

**INFINITE DIMENSIONAL NONLINEAR SYSTEMS
WITH STATE-DEPENDENT DELAYS AND STATE
SUPREMA: ANALYSIS, OBSERVER DESIGN AND
APPLICATIONS**

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INFINITE DIMENSIONAL NONLINEAR SYSTEMS WITH STATE-DEPENDENT DELAYS AND STATE SUPREMA: ANALYSIS, OBSERVER DESIGN AND APPLICATIONS

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To My Parents.

PREFACE

The dissertation revolves around a class of infinite dimensional systems characterized by state dependent delays or state suprema. Systems with state-dependent delays are ubiquitous in a lot of applications from physics, engineering, biology, ecology and epidemiology. The delay is not constant but is modulated by the state of the system itself. This feature makes the problem challenging. The difficulty lies in characterizing the state space and defining the Cauchy problem. Though systems with constant delays have been extensively studied, the literature on state-dependent delays is very sporadic. A unanimous framework for the analysis and design of systems with state-dependent delays is still lacking. One of the advantages of such systems is to use the delay information and invert it to recover the state of the system. This leads to observer design. The work carried out in this dissertation is a step towards the analysis and observer design for systems with state-dependent delays.

One needs to be very careful with variable delays whether time-varying or state dependent. The system no longer remains causal when the delay rate surpasses unity. Physically such systems make no sense. It is therefore mandatory to assume that the delay rate is less than unity in order to ensure the causality, consistency and well-posedness of the system.

There are systems where the evolution of the state depends not only on the current state of the system but also on the supremum of the state over a history of a finite or infinite memory length. Such infinite dimensional systems are a special class of systems with state-dependent delays. The appearance of the supremum operator makes them nonlinear. Here an effort is made to nail down the basic structure of such systems and solve the controller synthesis and observer design problems. The

discrete version is also studied and it is shown that such systems exhibit a character of auto-hybrid Multi-Mode Multi-Dimensional (M^3D) systems.

Practical applications are used to accompany the underlying theory of the controller synthesis and observer design of systems with state-dependent delays and state suprema. These include observer design for a subsonic rocket car, temperature control of a fluid in a tank, machine tool turning process, ocean navigation and analysis of gene regulatory networks.

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SUMMARY

The prime objective of this research is to investigate systems with state-dependent delays in a unified, well rounded, global and coherent framework from the very basic first principles. Such systems are inherently *nonlinear* and *infinite dimensional* in nature. The delay may either depend *explicitly* or *implicitly* on the state of the system. Our goal and contribution is four-fold.

Firstly, to give an information structure i.e., to define the Cauchy problem and characterize the state space for such systems. The state space should be a stationary construct, it should be fixed once and for all, and should encode the minimal sufficient information (statistic) for the evolution of the system. We use the framework of Banach function spaces with the topology of uniform convergence. Once the state space is well defined, stability analysis and controller synthesis is accomplished. In systems with time-varying and/or state-dependent delays, causality plays a crucial role and is lost when the delay rate exceeds unity. Throughout the analysis and synthesis problems in this research, causality of the system is ensured by keeping the time evolution rate of the delay less than unity.

Secondly, we perform the inversion of the state-dependent delay i.e., we use the information on the delay and retrieve the state vector of the system. We use our newly established technique of *Delay Injection* to recover the state of the system and solve the observer design problem.

Thirdly, we analyze and design controllers and observers for a special class of systems with state-dependent delays namely systems evolving with state suprema. We nail down the rich structure possessed by these systems. First we define the state space for such systems and then solve the controller synthesis and the observer

design problems. We also show that these systems can be expressed as *Multi-Mode Multi-Dimensional* (M^3D) systems.

Fourthly, we investigate the spectrum of higher order linear time delay systems in the framework of matrix Lambert W functions and give some *counter examples* to show that the already existing well established literature suffers from some discrepancies and limitations.

We support our theory with practical applications including estimator design for a subsonic rocket car for soft landing, nonlinear observer design for a turning process for high precision machining, observer design for the temperature control of a tank with state-dependent delay and analysis of gene expression. Throughout our analysis and synthesis problems, we use the exact models and nothing such as linearization or approximations by truncated Taylor series are used.

Glossary

Sets, Symbols and Operators

t	Continuous time
$s.t.$ or $ $	such that
$Co\{.\}$	Convex hull of $\{.\}$
\mathbb{R}^n	n -dimensional Euclidean space
\mathbb{R}^+	The set of real positive numbers
$\mathbb{R}^{n \times m}$	The set of all $n \times m$ real matrices
\mathbb{Z}	The set of integers
\mathbb{N}	The set of natural numbers or positive integers
$a \in \mathbb{A}$	a is an element or a member of set \mathbb{A}
\overline{A}	The closure of set A
\forall	For all
\Longleftrightarrow	If and only if
$\ .\ $	Vector or matrix norm
$\mathcal{R}(T)$	Range of the operator T
$\mathcal{N}(T)$	Null space or kernel T , same as $\ker(T)$
$\dim(X)$	Dimension of the space X
$\mathcal{C}([a, b]; \mathbb{R}^n)$	The Banach space of continuous functions mapping $[a, b]$ to \mathbb{R}^n
$\mathcal{C}^1([a, b]; \mathbb{R}^n)$	The space of continuously differentiable functions from $[a, b]$ to \mathbb{R}^n
$\mathcal{C}^{0,1}([a, b]; \mathbb{R}^n)$	The space of Lipschitz continuous functions $f : [a, b] \mapsto \mathbb{R}^n$
$\mathcal{W}^{k,p}([a, b]; \mathbb{R}^n)$	The Sobolev space with k -derivatives and p -th norm, $f : [a, b] \mapsto \mathbb{R}^n$
\mathcal{C}^ω	The space of analytic functions
\mathcal{H}	Hilbert space
$\ker(T)$	Kernel of the Operator T , same as $\mathcal{N}(T)$
$\sigma(T)$	Spectrum of the Operator T
$\sigma_p(T)$	Point Spectrum of the Operator T
$\sigma_c(T)$	Continuous Spectrum of the Operator T
$\sigma_r(T)$	Residual Spectrum of the Operator T
$\rho(T)$	Resolvent Set of the Operator T
$r_\sigma(T)$	Spectral Radius of the Operator T
D	Differential or Derivative Operator, $\frac{d}{dt}(\cdot)$
σ_t	Evaluation Functional, $\sigma_t x := x(t)$
S_α	Scaling Operator, $\sigma_t(S_\alpha x) = \sigma_{\alpha t} x = x(\alpha t)$

Signal Conventions

$u(t)$	Control input
$x(t)$	State vector of the system
$\tau(x(t))$	State-dependent delay
$\dot{x}(t)$	Derivative of $x(t)$ with respect to time t , $\frac{dx}{dt}$
$y(t)$	Measured output (from sensor/sensors)
$z(t)$	Controlled output
$\tau(t)$	Time varying delay
$\dot{\tau}(t)$	Time derivative of $\tau(t)$
h	Upper bound on the time varying delay $\tau(t)$
d	Upper bound on the time derivative $\dot{\tau}(t)$ of the time varying delay
$x(t - \tau(t))$	Delayed state vector (delayed by an amount $\tau(t)$)
$\dot{x}(t - \tau(t))$	Time derivative of the delayed state vector
$x_t(\theta)$	$x(t + \theta) \quad \forall \theta \in [-h \quad 0]$
l^2	The Hilbert sequence space $(x_n)_{n \in \mathbb{N}}$ of all square summable sequences
l^∞	The Banach sequence space $(x_n)_{n \in \mathbb{N}}$ of all bounded sequences

Matrix-related Notations

A^T or A^\top	Transpose of matrix A
$A > 0$	The matrix A is positive definite
$A \geq 0$	The matrix A is positive semi-definite
$A < 0$	The matrix A is negative definite
$A \leq 0$	The matrix A is negative semi-definite
$A > B, A \geq B$	$A - B > 0, A - B \geq 0$
$*$ or \star	Block induced easily due to symmetry
$\lambda(A), \lambda_i(A)$	An eigenvalue and the i th eigenvalue of matrix A
$\lambda_{\max}(A)$	The maximum eigenvalue of matrix A
$\lambda_{\min}(A)$	The minimum eigenvalue of matrix A
$\sigma_{\max}(A)$	The upper singular value of matrix A ; $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^\top A)}$
$\ A\ $	The induced norm or spectral norm of $A = \sigma_{\max}(A)$
$\text{vec}(A)$	The vectorization of the matrix A
$A \otimes B$	The Kronecker product of matrices A and B

Abbreviations

DDE	Delay Differential Equation
FDE	Functional Differential Equation
SD-DDE	State-Dependent Delay Differential Equation
RFDE	Retarded Functional Differential Equation
NFDE	Neutral Functional Differential Equation
TDS	Time Delay System
CQLF	Common Quadratic Lyapunov Function
ALE	Algebraic Lyapunov Equation
DARE	Delay Algebraic Riccati Equation
LMI	Linear Matrix Inequality
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
LTI	Linear Time Invariant
LQR	Linear Quadratic Regulator
LBI	Lagrange-Bürmann Invesrion
CSTR	Continuous Stirred-Tank Reactor
DOF	Degrees of Freedom
$\infty - D$	Infinite Dimensional
3-D	Three Dimensional
AM-GM	Arithmetic Mean-Geometric Mean
AUV	Autonomous Underwater Vehicle
PBH	Popov-Belevitch-Hautus
UGHF	Unscented Gauss-Helmert Filter
OCF	Optimal Control Problem
QPmR	Quasi-Polynomial mapping based Root finder
LK	Lyapunov-Krasovskii
LR	Lyapunov-Razumikhin
SDS	Scale Dynamic/Delay System
M^3D	Multi-Mode Multi-Dimensional
MPPT	Maximum Power Point Tracking
RNA	Ribo-Nucleic Acid
mRNA	messenger RNA

CHAPTER I

INTRODUCTION & MOTIVATION

1.1 Introduction

Delays are ubiquitous in nature. Whenever there is a transport of material, energy or information, there always exists a lag or latency from the cause to effect or in the states. This latency is referred to as delay in the system. The dynamic behavior and the evolution of such systems cannot be adequately captured and described by Ordinary Differential Equations (ODEs). Neglecting the delays in the models of such systems translate to neglecting the physical realities. Time Delay Systems (TDS) also called hereditary systems or systems with after-effects are described by Functional Differential Equations (FDEs) or Delay Differential Equations (DDEs). The initial history of such systems does not lie as a point in a finite dimensional space \mathbb{R}^n but is a function in some suitable function space. As a result, TDS are inherently *infinite dimensional* systems.

Time delays can be constant (fixed) as we have experienced while taking shower or can be time-varying as in the internet congestion systems or vision based control loops. The delays can be present in the system input, output, state or any combination of these three. Over the last three decades, the literature on the analysis and design of systems with fixed or constant delays has witnessed a huge proliferation. Both state space and frequency domain based approaches have been used to study systems with fixed delays.

There are a lot of practical examples where the delay is not constant but varies

with the state of the dynamical system. We call such systems as systems with state-dependent delays and the associated DDEs or FDEs as State-Dependent Delay Differential Equations (SD-DDEs). These are generally considered as very hard problems. The major difficulty lies in defining the state space, the information structure and the associated Cauchy problem. The literature on systems with state-dependent delays is very sporadic and scattered and a general theory is still lacking. Also, many times in the literature, linearization is used to translated the system to linear DDEs with fixed delays, which is not correct! The linearized system with constant delay is not a true replica of the original nonlinear system with state-dependent delay. To the best of the knowledge of the author, there is no previous work on the observer design for systems with state-dependent delays. The purpose of the work in this thesis is to fill and bridge this gap.

The prime objective of this research is to investigate systems with state-dependent delays in a unified, well rounded, global and coherent framework from first principles. Such systems are inherently *nonlinear* and *infinite dimensional* in nature. The delay may either depend *explicitly* or *implicitly* on the state of the system. Our goal and contribution is four-fold. First, to give an information structure i.e., to define the *Cauchy Problem* (Initial Value Problem) and characterize the state space for such systems. The state space should be a stationary construct, it should be fixed once and for all, and should encode the minimal sufficient information (statistic) for the evolution of the system. We use the framework of Banach function spaces with the topology of uniform convergence. Once the state space is well defined, stability analysis and controller synthesis is accomplished. In systems with time-varying and/or state-dependent delays, causality plays a crucial role and is lost when the delay rate exceeds unity. Throughout the analysis and synthesis problems in this research, causality of the system is ensured by keeping the time evolution rate of the delay less than unity.

Secondly, we perform the inversion of the state-dependent delay i.e., we use the information on the delay and retrieve the state vector of the system. We use our newly established technique of *Delay Injection* to recover the state of the system and solve the observer design problem.

Thirdly, we consider a special class of systems with state-dependent delays namely systems evolving with state suprema. We nail down the rich structure possessed by these systems. The state space, information structure and the Cauchy problem are defined. First, we give sufficient conditions for the asymptotic stability of such systems and then we synthesize the controllers and observers for such systems. The discrete time counterpart of such systems is also analyzed.

Fourthly, we investigate the spectrum of higher order linear time delay systems in the framework of matrix Lambert W functions and give some *counter examples* to show that the already existing well established literature suffers from some discrepancies and limitations. We support our theory with practical applications including estimator design for a subsonic rocket car for soft landing, nonlinear observer design for a turning process for high precision machining, observer design for the temperature control of a tank with state-dependent delay and analysis of gene expression. Throughout our analysis and synthesis problems, we use the exact models and nothing such as linearization or approximations by truncated Taylor series are used.

1.2 Motivation

State-dependent delays can be found in a plethora of engineering applications. Here we describe three important practical examples which will also be used as motivational case studies in the next chapters. These examples illustrate scenarios where the delay depends *implicitly* or *explicitly* on the state of the system under consideration.

1.2.1 Subsonic Rocket Car (Soft Landing)

Consider the rocket car shown in Fig. 1. Let the instantaneous position of the rocket car w.r.t. the wall be denoted by $x_1(t)$ and its velocity be $x_2(t)$. Let the thrust of the engine be denoted by $u(t)$. Now, assuming unit mass, it follows from Newtonian dynamics that the rocket car can be modeled as,

$$\dot{x}_1(t) = x_2(t) \quad (1)$$

$$\dot{x}_2(t) = u(t) \quad (2)$$

Now the transmitter Tx of the ultrasonic sensor transmits a sound wave signal which travels with the speed of sound v , it is echoed by the wall and is detected by the receiver Rx of the sensor. Let $\tau(t)$ denote the instantaneous round trip delay between the transmission and reception of the pulse. Let us assume that the rocket car had the position $x_1(t)$ at the moment when the pulse was received by the receiver Rx. This means that the position of the rocket car at the moment when the sound signal was transmitted was $x_1(t - \tau)$. Thus, the round trip delay from transmission to reception can be expressed by,

$$\tau(t) = \frac{x_1(t) + x_1(t - \tau(t))}{v}. \quad (3)$$

Notice that here the delay τ is not only state-dependent but also implicitly related to the state of the system. The objective is to use only the information on the *implicitly* related delay τ and its derivatives to recover the state vector of the system in the presence of known and unknown inputs. The details are given in [2] and [4].

1.2.2 Temperature Control of a Tank

Consider the problem of controlling the temperature of a fluid in a tank as illustrated in Fig. 2. We see that there are two different pipes containing fluid at two different temperatures. We call the ends of these pipes as hot junctions and cold junction respectively. Let t represent an arbitrary instant of time. The fluid flowing in the

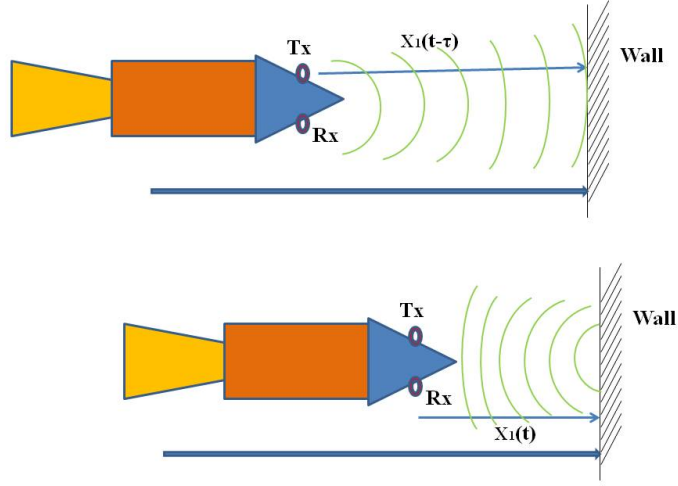


Figure 1: Positions of the rocket car at the the instants the ultrasonic signal is transmitted and received

hot junction is at temperature $T_h(t)$ whereas the fluid in the cold junction has a temperature $T_c(t)$ where $T_c(t) < T_h(t)$. The two fluids are mixed via a mixing valve and attain a temperature $T_v(t)$ at the output of the valve immediately after leaving the valve. The tank is located at a distance d from the valve. The temperature of the fluid in the tank is $T_0(t)$. Our objective is to control the temperature of the tank.

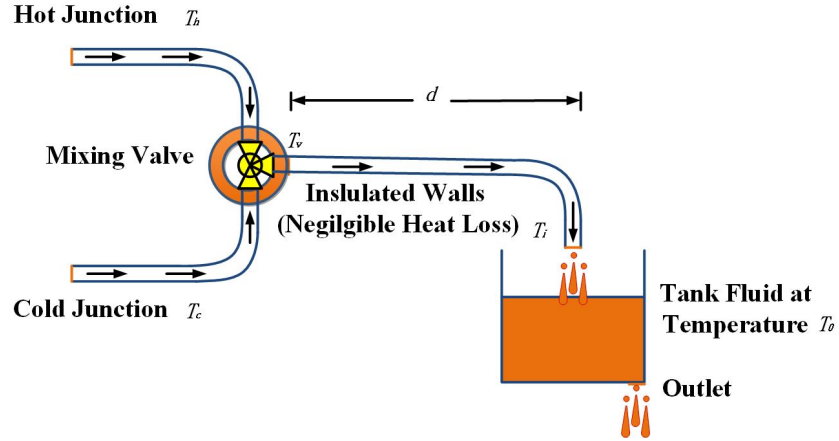


Figure 2: Temperature Control of a Fluid in a Tank

Because of the separation between the tank and the mixing valve and the fact that the velocity (flow rate) of the fluid is not infinite, the fluid takes a finite but nonzero

amount of time to reach the tank. This time τ is referred to as delay or more precisely the transport lag. Let $T_i(t)$ denote the temperature of the fluid entering the tank. This temperature at any instant t is the delayed version of the valve temperature at that particular time i.e.,

$$T_i(t) = T_v(t - \tau). \quad (4)$$

In this particular scenario, the control is achieved based on velocity or flow rate control. Let the velocity of the fluid be $q(t)$ at any time t . Here we control the tank temperature T_0 by means of velocity q that is $T_0 = f(q)$ and the actuator flow rate or velocity is varied by the output temperature i.e., $q = g(T_0)$. Notice that in this scenario, the delay no longer remains constant, rather it depends on the output temperature T_0 . Mathematically, we can write this delay as the ratio of the distance between sensor and actuating valve and velocity of the fluid i.e.,

$$\tau = \frac{d}{q(t)} = \frac{d}{g(T_0)}. \quad (5)$$

This equation shows that the delay is state-dependent. A simple and convenient choice would be to consider the velocity based control law as a linear or affine function of temperature i.e., $q(t) = g(T_0) = k_1 T_0 + k_2$ with k_1 and k_2 being constant gains. So, the output temperature dynamics are given as follows [9].

$$\dot{T}_0(t) = -\mu T_0(t) + \mu k T_0 \left(t - \frac{d}{g(T_0)} \right) \quad (6)$$

This motivates the analysis of the following state-dependent delay differential equation, where the delay is an *explicit* function of the state of the system.

$$\begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t - \tau(x(t))) \\ \tau(x) = \frac{d}{k_1 x(t) + k_2} \end{cases} \quad (7)$$

Here, $x(t) \in \mathbb{R}$ is the state and $\alpha, \beta \in \mathbb{R}$ are some constants. A more practical scenario is depicted below in Fig. 3. This shows a Continuous Stirred-Tank Reactor (CSTR) which is used as a famous ideal reactor test bench in chemical and process

engineering. This is also known as a vat reactor or backmix reactor. The top inlet delivers liquid to be mixed in the tank. The objective is to maintain a constant temperature by varying the amount of steam supplied to the heat exchanger (bottom pipe) via its control valve. Variations in the temperature of the inlet flow are the main source of disturbances in this process. See [9] for further details.

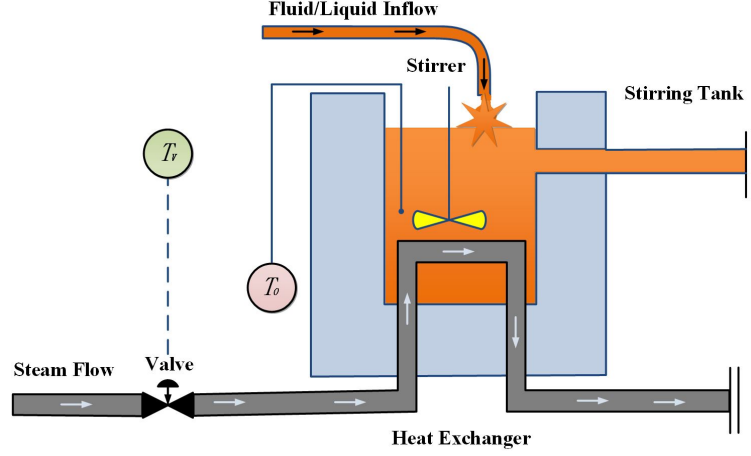


Figure 3: Continuous Stirred-Tank Reactor (CSTR)

1.2.3 Turning Process

Fig. 6 shows the standard schematic model of the turning process. In order to account for the machine tool vibrations (chatter phenomenon), the tool is modeled by a mass, spring and damper system. Fig. 4 gives a simple and conceptual diagram of the single Degree of Freedom (1-DOF) motion of the tool, clearly depicting the desired profile and the exaggerated actual turned profile. Because of the chatter of the machine tool, the workpiece (job) undergoes/suffers interrupted cutting. As a result the turned profile or surface is wavy. Fig. 5 portrays the Two Degree of Freedom (2-DOF) motion machine tool and an exaggerated view of the the regeneration mechanism in machine tool turning process as considered in [49].

We consider the 2-DOF motion of the machine tool as shown in Fig. 5. $U_x(t)$ and $U_y(t)$ represent the horizontal x and vertical y components of the force exerted by the

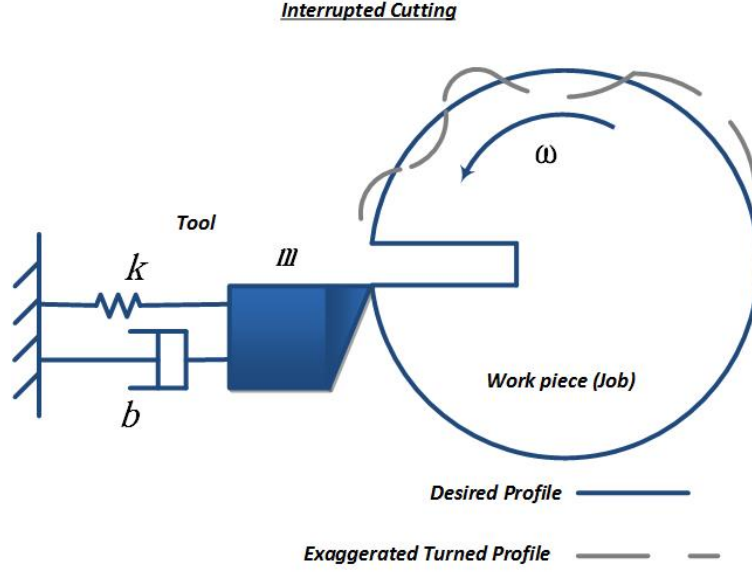


Figure 4: 1-DOF Motion of the Tool

machine tool on the workpiece respectively. Let $x(t)$ and $y(t)$ be the instantaneous displacements of the tool in x and y directions respectively. The tool is characterized by mass m , damping coefficient in the x direction b_x , damping coefficient in the y direction b_y , stiffness or (spring constant) in the x direction k_x and the same in the y direction k_y . The motion of the tool is described by the following equations.

$$m\ddot{x}(t) + b_x\dot{x}(t) + k_x x(t) = U_x(t) \quad (8)$$

$$m\ddot{y}(t) + b_y\dot{y}(t) + k_y y(t) = -U_y(t) \quad (9)$$

Referring to Fig. 5, if R is the radius of the workpiece, then the workpiece moves by a distance of $2\pi R$ units when one revolution of the spindle is completed. Let τ represent the lag (latency) or delay between the previous and current cut of the workpiece. The current cut position is $x(t)$, So the position at previous cut is $x(t - \tau)$. Thus, the delay between the past cut and a present cut is precisely given by,

$$\tau = \frac{2\pi R + x(t) - x(t - \tau)}{R\omega} \quad (10)$$

where ω is the angular velocity of the spindle.

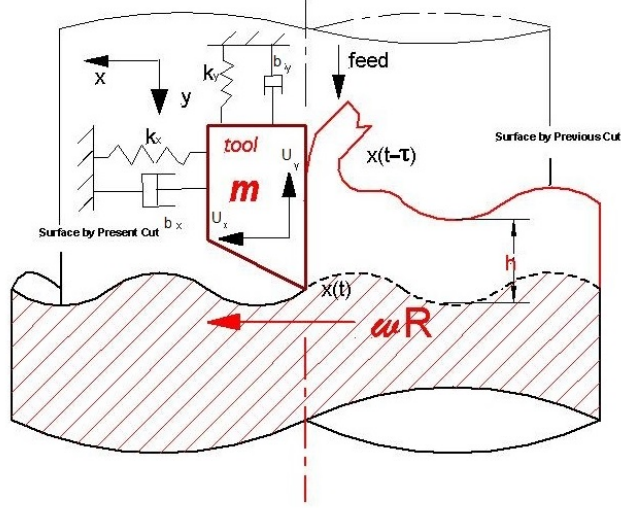


Figure 5: Exaggerated View of Regeneration Mechanism in Machine Tool Turning Process

Now we see from the above equation that the delay τ is a function of the state (position x). We also notice that (10) is an *implicit* relation between the delay and the state. This makes the inversion problem hard, challenging and intricate to analyze because position cannot be easily recovered from the delay measurement. This fact motivated us to estimate the position from delay measurement or observation which leads to the design of observer.

Our scheme is based on a novel idea of delay measurements in a realistic scenario using Fabry-Perot type interferometric sensors or high speed cameras. It is also emphasized here that the same technique can be used for position estimation not only in turning process discussed here as above but also in other machining techniques such as milling and drilling. Accurate mathematical modeling and state recovery guarantees high precision machining. The objective is to use the information on the delay and design an observer to estimate the position and velocity of the tool.

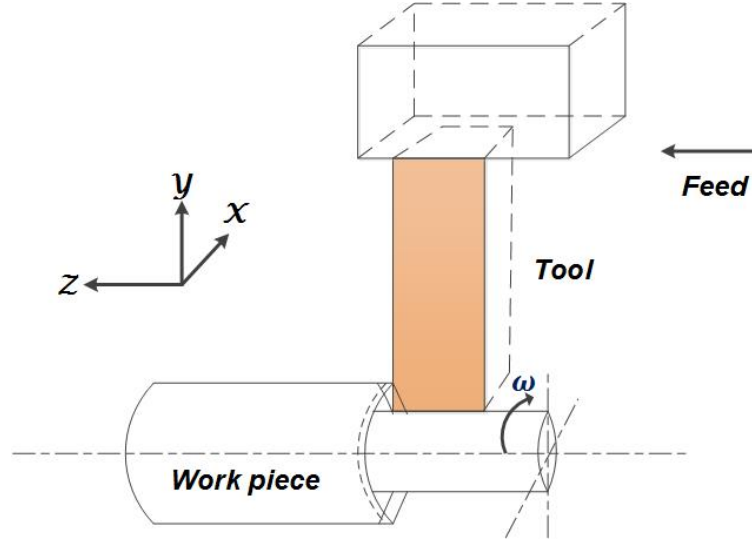


Figure 6: Standard Schematic Model of the Turning Process

1.3 Organization

The organization of the dissertation is as follows. In Chapter I, we give the basic introduction, motivation, contribution and goals of this work. It also gives some practical applications where systems with state-dependent delays arise. Chapter II presents a comprehensive literature survey. In Chapter III, we give the necessary mathematical machinery which involves concepts from functional analysis, operator theory and complex analysis. These tools will be used in the subsequent chapters. The state space characterization and the Cauchy problem for constant, time-varying and state-dependent delays is discussed in Chapter IV. This chapter also focusses on the causality and well-posedness of time delay systems. Chapter V describes the qualitative behavior of systems involving state-dependent delays. Here we show that truncation based on Taylor series approximations results in a lot of anomalies in SD-DDEs. We also show that systems with state-dependent delays exhibit the phenomenon of transcritical bifurcation. Chapter VI describes scale dynamic systems and the self-starting character as one of their salient features. The novel idea of delay injection for observer design is introduced in chapter VII and is applied to

the position estimation of a rocket car, machine tool and basic submarine model in a 3-D environment. Chapter VIII encapsulates the analysis and observer design technique for a class of scalar systems with explicit state-dependent delays. Chapter IX highlights a general framework for the analysis and synthesis of a class of nonlinear systems with *implicit* and *explicit* state-dependent delays. The generic framework in chapter IX is applied to gene expression regulation in chapter X. Chapter XI highlights the analysis and observer design of scalar and higher order systems evolving with state suprema. The discrete time counterpart is also discussed and M^3D features of these systems are also highlighted. The optimal control of a class of systems evolving with state suprema is the subject matter of chapter XII. Chapter XIII brings the spectrum analysis of linear time invariant systems in the Lambert W function based framework. Finally, Chapter XIV highlights the concluding remarks of the dissertation and future recommendations.

CHAPTER II

LITERATURE SURVEY

We divide the literature survey into three parts. The first section focusses on the work related to systems with state-dependent delays. The second section encompasses the literature survey relevant to systems evolving with state suprema which form a special class of systems with state-dependent delays. The third part highlights the previous work in the literature related to the spectrum analysis of linear time invariant systems with constant delay in the framework of Lambert W functions.

2.1 State-Dependent Delays

Time delay systems in general and State-Dependent Delay Differential Equations (SD-DDE) in particular have remained enigmatic in systems theory literature. A proper system theoretic and rigorous treatment of time delay systems starts from the seminal manuscript by Hale and Lunel [43]. In this book, the authors discuss systems with constant and time-varying delays. They make it clear that one should consider the state space as the function space consisting of the initial data on a delay interval and then consider such equations as evolutionary equations in this function space. The monographs [33] and [38] shed light on the robust stability and control of TDS with constant or time-varying delays. In general, SD-DDEs are notoriously hard problems, difficult to study mathematically and may possess some surprising dynamics [58]. In [73], Runge-Kutta based methods are used to solve numerically DDEs with time- and state-dependent delays. See [70] for a survey on TDS which mentions systems with state-dependent delays as one of the open problems.

The analysis and observation of systems with state-dependent varying delays brings a lot of intricacies in system and control theory. A lot of open problems

in this area include but are not limited to information structure development, state space characterization, solving Cauchy problem associated with state-dependent delay differential equations, and stability analysis. Some recent work in this field starts with toy systems as discussed in [85] and [86]. Besides the machine tool chatter problem under consideration, these state-dependent delays also arise for instance in measurement based on ultrasonic sensors in sonars, retarded potentials in electromagnetic theory, automatic milling machines, population dynamics, congestion control in communication networks and biological models to mention just but a few; see [60], [44], [49], [1, 2, 4, 32, 87] and [24] and the references therein. See [71] for a detailed survey of DDEs in single species population dynamics.

Some preliminary results on the asymptotic stability of systems with state-dependent delay were given in [81]. Lyapunov-Krasovskii (LK) theory is used to obtain stability conditions in the local framework. Hartung et al. in [44] illustrate the theory and applications of functional differential equations involving state-dependent delays, with emphasis on particular models and on the emerging theory from the dynamical systems point of view. In [98], Walther models soft landing by an SD-DDE and it is shown that soft landing occurs for an open set of initial data, which is determined by means of a smooth invariant manifold. In [65], the authors consider the effect of state-dependent delay on a weakly damped nonlinear oscillator. They also consider Hopf Bifurcation and persistent oscillations associated with SD-DDE.

Sipahi et al. in a featured article [75] shed light on a wide spectrum of applications of time delay systems ranging from variable-pitch milling dynamics, microscopic vehicular traffic flow, biochemical feedback in cell regulatory networks and epidemics to operations research. They show that delays can also be used for the stabilization of chaotic systems with unstable periodic orbits.

In [48], global stability lobes of turning processes with state-dependent delay are investigated and in [55], bifurcation analysis is performed on a turning system with

large and state-dependent time delay. Both the papers use approximate linearization techniques. None of them considers the estimation problem.

Integrator back-stepping technique is used for forward-complete nonlinear systems involving state-dependent delays [19]. A compensation technique known as predictor-feedback design is used for the system characterized by an SD-DDE. Regional stability results and an estimate of the domain of attraction is given. In [20], the authors investigate a compensation technique for compensating state-dependent input delays for both linear and nonlinear systems. For forward complete nonlinear systems with state-dependent input delay, the authors of [20] and [21] design a predictor-based compensator and give local results with a prescribed region of attraction.

In [3], the authors investigate the behavior of systems with state-dependent delays. Here, the authors emphasize the fact that the truncated Taylor series approximation or linearization of systems with state-dependent delays may lead to highly erroneous and anomalous results. Recently in [62], a real world implementation of a photonic dynamical system is presented and state-dependent delay is taken into account. The system comprises a semiconductor laser with two delay loops which are actively dependent on the laser's dynamical state. The authors emphasize that dynamical systems with *state-dependent* delays possess a different nature than the one with *time-dependent* delays.

We use the exact models of state-dependent delays without any linearization or Taylor series approximation. By first characterizing the state space, stability of systems with state dependent delays is analyzed. Both explicit and implicit state-dependent delays are considered. To the best of our knowledge, there is no work done on the observation with state-dependent delays. We give a novel observer design technique referred to as *Delay Injection* to estimate the state of of a system with state-dependent delay using the delay information.

2.2 Systems Evolving With State Suprema

Voltage Regulation of a Constant Current Generator: We start with the following system.

$$\dot{x}(t) = -\frac{1}{T}x(t) - \frac{q}{T} \sup_{t-\tau \leq \theta \leq t} x(\theta) + \frac{1}{T}w(t) \quad (11)$$

The above equation was derived by E. P. Popov while studying the voltage regulation problem of a constant current generator [69]. Here $T \in \mathbb{R}$, $q \in \mathbb{R}$ and $\tau \in \mathbb{R}_+$ are constants which characterize the object. The state $x(t)$ and the driving term (forcing function) $w(t)$ physically represent the regulated voltage and the perturbation effect respectively at any arbitrary instant of time t . Notice that (11) not only involves the unknown function x but also its maximum value over an interval of past history of length τ and, therefore, represents an *infinite dimensional* system. Furthermore, the presence of the sup functional makes the system *nonlinear*.

Hausrath Equation: One of the systems evolving with state suprema is the Hausrath equation [42],

$$\dot{x}(t) = -\zeta x(t) + \zeta \sup_{t-\tau \leq \theta \leq t} |x(\theta)|, \zeta > 0, t \geq 0 \quad (12)$$

This equation possesses a richer structure than the one in (11). Here the supremum over the past history can never go negative because of the modulus operator. However, the regularity and smoothness of its solution are inferior to that of (11).

Vision Process in a Compound Eye: The vision process in the compound eye of a horseshoe crab can be modeled by the following differential equation evolving with state suprema [40].

$$\dot{x}(t) = -\delta x(t) + p \sup_{t-\tau \leq \tau(t) \leq t} (x(\tau(t)), c); \delta, p \in \mathbb{R}, c < 0 \quad (13)$$

where the state x is related to the activation potential above a certain threshold produced in sensory cell by light. The reciprocal of δ accounts for the response time constant. τ stands for the lateral inhibition delay, which is about 100 msec. as determined experimentally. The function $\sup(x(\tau(t)), c)$ is called the rectifier function.

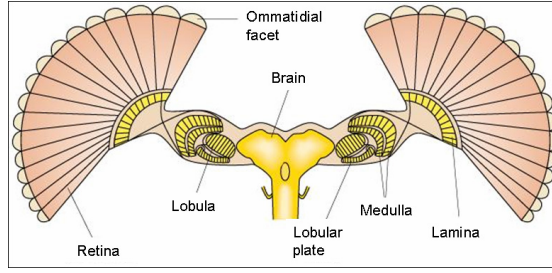


Figure 7: Compound Eye of an Arthropod

The schematic structure of the compound eye of an arthropod is shown in Fig. 7. Light stimulation creates depolarizing graded potentials in insect photoreceptors (as opposed to hyperpolarizing in vertebrate rods and cones). Action potentials do not exist, generally, although they may have a role in photoreceptors of some species (e.g. in the cockroach, [45]). The signals are processed in the first synaptic layer, the lamina, and in the further neural centers (e.g. the medulla) in a retinotopic fashion.

Fig. 8 portrays the cross sectional anatomy of the compound eye of an insect (arthropod). Notice that, unlike mammals and birds, it does not have a single lens. A compound eye of a horseshoe crab is characterized by a large number of small eyes (varying from a few to thousands) known as ommatidia, which function as independent photoreceptor units with an optical system (lens, cornea and some accessory structures) and normally eight photoreceptor cells. The compound eyes do not form an image like the large lens eyes of octopi and vertebrates, but a “neural picture” is formed by the photoreceptors in small eyes (ommatidia), which are oriented to receive light from different directions. The authors of [52] established some very useful and interesting links between the York 3/2 conditions, the Halanay inequality and scalar differential equations with maxima. In [36], the authors deduced asymptotic equilibrium for a certain class of scalar FDEs with maxima. They use the basic inequality of Gronwall-Bellman type, fixed point methods and contractive operators.

In [22], the authors investigate parametric stability for nonlinear differential equations with “maxima” in terms of two measures. The authors obtain several sufficient

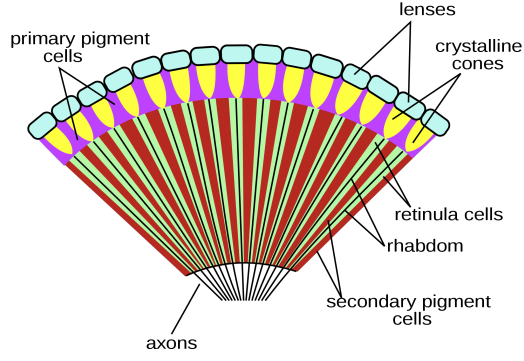


Figure 8: Cross Sectional Anatomy of the Compound Eye: Detailed View

conditions for parametric stability as well as uniform parametric stability. We refer to [17] for the qualitative treatment of differential equations with maxima.

In this work, we want to understand the systems with state suprema from first principles. Our objective is to unravel and explore the rich structure present in this particular class of infinite dimensional systems. Very little is known about such systems from systems and controls perspective. We want to analyze the stability and solve the controller synthesis and observer design problem for these systems. The discrete version is also studied and it is shown that these systems can be recast M^3D systems. The theoretical results and investigations of FDEs with “suprema” opens the door to enormous possibilities for their applications to real world processes and phenomena (see [17]).

2.3 Spectrum of Higher Order Systems With Fixed Delays

Consider the class of linear constant coefficient time delay systems with fixed delay,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_d\mathbf{x}(t - \tau) \quad (14)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector. The matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{A}_d \in \mathbb{R}^{n \times n}$ are constant matrices and $\tau > 0$ is the constant delay. The associated characteristic

equation is,

$$\det(s\mathbf{I} - \mathbf{A} - \mathbf{A}_d e^{-s\tau}) = 0; s \in \mathbb{C}. \quad (15)$$

In [53], a general matrix version of the Lambert W function was defined in terms of the Jordan blocks. Following this formulation, an explicit expression was found for the eigenvalues and spectrum of the above the system in the special case when the system matrices \mathbf{A} and \mathbf{A}_d are *simultaneously triangularizable* which included the special case of *commutativity*. It was shown that the matrix Lambert W function based approach fails if $\mathbf{A}\mathbf{A}_d \neq \mathbf{A}_d\mathbf{A}$.

In [105, 107], an algorithm is presented to extend and generalize the idea of finding the characteristic roots and spectrum analysis to higher order case using matrix Lambert W functions. The authors use the matrix Lambert W function based approach to find the characteristic roots and the spectrum; and analyze the stability, controllability and observability of linear time invariant systems with a constant delay. The methodology is general and is not restricted to a commuting pair of \mathbf{A} and \mathbf{A}_d matrices.

We investigate some pathological and degenerate cases in the spectrum analysis of higher order time delay systems using the idea of matrix Lambert W function. For the scalar case i.e., first order time delay systems, the Lambert-W function framework can be efficiently used. We show that the formulation carried out using matrix Lambert W functions, suffers from some limitations. We provide some counter examples to show that one needs to be very careful in drawing conclusions about the spectrum of higher order system using this approach. In particular, Yi and Ulsoy's algorithm in [105, 107] does not produce satisfactory results when the modes have multiplicity (repeated roots). In some cases, the algorithm produces unnecessary and redundant roots which are not the actual modes of the system under consideration; in other cases it fails to catch all the poles of the system. Also, the algorithm may give an incorrect judgement of the dominant modes of the system.

CHAPTER III

MATHEMATICAL MACHINERY

In this chapter we consolidate some mathematical tools which will be used in the forthcoming chapters. This mathematical machinery contains some definitions, lemmas and theorems from functional analysis, operator theory and complex analysis.

3.1 Functional Analytic & Operator Theoretic Concepts

Definition 1 Set Closure *The set consisting of the points of M and the accumulation points (limit points of M) i.e., the union of M and its limit points is called the closure of the set M . The closure of a set M , denoted by \bar{M} , is the smallest closed set containing the set M . A set M is closed iff $M = \bar{M}$*

Definition 2 Dense Set *A subset M of a metric space X is said to be dense in X if the closure of M is X i.e., $\bar{M} = X$.*

Definition 3 Separable Space *A metric space X is said to be separable if it has a countable subset which is dense in X .*

Definition 4 Compact Operator *Let X and Y be two normed linear spaces. An operator $T \in \mathcal{L}(X, Y)$ is said to be a compact operator if T maps bounded sets of X onto relatively compact sets of Y . Equivalently, a compact operator is defined as a linear operator $T : X \rightarrow Y$ such that for any bounded sequence $\{x_n\}, n \in \mathbb{N}$ in X , the image sequence $\{Tx_n\}$ has a convergent subsequence in Y .*

Compact operators behave in a similar fashion to those defined on finite-dimensional spaces. Compact operators are also termed as *completely continuous operators* or *finite-rank operators*. The following lemma is very useful in the context of compact operators [30].

Lemma 1 *Let X and Y be two normed linear spaces and let the operator $T : X \rightarrow Y$ be a linear operator. Then the following assertions hold:*

- (1). *If T is bounded and $\dim(T(X)) < \infty$ i.e., the range of T is finite-dimensional then the operator T is compact.*
- (2). *If the domain of T is finite i.e., $\dim(X) < \infty$ then the operator T is compact.*
- (3). *The range of T is separable if T is compact.*
- (4). *If S, R are elements of $\mathcal{L}(X_1, X)$ and $\mathcal{L}(Y, Y_1)$, respectively, and $T \in \mathcal{L}(X, Y)$ is compact, then so is the composite operator RTS .*
- (5). *If $\{T_n\}$ is a sequence of compact operators from X to the Banach space Y , that converges uniformly to T i.e., $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$, then the limit operator T is a compact operator.*
- (6). *The identity operator, I , on the Banach space X is compact if and only if $\dim(X) < \infty$. In other words, the identity operator I is not compact on infinite dimensional spaces.*
- (7). *If T is a compact operator in $\mathcal{L}(X, Y)$ whose range is a closed subspace of Y , then the range of T is finite-dimensional.*

Definition 5 *Let X and Y be two normed linear spaces and $T : D(T) \subset X \rightarrow Y$ a linear operator. The graph $\mathcal{G}(T)$ is the set*

$$\mathcal{G}(T) = \{(x, Tx) | x \in D(T)\}$$

in the product space $X \times Y$.

Definition 6 Closed Operator *A linear operator T is called a closed operator if its graph $\mathcal{G}(T)$ is a closed linear subspace of $X \times Y$. Alternatively, T is closed if whenever*

$$x_n \in D(T), n \in \mathbb{N} \text{ and } \lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} Tx_n = y,$$

it follows that $x \in D(T)$ and $Tx = y$.

The following can be easily established.

A linear operator $T : D(T) \rightarrow \mathcal{H}$ is closed if and only if the domain $D(T)$ endowed with the norm $\|f\|_T := \sqrt{\|f\|^2 + \|Tf\|^2}$ is a Banach space i.e., a linear, normed and complete space.

Definition 7 Resolvent Operator Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \rightarrow X$ a linear operator with domain $D(T) \subset X$. With T we associate the operator

$$T_\lambda = \lambda I - T$$

where $\lambda \in \mathbb{C}$ and I is the identity operator on $D(T)$. If T_λ has an inverse, we denote it by $R_\lambda(T)$, i.e.,

$$R_\lambda(T) = T_\lambda^{-1} = (\lambda I - T)^{-1}$$

and is called the Resolvent Operator of T or, simply, the Resolvent of T .

Definition 8 \mathbf{C}_0 -Semigroup A strongly continuous semigroup (\mathbf{C}_0 -semigroup) is an operator valued function $T(t)$ from \mathbb{R}^+ to X that satisfies the following properties:

- (1). $T(t+s) = T(t)T(s)$ for $t, s \geq 0$;
- (2). $T(0) = I$;
- (3). $\|T(t)x_0 - x_0\| \rightarrow 0$ as $t \rightarrow 0^+ \forall x_0 \in X$.

Definition 9 Infinitesimal Generator The infinitesimal generator \mathcal{A} of a \mathbf{C}_0 -semigroup on Banach space X is defined by

$$\mathcal{A}x = \lim_{t \rightarrow 0^+} \frac{1}{t}(T(t) - I)x$$

whenever the limit exists; the domain of \mathcal{A} , $D(\mathcal{A})$, being the set of elements in X for which the limit exists.

Next we present two very important generation theorems which give necessary and sufficient conditions for open operator to qualify as an infinitesimal generator of a semigroup.

Theorem 1 *Hille-Yosida Theorem* A necessary and sufficient condition for a closed, densely defined, linear operator \mathcal{A} on a Banach space X to be the infinitesimal generator of a \mathbf{C}_0 -semigroup is that there exist real numbers M, ω , such that for all real $\lambda > \omega$, $\lambda \in \rho(\mathcal{A})$, the resolvent set of \mathcal{A} , and

$$\|R(\lambda, \mathcal{A})^r\| \leq \frac{M}{(\lambda - \omega)^r}, \quad \text{for all } r \geq 1,$$

where $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ is the resolvent operator of \mathcal{A} . In this case

$$\|T(t)\| \leq Me^{\omega t}.$$

The Lumer-Phillips Theorem stated as follows gives a necessary and sufficient condition for a linear operator \mathcal{A} on a Banach space X to be the infinitesimal generator of a contraction semigroup.

Theorem 2 *Lumer-Phillips Theorem* Let a linear operator \mathcal{A} be defined on a domain $D(\mathcal{A})$ which is a linear subspace a Banach space X the \mathcal{A} is the infinitesimal generator of a contraction semigroup if and only if

- (1). \mathcal{A} is closed.
- (2). $D(\mathcal{A})$ is dense in X .
- (3). \mathcal{A} is dissipative i.e., $\|(\mathcal{A} - \lambda I)x\| \leq \lambda \|x\|$; $\forall \lambda > 0, \forall x \in D(\mathcal{A})$
- (4). $\mathcal{A} - \mu I$ is onto (surjective) for some $\mu > 0$.

Moreover,

$$\|T(t)\| \leq 1, \quad \forall t \geq 0.$$

Definition 10 *Regular Value, Resolvent Set, Spectrum* A regular value $\lambda \in \mathbb{C}$ of the operator T is a complex number λ such that

- (1). $R_\lambda(T)$ exists,
- (2). $R_\lambda(T)$ is a continuous/bounded operator,
- (3). $R_\lambda(T)$ is defined on a set which is dense in X i.e., the domain of R_λ is dense in

X .

The resolvent set $\rho(T)$ of T is the set of all regular values λ of T . Mathematically,

$$\begin{aligned}\rho(T) &= \left\{ \lambda \in \mathbb{C} : \lambda I - T \in \mathcal{G}(X) \right\} \\ &= \left\{ \lambda \in \mathbb{C} : \lambda I - T \text{ has an inverse in } \mathcal{L}(X) \right\} \\ &= \left\{ \mathcal{N}(\lambda I - T) = \{0\} \text{ and } \mathcal{R}(\lambda I - T) = X \right\}.\end{aligned}$$

The complement of the resolvent set of T in \mathbb{C} is called the spectrum $\sigma(T)$ of the operator T i.e.,

$$\sigma(T) = \rho^c(T) = \mathbb{C} - \rho(T),$$

and a $\lambda \in \sigma(T)$ is called the spectral value of the operator T .

The spectrum of a bounded operator T can be classified or partitioned into three disjoint components as follows.

Definition 11 Point Spectrum or Discrete Spectrum or Eigen Spectrum

The point spectrum or discrete spectrum of T i.e., $\sigma_p(T)$ is the set of spectral values λ such that $R_\lambda(T)$ does not exist which happens when $\lambda I - T$ fails to be injective or one-to-one. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T . Mathematically,

$$\sigma_p(T) = \left\{ \lambda \in \mathbb{C} : \mathcal{N}(\lambda I - T) \neq \{0\} \right\}.$$

Definition 12 Continuous Spectrum The continuous spectrum $\sigma_c(T)$ of the operator T is the set such that $R_\lambda(T)$ exists and is defined on a set which is dense in X but $R_\lambda(T)$ is not bounded or continuous. Mathematically,

$$\begin{aligned}\sigma_c(T) &= \left\{ \lambda \in \mathbb{C} : \mathcal{N}(\lambda I - T) = \{0\}, \overline{\mathcal{R}(\lambda I - T)} = X \right. \\ &\quad \left. \text{and } \mathcal{R}(\lambda I - T) \neq X \right\}.\end{aligned}$$

Definition 13 *Residual Spectrum* The residual spectrum $\sigma_r(T)$ of the operator T is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but its domain is not dense in X . Mathematically,

$$\sigma_c(T) = \left\{ \lambda \in \mathbb{C} : \mathcal{N}(\lambda I - T) = \{0\} \text{ and } \overline{\mathcal{R}(\lambda I - T)} \neq X \right\}.$$

Notice that there cannot be any other component in the spectrum of T besides these three constituent spectra.

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

Also, the three constituent spectra are mutually disjoint. i.e.,

$$\sigma_p(T) \cap \sigma_c(T) = \sigma_c(T) \cap \sigma_r(T) = \sigma_r(T) \cap \sigma_p(T) = \emptyset$$

and thus the whole (entire) complex plane \mathbb{C} be partitioned as follows.

$$\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) = \mathbb{C}.$$

It can be shown that the spectrum of a bounded operator is always compact. Fig. 9 shows various types of spectra of a bounded linear operator T defined on a Banach space X .

Definition 14 *Spectral Radius* The spectral radius r_T of a continuous operator $T : X \rightarrow X$ is defined as follows.

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} \{|\lambda|\}$$

It is the radius of the smallest disc centered at the origin that contains the spectrum of T .

Also, it can be shown that, the following Gelfand's formula can be used for computing the spectral radius

$$r_\sigma(T) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T^n\|}$$

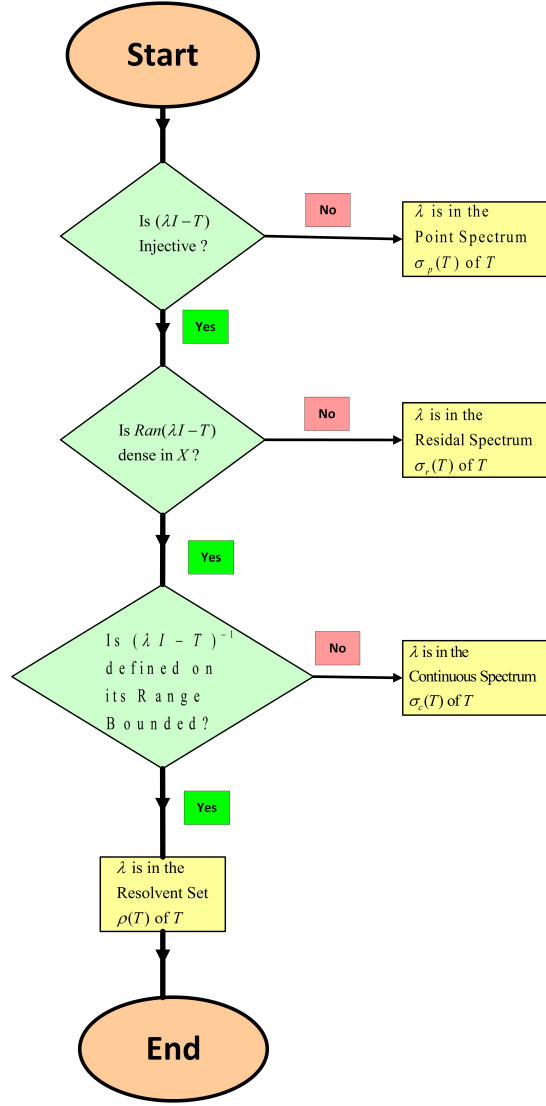


Figure 9: Spectrum Analysis of a Continuous Operator T defined on a Banach space X

where the norm in the above equation is any norm and, therefore, the spectral radius of an operator is independent of the norm of the operator.

The following theorem is very useful in the spectral analysis of compact operators [56], [67].

Theorem 3 *Spectral Theorem for Compact Operators* Let $T : X \rightarrow X$ be a compact linear operator a Banach space X . The following spectral properties of T hold:

- (1). Every nonzero $\lambda \in \sigma(T)$ is in the point or discrete spectrum σ_p of T i.e., it is an eigenvalue of T .
- (2). For all nonzero $\lambda \in \sigma(T)$, there exists $m \in \mathbb{N}$ such that $\mathcal{N}(\lambda I - T)^m = \mathcal{N}(\lambda I - T)^{m+1}$, and this null space is finite-dimensional.
- (3). The spectral values can only cluster or accumulate at 0. If $\dim(X) = \infty$ then $\{0\}$ must be in the spectrum of T .
- (4). The spectrum of T is at the most countably infinite.
- (5). Every nonzero spectral value $\lambda \in \sigma(T)$ is a pole of the resolvent operator $\mu \rightarrow (\mu I - T)^{-1}$.

Remark 1:

Notice that when X is an infinite-dimensional space and $T : X \rightarrow X$ is compact then $\lambda = 0$ must be in the spectrum of T . However it is not clear as to which part of the spectrum it should belong. All the three cases are possible i.e., $0 \in \sigma_p(T)$ or $0 \in \sigma_c(T)$ or $0 \in \sigma_r(T)$ as illustrated by the following examples. Consider the following systems.

$$\dot{x} = \mathcal{A}x$$

Example 1 Let $\mathcal{A} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be as follows.

$$\mathcal{A} = \begin{pmatrix} -1 & & & & \\ & -1/2 & & & \\ & & -1/3 & & \\ & & & -1/4 & \\ & & & & \ddots \end{pmatrix}$$

Here,

$$\sigma(\mathcal{A}) = \underbrace{\bigcup_{j=1}^{\infty} \left(-\frac{1}{j}\right)}_{\sigma_p(\mathcal{A})} \cup \underbrace{\{0\}}_{\sigma_c(\mathcal{A})}$$

Notice that here $\mathcal{A}x = 0$ implies $x = 0$. Therefore, 0 is not an eigenvalue of the compact infinitesimal generator \mathcal{A} and is, therefore, not in the point spectrum or

discrete spectrum. Rather, $\{0\}$ forms the continuous spectrum of \mathcal{A} . It can be seen that \mathcal{A} does not have a continuous or bounded inverse. The system is asymptotically stable.

Example 2 Let $\mathcal{A} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be as follows.

$$\mathcal{A} = \begin{pmatrix} 0 & & & & \\ & -1 & & & \\ & & -1/2 & & \\ & & & -1/3 & \\ & & & & \ddots \end{pmatrix}$$

Here,

$$\sigma(\mathcal{A}) = \underbrace{\bigcup_{j=1}^{\infty} \left(-\frac{1}{j}\right)}_{\sigma_p} \cup \underbrace{\{0\}}_{\sigma_p}.$$

Notice that here 0 is an eigenvalue of the compact infinitesimal generator \mathcal{A} and is, therefore, in the point spectrum or discrete spectrum. The system is stable but not asymptotically stable.

Example 3 Let $\mathcal{A} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be as follows.

$$\mathcal{A} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1/2 & 0 & & \\ 0 & 0 & 1/3 & 0 & \\ \vdots & & \vdots & & \ddots \end{pmatrix}$$

Here,

$$\sigma(\mathcal{A}) = \sigma_r(\mathcal{A}) = \{0\}.$$

Notice that here 0 is in the residual spectrum (the only spectrum here) of the compact infinitesimal generator \mathcal{A} and is, therefore, not an eigenvalue. The range of \mathcal{A} , $\mathcal{R}(\mathcal{A})$ is not dense in l^2 , although \mathcal{A} is injective. The system is unstable.

Example 4 Let $\mathcal{A} : l^2 \rightarrow l^2$ be as follows.

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1/2 & & \\ & 0 & 0 & 1/3 & \\ & 0 & 0 & 0 & 1/4 \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

Here,

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{0\}.$$

Notice that here 0 is in the point spectrum (the only spectrum here) or eigenvalue of the compact infinitesimal generator \mathcal{A} . The residual and the continuous spectrum are both empty. For all $\lambda \in \mathbb{C} \setminus \{0\}$, the operator $\mathcal{A} - \lambda I$ is always surjective as well as injective, hence bijective and $(\mathcal{A} - \lambda I)^{-1}$ exists and is bounded. The system is unstable.

3.2 The Lambert W Function

The Lambert W function is defined as the multi-valued function which solves the following transcendental equation:

$$W(z)e^{W(z)} = z, \quad z \in \mathbb{C}, \quad (16)$$

or, equivalently, as the multi-valued inverse of the function $f : z \mapsto ze^z$. It is also called the *product log* function or *omega* function. Equation (16) always has an infinite number of solutions, hence the multi-valuedness of the W function. These solutions are indexed by the integer variable j . Thus we say that (16) is solved by the branches of the W function, W_j , for $j \in \mathbb{Z}$. Of special relevance to physics and engineering applications are the solutions of (16) when the argument is purely real. In this case there can be at most two real solutions, corresponding to the branches W_0 and W_1 , where W_0 is the principal branch of the W function. For real solutions to exist, we must require that $z \in (-1/e, \infty)$, in which case $W_0(z) \in [-1, \infty)$ and

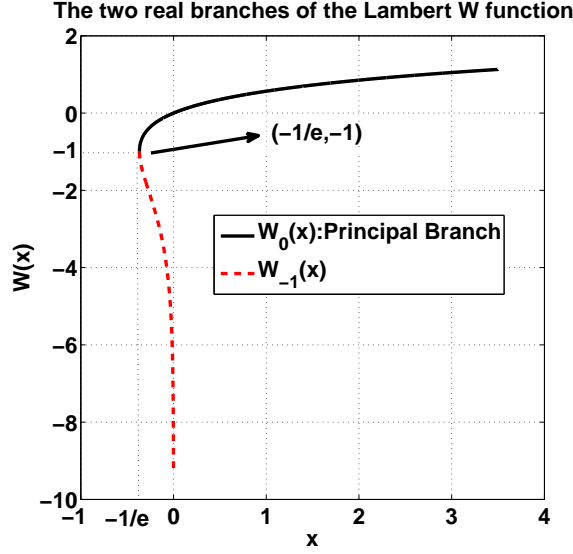


Figure 10: The two real branches of the W function. The solid line represents the principal branch W_0 and the dashed line represents W_{-1} .

$W_{-1}(z) \in (-\infty, -1]$. Moreover, $W_0(z) < 0$ if $z \in (-1/e, 0)$ and $W_0(z) \geq 0$ for $z \in [0, \infty)$. The branches W_0 and W_{-1} are monotonically increasing and monotonically decreasing, respectively.

Fig. 10 portrays the two real branches of the Lambert W function. The solid line represents W_0 and the dashed line represents W_{-1} . We can notice that $W_0(-\frac{1}{e}) = W_{-1}(-\frac{1}{e}) = -1$ and $W_0(0) = 0$. If $z < -\frac{1}{e}$ then there are no real solutions. All other branches of W are always complex [77]. Using the Lagrange inversion theorem on (16), the Taylor series expansion of the principal branch W_0 around 0 is given by,

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \dots \quad (17)$$

provided that $|x| < \frac{1}{e}$ i.e. the radius of convergence is $\frac{1}{e}$. This shows that the principal branch of the Lambert W function i.e. W_0 is analytic at 0.

Implicit differentiation of (16) yields,

$$\frac{dW}{dz} = \frac{1}{(1 + W(z))e^{W(z)}}; \quad z \neq -\frac{1}{e} \quad (18)$$

or

$$\frac{dW}{dz} = \frac{W(z)}{z(1+W(z))}; \quad z \notin \left\{0, -\frac{1}{e}\right\}. \quad (19)$$

Notice that,

$$\left. \frac{dW}{dz} \right|_{z=0} = 1. \quad (20)$$

Besides the theoretical advantages of providing an adequate analytical formalism for a given problem, another advantage of solving problems in terms of the W function is the availability of libraries in computer algebra systems, which allows for a convenient way to obtain values, expansions, plots, etc., of the quantity being solved for [77]. For a detailed survey on the Lambert W function and its applications we refer the reader to a nice article by [29].

3.3 *Razumikhin Theorem*

We state the Lyapunov-Razumikin (LR) theorem which is a very powerful theorem in the context of stability analysis of systems characterized by FDEs [43]. Unlike the Lyapunov-Krasovskii (LK) functional based approach in an infinite dimensional setting, this theorem uses functions which are relatively easier to handle with. This theorem also gives sufficient conditions for the stability.

Here, $\mathcal{C}_{n,\tau} = C([- \tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[- \tau, 0]$ into \mathbb{R}^n with the topology of uniform convergence and designate the norm of an element Φ in $\mathcal{C}_{n,\tau}$ by

$$\|\Phi\| = \sup_{\theta \in [-\tau, 0]} \|\Phi(\theta)\|_a \quad (21)$$

where $\|\cdot\|_a$ denotes any norm because in the finite dimensional space (\mathbb{R}^n), all the norms are equivalent.

Theorem 4 Consider the functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t); & t \geq t_0 \\ x_{t_0}(\theta) = \psi(t_0 + \theta), \forall \theta \in [-\tau, 0] \end{cases} \quad (22)$$

where $x_t(\theta), t \geq t_0$ denotes the restriction of $x(\cdot)$ to the interval $[t - \tau, t]$ translated to $[-\tau, 0]$, that is $x_t(\theta) = \psi(t + \theta), \forall \theta \in [-\tau, 0]$ with $\psi \in \mathcal{C}_{n,\tau}$. Let the function $f(t, \psi) : \mathbb{R} \times \mathcal{C}_{n,\tau} \rightarrow \mathbb{R}^n$ be continuous in t and Lipschitzian in ψ with $f(t, 0) = 0$. $\alpha, \beta, \gamma, \eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and nondecreasing functions with

$$\begin{cases} \alpha(r), \beta(r), \gamma(r) > 0; & r > 0 \\ \alpha(0) = \beta(0) = 0 \\ \eta(r) > r; & r > 0 \end{cases}$$

If there exists a continuous function $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\begin{cases} \text{(i). } \alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), t \in \mathbb{R}, x \in \mathbb{R}^n \\ \text{(ii). } \dot{V}(t, x(t)) \leq -\gamma(\|x\|) \text{ if} \\ V(t + \delta, x(t + \delta)) \leq \eta(V(t, x(t))), \forall \delta \in [-\tau, 0] \end{cases}$$

then the trivial solution of (22) i.e. the origin ($x(t) = 0$) is uniformly asymptotically stable.

3.4 Halanay's Inequality

The following lemma is very handy in the analysis and observer design of continuous systems evolving with state suprema. It will be referred to as Halanay's inequality [41].

Lemma 2 If $\dot{f}(t) \leq -k_1 f(t) + k_2 \sup_{t-\tau \leq \sigma \leq t} f(\sigma)$ for $t \geq t_0$ and if $k_1 > k_2 > 0$, then there exists $\mu > 0$ and $k > 0$ such that $f(t) \leq k e^{-\mu(t-t_0)}$ for $t \geq t_0$. Moreover, $k = \sup_{t_0-\tau \leq \sigma \leq t_0} f(\sigma)$ and $\mu = k_1 - k_2 e^{-\mu\tau}$.

CHAPTER IV

STATE SPACE CHARACTERIZATION & CAUCHY PROBLEM

The state of a dynamical system refers to the Minimal Sufficient Information (MSI) required for the evolution of the system or for characterizing the Cauchy problem. State space should be fixed. For system characterized by Ordinary Differential Equations (ODEs), the state space is the usual Euclidean space \mathbb{R}^n . For systems governed by Partial Differential Equations (PDEs) and/or Delay Differential Equations (DDEs), finite dimensional spaces like \mathbb{R}^n cannot suffice to be the state space. One needs to consider suitable function spaces which are inherently infinite-dimensional. Once we set the state space only then we can talk about the state, state trajectory and stability of the system. This chapter highlights the state space required for system with constant, time-varying and state-dependent delays.

4.1 State Space for the Fixed Delay System

The Time Delay System (TDS) characterized by the Retarded Functional Differential Equation (RFDE) is not a finite dimensional system. The state is not just a point on the real line \mathbb{R} but is a function defined over an interval of compact support. The Cauchy problem for the evolution of the infinite-dimensional state can be characterized as follows.

$$\Sigma_f : \begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t - \tau) & \forall t \geq 0 \\ x(t) = \psi(t). & \forall t \in [-\tau, 0] \end{cases} \quad (23)$$

Where $x(t) \in \mathbb{R}$ is the state variable, $\tau \in \mathbb{R}^+$ is the constant delay, $\psi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R})$ is the initial infinite dimensional history function living in the Banach function space.

Here $\mathcal{C}([-\tau, 0]; \mathbb{R})$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to \mathbb{R} with the topology of uniform convergence. This means that the norm of an element ϕ in this function space is defined by the following uniform norm.

$$\|\phi\| = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\| \quad (24)$$

In mathematical analysis, the above norm is termed as the uniform norm (or sup norm) assigns to real- or complex-valued bounded functions f defined on a set S the non-negative number $\|f\|_\infty = \|f\|_{\infty, S} = \sup \{ |f(x)| : x \in S \}$.

This norm is also called the supremum norm, the Chebyshev norm, or the infinity norm. The name "uniform norm" derives from the fact that a sequence of functions $\{\phi_n\}$ converges to ϕ under the metric derived from the uniform norm if and only if ϕ_n converges to ϕ uniformly.

From functional analysis, the space of continuous functions defined over compact support and equipped with the uniform is always a complete space and is therefore a Banach space. Also from Heine-Borel theorem, any set in \mathbb{R}^n is compact if and only if it is closed and bounded.

Since ϕ is a continuous function on a closed interval, or more generally a compact set, then it is bounded and the supremum in the above definition is attained by the Weierstrass extreme value theorem, so we can replace the supremum by the maximum. In this case, the norm is also called the maximum norm. Therefore, (228) simplifies to

$$\|\phi\| = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|. \quad (25)$$

The following theorem gives a *sufficient* condition for the asymptotic stability of the RFDE (23).

Theorem 5 [79] *The null solution (equilibrium) $x \equiv 0$ of the homogenous equation (23) is globally asymptotically stable if the following condition on the coefficients holds.*

$$\alpha + |\beta| < 0 \quad (26)$$

Note:

A more general higher order and robust version of the above theorem is proved in [79] and [94]. It is worthy to be mentioned here that [79] is the historic paper where the Riccati equation appears for the first time in the context of delay systems.

4.2 State Space for Time-Varying Delay System

Let the delay in (23) be a time-varying delay i.e., $\tau(t)$ so that the system becomes,

$$\dot{x}(t) = \alpha x(t) + \beta x(t - \tau(t)). \quad (27)$$

Notice that $\mathcal{C}([-\tau(t), 0], \mathbb{R})$ cannot be the state space. The state space should be fixed and time independent. Also the delay rate $\dot{\tau}$ cannot be arbitrary. The backward time $t - \tau(t)$ should be monotonically increasing in order to ensure causality, see [83]. This imposes $\dot{\tau} \leq 1$. This will be more emphasized in §4.4 when the causality and well-posedness of variable delay systems will be discussed.

Assuming that the time-varying delay is bounded i.e., $\tau(t) \leq h, h \in \mathbb{R}_+$, and also $\dot{\tau} \leq 1$, an appropriate state space for the time-varying delay system (27) is $\mathcal{C}([-h, 0], \mathbb{R})$ or $\mathbb{L}^2([-h, 0], \mathbb{R})$.

4.3 State Space for State-Dependent Delay System

In general, it is not an easy job to characterize the state space of an SD-DDE. The reason is that the delay depends on the dependent variable x i.e., the solution of the SD-DDE which we do not know a priori. One could be tempted to take the space $\mathcal{C}([-\tau(x), 0]; \mathbb{R}^n)$, but this makes no sense. State space should be a stationary construct. It should be fixed once and for all. It should be independent of time and space and it should require the minimal sufficient information or statistic for the evolution of the system or for the characterization of the Cauchy problem. Assuming that the state dependent delay $\tau(x)$ is bounded and $0 \leq \tau(x) \leq \tau_{\max}$, one natural choice is to take $\mathcal{C}([-\tau_{\max}, 0]; \mathbb{R}^n)$ as the space for the initial history function to reside

in. Unfortunately, this space is way too big for SD-DDEs and, as we elaborate in the next subsection, there are anomalies associated with this space.

4.3.1 Some Peculiarities in the space $\mathcal{C}([-\tau_{\max}, 0]; \mathbb{R}^n)$

From the two counter examples given below, one can see that the space of continuous functions $\mathcal{C}([-\tau_{\max}, 0]; \mathbb{R}^n)$ is not appropriate to make the Cauchy problem of the system characterized by the SD-DDE well posed. We need a higher degree of smoothness or regularity on the initial history function to make the Cauchy problem well defined.

Example 1: [32] Consider the following SD-DDE.

$$\dot{x}(t) = -2x(t - x(t)) + 5, \quad t > 0 \quad (28)$$

Consider the initial history function, $\phi(t) = x(t) = 2 + \sqrt{|t+4|}$ on $-\infty < t \leq 0$. Clearly, ϕ is continuous but not Lipschitz continuous. We can easily verify that for this ϕ , both of the following functions satisfy (28) and are thus the solutions of the SD-DDE.

$$\begin{aligned} x(t) &= 4 + t, \text{ for } t > 0 \\ \text{and } x(t) &= 4 + t - t^2, \text{ for } 0 \leq t \leq 2 \end{aligned}$$

Therefore, the solution is not unique on the space of continuous functions.

Example 2: [101] We consider the following SD-DDE.

$$\dot{x}(t) = -x(t - |x(t)|), \quad t > 0 \quad (29)$$

Consider the initial history function,

$$\phi(t) = x(t) = \begin{cases} -1, & \text{if } t \leq -1; \\ \frac{3}{2}\sqrt[3]{t+1} - 1, & \text{if } -1 \leq t \leq -\frac{7}{8}; \\ \frac{10}{7}t + 1, & \text{if } -\frac{7}{8} < t \leq 0. \end{cases}$$

Clearly, ϕ is continuous but not Lipschitz continuous. We can easily verify that for this ϕ , both of the following functions are the solutions of (29) for small $t > 0$.

- (i). $x(t) = t + 1$, for $t > 0$
- (ii). $x(t) = t + 1 - \sqrt{t^3}$, for $t > 0$

Therefore, the solution is not unique on the space \mathcal{C} .

In [86], the author analyzed the state space realization problem of the SD-DDE $\dot{x}(t) = ax(t - x(t))$ and constructed the infinitesimal generator for the equivalent system $\dot{x}(t + x(t)) = ax(t)$ in the \mathcal{C}^1 framework. The infinitesimal generator was constructed on the following state space.

$$X = \left\{ (x, \phi(\cdot)) \in \mathbb{R}_+ \times \mathcal{C}^1((0, 1), \mathbb{R}_+) \mid \dot{\phi} > 0, \phi(0) = 1 \right\}$$

Let $\mathcal{C}^{0,1} := \mathcal{C}^{0,1}([-\tau_{\max}, 0]; \mathbb{R})$ denote the Banach space of Lipschitz continuous functions mapping the interval $[-\tau_{\max}, 0]$ to \mathbb{R} with the topology of uniform convergence. This means that the norm of an element ϕ in this function space is defined by the following endowed norm.

$$\|\phi\|_{0,1} = \max\{\|\phi\|_{\infty}, \|\phi'\|_{L^{\infty}}\} \quad (30)$$

Now, we consider a more general form of the nonlinear SD-DDE as follows.

$$\Sigma_s : \begin{cases} \dot{x}(t) = f(x(t - \tau(x(t)))) & \forall t \geq 0 \\ x(t) = \phi(t) \in \mathcal{C}^{0,1}. & \forall t \in [-\tau_{\max}, 0] \end{cases} \quad (31)$$

where f is a function from \mathbb{R} to \mathbb{R} and τ is a function from \mathcal{C} to $[0, \tau_{\max}]$. Again here, $\mathcal{C} := \mathcal{C}([-\tau_{\max}, 0]; \mathbb{R})$ is the Banach space equipped with the norm topology as defined in one of the previous sections.

As a standard notation for the state in the literature on TDS, let x_t denote an element of \mathcal{C} defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in [-\tau_{\max}, 0]. \quad (32)$$

[60] wrote the SD-DDE (31) in the following form

$$\begin{cases} \dot{x}(t) = F(x_t) \\ x_0 = \phi, \end{cases} \quad (33)$$

where $F(\phi) = f(\phi(-\tau(\phi)))$, for $\phi \in \mathcal{C}$. Clearly, whatever the regularity or smoothness of f and τ may be, F is neither differentiable nor locally Lipschitz continuous.

The functions f and τ satisfy the following assumptions.

(A.1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous.

(A.2) $\tau : \mathcal{C} \rightarrow [0, \tau_{\max}]$ is Lipschitz continuous on the bounded sets of \mathcal{C} .

(A.3) $|f(x)| \leq \xi|x| + \zeta, \forall x \in \mathbb{R}$ with ξ and ζ being two constants.

(A.4) The functions f and τ are of class \mathcal{C}^1 .

where, $\mathcal{C}^1 := \mathcal{C}^1([-\tau_{\max}, 0]; \mathbb{R})$ denotes the Banach space of continuously differentiable functions mapping the interval $[-\tau_{\max}, 0]$ to \mathbb{R} with the topology of uniform convergence. This means that the norm of an element ϕ in this function space is defined by the following endowed norm.

$$\|\phi\|_1 = \|\phi\|_{\infty} + \|\phi'\|_{\infty} \quad (34)$$

Clearly, we have the following inclusion.

$$\mathcal{C}^1 \subseteq \mathcal{C}^{0,1} \subseteq \mathcal{C} \quad (35)$$

Notice that the Cauchy problem of the SD-DDE (31) is not well posed in the space \mathcal{C} whatever the regularity of the functions τ and f may be because of the fact that the solution is no longer unique. As we saw in Example:1, f is a \mathcal{C}^{ω} (analytic) function but the solution is not unique because the history is not a $\mathcal{C}^{0,1}$ function.

The solution on the space $\mathcal{C}^{0,1}$ is unique but a strongly continuous semigroup (\mathbf{C}_0 semigroup) cannot be constructed in this space. In order to create a C_0 semigroup and hence an infinitesimal generator of the semigroup, a closed subset of $\mathcal{C}^{0,1}$ is required [60].

Notice that $\mathcal{C}^{0,1}$ is in fact a particular class of Sobolev spaces. Generally, \mathcal{C}^1 or $\mathcal{C}^{0,1}$ is considered as a more friendly space for some classes of systems characterized by SD-DDEs [44], [60].

4.4 Causality and Well-Posedness

In order to ensure well-posedness of systems with time varying delays [82], [83], [64] and [90], we impose $\dot{\tau}(t) \leq 1$. This is not just a technical condition to make the analysis tractable but is an essential causality constraint and avoids inconsistencies arising in the time-varying delay system. Violation of this constraint will make the system non-causal and ill-posed. Non-causal systems are practically not realizable and do not make sense physically. In summary, the following five potential problems may arise if the delay has fast growth rate i.e., $\dot{\tau}(t) > 1$.

- (i). Lack of causality
- (ii). Lack of minimality
- (iii). Non-uniqueness of the state space
- (iv). Inconsistencies in the system
- (v). Failure of existence and/or uniqueness of the solution

4.4.1 Implications

We now analyze this constraint and show that it is in perfect agreement with the physical reality in the problems under consideration.

4.4.1.1 Rocket Car

From (3), in the rocket car case, we have

$$\begin{aligned}
 \dot{\tau}(t) &= \frac{\dot{x}_1(t) - (1 - \dot{\tau})\dot{x}_1(t - \tau)}{v} \\
 v\dot{\tau}(t) &= \dot{x}_1(t) - (1 - \dot{\tau})\dot{x}_1(t - \tau) \\
 \dot{\tau}(t) &= \frac{\dot{x}_1(t) - \dot{x}_1(t - \tau)}{v - \dot{x}_1(t - \tau)}.
 \end{aligned}
 \tag{36}$$

Now, $\dot{\tau}(t) \leq 1$ translates precisely to $\dot{x}_1(t) \leq v$ which is quite natural and has a reasonable physical meaning. How can the ultrasonic sensor measure the position of the rocket car if the speed of the rocket exceeds the speed of sound?

For all the analysis and observer design problems it will be assumed that the control effort is such that the rocket remains subsonic.

4.4.1.2 Turning Process

Now we come to the state-dependent delay in the turning process. From (10), we have

$$\begin{aligned}\tau &= \frac{2\pi R + x(t) - x(t - \tau)}{R\omega} \\ \Rightarrow \dot{\tau}(t) &= \frac{\dot{x}(t) - \dot{x}(t - \tau)(1 - \dot{\tau})}{R\omega}\end{aligned}$$

from which,

$$\dot{\tau}(t) = \frac{\dot{x}(t) - \dot{x}(t - \tau)}{R\omega - \dot{x}(t - \tau)}. \quad (37)$$

Now, using the causality and well-posedness condition,

$$\begin{aligned}\dot{\tau}(t) &\leq 1 \\ \Rightarrow \frac{\dot{x}(t) - \dot{x}(t - \tau)}{R\omega - \dot{x}(t - \tau)} &\leq 1 \\ \Rightarrow \dot{x}(t) &\leq R\omega.\end{aligned} \quad (38)$$

Notice that $\dot{\tau}(t) \leq 1$ translates precisely to $\dot{x}(t) \leq R\omega$. This dictates that the linear velocity of the workpiece cannot be smaller than the instantaneous horizontal velocity of the tool in magnitude. This makes a realistic physical sense.

Comment 1:

It will be assumed that the control effort $U_x(\cdot)$ and $U_y(\cdot)$ in (8) and (9) respectively are designed such that the *causality* and *well-posedness* condition is not violated.

CHAPTER V

THE BEHAVIOR OF SYSTEMS INVOLVING STATE-DEPENDENT DELAYS

This chapter focusses on the completely strange qualitative behavior associated with the systems characterized by State Dependent-Delay Differential Equations (SD-DDEs). We consider one of the most simple and innocently looking SD-DDEs $\dot{\mathbf{x}}(t) = \pm \mathbf{x}(t - \mathbf{x}(t))$. This retarded SD-DDE brings a lot of intricacies. It looks linear but is actually a nonlinear SD-DDE in disguise. It exhibits the phenomenon of bifurcation. Also there is a switch in the stability properties of this system. The type of bifurcation exhibited by the system $\dot{\mathbf{x}}(t) = -\mathbf{x}(t - \mathbf{x}(t)) + \mu$ is *transcritical*. Furthermore, its Taylor series approximation, based on truncation and partial sums, gives no idea of the response. We show that Taylorization of SD-DDEs, which is ubiquitously used in physics and engineering community, could be misleading. We also perform singular and regular perturbation analyses and derive the solution of the small signal perturbed system in terms of the Lambert-W function. We also demonstrate that the instability of SD-DDE is quite different from that of ODEs. Our simulation results reveal that serious errors may occur when SD-DDEs are approximated either by Taylorization or by constant delay systems.

5.1 Introduction

Here we explore and unravel the nature of systems with state-dependent delays. We primarily consider the most basic dynamical system with the state-dependent delay characterized by the functional differential equations as follows.

$$\dot{x}(t) = -x(t - x(t)) \tag{39}$$

and

$$\dot{x}(t) = x(t - x(t)) \quad (40)$$

The state space for (39) was studied in [86]. Notice that the delay, $x(t)$, should be non-negative for all time otherwise the system is an anticipatory or advance system which will not make any sense physically. Therefore, from the very start and scratch, we assume that $x(t) \geq 0, \forall t \geq 0$.

We also make it clear that both of the equations (39) and (40) look apparently simple and innocent but are *nonlinear* in nature because of the presence of state dependent delay. It is not difficult to show that these equations fail to satisfy both the properties of superposition (additivity and homogeneity) required for demonstrating the linearity of a system.

Motivation:

The state-dependent delay dynamical system (39) is motivated by crystal growth problems. Such state-dependent delays play a dominant role in the problem of formation of macrosteps on crystal surfaces by agglomeration of parallel monatomic surface steps during the growth process [66]. A rigorous analysis of the crystallography modeling leads to the dynamics which are represented by the following type of recursive functional differential equation.

$$\dot{x}(t) = F(x(t), x(t - x(t))) \quad (41)$$

5.2 Asymptotic Stability of the Equilibrium Point

We now use the Lyapunov Razumikhin (LR) theorem to show that the trivial steady state (equilibrium point: $x = 0$) of the SD-DDE system (39) is asymptotically stable.

Consider the Lyapunov function $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ described by $V(x(t)) = \frac{1}{2}x^2(t)$. We have for $x \in \mathbb{R}^+$, $\alpha(x) \leq V(x) \leq \beta(x)$, with $\alpha(x) = \frac{1}{4}x^2$ and $\beta(x) = 2x^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(x) = xe^{\sqrt{2x}}$, $x \in \mathbb{R}^+$. Let $x(t)$ be the solution trajectory

of (39), such that for $t \geq 0$, $\theta \in [-\tau, 0]$,

$$\begin{aligned}
V(x(t+\theta)) &< \eta(V(x(t))) \\
\Rightarrow |x(t-x(t))| &< |x(t)| \\
\Rightarrow x(t-x(t)) &< x(t); \quad (\text{Since the delay should be positive}). \tag{42}
\end{aligned}$$

Now, we have for $t \geq 0$

$$\begin{aligned}
\dot{V}(x(t)) &= x(t)\dot{x}(t) \\
&= -x(t)x(t-x(t)); \quad (\text{Using (42)}) \\
&< -x^2(t) \\
&< 0
\end{aligned} \tag{43}$$

Since $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$, we, therefore, conclude that the equilibrium point of (39) is asymptotically stable. Furthermore, since $V(x) = \frac{1}{2}x^2(t)$ i.e. $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Therefore, V is radially unbounded, and by definition this implies the global asymptotic stability of the equilibrium point.

5.3 Perturbation Analysis

In this section we perform two types of perturbation analyses to the SD-DDEs namely the singular perturbation and the regular perturbation.

5.3.1 Singular Perturbation Analysis

Consider the singularly perturbed SD-DDE, with the small parameter $\varepsilon > 0$ as the perturbation parameter and $\forall a, b \in \mathbb{R}$, as follows.

$$\varepsilon \dot{x}(t) = ax(t) + bx(t-x(t)) \tag{44}$$

Notice that for $\varepsilon = 0$, (44) no longer remains an SD-DDE but becomes a state-dependent delay *difference* equation. The small signal analysis to (44) yields,

$$\begin{aligned}
\varepsilon \dot{x}(t) &= ax(t) + bx(t) - b\dot{x}(t)x(t) \\
\Rightarrow (bx + \varepsilon)\dot{x} &= (a + b)x \\
\Rightarrow \int \frac{bx + \varepsilon}{x} dx &= \int (a + b) dt + C \\
\Rightarrow bx + \varepsilon \ln x &= (a + b)t + C \\
\Rightarrow \ln(e^{bx} x^\varepsilon) &= (a + b)t + C \\
\Rightarrow e^{bx} x^\varepsilon &= Be^{(a+b)t} \\
\Rightarrow xe^{\frac{bx}{\varepsilon}} &= De^{\frac{(a+b)}{\varepsilon}t} \\
\Rightarrow \frac{bx}{\varepsilon} e^{\frac{bx}{\varepsilon}} &= D \frac{b}{\varepsilon} e^{\frac{(a+b)}{\varepsilon}t} \\
\Rightarrow \frac{bx}{\varepsilon} &= \mathbf{W}_k(D \frac{b}{\varepsilon} e^{\frac{(a+b)}{\varepsilon}t}) \\
\Rightarrow x(t) &= \frac{\varepsilon}{b} \mathbf{W}_k(D \frac{b}{\varepsilon} e^{\frac{(a+b)}{\varepsilon}t}) \tag{45}
\end{aligned}$$

In the above equations B , C and D are arbitrary constants of integration and $\mathbf{W}_k(\cdot)$ represents the k – th branch of the Lambert W function.

Remark 2:

Using the Lyapunov-Razumikhin theorem, it can be shown that the singularly perturbed systems is asymptotically stable if $a + |b| < 0$, (see Theorem 14 in §8.1.1). This is only a sufficient condition. This is also evident from (45).

5.3.2 Regular Perturbation Analysis

By introducing a small perturbation parameter $\sigma > 0$ in the delay term, we get the following regularly perturbed version of the SD-DDE.

$$\dot{x}(t) = ax(t) + bx(t - \sigma x(t)); \quad \forall a, b \in \mathbb{R} \tag{46}$$

Again performing the small signal perturbation analysis yields the following.

$$\begin{aligned}
\dot{x}(t) &= ax(t) + bx(t) - b\sigma x(t)x(t) \\
\Rightarrow (1 + b\sigma x)\dot{x} &= (a + b)x \\
\Rightarrow \int \frac{(1 + b\sigma x)}{x} dx &= \int (a + b) dt + E \\
\Rightarrow \ln x + b\sigma x &= (a + b)t + E \\
\Rightarrow \ln(xe^{b\sigma x}) &= (a + b)t + E \\
\Rightarrow xe^{b\sigma x} &= Fe^{(a+b)t} \\
\Rightarrow (b\sigma x)e^{(b\sigma x)} &= b\sigma Fe^{(a+b)t} \\
\Rightarrow b\sigma x &= \mathbf{W}_k(b\sigma Fe^{(a+b)t}) \\
\Rightarrow x(t) &= \frac{1}{b\sigma} \mathbf{W}_k(b\sigma Fe^{(a+b)t}) \tag{47}
\end{aligned}$$

where E and F are arbitrary constants of integration and $\mathbf{W}_k(\cdot)$ represents the k -th branch of the Lambert W function.

Remark 3:

Using the Lyapunov-Razumikhin theorem, it can be shown that the regularly perturbed systems is asymptotically stable if $a + |b| < 0$, (see Theorem 14 in §8.1.1 where $\rho \rightarrow 1$). This is only a sufficient condition. This is also evident from (47). In the special case when $\sigma = 0$ (delay free case) $a + b < 0$ is a necessary and sufficient for the stability. In this case the SD-DDE reduces to the ODE $\dot{x}(t) = (a + b)x(t)$ which has the closed form solution $x(t) = x_0 e^{(a+b)t}$ with $x_0 \in \mathbb{R}$ being any arbitrary initial condition.

5.4 Bifurcation and Stability Switches

Here we demonstrate that the system (39) exhibits a strange phenomenon of bifurcation and stability switches. In order to establish this phenomenon, we introduce a bifurcation parameter μ to (39) so that the resulting perturbed SD-DDE becomes

$$\dot{x}(t) = -x(t - x(t)) + \mu \tag{48}$$

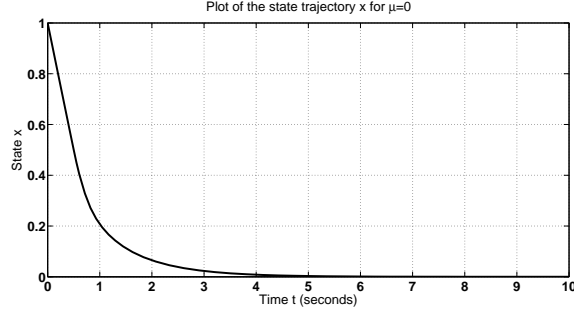


Figure 11: Profile of State $x(t)$ of the State-Dependent Delay System With $\mu = 0$

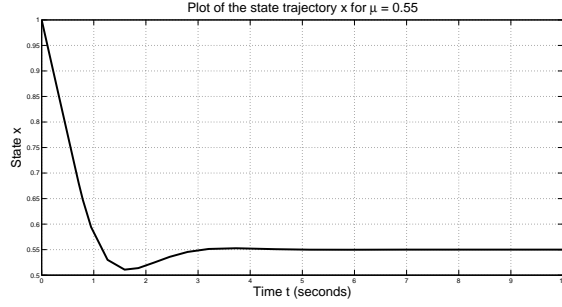


Figure 12: Profile of State $x(t)$ of the State-Dependent Delay System With $\mu = 0.55$

Now, we want to investigate the qualitative behavior of the SD-DDE (48) as the bifurcation parameter μ is varied from 0 to 2.

Fig. 11 shows the state trajectory of the nominal system (39) i.e. when the bifurcation parameter in (48) is 0. We see that the equilibrium point (origin) of the system is asymptotically stable. This is verified by the Lyapunov-Razumikhin theorem.

Now, we keep on increasing the bifurcation parameter μ . Fig. 12 depicts the profile of the state when $\mu = 0.55$. Again the same initial history is assumed that is $x(t) = 1 \forall t \in (-\infty, 0]$. We see that an undershoot has appeared revealing the fact that the relative stability has decreased as compared to the nominal case.

We further increase the bifurcation parameter to $\mu = 1.25$. The corresponding state trajectory is shown in Fig. 13. We observe that this time there are decaying oscillations. The system behaves pretty much like a second order under-damped system. This shows a further decrease in the relative stability.

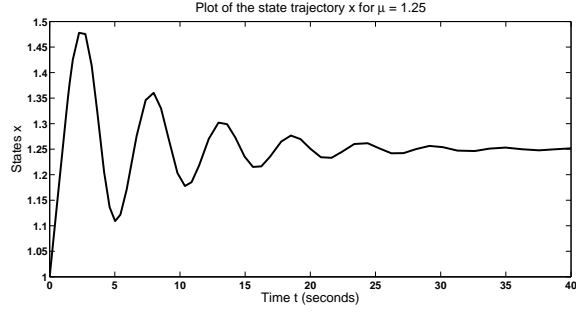


Figure 13: Profile of State $x(t)$ of the State-Dependent Delay System With $\mu = 1.25$

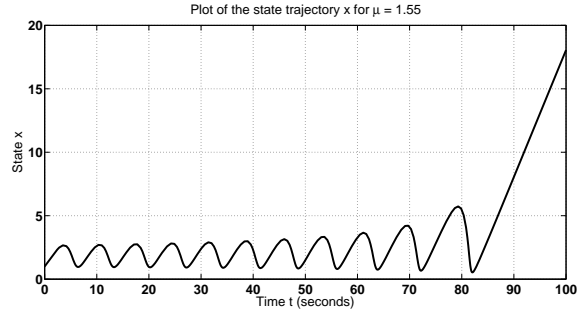


Figure 14: Profile of State $x(t)$ of the State-Dependent Delay System With $\mu = 1.55$

Fig. 14 demonstrates the state trajectory of the SD-DDE system (48) when μ catches a value of 1.55. In this case the moves from the oscillatory regime to the unstable regime. In the start, there are sustained oscillations with slowly increasing peaks and, then all of a sudden the system ramps up. We call it a stability switch.

Notice that for $\mu = 1.55$, the system at first exhibits oscillatory behavior for some time and then all of a sudden enters into an unstable regime,(see also [59] and [31]). We see that this instability is quite different from the one associated with linear time invariant unstable systems. Here the system is unstable but with a linear growth rate where as linear unstable systems we have exponential growth rate which is a more violent form of instability. Basically state-dependent delay system has some innate/inherent/intrinsic feedback in the time argument and thus has a self-regulation feature. This is the reason that we do not have violent instability (exponential growth rate of state trajectories).

Finally Fig. 15 reflects the profile or trajectory of the state when the bifurcation

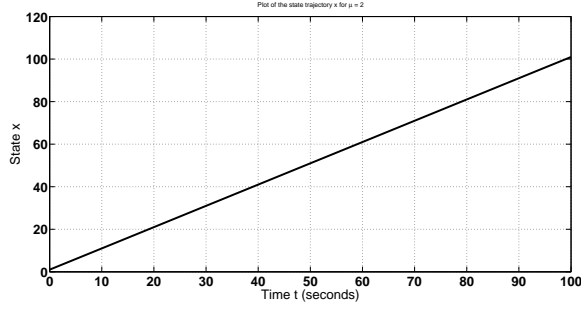


Figure 15: Profile of State $x(t)$ of the State-Dependent Delay System With $\mu = 2$

parameter $\mu = 2$. Again we mention that the same initial history is taken as for the previous cases. This time the system becomes unstable. Notice the ramp growth of the state. Further increase in μ will keep the system unstable.

So we observe that there is a qualitative change in the behavior of the system as the bifurcation parameter is varied. The system switches the stability status. This is referred to as bifurcation of the SD-DDE. Notice that this is not the usual Hopf bifurcation. Such a bifurcation is termed as transcritical bifurcation. The equilibrium point or fixed point exists for all values of a parameter and is never destroyed. Nevertheless, such a fixed point interchanges its stability with another fixed point as the parameter is varied. Another well known example of transcritical bifurcation is the famous Verhulst's logistic equation in population dynamics. We hereby mention that such a strange behavior can never occur in a perturbed Linear Time Invariant (LTI) system. An LTI system of the form $\dot{x}(t) = -x(t) + \mu$ never exhibits a stability switch and is *always stable* irrespective of the parameter μ .

5.5 Consequences of Taylorization

In this section, we show that approximation or truncation by the Taylor series expansion (we call it Taylorization) of SD-DDEs has very anomalous results. Let \mathcal{C}^ω denote the space of all holomorphic or real valued analytic functions defined on \mathbb{R}^+ . Now, assuming that the solution of (39) i.e. $x(t)$ is a \mathcal{C}^ω function, then so is $t - x(t)$ and

the composition of the two i.e. $x(t - x(t))$. Therefore, the Taylor series expansion of the state-dependent delay system (39) is given by,

$$\mathbf{D}x = - \sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!} \mathbf{D}^j x \quad (49)$$

or

$$\dot{x}(t) = -x(t) + x(t)\dot{x}(t) - \frac{1}{2!}x^2(t)\ddot{x}(t) + \dots \quad (50)$$

where $\mathbf{D} \triangleq \frac{d}{dt}(\cdot)$ is the usual Differential or Derivative operator.

Notice that this is not a linearization. All the terms in the expansion are nonlinear because of the state dependent delay in the original equation. This is the reason that we call it Taylorization.

A first order Taylorization of the state-dependent delay system is given as follows.

$$\dot{x}(t) = -\frac{x(t)}{1 - x(t)}; \quad x(0) = x_0 \quad (51)$$

Notice that this (51) has a singularity at $x(t) = 1$. It is a nonlinear differential equation. This system is clearly conditionally stable. It is stable when the initial state lies inside the unit ball and is unstable when $x > 1$. Whereas the original system characterized by the SD-DDE was asymptotically stable, Fig. 16. depicts the fact that the Taylorized nonlinear system of first order is unstable. The chosen initial condition is $x_0 = 1.0001$. We can also show the instability of the first order Taylor series approximated system by integrating (51) using separation of variables as follows.

$$\begin{aligned} \left(\frac{1-x}{x}\right)dx &= -dt \\ d(\ln x - x) &= -dt \\ \ln \frac{x}{x_0} - x + x_0 &= -t \\ \frac{x}{x_0} e^{-x} &= e^{-x_0 - t} \end{aligned}$$

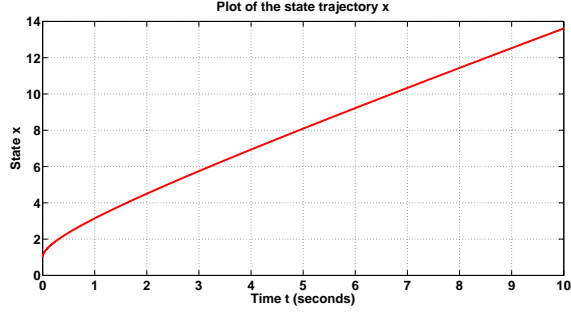


Figure 16: Taylorization of the State-Dependent Delay System By a First Order System

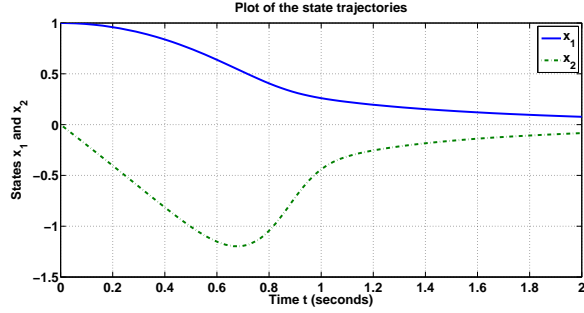


Figure 17: Taylorization of the State-Dependent Delay System By a Second Order System

This yields the following family of integral curves

$$x(t) = -W(-Ce^{-t}) \quad (52)$$

where $W(\cdot)$ represents the Lambert-W function and C is any arbitrary constant. Using the initial condition, $C = x_0 e^{-x_0}$ and thus $x(t) = -W(-x_0 e^{-t-x_0})$.

Now, we give the the second order Taylorization of the SD-DDE system as follows.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{2}{x_1^2(t)}[x_1(t)x_2(t) - x_1(t) - x_2(t)] \end{cases} \quad (53)$$

Fig. 17 shows the state trajectories with initial conditions $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

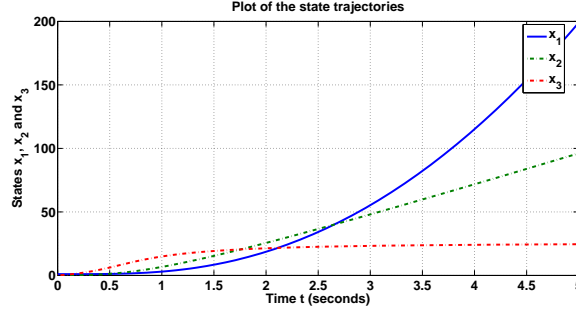


Figure 18: Taylorization of the State-Dependent Delay System By a Third Order System

Now, we Taylorize the SD-DDE system by a third order system as follows.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = \frac{3}{x_1^3(t)}[2x_1(t) + 2x_2(t) - 2x_1(t)x_2(t) \\ \quad + x_1^2(t)x_3(t)] \end{cases} \quad (54)$$

Fig. 18 shows the state trajectories with initial conditions $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

We can easily see that this system is unstable. In fact, the Taylorization has also invited a singularity at $x_1(t) = 0$.

Finally we construct the fourth order Taylorized nonlinear ODE system version of the SD-DDE expressed as given below.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \frac{24}{x_1^4(t)}[-x_1(t) - x_2(t) + x_1(t)x_2(t) \\ \quad - \frac{x_1^2(t)x_3(t)}{2} + \frac{x_1^3(t)x_4(t)}{6}] \end{cases} \quad (55)$$

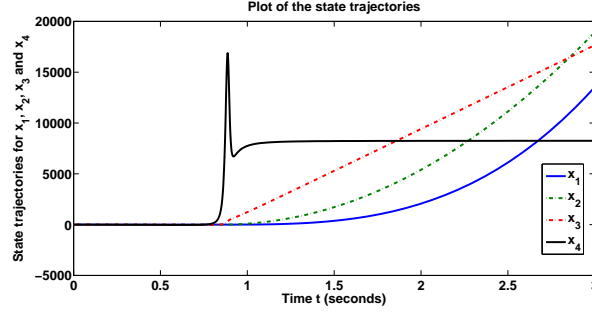


Figure 19: Taylorization of the SD-DDE Based System By a Fourth Order System

Fig. 19 shows the state trajectories with initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We notice that the trajectories of the resulting system blow up and the system becomes unstable. Again, Taylorization has introduced a fourth order singularity at $x_1(t) = 0$. Whereas, in reality, the original SD-DDE system (39) was stable as shown by using LR function based approach. From all the discussion and simulation results in this section, we infer that the conclusion drawn on the basis of Taylorization of infinite dimensional $(\infty - D)$ systems like SD-DDEs can lead to serious flaws, blunders and inaccuracies. Therefore, we proscribe the usage of Taylor series approximations in $\infty - D$ systems.

5.6 Decay Rates Comparisons of ODEs and SD-DDEs

Fig. 20 shows the comparison of the decay rate of the stable SD-DDE (39) and a stable Ordinary Differential Equation (ODE) of the form $\dot{x}(t) = -x(t)$. We notice that the SD-DDE decays at a faster rate as compared to the companion ODE. We explain this by the iterative and self-regulated nature of the SD-DDE. Notice that for the ODE the instantaneous rate at which the trajectory falls is proportional to the value of $x(t)$ at that particular instant whereas the rate at which the trajectory of SD-DDE falls is proportional to a value of x at a previous instant but that *previous instant* value

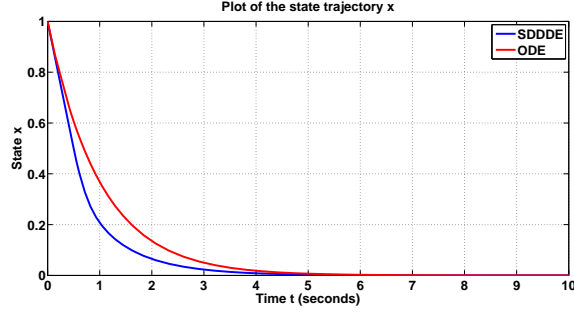


Figure 20: Decay Rates Comparisons of ODEs and SD-DDEs

is larger than $x(t)$ because $x(t)$ is continuous and monotonically decreasing (stable). This means that there exists a $\lambda > 1$ such that the trajectory of the SD-DDE falls proportional to $\lambda x(t)$. However, we mention that the asymptotic behavior of the two trajectories remains the same because the delay (state-dependent delay) itself decays as $t \rightarrow \infty$.

Fig. 21 gives a comparison of the state trajectories of the unstable SD-DDE system (40) and a comparison unstable ODE system $\dot{x}(t) = x(t)$. For ODE system, we get the well-known exponential growth. The behavior of the SD-DDE system is quite strange. It is unstable but it grows linearly. We explain this behavior by observing that for this system

$$\ddot{x}(t) = \dot{x}(t - x(t))(1 - \dot{x}(t)) \quad (56)$$

The causality constraint dictates that $\ddot{x}(t) \geq 0$ which in turn implies that $\dot{x}(t)$ is increasing. When $\dot{x}(t)$ reaches 1, this forces $\ddot{x}(t) = 0$ by (56). As a result a ramp behavior of the form $x(t) = at + b$ is achieved.

The bifurcation phenomenon in §5.4 can also be explained on similar footings as above. As the bifurcation parameter increases, the system behaves almost as $\dot{x}(t) = \mu$ in the asymptotic sense because the delay becomes smaller and smaller and also the decay is fast. This accounts for the ramp profile of the state when the bifurcation parameter is sufficiently large.

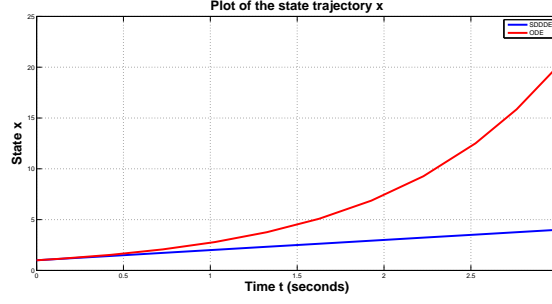


Figure 21: Comparison of Unstable ODE and SD-DDE

5.7 Concluding Remarks

In this chapter we make it clear that a special attention should be made while analyzing ∞ -D systems in general and systems involving state-dependent delays in particular. Highly erroneous, anomalous and strange results may arise if Taylorization (approximation by Taylor series partial sums) is attempted. The reason is that systems with state-dependent delays are inherently nonlinear infinite dimensional systems. Taylorization of such systems makes them finite dimensional but the nonlinear nature is still preserved. But this finite dimensional representation loses the inherent structural ingredients (e.g., the initial history which lies in a suitable function space) of the infinite dimensional systems. The Taylor series approximation is usually found ubiquitous in physics and engineering community. Therefore, we strongly proscribe the usage of this approximation to time delay systems in order to avoid inaccuracies.

We also demonstrated that systems with state-dependent delays exhibit bifurcation and stability switches. The system is stable for certain values of the bifurcation parameter μ and as μ increases, the systems first breaks into oscillations and then switches into an unstable regime. Perturbation analysis was also performed and the results were nicely expressed in terms of the Lambert-W function. A comparison of the decay rate of the SD-DDE and the companion ODE was also made. Furthermore, it was also emphasized that the instability of SD-DDE based systems is quite different from the violent (exponential growth) instability of LTI ODE based systems.

CHAPTER VI

SCALE DYNAMIC SYSTEMS & SELF-STARTING FEATURE

The main object of investigation in this chapter is the Scale Dynamic systems (SDS). These systems form a typical class of time-varying delay differential equations where the delay is unbounded but the delay rate is less than or equal to unity. The basic form is $\dot{x}(t) = \pm x(\alpha t); \alpha \in [0, 1]$. The central theme of this contribution is that the *infinitesimal generator* associated with such systems is a *compact operator* on the Hilbert space l^2 or the Banach space l^∞ . We investigate this behavior for the scalar and higher order SDS in an operator theoretic framework, analyze the spectrum and give necessary and sufficient conditions for the asymptotic stability [6].

6.1 Introduction

Consider the following system characterized by the functional differential equation (FDE),

$$\dot{x}(t) = bx(\alpha t) \tag{57}$$

where $b \in \mathbb{R}$ and $\alpha \in [0, 1]$. The boundary (endpoint) cases i.e., $\alpha = 0$ and $\alpha = 1$ are trivial cases and are of least interest since in both the case the FDE reduces to an ODE. In the literature, such systems are also termed as scale delay systems [88]. Such systems appear in physics and engineering e.g., Cherenkov radiation, light absorption by interstellar matter, the wavy motion exhibited by the overhead line attached to pantograph used for the collection of current of an electric locomotive, the theory of dielectric materials, number theory and continuum mechanics; see [54], [51], [88] and the references therein.

Using the Fundamental Theorem of Calculus, system (57) can be written in the form of an integral equation as follows.

$$x(t) = x(0) + b \int_0^t x(\alpha\zeta) d\zeta \quad (58)$$

$$\Leftrightarrow x(t) = x(0) + \frac{b}{\alpha} \int_0^{\alpha t} x(\theta) d\theta \quad (59)$$

Using Picard-Lindelöf iteration procedure, one can write the above equation in the following iterative form.

$$x_{n+1}(t) = x(0) + \frac{b}{\alpha} \int_0^{\alpha t} x_n(\theta) d\theta; \quad n = 0, 1, 2, \dots \quad (60)$$

Starting with $x_0(t) = x(0), \forall t \geq 0$, we get the following iterates for $x(t)$.

$$\begin{aligned} x_1(t) &= x(0) + \frac{b}{\alpha} \int_0^{\alpha t} x(0) d\theta = (1 + bt)x(0) \\ x_2(t) &= x(0) + \frac{b}{\alpha} \int_0^{\alpha t} x_1(\theta) d\theta = (1 + bt + \alpha \frac{(bt)^2}{2!})x(0) \\ x_3(t) &= x(0) + \frac{b}{\alpha} \int_0^{\alpha t} x_2(\theta) d\theta = (1 + bt + \alpha \frac{(bt)^2}{2!} \\ &\quad + \alpha^3 \frac{(bt)^3}{3!})x(0) \\ x_4(t) &= (1 + bt + \alpha \frac{(bt)^2}{2!} + \alpha^3 \frac{(bt)^3}{3!} + \alpha^6 \frac{(bt)^4}{4!})x(0) \\ x_5(t) &= (1 + bt + \alpha \frac{(bt)^2}{2!} + \alpha^3 \frac{(bt)^3}{3!} + \alpha^6 \frac{(bt)^4}{4!} \\ &\quad + \alpha^{10} \frac{(bt)^5}{5!})x(0) \\ &\vdots \\ x_n(t) &= (1 + bt + \alpha \frac{(bt)^2}{2!} + \alpha^3 \frac{(bt)^3}{3!} + \alpha^6 \frac{(bt)^4}{4!} + \alpha^{10} \frac{(bt)^5}{5!} \\ &\quad + \dots + \alpha^{\frac{n(n-1)}{2}} \frac{(bt)^n}{n!})x(0) \end{aligned}$$

Clearly, the above iterations converge and we have the following series expression for $x(t)$ as a closed form.

$$x(t) = \left(\sum_{n=0}^{\infty} \alpha^{\frac{n(n-1)}{2}} \frac{(bt)^n}{n!} \right) x(0) \quad (61)$$

The reason that the above infinite series is absolutely convergent can be justified by the fact that it is upper bounded by the exponential function i.e., $x(t) \leq e^{|bt|}$, by taking the upper limit on α i.e., $\alpha = 1$. Notice that the solution (61) reconciles with the one in [76] obtained using Frobenius power series method. The Frobenius power series method assumes that the solution is \mathcal{C}^ω (analytic).

Fundamentally, this is a delay system with the unbounded time-variant delay $\tau(t)$ satisfying $\tau(t) = t - \alpha t = (1 - \alpha)t$ and $\dot{\tau}(t) = 1 - \alpha \leq 1$. This bounded rate condition ($\dot{\tau} \leq 1$) makes the time-varying delay system well-posed and causal. See [82] and [83] for the well-posedness associated with time-varying delay systems.

Now, we consider the following system.

$$\dot{x}(t) = ax(t) + bx(\alpha t) \quad (62)$$

The above equation is also called pantograph equation, see e.g., Iserles [51] and the references therein. Using Picard-Lindelöf method, the following series solution can be obtained for the SDS of (62).

$$x(t) = \left(1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (a + b\alpha^j) \frac{t^n}{n!} \right) x(0) \quad (63)$$

Now, we are interested in the state space realization of (57). Let us defined our states as $x_1(t) = x(t), x_2(t) = x(\alpha t), x_3(t) = x(\alpha^2 t), \dots, x_N(t) = x(\alpha^{N-1} t), \dots$ where $N \in \mathbb{N}$ and \mathbb{N} represents the set of natural numbers. Therefore, the state dynamics

are governed by the following coupled first order ODEs.

$$\begin{aligned}
\dot{x}_1(t) &= \dot{x}(t) = bx(\alpha t) = bx_2(t) \\
\dot{x}_2(t) &= \alpha \dot{x}(\alpha t) = b\alpha x(\alpha^2 t) = b\alpha x_3(t) \\
\dot{x}_3(t) &= \alpha^2 \dot{x}(\alpha^2 t) = b\alpha^2 x(\alpha^3 t) = b\alpha^2 x_4(t) \\
&\vdots \\
\dot{x}_{N-1}(t) &= b\alpha^{N-2} x_N(t) \\
\dot{x}_N(t) &= b\alpha^{N-1} x_{N+1}(t) \\
&\vdots
\end{aligned}$$

In vector-matrix form the above equations translate as follows.

$$\dot{\mathfrak{X}} = \mathfrak{A}\mathfrak{X}$$

where,

$$\mathfrak{A} = b \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha^2 & 0 & \dots \\ \vdots & & & & & \\ 0 & 0 & \dots & \alpha^{N-1} & 0 & \dots \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

and

$$\mathfrak{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \\ \vdots \end{pmatrix}.$$

Notice that

$$\mathfrak{X}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \\ x_N(0) \\ \vdots \end{pmatrix} = \mathbf{1}x(0)$$

where $\mathbf{1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \end{pmatrix}^\top$ is a column vector with all entries being 1's. Here once again we see that only the germ or seed $x(0)$ is required for the system to evolve and hence the term self-starting. Also, observe that $\lim_{N \rightarrow \infty} \alpha^{N-1} = 0$. Therefore, $\dot{x}_N \rightarrow 0$ as $N \rightarrow \infty$. This means that when N is very very large, we get the following higher order finite dimensional equivalent version of the system (57).

$$\dot{\mathbf{x}} = \mathbf{A}_N \mathbf{x}$$

where $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_N \end{pmatrix}^\top$ and

$$\mathbf{A}_N = b \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha^2 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & \dots & 0 & 0 & \alpha^{N-2} \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \text{ It is noteworthy here that the matrix } \mathbf{A}_N$$

has a special structure of a strictly upper triangular matrix. All the eigenvalues of \mathbf{A}_N are zero. Therefore, $\text{tr}(\mathbf{A}_N) = \det(\mathbf{A}_N) = 0$ and $\text{Spec}(\mathbf{A}_N) = \{0\}$. In fact the matrix \mathbf{A}_N is a *nilpotent* matrix with the degree or index N i.e., $\mathbf{A}_N^N = 0$. The algebraic multiplicity of the zero eigenvalue is N and its geometric multiplicity (dimension of the eigenspace) is less than N . The corresponding eigenvectors cannot span \mathbb{R}^N . Thus, the matrix \mathbf{A}_N is a defective matrix and cannot be diagonalized. Because of the repeated eigenvalues at the origin, the system is unstable irrespective of the value

of $b \in \mathbb{R} - \{0\}$. The system behaves in a similar fashion to a chain of integrators in cascade.

The transformation matrix which brings \mathfrak{A}_N to the Jordan canonical form is as follows.

$$T_N = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1/\alpha & & & \\ & & & 1/\alpha^3 & & \\ & & & & \ddots & \\ & & & & & 1/\alpha^{\frac{(N-1)(N-2)}{2}} \end{pmatrix}$$

Clearly, $T_N \in Gl_n(\mathbb{R})$ where $Gl_n(\mathbb{R})$ represents the General linear group of all $n \times n$ real invertible (nonsingular) matrices. Notice that,

$$T_N^{-1} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \alpha & & & \\ & & & \alpha^3 & & \\ & & & & \ddots & \\ & & & & & \alpha^{\frac{(N-1)(N-2)}{2}} \end{pmatrix}$$

With this transformation, the matrix \mathfrak{A}_N can be transformed into the Jordan canonical form as follows.

$$\mathfrak{A}_N = T_N J_N T_N^{-1}$$

where the Jordan block matrix J_N is as follows.

$$J_N = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

The solution of the corresponding higher order ODE is as follows.

$$\mathbf{x}(t) = e^{A_N t} \mathbf{1} x(0) = \sum_{k=0}^{N-1} \frac{(A_N t)^k}{k!}$$

From which, again (61) follows. One can truncate the series upto N terms and ensure that the absolute error is less than certain tolerance ε for compute the solution for time T . So from (61), for ε -accurate simulations,

$$\alpha^{\frac{N(N-1)}{2}} \frac{(|b|T)^N}{N!} |x(0)| \leq \varepsilon.$$

This yields the upper bound T_{max} as a function of N as follows.

$$T_{\max} = \frac{1}{|b|} \sqrt[N]{\frac{N! \varepsilon}{\alpha^{\frac{N(N-1)}{2}} |x(0)|}} \quad (64)$$

By using the Stirling formula for the approximation of $N!$, we get the following.

$$T_{\max} \approx \frac{1}{|b|} \sqrt[N]{\frac{\sqrt{2\pi N} \left(\frac{N}{e}\right)^N \varepsilon}{\alpha^{\frac{N(N-1)}{2}} |x(0)|}} \quad (65)$$

Fig. 22 shows the state trajectories for $b = -1$ and different values of α . The initial condition is $x(0) = 1$. The value of N was taken as 10,000. Notice that the system is unstable. There are growing oscillations for $\alpha \in (0, 1)$.

6.2 Higher Order Case

Now, we consider the higher order version of (57) as follows.

$$\dot{x}(t) = Bx(\alpha t) \quad (66)$$

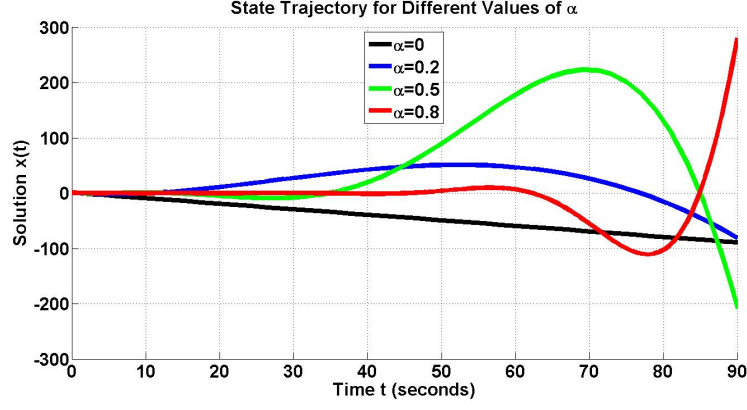


Figure 22: State Trajectories for Different Values of α and $b = -1$

where $x(t) \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$. Inspired by the series solution (61) of the corresponding scalar version, we have the following solution for the system (66).

$$x(t) = \left(\sum_{n=0}^{\infty} \alpha^{\frac{n(n-1)}{2}} \frac{(Bt)^n}{n!} \right) x(0) \quad (67)$$

Substitution in (66) yields that the equation is satisfied. Also, the series converges because the corresponding matrix exponential e^{Bt} converges. Likewise, extending the theory further, we consider the general vector-matrix version of the pantograph system (62) as follows. Now, we consider the following system,

$$\dot{x}(t) = Ax(t) + Bx(\alpha t) \quad (68)$$

where $x(t) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$. We have the following series solution for system (68).

$$x(t) = \left(I + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (A + B\alpha^j) \frac{t^n}{n!} \right) x(0) \quad (69)$$

The above solution converges for any pair of matrices (A, B) and satisfies the FDE (68). The following theorem provides sufficient conditions for the asymptotic stability of the system (68). The proof is based on the following Lyapunov-Krasovskii (LK) functional.

$$V(x(t)) = x^\top(t)Px(t) + \int_{\alpha t}^t x^\top(\xi)Q(\xi)d\xi; \quad P, Q > 0 \quad (70)$$

Theorem 6 [88] *The higher order pantograph SDS in (68) is globally asymptotically stable if any of the following three equivalent conditions are satisfied. (1). Given any symmetric and positive definite matrix $Q = Q^\top > 0$, there exists a matrix $P = P^\top > 0$ such that the following is satisfied.*

$$\begin{pmatrix} A^\top P + PA + Q & PB \\ B^\top P & -\alpha Q \end{pmatrix} < 0 \quad (71)$$

(2). There exist a pair of symmetric and positive definite matrices (P, Q) such that the following Algebraic Riccati Inequality (ARI) is satisfied.

$$A^\top P + PA + Q + PB(\alpha Q)^{-1} B^\top P < 0 \quad (72)$$

(3). There exists a triplet of symmetric and positive definite matrices (P, Q, R) such that the following Delay Algebraic Riccati Equation (DARE) holds.

$$A^\top P + PA + Q + PB(\alpha Q)^{-1} B^\top P + R = 0 \quad (73)$$

The following theorem gives a simple sufficient condition in terms of Algebraic Lyapunov equation which is a linear matrix equation and easy to solve as compared to the quadratic DARE.

Theorem 7 *The higher order system (68) is globally asymptotically stable if the following two conditions are satisfied.*

Given any symmetric and positive definite matrix $Q = Q^\top > 0$ and a scalar $\rho > 1$, there exists a symmetric and positive definite matrix $P = P^\top > 0$ such that the following is satisfied.

$$\lambda_{\min}(Q) - 2\rho\lambda_{\max}(P)\|B\| > 0 \quad (74)$$

where $\|B\| = \sigma_{\max}(B) = \sqrt{\lambda_{\max}(B^\top B)}$ and the matrix P is the solution of the following Algebraic Lyapunov Equation (ALE).

$$A^\top P + PA + Q = 0 \quad (75)$$

Proof. The proof is based on Razumikhin argument. Let the LR function be $V(x(t)) = x^\top(t)Px(t)$. Then $\|x(t - \tau(t))\|_P \leq \rho\|x(t)\|_P, \rho > 1$ implies $\dot{V}(x(t)) = (A^\top P + PA)\|x(t)\|^2 + 2x^\top PBx(\alpha t) \leq -\lambda_{\min}\|x(t)\|^2 + 2\rho\|P\|\|B\|\|x(t)\|^2$ and the proof is done. Q.E.D. ■

Remark 4:

Using the vectorization operator and Kronecker products, from the ALE, we have,

$$\begin{aligned} \text{vec}(A^\top P + PA) &= -\text{vec}(Q) \\ \Leftrightarrow (A^\top \otimes I + I \otimes A^\top)\text{vec}(P) &= -\text{vec}(Q) \\ \Leftrightarrow \text{vec}(P) &= -(A^\top \otimes I + I \otimes A^\top)^{-1}\text{vec}(Q) \end{aligned}$$

where the inverse exists if the matrix $(A^\top \otimes I + I \otimes A^\top)$ does not have a 0 as an eigenvalue.

For the scalar case, the solution of the Lyapunov equation yields $q = -2ap$ and we immediately get the following result as a corollary.

Corollary 1 *The scalar system $\dot{x}(t) = ax(t) + bx(\alpha t)$ is asymptotically stable if $a < 0$ and $|a| > |b|$.*

6.3 Necessary & Sufficient Condition for Asymptotic Stability

Let us reconsider the general SDS as follows.

$$\dot{x}(t) = Ax(t) + Bx(\alpha t) \quad (76)$$

where $x(t) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$. In the operator form, (76) can be written as follows.

$$(D - A - BS_\alpha)x = 0 \quad (77)$$

where D is the differential operator and S_α is the scaling operator i.e., $\sigma_t S_\alpha x = x(\alpha t)$, with σ_t being the evaluation functional defined as $\sigma_t x := x(t)$. We define our new

states as $y_1 := x$, $y_2 = S_\alpha x$, $y_3 = S_{\alpha^2} x, \dots, y_N = S_{\alpha^{N-1}} x, \dots$. In other words, $y_1(t) := x(t)$, $y_2(t) := x(\alpha t)$, $y_3(t) = x(\alpha^2 t)$, \dots , $y_N(t) = x(\alpha^{N-1} t), \dots$. Therefore, the new higher order state space realization will be as follows.

$$\dot{\mathbf{y}} = \mathcal{A} \mathbf{y}$$

where $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_N & \cdots \end{pmatrix}^\top$ and

$$\mathcal{A} = \begin{pmatrix} A & B & & & & \\ & \alpha A & \alpha B & & & \\ & & \alpha^2 A & \alpha^2 B & & \\ & & & \alpha^3 A & \alpha^3 B & \\ & & & & \ddots & \ddots \\ & & & & & \alpha^{N-1} A & \alpha^{N-1} B \\ & & & & & & \ddots & \ddots \end{pmatrix}. \quad (78)$$

Therefore, we can see that \mathcal{A} is the infinitesimal generator for the SDS (76). Let \mathcal{A}_N be the $N \times N$ truncated matrix of \mathcal{A} . Notice that the matrix \mathcal{A}_N has a bidiagonal structure and is a special case of an upper triangular matrix. The leading or principal diagonal contains $\alpha^j A$ and the superdiagonal contains block matrix entries of the form $\alpha^j B$.

Claim 1 $\mathcal{A} : l_2 \rightarrow l_2$ is a compact operator on the Hilbert space.

Proof: Consider the sequence of operators $A_N : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$ which are essentially finite dimensional operators ($\mathbb{R}^{Nn \times Nn}$ matrices). Then it is easy to show that $\mathcal{A}_N \rightarrow \mathcal{A}$ i.e., $\|\mathcal{A}_N - \mathcal{A}\| \rightarrow 0$ as $N \rightarrow \infty$ in the operator norm induced by l_2 norm. Since A_N is compact, so is \mathcal{A} . ■

Notice that $e^{\mathcal{A}t}, t \geq 0$ forms the \mathbf{C}_0 -semigroup with the associated infinitesimal generator \mathcal{A} . Moreover,

$$e^{\mathcal{A}t} = \sum_{n=0}^{\infty} \frac{(\mathcal{A}t)^n}{n!}. \quad (79)$$

The series converges in the operator norm, since \mathcal{A} is compact and bounded. One can also use theorems of Hille-Yosida and Lumer-Phillips as generation theorems. Since \mathcal{A} is an infinite dimensional compact operator, we can show using Fredholm alternative that the accumulation point $\{0\}$ is indeed in its spectrum. Notice that the spectrum of \mathcal{A} can be given as follows.

$$\text{Spec}(\mathcal{A}) = \bigcup_{j=0}^{\infty} \text{Spec}(\alpha^j A) \bigcup \{0\} \quad (80)$$

Since $\text{Spec}(\alpha^j A) = \alpha^j \text{Spec}(A)$, therefore, we get the following interesting result as a proposition.

Proposition 1

$$\text{Spec}(\mathcal{A}) = \begin{cases} \underbrace{\bigcup_{j=0}^{\infty} \alpha^j \text{Spec}(A)}_{\sigma_p} \bigcup \underbrace{\{0\}}_{\sigma_r}; & A \neq -B \\ \underbrace{\bigcup_{j=0}^{\infty} \alpha^j \text{Spec}(A)}_{\sigma_p} \bigcup \underbrace{\{0\}}_{\sigma_p}; & A = -B \end{cases} \quad (81)$$

Note:

In the special case, when $A = 0$, there is *only* point spectrum i.e.,

$$\text{Spec}(\mathcal{A}) = \{0\} = \sigma_p(\mathcal{A}) \quad (82)$$

Notice that in this case, the system is always unstable irrespective of the matrix B . Now, we give the following result as a necessary condition for the asymptotic stability of (68).

Theorem 8 *The system characterized by the self-starting dynamics given by (68) is asymptotically stable only if the matrix A is Hurwitz.*

Proof: Indeed if A is not Hurwitz, there is at least one eigenvalue of A in the right half plane and this mode will make the state of the system (68) unbounded for any

arbitrary initial condition.

Conjecture: The system characterized by the self-starting dynamics given by (68) is asymptotically stable if only if the matrix A is Hurwitz and $r_\sigma(A) > r_\sigma(B)$.

6.4 Operator Theoretic Treatment

Now, we want to extend linear algebraic concepts to infinite dimensions and enter the regime of operator theory with a more rigorous treatment. Let \mathbb{L}_2^n represent the Banach space of all the functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ which are square integrable (Lebesgue integrable), i.e., $\int_0^\infty \|f(t)\|^2 dt$ is well defined and finite. We define the \mathbb{L}_2 -norm as

$$\|f\|_{\mathbb{L}_2} \equiv \sqrt{\int_0^\infty \|f(t)\|_2^2 dt}. \quad (83)$$

The norm on the right hand side is the usual Euclidean norm.

Now, let's generalize this operator theoretic treatment to higher order systems. First, we define the two operators $\mathcal{P} : \mathbb{L}_2^n[0, 1] \rightarrow \mathbb{L}_2^n[0, 1]$ and $\mathcal{P}_\alpha : \mathbb{L}_2^n[0, 1] \rightarrow \mathbb{L}_2^n[0, 1]$ as follows.

$$\mathcal{P}x(t) = \int_0^t x(\theta) d\theta; \quad x(t) \in \mathbb{R}^n \quad (84)$$

$$\mathcal{P}_\alpha x(t) = \int_0^{\alpha t} x(\theta) d\theta; \quad x(t) \in \mathbb{R}^n \quad (85)$$

It can be shown that both \mathcal{P} and \mathcal{P}_α are not only bounded but also compact on the Banach space $\mathbb{L}_2^n[0, 1]$. Moreover,

$$\sigma(\mathcal{P}) = \sigma(\mathcal{P}_\alpha) = \{0\} = \sigma_c(\mathcal{P}). \quad (86)$$

From (68), we have,

$$\begin{aligned} x(t) &= x(0) + A \int_0^t x(\theta) d\theta + \frac{1}{\alpha} B \int_0^{\alpha t} x(\theta) d\theta \\ \Leftrightarrow x(t) &= x(0) + A\mathcal{P}x(t) + \frac{1}{\alpha} B\mathcal{P}_\alpha x(t) \Leftrightarrow x(t) = \left[I - \left(A\mathcal{P} + \frac{1}{\alpha} B\mathcal{P}_\alpha \right) \right]^{-1} x(0) \\ \Leftrightarrow x(t) &= \sum_{n=0}^{\infty} \left(A\mathcal{P} + \frac{1}{\alpha} B\mathcal{P}_\alpha \right)^n x(0) \end{aligned} \quad (87)$$

where the condition for the convergence of the composite Neumann series (87) is $\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| < 1$. In general, the operators \mathcal{P} and \mathcal{P}_α do not commute. Notice that,

$$\begin{aligned}\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| &\leq \|A\mathcal{P}\| + \|\frac{1}{\alpha}B\mathcal{P}_\alpha\| \\ &\leq \|A\|\|\mathcal{P}\| + \frac{1}{\alpha}\|B\|\|\mathcal{P}_\alpha\| \\ &= \frac{1}{\sqrt{2}}\|A\| + \frac{1}{\alpha}\|B\|\sqrt{\frac{\alpha}{2}} \\ &= \frac{1}{\sqrt{2\alpha}}(\sqrt{\alpha}\|A\| + \|B\|).\end{aligned}$$

Hence, if, $\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| < 1$, the solution to the initial value problem with $t_0 = 0$ only depends on $x(0)$, giving this a “self-starting character”. This means that the dynamics behave as a finite dimensional system whose evolution only depends on the *germ* $x(0)$, thus building up its own history. For all $t_0 \neq 0$, one needs the history $\mathcal{C}([\alpha t_0, t_0]; \mathbb{R}^n)$.

6.5 Conclusions

In this chapter, we investigated scale dynamic systems. We used Picard technique to find a closed form solution in the form of a sub-exponential series. We found that the evolution operator or in other words the associated *infinitesimal generator* for such systems is a *compact operator* on the Hilbert space l^2 or the Banach space l^∞ . This led us to ascertain the infinitesimal generator, the semigroup and the spectrum. Necessary and sufficient conditions for stability were also given. Starting from the basic scalar case, we extended the theory to the higher order case. Finally, we gave an operator theoretic treatment of scale dynamic systems and showed how such systems exhibit a *self starting* feature.

CHAPTER VII

DELAY INJECTION: A NOVEL OBSERVER DESIGN TECHNIQUE & ITS APPLICATIONS

Analysis and design of present machining methods used in mechanical engineering are based on Taylor series approximations and linearizations which cause erroneous and anomalous results as shown in chapter 5. High precision machining cannot tolerate such errors. We consider and analyze an exact mathematical model of the turning process without performing any kind of linearization of the model. We give a strategic design of an estimator for the position of a machining tool based on first principles. The system's dynamics are characterized by a State-Dependent Delay Differential Equation (SD-DDE) which is inherently *nonlinear*. This delay and state are *implicitly* related. The central tenet is to use inversion of the delay model and to extract the state vector given the delay. We use our recently developed observation technique referred to as *Delay Injection*, (see [2]). Both linear asymptotic and nonlinear observers with different architectures are designed for the state estimation of a machine tool based on state-dependent delay measurement. Simulation results are depicted at the end which portray the effectiveness, validity and usefulness of the proposed observer schemes.

7.1 Observation Using Delay Injection

We express the dynamics of the system and measurements in a *general* vector matrix form as follows,

$$\left\{ \begin{array}{l} \text{Dynamics of Plant/Process/System:} \\ \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t) \\ \text{Measurements or Observations:} \\ \tau = \tau_0 + \mathbf{M}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t - \tau) \end{array} \right. \quad (88)$$

The delay τ has the above described convoluted structure with the state. Notice that it not only has a constant (fixed) component but also carries an *implicit dependency* on the state \mathbf{x} of the system. Clearly, the delays given in (10) and (3) are special cases of the one in (88).

Here $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^p$ is the control input, both matrices \mathbf{A} , $\mathbf{A}_\tau \in \mathbb{R}^{n \times n}$; τ_0 is a scalar (constant) and \mathbf{M} , \mathbf{N} are both row vectors each of dimension n i.e., $\mathbb{R}^{1 \times n}$.

Observe that if $\mathbf{A}_\tau = \mathbf{O}$ and $\mathbf{N} = \mathbf{o}$, the system reduces to the classical observer design problem. Therefore, a *necessary condition* for the system (88) to be observable from the delay τ is that the pair (\mathbf{A}, \mathbf{M}) is completely state observable.

Notice that though the system seems to be apparently linear, it is in reality a nonlinear system because of the state-dependent delay in the dynamics as well as the observation or measurement. We measure the delay $\tau(t)$ at each instant of time t with the help of sensors and construct the observer for the state $\mathbf{x}(t)$. Let $\hat{\mathbf{x}}(t)$ represent the estimated state, the estimator will be governed by the dynamics as follows.

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{A}_\tau\hat{\mathbf{x}}(t - \tau) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\tau - \hat{\tau}) \quad (89)$$

where the last term is the correction term with $\mathbf{L} \in \mathbb{R}^n$ being the gain matrix of the estimator and $\hat{\tau}$ is precisely given by,

$$\hat{\tau} = \tau_0 + \mathbf{M}\hat{\mathbf{x}}(t) + \mathbf{N}\hat{\mathbf{x}}(t - \tau) \quad (90)$$

We call this *Delay Injection* in contrast to the so-called output injection. This is not the usual Luenberger Observer!

Fig. 23 gives a basic conceptual block diagram to illustrate the concept of estimation using delay injection.

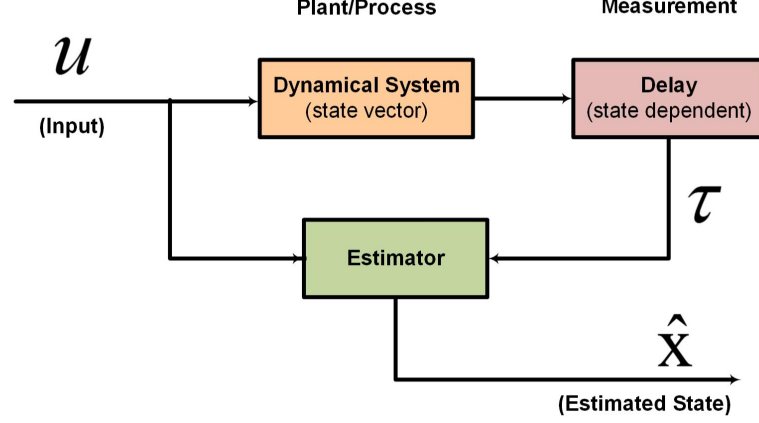


Figure 23: Conceptual Block Diagram for the Illustration of Delay Injection

Substitution for τ and $\hat{\tau}$ in (89) yields,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{A}_\tau\hat{\mathbf{x}}(t - \tau) + \mathbf{B}u(t) + \mathbf{L}(\mathbf{M}e(t) + \mathbf{N}e(t - \tau(t))) \quad (91)$$

where $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is the estimation error at any arbitrary time instant t .

Now we subtract the above equation from the original dynamic equation of the system as given by the top equation in (88) to get the error dynamics for the estimator as follows,

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{M})\mathbf{e}(t) + (\mathbf{A}_\tau - \mathbf{L}\mathbf{N})\mathbf{e}(t - \tau(t)) \quad (92)$$

From the above equation it is obvious that the error dynamics satisfy a Retarded Functional Differential Equation (RFDE) involving a time-varying delay. Now, the problem is tractable. The only design parameter is the matrix \mathbf{L} . The error will converge asymptotically to zero if the above delay differential equation is asymptotically stable or in other words the equilibrium point or fixed point (i.e., the origin $\mathbf{e}(t) = \mathbf{0}$) is asymptotically stable.

7.2 Uncertainty Characterization

The matrices of the system are not exactly known and may subject to uncertainties. This is because of the fact that the stiffness of the spring and the damping coefficient cannot be measured accurately. Also the accurate modeling of the machine tool involves nonlinearities of the spring which is typically a cubic nonlinearity. We will not use norm bounded uncertainties because they are unstructured and may sometimes assume the values of the parameters which the physical system never achieves in reality. This leads to highly conservative results. Though we do not know the exact values of mass, spring constant and damping coefficient, we do know the range or span of the parametric values of the individual components. Therefore, it is advantageous to characterize the uncertainty as a structured uncertainty. This will definitely help us in reducing the conservatism in the analysis and estimator design. In this chapter, we consider the polytopic type of uncertainty with respect to the matrices A , A_τ and B . i.e., there exist, say, N_v elements of the uncertainty set Δ ,

$$\Lambda^{(j)} = (A^{(j)}, A_\tau^{(j)}, B^{(j)}), \quad j = 1, 2, \dots, N_v. \quad (93)$$

known as vertices, such that Δ can be expressed as the convex hull of these vertices

$$\Delta = \mathcal{Co}\{ \Lambda^{(j)} \mid j = 1, 2, \dots, N_v \} \quad (94)$$

In other words, the uncertainty set Δ consists of all the convex linear combination of the vertices

$$\Delta = \left\{ \sum_{j=1}^{N_v} \lambda_j \Lambda^{(j)} \mid \lambda_j \geq 0, j = 1, 2, \dots, N_v ; \sum_{j=1}^{N_v} \lambda_j = 1 \right\} \quad (95)$$

7.3 Stability Analysis of the Error Dynamics

For the estimator to work, the asymptotic stability of the error dynamics equation (92) is of prime importance. Unless (92) is asymptotically stable, there is no question of estimator gain \mathbf{L} design. Since (92) is a time-varying delay differential equation, we

consider the linear system with a time-varying delay as characterized by the following form of retarded functional differential equation (RFDE) or differential-difference equation

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau(t)) \quad \forall t \geq 0 \\ x(t) &= \psi(t). \quad \forall t \in [-h, 0]\end{aligned}\tag{96}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $\psi(t) \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ is the initial infinite dimensional history function living in the Banach function space. Here $\mathcal{C}([-h, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-h, 0]$ to \mathbb{R}^n with the topology of uniform convergence. The matrices A_0 and A_1 each have size $n \times n$ and the time-varying delay $\tau(t)$ and its rate are characterized as follows.

$$\tau(t) \leq h; \quad \dot{\tau}(t) \leq d \leq 1.\tag{97}$$

i.e., h and d are the upper bounds on the size of the time-varying delay $\tau(t)$ and its time rate respectively. We give two approaches for the asymptotic stability analysis of error dynamics. The first approach uses the celebrated Riccati equation and the second one relies on Linear Matrix Inequalities (LMIs) [23].

7.3.1 Robust Stability Based on Riccati Equation

We directly use the following theorem.

Theorem 9 [80] *If there exists a triple of time-invariant positive definite matrices P, Q and R together with a scalar constant $\alpha \geq 1$ such that*

$$A_0^T P + P A + Q + R + \alpha^2 P A_1 Q^{-1} A_1^T P = 0\tag{98}$$

then the system (96) is robustly \mathcal{H}_α -asymptotically stable, where

$$\mathcal{H}_\alpha = \left\{ \tau(\cdot) \middle| 0 \leq \tau(\cdot) \leq h, \text{ and } \dot{\tau}(\cdot) \leq 1 - \frac{1}{\alpha^2} \right\}.\tag{99}$$

Replacing A_0 by $\mathbf{A} - \mathbf{L}\mathbf{M}$ and A_1 by $\mathbf{A}_\tau - \mathbf{L}\mathbf{N}$ in the above Riccati equation (98), our estimator gain \mathbf{L} must satisfy

$$(\mathbf{A} - \mathbf{L}\mathbf{M})^T P + P(\mathbf{A} - \mathbf{L}\mathbf{M}) + Q + R + \alpha^2 P(\mathbf{A}_\tau - \mathbf{L}\mathbf{N})Q^{-1}(\mathbf{A}_\tau - \mathbf{L}\mathbf{N})^T P = 0.\tag{100}$$

Comment 2:

Notice that the results given by the Riccati equation (186) are highly conservative. A full-fledged time-varying case for the matrices $P(t), Q(t)$ and $R(t)$ may give less restrictive results but was not attempted. Nevertheless, this conservatism is reduced by using the linear matrix inequality based approach in the next subsection which has more flexibility and extra degrees of freedom of in the choice of the matrix variables involved. Furthermore, the results not only involve the rate of the delay but also the upper bound of the delay.

7.3.2 Stability Bounds Analysis Using LMIs

Using auxiliary system based approach [68], we introduce a comparison system as follows.

$$\dot{w}(t) = A_2 w(t) + A_3 x(t), \quad w(t) \in \mathbb{R}^n \quad (101)$$

We construct the following augmented system which comprises (101) and the original delay-differential equation i.e., the top equation in (96).

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ A_3 & A_2 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ w(t - \tau(t)) \end{bmatrix}. \quad (102)$$

At this point, let us introduce $z(t)$ such that

$$\begin{aligned} z(t) &= x(t - h) - w(t) \\ &= x(t - h) - \int_{t-\tau(t)}^t \dot{w}(t) dt - w(t - \tau(t)) \\ &= x(t - h) - \int_{t-\tau(t)}^t (A_2 w(\beta) + A_3 x(\beta)) d\beta - w(t - \tau). \end{aligned} \quad (103)$$

Then system (96) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau(t)) + A_1 w(t) - A_1 w(t) \\ &= A_0 x(t) + A_1 w(t) + A_1 x(t - \tau(t)) - A_1 \\ &\quad \times \int_{t-\tau(t)}^t (A_2 w(\beta) + A_3 x(\beta)) d\beta \\ &\quad - A_1 w(t - \tau(t)). \end{aligned} \quad (104)$$

From (101) and (104) after re-organization of the terms, the following augmented coupled system is obtained.

$$\dot{\xi}(t) = \tilde{A}_0 \xi(t) + \tilde{A}_1 \xi(t - \tau(t)) + \Phi(t) \quad (105)$$

where

$$\begin{aligned} \tilde{A}_0 &= \begin{bmatrix} A_0 & A_1 \\ A_3 & A_2 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} A_1 & -A_1 \\ 0 & 0 \end{bmatrix}, \\ \Phi(t) &= \begin{bmatrix} A_1 \int_{t-\tau(t)}^t (A_2 w(\beta) + A_3 x(\beta)) d\beta \\ 0 \end{bmatrix}, \\ \xi(t) &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \end{aligned}$$

Finally, we invoke the following theorem to obtain a sufficient condition.

Theorem 10 *The time-delay system (96) is robustly uniformly asymptotically stable if there exists a quadruple of matrices (N_j, R_j, S_j, Z_j) where $N_j \in \mathbb{R}^{n \times 2n}$, $2n \times 2n$ symmetric positive definite matrices R_j, S_j and an $n \times n$ symmetric positive definite matrix Z_j , such that the following set (system) of Linear Matrix Inequalities hold (i.e., have a feasible solution):*

$$\Gamma^{(j)} := \begin{bmatrix} \Gamma_{11}^{(j)} & \star & \star \\ \Gamma_{21}^{(j)} & \Gamma_{22}^{(j)} & \star \\ \Gamma_{31}^{(j)} & 0 & \Gamma_{33}^{(j)} \end{bmatrix} < 0, \quad \forall j = 1, 2, \dots, N_v. \quad (106)$$

where an ellipsis \star represents an easily induced symmetric block, and

$$\begin{aligned}
\Gamma_{11}^{(j)} &= \begin{bmatrix} A_0^{(j)} & A_1^{(j)} \\ 0 & 0 \end{bmatrix} R_j + R_j \begin{bmatrix} A_0^{(j)} & A_1^{(j)} \\ 0 & 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} 0 \\ N_j \end{bmatrix} + \begin{bmatrix} 0 & N_j^T \end{bmatrix} + S_j \\
&\quad + h \begin{bmatrix} A_1^{(j)} \\ 0 \end{bmatrix} Z_j \begin{bmatrix} A_1^{(j)T} & 0 \end{bmatrix} \\
\Gamma_{21}^{(j)} &= N_j, \quad \Gamma_{22}^{(j)} = -\frac{1}{h} Z_j \\
\Gamma_{31}^{(j)} &= R_j^T \begin{bmatrix} A_1^{(j)T} & 0 \\ -A_1^{(j)T} & 0 \end{bmatrix}, \quad \Gamma_{33}^{(j)} = -(1-d)S_j
\end{aligned}$$

Proof:

The proof is based on the following three term positive definite Lyapunov-Krasovskii (LK) functional

$$V(\xi_t) = V_1(\xi(t)) + V_2(\xi_t) + V_3(\xi_t) \quad (107)$$

where,

$$V_1(\xi(t)) = \xi^T(t) P \xi(t), \quad P > 0, \quad (108)$$

$$V_2(\xi_t) = \int_{t-\tau(t)}^t \xi^T(\gamma) Q \xi(\gamma) d\gamma, \quad Q > 0, \quad (109)$$

$$\begin{aligned}
V_3(\xi_t) &= \int_{-h}^0 \int_{t+\alpha}^t \xi^T(\gamma) \begin{bmatrix} A_3^T \\ A_2^T \end{bmatrix} Z^{-1} \\
&\quad \times \begin{bmatrix} A_3 & A_2 \end{bmatrix} \xi(\gamma) d\gamma d\alpha, \quad Z > 0
\end{aligned} \quad (110)$$

Using the time derivative of this LK functional along the trajectory of the system, applying the matrix substitutions $R = P^{-1}$, $S = RQR$ and $N = \begin{bmatrix} A_3 & A_2 \end{bmatrix} R$, Schur complements and finally the convex polytopic uncertainty characterization in §7.2, the LMI is formulated. The detailed proof can be accomplished using the lines in [7]. ■

Again, substituting $\mathbf{A} - \mathbf{L}\mathbf{M}$ for A_0 and $\mathbf{A}_\tau - \mathbf{L}\mathbf{N}$ for A_1 in the above Linear Matrix Inequality of Theorem 1 and applying congruence transformation, the estimator gain matrix \mathbf{L} can be computed.

7.3.3 Observer Based Controller

Once the observer is designed, one can use the estimated state $\hat{\mathbf{x}}$ to design the static state feedback gain matrix $\mathbf{K} \in \mathbb{R}^{1 \times n}$ such that the control law becomes,

$$\mathbf{u} = \mathbf{K}\hat{\mathbf{x}}. \quad (111)$$

This renders the following closed-loop dynamics.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t - \tau) + \mathbf{K}\hat{\mathbf{x}}(t) \quad (112)$$

Using the definition of the estimation error, we get the following augmented dynamics for the observer based control system.

$$\begin{aligned} \begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{pmatrix} &= \begin{pmatrix} \mathbf{A} + \mathbf{B}\mathbf{K} & -\mathbf{A} \\ \mathbf{O} & \mathbf{A} - \mathbf{L}\mathbf{a}_1 \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{A}_\tau & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_\tau - \mathbf{L}\mathbf{a}_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}(t - \tau) \\ \mathbf{e}(t - \tau) \end{pmatrix} \end{aligned} \quad (113)$$

Given \mathbf{L} , the controller gain \mathbf{K} should be designed such that the above composite/augmented system is asymptotically stable.

Comment 3:

Notice that the estimator should be designed such that it is faster than the linear velocity of the spindle, otherwise the true position and velocity of the tool cannot be observed. The estimates will be faulty in this case.

Comment 4:

The observer designed as mentioned above is an infinite dimensional observer and it uses the time varying delay. Infinite dimensional observers are practically infeasible

because they require infinite memory. To circumvent this discrepancy, we develop finite dimensional nonlinear observers in the next section. See [95] for the observability of systems with delay convoluted observation.

7.4 *Nonlinear Observer Design: Unforced Case*

Expanding the delayed term on the right hand side of (10) as a Taylor series gives the following exact infinite dimensional expression of the the delay.

$$\tau = \frac{2\pi R + x(t) - (x(t) - \tau(t)\dot{x}(t) + \frac{\tau^2(t)}{2!}\ddot{x}(t) - \dots)}{R\omega} \quad (114)$$

Now, we have different case studies related to the the machining and the rocket car problems.

7.4.1 Case Study 1: Pure Inertia Case

For the unforced case of the system $u(t) = 0$ and therefore $\ddot{x}(t) = 0$ and all the third, fourth and higher derivatives of $x(t)$ vanish. Notice that we are not neglecting any higher order terms. This is neither an approximation nor a truncation of the Taylor series expansion. As a result we get the following exact and accurate implicit expression or representation of the delay $\tau(t)$ in terms of the state derivative (horizontal velocity of the tool),

$$\tau(t) = \frac{2\pi R + \tau(t)\dot{x}(t)}{R\omega} \quad (115)$$

Rearrangement of the terms yields the following exact expression of for the delay $\tau(t)$ as a nonlinear function of the state variable $\dot{x}(t) = x_2(t)$,

$$\tau(t) = \frac{2\pi R}{R\omega - x_2(t)}; \quad \dot{x}(t) = x_2(t) \neq R\omega \quad (116)$$

By analogy with the extended Kalman filter design, one may be tempted to use an estimate of $\tau(t)$ as follows,

$$\hat{\tau}(t) = \frac{2\pi R}{R\omega - \hat{x}_2(t)} \quad (117)$$

Now, using the concept of delay injection for the above $\hat{\tau}(t)$, the nonlinear observer equations are given as follows,

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) + l_1(\tau(t) - \hat{\tau}(t)) \quad (118)$$

$$\dot{\hat{x}}_2(t) = l_2(\tau(t) - \hat{\tau}(t)) \quad (119)$$

where l_1 and l_2 are the gains of the nonlinear observer. However, our simulation results reveal that the error does not decay to zero asymptotically unlike the linear asymptotic observer (292). This is because the delay only contains $x_2(t)$. One can easily see that the corresponding linear version is unobservable. One can think of using higher derivatives of $\tau(t)$ in our observation equations to get instantaneous observability conditions. However, we can easily see that $\dot{\tau} = 0$. Therefore, this case does not carry any practical significance and is worthless for the inversion.

7.4.2 Case Study 2: Undamped Case of the Tool

In this case we consider the undamped model of the tool i.e.,

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) \end{cases} \quad (120)$$

Again expanding the delay in (10) as a Taylor series and using the undamped dynamics of the tool in (120) gives the following exact infinite dimensional expression of the the delay,

$$\tau = \frac{2\pi R - (1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \dots)x_1 + x_1 + (\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \dots)x_2}{R\omega} \quad (121)$$

Simplification of (121) yields,

$$\tau R\omega = 2\pi R + (1 - \cos \tau)x_1 + (\sin \tau)x_2. \quad (122)$$

Notice that we have used the exact full Taylor series without any approximation or truncation using partial sums.

7.4.3 Exact Instantaneous Nonlinear Observer:

In the ideal (pure mathematical) case, exact knowledge of a differentiable delay τ implies knowledge of $\dot{\tau}$. Therefore, from (122) we obtain by differentiation,

$$\dot{\tau}R\omega = (1 - \cos \tau)x_2 + \dot{\tau} \sin \tau x_1 + \dot{\tau} \cos \tau x_2 - \sin \tau x_1 \quad (123)$$

Expressing (122) and (123) in vector matrix form, we get;

$$\mathcal{A}(\tau, \dot{\tau}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tau R\omega - 2\pi R \\ \dot{\tau} R\omega \end{pmatrix} \quad (124)$$

where $\mathcal{A}(\tau, \dot{\tau}) = \begin{pmatrix} 1 - \cos \tau & \sin \tau \\ (\dot{\tau} - 1) \sin \tau & 1 + (\dot{\tau} - 1) \cos \tau \end{pmatrix}$.

Finally, we get the following inversion equation for the extraction of states of the tool.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathcal{A}^{-1}(\tau, \dot{\tau}) \begin{pmatrix} \tau R\omega - 2\pi R \\ \dot{\tau} R\omega \end{pmatrix}. \quad (125)$$

We immediately get the observability condition for the state recovery as follows.

Given τ and $\dot{\tau}$, one can completely recover the state vector if and only if the matrix \mathcal{A} is invertible i.e., nonsingular. In other words $\ker(\mathcal{A}(\tau, \dot{\tau})) = \{0\}$ i.e., \mathcal{A} must have a trivial kernel. This leads to the following condition on the delay τ and its derivative $\dot{\tau}$.

$$(1 - \cos \tau)(1 + (\dot{\tau} - 1) \cos \tau) - (\dot{\tau} - 1) \sin^2 \tau \neq 0 \quad (126)$$

Simplification yields the following inequality,

$$(\dot{\tau} - 2)(\cos \tau - 1) \neq 0 \quad (127)$$

Therefore, the simplified observability conditions are expressed neatly as follows.

$$\begin{cases} \dot{\tau}(t) \neq 2 \\ \cos(\tau(t)) \neq 1 \end{cases} \quad \forall t \geq 0 \quad (128)$$

The top condition on the delay rate is already implied by the strong well-posedness and causality constraint. The second condition must be taken care of. It simply states that the delay at any instant of time should not be an integral multiple of 2π . Since delay is always nonnegative for all time (otherwise it becomes an advance problem which is noncausal and impractical), we can express this mathematically as,

$$\tau(t) \neq 2k\pi, k \in \mathbb{Z}^+ \quad \forall t \geq 0. \quad (129)$$

Using the above result in conjunction with the causality constraint, a more rigorous and stringent condition for the nonlinear observability will be expressed in the theorem as follows.

Theorem 11 *System (120) is observable from the state-dependent delay (122) if and only if the delay τ satisfies the following two conditions.*

$$\begin{cases} \dot{\tau}(t) \leq 1 \\ \tau(t) \neq 2k\pi, k \in \mathbb{Z}^+ \end{cases} \quad \forall t \geq 0 \quad (130)$$

7.4.4 Asymptotic Nonlinear Observer:

The nonlinear asymptotic observer takes up the following form.

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) + l_1(\tau(t) - \hat{\tau}(t)) + m_1(\dot{\tau}(t) - \dot{\hat{\tau}}(t)) \quad (131)$$

$$\dot{\hat{x}}_2(t) = -\hat{x}_1(t) + l_2(\tau(t) - \hat{\tau}(t)) + m_2(\dot{\tau}(t) - \dot{\hat{\tau}}(t)) \quad (132)$$

where l_1, l_2, m_1 and m_2 are the gains of the nonlinear observer; and $\hat{\tau}(t)$ and $\dot{\hat{\tau}}(t)$ are precisely given by the implicit relations as follows.

$$\hat{\tau}R\omega = 2\pi R + (1 - \cos \hat{\tau})\hat{x}_1 + (\sin \hat{\tau})\hat{x}_2 \quad (133)$$

$$\dot{\hat{\tau}}R\omega = (1 - \cos \hat{\tau})\hat{x}_2 + \hat{\tau} \sin \hat{\tau} \hat{x}_1 + \hat{\tau} \cos \tau \hat{x}_2 - \sin \hat{\tau} \hat{x}_1 \quad (134)$$

7.4.5 Case Study 3: Exact Nonlinear Observer for the Rocket Car

Notice that application of Taylor series expansion to the right hand side of (3) yields,

$$v\tau(t) = x_1(t) + x_1(t) - \tau(t)\dot{x}_1(t) + \frac{\tau^2(t)}{2!}\ddot{x}_1(t) - \dots \quad (135)$$

For the autonomous version of the system $u(t) = 0$ and therefore $\ddot{x}_1(t) = \dot{x}_2(t) = 0$ and all the subsequent higher derivatives of $x_1(t)$ in the above expansion are zero. This yields the following exact expression (no truncations),

$$\tau(t) = \frac{1}{v}(2x_1(t) - \tau(t)\dot{x}_1(t)) \quad (136)$$

Finally, we achieve an explicit expression for the delay $\tau(t)$ in terms of the state variables $x_1(t)$ and $x_2(t)$ as,

$$\tau(t) = \frac{2x_1(t)}{x_2(t) + v} \quad (137)$$

By analogy with the extended Kalman filter design, one may be tempted to use an estimate of $\tau(t)$ as follows,

$$\hat{\tau}(t) = \frac{2\hat{x}_1(t)}{\hat{x}_2(t) + v} \quad (138)$$

Now for the above $\hat{\tau}(t)$ the nonlinear observer equations with delay injection are given as follows,

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) + l_1(\tau(t) - \hat{\tau}(t)) \quad (139)$$

$$\dot{\hat{x}}_2(t) = l_2(\tau(t) - \hat{\tau}(t)) \quad (140)$$

where l_1 and l_2 are the gains of the nonlinear observer. However, our simulation results reveal that the error does not decay to zero asymptotically unlike the linear asymptotic observer. This is because of the nonlinear nature of the estimator dynamics. We also used higher derivatives of $\tau(t)$ in our observation equations to get

instantaneous observability conditions. In the ideal (pure mathematical) case, exact knowledge of τ implies knowledge of $\dot{\tau}$, as used in Kalman's original observability criterion.

$$\dot{\tau}(t) = \frac{2x_2(t)}{x_2(t) + v} \quad (141)$$

From this equation, the inversion equations for $x_1(t)$ and $x_2(t)$ in terms of τ and $\dot{\tau}$ are given by,

$$x_1(t) = \frac{\tau v}{2 - \dot{\tau}} \quad (142)$$

$$x_2(t) = \frac{\dot{\tau} v}{2 - \dot{\tau}} \quad (143)$$

We observe here that $\dot{\tau}(t) \neq 2$ is the observability condition. But this is already fulfilled by the strong causality constraint. See [95] for the observability of systems with state-dependent delays.

7.4.6 Case Study 4: Unknown Input Observability & Observer Design for the Rocket Car:

Here we assume that the input u in the rocket car equation is an unknown but a constant driving agency or perturbation. Now, we are not only interested in the observation of the states but also in the observation of this unknown input. Notice that here we will require more regularity conditions on the delay τ . We assume that $\tau \in \mathcal{C}^2$. Successive differentiation followed by some algebraic simplification yields the following set of equations.

$$\begin{cases} v\tau = 2x_1 - \tau x_2 + \frac{\tau^2}{2}u \\ v\dot{\tau} = 2x_2 - \tau u + \tau\dot{\tau}u - \dot{\tau}x_2 \\ v\ddot{\tau} = -\ddot{\tau}x_2 + (2 - 2\dot{\tau} + \ddot{\tau}\tau + \dot{\tau}^2)u \end{cases} \quad (144)$$

In a compact format, one can recast the above set of nonlinear equations as follows.

$$\begin{pmatrix} 2 & -\tau & \frac{\tau^2}{2} \\ 0 & 2 - \dot{\tau} & -\tau + \tau\dot{\tau} \\ 0 & -\ddot{\tau} & 2 - 2\dot{\tau} + \ddot{\tau}\tau + \dot{\tau}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = v \begin{pmatrix} \tau \\ \dot{\tau} \\ \ddot{\tau} \end{pmatrix} \quad (145)$$

From which,

$$\hat{\mathcal{X}} = \mathcal{T}^{-1}\mathcal{Y} \quad (146)$$

where,

$$\begin{aligned} \mathcal{T} &:= \begin{pmatrix} 2 & -\tau & \frac{\tau^2}{2} \\ 0 & 2 - \dot{\tau} & -\tau + \tau\dot{\tau} \\ 0 & -\ddot{\tau} & 2 - 2\dot{\tau} + \ddot{\tau}\tau + \dot{\tau}^2 \end{pmatrix}, \\ \hat{\mathcal{X}} &:= \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix}, \\ \mathcal{Y} &:= v \begin{pmatrix} \tau \\ \dot{\tau} \\ \ddot{\tau} \end{pmatrix}. \end{aligned}$$

The operator \mathcal{T} in (146) is invertible if and only if

$$\begin{aligned} &\det \begin{pmatrix} 2 - \dot{\tau} & \tau(\dot{\tau} - 1) \\ -\ddot{\tau} & 2 - 2\dot{\tau} + \ddot{\tau}\tau + \dot{\tau}^2 \end{pmatrix} \neq 0 \\ &\Leftrightarrow (2 - \dot{\tau})(2 - 2\dot{\tau} + \ddot{\tau}\tau + \dot{\tau}^2) + \ddot{\tau}\tau(\dot{\tau} - 1) \neq 0 \\ &\Leftrightarrow 4 - 6\ddot{\tau} + 4\dot{\tau}^2 - \dot{\tau}^3 + \tau\dot{\tau} - 2\tau\dot{\tau}\ddot{\tau} \neq 0. \end{aligned} \quad (147)$$

Notice that the system is observable from constant delays. Also, linearly varying delays of the form $\tau(t) = \alpha t + \beta$; $\forall \alpha, \beta \in \mathbb{R}$ ensure observability of the system except for $\alpha = 2$. In particular, the delay rate should not be identically equal to 2 for all the time. Notice that $\dot{\tau} = 2$ converts the above inequality into an equality. The above result can be expressed by the following theorem.

Theorem 12 *The rocket car system (1) with the unknown steady input u is observable from the delay $\tau \in \mathcal{C}^2$ given by (3) if and only if $4 - 6\ddot{\tau} + 4\dot{\tau}^2 - \dot{\tau}^3 + \tau\dot{\tau} - 2\tau\dot{\tau}\ddot{\tau} \neq 0$. Moreover, the exact/instantaneous nonlinear observer for the states and the unknown input is given by (146).*

Comment 5a:

Notice that if the unknown input u is a time varying function such that $D^n(u) = 0$, for some $n \in \mathbb{N}$, then we will require more regularity on τ such that $\tau \in \mathcal{C}^{n+1}$. This means that we must have the first $n + 1$ derivatives of the delay available to us in order to recover completely the states x_1, x_2 as well as the unknown input u .

Comment 5b:

In order to extend the above theory to closed loop systems, we use the concept of “borrowed state-feedback” (see [91]). Assume that a stabilizing feedback exists, then the closed loop is autonomous, hence the inversion can be done exactly. We do this inversion to obtain $\hat{\mathbf{x}}$, (we call it the “inverted state”) for the closed loop system. Now we break the feedback open and insert the inverted state where we used \mathbf{x} before. This puts the observer after the controller, instead of first observing then controlling, but in LTI, this is commutative anyway. It is a simple paradigm shift. This is motivated by the deterministic separation principle. The observer and controller design are completely independent in LTI systems. However, no such separation exists for nonlinear dynamical systems.

Comment 5c:

With the designed controller $u(t) = -k_1x_1(t) - k_2x_2(t)$, the closed loop system is autonomous, i. e. $\ddot{x}_1(t) = \dot{x}_2(t) = u(t) = -k_1x_1(t) - k_2\dot{x}_1(t)$. By an appropriate choice of the static state-feedback gains k_1 and k_2 , the transient behavior (overshoot and settling time or speed of response) of the machine tool or rocket car can be controlled. This is the so-called regulator design problem. Practically speaking, the tool or rocket car should move with uniform acceleration to avoid jerks in the motion.

Therefore, it is assumed that all the derivatives of $u(t)$ are zero and an exact expression like (116) holds. Once the regulator design is accomplished, the nonlinear estimator will give the estimated states \hat{x}_1 and \hat{x}_2 . Now, the control law will be practically implemented as $u(t) = -k_1\hat{x}_1(t) - k_2\hat{x}_2(t)$.

7.5 Observer Design for a Model Motivated by Submarine Dynamics in a 3-D Space

In this chapter, we design an estimator for the position of a basic submarine motivated dynamic model in 3-D framework. The scheme is based on a novel idea of delay measurements in a realistic scenario. Ultrasonic sensors or sonars are used to give the measurement of the delay between the transmitted and the received signal. The delay depends *implicitly* on the state of the system. We use Lagrange-Bürmann inversion (LBI) to transform the delay to an explicit nonlinear function of the state. Then an asymptotic nonlinear observer is designed.

7.5.1 Introduction of the Problem

Let $x(t)$, $y(t)$ and $z(t)$ represent respectively the x , y and z coordinates of the position of the prototype submarine at any instant of time t with respect to a stationary beacon. Let α_x , α_y and α_z be the drag coefficients or viscous damping coefficients along respective coordinates. Let u_x , u_y and u_z represent the thrusts or control efforts in the x , y and z -directions respectively. Now, assuming unit mass (for simplicity), the basic model of the submarine motivated dynamics in this 3-D framework are given by,

$$\begin{cases} \ddot{x}(t) + \alpha_x \dot{x}(t) = u_x(t) \\ \ddot{y}(t) + \alpha_y \dot{y}(t) = u_y(t) \\ \ddot{z}(t) + \alpha_z \dot{z}(t) = u_z(t) \end{cases} \quad (148)$$

It will be assumed that the control efforts $u(\cdot)$ is such that the submarine remains subsonic.

Let us denote our state variables as $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, $x_5 = z$, $x_6 = \dot{z}$. Furthermore, let us assume symmetric damping i.e., $\alpha_x = \alpha_y = \alpha_z = \alpha$. With this the 6th order state space realization of the system will be as follows.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\alpha x_2(t) + u_x(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = -\alpha x_4(t) + u_y(t) \\ \dot{x}_5(t) = x_6(t) \\ \dot{x}_6(t) = -\alpha x_6(t) + u_z(t) \end{cases} \quad (149)$$

Inspired from physics and using polar coordinates for the position, let the position vector of the submarine be $\vec{r}(t)$, where,

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \quad (150)$$

with \hat{i}, \hat{j} and \hat{k} being the unit vectors in the x, y and z-directions respectively; and the magnitude of $\vec{r}(t)$ is,

$$r(t) = \|\vec{r}(t)\| = \sqrt{x^2(t) + y^2(t) + z^2(t)} \quad (151)$$

or in terms of state variables,

$$r(t) = \sqrt{x_1^2(t) + x_3^2(t) + x_5^2(t)} \quad (152)$$

Active echo-location

Let the submarine be equipped with an ultrasonic sensor or sonar. Now the transmitter Tx of the ultrasonic sensor transmits a signal/pulse, it travels with the speed of sound c , it is echoed by the beacon and is detected by the receiver Rx of the sensor.

The delay or the lag experienced by the pulse will give us the information of the position of the submarine. Suppose the delay is denoted by τ and the submarine had the position $r(t)$ at the moment when the pulse was received by the receiver Rx. This means that the position of the submarine at the moment when the sound signal was transmitted was $r(t - \tau)$. Thus, the delay from transmission to reception is precisely given by,

$$\tau = \frac{r(t) + r(t - \tau)}{c} \quad (153)$$

The problem with active-echolocation is that not only the beacon but everything else will also reflect the signal.

Passive echo-location

Now, we consider the passive echo-location problem. Let the submarine emit a continuous time stamped signal $s(t_r)$ when it is at position $r(t_r)$ and detected by a stationary beacon located at the origin at time t . Notice that when the signal traverses a distance $r(t_r) = (t - t_r)v$, it reveals a past position of the submarine with respect to the beacon. If $\tau(t) = t - t_r$ is the delay then,

$$\tau(t) = \frac{r(t - \tau(t))}{c} \quad (154)$$

or equivalently in Cartesian coordinates,

$$\tau(t) = \frac{\sqrt{x^2(t - \tau(t)) + y^2(t - \tau(t)) + z^2(t - \tau(t))}}{c} \quad (155)$$

First, we use the following theorem for the inversion of the state-dependent delay [95] which is based on Lagrange-Bürmann Inversion (LBI) [37].

7.5.2 Lagrange-Bürmann Inversion

Now, we use the LBI to transform the delay to an explicit nonlinear function of the state in the form of a power series.

Theorem 13 [95] *Let the state and delay be related by the following general convoluted expression.*

$$G(\tau) = F(r(t), r(t - \tau)) \quad (156)$$

where F , G and τ (equivalently r) are all analytic. If $G'(\tau) + F_2(r(t), r(t)) \neq 0$ at t , then the Lagrange inversion of (156) at t is explicitly given by the following power series.

$$\tau(t) = \sum_{k=1}^{\infty} \frac{G(0) + F(r(t), r(t))^k}{k!} \left\{ \left(\frac{\partial}{\partial s} \right)^{k-1} \left(\frac{s}{f(s)} \right)^k \right\}_{s=0} \quad (157)$$

where,

$$f(\tau) = G(\tau) - G(0) - F(r(t), r(t - \tau)) + F(r(t), r(t)). \quad (158)$$

Now, we apply the above theorem to (154). Clearly in this particular case, by comparison, we have, $G(\tau) = c\tau$, $F(r(t), r(t - \tau)) = r(t - \tau)$ and $f(s) = cs - r(t - s) + r(t)$. Notice that,

$$\lim_{s \rightarrow 0} \frac{s}{f(s)} = \lim_{s \rightarrow 0} \frac{s}{cs - r(t - s) + r(t)} = \lim_{s \rightarrow 0} \frac{1}{c + r'(t - s)} = \frac{1}{c + \dot{r}(t)}$$

where we used de l'Hopital's rule to resolve the indeterminate form. Similarly,

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{\partial}{\partial s} \left(\frac{s}{f(s)} \right)^2 \\ &= 2 \lim_{s \rightarrow 0} \frac{s}{f(s)} \lim_{s \rightarrow 0} \frac{\partial}{\partial s} \left(\frac{s}{f(s)} \right) \\ &= 2 \frac{1}{c + \dot{r}(t)} \lim_{s \rightarrow 0} \frac{f(s) - sf'(s)}{f^2(s)} \\ &= \frac{2}{c + \dot{r}(t)} \lim_{s \rightarrow 0} \frac{r(t) - r(t - s) - sr'(t - s)}{(cs - r(t - s) + r(t))^2} \\ &= \frac{2}{c + \dot{r}(t)} \lim_{s \rightarrow 0} \frac{sr''(t - s)}{2(cs - r(t - s) + r(t))(c + r'(t - s))} \\ &= \frac{1}{c + \dot{r}(t)} \lim_{s \rightarrow 0} \frac{r''(t - s) - sr'''(t - s)}{(c + r'(t - s))^2 - r''(t - s)(cs - r(t - s) + r(t))} \\ &= \frac{\ddot{r}(t)}{(c + \dot{r}(t))^3}. \end{aligned}$$

Likewise, we can compute the other terms in the Lagrange inversion expansion. Finally, the following series for the inversion of the delay τ in (154) is obtained.

$$\tau = \frac{r}{c + \mathbf{D}r} + \frac{1}{2!} \frac{r^2 \mathbf{D}^2 r}{(c + \mathbf{D}r)^3} + \frac{1}{3!} \frac{3(\mathbf{D}^2 r)^2 - \mathbf{D}^3 r \mathbf{D}r - \mathbf{D}^3 r}{(c + \mathbf{D}r)^5} + \dots$$

where \mathbf{D}^k represents the k -th order differential operator. Let us denote the first k partial sums of the above series expansion by τ_k then,

$$\tau_1(t) = \frac{r(t)}{c + \dot{r}(t)} \quad \text{and} \quad \tau_2(t) = \frac{r(t)}{c + \dot{r}(t)} + \frac{1}{2} \frac{r^2(t) \ddot{r}(t)}{(c + \dot{r}(t))^3} \quad \text{etc.}$$

Notice that the condition for the convergence of the series is $\sigma_t \left\| \frac{\mathbf{D}r}{c} \right\| < 1, \forall t \geq 0$. This limits the scope of the LBI to local regimes only.

7.5.3 Direct Inversion of the State and Delay Map

Case 1: Passive echo-location

Suppose the delay $\tau(t)$ (along with its derivatives) is known to us for all $t \geq 0$. The question is how to retrieve the range information r from τ using only the following static implicit map between r and τ .

$$\tau(t) = \frac{r(t - \tau(t))}{c}$$

One can express $r(t)$ as follows.

$$r(t) = r((t - \tau(t)) + \tau(t))$$

Using the Lagrange/full Taylor series expansion, we get the following explicit expression for r in terms of τ and its subsequent derivatives;

$$r(t) = c \left(\tau(t) + \frac{\dot{\tau}(t)}{(1 - \dot{\tau}(t))} \cdot \tau(t) + \frac{\ddot{\tau}(t)}{(1 - \dot{\tau}(t))^3} \cdot \frac{\tau^2(t)}{2!} + \frac{(1 - \dot{\tau}(t)) \ddot{\tau}(t) + 3\ddot{\tau}^2(t)}{(1 - \dot{\tau}(t))^5} \cdot \frac{\tau^3(t)}{3!} + \dots \right) \quad (159)$$

Case 2: Active echo-location

Here we consider the following state and delay map.

$$\tau(t) = \frac{r(t) + r(t - \tau(t))}{c} \quad (160)$$

One can easily prove by contradiction that τ is analytic if and only if r is analytic. i.e., $r \in \mathcal{C}^\omega \Leftrightarrow \tau \in \mathcal{C}^\omega$. If τ is small, r is also small. Let $\tau = \varepsilon\tau_1$ where $\varepsilon = \frac{1}{c}$ is the perturbation parameter of interest. Consider the following perturbation expansion for r .

$$r(t) = \varepsilon r_1(t) + \varepsilon^2 r_2(t) + \varepsilon^3 r_3(t) + \varepsilon^4 r_4(t) + \dots \quad (161)$$

Substituting this in (160) yields the following.

$$\begin{aligned} \varepsilon(c\tau_1(t)) &= \varepsilon r_1(t) + \varepsilon^2 r_2(t) + \varepsilon^3 r_3(t) + \varepsilon^4 r_4(t) + \dots \\ + \quad &\varepsilon r_1(t - \varepsilon\tau_1(t)) + \varepsilon^2 r_2(t - \varepsilon\tau_1(t)) \\ + \quad &\varepsilon^3 r_3(t - \varepsilon\tau_1(t)) + \varepsilon^4 r_4(t - \varepsilon\tau_1(t)) + \dots \\ = \quad &\varepsilon r_1(t) + \varepsilon^2 r_2(t) + \varepsilon^3 r_3(t) + \varepsilon^4 r_4(t) + \dots \\ + \quad &\varepsilon r_1(t) - \varepsilon^2 \tau_1(t) \dot{r}_1(t) + \frac{\varepsilon^3}{2!} \tau_1(t) \ddot{r}_1(t) - \dots \\ + \quad &\varepsilon^2 r_2(t) - \varepsilon^3 \tau_1(t) \dot{r}_2(t) + \frac{\varepsilon^4}{2!} \tau_1(t) \ddot{r}_2(t) - \dots \\ + \quad &\varepsilon^3 r_3(t) - \varepsilon^4 \tau_1(t) \dot{r}_3(t) + \frac{\varepsilon^5}{2!} \tau_1(t) \ddot{r}_3(t) - \dots \\ + \quad &\varepsilon^4 r_4(t) - \varepsilon^5 \tau_1(t) \dot{r}_4(t) + \dots \end{aligned}$$

The analyticity of τ (and hence equivalently r) guarantees the convergence of the above infinite series. Using the method of matched asymptotic expansion, we get the following.

$$\begin{aligned} \mathcal{O}(\varepsilon^1) : \quad &c\tau_1 - 2r_1 = 0 \Leftrightarrow r_1(t) = \frac{c}{2}\tau_1(t) \\ \mathcal{O}(\varepsilon^2) : \quad &2r_2 - \tau_1 \dot{r}_1 = 0 \Leftrightarrow r_2(t) = \frac{1}{2}\tau_1(t) \dot{r}_1 \\ \mathcal{O}(\varepsilon^3) : \quad &2r_3 + \frac{1}{2!} \tau_1 \ddot{r}_1 - \tau_1 \dot{r}_2 = 0 \\ &\Leftrightarrow r_3(t) = \frac{1}{2} \tau_1(t) \left(\dot{r}_2(t) - \frac{1}{2!} \ddot{r}_1(t) \right) \\ \mathcal{O}(\varepsilon^4) : \quad &2r_4 - \frac{1}{3!} \tau_1 \ddot{r}_1 + \frac{1}{2!} \tau_1 \ddot{r}_2 - \tau_1 \dot{r}_3 = 0 \\ &\Leftrightarrow r_4(t) = \frac{1}{2} \tau_1(t) \left(\frac{1}{3!} \ddot{r}_1(t) - \frac{1}{2!} \ddot{r}_2(t) + \dot{r}_3(t) \right) \\ &\vdots \end{aligned}$$

From the above, given $\tau(t)$, one can successively find $r_1(t), r_2(t), r_3(t), \dots$ and hence $r(t)$ using (161).

7.5.4 Dynamic Observer Based On Delay Injection

One can easily see by using the Popov-Belevitch-Hautus (PBH) test that the linearized system is not observable if only one delay measurement is used. This is actually also obvious from the geometry. It turns out that one needs to have at least three computational platforms to estimate the position of the mobile unit (submarine) in a 3D environment. Let \vec{r}_1, \vec{r}_2 and \vec{r}_3 denote the instantaneous position vectors of the computational platforms i.e.,

$$\begin{aligned}\vec{r}_1(t) &= X_1(t)\hat{i} + Y_1(t)\hat{j} + Z_1(t)\hat{k} \\ \vec{r}_2(t) &= X_2(t)\hat{i} + Y_2(t)\hat{j} + Z_2(t)\hat{k} \\ \vec{r}_3(t) &= X_3(t)\hat{i} + Y_3(t)\hat{j} + Z_3(t)\hat{k}\end{aligned}$$

Let $\vec{r}(t)$ denote the instantaneous position vector of the submarine at any particular instant of time t then the measurements recorded at the three computational platforms are as follows.

$$\tau_j(t) = \frac{1}{c} \|\vec{r}(t - \tau_j(t)) - \vec{r}_j\|; \quad j = 1, 2, 3 \quad (162)$$

In terms of the state variables (in Cartesian form), the above measurement equation takes the following form.

$$\tau_j(t) = \left(\frac{1}{c}\right) \sqrt{(x_1(t - \tau_j(t)) - X_j)^2 + (x_3(t - \tau_j(t)) - Y_j)^2 + (x_5(t - \tau_j(t)) - Z_j)^2};$$

$j = 1, 2, 3$

Now, using the Lagrange inversion and taking $c = 1$ for simplicity, we get the following *explicit* form for each of the delayed measurements in (162).

$$\tau_j(t) = \sigma_t \left(\frac{\|\vec{r} - \vec{r}_j\|}{1 + \mathbf{D}\|\vec{r} - \vec{r}_j\|} + \frac{1}{2!} \frac{\|\vec{r} - \vec{r}_j\|^2 \mathbf{D}^2 \|\vec{r} - \vec{r}_j\|}{(1 + \mathbf{D}\|\vec{r} - \vec{r}_j\|)^3} + \dots \right) \quad ; j = 1, 2, 3$$

where σ_t represents the evaluation functional i.e., $\sigma_t x = x(t)$. Now, we compute $\mathbf{D}\|\vec{r} - \vec{r}_j\|$ and $\mathbf{D}^2\|\vec{r} - \vec{r}_j\|$ as follows. Let $\vec{u}(t) = \vec{r}(t) - \vec{r}_j$. Therefore,

$$\mathbf{D}\|\vec{u}\| = \frac{\vec{u}^\top \dot{\vec{u}}}{\|\vec{u}\|} \quad (163)$$

$$\therefore \sigma_t \mathbf{D}\|\vec{r} - \vec{r}_j\| = \frac{(\vec{r}(t) - \vec{r}_j) \cdot \frac{d}{dt}(\vec{r}(t) - \vec{r}_j)}{\|\vec{r}(t) - \vec{r}_j\|} \quad (164)$$

Using the state variables, in cartesian form, we get the following.

$$\sigma_t \mathbf{D}\|\vec{u}\| = \frac{(x(t) - X_j)\dot{x}(t) + (y(t) - Y_j)\dot{y}(t) + (z(t) - Z_j)\dot{z}(t)}{\sqrt{(x(t) - X_j)^2 + (y(t) - Y_j)^2 + (z(t) - Z_j)^2}},$$

$j = 1, 2, 3$

Likewise, we can compute $\mathbf{D}^2\|\vec{u}\|$ as follows.

$$\begin{aligned} \mathbf{D}^2\|\vec{u}\|^2 &= \frac{d^2}{dt^2}(\vec{u}^\top \vec{u}) \\ \Leftrightarrow \mathbf{D}^2\|\vec{u}\|^2 &= \frac{d}{dt}(2\vec{u}^\top \dot{\vec{u}}) \\ \Leftrightarrow 2\mathbf{D}(\|\vec{u}\|\mathbf{D}\|\vec{u}\|) &= 2(\vec{u}^\top \ddot{\vec{u}} + \dot{\vec{u}}^\top \dot{\vec{u}}) \\ \Leftrightarrow \|\vec{u}\|\mathbf{D}^2\|\vec{u}\| + (\mathbf{D}\|\vec{u}\|)^2 &= \vec{u}^\top \ddot{\vec{u}} + \dot{\vec{u}}^\top \dot{\vec{u}} \\ \Leftrightarrow \mathbf{D}^2\|\vec{u}\| &= \frac{\vec{u}^\top \ddot{\vec{u}} + \dot{\vec{u}}^\top \dot{\vec{u}} - \left(\mathbf{D}\|\vec{u}\|\right)^2}{\|\vec{u}\|} \end{aligned} \quad (165)$$

or substituting for $\mathbf{D}\|\vec{u}\|$ from (163), we get the following handy expression.

$$\mathbf{D}^2\|\vec{u}\| = \frac{1}{\|\vec{u}\|} \left\{ \vec{u}^\top \ddot{\vec{u}} + \dot{\vec{u}}^\top \dot{\vec{u}} - \left(\frac{\vec{u}^\top \dot{\vec{u}}}{\|\vec{u}\|} \right)^2 \right\} \quad (166)$$

Notice that various terms in the above expression can be computed as under.

$$\begin{aligned} \|\vec{u}\| &= \sqrt{(x(t) - x_j)^2 + (y(t) - y_j)^2 + (z(t) - z_j)^2} \\ \vec{u}^\top \dot{\vec{u}} &= (x(t) - x_j)\dot{x}(t) + (y(t) - y_j)\dot{y}(t) + (z(t) - z_j)\dot{z}(t) \\ \dot{\vec{u}}^\top \dot{\vec{u}} &= \dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t) \\ \vec{u}^\top \ddot{\vec{u}} &= (x(t) - x_j)\ddot{x}(t) + (y(t) - y_j)\ddot{y}(t) + (z(t) - z_j)\ddot{z}(t); \\ &j = 1, 2, 3. \end{aligned}$$

Now, we set up our dynamics and measurement equations as follows.

$$\left\{ \begin{array}{l} \textbf{Dynamics of the Mobile Unit in 3-D:} \\ \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \textbf{Measurements or Observations:} \\ \tau_j(\vec{u}(t)) = \frac{\|\vec{u}(t)\|}{1 + \mathbf{D}\|\vec{u}(t)\|} + \frac{1}{2!} \frac{\|\vec{u}(t)\|^2 \mathbf{D}^2 \|\vec{u}(t)\|}{(1 + \mathbf{D}\|\vec{u}(t)\|)^3} \\ \text{where } \vec{u}(t) = \vec{r}(t) - \vec{r}_j; j = 1, 2, 3 \end{array} \right. \quad (167)$$

$$\text{Here } \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\alpha \end{pmatrix}, \mathbf{x}(t) \in \mathbb{R}^6 \text{ is the state vector, } \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{and } \mathbf{u}(t) = \begin{pmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{pmatrix} \in \mathbb{R}^3 \text{ is the control input vector. We can write the nonlinear}$$

measurement vector as $\mathcal{T} = \begin{pmatrix} \tau_1(\vec{u}(t)) \\ \tau_2(\vec{u}(t)) \\ \tau_3(\vec{u}(t)) \end{pmatrix}$. Let $\hat{\mathbf{x}}(t) \in \mathbb{R}^6$ be the observed or the estimated state vector, then the dynamic observer based on *Delay Injection* is as follows.

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathcal{T} - \hat{\mathcal{T}}) \quad (168)$$

where the last term is the correction term with $\mathbf{L} \in \mathbb{R}^{6 \times 3}$ being the gain matrix of the estimator and $\hat{\mathcal{T}}$ is precisely given by,

$$\hat{\mathcal{T}} = \begin{pmatrix} \hat{\tau}_1(\vec{u}(t)) \\ \hat{\tau}_2(\vec{u}(t)) \\ \hat{\tau}_3(\vec{u}(t)) \end{pmatrix}. \quad (169)$$

Here,

$$\hat{\tau}_j(\vec{u}(t)) = \frac{\|\vec{\hat{u}}(t)\|}{1 + \mathbf{D}\|\vec{\hat{u}}(t)\|} + \frac{1}{2!} \frac{\|\vec{\hat{u}}(t)\|^2 \mathbf{D}^2 \|\vec{\hat{u}}(t)\|}{(1 + \mathbf{D}\|\vec{\hat{u}}(t)\|)^3}$$

$$\vec{\hat{u}}(t) = \vec{\hat{r}}(t) - \vec{r}_j; \quad j = 1, 2, 3.$$

Note:

Here we mention that in [102], a decoupled controller design based approach is used for the formation control of autonomous underwater vehicles (AUVs). However, the delays are taken as constant. Practically, as evident from our problem formation, the lags are state-dependent. In [103], unscented Gauss-Helmert filter (UGHF) is used for acoustic tracking with state-dependent propagation delay in a 2-D environment. The problem with the Lagrange inversion is the appearance of the singularities in the explicit form which makes the observer highly nonlinear. Another drawback is its computational complexity. This limits the method to local regime only.

7.6 Simulation Results

7.6.1 Example 1: Rocket Car

In this example, we design an asymptotic observer for the autonomous rocket car system. Clearly, for the rocket car discussed here, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Fig. 24 shows the original position of the rocket car and its estimate. The initial conditions for the system are $\begin{pmatrix} 7.5 \\ -0.5 \end{pmatrix}$ and those for the estimator are $\begin{pmatrix} 1.5 \\ -0.3 \end{pmatrix}$.

The gain of the asymptotic estimator was chosen as $\mathbf{L} = \begin{pmatrix} 90 \\ 100 \end{pmatrix}$. The speed of sound v in the air was taken as 332 m/s and the delay $\tau(t) = \frac{15-t}{v-0.5}$ in a realistic fashion. A set of positive definite and symmetric matrices R, S and Z which makes the inequality (106) feasible were found using Matlab and are given as follows.

$$\begin{aligned}
R &= \begin{pmatrix} 59.2052 & 12.4865 & 37.5752 & 0 \\ 12.4865 & 39.1904 & 25.2265 & 0 \\ 37.5752 & 25.2265 & 46.2072 & 0 \\ 0 & 0 & 0 & 32.9406 \end{pmatrix} \\
S &= \begin{pmatrix} 17.0862 & -2.2315 & -1.8066 & 0 \\ -2.2315 & 10.9154 & 7.1515 & 0 \\ -1.8066 & 7.1515 & 26.6621 & 0 \\ 0 & 0 & 0 & 23.7484 \end{pmatrix} \\
Z &= \begin{pmatrix} 4.1265 & 0 \\ 0 & 2.9230 \end{pmatrix}
\end{aligned}$$

The rectangular matrix N in (106) was found as follows.

$$N = \begin{pmatrix} -0.7365 & 20.4172 & -25.8251 & 0 \\ 0 & 0 & 0 & -25.8743 \end{pmatrix}$$

Fig. 25 shows the profiles for position error $e_1(t)$ and velocity error $e_2(t)$ of the rocket car. We see that the error decays down to zero as time progresses. In other words the estimated position approaches the actual position. Same is the case for the velocity.

7.6.2 Example 2: Machine Tool

In this academic example, we design an asymptotic observer for the autonomous system. Here we consider the following uncertain model for the horizontal motion of the machine tool,

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -0.5 + \delta_1 & -1 \end{pmatrix}$, $\mathbf{A}_\tau = \begin{pmatrix} 0 & 1 \\ -1 & -0.4 - \delta_2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here δ_1 and δ_2 are representing the uncertainties in the model (lack of perfect model or discrepancies in the parameters i.e., mass, spring and stiffness of the tool). In this

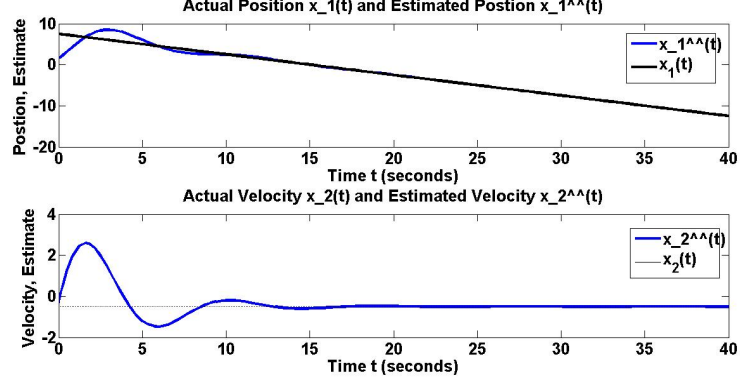


Figure 24: Actual Position $x_1(t)$, its estimate $\hat{x}_1(t)$ and Actual Velocity $x_2(t)$, its estimate $\hat{x}_2(t)$ of the rocket car

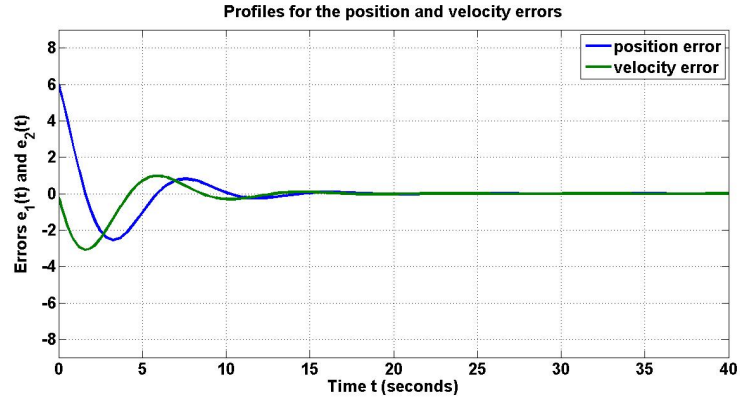


Figure 25: Profiles for Position Error $e_1(t)$ and Velocity Error $e_2(t)$ of the rocket car

example, we take $|\delta_1| \leq 0.21$ and $|\delta_2| \leq 0.16$. Therefore, \mathbf{A} and \mathbf{A}_τ can be recast in terms of the convex hull of the four vertices of polytope as described in section IV.

The initial conditions for the estimation error vector are taken as $\begin{pmatrix} -4.54 \\ 2.62 \end{pmatrix}$. The state-dependent delay is $\tau = 2 + x_1(t) + x_1(t - \tau)$. Therefore, in terms of the format in (6), we have $\tau_0 = 2$, $\mathbf{M} = \mathbf{N} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. The gain of the asymptotic estimator was chosen as $\mathbf{L} = \begin{pmatrix} 5.4573 \\ 0.2618 \end{pmatrix}$. This selection of the observer makes the inequalities in section VI feasible. All the feasible matrices are not reproduced here because of space limitations.

Fig. 26 shows the profiles for the estimation error $e_1(t)$ in the horizontal position

$x_1(t)$ of the machine tool for all the four polytopic vertices. Fig. 27 shows the profiles for the estimation error $e_2(t)$ in the horizontal velocity $x_2(t)$ of the machine tool for all the four polytopic vertices. We clearly notice that the error decays down to zero as time progresses. In other words the estimated position approaches the actual position. The same is true for the velocity case.

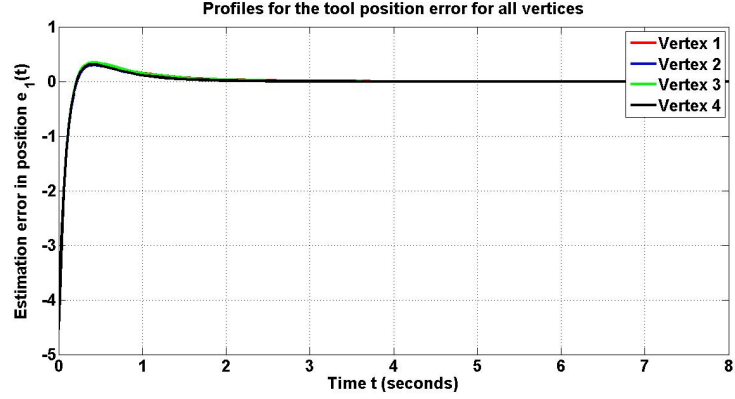


Figure 26: Estimation Error $e_1(t)$ in the horizontal position of the tool

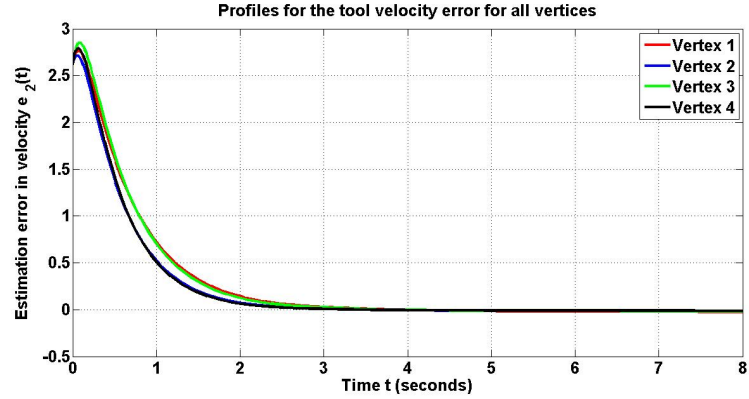


Figure 27: Estimation Error $e_2(t)$ in the horizontal velocity of the tool

7.6.3 Example 3: Exact Nonlinear Observer

This example manifests the power of the exact nonlinear observer based on delay inversion. This observer uses the delay τ and its time evolution rate $\dot{\tau}$ to exactly recover or estimate the state vector of the machine tool. The following *implicit* realistic delay model was used. The radius R of the workpiece was taken as 75 cm

and the angular velocity of the spindle ω was assumed to be 10 *radians/second*. Thus, the linear velocity being 7.5 *metres/second*. The following implicitly defined pair of τ and $\dot{\tau}$ were used as the observation/measurement equations.

$$\begin{cases} \tau = \frac{2\pi R + \cos t - \cos(t - \tau)}{R\omega} \\ \dot{\tau} = \frac{(1 - \dot{\tau}) \sin(t - \tau) - \sin t}{R\omega} \end{cases}$$

Fig. 28 shows the profiles of the delay and its rate. These profiles serve as our measurements or observations. We see that the delay and its rate fairly satisfy the nonlinear observability criteria for inversion given by (130). It is clear that there is no zero crossover of the delay at non-negative integral multiple of 2π . Also, the slope of the delay (delay rate) at any instant of time is less than unity. Therefore, exact instantaneous inversion is possible. Fig. 29 gives the estimated state trajectories of the position and velocity of the tool using the exact nonlinear estimator in (125). Notice that we have not shown the estimation errors e_1 and e_2 in this figure. Both of these errors are identically equal to zero because the observer is exact! The estimated states are the same as the true states.

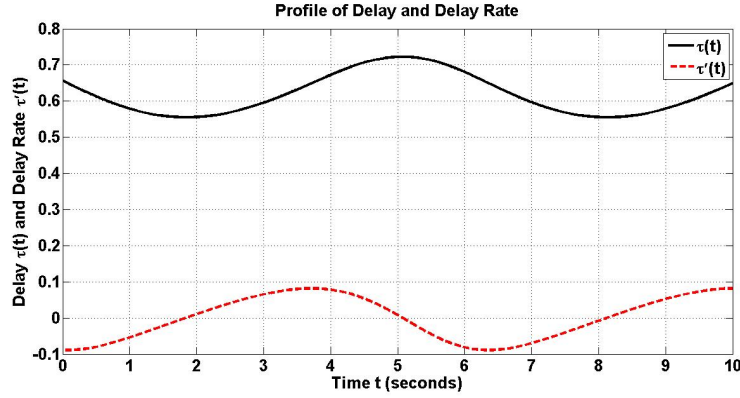


Figure 28: Delay τ and its Rate $\dot{\tau}$ Profiles

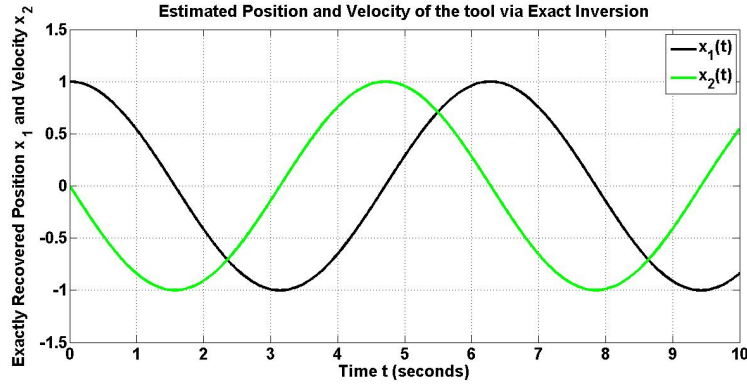


Figure 29: Tool Position and Velocity Recovery Using Exact Nonlinear Inversion Based Observer

7.6.4 Example 4: Basic Submarine (No Input Case)

In this example we take all the inputs as zero. Let the computational platforms be located at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in the 3-D space. For simplicity we took $c = 1$ and symmetric damping with $\alpha = 1$. Fig. 30 shows the actual and the estimated states and Fig. 31 portrays the estimation error profile for each of the states.

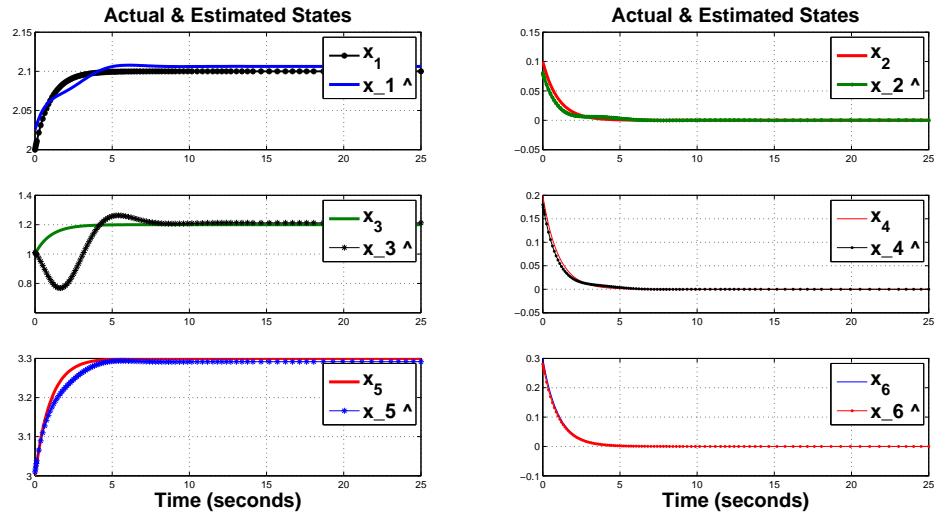


Figure 30: Actual States and Their Estimates

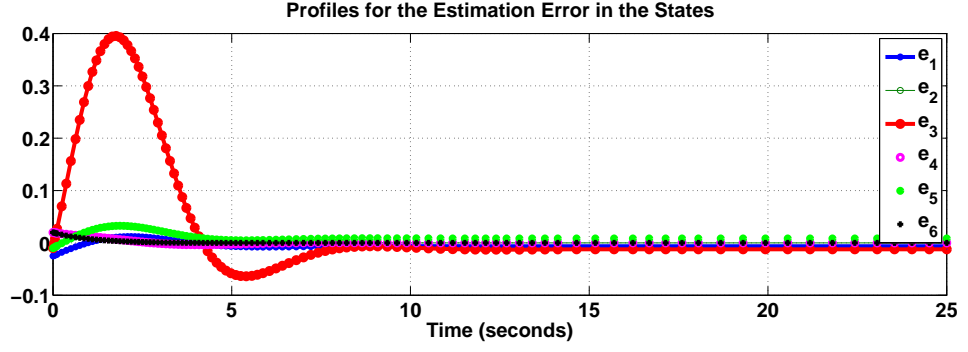


Figure 31: Profiles for the Estimation Error in the States

7.6.5 Example 5: Basic Submarine (Unknown Noisy Input)

In this example, all the parameters are the same as the previous example except that now there are unknown inputs. Here we take u_x, u_y and u_z as white noises of zero means and variances 0.04, 0.16 and 0.09 respectively. Fig. 32 shows the trajectories for the actual and the estimated states. Fig. 33 shows the trajectories of the estimation errors in each of the states. The observer gain matrix L was taken as follows.

$$L = \begin{pmatrix} 0.0001 & 0.0001 & 0.0001 \\ 0.01 & 0.01 & 0.01 \\ 0.23 & 0.23 & 0.23 \\ 0.02 & 0.02 & 0.02 \\ 0.1 & 0.1 & 0.1 \\ 0.002 & 0.002 & 0.002 \end{pmatrix}.$$

The initial conditions for actual and the observes state vectors are as follows.

$$x_0 = (2 \quad 0.1 \quad 1 \quad 0.2 \quad 3 \quad 0.3)^\top,$$

$$\hat{x}_0 = (2.025 \quad 0.08 \quad 1.01 \quad 0.18 \quad 3.01 \quad 0.28)^\top.$$

7.7 Concluding Remarks

The main focus of this chapter was to introduce the technique of delay injection for the observer design in a general setting. Two types of observers were considered

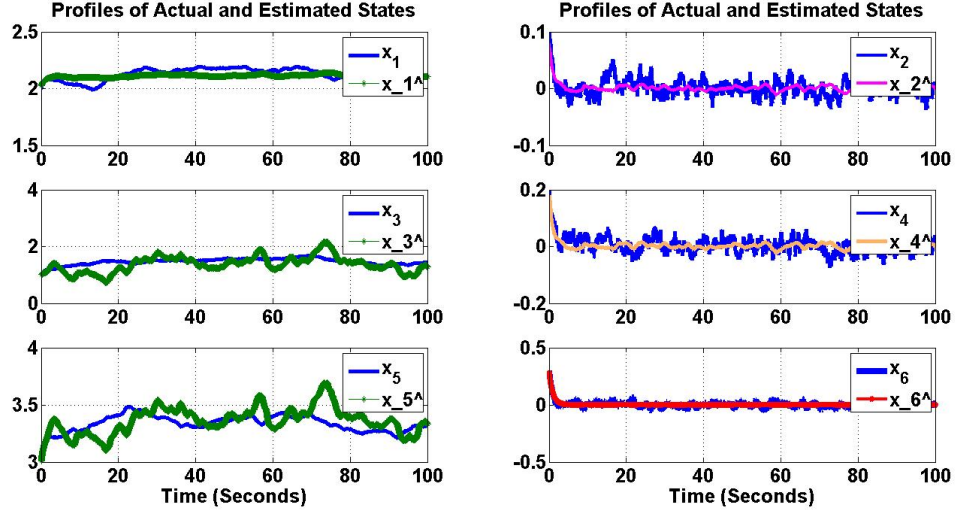


Figure 32: Actual States and Their Estimates in Presence of Noisy Inputs

namely asymptotic and exact. The observer can be designed via two approaches i.e., Riccati or LMIs. Both finite-dimensional and infinite-dimensional observers were designed. In a nutshell, we designed estimators for state-dependent delay systems and considered four case studies. These case studies involved the estimation of the states of a machine tool in a turning process and the position of a rocket car. We used the inversion of the state-dependent delay models (3) and (10) to retrieve or extract the state vector. We designed a robust asymptotic estimator which accounts for the parametric uncertainties of the system. Contrary to the existing methods in the literature, no approximation technique such as linearization or Taylorization of the system was used. All the work is based on the first principles. The central point is that using delay injection technique, the estimation error dynamics satisfy a Retarded Functional Differential Equation (RFDE) with a time-varying delay. The stability of such RFDEs can be ensured under the causality and well-posedness constraints. We also constructed two types of nonlinear observers. The first one, referred to as instantaneous observer, uses exact inversion of the delay model and showed that the nonlinear observer is much faster than the linear asymptotic case. The second type of observer is the nonlinear asymptotic observer and is based on delay and its derivative

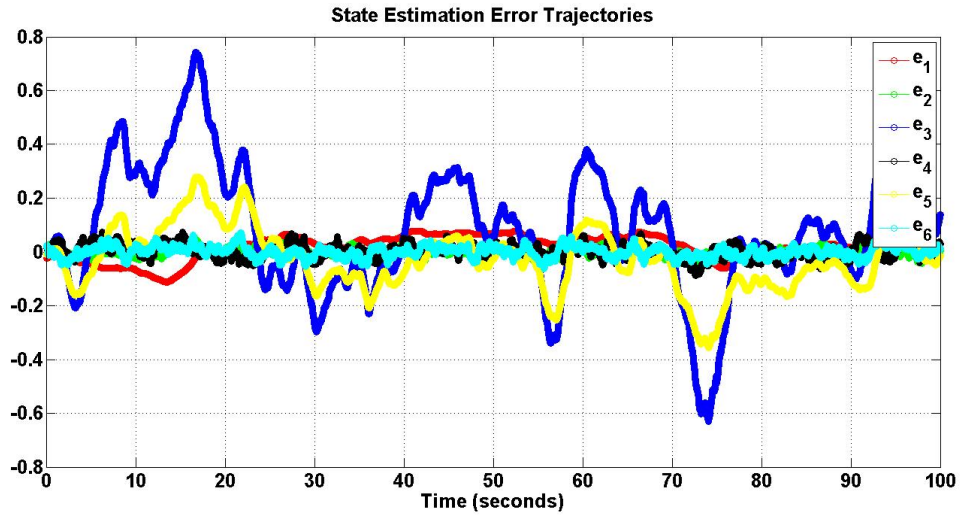


Figure 33: Profiles for the Estimation Error in the States in the Presences of Noisy Inputs

information. Finally we also gave observer design for a model motivated by submarine dynamics using LBI. Simulation results were presented to show the usefulness of the proposed observation and observer design schemes utilizing the inversion of the state-dependent delay.

CHAPTER VIII

SYSTEMS WITH EXPLICIT STATE-DEPENDENT DELAY: ANALYSIS & OBSERVER DESIGN

In this chapter, the delay is a an *explicit* nonlinear function of the state of the system and the resulting dynamics are governed by a nonlinear State-Dependent Delay Differential Equation (SD-DDE). The problem is inspired from the temperature control of a fluid in a tank as discussed in the preliminary chapter. We perform the stability analysis using Lyapunov-Razumikhin based approach. Using the new concept of *delay injection with inversion*, an infinite dimensional observer is designed to estimate the state of the system. We also shed some light on the state space required for characterizing the Cauchy problem and the evolution of the SD-DDE based system. Simulation results are depicted at the end which portray the effectiveness, validity and usefulness of the proposed observer scheme.

8.1 Problem Formulation

The DDE in the temperature control problem motivates us to analyze the stability properties of the following system.

$$\dot{x}(t) = \alpha x(t) + \beta x(t - \tau) + f(t) \quad (170)$$

Let us first investigate the unforced force where the delay τ depends explicitly on the state of the system. This motivates the analysis of the following explicit state-dependent delay differential equation.

$$\begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t - \tau(x(t))) \\ \tau(x) = \frac{d}{k_1 x(t) + k_2} \end{cases} \quad (171)$$

Notice that (171) is now no longer a linear constant coefficient delay differential equation. The dependence of the delay on the state makes this Functional Differential Equation (FDE) nonlinear. Also observe that the delay is a nonlinear function of the state. Even if the delay is a linear function of the state, the system is nonlinear because of the state-dependent delay. Such problems are very hard to handle.

Remark 5:

From the geometry and physical nature of the problem, it is evident that the delay τ is always non-negative. It cannot be zero because the sensor and actuator always have some separation. Also, mathematically $\tau < 0$ means an anticipatory and advanced system which is noncausal and physically not realizable. Therefore, it is reasonable to assume that $\tau(x) > 0$ and hence $k_1x(t) + k_2 > 0, \forall t$. Taking the state to be non-negative which physically means that the temperature is non-negative implies that $k_1 > 0$ and $k_2 > 0$. Also, $\frac{\partial \tau}{\partial x} = -\frac{k_1}{d}\tau^2 < 0$. This shows that the state-dependent delay $\tau(x)$ will be a monotonically decreasing function of the state x at any arbitrary instant of time. We can see that $\tau(x)$ is bounded on the domain $x \in [0, \infty)$. In fact,

$$\begin{cases} \inf_{x \in [0, \infty)} \tau(x) = 0 \\ \sup_{x \in [0, \infty)} \tau(x) = \frac{d}{k_1} \equiv \tau_{\max} \end{cases} \quad (172)$$

8.1.1 Asymptotic Stability of the Equilibrium Point of (171)

It was shown in [81], that the Lyapunov-Krasovskii (LK) functional based approach is not a suitable choice for the stability analysis of SD-DDEs governed systems.

We now use the LR theorem to show that equilibrium point ($x = 0$) of the SD-DDE system (171) is asymptotically stable.

Consider the Lyapunov function $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ described by $V(x) = \frac{1}{4}x^2$. We have for $x \in \mathbb{R}^+$, $\alpha(x) \leq V(x) \leq \beta(x)$, with $\alpha(x) = \frac{1}{16}x^2$ and $\beta(x) = 8x^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(r) = \rho^2 r$, $r \in \mathbb{R}^+$ and $\rho > 1$. This ensures one of the requirements of the LR theorem that $\eta(r) > r, \forall r > 0$. Let $x(t)$ be the solution

trajectory of (171), such that for $t \geq 0$, $\theta \in [-\tau_{\max}, 0]$,

$$V(x(t + \theta)) < \eta(V(x(t))) \quad (173)$$

$$\Rightarrow x^2(t + \theta) < \rho^2 x^2(t)$$

$$\Rightarrow |x(t - \tau(x))| < \rho |x(t)| \quad (174)$$

Now, we have for $t \geq 0$

$$\begin{aligned} \dot{V}(x(t)) &= \frac{\partial V}{\partial x} \frac{dx}{dt} \\ &= \frac{1}{2} \alpha x^2(t) + \frac{1}{2} \beta x(t) x(t - \tau(x(t))) \\ &\leq \frac{1}{2} \alpha x^2(t) + \frac{1}{2} |\beta| |x(t)| |x(t - \tau(x(t)))| \\ &= \frac{1}{2} (\alpha + \rho |\beta|) x^2(t) \\ &< 0 \text{ if } \alpha + \rho |\beta| < 0 \end{aligned} \quad (175)$$

Since $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ whenever $V(x(t + \delta)) \leq \eta(V(x(t)))$, $\forall \delta \in [-\tau, 0]$, all the conditions of LR theorem are satisfied. We, therefore, conclude that the equilibrium point (origin) of (171) is asymptotically stable. Furthermore, since $V(x) = \frac{1}{2} x^2(t) = \frac{1}{2} \|x\|^2$ i.e. $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Therefore, V is radially unbounded, and by definition this implies the global asymptotic stability of the equilibrium point.

The above general result is very useful and can be stated as the following theorem.

Theorem 14 *The SD-DDE system characterized by (171) is uniformly globally asymptotically stable if there exists a scalar $\rho > 1$ such that $\alpha + \rho |\beta| < 0$.*

Remark 6:

Notice that the application of Razumikhin theorem requires the hypothesis that the right hand side of the FDE must be Lipschitz in x . In our problem, τ is Lipschitz and because of the fact that the composition of two Lipschitz maps is again a Lipschitz map, this hypothesis is satisfied and the LR theorem can be safely applied.

Remark 7:

We can express $\dot{V}(x)$ in the above analysis as follows.

$$\dot{V}(x(t)) \leq 2(\alpha + \rho|\beta|)V(x(t)) \quad (176)$$

The corollary given below is an immediate consequence of the above result.

Corollary 2 *The SD-DDE system given by (171) is globally exponentially stable if there exists a scalar $\rho > 1$ such that $\alpha + \rho|\beta| < 0$.*

An even more generic and beautiful result can be obtained, by following the same lines as above in the spirit of LR theorem, for the nonlinear *time-varying* version of the SD-DDE i.e.

$$\begin{cases} \dot{x}(t) = \alpha(t)x(t) + \beta(t)x(t - \tau(x(t))) \\ \tau(x) = \frac{d}{k_1x(t) + k_2} \end{cases} \quad (177)$$

where α and β are continuous functions on \mathbb{R} such that $-\alpha(t) \geq \epsilon > 0$, and there is a κ , $0 \leq \kappa < 1$ such that $|\beta(t)| \leq \kappa\epsilon$.

Corollary 3 *The non-autonomous system (177) is globally uniformly asymptotically (and exponentially) stable if there exists a scalar $\rho > 1$ such that $1 - \rho\kappa > 0$.*

8.2 Perturbation Analysis

Here, we want to consider the effect of the parameter d on the dynamics of the SD-DDE. We nondimensionalize or normalize it and call it $\sigma = \frac{d}{d_0}$. By introducing a small perturbation parameter $\sigma > 0$ in the delay term, we get the following regularly perturbed version of the SD-DDE (171).

$$\dot{x}(t) = \alpha x(t) + \beta x \left(t - \sigma \frac{d_0}{k_1x(t) + k_2} \right); \forall \alpha, \beta \in \mathbb{R} \quad (178)$$

Performing the small signal perturbation analysis yields the following.

$$\begin{aligned}
\dot{x}(t) &= \alpha x(t) + \beta x(t) - \beta \sigma \frac{d_0}{k_1 x(t) + k_2} \dot{x}(t) \\
\Rightarrow (1 + \beta \sigma \frac{d_0}{k_1 x(t) + k_2}) \dot{x} &= (\alpha + \beta) x \\
\Rightarrow \int \left(\frac{1}{x} + \sigma \frac{\beta d_0}{x(k_1 x + k_2)} \right) dx &= \int (\alpha + \beta) dt + E \\
\Rightarrow \ln |x| + \frac{\sigma \beta d_0}{k_2} \ln |x| - \frac{\sigma \beta d_0}{k_2} \ln |k_1 x + k_2| \\
&= (\alpha + \beta) t + E \\
\Rightarrow \frac{x^{1 + \sigma \frac{\beta d_0}{k_2}}}{(k_1 x + k_2)^{\sigma \frac{\beta d_0}{k_2}}} &= F e^{(\alpha + \beta) t} \\
\Rightarrow x \left(\frac{x}{k_1 x + k_2} \right)^{\sigma \frac{\beta d_0}{k_2}} &= F e^{(\alpha + \beta) t} \tag{179}
\end{aligned}$$

where E and F are arbitrary constants of integration. This shows that if $\alpha + \beta < 0$, then as $t \rightarrow \infty$, $x(t) \rightarrow 0$. As expected, in the special and trivial case when $\sigma = 0$ (physically meaning that the pipe gets arbitrarily short), $x(t) = x_0 e^{(\alpha + \beta)t}$ because the SD-DDE reduces to an Ordinary Differential Equation (ODE).

8.3 Observation Using Delay Injection

We express the dynamics of the system and measurements as a setup or formulation for our observer problem as follows.

$$\left\{ \begin{array}{l} \text{Dynamics of Plant/Process/System:} \\ \dot{x}(t) = \alpha x(t) + \beta x(t - \tau(x(t))) + \theta u(t) \\ \text{Measurements or Observations:} \\ \tau(x) = \frac{d}{k_1 x(t) + k_2} \end{array} \right. \tag{180}$$

The delay has the above described *explicit* dependence on the state x of the system. The state is infinite dimensional and lives in the Banach function space $\mathcal{C}^{0,1}$. Here $x(t) \in \mathbb{R}$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, all the coefficients $\alpha, \beta, \theta, d, k_1$ and $k_2 \in \mathbb{R}$ are scalars (constants). Moreover, $d > 0, k_1 > 0$ and $k_2 > 0$.

We measure the delay $\tau(t)$ at each instant of time t with the help of a flow sensor. The measurement might be contaminated by noise. For feedback one needs the state of the system. Our objective here is to construct the observer for the recovery of the state $x(t)$. Since τ is a nonlinear function (inversely related function) of the state, we use the inverted delay $\frac{1}{\tau}$ in our observer equation and call this technique as *delay injection with inversion*. Since the delay is always nonzero, there is no harm in its inversion. In fact, the delay τ is always positive. Let $\hat{x}(t)$ represent the estimated state, the estimator (being a replica of the state) will be governed by the dynamics as follows.

$$\dot{\hat{x}}(t) = \alpha \hat{x}(t) + \beta \hat{x}(t - \hat{\tau}) + \theta u(t) + \mathbf{L} \left(\frac{1}{\tau} - \frac{1}{\hat{\tau}} \right) \quad (181)$$

where the last term is the correction term with $\mathbf{L} \in \mathbb{R}$ being the gain of the estimator and $\hat{\tau}$ is precisely given by,

$$\hat{\tau} = \frac{d}{k_1 \hat{x}(t) + k_2} \quad (182)$$

Fig. 34 gives a basic conceptual block diagram to illustrate the concept of estimation using delay injection with inversion. Substitution for τ and $\hat{\tau}$ in (291) yields,

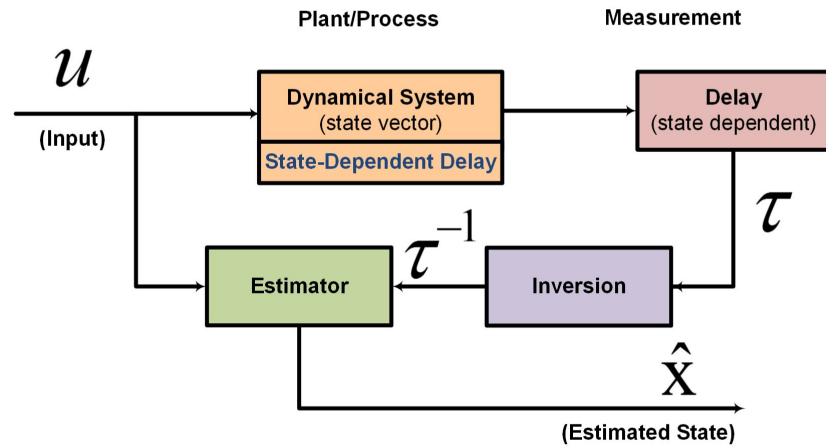


Figure 34: Conceptual Block Diagram for the Illustration of Delay Injection with Inversion

$$\begin{aligned}\dot{\hat{x}}(t) &= \alpha\hat{x}(t) + \beta\hat{x}(t - \tau) + \theta u(t) \\ &\quad + \mathbf{L} \left(\frac{k_1 x(t) + k_2}{d} - \frac{k_1 \hat{x}(t) + k_2}{d} \right)\end{aligned}\tag{183}$$

$$\Rightarrow \dot{\hat{x}}(t) = \alpha\hat{x}(t) + \beta\hat{x}(t - \tau) + \theta u(t) + \mathbf{L} \frac{k_1}{d} e(t)\tag{184}$$

where $e(t) = x(t) - \hat{x}(t)$ is the estimation error or observation error at any arbitrary time instant t .

Now we subtract the above equation from the original dynamic equation of the system as given by the top equation in (180) to get the error dynamics for the observer as follows,

$$\dot{e}(t) = (\alpha - \mathbf{L} \frac{k_1}{d})e(t) + \beta e(t - \tau(t))\tag{185}$$

From the above equation it is obvious that the error dynamics satisfy an autonomous Retarded Functional Differential Equation (RFDE) involving a *time-varying delay*. Now, the problem is tractable. The only design parameter is the gain \mathbf{L} . The error will converge asymptotically to zero if the above delay differential equation is asymptotically stable or in other words the equilibrium point or fixed point (i.e., the origin $e = 0$) is asymptotically stable.

Remark 8:

One might be tempted to use the static direct inversion of the delay model to recover the state. However, we want to mention here that if the measurement is impaired or contaminated by noise, the direct inversion might not be useful. Let $\frac{1}{\tilde{\tau}} = \frac{1}{\tau} + v(t)$ denote the noisy measurement where $v(t)$ is the Additive White Gaussian Noise (AWGN) that corrupts the observation. The estimate of the state by direct inversion would be $\tilde{x} = \frac{1}{k_1} (d \frac{1}{\tilde{\tau}} - k_2)$ rendering the state estimation error $\tilde{e} = -\frac{d}{k_1} v(t)$. Let $Var(n(t))$ denote the variance of the AWGN, then the variance of the estimation error will be $Var(\tilde{e}) = \frac{d^2}{k_1^2} Var(n(t))$. Therefore, the variance of the error increases linearly as the noise variance increases (or in other words as the Signal to Noise Ratio (SNR) decreases) i.e., the error escalates or worsens with noise. Another potential

problem with the direct inversion is its singularity. On the other hand, the *delay injection with inversion* based observer is a dynamic system and it acts like a filter for the incoming noise provided that the dynamics are stable. As we show in the simulation results, it nicely and uniformly tracks the original state even in the presence of noise. In the transient regime, the direct observer may compete the injection observer because of large initial error due to mismatch between the initial conditions of the actual and estimated state. To overcome that pathological case, the solution will be a hybrid observer. The dynamic observer outperforms the direct one in the steady state because of its inherent integrating, averaging and smoothing property.

8.4 Stability Analysis of the Error Dynamics & Observer Design

Replacing A_0 by the scalar $\alpha - \mathbf{L} \frac{k_1}{d}$ and A_1 by the scalar β in the above Riccati equation (98), our estimator gain \mathbf{L} must satisfy the following scalar Riccati equation.

$$2(\alpha - \mathbf{L} \frac{k_1}{d})p + q + r + \nu^2 \beta^2 \frac{p^2}{q} = 0 \quad (186)$$

8.4.1 Observer Design Using Riccati Equation

The scalar Riccati equation always has a positive definite solution $p > 0$ if

$$-(\alpha - \mathbf{L} \frac{k_1}{d}) > \nu |\beta| \quad (187)$$

which means that the observer gain must be chosen such that

$$\mathbf{L} > \frac{d}{k_1} (\alpha + \nu |\beta|). \quad (188)$$

8.4.2 Observer Design Using LR Theorem

Using the Lyapunov Razumikhin function, there should exist $\rho > 1$ such that $(\alpha - \mathbf{L} \frac{k_1}{d}) + \rho |\beta| < 0$ in order that the observer error dynamics are asymptotically stable. This gives the same condition on \mathbf{L} as in (188) and the two approaches reconcile to

give the same results.

Comment 6:

Notice that the estimator should be designed such that it is faster than the output temperature dynamics of the closed loop system, otherwise the true temperature cannot be observed. The estimates will be faulty in this case.

Comment 7:

The observer designed as mentioned above is an infinite dimensional observer and it uses the time-varying delay. Infinite dimensional observers are practically infeasible because they require infinite memory. To circumvent this discrepancy, we develop finite dimensional nonlinear observers which we leave as a future task.

8.5 Simulation Results

Example 1: Consider the nonlinear non-autonomous (time-varying) system with the state-dependent delay.

$$\begin{cases} \dot{x}(t) = -10(0.6 + e^{-0.25t})x(t) + 2.5 \cos(10t)x(t - \tau(x(t))) \\ \tau(x) = \frac{10}{x(t) + 10} \end{cases}$$

Here, $\phi(t) = 2.5t + 1.4 \cos(5t) + 3 \in \mathcal{C}^{0,1}$, $\forall t \in [-1, 0]$ was taken as the initial history function. It is obvious to see that the example meets all the condition of Corollary 3. Fig. 35 shows the variation of the state-dependent delay and Fig. 36 shows the state trajectory. We can clearly see that the system is asymptotically stable.

Example 2: In this example, we design an asymptotic observer for the autonomous system $\dot{x}(t) = -10.69x(t) + 2.6x(t - \tau(x(t)))$ with the nonlinear state-dependent delay $\tau(x(t)) = \frac{10}{x(t) + 10}$. Taking $\nu = 15$, the observer gain was computed as $\mathbf{L} = 8.5$. The $\mathcal{C}^{0,1}$ initial histories for the state and the estimate are $t + 1.4$ and 0.95 respectively for $t \in [-1, 0]$. We also consider the case of a noisy observation where the noise in the flow was assumed to be AWGN with a standard deviation of 0.25 .

Fig. 37 shows the profiles for the actual temperature, the estimated temperature

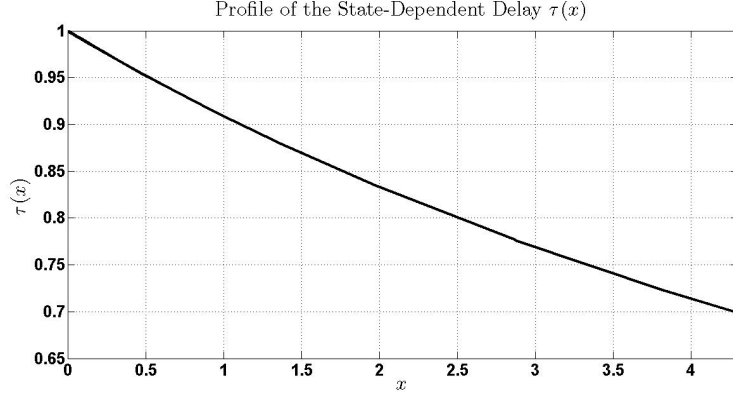


Figure 35: Profile of the State-Dependent Delay

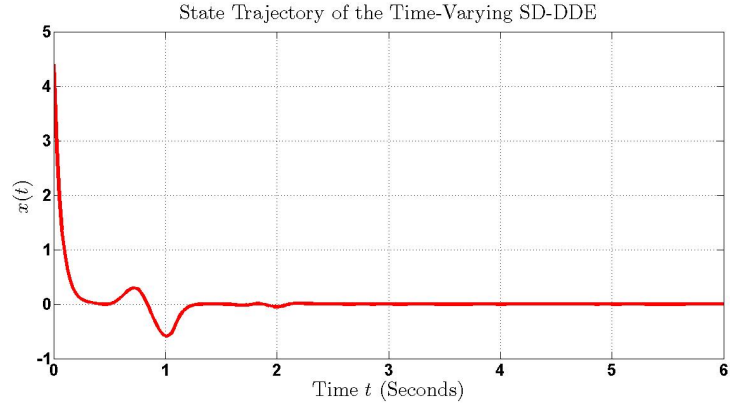


Figure 36: State Trajectory $x(t)$

and the observation error. We clearly notice that the error decays down to zero as time progresses in both the noisy and noiseless cases. In other words the estimated state approaches the actual state. Fig. 38 shows a comparison between the estimation error in the case of the dynamic observer based on delay injection and the direct observer for the same noise variance. Notice the spikes in the direct scheme and the smoothness of the delay injection observer. These spikes increase in magnitude as the noise variance increases.

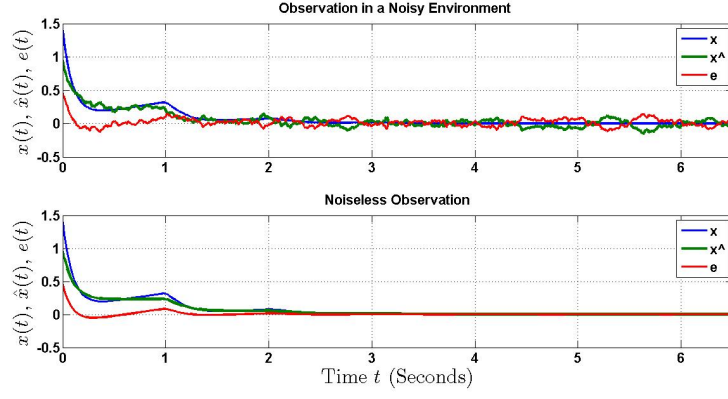


Figure 37: Trajectories of the Actual State, Observed State and Observation Error

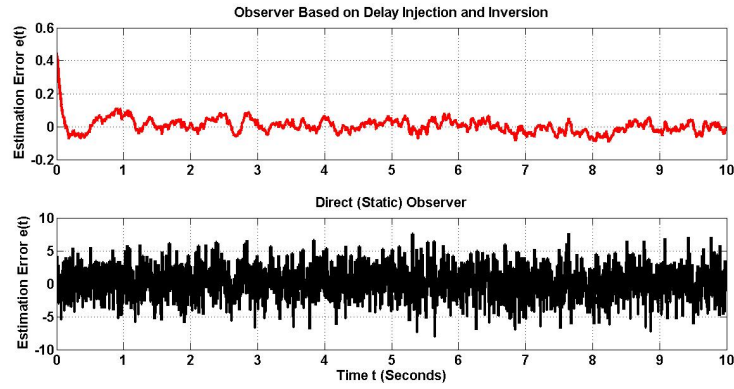


Figure 38: Comparison of the Estimation Error for the two Observation Schemes

8.6 Concluding Remarks

In this chapter we solved the problem of analysis and observer design for the state of temperature control of a fluid in a tank based on the observation of delay measurements using flow sensor. Unlike the usual cases in the literature, the delay considered here is neither fixed nor time-varying, rather it depends *explicitly* on the state of the system. The delay depends nonlinearly on the state and as a result the governing dynamics exhibit a nonlinear SD-DDE. To the best of our knowledge, no work is done for the observation or estimation of the states of temperature control systems involving state-dependent delays. For the state feedback control synthesis problem, one always needs the complete state vector of the system which is seldom (almost never) available. Therefore, it is always mandatory to estimate the state vector of

the system. Here we used our new idea of *delay injection with inversion* for the retrieval of the state from the knowledge of delay measurement. We also emphasized that the state space required for the SD-DDEs is not the usual function space of continuous functions over a compact support. Stability analysis was carried out using Razumikhin framework for the both the time-invariant and time-varying cases. We also performed regular perturbation analysis of the system. At the end, simulation results were shown to express the benefits of our analysis and observer design for the nonlinear SD-DDE based systems.

CHAPTER IX

GENERAL FRAMEWORK: ANALYSIS OF SYSTEMS WITH STATE-DEPENDENT DELAYS

The objective of this chapter is to develop a generic framework for the analysis and observer design of systems with state-dependent delays and state suprema. In the previous chapters, we analyzed the stability and designed observer in the general setting when the delay free system was linear. The only source of nonlinearity was the state-dependent delay. Now, we extend the idea to a more general setting where the dynamics as well as the observation are nonlinear as follows.

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + g(x(t - \tau(t))), \\ \tau(t) = h(x(t)), \\ x_0 = \phi \in \mathcal{C}^{0,1}([-\tau_{sup}, 0]; \mathbb{R}^n); \tau \in [\tau_{inf}, \tau_{sup}] \end{cases} \quad (189)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathcal{C}([-\tau_{sup}, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are all Lipschitz continuous functions. Moreover, we assume that $f(0) = g(0) = 0$. The delay function τ is assumed to be non-negative and $\tau_{sup} > \tau_{inf}$. After identifying the equilibrium solutions, the next step will be to determine the stability of the equilibria of (189). Notice that the Lyapunov-Krasovskii functional (LKF) of §7.3.2 cannot be used here because of the delay dependence on the state. The derivative of LKF results in a very convoluted expression from the stability condition is hard to conclude. We propose to use Lyapunov-Razumikhin function based approach to find the sufficient conditions for the asymptotic stability of (189). Likewise we can setup the nonlinear observer design problem with the state-dependent delay $\tau = h(x)$ as the observation or measurement and the dynamics as in (189). The objective is to extract the state vector using the delay injection concept. Let $L_f > 0$, $L_g > 0$ and $L_\tau > 0$ denote the

Lipschitz constants of the the vector fields (functions) f, g and τ respectively i.e.,

$$\|f(x) - f(y)\| \leq L_f \|x - y\|, \forall x, y \in \mathbb{R}^n \quad (190)$$

$$\|g(x) - g(y)\| \leq L_g \|x - y\|, \forall x, y \in \mathcal{C} \quad (191)$$

$$|\tau(x) - \tau(y)| \leq L_\tau \|x - y\|, \forall x, y \in \mathbb{R}^n \quad (192)$$

The general *implicit* form of the state-dependent delay system to be investigated is as follows.

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t - \tau(t))), \\ \dot{\tau}(t) = h(x(t), \tau(t)), \\ (x_0, \tau(0)) = (\phi, \tau) \in \mathcal{C}^{0,1}([-\tau_{sup}, 0]; \mathbb{R}^n) \times [\tau_{inf}, \tau_{sup}] \equiv C^{0,1} \times K \end{cases} \quad (193)$$

The analysis technique for the system (189) will be used to investigate the problem of gene expression regulation in the next chapter.

9.1 *Explicit State-Dependent Delay*

The following theorem gives sufficient conditions for the global asymptotic stability of the explicit state-dependent delay system (189).

Theorem 15 *The equilibrium solution of the nonlinear time delay system (189), with state dependent delay characterized as above, is uniformly globally asymptotically stable if given any symmetric and positive definite matrix R ($R = R^\top > 0$) there exists a symmetric and positive definite matrix $T = T^\top > 0$ together with a scalar ξ such that*

$$\begin{cases} A^\top T + TA + R = O; \\ \lambda_{\min}(R) - 2\|T\|(L_f + \xi L_g) > 0; \\ \xi > 1 \end{cases} \quad (194)$$

Proof:

Consider the Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ described by $V(x) = x^\top T x$ where $T =$

$T^\top > 0$. We have for $x \in \mathbb{R}^n$, $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$, with $\alpha(\|x\|) = \lambda_{\min}(T)\|x\|^2$ and $\beta(\|x\|) = \lambda_{\max}(T)\|x\|^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(r) = \xi^2 r$, $r \in \mathbb{R}^+$ and $\xi > 1$. This ensures one of the requirements of the LR theorem that $\eta(r) > r, \forall r > 0$. Let $x(t)$ be the solution trajectory of (227), such that for $t \geq 0$, $\theta \in [-\tau_{\max}, 0]$,

$$\begin{aligned} V(x(t+\theta)) &< \eta(V(x(t))) \\ \Rightarrow x^\top(t+\theta)Tx(t+\theta) &< \xi^2 x^\top(t)Tx(t) \\ \Rightarrow \|x(t+\theta)\|_P^2 &< \xi^2 \|x(t)\|_P^2 \\ \Rightarrow \|x(t-\tau)\|_P &< \xi \|x(t)\|_P; \forall \tau \in [\tau_{inf}, \tau_{sup}] \end{aligned}$$

Now, we have for $t \geq 0$,

$$\begin{aligned} \dot{V}(x(t)) &= \left(\frac{\partial V}{\partial x} \right)^\top \frac{dx}{dt} \\ &= \mathbf{D}(x^\top(t)Tx(t)) \\ &= \dot{x}^\top(t)Tx(t) + x^\top(t)T\dot{x}(t) \\ &= x^\top(t)(A^\top T + TA)x(t) + (f^\top(x(t))Tx(t) + x^\top(t)Tf(x(t))) \\ &\quad + (g^\top(x(t-h(x(t))))Tx(t) + x^\top(t)Tg(x(t-h(x(t))))) \\ &= -x^\top(t)Rx(t) + 2x^\top(t)Tf(x(t)) + 2x^\top(t)Tg(x(t-h(x(t)))) \\ &\leq -\lambda_{\min}(R)\|x(t)\|^2 + 2L_f\|T\|\|x(t)\|^2 + 2L_g\|T\|\|x(t)\|\|x(t-h(x(t)))\| \\ &\leq -(\lambda_{\min}(R) - 2\|T\|(L_f + \xi L_g))\|x(t)\|^2 \end{aligned} \tag{195}$$

This completes the proof. ■

9.2 Implicit State-Dependent Delay

Likewise, by using the Lyapunov Razumikhin function

$$V(x) = x^\top(t)Sx(t) + \frac{1}{2}\tau^2(t); S = S^\top > 0, \tag{196}$$

we have the following result for the system (193) with implicit delay.

Theorem 16 *The equilibrium solution of the nonlinear time delay system (193), with state dependent delay characterized as above, is uniformly globally asymptotically stable if given any symmetric and positive definite matrix R ($R = R^\top > 0$) there exists a symmetric and positive definite matrix $S = S^\top > 0$ together with a scalar ξ such that*

$$\begin{cases} A^\top S + SA + R = O; \\ \lambda_{\min}(R) - 2\xi\|S\|L_f - \tau_{\sup} > 0; \\ \xi > 1 \end{cases} \quad (197)$$

Proof:

By the causality constraint, $\dot{\tau} = h(x, \tau) < 1$. Now, using Razumikhin argument as in the previous theorem, from (196) we have,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^\top(t)Sx(t) + x^\top(t)S\dot{x}(t) + \tau(t)\dot{\tau}(t) \\ &= x^\top(t)(A^\top S + SA)x(t) + 2(f^\top(x(t - \tau(t))))Sx(t) + \tau(t)h(x(t), \tau(t)) \\ &= -x^\top(t)Rx(t) + (f^\top(x(t - \tau(t))))Sx(t) + \tau(t)h(x(t), \tau(t)) \\ &\leq -\lambda_{\min}(R) + 2L_f\|S\|\|x(t)\|\xi\|x(t)\| + \tau_{\sup}\|x(t)\|^2 \\ &\leq -(\lambda_{\min}(R) - 2\xi\|S\|L_f - \tau_{\sup})\|x(t)\|^2 \end{aligned}$$

This concludes the proof. ■

9.3 Controller Synthesis & Stabilization

Consider the following non-autonomous system corresponding to (189).

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + g(x(t - \tau(t))) + Bu(t), \\ \tau(t) = h(x(t)), \\ x_0 = \phi \in \mathcal{C}^{0,1}([-\tau_{\sup}, 0]; \mathbb{R}^n); \tau \in [\tau_{\inf}, \tau_{\sup}] \\ u(t) = Kx(t) \end{cases} \quad (198)$$

where $u(t) \in \mathbb{R}^m$ is the control law to be designed so that the resulting closed-loop systems is globally asymptotically stable. Here $B \in \mathbb{R}^{n \times m}$ and $K \in \mathbb{R}^{m \times n}$ is the static state feedback gain matrix. The following corollary is an immediate consequence of Theorem 15.

Corollary 4 *The equilibrium solution of the nonlinear time delay system (198), with state dependent delay characterized as above, is uniformly globally asymptotically stabilizable by the control law $u(t) = Kx(t)$ if given any symmetric and positive definite matrix R ($R = R^\top > 0$) there exists a symmetric and positive definite matrix $W = W^\top > 0$ together with a scalar ξ and a matrix $K \in \mathbb{R}^{m \times n}$ such that*

$$\begin{cases} (A + BK)^\top W + W(A + BK) + R = O; \\ \lambda_{\min}(R) - 2\|W\|(L_f + \xi L_g) > 0; \\ \xi > 1 \end{cases} \quad (199)$$

9.4 Observer Synthesis

Consider the following observer structure for the system (189), with $L \in \mathbb{R}^n$ being the observer gain matrix.

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + g(\hat{x}(t - \tau(t))) + L(\tau - \hat{\tau}), \\ \tau(t) = h(x(t)), \\ \hat{\tau}(t) = h(\hat{x}(t)), \\ \hat{x}_0 = \phi \in \mathcal{C}^{0,1}([-\tau_{sup}, 0]; \mathbb{R}^n); \tau \in [\tau_{inf}, \tau_{sup}] \end{cases} \quad (200)$$

The error dynamics are as follows.

$$\dot{e}(t) = Ae(t) + (f(x(t)) - f(\hat{x}(t))) + (g(x(t - \tau(t))) - g(\hat{x}(t - \tau(t)))) - L(h(x(t)) - h(\hat{x}(t))) \quad (201)$$

The following corollary can be used to design the observer.

Corollary 5 *The error dynamics (201) is globally asymptotically stable if given any symmetric and positive definite matrix R ($R = R^\top > 0$) there exists a symmetric and positive definite matrix $Z = Z^\top > 0$ together with a scalar ξ such that*

$$\begin{cases} A^\top Z + ZA + R = O; \\ \lambda_{\min}(R) - 2\|Z\|(L_f + \xi L_g + L_h\|L\|) > 0; \\ \xi > 1 \end{cases} \quad (202)$$

9.5 Concluding Remarks

In this chapter, we provided a generic framework for the stability analysis, controller synthesis and observer design for systems with state-dependent delays. Here, the delay free system is also nonlinear. The nonlinearities considered in this chapter are of Lipschitz type. Notice that the Lipschitzness is also necessary for ensuring the existence and uniqueness of the solutions. Both explicit and implicit state-dependent delays were considered in the analysis. The next chapter supplements these results for applications in genetic circuits.

CHAPTER X

MODELING & ANALYSIS OF GENE EXPRESSION AS A NONLINEAR FEEDBACK PROBLEM WITH STATE-DEPENDENT DELAY

This short chapter models the dynamics of gene expression as a nonlinear feedback system with a state-dependent delay. The delay accounts for the lag from the initiation of translation until the appearance of the mature protein messenger RNA (mRNA). We do not consider this delay to be constant. Rather, we take this delay to be dependent on the instantaneous concentration of the protein. This gives rise to a nonlinear biological system with a state-dependent delay. We consider the degradation of mRNA and protein in the mathematical model. We give conditions for the asymptotic stability of the system. We also give examples and simulation results for which periodic (oscillatory) solutions/limit cycles are obtained [5].

10.1 Modeling of Gene Expression Regulation

Generally, gene expression is modeled by the following coupled system of nonlinear ordinary differential equations (see e.g., [27], [46], [10], [75], [74] and the references therein).

$$\begin{cases} \dot{R}(t) = f(P(t)) - AR(t) \\ \dot{P}(t) = BR(t) - CP(t) \end{cases} \quad (203)$$

where $R(t)$ denotes the concentration of the messenger RNA (mRNA), $P(t)$ stands for the concentration of the protein, which is the end product of the reaction, at any arbitrary instant of time t . The rates $\dot{R}(t)$ and $\dot{P}(t)$ account for the balance between mRNA synthesis and the consumption of the end product. The real scalars

(constants) $A > 0$ and $C > 0$ respectively describe the degradation effects of mRNA and protein. $B > 0$ is the translational constant. f is the nonlinear feedback function which describes transcription. Generally, f is taken as the Hill function [10].

In [27], the authors use Taylor series approximation and propose a linear transcription model corresponding to (203). Experimentally, the linearized model with and without constant delay does not capture all the features (limit cycle, steady state equilibrium) of gene expression regulation. We modify the above model (203) by taking into consideration the fact that the process of transcription is not instantaneous. There is always a latency τ required for the transcription of protein to mRNA. Furthermore, we propose that this delay is not constant but varies with the concentration of protein i.e., $\tau = \tau(P)$. This is justified by the fact that the mechanisms which transport mRNA from the nucleus to the cytoplasm become saturated [78]. When the protein concentration is higher (overcrowded), the delay will be more and vice versa. We also assume that τ is not unbounded but is given by a monotone Hill function. In other words, both $\sup_P \tau(P)$ and $\inf_P \tau(P)$ exist. Therefore, we have the following model.

$$\begin{cases} \dot{R}(t) = f(P(t - \tau(P))) - AR(t) \\ \dot{P}(t) = BR(t) - CP(t) \end{cases} \quad (204)$$

The above equation has a delay which depends on the state of the system and is therefore an SD-DDE where the delay τ is given by,

$$\tau(P) = K_\tau \frac{P^n}{1 + P^n}; \quad n \in \mathbb{Z}^+. \quad (205)$$

Here, \mathbb{Z}^+ denotes the set of positive integers. Clearly, the state-dependent delay τ is bounded on the domain $P \in [0, \infty)$. In fact,

$$\begin{cases} \inf_{P \in [0, \infty)} \tau(P) = 0 \equiv \tau_{\inf} \\ \sup_{P \in [0, \infty)} \tau(P) = K_\tau \equiv \tau_{\sup} \end{cases} \quad (206)$$

Fig. 39 portrays the Hill function for the state-dependent delay $\tau(P)$ for different values of the index n .

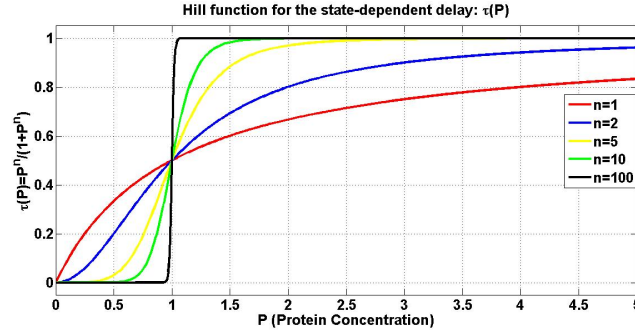


Figure 39: Hill Function for the Normalized ($K_\tau = 1$) State-Dependent Delay $\tau(P)$ for Different Values of the Index n

The nonlinear feedback network of genes, mRNAs and proteins with state-dependent delay is depicted in Fig. 40.

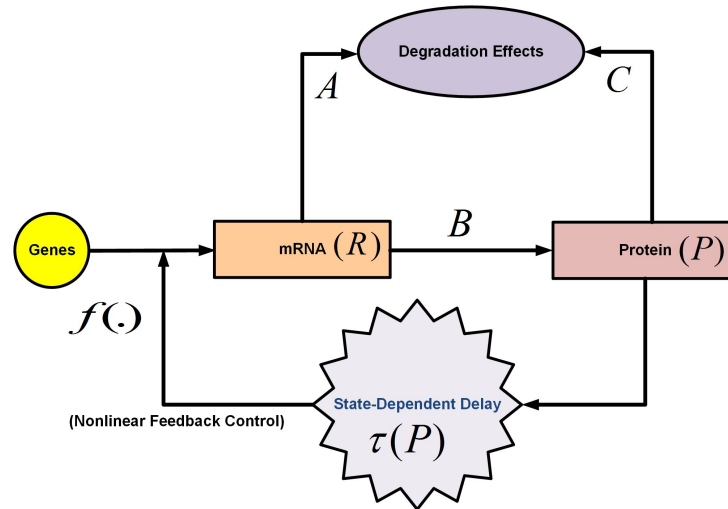


Figure 40: Block Diagram of Gene Regulation With Feedback Mechanism and State-Dependent Delay

The nonlinear feedback function f which accounts for the transcription is a monotone decreasing function and is given by the following Hill function with the Hill coefficient or sigmoidity index N .

$$f(P) = \frac{K_P}{1 + P^N}; \quad N \in \mathbb{Z}^+ \quad (207)$$

where $K_P \in \mathbb{R}^+$ is a positive constant. The greater the value of N , also called the Hill coefficient, the more the sigmoidity of the function f . Fig. 41 portrays the Hill function for the normalized ($K_P = 1$) nonlinear feedback: $f(P)$ for different values of the sigmoidity index or Hill exponent N .

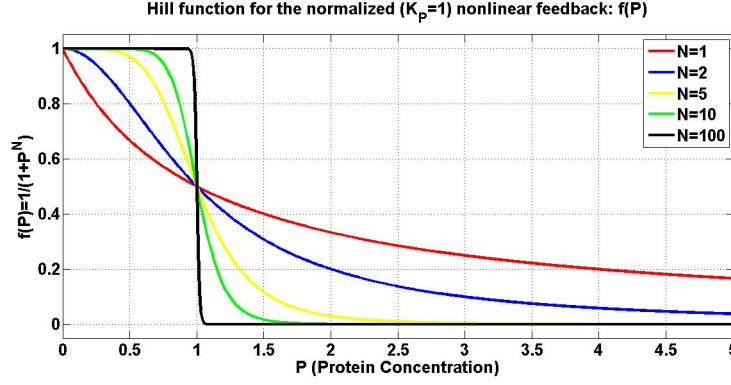


Figure 41: Hill Function for the Normalized ($K_P = 1$) Nonlinear Feedback: $f(P)$ for Different Values of the Sigmoidity Index N

Definition 15 *Schwarzian* Let $f \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ be a three times continuously differentiable function defined over real numbers then the Schwarzian or the Schwarzian derivative of f is defined as follows.

$$\mathcal{S}f(t) = \frac{D^3 f(t)}{Df(t)} - \frac{3}{2} \left(\frac{D^2 f(t)}{Df(t)} \right)^2 \quad (208)$$

where $D := \frac{d}{dt}$ is the usual differential operator.

Schwarzians are invariant under linear fractional transformations. Moreover, we use the Hill functions for modeling the state-dependent delay and for the feedback. Hill functions have the property that they have negative Schwarzian derivatives. The hyperbolic tangent functions are another example of such functions. It can be shown that for both the delay as well as the feedback functions in (205) and (207), $\mathcal{S}\tau(x) < 0$ and $\mathcal{S}f(x) < 0; \forall x \in \mathbb{R}$. The Schwarzian of any function is zero ($\mathcal{S}f = 0$) if and only if f is a linear fractional transformation i.e., $f(x) = \frac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$.

We can see that the derivative of f from (207) can be found as follows.

$$f'(P) = -\frac{NK_P P^{N-1}}{(1 + P^N)^2}; \quad P \geq 0 \quad (209)$$

From the above expression, it is easy to see that f' is bounded $\forall P \geq 0$ and therefore f is Lipschitz continuous. Let $L_f > 0$ denote the Lipschitz constant of f , then by definition,

$$|f(P_1) - f(P_2)| \leq L_f |P_1 - P_2| \quad \forall P_1 \neq P_2 \quad (210)$$

In fact, from (209)

$$\arg \max_{0 \leq P < \infty} |f'(P)| = \sqrt[N]{\frac{1}{N-1}}; \quad N \neq 1 \quad (211)$$

and

$$L_f = \max_{0 \leq P < \infty} |f'(P)| = K_P N^3 \sqrt[N]{(N-1)^{1-3N}}. \quad (212)$$

10.1.1 Bendixon's Criterion

If f_x and g_y are continuous in a region R which is simply-connected (i.e., without holes), and $f_x + g_y \neq 0$ at any point of R , then the system

$$\begin{cases} \dot{x}(t) = f(x(t), y(t)) \\ \dot{y}(t) = g(x(t), y(t)) \end{cases}$$

has no closed trajectories inside R .

Using this criterion it is easy to show that the delay free model (203) of the gene expression *cannot* have a limit cycle.

10.1.2 Causality Constraint:

From (205), we get the following.

$$\begin{aligned} \frac{1}{\tau} &= 1 + \frac{1}{P^n} \\ \Leftrightarrow -\frac{1}{\tau^2} \dot{\tau} &= -nP^{-n-1} \dot{P} \\ \Leftrightarrow \dot{\tau} &= \tau^2 (nP^{-n-1})(BR - CP) \\ \Leftrightarrow \dot{\tau} &= n \frac{(BR - CP)P^{n-1}}{(1 + P^n)^2} \end{aligned}$$

Therefore, the causality constraint translates to the following region in the $R - P$ plane.

$$\begin{aligned}\dot{\tau} \leq 1 &\Leftrightarrow n(BR - CP)P^{n-1} \leq (1 + P^n)^2 \\ &\Leftrightarrow R \leq \frac{(1 + P^n)^2 + nCP^n}{nB}\end{aligned}$$

Remark 9:

The above causality constraint reveals that mRNA and protein levels are always bounded. Notice that in [78], the authors assume the dependence of the delay on the mRNA concentration without taking into account the delay rate constraint. Moreover, the delay becomes unbounded as the mRNA concentration becomes larger and larger which seems to be unrealistic. In our model, the delay is a Hill function of protein concentration which makes more sense because protein is the end product of the reaction i.e., the whole gene-mRNA-protein cycle.

10.1.3 Equilibria of (204)

Notice that the equilibria (P^*, R^*) of (204) can be derived to satisfy the following polynomial equations.

$$\begin{cases} (P^*)^{N+1} + P^* - \frac{BK}{AC} = 0 \\ R^* = \frac{CP^*}{B} \end{cases} \quad (213)$$

Our objective will be to analyze the asymptotic stability of the equilibria of the system (204) and investigate the existence of a limit cycle (periodic solution).

10.2 Asymptotic Stability of the Equilibrium Point of (204)

We now use the above (LR) theorem to show that the steady state (equilibrium point) of the SD-DDE system (204) is asymptotically stable.

Consider the Lyapunov function $V : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ described by $V(R, P) = \frac{1}{2}R^2 + \frac{1}{2}P^2$. We have for $x \in \mathbb{R}^2$, $\alpha(x) \leq V(x) \leq \beta(x)$, with $\alpha(R, P) = \frac{1}{4}R^2 + \frac{1}{4}P^2$ and $\beta(R, P) = R^2 + P^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(r) = \rho^2 r$, $r \in \mathbb{R}^+$ and $\rho > 1$. This ensures one of the requirements of the LR theorem that $\eta(r) > r, \forall r > 0$. Let $x(t)$ be the solution trajectory of (204), such that for $t \geq 0$, $\theta \in [-\tau_{\max}, 0]$,

$$V(P(t+\theta), R(t)) < \eta(V(P(t), R(t))) \quad (214)$$

$$\begin{aligned} &\Leftrightarrow \frac{1}{2}P^2(t+\theta) + \frac{1}{2}R^2(t) < \rho^2 \frac{1}{2}P^2(t) + \frac{1}{2}R^2(t) \\ &\Leftrightarrow \frac{1}{2}P^2(t+\theta) < \rho^2 \frac{1}{2}P^2(t) \\ &\Leftrightarrow P^2(t+\theta) < \rho^2 P^2(t) \\ &\Leftrightarrow |P(t-\tau(P))| < \rho |P(t)| \end{aligned} \quad (215)$$

Now, we have for $t \geq 0$

$$\begin{aligned} &\dot{V}(R(t), P(t)) \\ &= \frac{\partial V}{\partial R} \frac{dR}{dt} + \frac{\partial V}{\partial P} \frac{dP}{dt} \\ &= R(t)\dot{R}(t) + P(t)\dot{P}(t) \\ &= R(t)f(P(t-\tau(P(t)))) - AR^2(t) - BP(t)R(t) - CP^2(t) \\ &\leq L_f R(t)P(t-\tau(P(t))) - AR^2(t) - BP(t)R(t) - CP^2(t) \\ &\leq L_f R(t)|P(t-\tau(P(t)))| - AR^2(t) - BP(t)R(t) - CP^2(t) \quad (216) \\ &\leq L_f \rho R(t)|P(t)| - AR^2(t) - BP(t)R(t) - CP^2(t) \\ &= - \begin{pmatrix} R(t) \\ P(t) \end{pmatrix}^T \begin{pmatrix} A & \frac{1}{2}(B - \rho L_f) \\ \frac{1}{2}(B - \rho L_f) & C \end{pmatrix} \begin{pmatrix} R(t) \\ P(t) \end{pmatrix} \\ &< 0 \text{ if } \begin{pmatrix} A & \frac{1}{2}(B - \rho L_f) \\ \frac{1}{2}(B - \rho L_f) & C \end{pmatrix} > 0 \end{aligned}$$

Since $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ whenever $V(x(t+\delta)) \leq \eta(V(x(t)))$, $\forall \delta \in [-\tau, 0]$, all the conditions of LR theorem are satisfied. We, therefore, conclude that the equilibrium point of (204) is asymptotically stable. Furthermore, since $V(x) = \frac{1}{2}x^2(t) = \frac{1}{2}\|x\|^2$ i.e. $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Therefore, V is radially unbounded, and by definition this implies the global asymptotic stability of the equilibrium point.

The above result is very useful and can be stated as the following theorem.

Theorem 17 *The SD-DDE system characterized by (204) is uniformly globally asymptotically stable if there exists a scalar $\rho > 1$ such that $(B - \rho L_f)^2 < 4AC$.*

10.3 Experimental Study From the Literature

In [47], an experimental data is provided which shows the periodic nature of mRNA and protein profiles. Fig. 42 shows the data which shows the trajectories of the concentrations of Hes1 mRNA and protein obtained from [47].

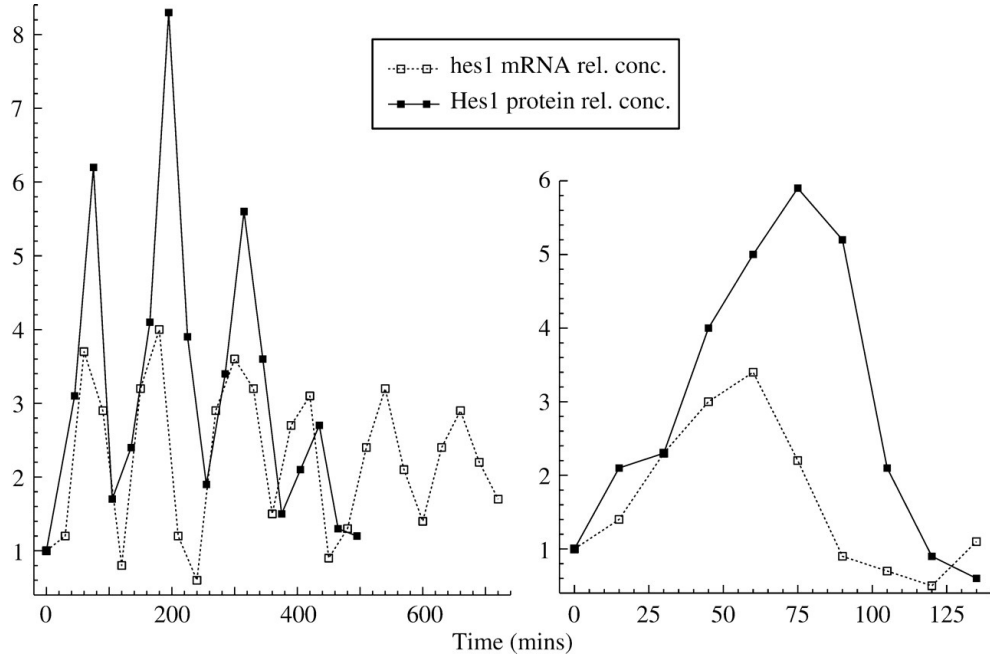


Figure 42: The left and right panel give two time courses of relative concentrations of Hes1 mRNA and protein obtained in Hirata et al. [47]. The observed data are given by the discrete points in the plots which are connected only for illustration. In one experiment (left panel), 17 discrete data points are available that describe the contemporaneous time course of the mRNA and protein at 30 min long time interval—except the first protein measurement was taken 45 min after an initial measurement at time 0. Protein and mRNA are not measured at the same time but are 15 min apart. A further time course for both variables (right panel) with 10 data points measured at 15 min interval length was also obtained by Hirata et al. [47]. Both data sets are used in the estimations [46].

10.4 Simulation Results

10.4.1 Example 1:

Consider the nonlinear biological feedback system with the state-dependent delay with the parameters $N = 2, A = B = C = K_p = 1$ and $n = 1$. The initial history function for the mRNA and protein concentration was chosen as $\phi(t) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \in \mathcal{C}^{0,1}, \forall t \in [-1, 0]$. Fig. 43 shows the state trajectories depicting the profiles of mRNA and protein concentrations. Clearly, the system is asymptotically stable.

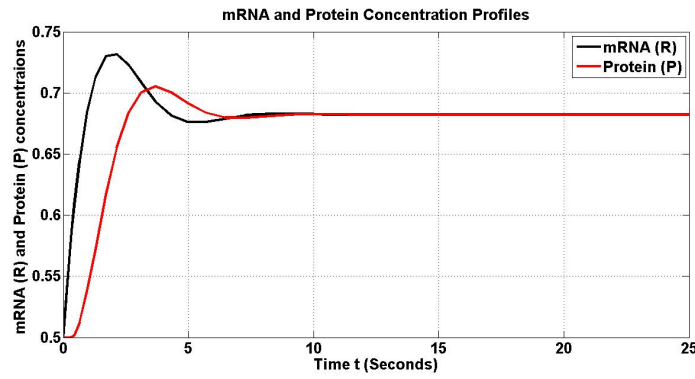


Figure 43: mRNA and Protein Concentrations: State Trajectories

10.4.2 Example 2:

Consider the nonlinear biological feedback system with the state-dependent delay with the parameters $N = 4, A = 1, B = 4.2, C = 1, K_p = 1$ and $n = 1$. The initial history function for the mRNA and protein concentration was chosen as $\phi(t) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \in \mathcal{C}^{0,1}, \forall t \in [-1, 0]$. Fig. 44 shows the state trajectories depicting the profiles of mRNA and protein concentrations. The graph shows that the system exhibits periodicity (oscillatory behavior). Fig. 45 shows the phase plane portrait of the system. We can easily see that there is a limit cycle.

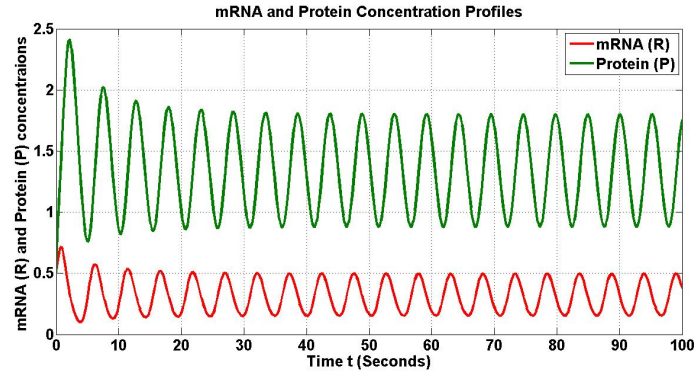


Figure 44: mRNA and Protein Concentrations: State Trajectories

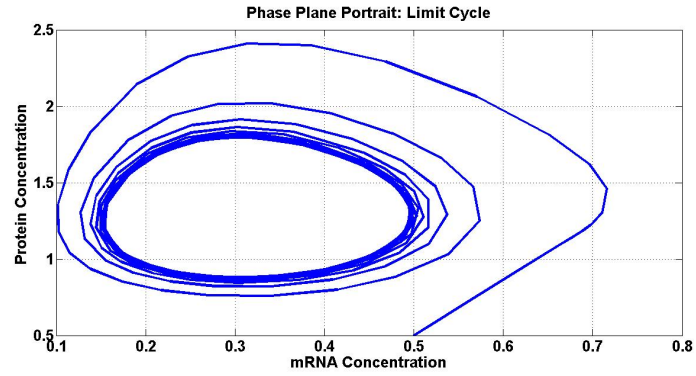


Figure 45: Phase Plane Portrait: A Limit Cycle

10.5 Concluding Remarks

In this chapter we solved the problem of modeling and analysis of gene expression as a nonlinear feedback system with a state-dependent delay. The delay accounts for the lag from the initiation of translation until the appearance of the mature protein messenger RNA (mRNA). We considered the delay to be dependent on the instantaneous concentration of the protein. This gave birth to a nonlinear biological system with a state-dependent delay. We considered the degradation of mRNA and protein in the mathematical model. We gave conditions for the asymptotic stability of the system. We also showed simulation results depicting the limit cycle.

CHAPTER XI

SYSTEMS EVOLVING WITH SUPREMA: STATE SPACE, EQUILIBRIA, STABILITY, CONTROL, OBSERVATION & M^3D STRUCTURE

In this chapter, we consider systems in which the evolution of the dynamics is governed by the supremum of the states in an interval containing past history. It is a class of nonlinear infinite-dimensional systems. We analyze the systems in the framework of Banach function spaces with the *topology of uniform convergence*. Our objective is to investigate the state space, equilibria, stability, control and observation of this class of systems. Three different kinds of equilibria, namely behavioral, Cauchy and asymptotic equilibria, are explained in the chapter. The state spaces associated with finite as well as infinite memory horizons are discussed in detail. First we start with the scalar case and then we extend the theory to higher order case where the partial state $x(t)$ of the system lies in \mathbb{R}^n . We use Razumikhin's framework for the stability analysis and controller design of these systems. The basic discrete systems counterpart of the continuous systems evolving with state suprema is also expounded in a fully detailed way. It is shown that such systems exhibit the features of Multi-Mode Multi-Dimensional (M^3D) systems. We derive conditions for the exponential solutions in terms of the Lambert-W function. We use Halanay's inequality for the observer design of scalar systems and Razumikhin argument for the higher order systems. At the end, simulation results are shown which shed light on the usefulness of the proposed theory.

11.1 Introduction & Motivation

We first investigate the behavior of the following scalar system evolving with state supremum,

$$\dot{x}(t) = \alpha x(t) + \beta \sup_{t-\tau \leq \theta \leq t} x(\theta) \quad (217)$$

where $x(t) \in \mathbb{R}$ is the scalar state of the system and α, β are constants. τ is the length of the memory of the sup functional which could be finite or infinite. This is the reason that we are using the sup and not the max throughout the chapter in order to maintain generality. Of course for finite memory τ , one can replace sup with *max*.

Very little is known about such systems. Our objective is to unravel the information in this system and exploit the structure present in this system. These systems, though complex, nonlinear and non-smooth in nature, form an important class of infinite-dimensional systems and are therefore worth researching. Examples of such systems are stuck float and ratchet where the decision is to be made by the maximum value of the state variable. When a float valve gets faulty or stuck, it can only give the indication of the maximum level of the liquid over a certain time interval. System (217) can be thought of as a feedback system in which the control policy at any instant of time t depends on the supremum of the state over the past history of length τ units of time. We consider systems with both finite and infinite memory horizons and give the state space characterization for each of them. We also give the symmetric version of the system. Three different kinds of equilibria related to these systems are expounded in the chapter. We refer to [17] for the qualitative treatment of differential equations with maxima.

11.1.1 Notations:

\mathbb{Z} and $\mathbb{R} = (-\infty, \infty)$ represent the set of integers and real numbers respectively. $\mathbb{R}_+ = [0, \infty)$ stands for the set of non-negative real numbers. W_0 denotes the principal

branch of the Lambert-W function. \mathbf{C} represents the space of continuous functions and \mathcal{C} denotes the Banach space of continuous functions over a compact support. $|\cdot|$ and $\|\cdot\|$ denote the absolute value (modulus) and norm respectively. \mathbf{D} represents the usual differential operator i.e., $\mathbf{D} := \frac{d}{dt}$. $P > 0$ means that the matrix P is positive definite.

11.2 System With State Suprema as a Nonlinear System

In this section, we list down some useful properties of the supremum (sup) functional and show that the sup is a nonlinear functional or operator. It can be seen using elementary real analysis that the sup functional exhibits the following properties. Let $\Theta \subset \mathbb{R}$ represent a set which may be a compact interval over a finite support in \mathbb{R} .

(1). The sup functional does not satisfy the additivity property. In fact, it is sub-additive. This means that for any two continuous functions x_1 and x_2 ,

$$\sup_{\theta \in \Theta} (x_1(\theta) + x_2(\theta)) \leq \sup_{\theta \in \Theta} x_1(\theta) + \sup_{\theta \in \Theta} x_2(\theta) \quad (218)$$

(2). The sup functional also fails to satisfy the homogeneity property. $\forall x \in \mathcal{C}, c \in \mathbb{R}$,

Case: 1). If $c \geq 0$,

$$\sup_{\theta \in \Theta} (cx(\theta)) \leq c \sup_{\theta \in \Theta} x(\theta) \quad (219)$$

Case: 2). If $c \leq 0$,

$$\sup_{\theta \in \Theta} (cx(\theta)) \geq c \sup_{\theta \in \Theta} x(\theta) \quad (220)$$

Notice that the equality holds only if the constant c is either zero or 1.

(3). Replacing x_1 by $x_1 - x_2$ in property 1 i.e., in (218), we get the following useful property.

$$\sup_{\theta \in \Theta} (x_1(\theta) - x_2(\theta)) \geq \sup_{\theta \in \Theta} x_1(\theta) - \sup_{\theta \in \Theta} x_2(\theta) \quad (221)$$

(4). The modulus of the supremum of a function is always dominated by the supremum of modulus of the function i.e., $\forall x \in \mathcal{C}, c \in \mathbb{R}$,

$$\|c \sup_{\theta \in \Theta} (x(\theta))\| \leq |c| \sup_{\theta \in \Theta} \|x(\theta)\| \quad (222)$$

In particular for $c = 1$, we have the following,

$$\|\sup_{\theta \in \Theta} (x(\theta))\| \leq \sup_{\theta \in \Theta} \|x(\theta)\| \quad (223)$$

Notice that the equality holds only if the function x is non-negative i.e., if $x(t) \geq 0, \forall t$ then,

$$\|c \sup_{\theta \in \Theta} (x(\theta))\| = |c| \|\sup_{\theta \in \Theta} x(\theta)\| = |c| (\sup_{\theta \in \Theta} x(\theta)). \quad (224)$$

(5). The supremum of the square of absolute value of a function is the same as the square of the supremum of the absolute value of a function i.e.,

$$\sup_{\theta \in \Theta} \|x(\theta)\|^2 = (\sup_{\theta \in \Theta} \|x(\theta)\|)^2. \quad (225)$$

(6). The supremum of the negative of a function is the negative infimum of that function i.e.,

$$\sup_{\theta \in \Theta} (-x(\theta)) = -\inf_{\theta \in \Theta} (x(\theta)). \quad (226)$$

(7). The supremum of a continuous function over a compact support is again a continuous function.

Nonlinear Nature:

In the light of the first two properties of the sup functional listed above, we establish that systems evolving with state suprema neither exhibit additivity nor homogeneity. This implies the absence of a superposition principle. Therefore, systems evolving with state suprema are *nonlinear* in nature. More precisely and rigorously, the sup functional is a *sublinear functional* or a *Banach functional*.

11.3 Information Structure

By information structure we mean the minimal sufficient statistic required, on the initial history, to make the problem well defined. We consider two different cases.

11.3.1 Finite Memory Horizon ($\tau < \infty$):

From a deeper look at (11), we see that the dynamic evolution of the system not only depends on its current state but also on previous states. In fact, the system requires a memory of length $\tau \in \mathbb{R}_+$ for its complete evolution. Therefore, its Cauchy Problem (Initial Value Problem) cannot be characterized just by considering a point on the real line ($\psi \in \mathbb{R}$) as the initial value. The system is no longer finite dimensional. We shall use the Banach space of continuous functions over an interval of compact support, which is a complete normed space, as our framework. The Cauchy problem for the evolution of the infinite dimensional state can be characterized as follows,

$$\Sigma_c : \begin{cases} \dot{x}(t) = \alpha x(t) + \beta \sup_{t-\tau \leq \theta \leq t} x(\theta); & \forall t \geq 0 \\ x(t) = \psi(t); & \forall t \in [-\tau, 0] \end{cases} \quad (227)$$

where $\alpha, \beta \in \mathbb{R}$ are constant scalars, $x(t) \in \mathbb{R}$ is the state variable and $\psi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R})$ is the initial infinite dimensional history function living in the Banach function space of continuous functions mapping the interval $[-\tau, 0]$ to \mathbb{R} with the topology of uniform convergence. This means that the norm of an element ϕ in this function space is defined by the following uniform norm.

$$\|\phi\|_c = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)| \quad (228)$$

One restatement of the Stone-Weierstrass theorem is that the set of all continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$. Therefore, \mathcal{C} is a *separable* Banach space.

11.3.2 Infinite Memory Horizon ($\tau \rightarrow \infty$):

In this case the memory of the *sup* is not finite. In other words we are interested in the information structure required when $\tau \rightarrow \infty$. It is worthy to be mentioned here that this not mere the limiting case of the first case. One might get deceived and could think of taking the interval $(-\infty, 0)$ and $\mathcal{C}((-\infty, 0]; \mathbb{R})$ as the initial history

function. Unfortunately, this is not correct! The common and major difficulty is that the interval $(-\infty, 0)$ is not compact, and the images of a solution map of closed and bounded sets in $\mathcal{C}((-\infty, 0]; \mathbb{R})$ with uniform norm may not be compact in the same space. Therefore, the space of continuous functions \mathcal{C} equipped with the uniform norm (228), as used in the first case, is not an appropriate space for the case of infinite memory horizon.

Another choice could be the space $\mathcal{BC}((-\infty, 0]; \mathbb{R})$ i.e., the space of bounded and continuous functions mapping the interval $(-\infty, 0]$ to \mathbb{R} and equipped with the uniform norm. Unfortunately, this space does not work either. In ([72]), it was shown that with such a space the delay differential equations with unbounded delay are not well posed.

Inspired by the idea of fading memory spaces for infinite (unbounded) delays as used by Haddock in [39] and [57], we use \mathcal{UC}_g spaces for the systems with infinite memory horizon. The details of these friendly fading memory space are given in the next subsection.

The Fading Memory Space: \mathcal{UC}_g : Let \mathcal{UC} denote the space of uniformly continuous functions on $(-\infty, 0]$. The fading memory space \mathcal{UC}_g is defined as follows.

$$\mathcal{UC}_g := \left\{ \begin{array}{l} \Phi \in \mathcal{C}((-\infty, 0]; \mathbb{R}) : \|\Phi\|_g < \infty, \\ \frac{\Phi(s)}{g(s)} \in \mathcal{UC}((-\infty, 0]; \mathbb{R}) \end{array} \right\} \quad (229)$$

where,

$$\|\Phi\|_g = \sup_{s \leq 0} \frac{\|\Phi(s)\|_{\mathbb{R}^n}}{g(s)} \quad (230)$$

and the parameterizing function $g : (-\infty, 0] \rightarrow [1, \infty]$ must possess the following three characteristics.

- (1). $g \in \mathcal{C}((-\infty, 0]; [1, \infty])$ is a continuous and nonincreasing (monotonically decreasing) function on $(-\infty, 0]$ such that $g(0) = 1$;
- (2). $\frac{g(s+u)}{g(s)} \rightarrow 1$ uniformly on $(-\infty, 0]$ as $u \rightarrow 0^-$;

(3). $g(s) \rightarrow \infty$ as $s \rightarrow -\infty$.

One example of a candidate parameterizing function g is the exponential function $g(s) = e^{-\gamma s}, \gamma > 0$.

Notice that the space \mathcal{UC}_g together with the uniform norm (230) is a normed space. It can be shown that any Cauchy sequence $\{f_n\}_{n=1}^\infty \in \mathcal{UC}_g$ will converge uniformly to the limit function $f^* \in \mathcal{UC}_g$. Therefore, \mathcal{UC}_g is complete and hence a Banach space.

11.4 *Symmetric sup System*

Notice that the system characterized by (227) has one serious limitation that it is not symmetric. This equation holds only for positive initial history and loses its structure for negative initial data. Though we can still define it! Clearly, replacing x by $-x$ in (227) we get the following.

$$\Sigma_c : \begin{cases} -\dot{x}(t) = -\alpha x(t) + \beta \sup_{t-\tau \leq \theta \leq t} (-x(\theta)); & \forall t \geq 0 \\ x(t) = -\psi(t); & \forall t \in [-\tau, 0]. \end{cases} \quad (231)$$

which is in fact equal to the following equation.

$$\Sigma_c : \begin{cases} -\dot{x}(t) = -\alpha x(t) - \beta \inf_{t-\tau \leq \theta \leq t} (x(\theta)); & \forall t \geq 0 \\ x(t) = -\psi(t); & \forall t \in [-\tau, 0]. \end{cases} \quad (232)$$

Clearly, the above equation does not possess the same *sup* structure as the original equation. In order to circumvent this problem, we symmetrize (227) as follows.

$$\Sigma_s : \begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(\theta^*(t)); & \forall t \geq 0 \\ \theta^*(t) = \arg(\sup_{t-\tau \leq \theta \leq t} |x(\theta)|), \\ x(t) = \psi(t); & \forall t \in [-\tau, 0]. \end{cases} \quad (233)$$

Since $\sup_{t-\tau \leq \theta \leq t} |-x(\theta)| = \sup_{t-\tau \leq \theta \leq t} |x(\theta)|$, we can observe that (233) is symmetric with respect to x as well as the initial history ψ .

The symmetrized version in the vector case will be as follows.

$$\Sigma_s : \begin{cases} \dot{x}(t) = Ax(t) + bc^\top x(\theta^*(t)); \\ \theta^*(t) = \arg(\sup_{t-\tau \leq \theta \leq t} |(c^\top x(\theta))|) \quad \forall t \geq 0; \\ x(t) = \psi(t); \quad \forall t \in [-\tau, 0]. \end{cases} \quad (234)$$

Where $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $b \in \mathbb{R}^n$, $c^\top \in \mathbb{R}^n$ are constant vectors. $x(t) \in \mathbb{R}^n$ is the state vector and $\psi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ is the initial history function living in the Banach function space. Here $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to \mathbb{R}^n with the topology of uniform convergence. i.e.,

$$\|\phi\|_{\mathcal{C}} = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_{\mathbb{R}^n}. \quad (235)$$

Notice that by using the symmetric version of the sup system, we are taking into account both the positive and negative excursions into account i.e., we can capture the maximal as well as the minimal symmetrical swings.

11.5 The Behavior of the sup Operator

In this section, we expound the behavior of the sup operator $\mathbf{Op} : \mathcal{C} \rightarrow \mathcal{C}$ and defined by:

$$\mathbf{Op} x(t) = \sup_{\theta \in [t-\tau, t]} x(\theta). \quad (236)$$

Observe that \mathbf{Op} is a bounded but nonlinear operator. Notice that continuity and boundedness of operators are equivalent for linear operators only. This is not true, in general, for nonlinear operators. The boundedness of the operator can be seen as follows.

$$\|\mathbf{Op}\| = \sup_{x \neq 0} \frac{\|\mathbf{Op} x\|_{\mathcal{C}}}{\|x\|_{\mathcal{C}}} = 1 \quad (237)$$

The nonlinearity of the operator is evident from the properties of the sup functional. \mathbf{Op} is a continuous operator. Also observe that $\forall x, y \in \mathcal{C}$,

$$\|\mathbf{Op} x - \mathbf{Op} y\| \leq \|x - y\| = L_{\mathbf{Op}} \|x - y\|. \quad (238)$$

This shows that \mathbf{Op} is a Lipschitz Operator with the Lipschitz constant $L_{\mathbf{Op}} = 1$. Therefore, \mathbf{Op} is a non-expansive map. We denote the kernel of the operator \mathbf{Op} by $\ker(\mathbf{Op})$ and define it as follows.

$$\ker(\mathbf{Op}) = \{x \in \mathcal{C} | \mathbf{Op} x = 0\} \quad (239)$$

We can see that $\ker(\mathbf{Op}) \neq \{0\}$. In fact $\{0\} \subset \ker(\mathbf{Op})$. The nontriviality of the kernel indicates that there are also non-zero functions x in the kernel of \mathbf{Op} besides the trivial zero ($x \equiv 0$) function. In fact, any monotone non increasing function defined on \mathbb{R}_+ with zero as the infimum serves as the example e.g., $(e^{-t} - 1)H(t)$, $-t^2H(t)$ etc., where H is the usual Heaviside unit step function. The point spectrum of the \mathbf{Op} contains the zero eigenvalue. This means that \mathbf{Op} is not bijective and therefore, the inverse operator \mathbf{Op}^{-1} does not exist!

We denote the image of the operator \mathbf{Op} by $Im(\mathbf{Op})$ and define it as follows.

$$Im(\mathbf{Op}) = \{y \in \mathcal{C} | \exists x \in \mathcal{C} \text{ s.t. } \mathbf{Op} x = y\} \quad (240)$$

When the memory is infinite, the image is a subset of \mathcal{C} and is the set of all monotone nondecreasing functions.

Here we remark that though the notion of kernel, image and spectrum make a good sense for linear operators, these concepts are vaguely defined for nonlinear operators. It is a hard problem in general and still an open area of research in pure mathematics and functional analysis. For the spectral theory of continuous nonlinear operators, see ([99]).

We consider two cases namely the infinite memory case and the finite memory case.

11.5.1 Infinite Memory Case:

Fig. 46 elucidates the action of the sup operator. Here we consider infinite length memory i.e., $\tau \rightarrow \infty$. The initial history is represented by ψ . $x(t)$ represents the

original function (solid line) and $z(t) = \sup_{t-\tau \leq \theta \leq t} x(\theta) \forall t \geq 0$ represents the action of the sup which is portrayed by a dotted line. In this particular figure, we have the maximum value of the history ψ at point A. Notice that when $x(t)$ is monotonically increasing i.e., $(\dot{x} \geq 0)$, $z(t)$ follows $x(t)$. However, when $x(t)$ starts decreasing, $z(t)$ remains held at the previous value x_B until x again starts increasing and reaches the value x_B at the point C whereafter $x(t) \geq z(t)$. From point C onwards $z(t)$ faithfully tracks $x(t)$. Observe that in this case, z is always a monotone nondecreasing function. This action can be simplified by observing that $z(t) = x(t - \tau(t))$ where $\tau(t)$ is a special type of time varying delay with rate unity as depicted in the bottom sub figure in Fig. 2. This special unit rate delay is either zero or of sawtooth type with unity slope.

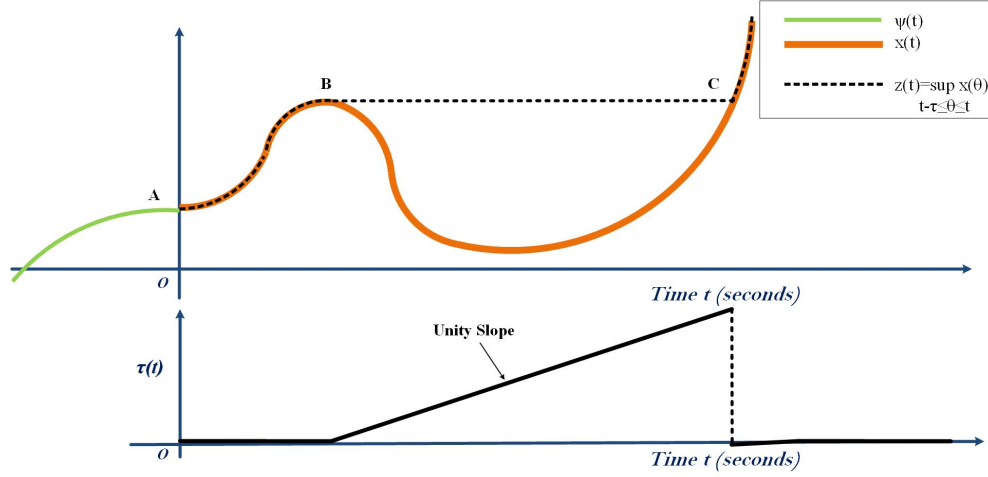


Figure 46: The action of the sup operator with infinite memory

11.5.2 Finite Length Memory Case:

Fig. 47 depicts the action of the sup operator. Here we consider a finite length memory of length τ units. This case is a bit more complex. The initial history is represented by ψ from A to B with the maximum at B. $x(t)$ represents the original function (solid line) and $z(t) = \sup_{t-\tau \leq \theta \leq t} x(\theta)$ represents the action of the sup which is shown by a dotted line. Notice that when $x(t)$ is monotonically increasing (from B to C) i.e., $(\dot{x} \geq 0)$, z follows x till $t \leq t_C$. However, when $x(t)$ starts decreasing

$z(t)$ remains held at the previous maximum value x_C until the memory is exhausted (point D). From point D onwards, z is a shifted version (τ delayed) of x till point E where $x \geq z$, now again x starts rising and z follows (overlaps) x . This action can be simplified by observing that $z(t) = x(t - \tau(t))$ where $\tau(t)$ is a special type of time varying delay with rate unity as depicted in the bottom sub figure in Fig. 3. This special unit rate delay is either zero or of sawtooth type with unity slope or constant. Observe that z is not always a monotone nondecreasing function in the finite memory horizon case. Here, z is always a continuous function which is differentiable a.e. In this case, we have an additional feature of constant delay. In general the time-varying delay function $\tau(t)$ looks of trapezoidal or sawtooth type or zero. Notice that the initial history $\psi \in \mathcal{C}$ does play a significant role in the output waveform of the sup functional.

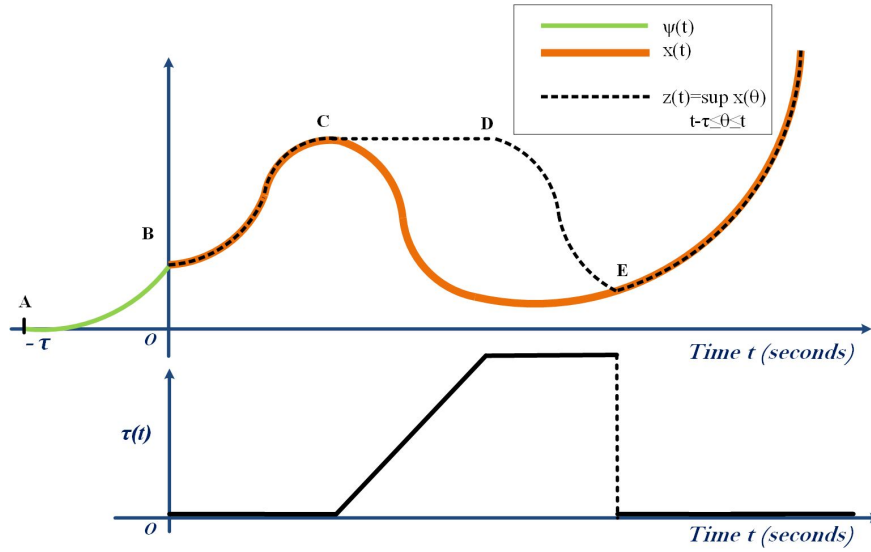


Figure 47: The action of the sup operator with finite memory of length τ time units

11.5.3 Natural Feature as a State-Dependent Delay System:

We notice that the behavior of the sup operator can be encapsulated by a special type of variable delay. This delay is continuous almost everywhere. The delay has jumps but the set of discontinuity is at the most countable. Moreover, the delay is

differentiable a.e. Let t^* be the argument of the sup then, $x(t^*) = x(t - (t - t^*))$ and the delay $\tau(t) = t - t^*$. However, t^* , the instant where the sup is attained, always depends on the state x_t . Here $x_t = x(t + \theta)$; $\forall \theta \in [-\tau, 0]$. Therefore, $\tau(x)$ is a state-dependent delay and systems evolving with state suprema can be regarded as a special class of systems with state-dependent delays. Systems governed by State-Dependent Delay Differential Equations (SD-DDEs) are inherently nonlinear and require more regularity on the initial data than continuity, see [9], [4] and [2] and the references therein.

11.6 Existence of Exponential Solutions

Consider a slightly more general version of (227) as follows.

$$\Sigma_s : \begin{cases} \dot{x}(t) = \alpha x(t) + \beta \sup_{t-\tau \leq \theta \leq t} x(\theta); & \forall t \geq t_0 \\ x(t) = \psi(t - t_0) \in \mathcal{C}. & \forall t \in [t_0 - \tau, t_0] \end{cases} \quad (241)$$

Now, we have the following theorem which gives the conditions on the parameters α and β for the existence of exponentially decaying solutions of (227). Notice that this is one of the many solutions.

Theorem 18 *If $\beta \geq 0$ and $\alpha < -\frac{1}{\tau}W_0(\tau\beta e^{-\alpha\tau})$, then there exists $M > 0$ and $\gamma > 0$ such that*

$$x(t) = Me^{-\gamma(t-t_0)}; t \geq t_0 \quad (242)$$

is a solution of (241) with $x(t_0) = \psi(t_0)$, where W_0 denotes the principal branch of the Lambert-W function.

Proof:

Let $x(t) = Me^{-\gamma(t-t_0)}; t \geq t_0$ be the solution of (241), then by definition it should

satisfy (241). This means that,

$$\begin{aligned}
& \mathbf{D}(Me^{-\gamma(t-t_0)}) = \alpha(Me^{-\gamma(t-t_0)}) + \beta \sup_{t-\tau \leq \theta \leq t} (Me^{-\gamma(\theta-t_0)}) \\
\Rightarrow & -\gamma(Me^{-\gamma(t-t_0)}) = \alpha(Me^{-\gamma(t-t_0)}) + \beta(Me^{-\gamma(t-\tau-t_0)}) \\
\Leftrightarrow & -\gamma = \alpha + \beta e^{\gamma\tau} \\
\Leftrightarrow & -(\gamma + \alpha)e^{-\gamma\tau} = \beta \\
\Leftrightarrow & -\tau(\gamma + \alpha)e^{-\tau(\gamma+\alpha)} = \tau\beta e^{-\alpha\tau} \\
\Leftrightarrow & -\tau(\gamma + \alpha) = W_k(\tau\beta e^{-\alpha\tau}); k \in \mathbb{Z} \\
\Leftrightarrow & \gamma = -\alpha - \frac{1}{\tau}W_k(\tau\beta e^{-\alpha\tau}); k \in \mathbb{Z}
\end{aligned}$$

In order to get a unique solution γ , we must require the principle branch of the Lambert- W function i.e., W_0 . To get a unique real solution, we should have the argument of the Lambert- W function to be non-negative. This dictates that,

$$\gamma = -\alpha - \frac{1}{\tau}W_0(\tau\beta e^{-\alpha\tau}); k \in \mathbb{Z} \quad (243)$$

and also,

$$\tau\beta e^{-\alpha\tau} \geq 0 \quad (244)$$

But we know that $\tau \in \mathbb{R}_+$ and the real exponential function is always positive i.e., $e^{-\alpha\tau} > 0$. This means that,

$$\beta \geq 0 \quad (245)$$

Also for $\gamma < 0$, it is mandatory to have that,

$$\alpha < -\frac{1}{\tau}W_0(\tau\beta e^{-\alpha\tau}). \quad (246)$$

Q.E.D. ■

The above theorem can be relaxed using the following Corollary which is an immediate consequence.

Corollary 6 *If $-\alpha \geq \beta > 0$, then there exists a decay rate $\gamma > 0$ and $M > 0$ such that for (241),*

$$x(t) = Me^{-\gamma(t-t_0)}$$

Notice that $M = \inf_{t_0-\tau \leq \theta \leq t_0} x(\theta)$. The above condition is independent of the memory length.

11.7 Equilibria

Before analyzing the stability of the system (233), it is primordial to investigate the equilibria or fixed points, also called singular points, of the system. We presents three different notions of the equilibria namely the equilibria in the behavioral sense, the equilibria in the Cauchy sense and the asymptotic equilibrium.

Equilibria in the Behavioral Sense: This definition is inspired from Willems's behavioral theory for open and interconnected systems ([100]). In this particular case, the time set $\mathbb{T} = \mathbb{R}$, the signal space $\mathbb{W} = \mathbb{R}$ in the scalar case; and $\mathbb{W} = \mathbb{R}^n$ in the higher order case. The set of all maps from \mathbb{T} to \mathbb{W} , denoted by $\mathbb{W}^{\mathbb{T}}$ is called the universum. Any behavior \mathfrak{B} of the system is a strict subset of the universum i.e., $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$.

Here an equilibrium solution is characterized by a constrained equation that satisfies a particular constraint for all time t . In other words, the equilibrium point is a particular behavior (a set or family of trajectories) exhibited by the dynamical system.

Definition 16 *An equilibrium point of the symmetrized system (233) is defined by the following behavior.*

$$\mathfrak{B} = \left\{ \begin{array}{l} w : \mathbb{R} \rightarrow \mathbb{W} \Big| w(t) = -\frac{\beta}{\alpha}(w(\theta^*(t))), \\ \theta^*(t) = \arg(\sup_{\theta \in [t-\tau, t]} |w(\theta)|), \\ \alpha \neq 0; \forall t \text{ a.e.} \end{array} \right\} \quad (247)$$

Clearly, the origin, $0 \in \mathfrak{B}$ is the trivial fixed point in $\mathbb{W}^{\mathbb{R}}$.

From (247), we get

$$(\alpha + \beta)|x^*|\text{sgn}(x^*) = 0 \quad (248)$$

where x^* is the equilibrium point. Using the definition of $\text{sgn}(\cdot)$, the above equation simplifies to,

$$(\alpha + \beta)x^* = 0 \quad (249)$$

Now there are two cases.

- (1). If $\alpha + \beta = 0$, any $x^* \in \mathbb{R}$ is an equilibrium point.
- (2). If $\alpha + \beta \neq 0$, only $x^* = 0$ is the unique equilibrium point.

Now, we consider the equilibrium points for the higher order version (234). Indeed, the equilibrium point x^* must satisfy the following equation.

$$\begin{cases} Ax^* = -bc^\top x^*(\theta^*), \\ \theta^*(t) = \arg(\sup_{t-\tau \leq \theta \leq t} |(c^\top x^*(\theta))|) \end{cases} \quad (250)$$

We have the following theorem.

Theorem 19 *Any $x^* \in \ker(A + bc^\top)$ is an equilibrium point of (250).*

Proof:

One can always use the state transformation (similarity transformation) and reduce (250) to a scalar version as follows. Let $c^\top x^* := y_1 = e_1^\top y$ be the first component of the vector $y \in \mathbb{R}^n$ (which is given by the following state transformation.)

$$y = Mx^* \Leftrightarrow x^* = M^{-1}y \quad (251)$$

where $M \in Gl_n(\mathbb{R})$. Here $Gl_n(\mathbb{R})$ represents the General Linear Group of $n \times n$ invertible matrices with elements in \mathbb{R} i.e., real entries. Since we have $y = Mx^*$ s.t. $y_1 = c^\top x^*$, i.e.,

$$e_1^\top M = c^\top \quad (252)$$

where, e_j represents the unit column vector with 1 at the $j - th$ position and rest of the entries zero. Therefore, (250) takes the form,

$$\begin{aligned}
Ax^* &= -bc^\top x^*(\theta^*(t)) \\
\Rightarrow Ax^* &= -b \sup_{t-\tau \leq \theta \leq t} |y_1(\theta)| \text{sgn}(y_1(\theta^*(t))) \\
\Rightarrow AM^{-1}Mx^* &= -b \sup_{t-\tau \leq \theta \leq t} |y_1(\cdot)| \text{sgn}(y_1(\cdot)) \\
\Rightarrow (AM^{-1})(Mx^*) &= -by_1 \\
\Rightarrow \bar{A}y &= -by_1 \\
\Rightarrow (\bar{A} + be_1^\top)y &= 0 \\
\Rightarrow (\bar{A} + be_1^\top)Mx^* &= 0 \\
\Rightarrow (A + bc^\top)x^* &= 0 \\
\Rightarrow x^* &\in \ker(A + bc^\top)
\end{aligned}$$

where, $\bar{A} \triangleq AM^{-1}$. This completes the proof. ■

Comment 8:

Notice that the equilibrium point x^* is unique if and only if $\ker(A + bc^\top)$ is trivial. That is the matrix $(A + bc^\top)$ is nonsingular (invertible).

Comment 9:

Using the Woodbury's lemma or matrix inversion lemma (ABCD lemma), we have

$$(A + bc^\top)^{-1} = A^{-1} - \frac{A^{-1}bc^\top A^{-1}}{1 + c^\top A^{-1}b}. \quad (253)$$

This means that a necessary condition for the system (250) to have a unique trivial equilibrium point is that A is invertible and $1 + c^\top A^{-1}b \neq 0$.

Equilibrium Points in the Cauchy Sense: Such an equilibrium point may depend on the initial history of the system.

Definition 17 x^* is called the equilibrium point of (233) in the Cauchy sense if $\dot{x}^* \equiv 0 \forall t \geq 0$ given $x(t) = \phi(t) \forall t \in [-\tau, 0]$.

For the scalar case with infinite memory horizon ($\tau = \infty$), $x^*(t)$ is an equilibrium in the Cauchy sense if there exists t such that

$$\begin{cases} x^*(t) = -\frac{\beta}{\alpha}(x(\theta^*(t))), \\ \theta^*(t) = \arg(\sup_{\theta < t} |x^*(\theta)|); \alpha \neq 0 \end{cases} \quad (254)$$

Likewise we can define the equilibria in the Cauchy sense for the vector case. Note that the set of equilibria in the behavioral sense is always contained in the set of the one in the Cauchy sense.

Equilibria in the Asymptotic Sense: This notion of equilibrium is inspired from the asymptotic or long term behavior of the system. The definition given below is inspired from ([36]). Here B_ρ represents an open ball of center 0 (the origin) and radius ρ i.e., $B_\rho = \{x \in \mathbb{R} \mid \|x\| < \rho\}$.

Definition 18 *The symmetrized differential equation with suprema (233) is said to exhibit the property of asymptotic equilibrium if:*

(i). *There exists $\rho > 0$ such that (233), with initial history $x(t) = \psi(t)$, for all $t \in [-\tau, 0]$ and $\|\psi\|_C < \rho$, has a solution defined for all $t \geq -\tau$ and there exists $\xi \in B_\rho$ which satisfies*

$$\lim_{t \rightarrow \infty} x(t) = \xi. \quad (255)$$

(ii). *There exists $\rho > 0$ such that for all $\xi \in B_\rho$, there exists a solution $x(t)$ of (233), which is defined on the interval $[-\tau, \infty)$ and verifies (255).*

We conjecture the equivalence of the asymptotic equilibria and the equilibria in the behavioral sense.

11.8 Global Existence and Uniqueness

Let us consider a more general nonlinear version of the system as follows.

$$\Sigma_g : \begin{cases} \dot{x}(t) = f(t, x(t), x(\theta^*(t))); \forall t \geq 0, \\ \theta^*(t) = \arg(\sup_{t-\tau \leq \theta \leq t} |x(\theta)|); \\ x(t) = \psi(t); \quad \forall t \in [-\tau, 0]. \end{cases} \quad (256)$$

Here $x(t) \in \mathbb{R}$ and $f \in \mathbf{C}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$. The theorem given below relates the conditions on the global existence of the solution and the asymptotic equilibrium of (256) and is the extension of ([36]) to the symmetrized systems with supremum.

Theorem 20 *If f in (256) satisfies the following conditions:*

(A1). $f(t, 0, 0)$ is integrable on $I = [0, \infty)$,

(A2). f is continuous on $I \times \mathbb{R} \times \mathbb{R}$,

(A3). There exists a function g which is integrable on I such that $0 \leq g(s) \leq Kg(s - \tau), \forall s \in I$, K a constant and, $\forall (t, x_1, y_1), (t, x_2, y_2) \in I \times \mathbb{R} \times \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq g(t)(|x_1 - x_2| + |y_1 - y_2|). \quad (257)$$

Then $\forall \xi \in \mathbb{R}$ there exists a solution of (256) over the whole interval $[-\tau, \infty)$ and satisfies the initial history i.e., $x(t) = \psi(t), \forall t \in [-\tau, 0]$. Moreover,

$$x(t) = \xi + \int_t^\infty f(s, 0, 0)ds + O(G(t)) \quad (258)$$

where $G(t) = \int_t^\infty g(s)ds$. Also, the solution is unique.

Proof:

The proof relies on the central idea of contraction mapping principle and Schauder/Banach fixed point theorem. Let \mathcal{C}_b denote Banach space of bounded and continuous real valued functions defined on $[-\tau, \infty]$; and equipped with the uniform norm. Let us define

the operator $T : \mathcal{C}_b \rightarrow \mathcal{C}_b$ as follows.

$$T(x)(t) = \begin{cases} \xi; & \forall t \in [-\tau, 0], \\ \xi - \int_t^\infty f(s, x(s), x(\arg(\sup_{v \in [s-\tau, s]} |x(v)|))) ds; & \\ \forall t \geq 0. \end{cases} \quad (259)$$

In order to show that T has a fixed point, it suffices to show that there exists a positive integer $n \in \mathbb{N}$ such that T^n is a contraction where $T^n = \underbrace{T \circ T \circ \dots \circ T}_{n\text{-times}}$ is the composition operator. This can be shown by applying the principle of mathematical induction to the following ([36]).

$$|(T^n x)(t) - (T^n y)(t)| \leq \frac{1}{Kn!} (2KG(t - \tau))^n \|x - y\| \quad (260)$$

By the assumption (A1) of integrability of the function g on I , therefore, one can find $n \in \mathbb{N}$ such that T^n is a contractive operator. This implies that indeed there exists a fixed point x of T and thus,

$$x(t) = \xi - \int_t^\infty f(s, x(s), \sup_{v \in [s-\tau, s]} |x(v)| \operatorname{sgn}(x(v))) ds; \quad \forall t \geq 0. \quad (261)$$

The uniqueness argument is simple by assuming two solutions w and z and showing that $\|w - z\| = 0$ by using the integral inequalities in ([17]). This completes the proof. ■

11.9 Stability Analysis Using Razumikhin Framework

Since the nature of the problem is infinite dimensional, simple Lyapunov functions cannot be used for stability analysis. Also, because of the *sup* functional in the dynamics, a direct use of the ubiquitous Lyapunov-Krasovskii (LK) functional, used in the delay literature, fails to work here. We now use the Razumikhin theorem ([43]) to ascertain the *sufficient conditions* such that the fixed point at the origin (equilibrium point: $x = 0$) or the trivial steady state of the suprema based system (233) is asymptotically stable. We express our sufficiency result as the following theorem.

Theorem 21 *The symmetric scalar system with state supremum characterized by (233) is globally asymptotically stable if there exists a scalar $\mathbb{R}_+ \ni \rho > 1$ such that $\alpha + \rho|\beta| < 0$.*

Proof: Consider the Lyapunov function $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ described by $V(x) = \frac{1}{2}x^2$. We have for $x \in \mathbb{R}^+$, $\vartheta(x) \leq V(x) \leq \zeta(x)$, with $\vartheta(x) = \frac{1}{4}x^2$ and $\zeta(x) = 2x^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(r) = \rho^2 r$, $\rho > 1$. This ensures one of the requirements of the Lyapunov-Razumikhin (LR) theorem that $\eta(r) > r, \forall r > 0$. Let $x(t)$ be the solution trajectory of (233), such that for $t \geq 0$, $\theta \in [-\tau, 0]$,

$$\begin{aligned} V(x(t + \theta)) &< \eta(V(x(t))) \\ \Rightarrow x^2(t + \theta) &< \rho^2 x^2(t) \\ \Rightarrow |x(t + \theta)| &< \rho |x(t)|; \forall \theta \in [-\tau, 0] \\ \Rightarrow \sup_{t-\tau \leq \theta \leq t} |x(\theta)| &< \rho |x(t)| \\ \Rightarrow |x(\theta^*(t))| &< \rho |x(t)| \end{aligned}$$

where the last inequality is the result of using the definition of $\theta^*(t) = \arg(\sup_{t-\tau \leq \theta \leq t} |x(\theta)|)$.

Now, we have for $t \geq 0$;

$$\begin{aligned} \dot{V}(x(t)) &= \frac{\partial V}{\partial x} \frac{dx}{dt} \\ &= x(t) \left(\alpha x(t) + \beta x(\theta^*(t)) \right) \\ &= \alpha x^2(t) + \beta x(t) x(\theta^*(t)) \\ &\leq \alpha x^2(t) + |\beta| |x(t)| |x(\theta^*(t))| \\ &< \alpha x^2(t) + |\beta| |x(t)| \cdot \rho |x(t)| \\ &= (\alpha + \rho |\beta|) x^2(t) \\ &< 0 \text{ if } \alpha + \rho |\beta| < 0 \end{aligned}$$

Since $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ whenever $V(x(t + \delta)) \leq \eta(V(x(t)))$, $\forall \delta \in [-\tau, 0]$, all the conditions of LR theorem are satisfied. We, therefore, conclude that the equilibrium point (origin) of (227) is asymptotically stable. Furthermore, since $V(x) =$

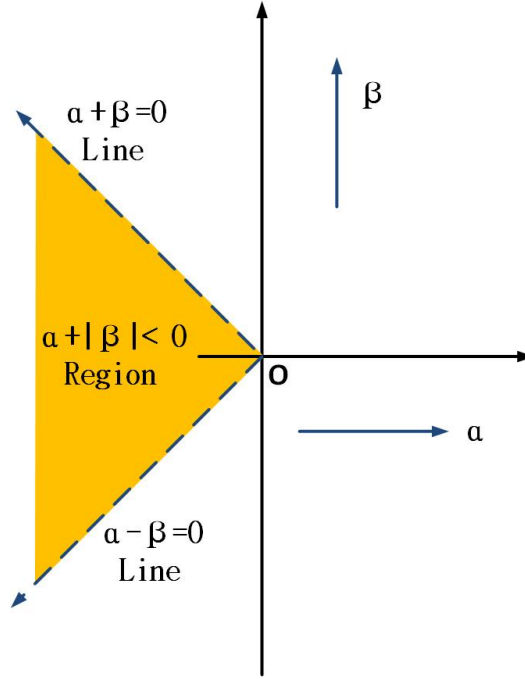


Figure 48: Stability Region $\alpha + |\beta| < 0$

$\frac{1}{2}x^2(t) = \frac{1}{2}\|x\|^2$ i.e., $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Therefore, V is radially unbounded, and by definition this implies the global asymptotic stability of the equilibrium point. This concludes the proof. \square

Fig. 48 shows the stability region for $\rho = 1$, in the parameter space i.e., $\alpha - \beta$ plane. Notice that as ρ increases, the stability region gets squeezed.

11.10 Discrete Systems With Suprema

Consider the following scalar discrete system with suprema.

$$\Sigma_d : \begin{cases} x_{k+1} = \alpha x_k + \beta \max_k(x_k, x_{k-1}); \forall k \in \mathbb{Z}, k \geq 0, \\ x_0, x_{-1} : \text{Given.} \end{cases} \quad (262)$$

The above system can be visualized as an auto-hybrid *state-dependent* switched system. It is in fact a Multi-Mode Multi-Dimensional (M^3D) system (see [84]) and can

be equivalently expressed as follows.

$$\Sigma_d^{M^3D} : \left\{ \begin{array}{l} \textbf{Mode I:} \\ x_{k+1} = (\alpha + \beta)x_k \text{ if } x_k \geq x_{k-1}; \forall k \in \mathbb{Z}, k \geq 0, \\ x_0 \in \mathbb{R} : \text{Given Mode History} \\ \textbf{Mode II:} \\ x_{k+1} = \alpha x_k + \beta x_{k-1} \text{ if } x_k \leq x_{k-1}; \forall k \in \mathbb{Z}, k \geq 0 \\ \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} \in \mathbb{R}^2 : \text{Given Mode History} \end{array} \right. \quad (263)$$

Theorem 22 *The discrete system with maxima characterized by (262) is globally asymptotically stable if given $0 < \epsilon < 1$, the parameters of the system satisfy the following set of algebraic inequalities.*

$$\left\{ \begin{array}{l} \alpha^2 < 1 - \epsilon, \\ \beta^2 - 1 < 0 \\ (\alpha + \beta)^2 < 1 - \epsilon \end{array} \right. \quad (264)$$

Proof:

Consider the following *common* quadratic Lyapunov function (CQLF) for each of the modes in (263).

$$V(x_k, x_{k-1}) = x_k^2 + \epsilon x_{k-1}^2; \epsilon > 0 \quad (265)$$

Now, we want to find the evolution of the above Lyapunov function V along the trajectories of each mode.

$$\begin{aligned} \Delta V &= V(x_{k+1}, x_k) - V(x_k, x_{k-1}) \\ \Leftrightarrow \Delta V &= x_{k+1}^2 + \epsilon x_k^2 - x_k^2 - \epsilon x_{k-1}^2 \\ \Leftrightarrow \Delta V &= x_{k+1}^2 - x_k^2 + \epsilon(x_k^2 - x_{k-1}^2) \end{aligned}$$

Now,

$$\begin{aligned}
\Delta V \Big|_{\text{Mode: I}} &= (\alpha + \beta)^2 x_k^2 - x_k^2 + \epsilon(x_k^2 - x_{k-1}^2) \\
\Leftrightarrow \Delta V \Big|_{\text{Mode: I}} &= ((\alpha + \beta)^2 + \epsilon - 1)x_k^2 - \epsilon x_{k-1}^2 \\
\Leftrightarrow \Delta V \Big|_{\text{Mode: I}} &= \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix}^\top \begin{pmatrix} (\alpha + \beta)^2 + \epsilon - 1 & 0 \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix}
\end{aligned}$$

Clearly,

$$\begin{aligned}
\Delta V \Big|_{\text{Mode: I}} < 0 &\Leftrightarrow \begin{pmatrix} (\alpha + \beta)^2 + \epsilon - 1 & 0 \\ 0 & -\epsilon \end{pmatrix} < 0 \\
&\Leftrightarrow \begin{cases} (\alpha + \beta)^2 + \epsilon - 1 < 0 \\ \epsilon > 0 \end{cases} \quad (266)
\end{aligned}$$

Similarly, using the very same Lyapunov function,

$$\begin{aligned}
\Delta V \Big|_{\text{Mode: II}} &= (\alpha x_k + \beta x_{k-1})^2 - x_k^2 + \epsilon(x_k^2 - x_{k-1}^2) \\
\Leftrightarrow \Delta V \Big|_{\text{Mode: II}} &= (\alpha^2 + \epsilon - 1)x_k^2 + 2\alpha\beta x_k x_{k-1} + (\beta^2 - 1)x_{k-1}^2 \\
\Leftrightarrow \Delta V \Big|_{\text{Mode: II}} &= \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix}^\top \begin{pmatrix} \alpha^2 + \epsilon - 1 & \alpha\beta \\ \alpha\beta & \beta^2 - 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix}
\end{aligned}$$

Clearly,

$$\begin{aligned}
\Delta V \Big|_{\text{Mode: II}} < 0 &\Leftrightarrow \begin{pmatrix} \alpha^2 + \epsilon - 1 & \alpha\beta \\ \alpha\beta & \beta^2 - 1 \end{pmatrix} < 0 \\
&\Leftrightarrow \begin{cases} \alpha^2 + \epsilon - 1 < 0 \\ \beta^2 - 1 < 0 \end{cases} \quad (267)
\end{aligned}$$

Since the parameter of the system α and β must be real, so that the system makes sense physically and practically, we require that $\epsilon < 1$. This condition in conjunction with the conditions obtained in (266) and (267) shows that $\Delta V < 0$ for both the

modes using the common Lyapunov function V . This ensures asymptotic stability. Moreover, since V is radially unbounded, the system is asymptotically stable in the large. This completes the proof. \square

Fig. 49 shows the stability region for $\epsilon = 0^+$. Notice that as (the free parameter) ϵ increases, the stability region gets squashed.

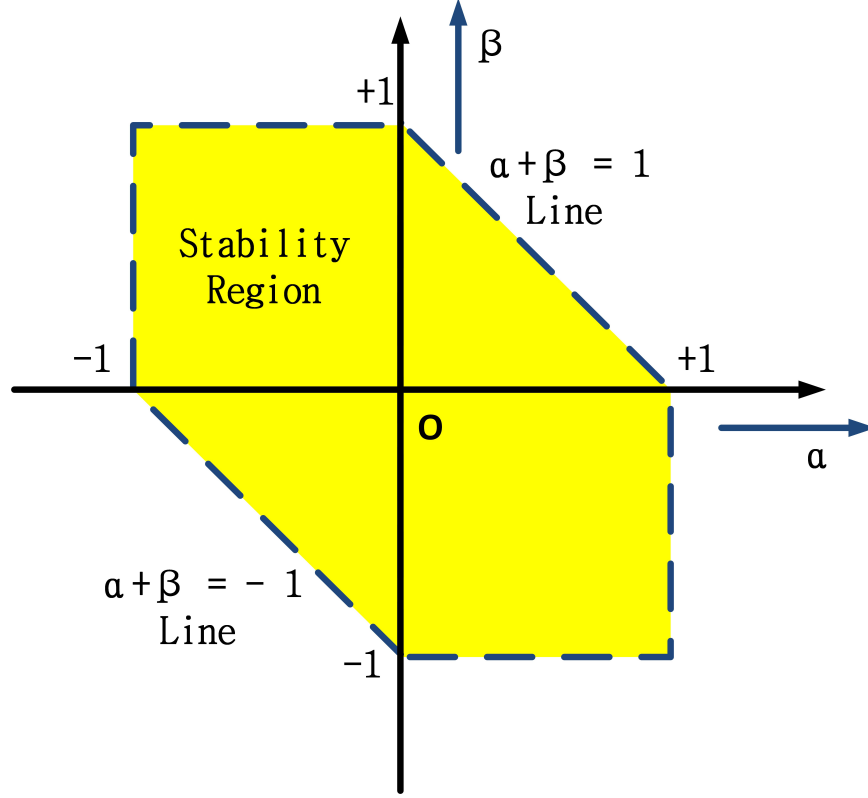


Figure 49: Stability Region for the Discrete System With Maxima

Observe that the the necessary and sufficient condition for stability for the Mode: I alone, irrespective of Mode: II, is given by

$$-1 < \alpha + \beta < 1. \quad (268)$$

Similarly by using the Möbius transformation (bilinear transformation) $z = \frac{s+1}{s-1}$ to the characteristic equation of Mode: II, $z^2 - \alpha z - \beta = 0$, one gets the following transformed characteristic equation.

$$s^2 + 2 \left(\frac{\beta + 1}{1 - \alpha - \beta} \right) s + \frac{1 + \alpha - \beta}{1 - \alpha - \beta} = 0$$

From, this the necessary and sufficient conditions for the global asymptotic stability of Mode: II can be readily ascertained as follows.

$$\Omega = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \left| \begin{array}{l} \beta > -1; \\ \alpha + \beta < 1; \\ \alpha - \beta > -1 \end{array} \right. \right\} \quad (269)$$

Remark 10:

Fig. 50 illustrates a comparison of the stability regions of the individual modes and that of the overall switched system. Notice that the stability region for Mode: I is the shaded region bounded by the parallel lines \overleftrightarrow{PQ} and \overleftrightarrow{RS} . The stability region for Mode: II is the region bounded by the red triangle $\triangle ABC$. It is noteworthy here that the stability region of the discrete switched system (system with maxima) is *not* the intersection of the stability regions of the individual modes. Another remarkable point is that there is a portion of the region of stability where the switched system is stable but Mode: II is unstable. Likewise, there is region of stability where the switched system is unstable but both of the modes are stable. Finally, notice that the stability of Mode: I is a necessary condition for the stability of the switched system.

11.11 Higher Order Nonlinear Systems Evolving with State Suprema

Now, we consider the general *higher order* case where the partial state, $x(t)$, lies in \mathbb{R}^n . The state space for such systems is an infinite-dimensional Banach space equipped with the *uniform convergence topology*. The stability and observer design for such systems is studied, the first using Razumikhin's framework, the second using the new concept of *sup based output injection*. Simulation results are shown at the end which demonstrate the effectiveness, validity and usefulness of the proposed observer scheme.

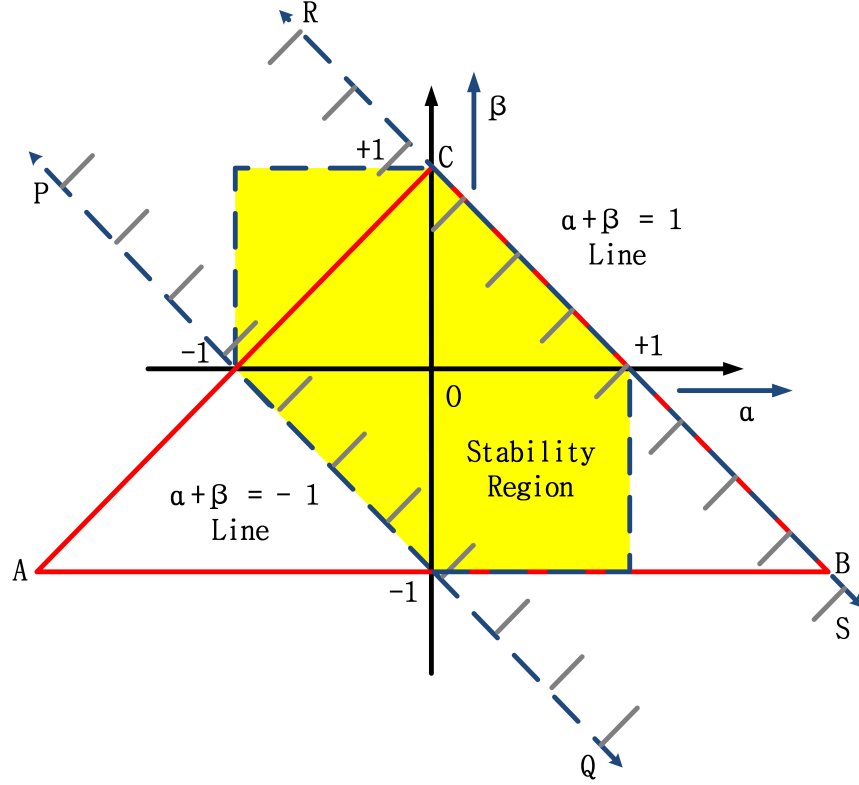


Figure 50: Comparison of Stability Regions for the Individual Modes and the Overall Discrete System With Maxima

11.11.1 Problem Formulation

We investigate the behavior of the following system evolving with state suprema,

$$\dot{x}(t) = Ax(t) + B \sup_{t-\tau \leq \theta \leq t} C^T x(\theta) \quad (270)$$

where $x(t) \in \mathbb{R}^n$ represents the state vector of the system, $A \in \mathbb{R}^{n \times n}$ is a constant time-invariant matrix and $B \in \mathbb{R}^n$, $C \in \mathbb{R}^n$ are constant vectors. $\tau \in \mathbb{R}_+$ is the length of the memory of the sup functional. Notice that (270) is not an Ordinary Differential Equation (ODE) but is indeed a Functional Differential Equation (FDE). System (270) can also be equivalently written in the following fashion.

$$\Sigma_u : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ u(t) = \sup_{t-\tau \leq \theta \leq t} C^T x(\theta). \end{cases} \quad (271)$$

In this format, it can be thought of as a closed-loop feedback system in which the control policy $u(t) \in \mathbb{R}$ at any instant of time t depends on the supremum of the weighted linear combination of the states over the past history interval of length τ units of time.

11.11.2 State Space & The Cauchy Problem

The Cauchy problem for the evolution of the infinite dimensional state can be characterized as follows.

$$\Sigma_c : \begin{cases} \dot{x}(t) = Ax(t) + B \sup_{t-\tau \leq \theta \leq t} C^T x(\theta); & \forall t \geq 0 \\ x(t) = \psi(t); & \forall t \in [-\tau, 0]. \end{cases} \quad (272)$$

Where $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $B \in \mathbb{R}^n$, $C \in \mathbb{R}^n$ are constant vectors. $x(t) \in \mathbb{R}^n$ is the state vector and $\psi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ is the initial infinite dimensional history function. Here $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to \mathbb{R}^n with the topology of uniform convergence. This means that the norm of an element ϕ in this function space is defined by the following norm.

$$\|\phi\|_c = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\| \quad (273)$$

11.11.3 Asymptotic Stability of the Equilibrium Point of (272)

We now use the Lyapunov-Razumikhin (LR) theorem which is a very powerful theorem in the context of stability analysis of systems characterized by FDEs ([43]) and derive the sufficient conditions for the asymptotic stability of the trivial steady state (equilibrium point: $x = 0$) of the sup based system (272). Unlike the Lyapunov-Krasovskii (LK) functional based approach in an infinite dimensional setting, this theorem uses functions which are relatively easier to handle with. This theorem also gives sufficient conditions for the stability.

Consider the Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ described by $V(x) = x^T P x$ where $P = P^T > 0$. We have for $x \in \mathbb{R}^n$, $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$, with $\alpha(\|x\|) = \lambda_{\min}(P)\|x\|^2$ and $\beta(\|x\|) = \lambda_{\max}(P)\|x\|^2$.

Define $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\eta(r) = \rho^2 r$, $r \in \mathbb{R}^+$ and $\rho > 1$. This ensures one of the requirements of the LR theorem that $\eta(r) > r, \forall r > 0$. Let $x(t)$ be the solution trajectory of (272), such that for $t \geq 0$, $\theta \in [-\tau_{\max}, 0]$,

$$\begin{aligned} V(x(t+\theta)) &< \eta(V(x(t))) \\ \Rightarrow x^T(t+\theta)Px(t+\theta) &< \rho^2 x^T(t)Px(t) \\ \Rightarrow \|x(t+\theta)\|_P^2 &< \rho^2 \|x(t)\|_P^2 \\ \Rightarrow \sup_{t-\tau \leq \theta \leq t} \|x(\theta)\| &< \rho \|x(t)\| \end{aligned}$$

Now, we have for $t \geq 0$,

$$\begin{aligned} \dot{V}(x(t)) &= \left(\frac{\partial V}{\partial x} \right)^T \frac{dx}{dt} \\ &= \nabla V(x) \cdot \frac{dx}{dt} \\ &= \mathbf{D}(x^T(t)Px(t)) \\ &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &= x^T(t)(A^T P + PA)x(t) + (B^T Px + x^T PB) \sup_{t-\tau \leq \theta \leq t} C^T x(\theta) \\ &\leq x^T(t)(A^T P + PA)x(t) + \|(B^T Px + x^T PB) \sup_{t-\tau \leq \theta \leq t} C^T x(\theta)\| \\ &\leq x^T(t)(A^T P + PA)x(t) + \|(B^T Px + x^T PB)\| \|C\| \sup_{t-\tau \leq \theta \leq t} \|x(\theta)\| \\ &\leq x^T(t)(A^T P + PA)x(t) + \|(B^T Px + x^T PB)\| \|C\| \rho \|x(t)\| \\ &\leq x^T(t)(A^T P + PA)x(t) + 2\rho \|B\| \|C\| \|P\| \|x(t)\|^2 \\ &= -x^T(t)Qx(t) + 2\rho \|B\| \|C\| \|P\| \|x(t)\|^2 \\ &\leq -(\lambda_{\min}(Q) - 2\rho \|B\| \|C\| \|P\|) \|x(t)\|^2 \\ &< 0 \text{ if } \lambda_{\min}(Q) > 2\rho \|B\| \|C\| \|P\|. \end{aligned} \tag{274}$$

Since $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ whenever $V(x(t + \delta)) \leq \eta(V(x(t)))$, $\forall \delta \in [-\tau, 0]$, all the conditions of LR theorem are satisfied. We, therefore, conclude that the equilibrium point (origin) of (272) is asymptotically stable. Furthermore, since $V(x) = x^T P x = \|x\|_P^2$ i.e. $V \rightarrow \infty$ as $\|x\|_P \rightarrow \infty$. Therefore, V is radially unbounded, and by definition this implies that the origin is a global attractor.

The above result is very useful and can be stated as the following theorem.

Theorem 23 *The infinite-dimensional system evolving with state suprema characterized by (272) is uniformly globally asymptotically stable if given any symmetric and positive definite matrix Q ($Q = Q^T > 0$) there exists a scalar $\rho > 1$ such that*

$$\lambda_{\min}(Q) > 2\rho\|B\|\|C\|\|P\| \quad (275)$$

where the matrix $P = P^T > 0$ precisely satisfies the following algebraic Lyapunov equation.

$$A^T P + P A + Q = O \quad (276)$$

Remark 11:

Notice that the above theorem requires that it is mandatory that the sup free system is asymptotically stable. In other words, the matrix A must be Hurwitz. Also since $\|P\| = \sqrt{\lambda_{\max}(P^T P)}$ is the induced or spectral norm of P , the upper bound on ρ in (310) can also be stated as follows.

$$\rho < \frac{\lambda_{\min}(Q)}{2\|B\|\|C\|\sqrt{\lambda_{\max}(P^T P)}} \quad (277)$$

Remark 12:

We can express $\dot{V}(x)$ in the above analysis as follows.

$$\dot{V}(x(t)) \leq -\zeta V(x(t)) \quad (278)$$

where $\zeta = \lambda_{\min}(Q) - 2\rho\|B\|\|C\|\|P\|$.

The corollary given below is an immediate consequence of the above result.

Corollary 7 *The dynamical system with suprema given by (272) is exponentially stable if A is Hurwitz and there exists a scalar $\rho > 1$ such that (310) and (310) hold.*

11.11.4 Controller Synthesis and the Stabilization Problem

Consider the following problem.

$$\Sigma_{fb} : \begin{cases} \dot{x}(t) = Ax(t) + B \sup_{t-\tau \leq \theta \leq t} C^T x(\theta) + \Gamma u(t); & \forall t \geq 0 \\ u(t) = Kx(t) \\ x(t) = \psi(t); & \forall t \in [-\tau, 0]. \end{cases} \quad (279)$$

Suppose that the unforced system (272) is not stable and we want to solve the stabilization problem i.e., we want to design a control policy $u = Kx$ such that the resulting closed-loop feedback system (279) is asymptotically stable. Here K represents the static state feedback gain matrix. Notice that a necessary condition for this problem is that the pair (A, Γ) must be controllable. We express our result as the following theorem.

Theorem 24 *The infinite-dimensional system evolving with state suprema characterized by (279) is uniformly globally asymptotically stabilizable if (A, Γ) is controllable and given any symmetric and positive definite matrix $Q \in \mathbb{R}^{n \times n}$ ($Q = Q^T > 0$) and a row vector $K \in \mathbb{R}^{1 \times n}$ there exists a scalar $\rho > 1$ such that*

$$\lambda_{\min}(Q) > 2\rho \|B\| \|C\| \|P\| \quad (280)$$

where the matrix $P = P^T > 0$ precisely satisfies the following algebraic Lyapunov equation.

$$(A + \Gamma K)^T P + P(A + \Gamma K) + Q = O \quad (281)$$

Proof:

The proof is an immediate consequence of the application of *Theorem 23* to the closed loop system (279). ■

By using a change of variable and setting $K = r\Gamma^T P$ for some scalar $r \neq 0$. we get the following result in the form of Riccati equation.

Corollary 8 *The infinite-dimensional system evolving with state suprema characterized by (279) is uniformly globally asymptotically stabilizable if (A, Γ) is controllable and given any symmetric and positive definite matrix $Q \in \mathbb{R}^{n \times n}$ ($Q = Q^T > 0$) there exists a scalar $\rho > 1$ such that*

$$\lambda_{\min}(Q) > 2\rho\|B\|\|C\|\|P\| \quad (282)$$

where the matrix $P = P^T > 0$ precisely satisfies the following delay algebraic Riccati equation

$$A^T P + PA + 2rP\Gamma\Gamma^T P + Q = O \quad (283)$$

and the controller (gain matrix) is given by:

$$K = r\Gamma^T P. \quad (284)$$

Remark 13:

It is noteworthy here that the Algebraic Riccati Equation (283) is different from the usual one obtained in the optimal control problem i.e., the infinite horizon Linear Quadratic Regulator (LQR) problem. In the LQR case, the ARE has a negative sign in front of the nonlinear (quadratic) term. The ARE in (283) is called the delay Riccati equation because of its ubiquitous appearance in the stability analysis of Time Delay Systems (TDS). By using Schur complements the delay ARE can be converted into the equivalent LMI as follows:

$$\begin{pmatrix} A^T P + PA + Q & P\Gamma \\ \Gamma^T P & -\frac{1}{2}r^{-1}I \end{pmatrix} < 0. \quad (285)$$

Synthesis of Output Feedback Controller: Consider the following output feedback control synthesis problem,

$$\Sigma_{ofb} : \begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = \sup_{t-\tau \leq \theta \leq t} (c^\top x(\theta^*)); \forall t \geq 0 \\ u(t) = ky(t) \\ x(t) = \psi(t); \forall t \in [-\tau, 0]. \end{cases} \quad (286)$$

where $u(t) \in \mathbb{R}$ is the control effort, $y(t) \in \mathbb{R}$ is the output and $k \in \mathbb{R}$ is the output feedback gain. The controlled closed loop system exhibits the following dynamics.

$$\dot{x}(t) = Ax(t) + bk \sup_{t-\tau \leq \theta \leq t} (c^\top x(\theta^*)); \forall t \geq 0 \quad (287)$$

Using LR framework, we want to design the controller k such that (287) is asymptotically stable. The following corollary gives the result.

Corollary 9 (287) is globally asymptotically stable if (A, b) is controllable and given any $Q \in \mathbb{R}^{n \times n}$ such that $Q = Q^\top > 0$ and $k \in \mathbb{R}$ there exists a scalar $\varrho > 2$ such that

$$\lambda_{\min}(Q) > \varrho \|k\| \|b\| \|c\| \sigma_{\max}(P) \quad (288)$$

where the matrix $P = P^\top > 0$ precisely is a solution of algebraic Lyapunov equation,

$$A^\top P + PA + Q = O \quad (289)$$

and $\sigma_{\max}(P)$ denotes the upper singular value of P .

11.12 The Observation Problem

In this section we present the observation problem for systems involving state suprema. We express the dynamics of the system and measurements as a setup or formulation for the observer problem as follows.

$$\left\{ \begin{array}{l} \text{Dynamics of the Plant Model:} \\ \dot{x}(t) = Ax(t) + \Gamma w(t) \\ \text{Observation or Measurement Equation:} \\ y(t) = \sup_{t-\tau \leq \theta \leq t} C^T x(\theta) \end{array} \right. \quad (290)$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^p$ is the driving force (control input or some known perturbation or disturbance), the state coupling matrix $A \in \mathbb{R}^{n \times n}$, the input coupling matrix (read in matrix) $\Gamma \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^n$ the readout or measurement matrix. The scalar $y(t)$ represents the nonlinear measurement at any arbitrary instant of time t and $\tau > 0$ denotes the memory associated with the sup functional. Notice that though the system seems to be apparently linear, it is in reality a nonlinear system because of the presence of the sup operator in the measurements.

The goal is the inversion of the measurement model i.e., our objective is to observe or estimate the state vector given the plant model, input and output of the system. For controller synthesis (state feedback), we require the complete state vector of the system to be available to us.

11.12.1 Observer Design

We will use the concept of sup *Based Output Injection* to construct the observer for the recovery of the state $x(t)$ in the same spirit as in [9]. Let $\hat{x}(t)$ represent the estimated state, the infinite-dimensional estimator (being a replica of the state dynamics) will be governed by the dynamics as follows.

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \Gamma w(t) + \mathbf{L} (y(t) - \hat{y}(t)) \quad (291)$$

where the last term is the correction term with $\mathbf{L} \in \mathbb{R}^n$ being the gain matrix of the estimator and the estimated output $\hat{y}(t)$ is precisely given by,

$$\hat{y}(t) = \sup_{t-\tau \leq \theta \leq t} C^T \hat{x}(\theta) \quad (292)$$

Fig. 51 gives a basic conceptual block diagram to elucidate the observation scheme using the concept of *sup based output injection*. Notice that the observer is nonlinear because of the presence of the sup block. Substituting the expressions for y and \hat{y} in

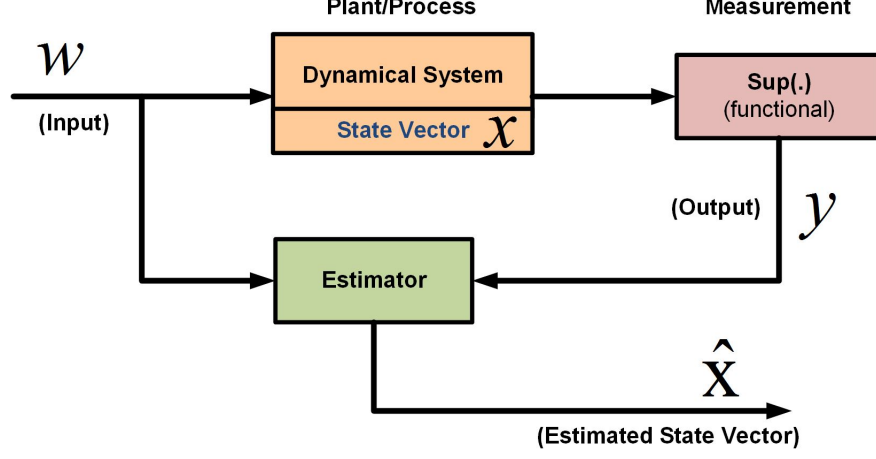


Figure 51: Illustrative Block Diagram of sup Based Observer

(291) yields,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \Gamma w(t) + \mathbf{L} \left(\sup_{t-\tau \leq \theta \leq t} C^T x(\theta) - \sup_{t-\tau \leq \theta \leq t} C^T \hat{x}(\theta) \right) \quad (293)$$

Defining $e(t) = x(t) - \hat{x}(t)$ as the estimation error or observation error at any arbitrary time instant t and subtracting the observer dynamics (293) from the plant dynamics equation in (290) , we get the observer error dynamics as follows.

$$\begin{aligned} \dot{e}(t) &= Ae(t) - \mathbf{L} \left(\sup_{t-\tau \leq \theta \leq t} C^T x(\theta) - \sup_{t-\tau \leq \theta \leq t} C^T \hat{x}(\theta) \right) \\ \Rightarrow \dot{e}(t) &\leq Ae(t) + \mathbf{L}^+ \sup_{t-\tau \leq \theta \leq t} (C^T x(\theta) - C^T \hat{x}(\theta)) \end{aligned} \quad (294)$$

$$\Rightarrow \dot{e}(t) \leq Ae(t) + \mathbf{L}^+ \sup_{t-\tau \leq \theta \leq t} (C^T e(\theta)) \quad (295)$$

Notice that the first inequality (294) is obtained from the preceding equation by using the subadditivity/difference property of the sup functional i.e.,

$$\sup_{\theta \in \Theta} (x_1(\theta) - x_2(\theta)) \geq \sup_{\theta \in \Theta} x_1(\theta) - \sup_{\theta \in \Theta} x_2(\theta).$$

In all the above inequalities, \mathbf{L}^+ stands for the vector \mathbf{L} but with a restriction that all the entries strictly positive. So our observer gain matrix \mathbf{L} cannot have zero or

negative elements. This positivity preserves the error inequality and is needed to make the observer design proof tractable using *Theorem 23*.

From (295), we can see that the error dynamics follow an infinite-dimensional vector differential inequality. The only design parameter is the observer gain matrix \mathbf{L}^+ . The error will converge asymptotically to zero if the above differential inequality is asymptotically stable or in other words the equilibrium point or fixed point (i.e., the origin $e = 0$) is asymptotically stable. Also notice that when there is no memory in the sup function (no past history taken into account) i.e., $\tau = 0$, we require the complete state observability of the pair (C^T, A) as a necessary and mandatory condition for the observer design. We give the following theorem for the observer design.

Theorem 25 *The infinite-dimensional system of error dynamics evolving with error suprema characterized by (295) is asymptotically stable if the pair (C^T, A) is observable and given any symmetric and positive definite matrix S ($S = S^T > 0$) there exists a scalar $\rho > 1$ such that*

$$\|\mathbf{L}^+\| < \frac{\lambda_{\min}(S)}{2\rho\|C\|\sqrt{\lambda_{\max}(R^T R)}} \quad (296)$$

where the matrix $R = R^T > 0$ is precisely a solution the following algebraic Lyapunov equation.

$$A^T R + RA + S = O \quad (297)$$

Proof:

The proof relies on the fact that the observer design is basically a dual problem of the control synthesis. The result is based on Razumikhin framework and application of *Theorem 23* to the error dynamics (295). ■

Remark 14:

We can see that there are two important conditions which need to be fulfilled before the observer is designed. The first condition is that the system must be asymptotically stable i.e. the matrix A must be Hurwitz. The second one is the observability

of the (C^T, A) pair. If any of these conditions fail, the observer design is out of question. Secondly, because of the nonlinearity of the sup functional, the deterministic separation principle may not hold. Also, from a practical point of view, the error dynamics of the observer must decay faster as compared to the plant dynamics.

Remark 15:

Notice that the observer above in (293) is an *infinite-dimensional* nonlinear system. It will require a memory of length τ units. The reason is evident from the measurement equation. Though the plant is finite dimensional, the observer cannot be. We do need to store the past history of measurement in order for the observer dynamics to evolve with the passage of time.

Remark 16:

One potential practical application of the above observation scheme can be the water level (water head) control of a hydroelectric power dam as shown in Fig. 52. One can make observations of the maximum intake water level from the reservoir over a period of week or a month and, from this maximum, can reconstruct the instantaneous level which can be used for the head control. The head control mechanism can be employed for the turbine speed which can be consequently used for the generation of desired power.

11.12.2 Observer Design Technique 2: Scalar Case

Here, we give another observer design technique which is based on Halanay's inequality [41]. This technique is only valid for scalar plants.

$$\left\{ \begin{array}{l} \text{Plant:} \\ \dot{x}(t) = \alpha x(t) + \xi w(t) \\ \text{Measurements:} \\ y(t) = \beta \sup_{t-\tau \leq \theta \leq t} x(\theta) \end{array} \right. \quad (298)$$

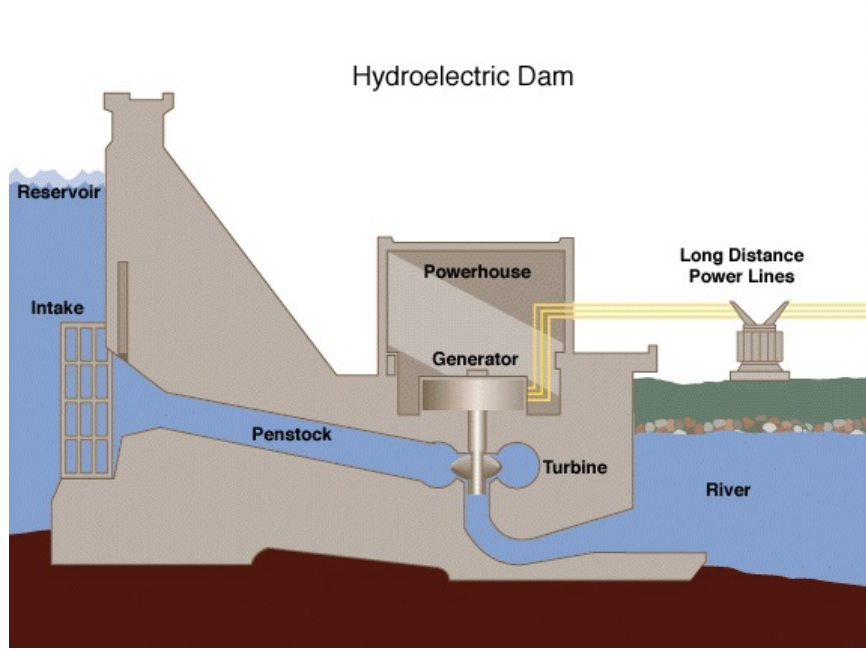


Figure 52: Hydroelectric Power Dam Water Level Control

Here $x(t) \in \mathbb{R}$ is the state, $w(t) \in \mathbb{R}$ is the driving agency (could be the control input or any known disturbance or perturbation), all the coefficients $\alpha, \xi, \beta \in \mathbb{R}$ are scalars (constants). $y(t) \in \mathbb{R}$ is the output of the system. We propose the following structure for our infinite dimensional observer to compute the estimated state $\hat{x}(t)$.

$$\dot{\hat{x}}(t) = \alpha \hat{x}(t) + \xi w(t) + l(y(t) - \hat{y}(t)) \quad (299)$$

where the last term accounts for the correction in which l is the observer gain and $\hat{y}(t)$ is the estimated output given as follows.

$$\hat{y}(t) = \beta \sup_{t-\tau \leq \theta \leq t} \hat{x}(\theta). \quad (300)$$

With the above substitution, (299) takes the following form.

$$\dot{\hat{x}}(t) = \alpha \hat{x}(t) + \xi w(t) + l \left(\beta \sup_{t-\tau \leq \theta \leq t} x(\theta) - \beta \sup_{t-\tau \leq \theta \leq t} \hat{x}(\theta) \right). \quad (301)$$

Subtracting the the observer equation (301) from the plant equation (298) and denoting the observation error $e(t) = x(t) - \hat{x}(t)$, we obtain the error dynamics of the

observer as follows.

$$\dot{e}(t) = \alpha e(t) - l\beta \left(\sup_{t-\tau \leq \theta \leq t} x(\theta) - \sup_{t-\tau \leq \theta \leq t} \hat{x}(\theta) \right). \quad (302)$$

Using the difference property of the sup functional and assuming that $l\beta < 0$, the above error equation translates to the inequality as under.

$$\dot{e}(t) \leq \alpha e(t) - l\beta \sup_{t-\tau \leq \theta \leq t} (x(\theta) - \hat{x}(\theta)) \quad (303)$$

or equivalently,

$$\dot{e}(t) \leq \alpha e(t) - l\beta \sup_{t-\tau \leq \theta \leq t} e(\theta). \quad (304)$$

Using Halanay's inequality, we give the following theorem.

Theorem 26 *If $-\alpha > -l\beta > 0$ then there exists $\delta > 0$ and $\kappa > 0$ such that*

$$e(t) \leq \kappa e^{-\delta t} \quad (305)$$

and the decay rate satisfies the following nonlinear transcendental equation.

$$\delta + \alpha - l\beta e^{-\delta\tau} = 0. \quad (306)$$

Fig. (51) portrays the block diagram of the proposed observer architecture. Notice that the observer is nonlinear because of the presence of the sup block.

Comment 10:

The above theorem requires that $\alpha < 0$ i.e., the homogeneous version of the plant (unforced system) should be asymptotically stable. Now, given any $\alpha < 0$ and β as the system parameters, the observer can be designed as follows.

$$\begin{cases} |l| < \left| \frac{\alpha}{\beta} \right|, \\ \text{sgn}(l) = -\text{sgn}(\beta) \end{cases} \quad (307)$$

where $\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ represents the usual signum or sign function defined as follows.

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (308)$$

Clearly,

$$\begin{aligned}\operatorname{sgn}(x) &:= \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \\ &= 2H(x) - 1\end{aligned}\tag{309}$$

where H is the usual Heaviside function or unit step function. Also, the sign function is an odd function i.e., $\operatorname{sgn}(-x) = -\operatorname{sgn}(x)$.

Also, from the quasi-polynomial equation (306), we can get the following *explicit expression* for the worst case decay rate δ of the estimation error e as,

$$\begin{aligned}\delta + \alpha - l\beta e^{-(\delta+\alpha)\tau} \cdot e^{\alpha\tau} &= 0 \\ \Leftrightarrow \tau(\delta + \alpha) &= l\beta\tau e^{\alpha\tau} e^{-(\delta+\alpha)\tau} \\ \Leftrightarrow \tau(\delta + \alpha) &= W_k(l\beta\tau e^{\alpha\tau}); k \in \mathbb{Z} \\ \Leftrightarrow \delta &= -\alpha + \frac{1}{\tau} W_k(l\beta\tau e^{\alpha\tau}); k \in \mathbb{Z}.\end{aligned}$$

Indeed for a unique and real value of δ , we must take into account only the principal branch W_0 i.e.,

$$\delta = -\alpha + \frac{1}{\tau} W_0(l\beta\tau e^{\alpha\tau})\tag{310}$$

11.13 Simulation Results

We illustrate the applicability and effectiveness of our sup based output injection technique for the observer design by the following academic examples.

Example 1:

Consider the following example for the observer design problem.

$$\begin{cases} \dot{x}(t) = -0.5x(t) + \xi w(t) \\ \psi(t) = 6.5 \cos(t); t \in [-2, 0] \\ y(t) = 1.25 \sup_{t-2 \leq \theta \leq t} x(\theta) \end{cases}\tag{311}$$

For simplicity and “proof of concept”, we consider the unforced version of the plant. Clearly, $\alpha = -0.5$, $\beta = 1.25$, $\tau = 2$ and $\xi = 0$. The observer gain satisfying the bound was selected as $l = -0.005$. This yields the worst case estimation error decay rate $\delta = 0.4977$. Fig. 53 shows the actual state x , output y , initial history function $\psi \in \mathcal{C}([-2, 0]; \mathbb{R})$, the worst case estimation error e and the estimate \hat{x} .

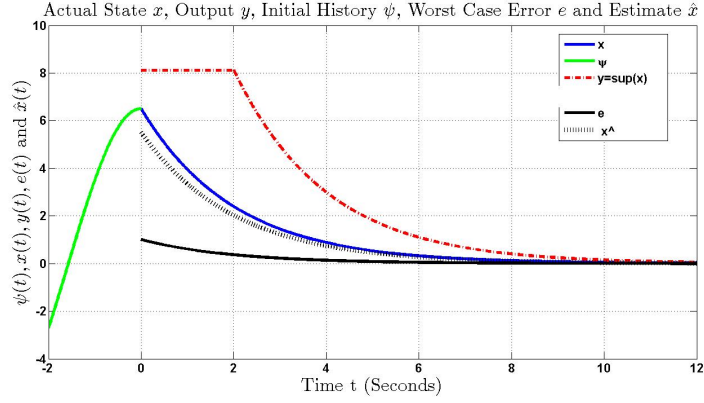


Figure 53: Actual State x , Output y , Initial History ψ , the Worst Case Error e and Estimate \hat{x}

Example 2:

Consider the following second order non-autonomous coupled system with sup based measurement.

$$\begin{cases} \dot{x}_1(t) = x_1(t) - x_2(t) \\ \dot{x}_2(t) = 5x_1(t) - 3x_2(t) + 0.5w(t) \\ y(t) = 1.75 \sup_{t-1 \leq \theta \leq t} x_1(\theta) \end{cases}$$

We take $\phi(t) = \begin{pmatrix} t+1 \\ 1 + 0.25 \cos(4.5t) \end{pmatrix} \in \mathcal{C}, \forall t \in [-1, 0]$ and

$\hat{\phi}(t) = \begin{pmatrix} 0.85 - 0.1 \sin(t) \\ t + 1.75 \end{pmatrix} \in \mathcal{C}, \forall t \in [-1, 0]$ as the initial infinite-dimensional history functions for the actual states and the observed states respectively. Clearly, our system and measurement matrices are as follows:

$$A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 1.75 \\ 0 \end{pmatrix}.$$

It is easy to check that A is Hurwitz and the pair (C^T, A) is observable. For simplicity, the matrix S was taken as the Identity matrix and $\rho > 1$ was picked as $\rho = 2$. Solving the Lyapunov equation yields $R = \begin{pmatrix} 1.5 & 2 \\ 2 & 3.5 \end{pmatrix}$ with the spectral norm, $\|R\| = 4.7361$. The unit Heaviside function was taken as the known perturbation/disturbance $w(t)$. Using *Theorem 25*, we designed the observer as $\mathbf{L} = \mathbf{L}^+ = \begin{pmatrix} 0.0125 \\ 0.0185 \end{pmatrix}$. This observer gain matrix \mathbf{L}^+ ensures that $\|\mathbf{L}^+\| = 0.023 < 0.03016$ and thus all the requirements of *Theorem 25* are totally satisfied.

Fig. 54 portrays the actual state trajectories, the observed state trajectories and the estimation errors of the states. We can clearly see that the error profiles are asymptotically stable. In other words, the estimated states are faithfully tracking the actual states as time progresses.

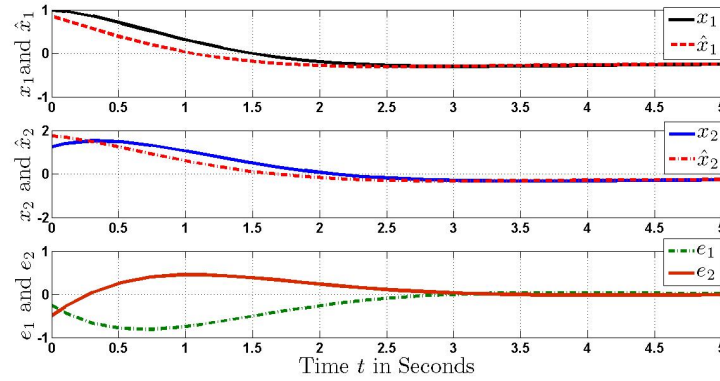


Figure 54: Actual, Observed State Trajectories and Estimation Error Profiles

Example 3:

Consider the following discrete time system with maxima.

$$\Sigma_d : \begin{cases} x_{k+1} = -0.48x_k + 0.85 \max_k(x_k, x_{k-1}); \forall k \in \mathbb{Z}, k \geq 0, \\ x_0 = 1.5, x_{-1} = 1.0 \end{cases} \quad (312)$$

It is easy to see that Mode: II in itself is unstable but the overall switched system is stable as expected from the stability region in Fig. 49. Fig. 55 shows the state trajectory of the discrete system with maxima. It verifies that the overall switched system is asymptotically stable.

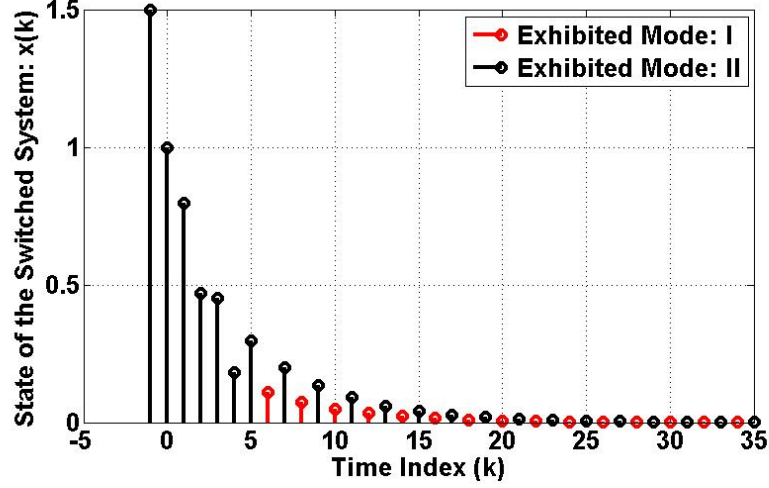


Figure 55: Discrete System with Maxima

11.14 Concluding Remarks

In this chapter we investigated and solved the problem of stability analysis, controller synthesis and observer design for scalar as well as general *higher order systems* with dynamics evolving with state suprema. We used Razumikhin function based platform for the analysis and synthesis. The state space for the characterization of trajectories and Cauchy problem is the infinite dimensional Banach space endowed with the *topology of uniform convergence*. The system was symmetrized and three different types of equilibria of the system were investigated. Conditions on global existence and uniqueness were also given using contraction mapping principle. Finally, some simulation results were shown to express the benefits of our analysis and observer design for the higher order systems evolving with state suprema.

CHAPTER XII

OPTIMAL CONTROL OF SYSTEMS EVOLVING WITH STATE SUPREMA

This chapter studies the optimal control of processes governed by a specific family of systems described by functional differential equations (FDEs) involving the sup-operator. Systems evolving with the state suprema constitute a useful abstraction for various models of technological and biological processes. The specific theoretic framework incorporates state suprema in the right hand side of the initially given differential equation and finally leads to a FDE with the state-dependent delays. We study a class of nonlinear FDE-featured optimal control problems (OCPs) in the presence of some additional control constraints. Our aim is to develop implementable first-order optimality conditions for the retarded OCPs under consideration. In [12], Azhmyakov used the celebrated Lagrange approach and prove a variant of the Pontryagin-like Minimum Principle for the given OCPs. Moreover, Azhmyakov discussed a computational approach to the main dynamic optimization problems and also considered a possible application of the developed methodology to the Maximum Power Point Tracking (MPPT) control of solar energy plants.

12.1 Problem Formulation and Motivation

Consider the following initial-value problem for the differential equation involving the sup-operator in the right hand side

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bk \sup_{t-\tau \leq \theta \leq t} c^T(t)x(\theta) \quad \forall t \in [0, t_f]. \\ x(t) &= \phi(t) \quad \forall t \in [-\tau, 0].\end{aligned}\tag{313}$$

Here $t_f \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$ is a state vector, $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given system matrix and system vector. By $\tau \in \mathbb{R}_+$ we denote a “memory length” associated with the given sup-functional. The control vector $u(t) := (k, c^T(t))^T \in U \subset \mathbb{R}^{n+1}$ constitutes an input in system (313). The admissible (vector) control function $c(\cdot)$ is assumed to be square integrable on the given time interval $c(\cdot) \in \mathbb{L}_n^2(0, t_f)$ and the gain k is bounded. We also assume that the admissible control region (a compact) U is determined by the following box-type restrictions

$$|k| \leq \Delta_1, \quad \|c(t)\|_{\mathbb{R}^n} \leq \Delta_2 \quad \forall t \in [0, t_f], \quad \Delta_1, \Delta_2 \in \mathbb{R}_+.$$

Let us denote by $\mathbb{C}(-\tau, 0)$ the Banach space of all continuous functions from the interval $[-\tau, 0]$ into \mathbb{R}^n equipped with the usual (uniform convergence) norm

$$\|\phi(\cdot)\|_{\mathbb{C}(-\tau, 0)} := \max_{s \in [-\tau, 0]} \|\phi(s)\|_{\mathbb{R}^n}.$$

Suppose $\phi(\cdot) \in \mathbb{C}(-\tau, 0)$. We call a function $x(\cdot)$ a solution (or trajectory) of (313), if there is $t_f > 0$ such that the function $x : [\tau, t_f) \rightarrow \mathbb{R}^n$ is absolutely continuous and satisfies conditions (313) for almost all $t \in [-\tau, t_f)$. For a given trajectory $x : I \rightarrow \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$ with $[t - \tau, t] \subset I$, a segment $x_t(\cdot)$ of $x(\cdot)$ can be defined by the relation

$$x_t(\vartheta) := x(t + \vartheta),$$

where $\vartheta \in [-\tau, 0]$. In fact, $x_t(\cdot)$ is a restriction of the function $x(\cdot)$ on the interval $[t - \tau, t]$. Using the presented notation, we can interpret (313) as an initial-value problem for a time-delayed differential equation with state-dependent delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bkc^T(t)x(t - r[x_t(\cdot)]) \quad \forall t \in [0, t_f] \\ x(t) &= \phi(t) \quad \forall t \in [-\tau, 0]. \end{aligned} \tag{314}$$

By $r : \mathbb{C}(-\tau, 0) \rightarrow [0, \tau]$ we denote here the “delay functional”. This functional expresses the action of the sup-operator in the original setting (313)

$$c^T(t)x(t - r[x_t(\cdot)]) \equiv \sup_{t-\tau \leq \theta \leq t} c^T(t)x(\theta).$$

Note that the linear-affine structure of the dynamic equation from (314) guarantees the existence of an absolutely continuous solution to the initial value problems (314) and (313) (see e.g., [17]). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function. By $x^u(\cdot)$ we denote a solution of (313) generated by an admissible control $u(\cdot)$ with $u(t) \in U$. The corresponding pair $(u(\cdot), x^u(\cdot))$, is called an admissible pair. We are now ready to formulate our main optimization problem, namely, the following OCP

$$\begin{aligned} & \text{minimize } J(u(\cdot)) \\ & \text{subject to } (313), |k| \leq \Delta_1, \|c(t)\|_{\mathbb{R}^n} \leq \Delta_2, t \in [0, t_f], \end{aligned} \tag{315}$$

where $J(u(\cdot)) := f(x^u(t_f))$ is a terminal (Mayer-type) functional. Let us note that using the simple coordinate transformations the given Mayer-type problem can be transformed into the general Bolza OCP (see e.g., [34, 50]).

Definition 19 *An admissible pair $(u^{opt}(\cdot), x^{opt}(\cdot))$, where $u^{opt}(\cdot) = (k^{opt}, c^{optT}(\cdot))^T$, is called a local (locally optimal) solution for OCP (315), if*

$$J(u^{opt}(\cdot)) \leq J(u(\cdot))$$

for all admissible control (vector) functions $u(\cdot)$ with $\|u^{opt}(\cdot) - u(\cdot)\|_{\mathbb{R} \times \mathbb{L}_n^2(0, t_f)} \leq \epsilon$, where $\epsilon > 0$ is a sufficiently small number.

Note that a neighborhood of an optimal control $u^{opt}(\cdot)$ from Definition 19 is in fact determined by the inequality

$$|k^{opt} - k| + \|c^{opt}(\cdot) - c(\cdot)\|_{\mathbb{L}_n^2(0, t_f)} \leq \epsilon.$$

where k and $c(\cdot)$ are admissible components of $u(\cdot)$. We next assume that OCP (315) has an optimal solution in the sense of Definition 19.

Note that the celebrated Pontryagin Minimum Principle as well some useful first-order optimality conditions are not sufficiently advanced to optimal control processes

governed by some newly established classes of differential equations (see e.g. [13–15] for the optimality conditions in hybrid and switched systems. The same is also true with respect to the OCPs associated with the general delayed differential equation involving state-dependent delays. In fact one has the adequate results only for the case of constant delays in a specific LQ type OCP (see e.g., [18, 61] and the references therein). Let us also mention some initial results that generalizes the conventional Bellman approach for the case of retarded OCPs. The usual equivalent transformations techniques that permit to reduce the differential equations with retarded arguments to the non-delayed FDEs may lead to a significant lack of smoothness (see [31, 43]). This is exactly the case of the state-dependent delays in the right hand sides of systems of the type (314). Additionally, the main analytic tool for the proof of the classic variants of the Pontryagin Minimum Principle, namely, the needle variations technique can not be applied in the case of delayed differential equations with state-dependent delays. Evidently, the delay functional $r[x_t(\cdot)]$ introduced above is determined by an a priori unknown relation. The formal situation here is in some sense similar to the general feedback optimal control problem. This fact and the infinite-dimensional nature of the FDE (314) make it difficult to derive the necessary adjoint equations and the related Pontryagin-like formalism.

Let us finally note that the abstract system (313) with a state suprema term in the right hand side can be naturally interpreted as an initially given linear control system

$$\dot{x} = Ax + bw$$

with an output $y(t) = c^T(t)x(t)$ whose measurement only consists of the suprema over some past time interval $\sup_{t-\tau \leq \theta \leq t} y(\theta, \cdot)$. This non-linear measurement information can be fed back in order to control the given system. Finally one gets the resulting

system (313) closed with the specific feedback control

$$w(t, x_t) := k \sup_{t-\tau \leq \theta \leq t} c^T(t)x(\theta).$$

Note that this feedback depends on the “memory” $x_t(\cdot)$ of the given dynamic system.

Let us now consider a practically oriented example of the obtained OCP (315) for the MPP-based solar plant system.

Example 5 *A simple mathematical model of the photovoltaic (PV) system is given by the following relations*

$$\begin{aligned} \frac{dP}{dV_{pv}}(t) &= I_{pv}(t) + V_{pv}(t) \frac{dI_{pv}}{dV_{pv}}(t), \\ \frac{d\phi(t)}{dt} &= A\phi(t) + bk \sup_{t-\tau \leq \theta \leq t} c^T(t)P(\theta), \\ \frac{d^2\phi(t)}{dt^2} &= F(t, \phi(t)). \end{aligned} \tag{316}$$

Here $\frac{dP}{dV_{pv}}$ is a power-voltage characteristic of a PV cell, V and I constitutes a voltage current characteristic of a solar plant. The presented model is a simple consequence of the conductance of PV cells (see [63]). Second and third equations in (316) represent a mechanical part of the solar system, namely, an orientation angle ϕ and the corresponding angular velocity $\frac{d\phi(t)}{dt}$ and acceleration $\frac{d^2\phi(t)}{dt^2}$ of the orientation system. The measured power is given by $c^T P$ such that the control gain k implements an instrumentation based non-linear feedback with “memory”. Following the MPP methodology, the above feedback-type control depends on the maximum of the (measured) power. An optimized (in the sense of the energy) solar plant can now be formalized by the following variant of the main OCP (315):

$$\begin{aligned} \text{minimize } J(u(\cdot)) &:= -P^2(t_f) + \phi^2(t_f) + \left(\frac{d\phi(t_f)}{dt}\right)^2 \\ \text{subject to } &(316), \quad |k| \leq \Delta_1, \quad \|c(t)\|_{\mathbb{R}^n} \leq \Delta_2, \quad t \in [0, t_f], \end{aligned} \tag{317}$$

for some given instrumentation-control parameters k and $c(\cdot)$ and corresponding bounds Δ_1, Δ_2 . Note that the objective functional in (317) corresponds to the output power maximization under simultaneous resources optimization.

Let us finally note that the conventional MPP algorithm discussed in Example 5 requires a heavy computation of the signature of the values $\frac{dP}{dV_{pv}}$ at every control time instant. The mathematical model of the OCP associated with a concrete dynamic system evolving with state suprema makes it possible to control optimally a sophisticated engineering process (for example, a solar plant) using the main OCP (315) and a suitable numerical solution procedure.

In [12], Azhmyakov proposes a new numerical approach for the optimal control processes governed by dynamic systems evolving with state suprema. The developed computational scheme is based on the extended abstract variant of the conventional Lagrange approach.

CHAPTER XIII

SPECTRUM ANALYSIS OF HIGHER ORDER LTI TDS

We investigate some pathological and degenerate cases in the spectrum analysis of higher order time delay systems using the idea of matrix Lambert W function. For the scalar case i.e., first order time delay systems, the Lambert-W function framework can be efficiently used. In (Yi and Ulsoy, 2006 and Yi et al. 2010, 2014), an algorithm (hereafter called Yi & Ulsoy's Algorithm) is presented to extend and generalize the idea to higher order case using matrix Lambert W functions. The aim of this chapter is to show that the formulation carried out using matrix Lambert W functions, suffers from some limitations. We provide some counter examples to show that one needs to be very careful in drawing conclusions about the spectrum of higher order system using this approach. In particular, Yi and Ulsoy's algorithm does not produce satisfactory results when the modes have multiplicity (repeated roots). In some cases, the algorithm produces spurious characteristic roots (poles) which are not the actual modes of the system under consideration.

13.1 Introduction & Motivation

We consider the class of linear constant coefficient time delay systems with fixed delay of the following form.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_d\mathbf{x}(t - \tau) \quad (318)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector. The matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{A}_d \in \mathbb{R}^{n \times n}$ are constant matrices and $\tau > 0$ is the constant delay.

The characteristic equation associated with system (318) is as follows.

$$\det(s\mathbf{I} - \mathbf{A} - \mathbf{A}_d e^{-s\tau}) = 0; s \in \mathbb{C} \quad (319)$$

In [53], a general matrix version of the Lambert W function was defined in terms of the Jordan blocks. Following this formulation, an explicit expression was found for the eigenvalues and spectrum of the above the system in the special case when the system matrices \mathbf{A} and \mathbf{A}_d are *simultaneously triangularizable* which included the special case of *commutativity*. It was shown that the matrix Lambert W function based approach fails if the system matrices \mathbf{A} and \mathbf{A}_d do not commute.

In [106], the same idea of matrix Lambert W functions has been extended to the solution of a system of Delay Differential Equations (DDEs) when the coefficient matrices in the system do not commute. In [104], point-wise controllability and observability of linear systems of DDEs is ascertained in the framework of matrix Lambert W functions.

In [105], an algorithm is presented to extend and generalize the idea of finding the characteristic roots and spectrum analysis to higher order case using matrix Lambert W functions. The authors use the matrix Lambert W function based approach to find the characteristic roots and the spectrum; and analyze the stability, controllability and observability of linear time invariant systems with a constant delay. The methodology is general and is not restricted to a commuting pair of \mathbf{A} and \mathbf{A}_d matrices. In [25], some special cases of this class of systems were considered and it was shown that in general there is no one to one correspondence between the branches of the matrix Lambert W function and the roots of the characteristic equation.

In [96,97], the authors present a method for computing all the zeros of a retarded quasi-polynomial that are located in a large region of the complex plane. The method is based on mapping the quasi-polynomial and on utilizing asymptotic properties of the chains of zeros. The method is called Quasi-Polynomial mapping based Root finder (QPmR). We use QPmR as a double check for the validation and verification of our results.

The aim and objective of this chapter is to show that the formulation carried

out in the book and the papers [105], [107], [106], [104] and [11], which is about the analysis of systems of linear delay differential equations using matrix Lambert W functions, suffers from some serious limitations. We provide some counter examples to show that one needs to be very careful in drawing conclusions about the spectrum of higher order system using this approach. In particular, Yi and Ulsoy's algorithm does not produce satisfactory results when the modes have multiplicity (repeated roots). In some cases, the algorithm produces spurious roots which are not the actual modes of the system under consideration and hence do not constitute the actual spectrum of the system. The claim is supported by counter examples which contradict the results obtained using the algorithm presented in [105], [104] and [11].

13.2 *Spectrum Analysis*

13.2.1 Scalar Case

Consider the following scalar DDE,

$$\Sigma_f : \begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t - \tau) & \forall t \geq 0 \\ x(t) = \psi(t). & \forall t \in [-\tau, 0] \end{cases} \quad (320)$$

where $x(t) \in \mathbb{R}$ is the state variable, $\psi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R})$ is the continuous initial infinite dimensional history function living in the Banach function space. We use the method of characteristics to ascertain the spectrum and closed form general solution of the homogeneous time delay system (320). Suppose that $x(t) = Ne^{\lambda t}$, $N \neq 0$ is mode of the system where $N, \lambda \in \mathbb{C}$ are constants. Then, substitution in the DDE

follows that,

$$\begin{aligned}
\mathbf{D}(Ne^{\lambda t}) &= \alpha Ne^{\lambda t} + \beta Ne^{\lambda(t-\tau)} \\
\Leftrightarrow \lambda(Ne^{\lambda t}) &= (\alpha + \beta e^{-\lambda\tau})(Ne^{\lambda t}) \\
\Leftrightarrow \lambda - \alpha &= \beta e^{-\tau\lambda} \\
\Leftrightarrow e^{\tau(\lambda-\alpha)}(\lambda - \alpha) &= \beta e^{-\tau\alpha} \\
\Leftrightarrow e^{\tau(\lambda-\alpha)}\tau(\lambda - \alpha) &= \tau\beta e^{-\tau\alpha} \\
\Leftrightarrow \tau(\lambda_k - \alpha) &= \mathbf{W}_k(\tau\beta e^{-\tau\alpha}) \\
\Leftrightarrow \lambda_k &= \alpha + \frac{1}{\tau}\mathbf{W}_k(\tau\beta e^{-\tau\alpha})
\end{aligned}$$

where in the above equations, $k \in \mathbb{Z}$ is an integer index, \mathbf{W}_k represents the k -th branch of the Lambert W function, \mathbf{D} is the usual differential operator and λ_k denotes the k -th eigenvalue associated with the k -th eigen mode $e^{\lambda_k t}$ of the system. The Lambert W function is defined as the multi-valued function which solves the following transcendental equation:

$$W(z)e^{W(z)} = z, \quad z \in \mathbb{C}, \quad (321)$$

Therefore, the spectrum of the system is only a point spectrum or discrete spectrum and is given by the following theorem.

Theorem 27 *The spectrum of the system (320) is*

$$Spec(\Sigma_f) = \left\{ \lambda_k \middle| k \in \mathbb{Z}, \lambda_k = \alpha + \frac{1}{\tau}\mathbf{W}_k(\tau\beta e^{-\tau\alpha}) \right\} \quad (322)$$

where \mathbf{W}_k denotes the k -th branch of the Lambert W function.

No continuous spectrum or residual spectrum is present for this infinite dimensional system Σ_f with discrete or fixed or constant delay.

The closed form general solution of the homogeneous system Σ_f given by (320) will be given as a linear combination of the countably infinite eigen modes (assuming

simple poles i.e., of multiplicity one) as follows.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{\lambda_k t}, \quad \forall k \in \mathbb{Z} \quad (323)$$

$$= \sum_{k=-\infty}^{\infty} C_k e^{(\alpha + \frac{1}{\tau} \mathbf{W}_k(\tau \beta e^{-\tau \alpha}))t} \quad (324)$$

where C_k 's, $k \in \mathbb{Z}$ are arbitrary constants which can be determined by the initial history function $\psi(t)$ using the Stone-Weirstrass theorem. Notice from (323) that the necessary and sufficient condition for the asymptotic stability of (320) is,

$$\Re \left(\alpha + \frac{1}{\tau} \mathbf{W}_k(\tau \beta e^{-\tau \alpha}) \right) < 0, \quad \forall k \in \mathbb{Z} \quad (325)$$

where $\Re(\cdot)$ represents the Real part of the complex number.

13.2.2 Higher Order Case

Now, we consider the higher order case as follows.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_d\mathbf{x}(t - \tau) \quad (326)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector. The matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{A}_d \in \mathbb{R}^{n \times n}$ are constant matrices and $\tau > 0$ is the constant delay.

The method of spectrum analysis of the higher order system (326), suggested in [105], is based on finding a matrix $\mathbf{S} \in \mathbb{C}^{n \times n}$ which satisfies the following transcendental matrix equation.

$$\mathbf{S} - \mathbf{A} - \mathbf{A}_d \exp(-\mathbf{S}\tau) = \mathbf{0}. \quad (327)$$

In order to generalize the theory and circumvent the highly restricted commutativity requirement of the matrices \mathbf{A} and \mathbf{A}_d , a matrix \mathbf{Q} is introduced in [105] which satisfies the following equation.

$$\tau(\mathbf{S} - \mathbf{A}) \exp(\tau(\mathbf{S} - \mathbf{A})) = \tau \mathbf{A}_d \mathbf{Q} \quad (328)$$

The above equation is in matrix Lambert-W friendly format and its solution is given as follows.

$$\mathbf{S}_k = \frac{1}{\tau} \mathbf{W}_k(\mathbf{M}) + \mathbf{A} \quad (329)$$

where $k \in \mathbb{Z}$ and $\mathbf{M} \triangleq \tau \mathbf{A}_d \mathbf{Q}$.

The set of all the eigenvalues of the matrix $\mathbf{S}_k, k \in \mathbb{Z}$ constitutes the spectrum of the higher order time delay system. Substitution of (329) in (326) yields the following equation.

$$\mathbf{W}_k(\mathbf{M}) \exp(\mathbf{W}_k(\mathbf{M}) + \mathbf{A}\tau) - \tau \mathbf{A}_d = \mathbf{0} \quad (330)$$

13.2.3 Yi and Ulsoy's Algorithm

The above method can be expressed by the following Yi and Ulsoy's Algorithm [105].

Repeat for $k = 0, \pm 1, \pm 2, \dots$

Step 1: Solve the nonlinear transcendental equation.

$$\mathbf{W}_k(\mathbf{M}_k) \exp(\mathbf{W}_k(\mathbf{M}_k) + \mathbf{A}\tau) - \tau \mathbf{A}_d = \mathbf{0} \quad (331)$$

for $\mathbf{M}_k = \tau \mathbf{A}_d \mathbf{Q}_k$.

Step 2: Compute \mathbf{S}_k corresponding to \mathbf{M}_k as

$$\mathbf{S}_k = \frac{1}{\tau} \mathbf{W}_k(\mathbf{M}_k) + \mathbf{A}. \quad (332)$$

Step 3: Compute the eigenvalues of \mathbf{S}_k .

13.2.4 Main Problem With Yi and Ulsoy's Algorithm:

Notice that (327) in [105] was derived by assuming the solution $\mathbf{x}(t) = \exp(\mathbf{S}t)\mathbf{x}_0$ to the system (318) which, after substitution in (318), yields the following.

$$(\mathbf{S} - \mathbf{A} - \mathbf{A}_d \exp(-\mathbf{S}\tau))\mathbf{x}(t) = 0 \quad (333)$$

If $\mathbf{P} \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{P}\mathbf{x} = \mathbf{0}$ then the nontrivial solution for \mathbf{x} exists iff $\ker(\mathbf{P}) \neq \mathbf{0}$, equivalently iff \mathbf{P} is singular i.e., $\det(\mathbf{P}) = 0$. Here, $\ker(\mathbf{P})$ denotes the kernel or Null space of the matrix \mathbf{P} .

In the light of the above fact, equation (327) is an incorrect characteristic equation. The correct characteristic equation should be,

$$\det(\mathbf{S} - \mathbf{A} - \mathbf{A}_d \exp(-\mathbf{S}\tau)) = 0 \quad (334)$$

Unfortunately when equation (334) is employed in the analysis, it does not boil down to *matrix Lambert W* functions, in general. Our strategy of finding the spectrum, in the counter examples below, is based on the scalar characteristic equation (319). We use factorization of the left hand side of (319) and *scalar Lambert W* function based approach rather than the matrix one.

In [28], it was shown that the matrix Lambert W function evaluated at the matrix A does not represent all possible solutions of $\mathbf{S} \exp(\mathbf{S}) = \mathbf{A}$.

13.3 Counter Examples

We give the following counter examples as our main results. Our counter examples bring some pathological cases where Yi and Ulsoy's algorithm does not produce satisfactory results. We have meticulously designed these examples so that the characteristic equation associated with the higher order system can be easily factorized into scalar systems so that one use the tools of the scalar Lambert W function. Notice that there is no problem with the usage of scalar Lambert W functions. Some of the examples deal with the repeated root (multiple eigenvalue) cases where Yi and Ulsoy's algorithm fails to produce the multiple eigen modes.

Example 1: 4th Order System

Consider the following 4th order system of DDEs in which,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix}, \mathbf{A_d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the delay $\tau = 4$.

This system has the following characteristic equation.

$$(\lambda + 1)^4 - e^{-4\lambda} = 0 \quad (335)$$

The above characteristic equation can be easily split and factorized into four linear

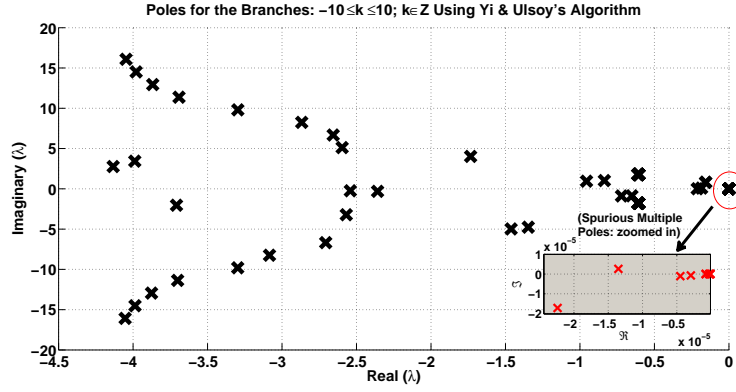


Figure 56: Poles Using Yi and Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 1

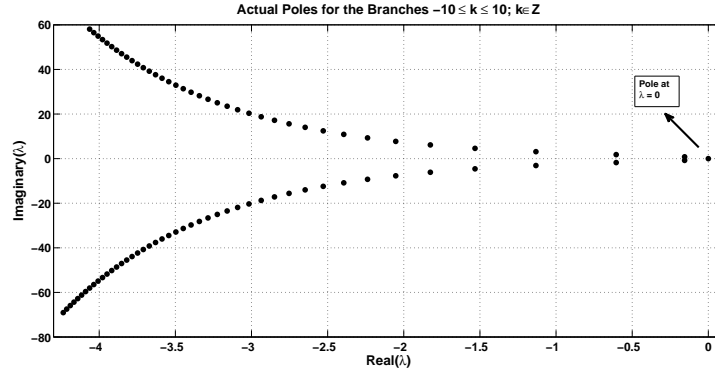


Figure 57: Actual Poles for Example: 1

pseudo-polynomials or quasi-polynomials (scalar Lambert-W friendly format factors) as follows.

$$(\lambda + 1 + e^{-\lambda})(\lambda + 1 - e^{-\lambda})(\lambda + 1 + je^{-\lambda})(\lambda + 1 - je^{-\lambda}) = 0 \quad (336)$$

Fig. 56 shows the eigenvalues (poles) of the system found using Yi and Ulsoy Algorithm for the branches $-10 \leq k \leq 10; k \in \mathbb{Z}$ using the hybrid branch based approach. The actual eigenvalues are shown in Fig. 57. Notice that there is indeed at eigenvalue at the origin as also evident from the characteristic equation as well. The original system is stable. One can easily see the difference between the actual poles and the poles captured by Yi and Ulsoy's Algorithm.

Example 2: Triple Root at the Origin

Consider the following 2nd order time delay system.

$$\ddot{y}(t) - 2\dot{y}(t) + 2y - 2y(t-1) = 0. \quad (337)$$

By defining our states as $x_1 = y$ and $x_2 = \dot{y}$, the above system can be expressed in the state space format of (326) by recognizing that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$, $\mathbf{A_d} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the delay is unity i.e. $\tau = 1$. The characteristic equation for this system is $\lambda^2 - 2\lambda + 2 - 2e^{-\lambda} = 0$. By using the Taylor series expansion of $e^{-\lambda}$, we observe the following scenario.

$$\begin{aligned} \lambda^2 - 2\lambda + 2 - 2e^{-\lambda} &= 0 \\ \Leftrightarrow 1 - \lambda + \frac{1}{2}\lambda^2 &= e^{-\lambda} \\ \Leftrightarrow 1 - \lambda + \frac{1}{2}\lambda^2 &= 1 - \lambda + \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3 + \dots \\ \Leftrightarrow \lambda^3\left(\frac{1}{3!} + \lambda\frac{1}{4!} + \dots\right) &= 0 \end{aligned}$$

This shows that the system has a repeated eigenvalue (characteristic root) of multiplicity 3 at the origin ($\lambda = 0$). Fig. 58 depicts the actual roots for this system when $-7 \leq \Re(\lambda) \leq 1$ using QPmR. Fig. 59 shows that Yi and Ulsoy's Algorithm not only fails to detect this repeated root but also produces a lot of extraneous poles (a whole cluster instead of a triple root) which are not part of the actual spectrum of the system under consideration. In particular, notice the spurious poles in the region $0 < \Re(\lambda) < 1$ of the complex plane.

Example 3: All Repeated Roots Except the One at the Origin

Consider the following 3rd order time delay system.

$$(D - T)(D - T)Dy = 0. \quad (338)$$

where D is the usual Differential Operator and T is the unit delay operator. By defining our states as $x_1 = y$, $x_2 = Dy = \dot{y}$ and $x_3 = (D - T)Dy = \ddot{y} - \dot{y}(t-1)$,

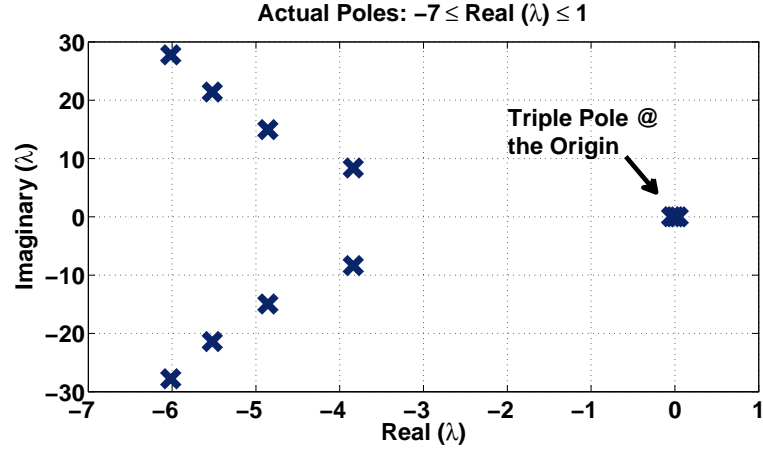


Figure 58: Actual Poles: $-7 \leq \Re(\lambda) \leq 1$ for Example: 2

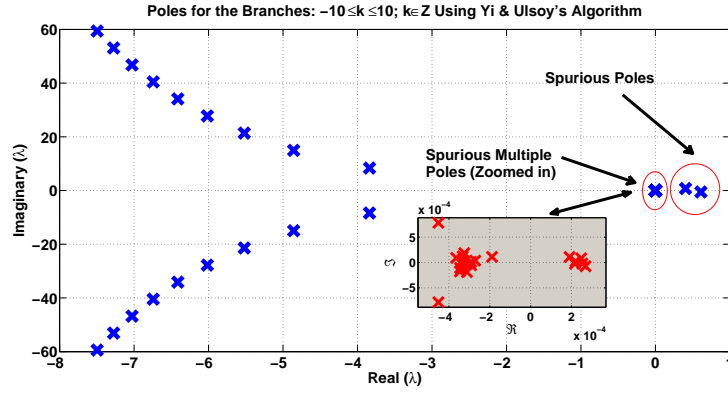


Figure 59: Poles Using Yi and Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 2

the above system can be expressed in the vector-matrix form of (326) with $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the delay is unity i.e., $\tau = 1$. Clearly, the matrices \mathbf{A} and \mathbf{A}_d do not commute. The characteristic equation for this system is $\lambda(\lambda - e^{-\lambda})^2 = 0$. This system has a simple eigenvalue $\lambda = 0$ and repeated eigenvalues $\lambda_k = W_k(1); \forall k \in \mathbb{Z}$ of multiplicity 2. Fig. 60 shows the actual poles of this system for the branches $-10 \leq k \leq 10; k \in \mathbb{Z}$ resembling a swallow-tail pattern. Fig. 61 shows the poles generated by Yi and Ulsoy's Algorithm. The initial value of the matrix \mathbf{Q}

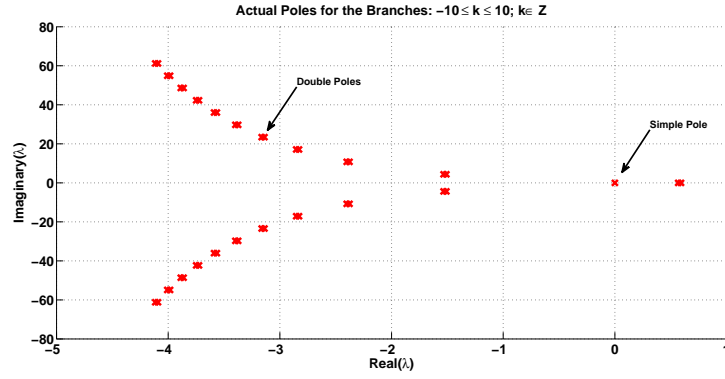


Figure 60: Actual Poles: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 3

was chosen such that all the entries are unity.

It is clear from this figure that the algorithm not only fails to produce repeated poles but also generates some spurious poles which are not the characteristic roots of the original system. One can observe the extraneous roots generated by the algorithm in the region $-1 < \Re(\lambda) < 0$ of the complex plane.

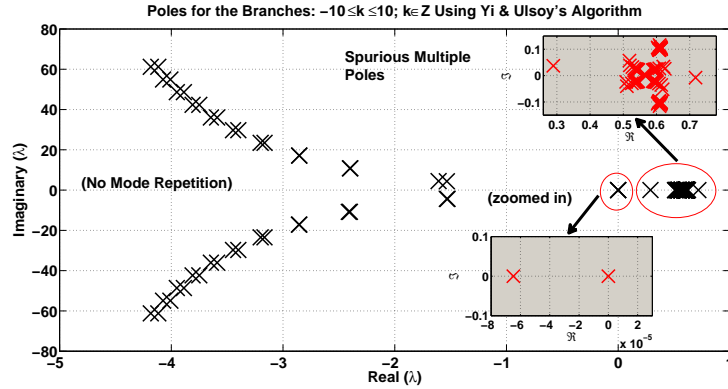


Figure 61: Poles for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ using Yi & Ulsoy's Algorithm for Example: 3

Example 4: Antipodal System With a Delay

Consider the following system.

$$\Sigma_a : \ddot{y} - y(t-1) = 0. \quad (339)$$

By defining our states as $x_1 = y$ and $x_2 = \dot{y}$, the above system can be expressed in

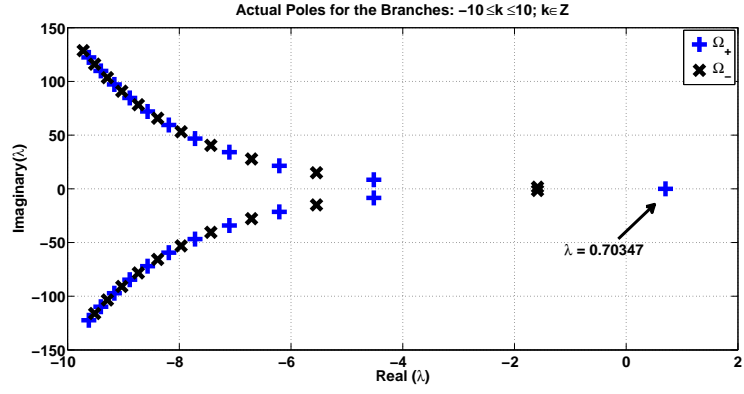


Figure 62: Actual Poles: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 4

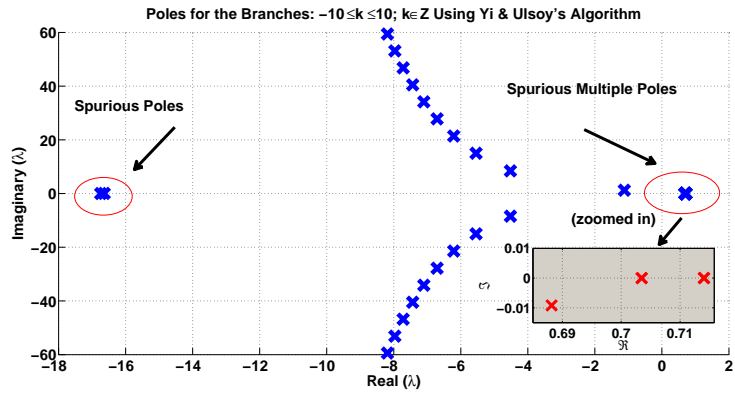


Figure 63: Poles Generated by Yi and Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 4

the state space format of (326) by recognizing that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and the delay is unity i.e., $\tau = 1$. Clearly, here \mathbf{A} and \mathbf{A}_d do not commute. The characteristic equation of (345) is $\lambda^2 - e^{-\lambda} = 0$. From which, $\lambda = \pm e^{-\frac{\lambda}{2}}$. Therefore, the spectrum associated with this system is,

$Spec(\Sigma_a) = \Omega_+ \cup \Omega_-$ where,

$$\Omega_{\pm} = \left\{ \lambda_k \mid \lambda_k = 2W_k \left(\pm \frac{1}{2} \right); k \in \mathbb{Z} \right\}. \quad (340)$$

Fig. 62 and Fig. 63 can be compared and contrasted for actual poles and the one generated by Yi and Ulsoy's algorithm respectively. Notice the false poles generated

by the later as shown enclosed in the red ellipse in the region $-18 \leq \Re(\lambda) \leq -16$ and the spurious multiple poles in the region $0 \leq \Re(\lambda) \leq 2$.

Example 5: Double Root at the Origin

Consider the following 2nd order system with the delay $\tau = 2$ units.

$$\Sigma_d : \ddot{y}(t) - 2\dot{y}(t) + y(t) - y(t-2) = 0. \quad (341)$$

By defining our states as $x_1 = y$ and $x_2 = \dot{y}$, the above system can be expressed in the state space format of (326) by recognizing that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and the delay is 2 units i.e., $\tau = 2$. Clearly, \mathbf{A} and \mathbf{A}_d do not commute. The characteristic equation of the system (341) is the following.

$$\begin{aligned} \lambda^2 - 2\lambda + 1 - e^{-2\lambda} &= 0 \Leftrightarrow (\lambda - 1)^2 = e^{-2\lambda} \\ \Leftrightarrow (\lambda - 1 + e^{-\lambda})(\lambda - 1 - e^{-\lambda}) &= 0 \\ \Leftrightarrow \lambda^2 \left(\frac{1}{2!} - \lambda \frac{1}{3!} + \dots \right) (\lambda - 1 - e^{-\lambda}) &= 0 \end{aligned}$$

This clearly shows that the transcendental characteristic equation has a *double root at the origin*. Furthermore, the spectrum associated with this system is,

$$Spec(\Sigma_d) = \Omega_+ \bigcup \Omega_- \quad (342)$$

where, $\Omega_{\pm} = \left\{ \lambda_k \mid \lambda_k = 1 + W_k(\pm e^{-1}); k \in \mathbb{Z} \right\}$.

Fig. 64 shows the actual poles of this system for the branches $-10 \leq k \leq 10; k \in \mathbb{Z}$. Notice that there is only one right half plane pole at $\lambda = 1.27846$. There are no poles in the region of the complex s -plane where $0 \leq \Re(\lambda) \leq 1$. The poles produced using Yi and Ulsoy's algorithm are depicted in Fig. 65. One can see that there is a clear difference between the actual spectrum and the one captured in Fig. 65.

Example 6: Two Realizations

Consider the following 2nd order time delay system.

$$(D + T)(D - T)y = 0. \quad (343)$$

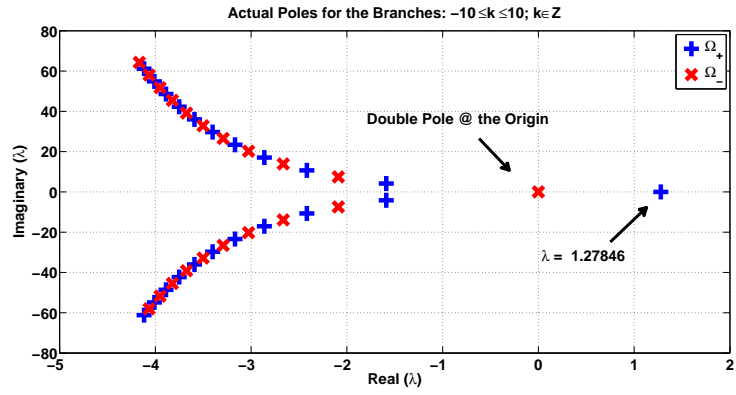


Figure 64: Actual Poles: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 5

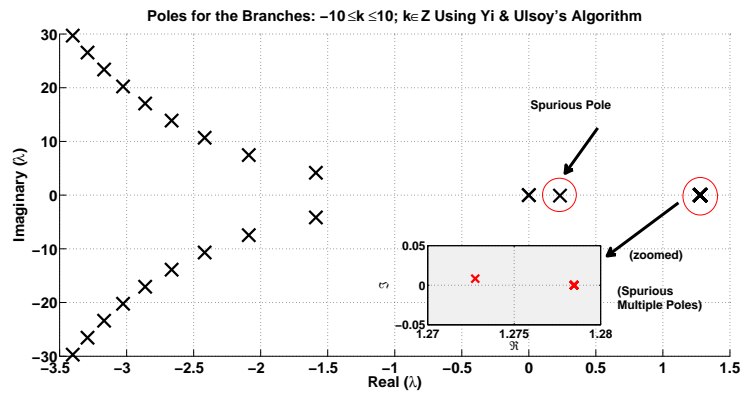


Figure 65: Poles Generated by Yi and Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 5

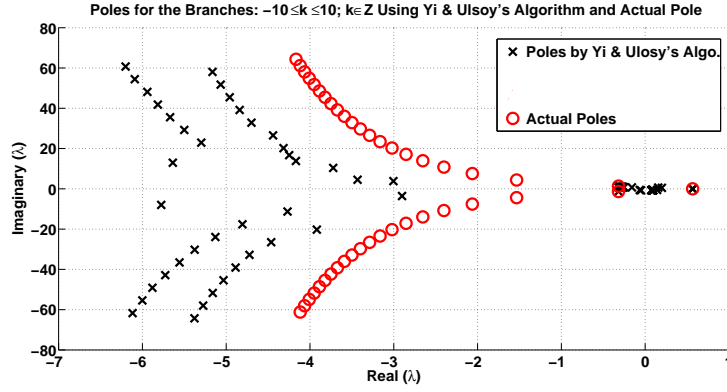


Figure 66: Actual Poles and Poles Generated by Yi & Ulosoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 6 With First State Space Realization

where D is the usual Differential Operator and T is the unit delay operator. By defining our states as $x_1 = y$ and $x_2 = (D - T)x_1$, the above system can be expressed in the vector-matrix form of (326) with $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the delay is unity i.e., $\tau = 1$. Clearly, the matrices \mathbf{A} and \mathbf{A}_d do not commute. The characteristic equation for this system is $(\lambda + e^{-\lambda})(\lambda - e^{-\lambda}) = 0$. This system has the spectrum $Spec = \{\lambda_k = W_k(\pm 1), k \in \mathbb{Z}\}$. One can also write the system in (343) as follows.

$$(D - T)(D + T)y = 0. \quad (344)$$

By defining our new states as $x_1 = y$ and $x_2 = (D + T)y$, the above system can be expressed in the vector-matrix form of (326) with a new state space realization where, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the delay is unity i.e., $\tau = 1$. Clearly, the matrices \mathbf{A} and \mathbf{A}_d do not commute.

Example 7: Delayed Resonator

Consider the following system which is in fact the "delayed" version of the well known harmonic oscillator. The system is characterized by,

$$\Sigma_o : \ddot{y} + y(t - 1) = 0. \quad (345)$$

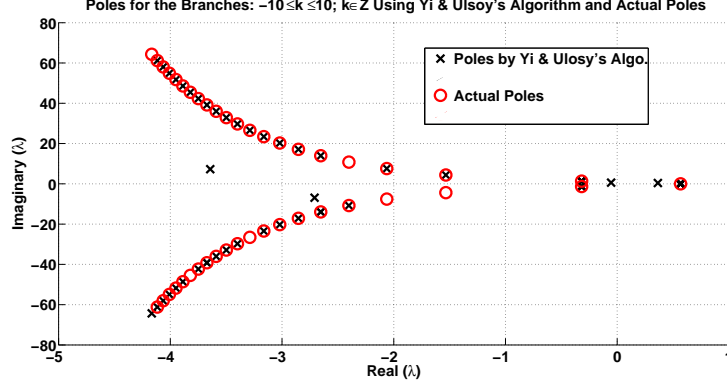


Figure 67: Actual Poles and Poles Generated by Yi & Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 6b With Second State Space Realization

By defining our states as $x_1 = y$ and $x_2 = \dot{y}$, the above system can be expressed in the state space format of (326) by recognizing that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{A}_d = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ and the delay is unity i.e., $\tau = 1$. Clearly, $\mathbf{A}\mathbf{A}_d \neq \mathbf{A}_d\mathbf{A}$. The characteristic equation of (345) is $\lambda^2 + e^{-\lambda} = 0$. From which, $\lambda = \pm j e^{-\frac{\lambda}{2}}$. Therefore, the spectrum associated with this system is,

$$Spec(\Sigma_o) = \Omega_+ \cup \Omega_- \quad (346)$$

where,

$$\Omega_+ = \left\{ 2W_k \left(\frac{j}{2} \right) \right\}; k \in \mathbb{Z} \quad (347)$$

$$\Omega_- = \left\{ 2W_k \left(-\frac{j}{2} \right) \right\}; k \in \mathbb{Z}. \quad (348)$$

Fig. 68 shows the actual poles of this system for the branches $-10 \leq k \leq 10; k \in \mathbb{Z}$. Notice that there is a complex conjugate pair of dominant poles in the region $-4 \leq \Re(\lambda) \leq -3$ which is not captured by the Yi and Ulsoy's algorithm in Fig. 69. One can see that there is a clear difference between the actual spectrum and the one captured in Fig. 69. Moreover, here, Yi and Ulsoy's algorithm fails to capture the two distinct characteristic spectra Ω_+ and Ω_- .

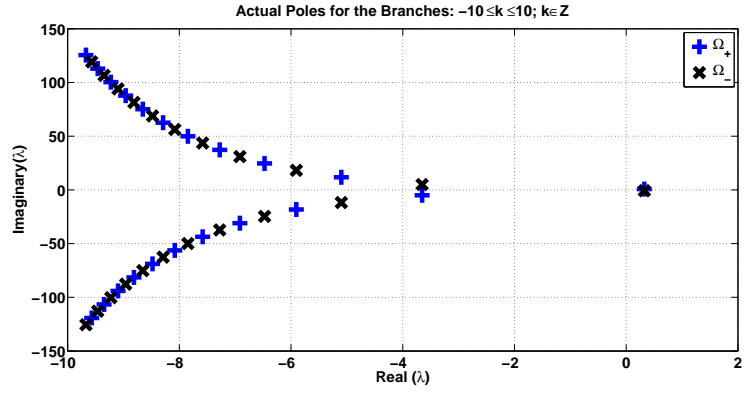


Figure 68: Actual Poles for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 7

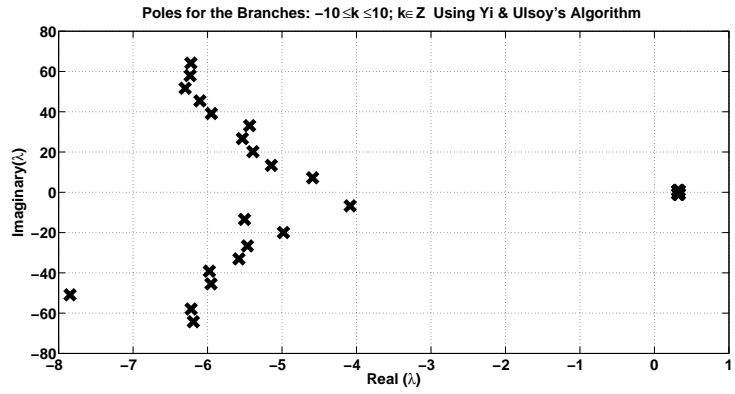


Figure 69: Poles Generated by Yi and Ulsoy's Algorithm for the Branches: $-10 \leq k \leq 10; k \in \mathbb{Z}$ for Example: 7

13.4 Concluding Remarks

We investigated some pathological and degenerate cases in the spectrum analysis of higher order time delay systems using the idea of matrix Lambert W function. The idea of matrix Lambert functions presented in [11], [106], [104], [107] and [105] has some limitations and one needs to be very careful in drawing conclusions about the spectrum of higher order system using this approach. In particular, Yi and Ulsoy's algorithm does not produce satisfactory results when the modes have multiplicity (repeated roots). In some cases, the algorithm produces unnecessary and redundant roots which are not the actual modes of the system under consideration; in other cases it fails to catch all the poles of the system. Also, the algorithm may give an incorrect judgement of the dominant poles. Our counter examples show that a lot of care must be taken while drawing conclusion about the spectrum and eigenvalues of higher order time delay systems using Yi and Ulsoy's Algorithm in [107] and [105]. All the examples can be reduced via factorization [89] to the scalar Lambert W friendly format. The authors also verified their results using the QPmR algorithm given in [97].

CHAPTER XIV

CONCLUSIONS & FUTURE RECOMMENDATIONS

14.1 *Conclusions*

Systems with constant time delays are very well understood. A well defined state space exists for such systems. The state space is either a Banach space of continuous functions $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ with the uniform norm topology or a Hilbert space of square integrable functions $\mathbb{L}^2([-\tau, 0]; \mathbb{R}^n)$. Inconsistencies arise when the delay is not constant or fixed. When the delay is time-varying, among other inconsistencies, causality is a prime problem which needs to be addressed. In order to have a well defined state space and ensure causality, the rate of the delay must be bounded by unity i.e., $\dot{\tau} \leq 1$. When the time-varying delay is bounded and its rate is bounded by unity, only then one can use $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ or $\mathbb{L}^2([-\tau, 0]; \mathbb{R}^n)$ as the state space and characterize the Cauchy problem. When the delay depends on the state of the system, the state space is not well defined in general. The space \mathcal{C} fails to define the Cauchy problem and characterize the state space. Extra regularity on the initial data is required e.g., in some cases $\mathcal{C}^1([-\tau, 0]; \mathbb{R}^n)$ or a closed subspace of $\mathcal{C}^{0,1}([-\tau, 0]; \mathbb{R}^n) = \mathcal{W}^{1,\infty}([-\tau, 0]; \mathbb{R}^n)$ (Sobolev space) can be used as a state space. In a particular case, when the state-dependent delay incurs a zero crossover, the system behaves as a finite-dimensional system called self-starting system. This is a special case of the scale dynamic systems or scale delay systems.

Some salient features of the problems analyzed and discussed in the dissertation are as follows.

- Stability analysis of a class of systems with explicit state-dependent delays
- Stability analysis of a class of systems with implicit state-dependent delays

- Observer design for a class of systems with explicit state-dependent delays
- Observer design for a class of systems with implicit state-dependent delays
- State space characterization of a class of systems with state-dependent delay
- Analysis of scale dynamic systems and the self-starting features
- Spectrum analysis of higher order linear time delay systems in the Lambert W function based framework
- Stability analysis and controller synthesis for systems evolving with state suprema
- Observer design for systems evolving with state suprema
- Characterization of discrete systems evolving with state suprema as M^3D systems
- Applications to position estimation of rocket car and machine tool with state-dependent delay
- Application to tank temperature observation with state-dependent delay
- Application to position estimation of a submarine in a 3-D environment and ocean navigation
- Application of state-dependent delays in gene regulatory networks

14.2 Future Recommendations

There are still a lot of challenging areas and open problems in the context of systems with state-dependent delays and state suprema. Some of the avenues for future research are listed as follows.

- In all the state-dependent delay systems, we assumed a single delay. One may also extend the developed theory to multiple delays.

- Likewise, one may consider the problem of system evolving with state suprema with multiple memory elements.
- In the dissertation we considered only RFDEs with state-dependent delays. The study of neutral systems with state dependent delays i.e., NFDEs is a future research avenue.
- The case when the memory τ of the sup functional is time-varying or state dependent is highly complex. It requires more regularity and smoothness assumptions on the initial history function. This is under the umbrella of our ongoing research.
- Throughout the dissertation we considered continuous systems with state-dependent delays. It will be useful to study the discrete counterpart of such systems.
- The problem of state space characterization when the state-dependent delay is unbounded (infinite memory) is an open problem for research.
- Optimal control of systems involving state suprema by the M^3D characterization is also a future research direction.

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VITA

Aftab Ahmed (Khattak) was born in Nowshera, Khyber Pakhtunkhwa province of Pakistan. He acquired initial education from his uncle, Said Afzal (late), who taught him counting and basic mathematics. He got first position each in SSC and HSSC examinations of the Federal Board of Intermediate & Secondary Education (FBISE), Islamabad and topped his school and college. He then got admission in University of Engineering & Technology Peshawar from where he got BS Electrical Engineering with honors and with distinction. Throughout the four years, he has been the recipient of University Merit Scholarships and awards for the top position holders. He then did MS in Systems Engineering from PIEAS where he was awarded the Best Thesis Gold Medal. He is also the recipient of the Fauji Foundation Academic Distinction Award and Excellence Award. He then came to the United States after getting the Fulbright Scholarship in Fall 2012. After passing the preliminary examination in Fall 2012, he started his Ph.D. research with Prof. Verriest in Spring 2013. He completed the MS in Electrical Engineering in Summer 2014 from Georgia Tech. In Fall 2015, he completed the MS Mathematics degree from Georgia Tech. The work in this dissertation is a part of the Ph.D. degree. He was affiliated with MAST Lab. and Decision & Control Laboratory (DCL). He also organized the Annual Student Symposium of the DCL and served as general chair and program chair for two years. Apart from ECE and DCL seminars, he was also an active participant and attendee of the seminars in the schools of mathematics and physics at Georgia Tech. He also served as the elected senator from ECE in the Student Government Association (SGA) at Georgia Tech. In his free time, he enjoys doing calligraphy in Pashto, Urdu, Arabic and English. His research interests include Mathematical Systems Theory, Systems and Controls,

A. Related Publications

1. Ahmed, A., Farooq, S., Khan, A. Q., and Abid, M., “An LMI based approach to passivity analysis and robust passification of uncertain neutral systems with time varying delays,” in 51st IEEE Conference on Decision and Control (CDC), Maui, HI, pp. 7799-7804, 2012.
2. Ahmed, A. and Verriest, E. I., “Estimator design for a subsonic rocket car (soft landing) based on state-dependent delay measurement,” Proceedings of the 52-nd IEEE Conference on Decision and Control (CDC), Florence, Italy, pp. 5698-5703, Dec. 2013.
3. Ahmed, A. and Verriest, E. I., “Nonlinear systems evolving with state suprema as multi-mode multi-dimensional (M^3D) systems: Analysis observation,” IFAC-PapersOnLine, vol. 48, no. 27, pp. 242-247, 2015.
4. Ahmed, A. and Verriest, E. I., “The behavior of systems involving state-dependent delays,” Proc. 10-th IEEE Asian Control Conference (ASCC), Kota Kinabalu, Malaysia, pp. 2974-2979, 2015.
5. Ahmed, A. and Verriest, E. I., “Design of linear & nonlinear observers for a turning process with implicit state-dependent delay,” Proc. 53rd IEEE Annual Allerton Conference (UIUC), Monticello, IL, USA, pp. 546-551, 2015.
6. Ahmed, A. and Verriest, E. I., “Modeling & analysis of gene expression as a nonlinear feedback problem with state-dependent delay,” IFAC 2017 World Congress, Toulouse, France, pp. -, 2017 (Accepted).
7. Ahmed, A. and Verriest, E. I., “Solution of scale dynamic systems,” ENOC 2017, Budapest, Hungary, pp. -, 2017 (Accepted).
8. Ahmed, A. and Iqbal, N., “Synthesis of a robust dynamic memoryless state feedback controller for systems with time-varying delays using LMIs,” in Proceedings of the 26th IASTED International Conference on Modeling, Identification, and Control, pp. 158-163, ACTA Press, 2007.
9. Ahmed, A. and Verriest, E. I., “Observer based temperature control of a tank: From constant to nonlinear state-dependent delay,” IFAC-PapersOnLine, vol. 48, no. 12, pp. 197-202, 2015.
10. Azhmyakov, V., Ahmed, A., and Verriest, E. I., “On the optimal control of systems evolving with state suprema,” in 2016 IEEE 55th Conference on Decision and Control (CDC), pp. 3617-3623, 2016.

B. Presentations, Posters, Seminars and Symposia

1. Ahmed, A. and Verriest, E. I., “Weird behaviour associated with systems involving state-dependent delays,” 1st International Conference on Dynamics of Differential Equations, Atlanta, Georgia, 2013.
2. Ahmed, A. and Verriest, E. I., “Chaos quenching in the Lorenz system via memory controller using delayed feedback,” Georgia Tech Research & Innovation Conference (GTRIC), Atlanta, Georgia, 2014.
3. Ahmed, A. and Verriest, E. I., “Synthesis of observers for systems with implicit state-dependent delays,” Georgia Tech Annual Decision & Control Laboratory (DCL) Student Symposium, 2014.
4. Ahmed, A. and Verriest, E. I., “Infinite-dimensional systems evolving with state suprema: Analysis & observation,” Georgia Tech Annual Decision & Control Laboratory (DCL) Student Symposium, 2015.
5. Ahmed, A. and Verriest, E. I., “State observation in systems with explicit or implicit state-dependent delay,” Seventh International Conference on Dynamic Systems and Applications & Fifth International Conference on Neural, Parallel, and Scientific Computations, GA, USA, 2015.
6. Ahmed, A. and Verriest, E. I., “Analysis of Gene Regulatory Networks (GRNs) with state-dependent delay,” Career, Research, Innovation and Development Conference (CRIDC), Georgia Tech, 2016.
7. Ahmed, A. and Verriest, E. I., “Stability and Passivity Analysis of Systems with State-Dependent Delays,” 7th International Conference on Dynamic Systems and Applications, GA, USA, 2015.
8. Ahmed, A. and Verriest, E. I., “Inversion of the state-dependent delay: Observer design & applications,” Georgia Tech Annual Decision & Control Laboratory (DCL) Graduate Student Symposium, 2016.
9. Ahmed, A. and Verriest, E. I., “Observer based temperature control of a uid in a tank with explicit state-dependent delay,” 13th Annual Graduate Technical Symposium, Atlanta, Georgia, 2016.
10. Ahmed, A. and Verriest, E. I., “Scale Dynamic Systems: Self-Starting Character,” GTMAP Seminar, GA, 2017.