Institute of Paper Science and Technology
Atlanta, Georgia

IPST Technical Paper Series Number 756

On Monodromy Matrix Computation

X. Wang and J.K. Hale

October 1998

Submitted to
SIAM Journal on Numerical Analysis

INSTITUTE OF PAPER SCIENCE AND TECHNOLOGY
PURPOSE AND MISSIONS

The Institute of Paper Science and Technology is an independent graduate school, research organization, and information center for science and technology mainly concerned with manufacture and uses of pulp, paper, paperboard, and other forest products and byproducts. Established in 1929, the Institute provides research and information services to the wood, fiber, and allied industries in a unique partnership between education and business. The Institute is supported by 52 North American companies. The purpose of the Institute is fulfilled through four missions, which are:

- to provide a multidisciplinary education to students who advance the science and technology of the industry and who rise into leadership positions within the industry;
- to conduct and foster research that creates knowledge to satisfy the technological needs of the industry;
- to serve as a key global resource for the acquisition, assessment, and dissemination of industry information, providing critically important information to decision-makers at all levels of the industry; and
- to aggressively seek out technological opportunities and facilitate the transfer and implementation of those technologies in collaboration with industry partners.

ACCREDITATION

The Institute of Paper Science and Technology is accredited by the Commission on Colleges of the Southern Association of Colleges and Schools to award the Master of Science and Doctor of Philosophy degrees.

NOTICE AND DISCLAIMER

The Institute of Paper Science and Technology (IPST) has provided a high standard of professional service and has put forth its best efforts within the time and funds available for this project. The information and conclusions are advisory and are intended only for internal use by any company who may receive this report. Each company must decide for itself the best approach to solving any problems it may have and how, or whether, this reported information should be considered in its approach.

IPST does not recommend particular products, procedures, materials, or service. These are included only in the interest of completeness within a laboratory context and budgetary constraint. Actual products, procedures, materials, and services used may differ and are peculiar to the operations of each company.

In no event shall IPST or its employees and agents have any obligation or liability for damages including, but not limited to, consequential damages arising out of or in connection with any company's use of or inability to use the reported information. IPST provides no warranty or guaranty of results.

On Monodromy Matrix Computation

by

Xiaodong Wang † and Jack K. Hale ‡

Abstract

We present a study on the critical time step for the numerical integration based on the Runge-Kutta method of the monodromy matrix (the fundamental matrix solution) associated with a set of n first-order linear ordinary differential equations with periodic coefficients. By applying the Liapunov-Schmidt method, for any dimension n and systems which are perturbations of autonomous systems, we give an approximation to the critical time step which involves the autonomous part as well as the periodic perturbation.

---

† Assistant Professor, Institute of Paper Science and Technology, Atlanta, GA 30318

‡ Professor, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332

Center for Dynamical Systems & Nonlinear Studies, GIT, Report CDSNS98-305

SIAM Journal on Numerical Analysis (submitted)

1 Introduction

In many engineering practices, with problems such as the vibration of pipes conveying pulsatile flows, column structures under periodic axial loading, and moving webs subjected to periodic excitation forces on the boundary, we encounter linear ordinary differential equations with periodic coefficients [2] [9] [13] [14] [15].

For low dimensional systems, the traditional approach of the Galerkin-Ritz method with one or two terms of the series expansions can be effective in the determination of the monodromy matrix (see, for example, [1]). For systems of very large dimensions, we often incorporate the direct time integration for the computation of the monodromy matrix. Because we deduce the dynamic stability information from the eigenvalues of the monodromy matrix [5] [8] [12], it is critical to understand the accuracy and stability of the numerical schemes used to derive the monodromy matrix. In practice, especially when dealing with large dynamical systems, the efficiency of the numerical algorithm depends on the choice of the time step [14] [15]. Although the critical time steps of various numerical schemes used for linear ordinary differential equations with constant coefficients are well understood [10], no available literature exists on *a priori* selection of the time step in the numerical integration of the monodromy matrix. This paper pursues this topic and aims to provide the critical time step as a function of key parameters of the dynamical systems with periodic perturbations.

Let us describe briefly the ideas in this paper. Suppose that $\mathbf{A}(t)$, a continuous matrix $\in \mathcal{R}^{n\times n}$, is periodic in t of period T_o, and consider the first order differential system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}. \tag{1}$$

If $\mathbf{X}(t)$, with $\mathbf{X}(0) = \mathbf{I}$, is an $n \times n$ matrix solution of Eq. (1), then the monodromy matrix is defined to be $\mathbf{X}(T_o)$. The eigenvalues of this matrix are the Floquet multipliers of Eq. (1). If each Floquet multiplier has modulus less than one, then the origin is exponentially stable. If at least one multiplier has modulus greater than one, then the origin is unstable. A complex number $\rho = e^{\lambda T_o}$ is a Floquet multiplier of Eq. (1) if and only if there is a non-zero n-dimensional vector function $\mathbf{p}(t)$, periodic of period T_o, such that $\mathbf{x}(t) = e^{\lambda t}\mathbf{p}(t)$ is a solution of Eq. (1). In applications, the matrix in Eq. (1) is given as $\mathbf{A}(t, \mathbf{c})$, where $\mathbf{c} \in \mathcal{R}^s$ designates physical parameters. The problem is then to find the regions of stability in the parameter space and, especially, to find the surfaces in the parameter space that represent surfaces of transition from stability to instability. In this paper, we consider the dynamical systems with two parameters, i.e., $s = 2$.

For the determination of the approximate Floquet multipliers of the perturbed system, we often use the Liapunov-Schmidt method, which works as follows. If, for example, the Floquet multipliers of the unperturbed system are simple, then the determination of the Floquet multipliers of the perturbed system reduces the determination of the eigenvalues of an $n \times n$ matrix to the solution of n one-dimensional problems. Furthermore, each Floquet multiplier can be given to any desired degree of accuracy. In this paper, we demonstrate the procedure with a second-order expansion. Although theoretically, based on the Liapunov-Schmidt method, we can obtain explicitly the analytical approximations for the Floquet multipliers and the solutions of the perturbed system up to any desired order of accuracy, the evaluation of these algebraic expressions as functions of both $\mathbf{c}$ and t is an insurmountable task, especially for problems with large dimensions. Therefore, instead we use numerical schemes to determine the monodromy matrix $\mathbf{X}(T_o)$. The monodromy matrix

computation is also a very difficult and time-consuming task if the dimension n of Eq. (1) is very large and the time step is very small. Moreover, to obtain dynamical stability regions within the parameter space of interest, we have to evaluate the monodromy matrices and their eigenvalues at various points within the parameter space. As a consequence, one would like to take the largest time step possible which preserves stability of the numerical method and provides correct dynamical stability information. We refer to this number as the critical time step, defined as Δt_c.

In many applications, the system (1) is the perturbation of an autonomous one; that is, the perturbation of a linear system with constant coefficients. Knowing the complete behavior of the autonomous system should give some information about the critical step size that can be used when the perturbed system is considered. Of course, there is a critical step size for the autonomous problem and it is not unreasonable to take this as the step size for the perturbed equation. However, such a step size uses no information whatsoever about the nature of the perturbation. In addition, it assumes that the perturbation is very small relative to Δt which, in general, is not true. On the other hand, if we can find a critical step size which incorporates some information about the perturbation, then it is to be expected that this numerical approach can lead to more efficient schemes and perhaps can be guides to problems for which the perturbation is not so small.

The approach taken in this paper is to obtain, using the Liapunov-Schmidt method, the critical time step Δt_c as a function of the perturbation terms and as a consequence, to select an optimum time step $\Delta t \leq \Delta t_c$ for the numerical integration of the monodromy matrix $\mathbf{X}(T_o)$. We note that the first few terms in the Taylor expansions of the Floquet multipliers in terms of the perturbation are easy to obtain and yield reasonable approximations for the critical time step. As noted before, it

is the general feeling that the step size chosen will be valid for a much wider range of perturbation. This point will be demonstrated in the third numerical example in this paper.

We begin with the governing differential equations and relevant theorems in Section 2, and then discuss in Section 3 the Liapunov-Schmidt method. The numerical procedures for the monodromy matrix computation and the derivation of the critical time step are presented in Sections 4 & 5. In Section 6, we confirm the proposed method with some numerical studies, and in the concluding section, we reiterate the important findings.

2 Preliminaries

Consider the following perturbed n-dimensional linear differential equations

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_o + \epsilon \mathbf{A}_1(t))\mathbf{x}(t), \tag{2}$$

where $\mathbf{A}_o$ is a constant matrix $\in \mathcal{R}^{n\times n}$, $\mathbf{A}_1(t)$ is a T_o-periodic continuous coefficient matrix $\in \mathcal{R}^{n\times n}$, and the perturbation constant ϵ is small.

We assume for simplicity that the monodromy matrix $e^{\mathbf{A}_o T_o}$ for Eq. (2) at $\epsilon = 0$ has simple eigenvalues; that is

$$e^{(-\lambda_j+\lambda_k)T_o} \neq 1, \quad j \neq k, k \geq 1, \tag{3}$$

where $\lambda_m \in \mathcal{C}$, with $m = 1, \ldots, n$, represent the eigenvalues of the matrix $\mathbf{A}_o$.

Introducing the modal matrix $\mathbf{\Phi} \in \mathcal{C}^{n\times n}$ and the generalized solution vector $\boldsymbol{\xi}(t) \in \mathcal{C}^n$, with

$$\mathbf{\Phi}^{-1}\mathbf{A}_o\mathbf{\Phi} = \mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_n), \tag{4}$$

and $\mathbf{x}(t) = \mathbf{\Phi}\boldsymbol{\xi}(t)$, we obtain

$$\dot{\boldsymbol{\xi}}(t) = (\mathbf{\Lambda} + \epsilon\bar{\mathbf{A}}_1(t))\boldsymbol{\xi}(t), \tag{5}$$

with $\bar{\mathbf{A}}_1(t) = \mathbf{\Phi}^{-1}\mathbf{A}_1(t)\mathbf{\Phi}$.

If $\mathbf{X}_\epsilon(t)$, with $\mathbf{X}_\epsilon(0) = \mathbf{I}$, is a fundamental matrix solution of Eq. (2), then $\mathbf{X}_\epsilon(T_o)$ is the monodromy matrix. Moreover, $\mathbf{\Xi}(t) = \mathbf{\Phi}^{-1}\mathbf{X}_\epsilon(t)$ is a fundamental matrix solution of Eq. (5). Note that the monodromy matrix $\mathbf{X}_\epsilon(T_o)$ for Eq. (2) is an analytical function of ϵ, and the eigenvalues of $\mathbf{X}_\epsilon(T_o)$ are the Floquet multipliers which we write as $e^{\mu_j(\epsilon)T_o}$. We introduce here the implicit function theorem.

Theorem 1: Suppose that $\mathcal{F} : \mathcal{R}^k \times \mathcal{R} \to \mathcal{R}$; and $(\mathbf{c}, x) \mapsto \mathcal{F}(\mathbf{c}, x)$, is a C^1 function satisfying

$$\mathcal{F}(\mathbf{0}, 0) = 0 \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial x}(\mathbf{0}, 0) \neq 0. \tag{6}$$

Then there are constants $\delta_1 > 0$ and $\delta_2 > 0$, and a C^1 function

$$\psi : \{\mathbf{c} : \|\mathbf{c}\| < \delta_1\} \to \mathcal{R}$$

such that

$$\psi(\mathbf{0}) = 0 \quad \text{and} \quad \mathcal{F}(\mathbf{c}, \psi(\mathbf{c})) = 0 \quad \text{for} \quad \|\mathbf{c}\| < \delta_1.$$

Moreover, if there is a $(\mathbf{c}_o, x_o) \in \mathcal{R}^k \times \mathcal{R}$ such that $\|\mathbf{c}_o\| < \delta_1$ and $|x_o| < \delta_2$, and satisfies the equation $\mathcal{F}(\mathbf{c}_o, x_o) = 0$, then $x_o = \psi(\lambda_o)$.

Since we are assuming that the Floquet multipliers of Eq. (2) for $\epsilon = 0$ are simple, it follows from Theorem 1 that the Floquet multipliers of Eq. (2) for $\epsilon \neq 0$ and sufficiently small are analytic functions of ϵ. Therefore, we can choose the exponents $\mu_j(\epsilon)$ as analytic functions of ϵ where $\mu_j(0) = \lambda_j$, with $j = 1, \ldots, n$.

As remarked in the introduction, $e^{\mu_j(\epsilon)T_o}$ is a Floquet multiplier of Eq. (2) if and only if there is a non-zero solution $e^{\mu_j(\epsilon)T_o}\mathbf{p}_\epsilon(t)$ of Eq. (2) with $\mathbf{p}_\epsilon(t)$ a T_o-periodic n-dimensional vector function.

If we introduce the following transformation

$$\boldsymbol{\xi}(t) = e^{\mu_j(\epsilon)t}\mathbf{w}(t), \tag{7}$$

we change the original problem to the determination of the constant $\mu_j(\epsilon) \in \mathcal{C}$, as a function of ϵ, and the T_o-periodic vector function $\mathbf{w}(t) \in \mathcal{C}^n$. Substituting Eq. (7) into Eq. (5), we obtain,

$$\dot{\mathbf{w}}(t) = (\mathbf{\Lambda} - \mu_j(\epsilon)\mathbf{I})\mathbf{w}(t) + \epsilon\bar{\mathbf{A}}_1(t)\mathbf{w}(t), \tag{8}$$

with the identity matrix $\mathbf{I} \in \mathcal{R}^{n\times n}$.

For the application of the Liapunov-Schmidt method in the next section, we need the following well-known results.

Theorem 2: Suppose that $\mathbf{B}_o(t)$ is a T_o-periodic continuous $n \times n$ matrix, $\mathbf{f}(t)$ is a continuous T_o-periodic n-dimensional vector function and consider the equation

$$\dot{\mathbf{w}}(t) = \mathbf{B}_o(t)\mathbf{w}(t) + \mathbf{f}(t). \tag{9}$$

There exists a T_o-periodic solution $\mathbf{w}(t)$ of Eq. (9) if and only if

$$\int_0^{T_o} \hat{\mathbf{w}}(s)\mathbf{f}(s)ds = 0 \tag{10}$$

for all T_o-periodic solutions of the adjoint equation

$$\dot{\hat{\mathbf{w}}}(t) = -\hat{\mathbf{w}}(t)\mathbf{B}_o(t), \tag{11}$$

where $\hat{\mathbf{w}}(t)$ is an n-dimensional row vector.

In addition, if there is a T_o-periodic solution $\mathbf{w}^*(t)$ of Eq. (9), and $\mathbf{w}_1(t), \ldots, \mathbf{w}_k(t)$ are the T_o-periodic solutions of the equation

$$\dot{\mathbf{w}}(t) = \mathbf{B}_o(t)\mathbf{w}(t), \tag{12}$$

then every T_o-periodic solution of Eq. (9) has the form $\mathbf{w}(t) = \sum_{j=1}^{k} c_j \mathbf{w}_j(t) + \mathbf{w}^*(t)$.

Theorem 3: There exists a unique T_o-periodic solution of Eq. (9) for every $\mathbf{f}(t)$ if and only if there is no non-trivial T_o-periodic solution of Eq. (12). The resulting T_o-periodic solution $\mathbf{w}^*(t, \mathbf{f})$ is a continuous linear functional on $\mathbf{f}(t)$; that is, there is a constant K such that $\sup_t \|\mathbf{w}^*(t, \mathbf{f})\| \leq K \sup_t \|\mathbf{f}(t)\|$ for any function $\mathbf{f}(t)$.

The first part of this result is obvious from Theorem 2. The second part requires additional information about the T_o-periodic solution and is not completely trivial.

3 Liapunov-Schmidt Method

In preparation for the reduction from a multi-dimensional problem to a scalar one, let

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \xi_1(t) \\ \boldsymbol{\eta}(t) \end{bmatrix}, \tag{13}$$

with $\xi_1(t) \in \mathcal{C}$ and $\boldsymbol{\eta}(t) \in \mathcal{C}^{n-1}$, and rewrite matrices $\bar{\mathbf{A}}_1(t)$ and $\boldsymbol{\Lambda}$ as

$$\bar{\mathbf{A}}_1(t) = \begin{bmatrix} \bar{A}_{11}(t) & \bar{\mathbf{A}}_{12}(t) \\ \bar{\mathbf{A}}_{21}(t) & \bar{\mathbf{A}}_{22}(t) \end{bmatrix}, \tag{14}$$

$$\boldsymbol{\Lambda} = diag(\lambda_1, \bar{\boldsymbol{\Lambda}}) \text{ and } \bar{\boldsymbol{\Lambda}} = diag(\lambda_2, \ldots, \lambda_n). \tag{15}$$

Thus, the original system depicted with Eq. (5) can be decomposed into one scalar and one vector equation

$$\dot{\xi}_1(t) = \lambda_1 \xi_1(t) + \epsilon \bar{A}_{11}(t)\xi_1(t) + \epsilon \bar{\mathbf{A}}_{12}(t)\boldsymbol{\eta}(t), \tag{16}$$

$$\dot{\boldsymbol{\eta}}(t) = \bar{\boldsymbol{\Lambda}}\boldsymbol{\eta}(t) + \epsilon \bar{\mathbf{A}}_{21}(t)\xi_1(t) + \epsilon \bar{\mathbf{A}}_{22}(t)\boldsymbol{\eta}(t). \tag{17}$$

Furthermore, we substitute in Eqs. (16) and (17) the following change of variables,

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \xi_1(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} = e^{\tau_1 t} \begin{bmatrix} \bar{\xi}_1(t) \\ \bar{\boldsymbol{\eta}}(t) \end{bmatrix} \tag{18}$$

and obtain

$$\dot{\bar{\xi}}_1(t) = (\lambda_1 - \tau_1)\bar{\xi}_1(t) + \epsilon \bar{A}_{11}(t)\bar{\xi}_1(t) + \epsilon \bar{\mathbf{A}}_{12}(t)\bar{\boldsymbol{\eta}}(t), \tag{19}$$

$$\dot{\bar{\boldsymbol{\eta}}}(t) = (\bar{\boldsymbol{\Lambda}} - \tau_1 \mathbf{I}_{n-1})\bar{\boldsymbol{\eta}}(t) + \epsilon \bar{\mathbf{A}}_{21}(t)\bar{\xi}_1(t) + \epsilon \bar{\mathbf{A}}_{22}(t)\bar{\boldsymbol{\eta}}(t), \tag{20}$$

with the identity matrix $\mathbf{I}_{n-1} \in \mathcal{R}^{(n-1)\times(n-1)}$.

For the original problem (2) to have $e^{\tau_1 T_o}$ as a Floquet multiplier, we need to determine τ_1 so that Eqs. (19) and (20) have a non-trivial T_o-periodic solution $(\bar{\xi}_1(t), \bar{\boldsymbol{\eta}}(t))$.

Denote by $\mathcal{P}_{T_o}$ the space of continuous T_o-periodic functions. For any $\hat{\xi}_1 \in \mathcal{P}_{T_o}$ consider the equation

$$\begin{aligned} \dot{\bar{\boldsymbol{\eta}}}(t) &= [\bar{\boldsymbol{\Lambda}} - \tau_1 \mathbf{I}_{n-1} + \epsilon \bar{\mathbf{A}}_{22}(t)]\bar{\boldsymbol{\eta}}(t) + \epsilon \bar{\mathbf{A}}_{21}(t)\hat{\xi}_1(t) \\ &\stackrel{def}{=} \mathbf{B}_o(t, \epsilon, \tau_1)\bar{\boldsymbol{\eta}}(t) + \epsilon \bar{\mathbf{A}}_{21}(t)\hat{\xi}_1(t). \end{aligned} \tag{21}$$

For $\epsilon = 0$ and $\tau_1 = \lambda_1$, we have $\mathbf{B}_o(t, 0, \lambda_1) = \bar{\boldsymbol{\Lambda}} - \lambda_1 \mathbf{I}_{n-1}$. In addition, assumption (3) implies that there is no T_o-periodic solution of the equation

$$\dot{\mathbf{w}}(t) = (\bar{\mathbf{\Lambda}} - \lambda_1 \mathbf{I}_{n-1})\mathbf{w}(t) \tag{22}$$

that is, there is no Floquet multiplier equal to one. Furthermore, Theorem 1 implies that the monodromy matrix of the equation

$$\dot{\mathbf{w}}(t) = \mathbf{B}_o(t, \epsilon, \tau_1)\mathbf{w}(t) \tag{23}$$

is continuous in ϵ and τ_1, and that there is no Floquet multiplier of Eq. (23) equal to one if $|\epsilon|$ and $|\tau_1 - \lambda_1|$ are small. Therefore, Equation (23) has no non-trivial T_o-periodic solution. Based on Theorems 1, 2, and 3, Equation (21) has a unique T_o-periodic solution $\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(t)$ which is linear and continuous in $\hat{\xi}_1$, and continuous and analytic in ϵ and τ_1, with $\bar{\boldsymbol{\eta}}(\hat{\xi}_1, 0, \tau_1)(t) = 0$.

As a consequence of these remarks, the functions $\hat{\xi}_1(t)$ and $\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(t)$ will be T_o-periodic solutions of Eqs. (19) and (20) if and only if $\hat{\xi}_1(t)$ is a T_o-periodic solution of the equation

$$\begin{aligned} \dot{\hat{\xi}}_1(t) &= [\lambda_1 - \tau_1 + \epsilon \bar{A}_{11}(t)]\hat{\xi}_1(t) + \epsilon \bar{\mathbf{A}}_{12}(t)\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(t) \\ &\stackrel{def}{=} \alpha(\lambda_1 - \tau_1, \epsilon, t)\hat{\xi}_1(t) + \epsilon \bar{\mathbf{A}}_{12}(t)\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(t). \end{aligned} \tag{24}$$

We must now determine $\tau_1 = \tau_1(\epsilon)$ in such a way that Eq. (24) has a T_o-periodic solution. Note that Eq. (24) is not a differential equation, but we can still analyze its properties.

Equation (24) has a T_o-periodic solution $\hat{\xi}_1$ if and only if

$$\int_0^{T_o} [\alpha(\lambda_1 - \tau_1, \epsilon, s)\hat{\xi}_1(s) + \epsilon \bar{\mathbf{A}}_{12}(s)\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(s)]ds = 0. \tag{25}$$

Based on the previous discussions on $\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1)(t)$, it is clear that Eq. (25) is linear in $\hat{\xi}_1$, and can be used to determine $\tau_1 = \tau_1(\epsilon)$ and $\hat{\xi}_1(t) = \hat{\xi}_1(\epsilon, t)$. Notice that, based on Theorem 1, all functions are analytical in ϵ and we may theoretically determine the power series of these quantities to any accuracy desired. To reiterate, we want to determine $\tau_1 = \tau_1(\epsilon)$ so that $e^{\tau_1(\epsilon)T_o}$ satisfies $\tau_1(0) = \lambda_1$; that is, the Floquet multiplier is close to $e^{\lambda_1 T_o}$. As remarked earlier, $\tau_1(\epsilon)$ is also an analytical function of ϵ and can be written as

$$\tau_1(\epsilon) = \lambda_1 + \beta_1\epsilon + \beta_2\epsilon^2 + O(\epsilon^3), \tag{26}$$

where β_1 and β_2 are constants to be determined. Furthermore, at $\epsilon = 0$, Equation (24) becomes $\dot{\hat{\xi}}_1(t) = 0$, which means that $\hat{\xi}_1$ is a constant. In order to normalize $\hat{\xi}_1(\epsilon, t)$ so that $\hat{\xi}_1(0, t) = 1$, we let

$$\hat{\xi}_1(\epsilon, t) = 1 + \epsilon\hat{\xi}_{11}(t) + \epsilon^2\hat{\xi}_{12}(t) + O(\epsilon^3) \tag{27}$$

and deduce from Eq. (21) (refer to Hale [4] for details)

$$\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1(\epsilon))(t) = \epsilon\bar{\boldsymbol{\eta}}_1(t) + O(\epsilon^2), \tag{28}$$

with

$$\bar{\boldsymbol{\eta}}_1(t) = e^{(\bar{\boldsymbol{\Lambda}} - \lambda_1\mathbf{I}_{n-1})t}[e^{-(\bar{\boldsymbol{\Lambda}} - \lambda_1\mathbf{I}_{n-1})T_o} - \mathbf{I}]^{-1} \int_t^{t+T_o} e^{-(\bar{\boldsymbol{\Lambda}} - \lambda_1\mathbf{I}_{n-1})s}\bar{\mathbf{A}}_{21}(s)ds.$$

Note that, if $\bar{\mathbf{A}}_{21}(t)$ is not a function of t, we can simply have

$$\bar{\boldsymbol{\eta}}(\hat{\xi}_1, \epsilon, \tau_1(\epsilon))(t) = -\epsilon(\bar{\boldsymbol{\Lambda}} - \lambda_1\mathbf{I}_{n-1})^{-1}\bar{\mathbf{A}}_{21} + O(\epsilon^2). \tag{29}$$

Furthermore, substituting Eq. (28) in Eq. (25), we obtain,

$$\beta_1 = \frac{1}{T_o}\int_0^{T_o} \bar{A}_{11}(s)ds, \tag{30}$$

$$\beta_2 = \frac{1}{T_o}\int_0^{T_o} \bar{\mathbf{A}}_{12}(s)\bar{\boldsymbol{\eta}}_1(s)ds. \tag{31}$$

Therefore, we see that Eq. (24) up to terms of order ϵ^3 is equivalent to the scalar ordinary differential equation

$$\dot{\bar{\xi}}_1(t) = \{\epsilon[\bar{A}_{11}(t) - \beta_1] + \epsilon^2[\bar{\mathbf{A}}_{12}(t)\bar{\boldsymbol{\eta}}_1(t) - \beta_2]\}\bar{\xi}_1(t) + O(\epsilon^3). \tag{32}$$

The first-order approximation is then given by

$$\dot{\xi}_1(t) = a(\epsilon, t)\xi_1(t) + O(\epsilon^2), \tag{33}$$

with

$$a(\epsilon, t) = \lambda_1 + \epsilon\bar{A}_{11}(t),$$

which is much easier to calculate than Eq. (32) for systems of high dimensions.

Note that the same approaches should be applied to all eigenvalues of the system. It is clear at this point that although we can derive approximations of the solutions of Eqs. (16) and (17) with any desired order of accuracy, the analytical evaluation of the matrices $\bar{\mathbf{A}}_{21}(t)$, $\bar{\mathbf{A}}_{12}(t)$, and $\bar{\mathbf{A}}_{22}(t)$ can be very challenging, if at all feasible, especially for high dimensional problems. Therefore, we choose to introduce the following numerical algorithm.

4 Numerical Algorithm

Let $\mathbf{N} = \mathbf{X}(T_o)$ be the monodromy matrix. With $\mathbf{X}(0) = \mathbf{I}$, we can determine the monodromy matrix $\mathbf{N}$, by numerically integrating Eq. (1) for $t \in [0, T_o]$. However,

due to the presence of the time-varying coefficient matrix $\mathbf{A}(t) = \mathbf{A}_o + \epsilon\mathbf{A}_1(t)$, we cannot use full implicit numerical schemes. This limits us to explicit schemes which often require small time steps. In this paper, we investigate the use of the second-order Runge-Kutta (RK2) scheme depicted as follows:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta t(\mathbf{k}_1 + \mathbf{k}_2)/2, \tag{34}$$

where

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{A}^k\mathbf{x}^k, \\ \mathbf{k}_2 &= \mathbf{A}^{k+1}(\mathbf{x}^k + \Delta t\mathbf{A}^k\mathbf{x}^k), \end{aligned}$$

with $\mathbf{A}^k = \mathbf{A}(k\Delta t)$ and $0 \leq k \leq T_o/\Delta t$.

Note that the m^{th} column of the matrix $\mathbf{N}$ corresponds to the numerical solution of Eq. (1) with the m^{th} column of the identity matrix $\mathbf{I}$ as the initial condition. Therefore, the numerical integration illustrated in (34) has to be performed n times to form a monodromy matrix. In general, the explicit nature of the Runge-Kutta scheme requires the use of excessively small time steps and the construction of the monodromy matrix can be very expensive. Furthermore, in order to obtain the dynamical stability regions, we have to compute the monodromy matrices and their eigenvalues at all parameter spatial grid points within a parameter space subdivided into parameter spatial divisions. Because we do not know *a priori* the structure of the matrix $\mathbf{N}$, and because a poorly constructed matrix $\mathbf{N}$ can lead to incorrect conclusions of the dynamical instability, we need to know the critical time step Δt_c before the numerical integration, and try to avoid the costly trial and error process.

5 Critical Time Step

Because we obtain Eq. (5) from Eq. (2) through the transformation with the constant modal matrix $\mathbf{\Phi}$, the scheme presented in (34) can be applied equivalently to Eq. (5).

Hence, for the unperturbed system $\mathbf{A}^k = \mathbf{A}_o$, the equivalent scheme is written as: for $m = 1, \ldots, n$

$$\xi_m^{k+1} = \xi_m^k + \Delta t(\tilde{k}_1 + \tilde{k}_2)/2, \tag{35}$$

where

$$\begin{aligned} \tilde{k}_1 &= \lambda_m \xi_m^k, \\ \tilde{k}_2 &= \lambda_m(\xi_m^k + \Delta t \lambda_m \xi_m^k), \end{aligned}$$

or

$$\xi_m^{k+1} = G_m \xi_m^k, \tag{36}$$

with $G_m = 1 + \lambda_m \Delta t + (\lambda_m \Delta t)^2/2$.

Furthermore, the critical time step Δt_c governed by the stability requirement of the RK2 scheme, corresponds to

$$|G_m| \leq 1, \quad \forall \lambda_m, \text{ with } m = 1, \ldots, n. \tag{37}$$

Of course, we need to have $Re(\lambda_m) < 0$, based on the stability of the unperturbed system. For the perturbed system, the matrix $\bar{\mathbf{A}}_1(t)$ is, in general, not diagonal, and a direct study of the scheme in the form of (34) becomes very difficult. Based on the discussion in previous sections, we can use the Liapunov-Schmidt method to transform the original non-autonomous linear system in Eq. (5) to n one-dimensional problems in the form of (33), the equivalent scalar non-autonomous equation as the first-order approximation of ϵ. Again, because it is equivalent to apply the scheme in (34) to Eq. (2) as to Eqs. (5) and (33), we are prepared to discuss the numerical stability issues for the perturbed systems.

Introduce the RK2 scheme in (34) to Eq. (33), we obtain

$$\xi_1^{k+1} = G_1^k \xi_1^k + O(\epsilon^2), \tag{38}$$

where

$$G_1^k = 1 + [a^k(\epsilon) + a^{k+1}(\epsilon)]\Delta t/2 + a^k(\epsilon)a^{k+1}(\epsilon)\Delta t^2/2, \tag{39}$$

with $\bar{A}_{11}^k = \bar{A}_{11}(k\Delta t)$, $a^k(\epsilon) = a(\epsilon, k\Delta t) = \lambda_1 + \epsilon \bar{A}_{11}^k$, and $0 \leq k \leq T_o/\Delta t$.

We introduce the following supnorm $\|G_1\|_{sup}$, such that,

$$\|G_1\|_{sup} = \sup_{0 \leq k \leq T_o/\Delta t} |G_1^k|, \tag{40}$$

and without loss of generality, if we apply the same approach to all the eigenvalues λ_m, with $m = 1, \ldots, n$, the critical time step Δt_c satisfies

$$\|G_m\|_{sup} \leq 1, \quad \forall \lambda_m, \text{ with } m = 1, \ldots, n. \tag{41}$$

Notice that the derivation of Δt_c is of $O(\epsilon^2)$. With $\epsilon = 0$, (41) is equivalent to (37). Hence, we denote $\Delta t_c(0)$ as the critical time step of the corresponding autonomous system, and $\Delta t_c(\epsilon)$ as the critical time step of the non-autonomous system with the perturbation ϵ.

6 Numerical Examples

To confirm the proposed critical time step as a function of the autonomous part as well as the periodic perturbation, we present three paradigms of Mathieu-Hill equations. We begin by introducing the following second-order linear system

$$\mathbf{M}_o \ddot{\mathbf{Y}}(t) + \mathbf{C}_o \dot{\mathbf{Y}}(t) + \mathbf{K}_o \mathbf{Y}(t) = \mathbf{0}, \tag{42}$$

with solution vector $\mathbf{Y}(t) \in \mathcal{R}^r$, and constant coefficient matrices $\mathbf{M}_o$, $\mathbf{C}_o$, and $\mathbf{K}_o \in \mathcal{R}^{r\times r}$. If we assume a characteristic solution, $\mathbf{Y}(t) = e^{i\omega t}\widehat{\mathbf{Y}}$, with $i = \sqrt{-1}$, where $\widehat{\mathbf{Y}}$ represents the mode shape of the natural frequency $\omega = 2\pi f$, the stable system corresponds to $Im(\omega) \geq 0$ with $Re(\omega) \neq 0$. In engineering practice, we often define the buckling instability as $Re(\omega) \to 0$ with $Im(\omega) \geq 0$, and the flutter instability as $Im(\omega) < 0$ with $Re(\omega) \neq 0$. Moreover, having the set of r second-order linear ordinary differential equations in Eq. (42), if we introduce a new solution vector, $\mathbf{x}(t) = (\mathbf{Y}(t), \dot{\mathbf{Y}}(t)) \in \mathcal{R}^n$, with $n = 2r$, we can replace Eq. (42) with a system of n first-order linear ordinary differential equations in the form of Eq. (2) with

$$\mathbf{A}(t) = \mathbf{A}_o = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_o^{-1}\mathbf{K}_o & -\mathbf{M}_o^{-1}\mathbf{C}_o \end{bmatrix}.$$

Now, let us consider a perturbation of Eq. (42),

$$\mathbf{M}_o\ddot{\mathbf{Y}}(t) + (\mathbf{C}_o + \epsilon\mathbf{C}_1(t))\dot{\mathbf{Y}}(t) + (\mathbf{K}_o + \epsilon\mathbf{K}_1(t))\mathbf{Y}(t) = \mathbf{0}, \tag{43}$$

where $\mathbf{C}_1(t)$ and $\mathbf{K}_1(t)$ are T_o-periodic coefficient matrices, and the perturbation constant ϵ is small. Similarly, Eq. (43) can be written in the form of Eq. (2) with

$$\mathbf{A}(t) = \mathbf{A}_o + \epsilon\mathbf{A}_1(t) \text{ and } \mathbf{A}_1(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_o^{-1}\mathbf{K}_1(t) & -\mathbf{M}_o^{-1}\mathbf{C}_1(t) \end{bmatrix}.$$

To implement the proposed critical time step presented in (41), we introduce the first problem illustrated as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2(1 + \epsilon\cos\omega_o t)x = 0, \tag{44}$$

with $T_o = 2\pi/\omega_o$, $\omega = \sqrt{2}$, $\zeta \in [0, 1]$, $\omega_o \in [2.0, 3.6]$, and $\epsilon \in [0, 1]$.

For simplicity, in order to compare with analytical solutions, we consider the damping ratio $\zeta = 0$. Hence, we have $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$. In general,

we use $\lambda_{2m-1} = i\omega_m$ and $\lambda_{2m} = -i\omega_m$, with $m = 1, \ldots, r$. To determine the critical time step according to (41), we implement a simple program to evaluate $\|G_m\|_{sup}, \quad \forall \lambda_m$, with $m = 1, \ldots, n$, in which we compute the eigenvalues and eigenvectors of the constant matrix $\mathbf{A}_o$, and the corresponding $\bar{A}_{11}$. It is important to point out that we only have to evaluate the eigensolutions of the matrix $\mathbf{A}_o$ once, and the computation effort is comparable to the determination of the eigensolutions of one monodromy matrix. Of course, prior to the expensive monodromy matrix computation for every parameter spatial grid point, using the simple program, we can conveniently determine the maximum time step satisfying (41) as a function of parameters. For instance, we can easily obtain $\lambda_2 = -i\omega$, $\bar{A}_{11}(t) = -i\omega \cos \omega_o t/2$, and the corresponding $a(t) = -i\omega(1 + \epsilon \cos \omega_o t/2)$. Using Eq. (39), it is straightforward to derive $\Delta t_c(\epsilon)$ from (41) for various values of ϵ and ω_o. Although we should check every eigensolution of the matrix $\mathbf{A}_o$ in (41), intuitively, we understand, the numerical stability requirement is mainly governed by the perturbation ϵ, the highest natural frequency ω_r, or λ_n, and its corresponding $\bar{A}_{11}(t)$.

As shown in Fig. 1, the critical time step is a function of the perturbation ϵ and does not seem to be dependent on the period T_o for periodic perturbations. Notice that the critical time step is normalized with the critical time step for the corresponding autonomous system. If we take a time step $\Delta t = 0.66\Delta t_c(0)$, satisfying $\Delta t \leq \Delta t_c(\epsilon), \quad \forall \epsilon \in [0, 1]$, according to (41) and the discussion in the previous sections, the numerical integration scheme should be stable, and from the eigenvalues of monodromy matrices at various values of ϵ and ω_o, we should obtain the correct dynamic stability regions within the parameter space of $\epsilon \in [0, 1]$ and $\omega_o \in [2.6, 3.6]$. Figure 2 confirms our predictions, and by comparing with the asymptotic solutions discussed in Ref. [3], it is clear that the time step selection based on (41) is appro-

priate and sufficient. As expected, with damping effects, the stability boundaries are smoothed and the instability regions are reduced. Figure 2 also shows that when we select $\Delta t = \Delta t_c(0)$, a critical time step of the corresponding autonomous system, the dynamical stability results are very poor.

For the first example, with the correct answers in mind, we can afford to try out various time steps to empirically determine the convergence of the numerical integration of the monodromy matrix. Unfortunately, in practice we neither know *a priori* stability solutions nor can we afford the trial and error procedures for multi-dimensional problems. To further explore the main points of this paper, we consider the second example, Eq. (43), with

$$\begin{aligned} \mathbf{M}_o &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{C}_o &= \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & 0.2 \end{bmatrix}, \\ \mathbf{K}_o &= \begin{bmatrix} 2 & 1 \\ 1 & 30 \end{bmatrix}, & \mathbf{C}_1 &= \begin{bmatrix} 0.1\cos\omega_o t & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{K}_1 &= \begin{bmatrix} 2\cos\omega_o t & 0 \\ 0 & 30\cos\omega_o t \end{bmatrix}, \end{aligned} \tag{45}$$

for $\epsilon \in [0, 1]$ and $\omega_o \in [1.0, 3.6]$.

From Eq. (42), based on the theorems on real operators with complex eigenvalues and eigenvectors [6] [7], we can easily obtain the eigenvalues and eigenvectors of this dynamical system:

$$\mathbf{\Phi} = \begin{bmatrix} -0.002 & 0.626 & -0.000 & 0.008 \\ -0.006 & -0.022 & 0.111 & 0.103 \\ 0.876 & -0.028 & 0.044 & 0.000 \\ -0.031 & 0.010 & 0.554 & -0.617 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{bmatrix},$$

$$\begin{aligned} \lambda_1 &= -0.050 + 1.400i, & \lambda_2 &= -0.050 - 1.400i, \\ \lambda_3 &= -0.100 + 5.483i, & \lambda_4 &= -0.100 - 5.483i. \end{aligned}$$

It is then clear that the original system with the constant matrices defined in (45) is stable. Moreover, for λ_4, we have

$$\bar{A}_{11}(t) = (0.0002 - 2.7325i)\cos\omega_o t,$$
$$a(t) = (-0.100 - 5.483i) + \epsilon(0.0002 - 2.7325i)\cos\omega_o t.$$

Similar to the approach for the first example, with the aid of a simple program, we can easily obtain the critical time step as a function of the perturbation. As shown in Fig. 3, we select a time step $\Delta t = 0.56\Delta t_c(0)$ prior to the monodromy matrix computation, such that $\Delta t < \Delta t_c(\epsilon), \quad \forall\epsilon \in [0,1]$. Figure 3 also indicates that the critical time step does not depend on the period of the perturbation. With the dynamical stability results obtained with a sufficiently small time step as a reference solution, it is demonstrated that the time step $\Delta t = \Delta t_c(0)$ does not provide an accurate solution, while the time step $\Delta t = 0.56\Delta t_c(0)$, judiciously chosen according to (41), yields much better results. In addition, the second example demonstrates that the maximum system eigenvalue λ_n and its corresponding perturbation terms contribute significantly to the critical time step Δt_c, and in general, if we are interested in the dynamical instability region around the lower harmonics and their resonances, because of the nature of the explicit scheme, the time step selection $\Delta t \leq \Delta t_c$ often satisfies the accuracy requirement of $\Delta t \leq T_o/20$.

As depicted in Figs. 1 and 3, with small perturbations in the first two examples, the critical time step can be reduced to 50% of the critical time step of the corresponding autonomous system. In many engineering practices, the periodic perturbation could be significant. Because the numerical scheme illustrated in (34) does not limit itself to small perturbations, and is in fact applicable to general non-autonomous first-order linear ordinary differential equations, in order to verify the possible extension of the approximation results of the critical time step, we introduce the third problem illustrated as

$$\ddot{x} + (a + b\cos t)x = 0, \tag{46}$$

with $a \in [0, 2]$ and $b \in [0, 4]$.

Similar to the procedures taken in the first two examples, consider $\lambda_2 = -i\sqrt{a}$, we obtain the corresponding $\bar{A}_{11}(t) = -i\sqrt{a}\cos t/2$ and $a(t) = -i\sqrt{a}(1 + b/2a\cos t)$. Figure 5 presents the critical time step Δt_c as a function of the parameters a and b. It is obvious that as a increases, $\Delta t_c(a, b)$ decreases, and for a fixed value of a, with the increase of b, i.e., the perturbation $\epsilon = b/a$, the critical time step $\Delta t_c(a, b)$ can be reduced to a mere 10% of the critical time step of the corresponding autonomous system. As shown in Fig. 5, the time step $\Delta t = 0.075$ satisfies (41), $\forall a \in [0, 2]$ and $b \in [0, 4]$, and should provide a sufficiently accurate dynamical stability result. With the reference of the result derived from a very small time step $\Delta t = 0.01\pi$ and the graph presented in Ref. [11], the results shown in Fig. 6 confirm our predictions. In addition, when we select $\Delta t = \Delta t_c(2, 0)$, the dynamical stability results are only accurate within the region of $a \in [0, 0.5]$ and $b \in [0, 1]$, where $\Delta t_c(2, 0)$ is smaller than $\Delta t_c(a, b)$. Therefore, we conjecture that (41) can be useful to problems with significant periodic perturbations.

As a final remark, although the number of parameter spatial divisions does not directly affect the monodromy matrix computation, when the stability zones are very narrow, sufficiently refined parameter spatial divisions are required. Notice that in order to obtain the dynamical stability information, we have to find the eigenvalues of the monodromy matrices at all parameter spatial grid points, which in fact, further demonstrates the need to obtain an optimum time step *a priori*.

7 Conclusion

In summary, we present in this paper the critical time step for the numerical construction of the monodromy matrix, and point out that it is essential to have *a priori* estimate of the critical time step to achieve accurate analyses of the system dynamical stability.

Although, in general, the critical time step is governed by the largest eigenvalue λ_n, and the corresponding perturbation term $\epsilon\bar{A}_{11}$, a simple program for the computation of $a(\epsilon,t)\Delta t = (\lambda_m + \epsilon\bar{A}_{11})\Delta t$ and $\|G_m\|_{sup}, \quad \forall\lambda_m$, with $m = 1,\ldots,n$, can easily be implemented and applied to find an optimum time step $\Delta t \leq \Delta t_c$ prior to the monodromy matrix computation. Thus, the general procedure for the monodromy matrix computation contains two steps: firstly, we search for the critical time step as a function of parameters and select a time step smaller than the minimum value of the critical time step within the region of the parameter space of interest; secondly, we perform the numerical integration for $t \in [0, T_o]$ to obtain $\mathbf{X}(T_o)$, with $\mathbf{X}(0) = \mathbf{I}$, for all parameter spatial grid points. Of course, to derive the dynamical stability information, we need to compute the eigenvalues of the monodromy matrices.

Although we discuss extensively the second-order Runge-Kutta scheme, the same approaches can be directly applied to the other numerical integration schemes. For the dynamical systems with small damping ratio, we should strive to choose the numerical scheme (explicit) which covers more areas of the imaginary axis in the complex $a(\epsilon,t)\Delta t$ plane.

8 Acknowledgment

This work was partially supported by ARO Grant DAAG-55-98-1-0364 and NSF DMS-9704853. In addition, the authors would like to thank the Institute of Paper Science and Technology and the School of Mathematics at the Georgia Institute of Technology for their support.

References

[1] V.V. Bolotin. *The Dynamic Stability of Elastic Systems.* Holden-Day, 1964.

[2] S.S. Chen. Dynamic stability of tube conveying fluid. *Journal of the Engineering Mechanics Division ASCE*, 97:1469–1485, 1971.

[3] P. Hagedorn. *Non-linear Oscillations.* Oxford University Press, 1988.

[4] J.K. Hale. *Oscillations in Nonlinear Systems.* Dover, 1992.

[5] J.K. Hale and H. Koçak. *Dynamics and Bifurcations.* Springer-Verlag, 1991.

[6] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, 1974.

[7] L. Meirovitch. *Methods of Analytical Dynamics.* McGraw-Hill, 1988.

[8] A.H. Nayfeh and B. Balachandran. *Applied Nonlinear Dynamics - Analytical, Computational, and Experimental Methods.* John Wiley & Sons, 1995.

[9] M.P. Paidoussis and N.T. Issid. Dynamic stability of pipes conveying fluid. *Journal of Sound and Vibration*, 33(3):267–294, 1974.

[10] J.C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations.* Wadsworth & Brooks/Cole Advanced Books & Software, 1989.

[11] S.P. Timoshenko and J.M. Gere. *Theory of Elastic Stability.* McGraw-Hill, 1988.

[12] F. Verhulst. *Nonlinear Differential Equations and Dynamical Systems.* Springer, 1996.

[13] X. Wang. Numerical analysis of moving orthotropic thin plates. *Computers & Structures*, 1998. In press.

[14] X. Wang and F. Bloom. Dynamics of a submerged and inclined concentric pipe system with internal and external flows. *Journal of Fluids and Structures*, 1998. In press.

[15] X. Wang and F. Bloom. Stability issues of concentric pipes conveying steady and pulsatile fluids. *Journal of Fluids and Structures*, 1998. Submitted.

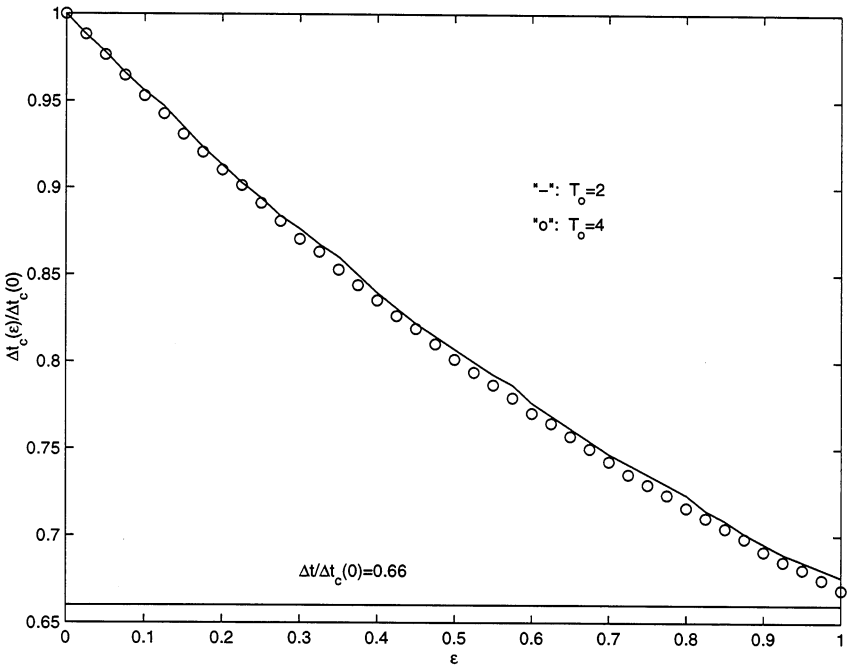

Figure 1: The effects of the perturbation term on the critical time step $\Delta t_c(\epsilon)$, which corresponds to $||G_2||_{sup} = 1$ of the first example with $\Delta t_c(0) = 0.2114$.

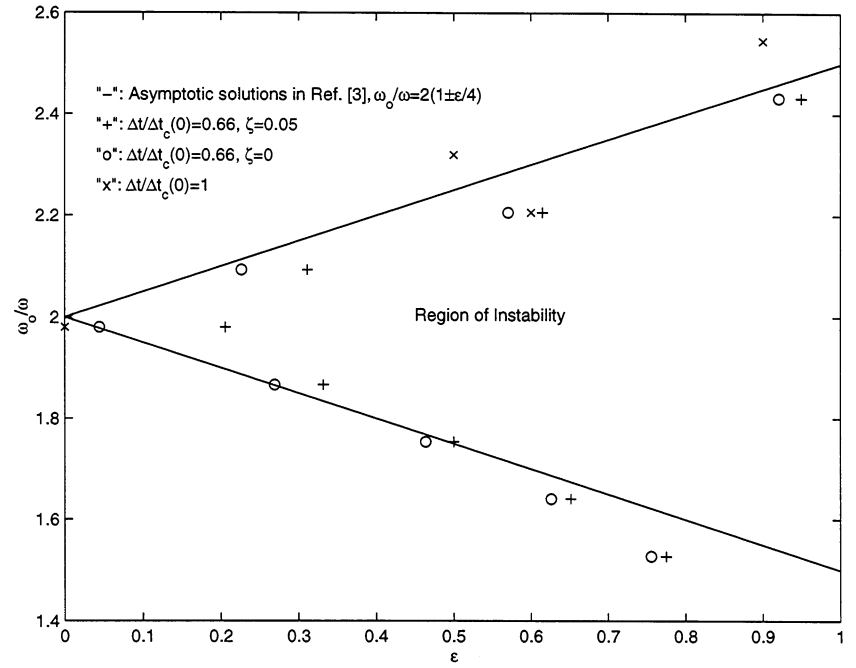

Figure 2: The regions of instability of the first example problem. ($\omega^2 = 2$ and 10×10 divisions within the parameter space of $\epsilon \in [0, 1]$ and $\omega_o \in [2.0, 3.6]$)

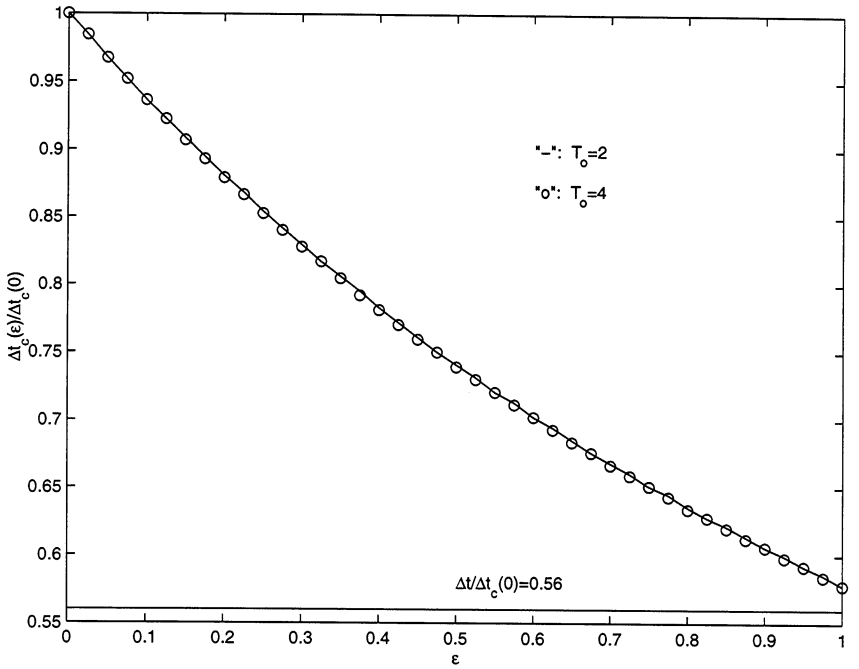

Figure 3: The effects of the perturbation term on the critical time step $\Delta t_c(\epsilon)$, which corresponds to $\|G_4\|_{sup} = 1$ of the second example with $\Delta t_c(0) = 0.1003$.

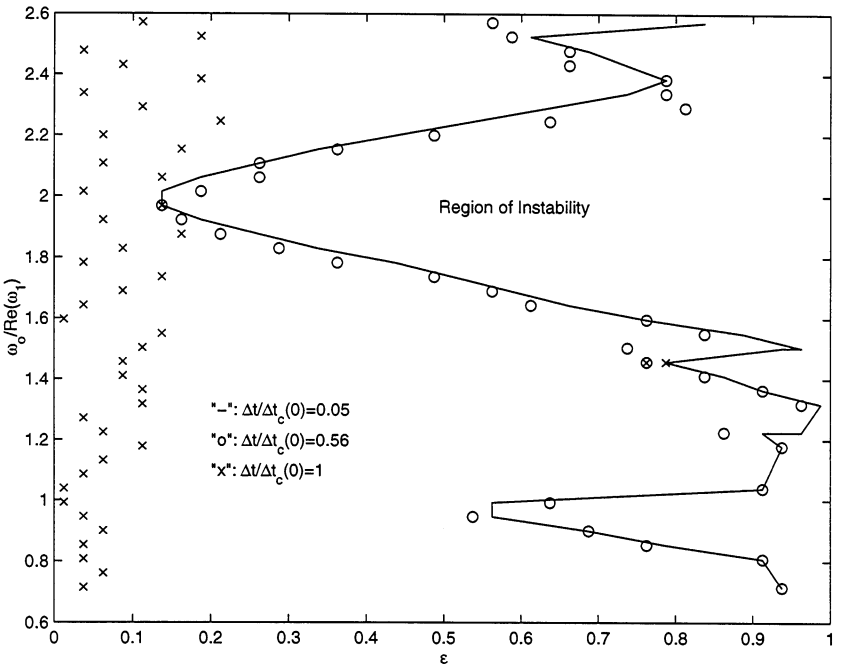

Figure 4: The regions of instability of the second example problem. (40×40 divisions within the parameter space of $\epsilon \in [0, 1]$ and $\omega_o \in [1.0, 3.6]$)

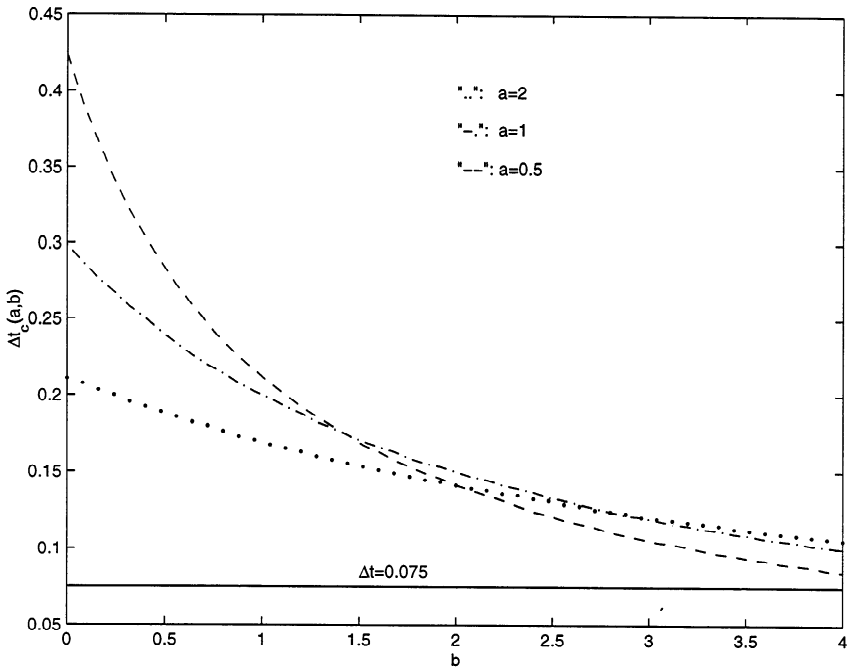

Figure 5: The effects of the perturbation term on the critical time step $\Delta t_c(a,b)$, which corresponds to $\|G_2\|_{sup} = 1$ of the third example with $\Delta t_c(2,0) = 0.2111$, $\Delta t_c(1,0) = 0.2989$, and $\Delta t_c(0.5,0) = 0.4233$.

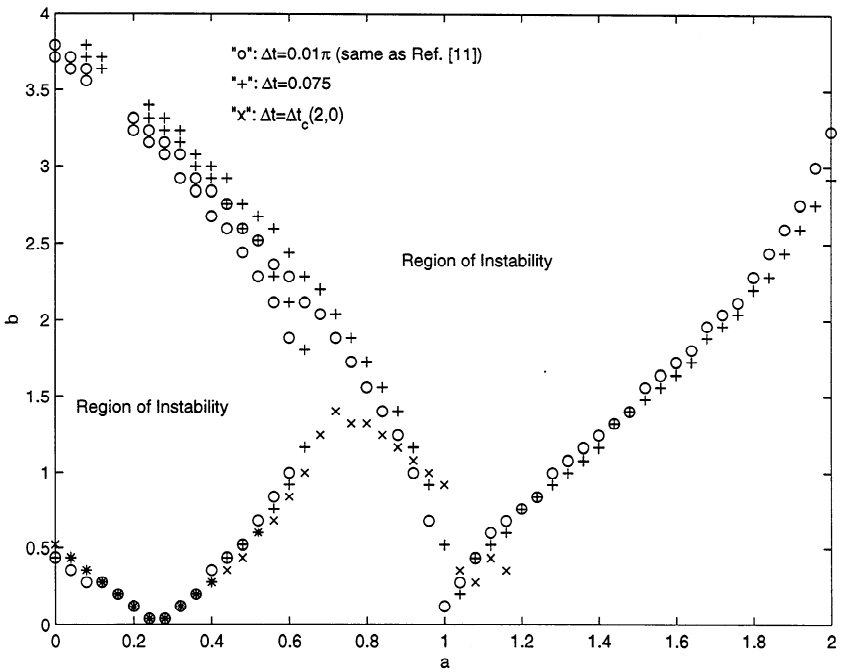

Figure 6: The regions of instability of the third example problem. (50×50 divisions within the parameter space of $a \in [0,2]$ and $b \in [0,4]$)