)) &\leq \mathbb{E}(b^2 - (t-1)b + mt + mb) \\
&= \mathbb{E}\left(\left(\sum_{i=1}^t x_i\right)^2 - (t-1-m)\left(\sum_{i=1}^t x_i\right) + mt\right) \\
&\leq \left(1 + \frac{1}{4k}\right) p^2(t^2 - t) + pt - (t-1-m)pt + mt \quad (\text{by (4)}).
\end{aligned}$$

Hence, combining the above bounds on $\mathbb{E}(h(x, b))$, we have

$$\left(1 + \frac{1}{4k}\right) p^2(t^2 - t) + pt - (t-1-m)pt + mt > mt(1 - \sigma).$$

Thus,

$$\begin{aligned}
\sigma mt &> pt \left(t - m - \left(1 + \frac{1}{4k}\right) pt - 2 + \left(1 + \frac{1}{4k}\right) p \right) \\
&> pt \left(\left(1 - \left(1 + \frac{1}{4k}\right) p\right) t - m - 2 \right) \\
&\geq pt \left(\left(\left(1 - \frac{1}{2} \left(1 + \frac{1}{4k}\right)\right) \left(2 + \frac{1}{k}\right) - 1 \right) m - 2 \right) \quad (\text{by (5) and the definition of } t) \\
&= pt \left(\frac{2k-1}{8k^2} m - 2 \right) \\
&> ptm/(9k) \quad (\text{since } m \geq cn \text{ and } n \text{ is large}).
\end{aligned}$$

Therefore, $p < 9k\sigma$, contradicting (3). Hence H must be $\sqrt{\xi}$ -close to $H_k^k([n] \setminus [m], [m])$. ■

Remark. In the proof of Lemma 7.3 we require $m \leq n/(2k)$ (e.g., when we define t and \mathcal{M} before (3)). We will see in the next chapter that we can replace it with $n/2 - 1$ when $k = 3$ and $l = 1$.

7.2 Perfect Fractional Matchings

For a hypergraph H , let

$$\nu^*(H) = \max \left\{ \sum_{e \in E(H)} w(e) : w \text{ is a fractional matching in } H \right\}.$$

A *fractional vertex cover* of H is a function $w : V(H) \rightarrow [0, 1]$ such that, for each $e \in E$, $\sum_{v \in e} w(v) \geq 1$. Let

$$\tau^*(H) = \min \left\{ \sum_{v \in V(H)} w(v) : w \text{ is a fractional vertex cover of } H \right\}.$$

Then the strong duality theorem of linear programming gives

$$\nu^*(H) = \tau^*(H).$$

We are now ready to prove the existence of a perfect fractional matching in a uniform hypergraph whose independence number is not too large.

Lemma 7.4. *Let k, l be integers with $k \geq 3$ and $k/2 \leq l < k$, and let ε, ξ be positive reals with $\xi < (\varepsilon/5)^2(3k)^{-4(k-l)}$. Let n be a positive integer such that n is sufficiently large and $n \in k\mathbb{Z}$. Let H be a k -graph of order n such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-n/k}{k-l} - \xi n^{k-l}$ and $\alpha(H) \leq (1 - 1/k - \varepsilon/5)n$. Then H contains a perfect fractional matching.*

Proof. For convenience, let $V(H) = [n]$. Let ω be a minimum fractional vertex cover of H and we may assume that $\omega(1) \geq \omega(2) \geq \dots \geq \omega(n)$. Let $E' = \{e \in \binom{[n]}{k} : e \notin E(H) \text{ and } \sum_{i \in e} \omega(i) \geq 1\}$ and let H' be obtained from H by adding the edges in E' . Then H' is stable and $\tau^*(H') = \tau^*(H)$. Thus $\nu^*(H) = \nu^*(H') \geq \nu(H')$, and it suffices to show that $\nu(H') = n/k$, i.e., H' contains a perfect matching.

Let $S = [n] \setminus [n-l]$, and let G be the hypergraph with $V(G) = [n]$ and $E(G) = N_{H'}(S)$,

which is a $(k-l)$ -graph on $[n]$. Since H' is stable, G is also stable. We may assume that

$$(1) \nu(G) \leq n/k - 1.$$

For, otherwise, let $f_1, \dots, f_{n/k}$ be a matching in G . Now $[n] \setminus \left(\bigcup_{i=1}^{n/k} f_i\right)$ is a set of size $(n/k)l$ and, hence, can be partitioned into l -sets, say $S_1, \dots, S_{n/k}$. Since H' is stable and $S \cup f_i \in E(H')$ for $i \in [n/k]$, we have $S_i \cup f_i \in E(H')$ for $i \in [n/k]$. Hence, $\{S_i \cup f_i : i \in [n/k]\}$ is a perfect matching in H' . Hence, we may assume (1). \square

We may also assume that

$$(2) l \leq k - 2.$$

For, suppose $l = k - 1$. Then G is a 1-graph. Since H' is stable and $e(G) \geq \delta_{k-1}(H) \geq n/k - \lceil \xi n \rceil$, the first $n/k - \lceil \xi n \rceil$ vertices of G are edges of G .

Note that $H' - [n/k - \lceil \xi n \rceil]$ has $n - n/k + \lceil \xi n \rceil$ vertices. Since $\alpha(H) \leq (1 - 1/k - \varepsilon/5)n$, $H' - [n/k - \lceil \xi n \rceil]$ has an edge. In fact, since $\xi < (\varepsilon/5)^2(3k)^{-4(k-l)}$, we can greedily find pairwise disjoint edges $f_1, \dots, f_{\lceil \xi n \rceil}$ in $H' - [n/k - \lceil \xi n \rceil]$. Since

$$n - (n/k - \lceil \xi n \rceil) - \lceil \xi n \rceil k = (k-1)(n/k - \lceil \xi n \rceil),$$

we can partition $[n] \setminus \left([n/k - \lceil \xi n \rceil] \cup \bigcup_{i=1}^{\lceil \xi n \rceil} f_i\right)$ into $(k-1)$ -sets $S_1, \dots, S_{n/k - \lceil \xi n \rceil}$. Now $S_i \cup \{i\}$, $i \in [n/k - \lceil \xi n \rceil]$, form a matching in H' . These edges and $\{f_1, \dots, f_{\lceil \xi n \rceil}\}$ form a perfect matching in H' . So we may assume (2). \square

Let $\eta = \varepsilon/(5k)$ and let $t = n/k - \lfloor \eta n \rfloor$. For $i \in [n]$, we use $d_G(i)$ to denote the degree of i in G . We claim that

$$(3) d_G(t) > \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}.$$

For suppose $d_G(t) \leq \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}$. Since H' is stable, $d_G(i) \leq \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}$ for

$t \leq i \leq n/k$. Note that the degree of t in $H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])$ is $\binom{n-1}{k-l-1}$. Thus,

$$\begin{aligned}
& |E(H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])) \setminus E(G)| \\
& \geq \frac{1}{k-l} \left(\sum_{i=t}^{n/k} (d_{H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])}(i) - d_G(i)) \right) \\
& \geq \frac{1}{k-l} (n/k - t + 1) \binom{n/(2k)}{k-l-1} \\
& > \frac{1}{k-l} \eta n (3k)^{-(k-l-1)} \binom{n}{k-l-1} \\
& > \sqrt{\xi} n^{k-l},
\end{aligned}$$

as $\xi < (\varepsilon/5)^2 (3k)^{-4(k-l)}$.

Hence G is not $\sqrt{\xi}$ -close to $H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])$. However, since G is stable and $n/k \leq n/(2(k-l))$ (as $l \geq k/2$), we may apply Lemma 7.3 with $n/k, k-l, \xi$ as m, k, ξ , respectively. So $\nu(G) \geq n/k$, contradicting (1) and completing the proof of (3). \square

Note that $H' - [t]$ has $n - n/k + \lfloor \eta n \rfloor$ vertices. Since $\alpha(H) \leq (1 - 1/k - \varepsilon/5)n$, $H' - [t]$ has an edge. In fact, since $\varepsilon n = 5k\eta n$, $H' - [t]$ has $\lfloor \eta n \rfloor$ pairwise disjoint edges, say $f_1, \dots, f_{\lfloor \eta n \rfloor}$. Let $T = \bigcup_{i=1}^{\lfloor \eta n \rfloor} f_i$.

Next we find disjoint edges e_1, \dots, e_t of G such that $|e_i \cap [t]| = 1$ and $e_i \cap T = \emptyset$ for all $i \in [t]$. Suppose for some $s \in [t-1]$ we have found pairwise disjoint edges e_1, \dots, e_s of G such that, for $i \in [s]$, $e_i \cap [t] = \{i\}$ and $e_i \cap T = \emptyset$. The number of edges of G containing $s+1$ and intersecting $T \cup ([t] \setminus \{s+1\}) \cup (\bigcup_{i=1}^s e_i)$ is at most $\binom{n-1}{k-l-1} - \binom{n-|T|-t-(k-l)s}{k-l-1}$. Note that $n - |T| - t - (k-l)s \geq n/(2k)$, as $l \geq k/2$. Hence, by (3), there exists $e_{s+1} \in E(G)$ such that $e_{s+1} \cap [t] = \{s+1\}$, $e_{s+1} \cap T = \emptyset$, and e_{s+1} is disjoint from $\bigcup_{i=1}^s e_i$.

Since $t = n/k - \lfloor \eta n \rfloor$, $(H' - T) - \bigcup_{i=1}^t e_i$ has exactly tl vertices (as $|e_i \cap [t]| = 1$ for $i \in [t]$). Partition the vertices in $(H' - T) - \bigcup_{i=1}^t e_i$ to pairwise disjoint l -sets S_1, \dots, S_t . Then, since H' is stable, $S_i \cup e_i \in E(H')$ for $i \in [t]$. Hence, $\{f_i : i \in [\lfloor \eta n \rfloor]\} \cup \{S_j \cup e_j : j \in [t]\}$ is a perfect matching in H' . \blacksquare

Remark. When we apply Lemma 7.3 in the end of the proof of (3), we require $l \geq k/2$ so that $n/k \leq n(2(k-l))$ (which amounts to $m \leq n/(2k)$ in Lemma 7.3). This is not necessary when $k = 3$ and $l = 1$, as we can use Lemma 8.2 (see the next chapter) which is the same as Lemma 7.3 except with $m \leq n/(2k) = n/4$ replaced by $m \leq n/2 - 1$.

CHAPTER 8

PERFECT FRACTIONAL MATCHINGS IN $(1, 3)$ -PARTITE 4-GRAPHS

In this chapter, we will prove the corresponding perfect fractional matching results in $(1, 3)$ -partite 4-graphs. We first show a lemma for stable graphs.

8.1 Stable Graphs

We need a result of Berge [7] on maximum matchings. For a graph G , we use $c_o(G)$ to denote the number of odd components in G .

Lemma 8.1 (Berge, 1958). *Let G be a graph on n vertices. Then*

$$\nu(G) = \min \{ (n - c_o(G - W) + |W|) / 2 : W \subseteq V(G) \}.$$

We prove the following result for stable graphs as an analog of Lemma 7.3.

Lemma 8.2. *Let c, ρ be constant with $0 < \rho \ll 1$ and $0 < c < 1/2$, and let m, n be positive integers with n sufficiently large and $cn \leq m \leq n/2 - 1$. Let G be a 2-graph with $V(G) = [n]$ such that $\nu(G) \leq m$ and G is stable with respect to the natural order on $[n]$. If $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, then G is $2\sqrt{\rho}$ -close to $H_2^2([n] \setminus [m], [m])$.*

Proof. Since G is stable, we have

$$(1) \quad N_G(i) \setminus \{j\} \subseteq N_G(j) \setminus \{i\} \text{ for any } i, j \in [n] \text{ with } i > j.$$

By Lemma 8.1, there exists $W \subseteq V(G)$ such that

$$\nu(G) = (n - c_o(G - W) + |W|) / 2.$$

We choose maximal such W , and let C_1, \dots, C_q denote the components of $G - W$. Without loss of generality, assume $|V(C_1)| \geq \dots \geq |V(C_q)|$, and let $c_i := |V(C_i)|$ for $i \in [q]$. Then

(2) $q = c_o(G - W)$, i.e., c_i is odd for all $i \in [q]$.

For, otherwise, suppose that c_i is even for some $i \in [q]$. Let $x \in V(C_i)$ and $W' := W \cup \{x\}$. Then $c_o(G - W') \geq c_o(G - W) + 1$. This forces $(n - c_o(G - W) + |W|)/2 = (n - c_o(G - W') + |W'|)/2$, as $\nu(G) = (n - c_o(G - W) + |W|)/2$. But then, W' contradicts the choice of W , completing the proof of (2). \square

Next, we claim that

(3) $c_i = 1$ for $i = 2, \dots, q$.

For, suppose $c_2 \geq 2$. Then $c_1 \geq c_2 \geq 2$; so there exist $a_1b_1 \in E(C_1)$ and $a_2b_2 \in E(C_2)$. If $a_1 > a_2$ then $a_1b_2 \in E(G)$ by (1), and if $a_1 < a_2$ then $b_1a_2 \in E(G)$ by (1). So there is edge between C_1 and C_2 , contradicting the fact that C_1 and C_2 are different components of $G - W$. This completes the proof of (3). \square

By (3), we have

$$\begin{aligned} m &\geq \nu(G) = (n - (c_o(G - W) - |W|))/2 \\ &= ((c_1 + |W| + q - 1) - (q - |W|))/2 \\ &= (c_1 - 1)/2 + |W|. \end{aligned}$$

Thus, $|W| \leq m - (c_1 - 1)/2$. Hence,

$$e(G) \leq \binom{n}{2} - \binom{n - |W|}{2} + \binom{c_1}{2} \leq \binom{n}{2} - \binom{n - m + (c_1 - 1)/2}{2} + \binom{c_1}{2}.$$

Since $e(G) > \binom{n}{2} - \binom{n - m}{2} - \rho n^2$, we have

$$\begin{aligned} \binom{n - m}{2} + \rho n^2 &> \binom{n - m + (c_1 - 1)/2}{2} - \binom{c_1}{2} \\ &= \binom{n - m}{2} + \frac{1}{8}(c_1 - 1)^2 + \frac{1}{4}(c_1 - 1)(2n - 2m - 1) - \binom{c_1}{2}, \end{aligned}$$

which gives

$$-\frac{3}{8}(c_1 - 1)^2 + \frac{1}{4}(c_1 - 1)(2n - 2m - 3) < \rho n^2.$$

Hence, $c_1 < \sqrt{\rho}n$, since $\rho \ll 1$ and $m \leq n/2 - 1$.

Note that every edge of G intersects $W \cup V(C_1)$. So by (1), every edge of G intersects $[|W| + c_1] \subseteq [m + (c_1 + 1)/2] \subseteq [m + \sqrt{\rho}n/2]$. Since $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, we have

$$|E(H_2^2([n] \setminus [m], [m])) \setminus E(G)| \leq 2\sqrt{\rho}n^2.$$

This completes the proof of the lemma. ■

8.2 Perfect Fractional Matchings

Now we show the main result of this chapter.

Lemma 8.3. *Let ρ, ε be constants with $0 < \varepsilon \ll 1$ and $0 < \rho < \varepsilon^{12}$, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Suppose $d_H(\{u, v\}) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for any $v \in Q$ and $u \in P$. If H contains no independent set S with $|S \cap Q| \geq n/3 - \varepsilon^2 n$ and $|S \cap P| \geq 2n/3 - \varepsilon^2 n$, then H contains a perfect fractional matching.*

Proof. Let $\omega : V(H) \rightarrow \mathbb{R}^+ \cup \{0\}$ be a minimum fractional vertex cover of H , i.e., $\sum_{x \in e} \omega(x) \geq 1$ for $e \in E(H)$ and, subject to this, $\sum_{x \in V(H)} \omega(x)$ is minimum. Let $P = \{u_1, \dots, u_n\}$ and $Q = \{v_1, \dots, v_{n/3}\}$, such that $\omega(v_1) \geq \dots \geq \omega(v_{n/3})$ and $\omega(u_1) \geq \dots \geq \omega(u_n)$. Let H' be the $(1, 3)$ -partite 4-graph with vertex set $V(H)$ and edge set $E(H') = E'$, where

$$E' = \left\{ e \in \binom{V(H)}{4} : |e \cap Q| = 1 \text{ and } \sum_{x \in e} \omega(x) \geq 1 \right\}.$$

We claim that ω is a minimum fractional vertex cover of H' . Since ω is fractional vertex cover of H , $e \in E(H)$ implies that $e \in E(H')$; so $E(H) \subseteq E(H')$ and ω is also

a fractional vertex cover of H' . Let ω' be a minimum fractional vertex cover of H' . Then $\omega(H) \geq \omega'(H')$, where $\omega(H) := \sum_{v \in V(H)} \omega(v)$ and $\omega'(H') := \sum_{v \in V(H')} \omega'(v)$. On the other hand, ω' is also a vertex cover of H ; so $\omega'(H') \geq \omega(H)$. Hence, $\omega(H) = \omega'(H')$, i.e., ω is a minimum fractional vertex cover of H' .

Let $\nu_f(H)$ and $\nu_f(H')$ denote the maximum fractional matching numbers of H and H' , respectively; then by the Strong Duality Theorem of linear programming, $\nu_f(H) = \omega(H)$ and $\nu_f(H') = \omega(H')$. Thus $\nu_f(H) = \nu_f(H')$ and, hence, it suffices to show that H' has a perfect matching.

Next, we observe that the edges of H' form a stable family with respect to the above ordering of vertices in P and Q : for any $e_1 = \{v_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}\}$ and $e_2 = \{v_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}\}$ with $i_l \geq j_l$ for $1 \leq l \leq 4$, $e_2 \in E(H')$ implies $e_1 \in E(H')$. To see this, note that, since $i_l \geq j_l$ for $1 \leq l \leq 4$, we have $\omega(v_{i_1}) \geq \omega(v_{j_1})$ and $\omega(u_{i_l}) \geq \omega(u_{j_l})$ for $2 \leq l \leq 4$. If $e_2 \in E(H')$ then $\sum_{x \in e_2} \omega(x) \geq 1$; so $\sum_{x \in e_1} \omega(x) \geq 1$ and, hence, $e_1 \in E(H')$.

Let G denote the graph with vertex set P and edge set formed by $N_{H'}(\{v_{n/3}, u_n\})$. Then G is stable with respect to u_1, \dots, u_n . Note that $e(G) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ (by assumption). Since the edges of H' form a stable family, $\{u, v\} \cup e \in E(H')$ for all $u \in P, v \in Q$, and $e \in E(G)$. Thus, if G contains a matching $M := \{e_1, \dots, e_{n/3}\}$ then let $x_1, \dots, x_{n/3} \in P \setminus V(M)$; we see that $\{\{v_i, x_i\} \cup e_i \in E(H') : i \in [n/3]\}$ is a perfect matching in H' .

Thus, we may assume $\nu(G) < n/3$. Recall that $e(G) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ and G is a stable 2-graph. Hence, by Lemma 8.2, G is $2\sqrt{\rho}$ -close to the graph with vertex $V(G)$ and edge set $\{e \in \binom{V(G)}{2} : e \cap \{u_i : i \in [n/3 - 1]\} \neq \emptyset\}$. Therefore, G has at most $\sqrt{2\sqrt{\rho}}n$ vertices in $\{u_j : j \in [n/3 - 1]\}$ of degree less than $n - 1 - \sqrt{2\sqrt{\rho}}n$. Since G is stable with respect to u_1, \dots, u_n , we have $d_G(u_{n/3 - \sqrt{2\sqrt{\rho}}n}) \geq n - 1 - \sqrt{2\sqrt{\rho}}n$.

Since $\rho < \varepsilon^{12}$ and H contains no independent set S such that $|S \cap Q| \geq n/3 - \varepsilon^2 n$ and $|S \cap P| \geq 2n/3 - \varepsilon^2 n$, we may form a matching M_0 of size $\sqrt{2\sqrt{\rho}}n$ in $H - \{u_1, \dots, u_{n/3}\}$ by greedily choosing edges.

Since $d_G(u_{n/3-\sqrt{2\sqrt{\rho}n}}) \geq n - 1 - \sqrt{2\sqrt{\rho}n}$, $G - V(M_0)$ has a matching M of size $n/3 - \sqrt{2\sqrt{\rho}n}$ which can be found by greedily choosing distinct neighbors of u_i , $1 \leq i \leq n/3 - \sqrt{2\sqrt{\rho}n}$, in $V(G) \setminus V(M_0)$. Since $\{u, v\} \cup e \in E(H')$ for $u \in P$, $v \in Q$, and $e \in M$, we may extend M to a matching M' of size $|M|$ in $H' - M_0$. Then $M' \cup M_0$ gives a perfect matching in H' . ■

CHAPTER 9

ALMOST PERFECT MATCHINGS

In this chapter we prove the existence of an almost perfect matching in the hypergraph obtained from the k -graph (or $(1, 3)$ -partite 4-graph, respectively) H after deleting the absorbing matching, in the cases when the H is not close to $H_k^{k-l}(U, W)$ (or $H_{1,3}(n, n/3)$, respectively) for any partition of $V(H)$ into U, W with $|W| = m$.

9.1 Almost Perfect Matchings in k -Graphs

To find such almost perfect matching, we will find an almost regular spanning subgraph of the hypergraph (obtained from H after deleting the absorbing matching) with bounded maximum 2-degree, so that the following result of Frankl and Rödl [13] can be applied.

Lemma 9.1 (Frankl and Rödl, 1985). *For every integer $k \geq 2$ and any real $\varepsilon > 0$, there exist $\tau = \tau(k, \varepsilon)$ and $d_0 = d_0(k, \varepsilon)$ such that, for every $n \geq D \geq d_0$ the following holds: Every k -graph on n vertices with $(1 - \tau)D < d_H(v) < (1 + \tau)D$ and $\Delta_2(H) < \tau D$ contains a matching covering all but at most εn vertices.*

In order to find a subgraph in a k -graph satisfying conditions in Lemma 9.1, we use the two-round randomization technique in [3]. Note that the only difference in the first round is that we also need to bound the independence number of the subgraph, as discussed in Chapter 6.

The following result is the outcome of the first round of the two-round randomization procedure in [3]. We summarize this round as a lemma (see the proof of Claim 4.1 in [3]) and outline a proof, since we need to make some small adjustments. Here we adopt the notation in [3].

Lemma 9.2. *Let $k > d > 0$ be integers with $k \geq 3$ and let H be a k -graph on n vertices. Let R be chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then arbitrarily deleting less than k vertices so that $|R| \in k\mathbb{Z}$. Take $n^{1.1}$ independent copies of R and denote them by R^i , $1 \leq i \leq n^{1.1}$. For each $S \subseteq V(H)$ with $|S| \leq k$, let $Y_S := |\{i : S \subseteq R^i\}|$ and $\text{DEG}_S^i := |N_H(S) \cap \binom{R^i}{k-|S|}|$. Then with probability at least $1 - o(1)$, all of the following statements hold:*

- (i) *for every $v \in V(H)$, $Y_{\{v\}} = (1 + o(1))n^{0.2}$*
- (ii) *$Y_{\{u,v\}} \leq 2$ for every pair $\{u, v\} \subseteq V(H)$,*
- (iii) *$Y_e \leq 1$ for every edge $e \in E(H)$,*
- (iv) *for all $i = 1, \dots, n^{1.1}$, we have $|R^i| = (1 + o(1))n^{0.1}$, and*
- (v) *if μ, ρ' are constants with $0 < \mu \ll \rho'$, $n/k - \mu n \leq m \leq n/k$, and $\delta_d(H) \geq \binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-l}$, then for all $i = 1, \dots, n^{1.1}$ and all $D \in \binom{V(H)}{d}$ and for any positive real $\xi \geq 2\rho'$, we have*

$$\text{DEG}_D^i > \binom{|R^i| - d}{k - d} - \binom{|R^i| - d - |R^i|/k}{k - d} - \xi |R^i|^{k-d}.$$

Proof. Note that the removal of less than k vertices from each R^i does not affect (i) – (iv). Also note that $|Y_S| \sim \text{Bi}(n^{1.1}, n^{-0.9|S|})$ for $S \subseteq V(H)$.

Thus, $\mathbb{E}(|Y_{\{v\}}|) = n^{0.2}$ for $v \in V(H)$, and it follows from Lemma 5.1 that

$$\mathbb{P}(|Y_{\{v\}} - n^{0.2}| > n^{0.15}) \leq e^{-\Omega(n^{0.1})}$$

Hence (i) holds with probability at least $1 - e^{-\Omega(n^{0.1})}$.

To prove (ii), let

$$Z_2 = \left| \left\{ \{u, v\} \in \binom{V(H)}{2} : Y_{\{u,v\}} \geq 3 \right\} \right|,$$

and for $k \geq 3$, let

$$Z_k = \left| \left\{ S \in \binom{V(H)}{k} : Y_S \geq 2 \right\} \right|.$$

Then $\mathbb{E}(Z_2) < n^2(n^{1.1})^3(n^{-0.9})^6 = n^{-0.1}$ and $\mathbb{E}(Z_k) < n^k(n^{1.1})^2(n^{-0.9})^{2k} = n^{2.2-0.8k} \leq n^{-0.2}$ (for $k \geq 3$). By Markov's inequality,

$$\mathbb{P}(Z_2 = 0) > 1 - n^{-0.1} \text{ and, for } k \geq 3, \mathbb{P}(Z_k = 0) > 1 - n^{-0.2}.$$

Thus (ii) and (iii) hold with probability at least $1 - n^{-0.1}$ and $1 - n^{-0.2}$, respectively.

By Lemma 5.1 (with $\lambda = n^{0.095}$), we have

$$\mathbb{P}(|R^i| - n^{0.1} \geq n^{0.095}) \leq e^{-\Omega(n^{0.09})}$$

for each i . Thus by union bound, (iv) holds with probability at least $1 - n^{1.1}e^{-\Omega(n^{0.09})}$.

Next, we prove (v). Conditioning on $||R^i| - n^{0.1}| < n^{0.095}$ for all i and using the assumption that, $0 < \mu \ll \rho'$, $n/k - \mu n \leq m \leq n/k$, and n is large, we have

$$\begin{aligned} & \left(\binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-d} \right) (n^{-0.9})^{k-d} \\ & \geq \binom{|R^i| - d}{k-d} - \binom{|R^i| - d - |R^i|/k}{k-d} - 1.5\rho'|R^i|^{k-d}. \end{aligned}$$

So for each $D \in \binom{V(H)}{d}$ and each fixed R^i ,

$$\begin{aligned} \mathbb{E}(\text{DEG}_D^i) &= (1 - o(1))d_H(D)(n^{-0.9})^{k-d} \\ &\geq (1 - o(1)) \left(\binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-d} \right) (n^{-0.9})^{k-d} \\ &\geq (1 - o(1)) \left(\binom{|R^i| - d}{k-d} - \binom{|R^i| - d - |R^i|/k}{k-d} - 1.5\rho'|R^i|^{k-d} \right) \\ &\geq \binom{|R^i| - d}{k-d} - \binom{|R^i| - d - |R^i|/k}{k-d} - 1.8\rho'|R^i|^{k-d}. \end{aligned}$$

In particular,

$$\mathbb{E}(\text{DEG}_D^i) = \Omega(n^{0.1(k-d)}).$$

We apply Janson's Inequality (Theorem 8.7.2 in [5]) to bound the deviation of DEG_D^i .

Write $\text{DEG}_D^i = \sum_{e \in N_H(D)} X_e$, where $X_e = 1$ if $e \subseteq R^i$ and $X_e = 0$ otherwise. Then

$$\Delta = \sum_{e \cap f \neq \emptyset} \mathbb{P}(X_e = X_f = 1) \leq \sum_{l=1}^{k-d-1} p^{2(k-d)-l} \binom{n-d}{k-d} \binom{k-d}{l} \binom{n-k}{k-d-l}$$

and, thus, $\Delta = O(n^{0.1(2(k-d)-1)})$. By Janson's inequality, for any $\gamma > 0$,

$$\mathbb{P}(\text{DEG}_D^i \leq (1 - \gamma)\mathbb{E}(\text{DEG}_D^i)) \leq e^{-\gamma^2 \mathbb{E}(\text{DEG}_D^i)/(2 + \Delta/\mathbb{E}(\text{DEG}_D^i))} = e^{-\Omega(n^{0.1})}.$$

Since $\xi \geq 2\rho'$, by taking γ small, the union bound shows that, with probability at least $1 - n^{d+1.1}e^{-\Omega(n^{0.1})}$,

$$\text{DEG}_D^i \geq \binom{|R^i| - d}{k - d} - \binom{|R^i| - d - |R^i|/k}{k - d} - \xi |R^i|^{k-d}.$$

Thus, (v) holds with probability at least

$$(1 - n^{1.1}e^{-\Omega(n^{0.09})})(1 - n^{d+1.1}e^{-\Omega(n^{0.1})}) > 1 - n^{1.1}e^{-\Omega(n^{0.09})} - n^{d+1.1}e^{-\Omega(n^{0.1})}.$$

Hence, it follows from union bound that, with probability at least

$$1 - e^{-\Omega(n^{0.1})} - n^{-0.1} - n^{-0.2} - n^{1.1}e^{-\Omega(n^{0.09})} - n^{1.1}e^{-\Omega(n^{0.09})} - n^{d+1.1}e^{-\Omega(n^{0.1})} = 1 - o(1),$$

(i)-(v) hold. ■

We summarize the second round randomization in [3] as the following lemma (again, see the proof of Claim 4.1 in [3]).

Lemma 9.3. Assume R^i , $i = 1, \dots, n^{1.1}$, satisfy (i)-(v) in Lemma 9.2, and that each R^i has a perfect fractional matching w^i . Then there exists a spanning subgraph H'' of H such that $d_{H''}(v) = (1 + o(1))n^{0.2}$ for each $v \in V$, and $\Delta_2(H'') \leq n^{0.1}$.

We are now ready to show that for any H satisfying the conditions of Theorem 1.3 and not ε -close to $H_k^{k-l}(U, W)$, $H - V(M_a)$ has an almost perfect matching, where M_a is an absorbing matching from Lemma 5.3.

Lemma 9.4. Let k, l be integers with $k \geq 3$ and $k/2 \leq l \leq k - 1$. Let $\rho', \varepsilon, \sigma, \mu$ be positive reals with $\rho' < \varepsilon^2(3k)^{-4(k-l)}/100$ and $\mu \leq \varepsilon/40$. Let n, m be sufficiently large integers such that $n/k - \mu n \leq m \leq n/k$. Suppose H is a k -graph on n vertices such that $\delta_l(H) \geq \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$, and H is not ε -close to $H_k^{k-l}(U, W)$ for any partition of $V(H)$ into U, W with $|W| = m$. Then H contains a matching covering all but at most σn vertices.

Proof. By Lemma 6.7, $e(H[S]) \geq (\varepsilon/(2k!))e(H)$ for all $S \subseteq V(H)$ with $|S| \geq \alpha n$, where $\alpha = 1 - 1/k - \varepsilon/4$. Note that

$$e(H) = \delta_0(H) \geq \binom{n}{l} \delta_l(H) / \binom{k}{l} \geq cn^k,$$

where $c > 0$ is a constant and $c \ll 1/\binom{k}{l}$.

Let R, R^i be given as in Lemma 9.2. Then it follows from Lemma 6.8 that, with probability $1 - o(1)$, we have $\alpha(H[R^i]) \leq (\alpha + \gamma + o(1))n^{0.1}$ for all i , where $\gamma \ll \alpha$. Additionally, by (v) of Lemma 9.2, $\delta_d(H[R^i]) > \binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - \xi |R^i|^{(k-d)}$ for any $\xi \geq 2\rho'$. Thus by Lemma 7.4, with probability $1 - o(1)$, for each i , $H[R^i]$ has a perfect fractional matching.

Hence by Lemma 9.3, H has a spanning subgraph H'' such that $d_{H''}(v) = (1 + o(1))n^{0.2}$ for each $v \in V$, and $\Delta_2(H'') \leq n^{0.1}$. Thus we may apply Lemma 9.1 to find a matching covering all but at most σn vertices in H'' , for sufficiently large n . ■

9.2 Balancing in $(1, 3)$ -Partite 4-Graphs

Our idea to find an almost perfect matching in a $(1, 3)$ -partite 4-graph H not close to any isomorphic copy of $H_{1,3}(n, n/3)$ is quite similar to the idea in the previous section. That is, we need to find a sequence of random subgraphs in the remaining $(1, 3)$ -partite 4-graph after deleting the absorber and use them to find a spanning subgraph on which a “Rödl nibble” type result can be applied.

Since we are in fact looking for perfect matchings in some supergraphs of balanced $(1, 3)$ -partite 4-graphs in Lemma 8.3, we need to make sure each random subgraph taken is also balanced. So we slightly modify the randomization process in Lemma 9.2. We first fix an arbitrary small set $S \subseteq V(H)$. Each time we obtain a random copy R , we delete some vertices in $R \cap S$ so that the resulting graph is balanced. We can do so in a way that, with high probability, all properties in Lemma 9.2 remain (approximately) true. More specifically, we need the following two lemmas.

Lemma 9.5. *Let n be a sufficiently large positive integer, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Let $S \subseteq V(H)$ be a set of vertices such that $|S \cap Q| = n^{0.99}/3$ and $|S \cap P| = n^{0.99}$. Take $n^{1.1}$ independent copies of R_+ and denote them by R_+^i , $1 \leq i \leq n^{1.1}$, where R_+ is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$. Define $R_-^i = R_+^i \setminus S$ for $1 \leq i \leq n^{1.1}$.*

Then, with probability $1 - o(1)$, for any sequence R^i , $1 \leq i \leq n^{1.1}$, satisfying $R_-^i \subseteq R^i \subseteq R_+^i$, all of the following hold:

- (i) $|R^i| = (4/3 + o(1))n^{0.1}$ for all $i = 1, \dots, n^{1.1}$.
- (ii) For each $X \subseteq V(H)$, let $Y_X := |\{i : X \subseteq R^i\}|$, then,
 - (iia) $Y_{\{v\}} \leq (1 + o(1))n^{0.2}$ for $v \in V(H)$,
 - (iib) $Y_{\{v\}} = (1 + o(1))n^{0.2}$ for $v \in V(H) \setminus S$,
 - (iic) $Y_{\{u,v\}} \leq 2$ for distinct $u, v \in V(H)$, and
 - (iid) $Y_e \leq 1$ for $e \in E(H)$.

(iii) For each $X \in \binom{V(H)}{2}$, let $\text{DEG}_X^i = |N_H(X) \cap \binom{R^i}{2}|$. If $\rho > 0$ is a constant and $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for all $v \in Q$ and $u \in P$, then for any constant $\xi \geq 5\rho$, we have

$$\text{DEG}_{\{u,v\}}^i > \binom{|R^i \cap P| - 1}{2} - \binom{2|R^i \cap P|/3}{2} - \xi |R^i \cap P|^2.$$

for all $i = 1, \dots, n^{1.1}$, $v \in Q$, and $u \in P$.

Proof. Note that $\mathbb{E}(|R_+^i|) = (4n/3) \cdot n^{-0.9} = 4n^{0.1}/3$, and

$$\mathbb{E}(|R_-^i|) = (4n/3 - 4n^{0.99}/3) \cdot n^{-0.9} = 4n^{0.1}/3 - 4n^{0.09}/3.$$

By Lemma 5.1,

$$\mathbb{P}(|R_+^i| - 4n^{0.1}/3 \geq n^{0.095}) \leq e^{-\Omega(n^{0.09})}$$

and

$$\mathbb{P}(|R_-^i| - (4n^{0.1}/3 - 4n^{0.09}/3) \leq -n^{0.095}) \leq e^{-\Omega(n^{0.09})}.$$

In particular, (i) holds with probability at least $1 - e^{-\Omega(n^{0.09})}$.

Let $Y_X^+ := |\{i : X \subseteq R_+^i\}|$ for $X \subseteq V(H)$. Then $Y_X^+ \sim \text{Bi}(n^{1.1}, n^{-0.9|X|})$ and $Y_X \leq Y_X^+$ for all $X \subseteq V(H)$, and $Y_X = Y_X^+$ for all $X \subseteq V(H) \setminus S$. Then by Lemma 9.2, (iic) and (iid) hold with probability $1 - o(1)$.

For each $v \in V(H)$, $\mathbb{E}(Y_{\{v\}}^+) = n^{0.2}$, thus by Lemma 5.1,

$$\mathbb{P}\left(\left|Y_{\{v\}}^+ - n^{0.2}\right| \geq n^{0.15}\right) \leq e^{-\Omega(n^{0.1})}.$$

Thus (iia) and (iib) hold with probability at least $1 - e^{-\Omega(n^{0.1})}$.

Let $\deg_X^i = \left|N_H(X) \cap \binom{R^i}{2}\right|$. To prove (iii), since n is sufficiently large, it suffices to

show that for all $v \in Q$ and $u \in P$,

$$\deg_{\{u,v\}}^i > \binom{n^{0.1} - 1}{2} - \binom{2n^{0.1}/3}{2} - \xi n^{0.2}/2.$$

Conditioning on $|R_+^i| < 4n^{0.1}/3 - n^{0.095}$ and $|R_-^i| > (4n^{0.1}/3 - 4n^{0.01}/3) - n^{0.095}$ for all i , we have, for all $v \in Q$ and $u \in P$,

$$\begin{aligned} \mathbb{E}(\deg_{\{u,v\}}^i) &= d_{H-S}(\{u,v\})(n^{-0.9})^2 \\ &\geq (1 - o(1)) \left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2 \right) (n^{-0.9})^2 \\ &\geq \binom{n^{0.1} - 1}{2} - \binom{2n^{0.1}/3}{2} - 2\rho n^{0.2}, \end{aligned}$$

where the first inequality holds because $|S| = 4n^{0.99}/3$ (and, hence, $d_{H-S}(\{u,v\}) = (1 - o(1))d_H(\{u,v\})$). In particular, $\mathbb{E}(\deg_{\{u,v\}}^i) = \Omega(n^{0.2})$. Next, we apply Janson's Inequality (Theorem 8.7.2 in [5]) to bound the deviation of $\deg_{\{u,v\}}^i$. Write $\deg_{\{u,v\}}^i = \sum_{e \in N_H(\{u,v\})} X_e$, where $X_e = 1$ if $e \subseteq R_-^i$ and $X_e = 0$ otherwise. Then

$$\Delta := \sum_{e \cap f \neq \emptyset} \mathbb{P}(X_e = X_f = 1) \leq \binom{n-1}{2} \binom{2}{1} \binom{n-3}{1} (n^{-0.9})^3$$

and, thus, $\Delta = O(n^{0.3})$. By Janson's inequality, for any constant $\gamma > 0$,

$$\mathbb{P}(\deg_{\{u,v\}}^i \leq (1 - \gamma)\mathbb{E}(\deg_{\{u,v\}}^i)) \leq e^{-\gamma^2 \mathbb{E}(\deg_{\{u,v\}}^i)/(2 + \Delta/\mathbb{E}(\deg_{\{u,v\}}^i))} = e^{-\Omega(n^{0.1})}.$$

Since $\xi \geq 5\rho$ (and taking γ sufficiently small), the union bound implies that, with probability at least $1 - n^{2+1.1}e^{-\Omega(n^{0.1})}$, for all $v \in Q$ and $u \in P$ and for all $i \in [n^{1.1}]$,

$$\deg_{\{u,v\}}^i > \binom{n^{0.1} - 1}{2} - \binom{2n^{0.1}/3}{2} - \xi n^{0.2}/2.$$

Thus, (iii) holds with probability at least

$$(1 - n^{1.1}e^{-\Omega(n^{0.09})})(1 - n^{2+1.1}e^{-\Omega(n^{0.1})}) > 1 - n^4e^{-\Omega(n^{0.09})}.$$

Hence, it follows from union bound that, with probability at least $1 - o(1)$, (i)-(iii) hold for any sequence R^i , $1 \leq i \leq n^{1.1}$, satisfying $R_-^i \subseteq R^i \subseteq R_+^i$. \blacksquare

Lemma 9.6. *Let n, H, P, Q, S and R_+^i, R_-^i , $i \in [n^{1.1}]$, be given as in Lemma 9.5. Then, with probability $1 - o(1)$, for every $i \in [n^{1.1}]$, there exist subgraphs R_i such that $R_-^i \subseteq R^i \subseteq R_+^i$ and R^i is balanced.*

Proof. Recall that $|P| = n$, $|Q| = n/3$, $|S \cap P| = n^{0.99}$, and $|S \cap Q| = n^{0.99}/3$, and that R_+^i is formed by taking each vertex of H independently and uniformly at random with probability $n^{-0.9}$. So for $i \in [n^{1.1}]$,

$$\begin{aligned}\mathbb{E}(|R_+^i \cap P|) &= n^{0.1}, \\ \mathbb{E}(|R_+^i \cap P \cap S|) &= n^{0.09}, \\ \mathbb{E}(|R_+^i \cap Q|) &= n^{0.1}/3, \quad \text{and} \\ \mathbb{E}(|R_+^i \cap P \cap S|) &= n^{0.09}/3.\end{aligned}$$

By Lemma 5.1,

$$\begin{aligned}\mathbb{P}(|R_+^i \cap P| - n^{0.1}| \geq n^{0.08}) &\leq e^{-\Omega(n^{0.06})}, \\ \mathbb{P}(|R_+^i \cap P \cap S| - n^{0.09}| \geq n^{0.08}) &\leq e^{-\Omega(n^{0.07})}, \\ \mathbb{P}(|R_+^i \cap Q| - n^{0.1}/3| \geq n^{0.08}) &\leq e^{-\Omega(n^{0.06})}, \quad \text{and} \\ \mathbb{P}(|R_+^i \cap P \cap S| - n^{0.09}/3| \geq n^{0.08}) &\leq e^{-\Omega(n^{0.07})}.\end{aligned}$$

Thus, with probability $1 - o(1)$, for all $i \in [n^{1.1}]$,

$$\begin{aligned} |R_+^i \cap P| &\in [n^{0.1} - n^{0.08}, n^{0.1} + n^{0.08}], \\ |R_+^i \cap P \cap S| &= (1 + o(1))n^{0.09}, \\ |R_+^i \cap Q| &\in [n^{0.1}/3 - n^{0.08}, n^{0.1}/3 + n^{0.08}], \quad \text{and} \\ |R_+^i \cap Q \cap S| &= (1 + o(1))n^{0.09}. \end{aligned}$$

Therefore,

$$||R_+^i \cap P| - 3|R_+^i \cap Q|| \leq 4n^{0.08} < \min\{|R_+^i \cap P \cap S|, |R_+^i \cap Q \cap S|\}.$$

Hence, with probability $1 - o(1)$, R^i can be taken to be balanced for all $i \in [n^{1.1}]$. ■

9.3 Second Round of Sampling in $(1, 3)$ -Partite 4-Graphs

Another small difference between here and [3] is that condition (ii) in Lemma 9.5 is slightly weaker than the corresponding condition in [3]. In [3] all vertices have almost the same degree, but here a small portion of the vertices could have smaller degree. The following lemma reflects a slightly weaker conclusion due to this difference, and the proof mainly follows that of Claim 4.1 in [3].

Lemma 9.7. *Let $n, H, S, R^i, i = 1, \dots, n^{1.1}$ be given as in Lemma 9.6 such that each $H[R^i]$ is a balanced $(1, 3)$ -partite 4-graph and has a perfect fractional matching w^i . Then there exists a spanning subgraph H'' of $H' := \bigcup_{i=1}^{n^{1.1}} H[R^i]$ such that*

- (i) $d_{H''}(u) \leq (1 + o(1))n^{0.2}$ for $u \in S$,
- (ii) $d_{H''}(v) = (1 + o(1))n^{0.2}$ for $v \in V(H) \setminus S$, and
- (iii) $\Delta_2(H'') \leq n^{0.1}$.

Proof. Let $H' = \bigcup_{i=1}^{n^{1.1}} H[R^i]$. By (iid) of Lemma 9.5, each edge of H is contained in at most one R^i . Let i_e denote the index i such that $e \subseteq R^i$ (if exists); and let $w^{i_e}(e) = 0$ when i_e is not defined. Let H'' be a spanning subgraph of H' obtained by independently selecting each edge e at random with probability $w^{i_e}(e)$.

For $v \in V(H'')$, let $I_v = \{i : v \in R^i\}$, $E_v = \{e \in H' : v \in e\}$, and $E_v^i = E_v \cap H[R^i]$. Then E_v^i , $i \in I_v$, form a partition of E_v . Hence, for $v \in V(H'')$,

$$d_{H''}(v) = \sum_{e \in E_v} 1 = \sum_{i \in I_v} \sum_{e \in E_v^i} X_e,$$

where $X_e \sim \text{Be}(w^{i_e}(e))$ is the Bernoulli random variable with $X_e = 1$ if $e \in E(H'')$ and $X_e = 0$ otherwise. Thus, since $\sum_{e \in E_v^i} w^i(e) = 1$ (as w^i is a perfect fractional matching in $H[R^i]$),

$$\mathbb{E}(d_{H''}(v)) = \sum_{i \in I_v} \sum_{e \in E_v^i} w^i(e) = \sum_{i \in I_v} 1.$$

Hence, $\mathbb{E}(d_{H''}(v)) = (1 + o(1))n^{0.2}$ for $v \in V(H) \setminus S$ (by (iib) of Lemma 9.5), and $\mathbb{E}(d_{H''}(v)) \leq (1 + o(1))n^{0.2}$ for $v \in S$ (by (iia) of Lemma 9.5). Now by Lemma 5.1, for $v \in V(H) \setminus S$,

$$\mathbb{P}(|d_{H''}(v) - n^{0.2}| \geq n^{0.15}) \leq e^{-\Omega(n^{0.1})},$$

and for $v \in S$,

$$\mathbb{P}(d_{H''}(v) - n^{0.2} \geq n^{0.15}) \leq e^{-\Omega(n^{0.1})}.$$

Thus by taking union bound over all $v \in V(H)$, we have that, with probability $1 - o(1)$, $d_{H''}(v) = (1 + o(1))n^{0.2}$ for all $v \in V(H) \setminus S$ and $d_{H''}(v) \leq (1 + o(1))n^{0.2}$ for all $v \in S$.

Next, note that for distinct $u, v \in V(H)$,

$$d_{H''}(\{u, v\}) = \sum_{e \in E_u \cap E_v \cap E(H'')} 1 = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} X_e$$

and $\mathbb{E}(d_{H''}(\{u, v\})) = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} w^i(e)$. By (iic) in Lemma 9.5, $|I_u \cap I_v| \leq 2$.

So $\mathbb{E}(d_{H''}(\{u, v\})) \leq |I_u \cap I_v| \leq 2$. Thus by Lemma 5.1,

$$\mathbb{P}(d_{H''}(\{u, v\}) \geq n^{0.1}) \leq e^{-\Omega(n^{0.2})}.$$

Hence by a union bound $\Delta_2(H'') \leq n^{0.1}$ with probability $1 - o(1)$.

Therefore, with probability $1 - o(1)$, H'' satisfies (i), (ii), and (iii). ■

We also need the following result attributed to Pippenger [38], stated as Theorem 4.7.1 in [5]. A *cover* in a hypergraph H is a set of edges whose union is $V(H)$.

Lemma 9.8 (Pippenger and Spencer, 1989). *For every integer $k \geq 2$ and reals $r \geq 1$ and $a > 0$, there are $\gamma = \gamma(k, r, a) > 0$ and $d_0 = d_0(k, r, a)$ such that for every n and $D \geq d_0$ the following holds: Every k -uniform hypergraph $H = (V, E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:*

- (1) *For all vertices $x \in V$ but at most γn of them, $d(x) = (1 \pm \gamma)D$;*
- (2) *For all $x \in V$, $d(x) < rD$;*
- (3) *For any two distinct $x, y \in V$, $d(x, y) < \gamma D$;*

contains a cover of at most $(1 + a)(n/k)$ edges.

Note that H contains a cover of at most $(1 + a)(n/k)$ edges implies that H contains a matching of size at least $(1 - (k - 1)a)(n/k)$ (see, for example, [38]). Now we are ready to state and prove the main result of this section, which will be used to find an almost perfect matching after deleting an absorber.

Lemma 9.9. *Let $\sigma > 0$ and $0 < \rho \leq \varepsilon/4 \ll 1$, let n be a sufficiently large positive integer, and let H be a $(1, 3)$ -partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Suppose H is not ε -close to any $H_{1,3}(n, n/3)$ with $V(H_{1,3}(n, n/3))$ and $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for all $v \in Q$ and $u \in P$. Then H contains a matching covering all but at most σn vertices.*

Proof. By Lemmas 9.5 and 9.6, we have the random subgraphs R^i , $i \in [n^{1.1}]$, such that, with probability $1 - o(1)$, all R^i satisfies the properties in Lemmas 9.5 and 9.6. In particular, $H[R^i]$ is balanced with respect to the partition classes Q, P .

Next, by Lemma 6.5, H is $(\mathcal{F}, \varepsilon/6)$ -dense, where

$$\mathcal{F} = \{A \subseteq V(H) : |A \cap Q| \geq (1/3 - \varepsilon/8)n \text{ and } |A \cap P| \geq (2/3 - \varepsilon/8)n\}.$$

Note that

$$e(H) = \sum_{v \in Q} \sum_{u \in P} d_H(\{u, v\})/3 \geq (n/3)(n/3) \left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2 \right) \geq n^4/100.$$

Hence by Lemma 6.6 (and choosing suitable $\alpha_1, \alpha_2, \gamma$), we see that, with probability $1 - o(1)$, for all $i \in [n^{1.1}]$ and for all independent sets J in $H[R^i]$, $|J \cap P| \leq (\alpha_1 + \gamma + o(1))n^{0.1} < n/3 - \varepsilon^2 n$ or $|J \cap Q| \leq (\alpha_2 + \gamma + o(1))n^{0.1} < 2n/3 - \varepsilon^2 n$

Moreover, by (iii) of Lemma 9.5, with probability $1 - o(1)$, $d_{H[R^i]}(\{u, v\}) > \binom{|R^i \cap P| - 1}{2} - \binom{2|R^i \cap P|/3}{2} - \xi |R^i \cap P|^2$ for all $u \in P$ and $v \in Q$. Hence, by Lemma 8.3, $H[R^i]$ contains a perfect fractional matching for all $i \in [n^{1.1}]$.

Thus by Lemma 9.7, there exists a spanning subgraph H'' of $\bigcup_{i=1}^{n^{1.1}} H[R^i]$ such that $d_{H''}(u) \leq (1 + o(1))n^{0.2}$ for each $u \in S$, $d_{H''}(v) = (1 + o(1))n^{0.2}$ for each $v \in V(H) \setminus S$, and $\Delta_2(H'') \leq n^{0.1}$. Hence, by Lemma 9.8 (by setting $D = n^{0.2}$), H'' contains a cover of at most $(1 + a)(n/3)$ edges, where a is a constant satisfying $0 < a < \sigma/3$.

Now by greedily deleting intersecting edges, we obtain a matching of size at least $(1 - 3a)(n/3)$. Hence H contains a matching covering all but at most σn , provided n is sufficiently large. ■

CHAPTER 10

CONCLUSIONS AND REMARKS

In this chapter, we complete the proof of Theorem 1.3 and Theorem 2.8, and make some concluding remarks.

10.1 Proof of Theorem 1.3

Proof. By Lemmas 3.2 and 4.6, we may assume that for any $0 < \varepsilon < (8^{k-1}k^{5(k-1)}k!)^{-3}$, H is not ε -close to $H_k^{k-l}(U, W)$ for any partition of $V(H)$ into U, W with $|W| = m$.

By Lemma 5.3, there exist constants $c' = c'(k, l)$ and $\rho = \rho(c', k, l, \varepsilon)$ small enough, satisfying the following property: For positive integers a, h satisfying $h \leq l$, $a \leq k - l$, and $al \geq a(k - l) + (k - h)$, there exists a matching M_a such that $|M_a| \leq 2k\rho n$ and, for any subset $S \subseteq V(H)$ with $|S| \leq c'\rho n$, $H[V(M_a) \cup S]$ has a matching covering all but at most $al + h - 1$ vertices.

Now consider $H_1 = H - V(M_a)$. Then $\delta_l(H_1) \geq \delta_l(H) - (2k^2\rho n)n^{k-l-1}$. Let $\rho_1 = 4k^2\rho$ and $n_1 = n - k|M_a|$. Then, since n is large enough and $\rho \ll \varepsilon$,

$$\delta_l(H_1) \geq \binom{n_1 - l}{k - l} - \binom{n_1 - l - m}{k - l} - \rho_1 n_1^{k-l}$$

and H_1 is not $(\varepsilon/2)$ -close to $H_k^{k-l}(U, W)$ for any partition of $V(H_1)$ into U, W with $|W| = m$.

By Lemma 9.4, H_1 has a matching M_1 such that $|V(H_1) \setminus V(M_1)| < c'\rho n_1 \leq c'\rho n$. Then there exists a matching M_2 in $H_2 := H[V(M_a) \cup (V(H_1) \setminus V(M_1))]$ such that $|V(H_2) \setminus V(M_2)| \leq al + h - 1$.

Now $M_1 \cup M_2$ is a matching in H covering all but at most $al + h - 1$ vertices of H . By taking $a = \lceil (k - l)/(2l - k) \rceil$ and $h = k - a(2l - k)$, which minimizes $al + h - 1$, we see

that $M_1 \cup M_2$ is a matching in H of size $n/k - 1 - (1 - l/k)\lceil(k - l)/(2l - k)\rceil$. ■

10.2 Proof of Theorem 2.8

Proof. By Lemma 4.5, we may assume H is not ε -close to any $H_{1,3}(n, n/3)$, where $\varepsilon \ll$

1. By Lemma 5.2, $H_{1,3}(\mathcal{F})$ has a matching M' such that, for some $0 < \rho' \ll \rho \ll \varepsilon$, $|M'| \leq \rho n/4$ and, for any $S \subseteq V(H_{1,3}(\mathcal{F}))$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap P|$, $H_{1,3}(\mathcal{F})[S \cup V(M')]$ has a perfect matching.

Let $H_1 = H - V(M')$. Then $d_{H_1}(\{u, v\}) \geq \binom{n'-1}{2} - \binom{2n'/3}{2} - \rho(n')^2$ for all $v \in Q \cap V(H_1)$ and $u \in P \cap V(H_1)$, and H_1 is not (2ε) -close to $H_{1,3}(n', n'/3)$, where $n' = (1 - o(1))n$.

By Lemma 9.9, H_1 contains a matching M_1 covering all but at most σn vertices, where we choose σ so that $0 < \sigma < \rho'$. Now $H[(V(H_1) \setminus V(M_1)) \cup V(M)]$ has a perfect matching M_2 . Clearly, $M_1 \cup M_2$ forms a perfect matching in H . ■

10.3 Remarks

There are two places in the proof of Theorem 1.3 where we require $l > k/2$: Lemma 5.3 for absorbing matching and Lemma 7.4 for perfect fractional matchings. We do not know how to derive such results for $l \leq k/2$.

However, for $k = 3$ and $l = 1$, the absorbing part can be taken care of by the following result of Hàn, Person, and Schacht [22].

Lemma 10.1 (Hàn, Person, and Schacht, 2009). *Given any $\gamma > 0$, there exists an integer $n_0 = n_0(\gamma)$ such that the following holds. Suppose that H is a 3-graph on $n \geq n_0$ vertices such that $\delta_1(H) \geq (1/2 + 2\gamma)\binom{n}{2}$. Then there is a matching M in H of size $|M| \leq \gamma^3 n/3$ such that for every subset $V' \subseteq V(H) \setminus V(M)$ with $|V'| \in 3\mathbb{Z}$ and $|V'| \leq \gamma^6 n$, there is a matching in H covering precisely the vertices in $V' \cup V(M)$.*

Using Lemma 8.2 instead of Lemma 7.3 in the end of the proof of (3) for Lemma 7.4,

we see that Lemma 7.4 holds in the case when $k = 3$ and $l = 1$. Thus, our approach (using Lemma 10.1 instead of Lemma 5.3) gives an alternative proof of the following result of Kühn, Osthus, and Townsend [31] (and independently by Khan[28]) on perfect matchings in 3-graphs.

Theorem 10.2 (Kühn, Osthus, and Townsend, 2014; Khan, 2013). *There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geq n_0$, $m \leq n/3$, and $\delta_1(H) > \binom{n-1}{2} - \binom{n-m}{2}$, then $\nu(H) \geq m$.*

For the general case, Hàn, Person, and Schacht [22] and, independently, Kühn, Osthus, and Townsend [30] conjectured that the asymptotic l -degree threshold for a perfect matching in a k -graph with n vertices is

$$\left(\max \left\{ \frac{1}{2}, 1 - \left(1 - \frac{1}{k} \right)^{k-l} \right\} + o(1) \right) \binom{n-l}{k-l}.$$

The first term $(1/2 + o(1)) \binom{n-l}{k-l}$ comes from a parity construction: Take disjoint nonempty sets A and B with $||A| - |B|| \leq 2$, form a hypergraph H by taking all k -subsets f of $A \cup B$ with $|f \cap A| \not\equiv |A| \pmod{2}$. The second term is given by the hypergraph obtained from K_n^k (the complete k -graph on n vertices) by deleting all edges from a subgraph $K_{n-n/k+1}^k$.

REFERENCES

- [1] R. Aharoni and D. Howard, “Size conditions for the existence of rainbow matchings,” preprint, <http://math.colgate.edu/~dmhoward/rsc.pdf>.
- [2] J. Akiyama and P. Frankl, “On the size of graphs with complete factors,” *J. Graph Theory*, vol. 9, no. 1, pp. 197–201, 1985.
- [3] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov, “Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels,” *J. Combin. Theory Ser. A*, vol. 119, no. 6, pp. 1200–1215, 2012.
- [4] N. Alon, H. Huang, and B. Sudakov, “Nonnegative k -sums, fractional covers, and probability of small deviations,” *J. Combin. Theory Ser. B*, vol. 102, no. 3, pp. 784–796, 2012.
- [5] N. Alon and J. H. Spencer, *The probabilistic method*. John Wiley & Sons, 2016.
- [6] J. Balogh, R. Morris, and W. Samotij, “Independent sets in hypergraphs,” *J. Amer. Math. Soc.*, vol. 28, no. 3, pp. 669–709, 2015.
- [7] C. Berge, “Sur le couplage maximum d’un graphe,” *C. R. Acad. Sci. Paris*, vol. 247, pp. 258–259, 1958.
- [8] B. Bollobás, D. E. Daykin, and P. Erdős, “Sets of independent edges of a hypergraph,” *Quart. J. Math. Oxford Ser. (2)*, vol. 27, no. 105, pp. 25–32, 1976.
- [9] J. Edmonds, “Paths, trees, and flowers,” *Canadian Journal of Mathematics*, vol. 17, pp. 449–467, 1965.
- [10] D. Ellis, N. Keller, and N. Lifshitz, “Stability versions of Erdős-Ko-Rado type theorems via isoperimetry,” *J. Eur. Math. Soc. (JEMS)*, vol. 21, no. 12, pp. 3857–3902, 2019.
- [11] P. Erdős, “A problem on independent r -tuples,” *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, vol. 8, pp. 93–95, 1965.
- [12] L. R. Ford and D. R. Fulkerson, “Maximal flow through a network,” *Canadian Journal of Mathematics*, vol. 8, pp. 399–404, 1956.
- [13] P. Frankl and V. Rödl, “Near perfect coverings in graphs and hypergraphs,” *European J. Combin.*, vol. 6, no. 4, pp. 317–326, 1985.

- [14] P. Frankl, “Improved bounds for Erdős’ matching conjecture,” *J. Combin. Theory Ser. A*, vol. 120, no. 5, pp. 1068–1072, 2013.
- [15] —, “On the maximum number of edges in a hypergraph with given matching number,” *Discrete Appl. Math.*, vol. 216, no. part 3, pp. 562–581, 2017.
- [16] P. Frankl and A. Kupavskii, “Simple juntas for shifted families,” *Discrete Anal.*, Paper No. 14, 18, 2020.
- [17] —, “The Erdős matching conjecture and concentration inequalities,” arXiv preprint, arXiv:1806.08855.
- [18] P. Frankl, T. Łuczak, and K. Mieczkowska, “On matchings in hypergraphs,” *Electron. J. Combin.*, vol. 19, no. 2, Paper 42, 5, 2012.
- [19] P. Frankl, V. Rödl, and A. Ruciński, “On the maximum number of edges in a triple system not containing a disjoint family of a given size,” *Combinatorics, Probability and Computing*, vol. 21, no. 1-2, pp. 141–148, 2012.
- [20] J. Gao, H. Lu, J. Ma, and X. Yu, “On the rainbow matching conjecture for 3-uniform hypergraphs,” *Science China Mathematics*, 2021.
- [21] P. Hall, “On representatives of subsets,” *Journal of the London Mathematical Society*, vol. s1-10, no. 1, pp. 26–30, 1935. eprint: <https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jlms/s1-10.37.26>.
- [22] H. Hàn, Y. Person, and M. Schacht, “On perfect matchings in uniform hypergraphs with large minimum vertex degree,” *SIAM J. Discrete Math.*, vol. 23, no. 2, pp. 732–748, 2009.
- [23] J. Han, “Near perfect matchings in k -uniform hypergraphs,” *Combin. Probab. Comput.*, vol. 24, no. 5, pp. 723–732, 2015.
- [24] H. Huang, P.-S. Loh, and B. Sudakov, “The size of a hypergraph and its matching number,” *Combin. Probab. Comput.*, vol. 21, no. 3, pp. 442–450, 2012.
- [25] R. M. Karp, “Reducibility among combinatorial problems,” in *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, 1972, pp. 85–103.
- [26] G. Katona, “Intersection theorems for systems of finite sets,” *Acta Math. Acad. Sci. Hungar.*, vol. 15, pp. 329–337, 1964.
- [27] N. Keller and N. Lifshitz, “The junta method for hypergraphs and the Erdős-Chvátal simplex conjecture,” *Adv. Math.*, vol. 392, Paper No. 107991, 95, 2021.

- [28] I. Khan, “Perfect matchings in 3-uniform hypergraphs with large vertex degree,” *SIAM J. Discrete Math.*, vol. 27, no. 2, pp. 1021–1039, 2013.
- [29] D. Kühn and D. Osthus, “Matchings in hypergraphs of large minimum degree,” *J. Graph Theory*, vol. 51, no. 4, pp. 269–280, 2006.
- [30] D. Kühn, D. Osthus, and T. Townsend, “Fractional and integer matchings in uniform hypergraphs,” *European J. Combin.*, vol. 38, pp. 83–96, 2014.
- [31] D. Kühn, D. Osthus, and A. Treglown, “Matchings in 3-uniform hypergraphs,” *J. Combin. Theory Ser. B*, vol. 103, no. 2, pp. 291–305, 2013.
- [32] H. Lu, Y. Wang, and X. Yu, “A better bound on the size of rainbow matchings,” arXiv preprint, arXiv:2004.12561.
- [33] H. Lu, X. Yu, and X. Yuan, “Nearly perfect matchings in uniform hypergraphs,” *SIAM J. Discrete Math.*, vol. 35, no. 2, pp. 1022–1049, 2021.
- [34] —, “Rainbow matchings for 3-uniform hypergraphs,” *J. Combin. Theory Ser. A*, vol. 183, Paper No. 105489, 21, 2021.
- [35] T. Łuczak and K. Mieczkowska, “On Erdős’ extremal problem on matchings in hypergraphs,” *J. Combin. Theory Ser. A*, vol. 124, pp. 178–194, 2014.
- [36] M. Mitzenmacher and E. Upfal, *Probability and computing*. Cambridge University Press, Cambridge, 2005, pp. xvi+352, Randomized algorithms and probabilistic analysis, ISBN: 0-521-83540-2.
- [37] O. Pikhurko, “Perfect matchings and K_4^3 -tilings in hypergraphs of large codegree,” *Graphs Combin.*, vol. 24, no. 4, pp. 391–404, 2008.
- [38] N. Pippenger and J. Spencer, “Asymptotic behavior of the chromatic index for hypergraphs,” *J. Combin. Theory Ser. A*, vol. 51, no. 1, pp. 24–42, 1989.
- [39] V. Rödl, A. Ruciński, and E. Szemerédi, “Perfect matchings in large uniform hypergraphs with large minimum collective degree,” *J. Combin. Theory Ser. A*, vol. 116, no. 3, pp. 613–636, 2009.
- [40] —, “Perfect matchings in uniform hypergraphs with large minimum degree,” *European J. Combin.*, vol. 27, no. 8, pp. 1333–1349, 2006.
- [41] D. Saxton and A. Thomason, “Hypergraph containers,” *Invent. Math.*, vol. 201, no. 3, pp. 925–992, 2015.

- [42] A. Treglown and Y. Zhao, “Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II,” *J. Combin. Theory Ser. A*, vol. 120, no. 7, pp. 1463–1482, 2013.
- [43] W. T. Tutte, “The factorization of locally finite graphs,” *Canadian Journal of Mathematics*, vol. 2, pp. 44–49, 1950.