QMath13, 10th October 2016

Classification theory of topological insulators with Clifford algebras and its application to interacting fermions

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GORDON AND BETTY
MOORE
FOUNDATION

Collaborators

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- Christopher Mudry (PSI)

Morimoto, Furusaki, PRB 88, 125129 (2013)
Morimoto, Furusaki, Mudry, Phys. Rev. B 91, 235111 (2015)

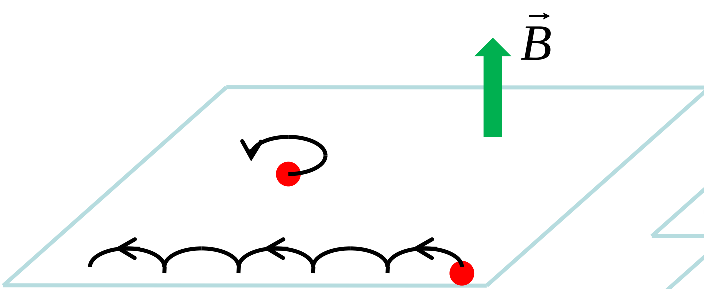
Plan of this talk

- Introduction
  - Topological insulators and superconductors
  - Ten fold way classification
- Classification theory of topological insulators
  - Massive Dirac Hamiltonian
  - Clifford algebras and classifying spaces
  - Application to topological crystalline insulators
- Breakdown of ten fold way classification with interactions
  - Dynamical mass terms
  - Nonlinear sigma model

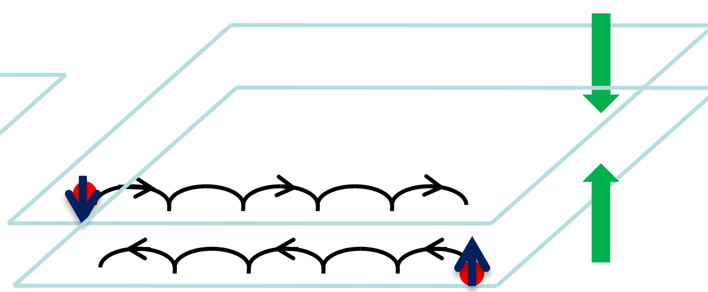
Topological insulator/superconductor is :

- A system of non-interacting fermions with a band gap.
  - Band insulator
  - Superconductor with a full gap (BdG equation)
- Characterized by a non-trivial topological number (Z or Z2).
- Accompanied with a gapless surface state.

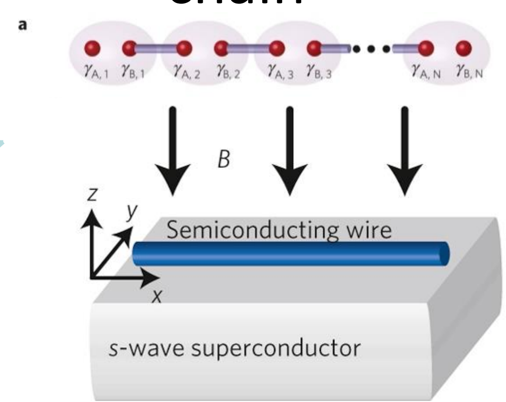
IQHE



QSHE



Majorana chain

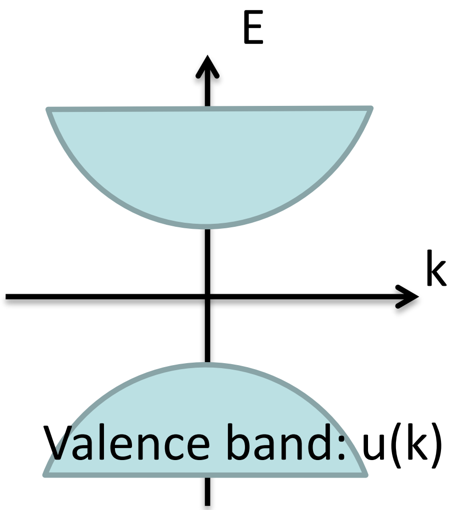


Topology of energy band

Energy band $\quad k \rightarrow E(k)$

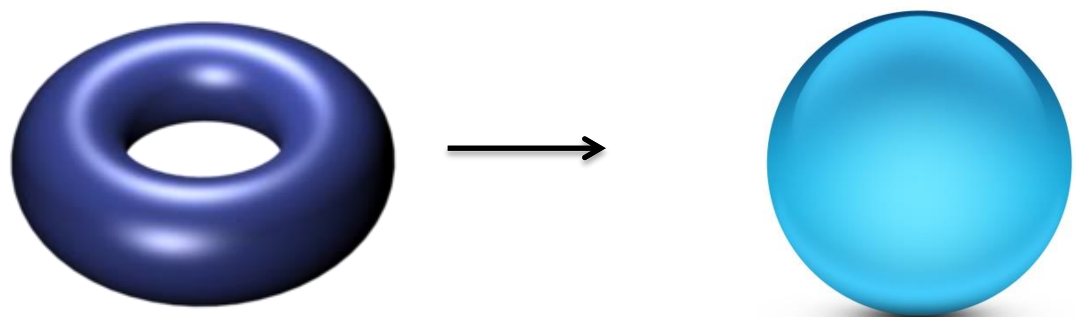
Bloch wavefunction $\quad k \rightarrow |u(k)>$

<u>Insulator</u>



Brillouin zone

Valence wavefunctions



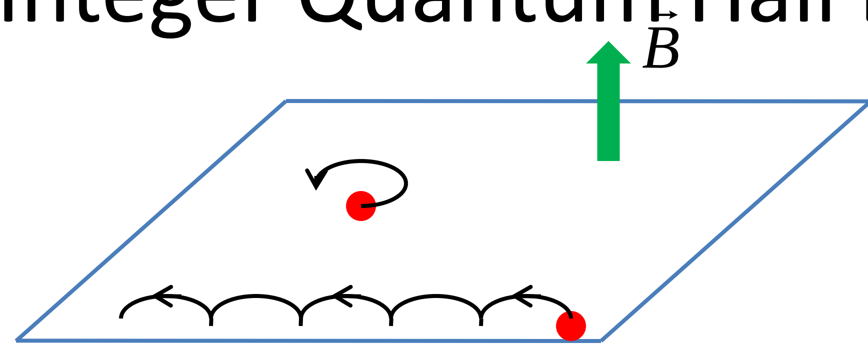
k

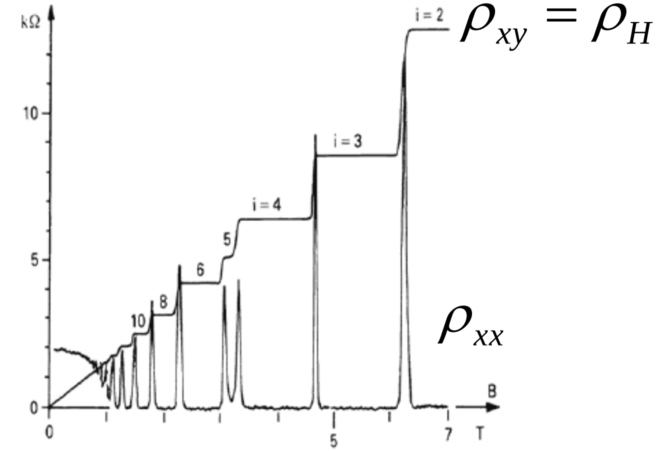
|u(k) >

Non trivial way that the Brillouin zone wraps the space of valence wavefucntion.

= Topological insulators and superconductors

Integer Quantum Hall Effect





TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

TKNN (1982); Kohmoto (1985)

1st Chern number

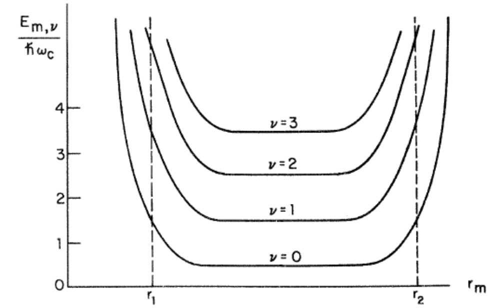
integer valued

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k \; \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$ = number of edge modes crossing E_F

bulk-edge correspondence

$$\vec{A}(k_x, k_y) = \langle \vec{k} | \vec{\nabla}_k | \vec{k} \rangle$$ Berry connection

$$\vec{\nabla}_k = \left(\partial_{k_x}, \partial_{k_y}\right)$$



- ☐ Systematic understanding of topological phases?
- ☐ Relationships to the symmetry and the dimensionality?

- ● A system of non-interacting fermion is classified into 10 Altland-Zirnbauer classes
- ● 5 classes of non-trivial TI/TSC for each dimension

Table of topological insulators/superconductors

		TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	IQHE --
	AI (orthogonal)	+1	0	0	--	-- QSHE	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2 Z_2TPI
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z polyacetylene (SSH)	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	p SC Z_2	Z p+ip SC	--
	C (d-wave SC)	0	-1	0	--	Z d+id SC	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z ^{3}He-B
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Schnyder, Ryu, Furusaki, and Ludwig, PRB (2008)

Ten Altland-Zirnbauer symmetry classes

Bilinear Hamiltonian: $$\mathcal{H} = \Psi^\dagger H \Psi$$

Fully block-diagonalized Hamiltonian matrix

Three generic symmetries:

-- Time-reversal symmetry ← Without B or magnetization

$$THT^{-1} = H \qquad TiT^{-1} = -i$$

-- Particle hole symmetry ← BdG equation, (superconductors)

$$CHC^{-1} = -H \qquad CiC^{-1} = -i$$

-- Chiral symmetry ← Sublattice symmetry, combination of TC

$$\Gamma H = -H\Gamma \qquad \Gamma i \Gamma^{-1} = i$$

class	T	C	Γ
A	0	0	0
AIII	0	0	1
AI	+1	0	0
BDI	+1	+1	1
D	0	+1	0
DIII	−1	+1	1
AII	−1	0	0
CII	−1	−1	1
C	0	−1	0
CI	+1	−1	1

Derivation of the topological periodic table: Dirac Hamiltonian and topological phase

$$H = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$$

(Assumption: Any gapped Hamiltonian can be deformed into the Dirac form)

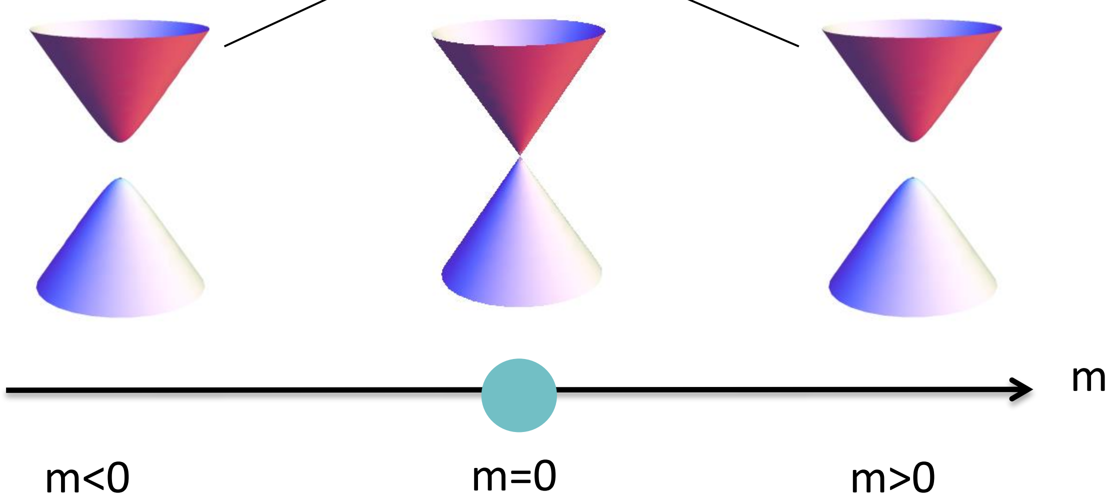
Gapped phase

⇕

Massive Dirac Hamiltonian

- If Dirac mass $\gamma 0$ is unique,

Topologically distinct phases



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Gapped phase

⇕

Massive Dirac Hamiltonian

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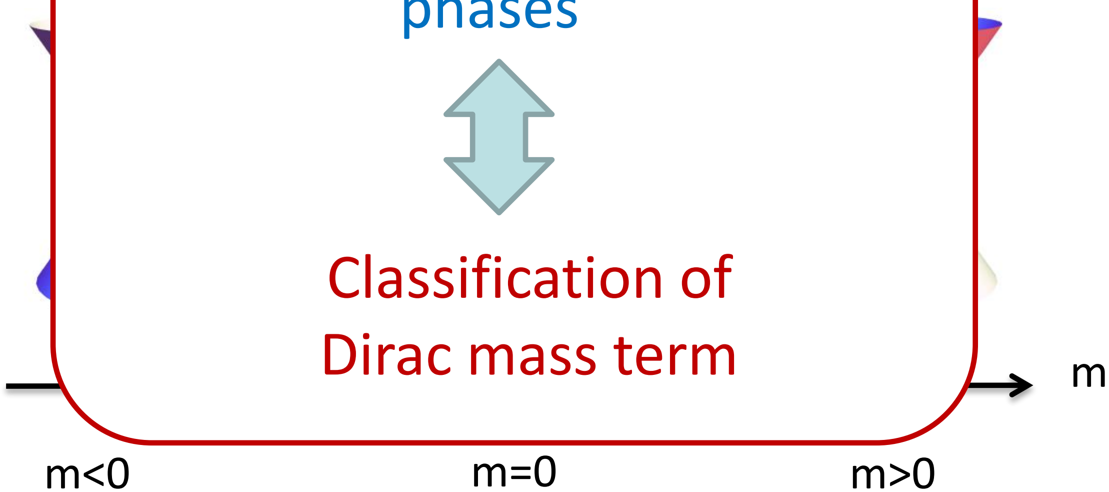
Classification of topological phases

⇕

Classification of Dirac mass term



Clifford algebras

Complex Clifford algebra: Cl_n

n generators: e_i $\{e_i, e_j\} = 2\delta_{i,j}.$

Real Clifford algebra: $Cl_{p,q}$

p+q generators: e_i

$$\{e_i, e_j\} = 0 \quad i \neq j$$

$$e_i^2 = \begin{cases} -1 & (1 \leq i \leq p) \\ +1 & (p+1 \leq i \leq p+q) \end{cases}$$

$2^{(p+q)}$ –dim real vector space spanned by bases of combinations of e_i's

Clifford algebra of real symmetry classes

$$H(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$$

Time-reversal symmetry T $\quad \{T, \gamma_i\} = [T, \gamma_0] = 0, \quad T^2 = \pm 1$

Particle-hole symmetry C $\quad [C, \gamma_i] = \{C, \gamma_0\} = 0, \quad C^2 = \pm 1$

T and C are antiunitary $\quad \{T, i\} = \{C, i\} = 0$

J represents for "i", $\quad J^2 = -1$

(i) T only (AI & AII): $e_0=J\gamma_0$, $e_1=T$, $e_2=TJ$, $e_3=\gamma_1$, … , $e_{2+d}=\gamma_d$

(ii) C only (D & C): $e_0=\gamma_0$, $e_1=C$, $e_2=CJ$, $e_3=J\gamma_1$, … , $e_{2+d}=J\gamma_d$

(iii) T and C (BDI, DIII, CII & CI): $e_0=\gamma_0$, $e_1=C$, $e_2=CJ$, $e_3=TCJ$, $e_4=J\gamma_1$, … , $e_{3+d}=J\gamma_d$

Symmetry constraints for class D:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \quad [C, \gamma_i] = \{C, \gamma_0\} = 0, \quad \{C, i\} = 0$$

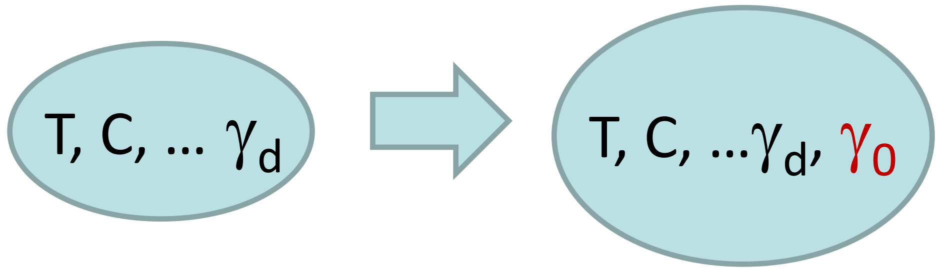
$$e_0 = \gamma_0, \quad e_1 = C, \quad e_2 = iC, \quad e_3 = i\gamma_1, \dots, e_{2+d} = i\gamma_d$$

Classification of topological insulators

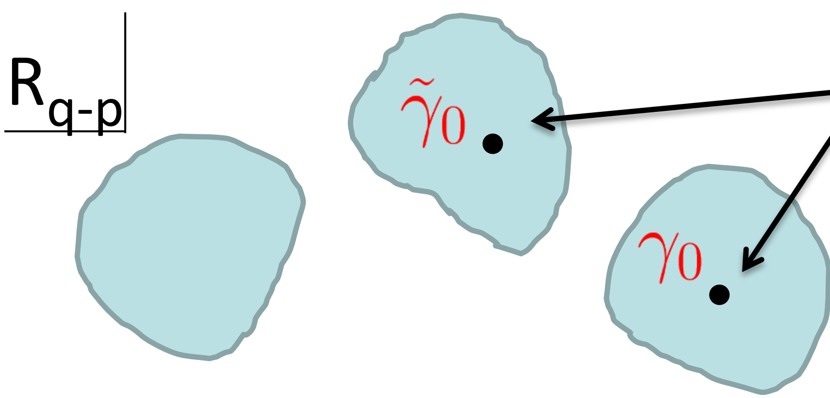
= Classification of Dirac mass term

(i) We consider extension of Clifford algebra without γ_0 into algebra with γ_0

$$Cl_{p,q} \rightarrow Cl_{p,q+1}$$



(ii) All the possible extensions ("classifying space") = space of γ_0 R_{q-p}



γ_0 and $\tilde{\gamma}_0$ in disconnected parts

➔ Distinct topological phases

(iii) Topological classification = $\pi_0(R_{q-p})$

Kitaev 09

				(i)	(ii)	(iii)
class	T	C	Γ	extension	V	$\pi_0(V_{d=0})$
A	0	0	0	$Cl_d \to Cl_{d+1}$	C_{0+d}	$\mathbb{Z}$
AIII	0	0	1	$Cl_{d+1} \to Cl_{d+2}$	C_{1+d}	0
AI	$+1$	0	0	$Cl_{0,d+2} \to Cl_{1,d+2}$	R_{0-d}	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$Cl_{d+1,2} \to Cl_{d+1,3}$	R_{1-d}	$\mathbb{Z}_2$
D	0	$+1$	0	$Cl_{d,2} \to Cl_{d,3}$	R_{2-d}	$\mathbb{Z}_2$
DIII	-1	$+1$	1	$Cl_{d,3} \to Cl_{d,4}$	R_{3-d}	0
AII	-1	0	0	$Cl_{2,d} \to Cl_{3,d}$	R_{4-d}	$\mathbb{Z}$
CII	-1	-1	1	$Cl_{d+3,0} \to Cl_{d+3,1}$	R_{5-d}	0
C	0	-1	0	$Cl_{d+2,0} \to Cl_{d+2,1}$	R_{6-d}	0
CI	$+1$	-1	1	$Cl_{d+2,1} \to Cl_{d+2,2}$	R_{7-d}	0

Bott periodicity

Complex and real K-theory

Topological periodic table from Clifford algebra

$\pi_0(C_{q-d})$

Cartan	d 0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	period
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	d = 2
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	period
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	d = 8
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

$\pi_0(R_{q-d})$

Topological crystalline insulator

- Topological insulator with

  time-reversal + Reflection symmetry

  ($Z_2 \rightarrow Z$)

  -- SnTe compounds

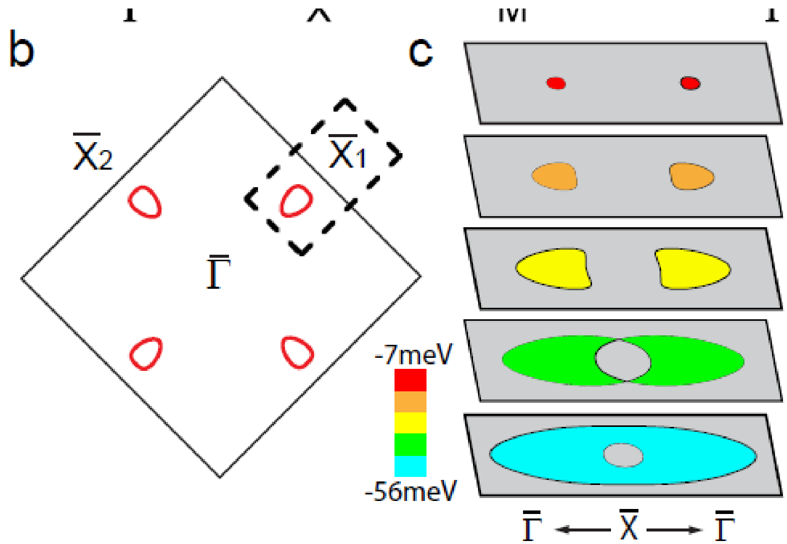
  with even # of surface Dirac cones

Experiments （SnTe/PbTe）:

Tanaka et al., Nat. Phys. 2012

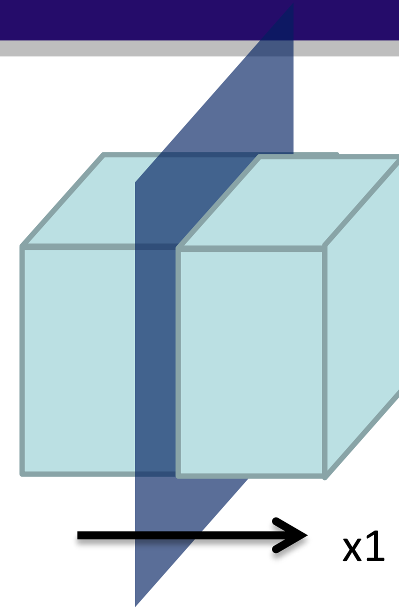
Xu et al., Nat. Commun. 2012

Dziawa et al., Nat. Mat. 2012



Hsieh et al. Nat. Commun. 2012

Reflection symmetry



Reflection symmetry with 1-direction

$$R^{-1}H(-k_1,k_i)R = H(k_1,k_i)$$

Dirac Hamiltonian $$H = m\gamma_0 + \sum_{i=1}^{d} k_i\gamma_i,$$

$$\{R,\gamma_1\}=0, \quad [R,\gamma_i]=0 \quad (i\neq 1)$$

R^{η_T,η_C} gives an additional chiral symmetry M

$$M = i\gamma_1 R^{\eta_T,\eta_C}$$ anti-commutes with all γ_i's

→ New generator of Clifford algebra

Topological periodic table with a reflection symmetry

$$R^{\eta_T} T = \eta_T T R^{\eta_T} \quad \text{and} \quad R^{\eta_C} C = \eta_C C R^{\eta_C}$$

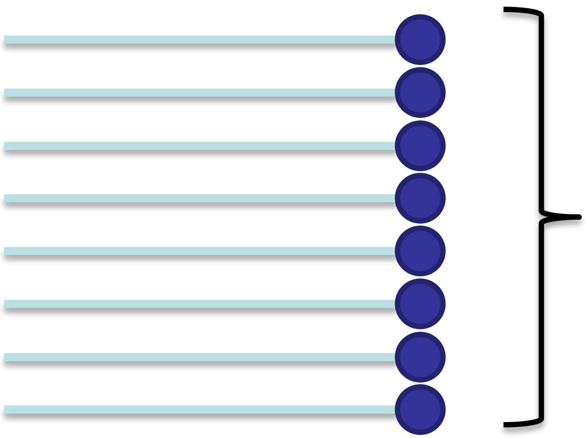
Reflection	Class	C_q or R_q	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
R	A	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^+	AIII	C_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R^-	AIII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	R_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
R^+	D	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	DIII	R_4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
R^{++}	AII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	R_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	AI	R_7	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$
R^-	D	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"
	DIII	R_2	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
R^{--}	AII	R_3	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	R_4	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	R_5	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	R_6	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0
R^{+-}	BDI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
R^{-+}	DIII	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^{+-}	CII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
R^{-+}	CI	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
R^{-+}	BDI, CII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+-}	DIII, CI	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

SnTe
(class AII+R^- d=3)

Interaction effects on topological insulators and superconductors

Breakdown of non-interacting topological phases labeled by Z with interactions

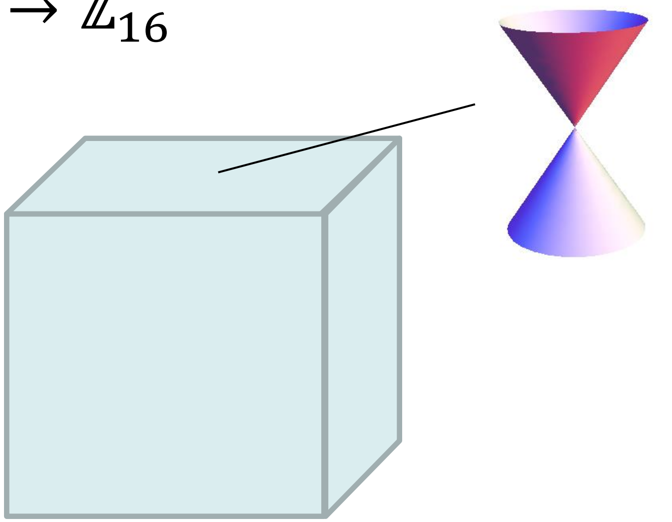
Time-reversal symmetric Majorana chain (1D class BDI)
$\mathbb{Z} \to \mathbb{Z}_8$



8 Majorana zero modes at the boundary can be gapped without breaking TRS.

Fidkowski and Kitaev, PRB (2010), PRB (2011)

Time-reversal symmetric 3D topological SC (3D class DIII)
$\mathbb{Z} \to \mathbb{Z}_{16}$



16 Dirac surface fermions can be gapped without breaking TRS.

Kitaev (2011), Fidkowski etal. PRX (2013), Metlitski, Kane & Fisher. (2014), ….

Aim: Systematic study of the breakdown of Z classification

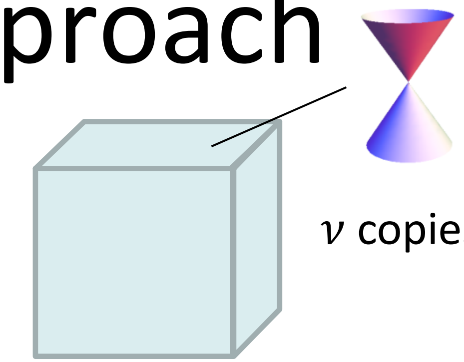
- Stability analysis of boundary gapless states against interactions in any dimension and any symmetry class
  ← Nonlinear sigma model

- Applications to the tenfold way and other topological phases (topological crystalline insulators)

Result:

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

Nonlinear sigma model approach



ν copies of gapless boundary states

Boundary massless Dirac fermions + quartic interactions

$$\mathcal{L}_{\mathrm{bd}} := \Psi^{\dagger}\left(\partial_{\tau} + \mathcal{H}^{(0)}_{\mathrm{bd}}\right)\Psi + \lambda \sum_{\{\beta\}} \left(\Psi^{\dagger}\beta\,\Psi\right)^{2}.$$

$$\mathcal{H}^{(0)}_{\mathrm{bd}} = \sum_{i=1,\ldots,d-1} (-i)\partial_i \alpha_i \qquad \{\beta_1, \beta_2, \ldots, \beta_N\}$$

α, β: anti-commuting gamma matrices

α respects symmetries, β can be odd under some symmetry operations.

Hubbard-Stratonovich transformtation

$$\mathcal{L}'_{\mathrm{bd}} := \Psi^{\dagger}\left(\partial_{\tau} + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}\right)\Psi + \frac{1}{\lambda}\sum_{\{\beta\}} \phi_{\beta}^{2},$$

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(\tau, \boldsymbol{x}) := \mathcal{H}^{(0)}_{\mathrm{bd}}(\boldsymbol{x}) + \sum_{\{\beta\}} \underline{2\mathrm{i}\,\beta\,\phi_{\beta}(\tau, \boldsymbol{x})},$$

Dynamical Dirac masses

cf. You and Xu, PRB (2014), Kitaev's talk (2015)

Integration of fermions

$$\mathcal{S}_{\mathrm{eff}}[\boldsymbol{\phi}] := (-1)\mathrm{Tr}\,\log\left[\partial_\tau + \sum_{j=1}^{d-1}(-\mathrm{i}\partial_j)\,\alpha_j + \sum_{\{\beta\}} 2\mathrm{i}\beta\,\phi_\beta\right] + \frac{1}{\lambda r}\sum_{\{\beta\}}\mathrm{Tr}\,(\phi_\beta^2).$$

Saddle point approximation

+ including fluctuations about the direction in which ϕ freezes

Nonlinear sigma model + topological term

Abanov, Wiegmann
Nucl. Phys. B (2000)

$$Z_{\mathrm{bd}} \approx \int \mathcal{D}[\boldsymbol{\phi}]\,\delta(\boldsymbol{\phi}^2 - 1)\,e^{-S_{\mathrm{QNLSM}}\boxed{-S_{\mathrm{top}}}},$$

$$S_{\mathrm{QNLSM}} = \frac{1}{2\,g}\int \mathrm{d}\tau \int \mathrm{d}^{d-1}\boldsymbol{x}\;(\partial_i\boldsymbol{\phi})^2$$

$\boldsymbol{\phi} \in S^{N(\nu)-1}$ Target space of NLSM is a sphere generated by $N(\nu)$ anticommuting dynamical masses β's

Topological term in NLSM

The presence or absence of a topological term is determined by the homotopy group of the target space.

$\pi_0(S^{N(\nu)-1}) \neq 0$ domain wall

$\pi_1(S^{N(\nu)-1}) \neq 0$ vortex

$\pi_d(S^{N(\nu)-1}) \neq 0$

$\pi_{d+1}(S^{N(\nu)-1}) \neq 0$ Wess-Zumino term

Nontrivial homotopy group

⇓

Topological term in NLSM

⇓

Boundary states remain gapless

Condition for the breakdown

$$\pi_D(S^{N(\nu)-1}) = 0 \quad \text{for } D = 0, \ldots, d+1$$

Topological defects in the dynamical mass bind fermion zero-energy states.

$\nu_{\min}$: the minimum ν satisfying the above condition, $\mathbb{Z} \to \mathbb{Z}_{\nu_{\min}}$

Example: 3D class DIII (3He-B phase)

$\nu = 1$

Bulk: $\mathcal{H}^{(0)}(\boldsymbol{x}) := -\mathrm{i}\partial_1 X_{31} - \mathrm{i}\partial_2 X_{02} - \mathrm{i}\partial_3 X_{11} + m(\boldsymbol{x}) X_{03}.$

$\mathcal{T} := \mathrm{i}X_{20}\,\mathsf{K},$
$\mathcal{C} := X_{01}\,\mathsf{K}.$

Boundary: $\mathcal{H}^{(0)}_{\mathrm{bd}}(x,z) = -\mathrm{i}\partial_x \tau_3 - \mathrm{i}\partial_z \tau_1,$

$\mathcal{T}_{\mathrm{bd}\,\nu} := \mathrm{i}\tau_2 \otimes \mathbb{1}\,\mathsf{K}$
$\mathcal{C}_{\mathrm{bd}\,\nu} := \tau_0 \otimes \mathbb{1}\,\mathsf{K}$

ν copies

Dynamical mass: $\tau_2 \otimes \underline{M(\tau, x, z)}$

$\nu \times \nu$ Real symmetric matrix

Dynamical masses break T, but preserves C.

Space of masses (2D class D)
= real Grassmannians:

$R_0 = \bigcup_{k=1}^{\nu} O(\nu)/\left[O(k) \times O(\nu-k)\right]$

v=1: $M = \pm 1$

v=2: $M = X_1, X_3$

v=4: $M = X_{13}, X_{33}, X_{01}$

v=8: $M = X_{133}, X_{333}, X_{013}, X_{001}, X_{212}$

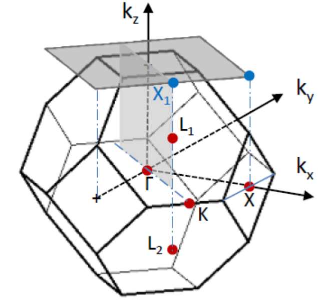
D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex line
2	$\mathbb{Z}_2$	4	Monopole
3	0		
4	$\mathbb{Z}$	8	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	$\boxed{16}$	None

Higher dimensions

- Z_2 entries are stable.
- Z in even dimensions is stable.
- Z in odd dimensions is unstable.

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	$+1$	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$+1$	$+1$	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	$+1$	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	$+1$	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	-1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	-1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	$+1$	-1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

3D topological crystalline insulator (SnTe) (TRS+ reflection → Z classification)



Hsieh et al. 2012

Boundary: $\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})}(x,y) = -\mathrm{i}\partial_x\,\sigma_2 - \mathrm{i}\partial_y\,\sigma_1 + M(\tau,x,y)\,\sigma_3$

BdG: $\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})} = \left(-\mathrm{i}\partial_x\,\sigma_2\otimes\rho_3 - \mathrm{i}\partial_y\,\sigma_1\otimes\rho_0\right)\otimes\mathbb{1} + \gamma'(\tau,x,y),$

$R_0 = \bigcup^{\nu} O(\nu)/\left[O(k)\times O(\nu-k)\right]$

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$		
1	$\mathbb{Z}_2$	1	Vortex
2	$\mathbb{Z}_2$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	$\boxed{8}$	None

Minimal TCI
= 2copies of 3He-B phase

$\mathbb{Z} \to \mathbb{Z}_8$

Summary and outlook

- Classification of topological crystalline insulators
  - General spatial symmetry that cannot fit into Clifford algebras? Twisted equivariant K-theory?

  T. Morimoto, A.Furusaki, Phys. Rev. B 88, 125129 (2013).

- Effects of interactions on topological insulators
  - Nonlinear sigma model analysis over spherical target spaces
  - More general interactions?
  - New topological phases that emerges with interactions?

  T. Morimoto, A.Furusaki, C.Mudry, Phys. Rev. B 92, 125104 (2015)