AN EXPERIMENTAL VERIFICATION
OF THE
LATERAL STABILITY OF I-BEAMS

A THESIS

Presented to

the Faculty of the Graduate Division

Georgia Institute of Technology

in Partial Fulfillment

of the Requirements for the Degree

Master of Science in Engineering Mechanics

By

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May 1956

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Date Approved by Chairman: June 6 1956

PREFACE AND ACKNOWLEDGEMENT

Among the many discussions which the writer had with Professor F. M. Hill, his thesis advisor, the problem of lateral stability became more and more intriguing. After preliminary research had revealed the fact that little or no experimental work had been carried out in verifying the theories of lateral stability, this topic was chosen as a thesis. In narrowing this vast field, it was decided to concentrate upon a cantilever beam with a single concentrated load at the free end. To the end of presenting some experimental work in the verification of the theories of lateral stability of I-beams, this work is submitted.

It is with all sincerity that heartfelt thanks and appreciation are extended to Professor Hill. Without his encouragement, his constant assistance, and his determined efforts, this thesis would not now be a reality.

It is also well to express thanks to Dr. D. M. Smith, who gave so much of his time and assistance with mathematical calculations.

Mr. Walter H. Witzel, a mechanical engineer assigned to the Department of Mechanics at West Point, N. Y., made the suggestion concerning logarithmic plotting used in this thesis. Grateful acknowledgement is made to Mr. Witzel at this time for his suggestion and assistance.

The financial backing of the United States Air Force Institute of Technology and of the Department of Mechanics at West Point, N. Y. as well as the extreme cooperation of the Department of Mechanics at West Point, N. Y. are duly acknowledged at this time.

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ABSTRACT

AN EXPERIMENTAL VERIFICATION
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By

Frank Winthrop Draper

After looking into the topic of lateral stability in I-beams, it became apparent that very little experimental work had been carried out in any attempt to verify the theoretical results of Dr. Stephen P. Timoshenko. This, then, became the subject for this thesis. It was decided to gather the material to compare the theoretical results of Dr. Timoshenko's equation for predicting the critical load at which an I-beam would no longer be laterally stable. In checking the many possibilities for supporting the I-beam and loading it, it was further decided to limit this experimental work to a cantilever I-beam with a single concentrated load at the free end.

Due to the fact that neither the space nor the equipment was available to make use of standard structural sizes, it was decided

that a model I-beam could be made to do the work of an I-beam of standard structural size. If this were to be the case, the model beam was of necessity to be as nearly exact throughout in dimension as it was possible to make it. Two of these model I-beams were manufactured by the DeLavall Machine Company of Poughkeepsie, N. Y. The model I-beam was constructed well within the specified tolerances of 0.005 inch.

After the model I-beam was constructed, careful measurements were made to determine all of the constants which applied to those used in Timoshenko's equation for P_{cr}. Graphs were drawn in which P_{cr} for thin rectangular beams and for I-beams were plotted against the length.

It then became necessary to design and make a device for clamping the model I-beam, loading it, and making the necessary measurements. The faculty advisor for this thesis gave great inspiration and assistance in the design and machining of this equipment.

The beam was loaded at its most extreme length, and each successive operation was made upon a shorter beam. The beam was moved into the clamp a distance of one inch after each loading in order that that portion of the beam in which the elastic limit of the material was at times exceeded would not enter into subsequent calculations.

By means of a lateral adjustment in the loading head, and a lateral positioning screw in the load restrainer, the beam was constantly adjusted to keep it untwisted and laterally stable. P_{cr_e} was assumed to have been reached when no further adjustment could hold the beam in a stable untwisted state.

The values of P_{cr_e}, so determined, were plotted against length on both standard graph paper and logarithmic graph paper. It was of interest to find that Timoshenko's equation for I-beams, Prandtl's equation for thin rectangular beams, and the experimentally determined values of P_{cr_e} all produced straight lines on logarithmic graph paper. It was of further interest to find that the curve described by Timoshenko's predicted values of P_{cr} and the curve of the experimental values of P_{cr_e} were exactly parallel on logarithmic paper.

Within the accuracy and completeness of this investigation, it was found that experimental values of P_{cr_e} ran consistently lower than, but within approximately 10 per cent of, Timoshenko's equation. It was also found that the experimental values of P_{cr_e} ran about 10 per cent above Prandtl's values for lengths around 40 inches, and about 25 per cent above Prandtl's values for lengths around 24 inches. This was due to the added resistance to bending, in the planes of the flanges, given to the beam by the flanges, and, as the length of the beam increased, the effect of this added resistance to bending was minimized.

ABBREVIATIONS AND DEFINITIONS OF SYMBOLS

Symbol	Definition
a^2	A constant for the term $\frac{Dh^2}{2C}$
B_1	A constant for the term EI_1
B_2	A constant for the term EI_2
C	A constant for the term JG
D	A constant for the Flexural Rigidity of one flange of the I-beam in the plane of the flange. This constant may usually be taken with sufficient accuracy to be equal to $\frac{1}{2} B_1$
E	Modulus of elasticity
G	Modulus of Rigidity
I_1	Moment of inertia of the cross section of the I-beam with respect to the vertical centroidal axis
I_2	Moment of inertia of the cross section of the I-beam with respect to the horizontal centroidal axis
J	Polar moment of inertia of the cross section of the I-beam with respect to the centroid
L	Length of the beam
m	A coefficient which varies with L and a
P_{cr}	Predicted value of critical load
P_{cr_e}	Experimentally determined values of critical load

CHAPTER I

INTRODUCTION

Prior to 1897, little interest had been given to the problem of lateral stability of beams. Attention, however, was quickly focused on this question when the bridge disaster occurred near Tarbes, France. On July 17, 1897, the supporting spans of this bridge failed, and the bridge was destroyed. Failure was due to lateral buckling of the spanning beams, and a report of this failure appeared in a French Engineering Publication.[1] After the appearance of this article, interest in the investigation of lateral stability in beams gradually developed.

Among the first men to inquire into the problem of lateral stability of beams were Dr. L. Prandtl,[2] and A. G. M. Mitchell.[3] These men directed their efforts towards the theoretical aspects of this investigation, and dealt with beams of narrow rectangular cross sections. The first man to extend these theories and to apply them to

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[1] La Revue Technique, vol. 18, Nov. 15, 1897

[2] L. Prandtl, "Kipperscheinungen," Nuremberg, 1899

[3] A. G. M. Mitchell, Philosophical Magazine, vol. 48, 1899

structural sections of "I" shape was Dr. Stephen Timoshenko.[4]

There has been a considerable amount of discussion and investigation of the theory of lateral stability of beams since 1900. However, until very recently, little experimental work has been carried out to verify these theories. In January, 1952, Dr. A. R. Flint, of London, England, presented some interesting experimental results based upon his tests with simply supported I-beams.[5] Dr. Flint used a very unique arrangement of wires to support his model I-beams, and his work substantially verified the theoretical analysis of simply supported I-beams which Dr. Timoshenko had presented in 1924.

The purpose of this thesis is to present the results of experimental work with structural sections of I shape. These results were based upon investigation of an I-beam, supported as a cantilever, and loaded with a single concentrated load at the free end. The cantilever was made as a model, and its dimensions were made to conform as nearly as possible to standard structural I-beams.

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[4]Stephen Timoshenko, "Beams Without Lateral Support," Transactions ASCE, vol. 87, 1924, p. 1247

[5]A. R. Flint, "The Lateral Stability of Unrestrained Beams," Engineering, vol. 173, No. 4487, London, Friday, Jan 25, 1952, p. 99

CHAPTER II

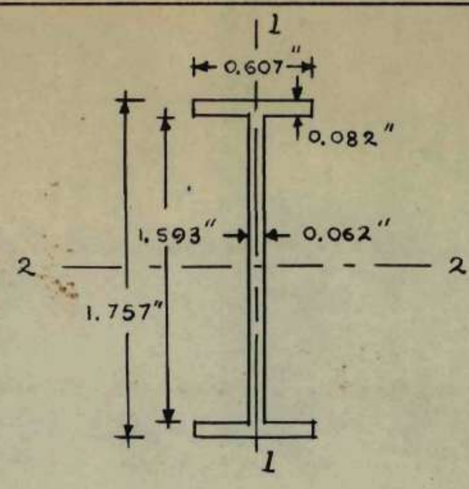
EQUIPMENT AND INSTRUMENTATION

The equipment used in this experiment consists of the model, the loading equipment, and the measuring devices. Each of these will here be presented and discussed in turn.

The Model Beam

Due to the fact that neither space nor the facilities to make use of heavy loads were available, it was decided that a model beam closely approximating proper ratios of dimensions of Standard Structural sizes would be used. To determine the suitable dimensions, a standard table of American Standard I-beams was closely scrutinized, and it was found that the overall depth of the model I-beam should be approximately 28 times the thickness of the web. The depth of beam used in this work was approximately 1.75 inches.

The beam itself was machined from a flat bar of hot rolled, mild steel with the dimensions $\frac{3}{4}$" x $2\frac{1}{4}$" x 10'. The flat bar was first cut into five pieces. Two pieces were cut into lengths of five inches each. These five pieces of steel were then raised to 1500° F,



VERTICAL AXIS - 1-1

HORIZONTAL AXIS - 2-2

$$I_1 = 3,088 \times 10^{-6} \text{ in.}^4$$

$$I_2 = 90,800 \times 10^{-6} \text{ in.}^4$$

$$C = 4,201 \text{ lb.-in.}^2$$

$$E = 30 \times 10^6 \text{ psi.}$$

$$B_1 = 92,950 \text{ lb.-in.}^2$$

$$B_2 = 2,733,100 \text{ lb.-in.}^2$$

$$a^2 = 17.07$$

Cross Section of Model I-Beam With Values of Constants

Fig. 1

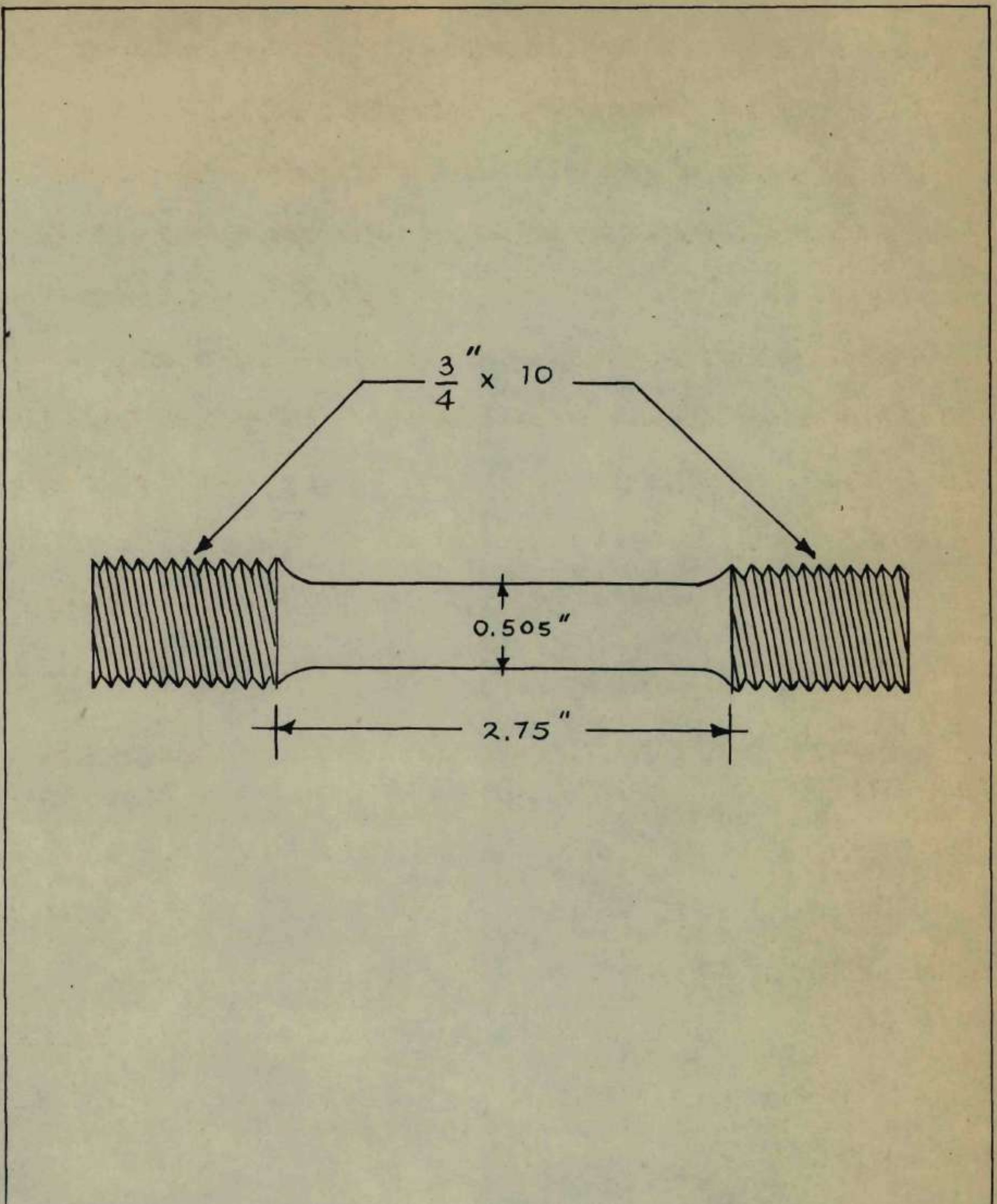
held there for one hour, removed, and allowed to cool in the air.

After cooling, the two four foot lengths of steel were machined to within 0.005 inch of the dimensions shown in Fig. 1, and the three five-inch pieces were machined to within 0.005 inch of the dimensions shown in Fig. 2.

After these initial cuts had been finished, the five pieces of machined steel were again raised to 1500° F and allowed to cool in the air. This process was employed in order to relieve the residual and machining stresses to as great an extent as possible. The I-beams and test specimens were cut from the same bar and subjected to the same heat treating process in order to determine the actual mechanical properties for the I-beams. The two four-foot beams, now having the shape of an I-beam section, were ground and polished to their final dimensions. The three test specimens were also ground and polished to their required dimensions.

After all the machine work had been completed, the dimensions of the three test specimens were found to be exactly as shown in Fig. 2. The I-beams, however, were found to vary from the specified dimensions of the beam by amounts indicated in Fig. 15 and Tab. 1 (in appendix). As may be seen, the average variation in the thickness of the flanges was 0.003 inch, and there was a maximum variation at one point of 0.007 inch.

The web of the I-beam was found to vary only 0.001 inch in its



Test Specimen

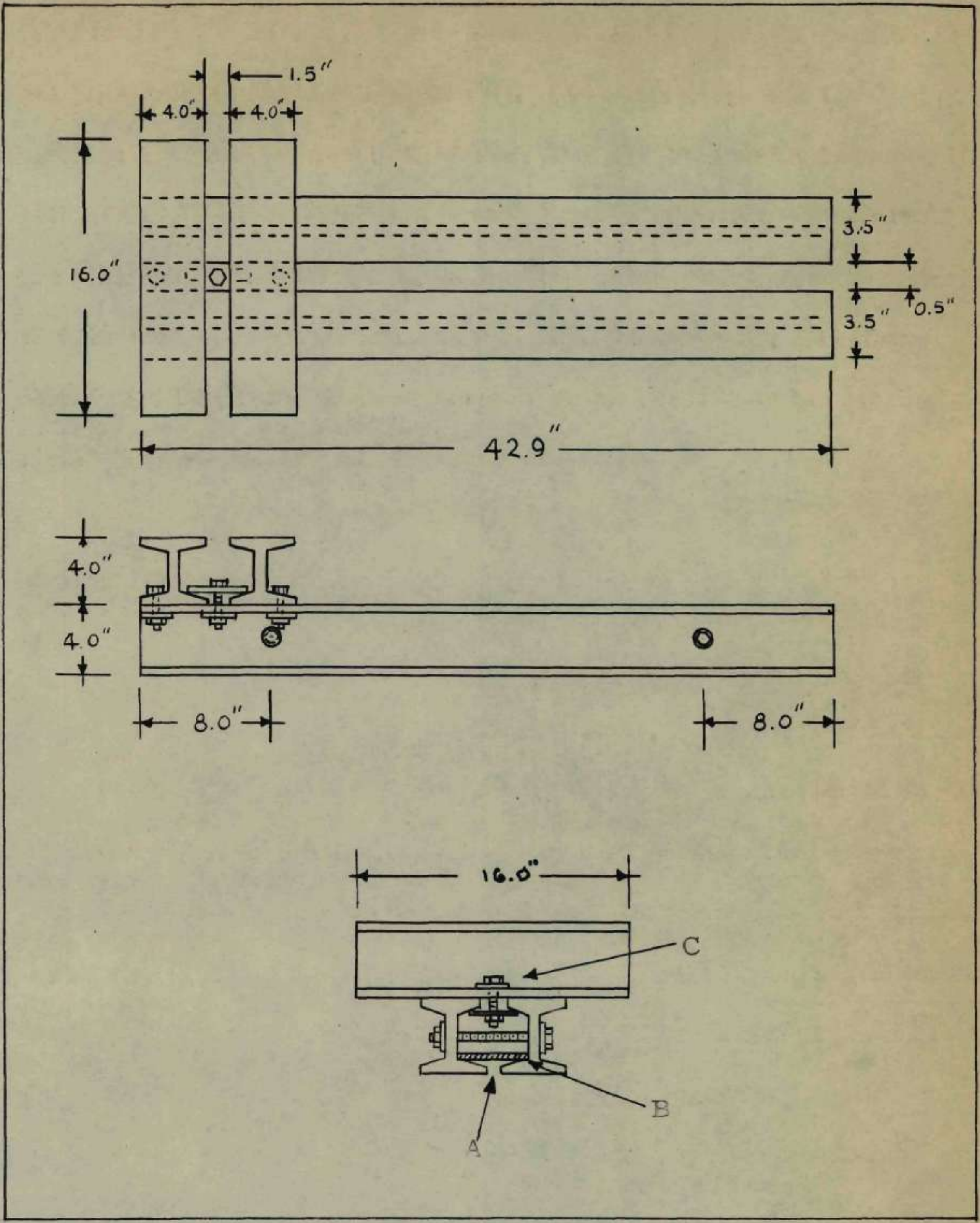
Fig. 2

thickness at any section. It was then decided to determine how much the beam varied from a straight line. This test was made by placing a straight edge along the edge of each flange in turn and using a combination of shims to determine the variation. The results of this test are shown in Fig. 15 and Tab. 1. A similar investigation was conducted with each side of the web, along its longitudinal axis, and it was discovered that there was no measurable variation from the straight edge.

Loading Equipment

Base, --The base itself (A, Fig. 3) was made from two 4" x 4" x 42.9" H-beams. In fastening the two beams together, a spacing of $\frac{1}{2}$ inch between the top flanges of each was used. This spacing was maintained by means of two separators. This separation between the beams was for the purpose of providing a means to bolt the remaining units to the base. At a position eight inches from each end of the beams and two inches from the bottom of the beams, two bolts were used to hold the beams together against the separators (B, Fig. 3). The top surface of the base was then milled 0.015 inch in depth in order that the surface would be flat and parallel to the bottom surface.

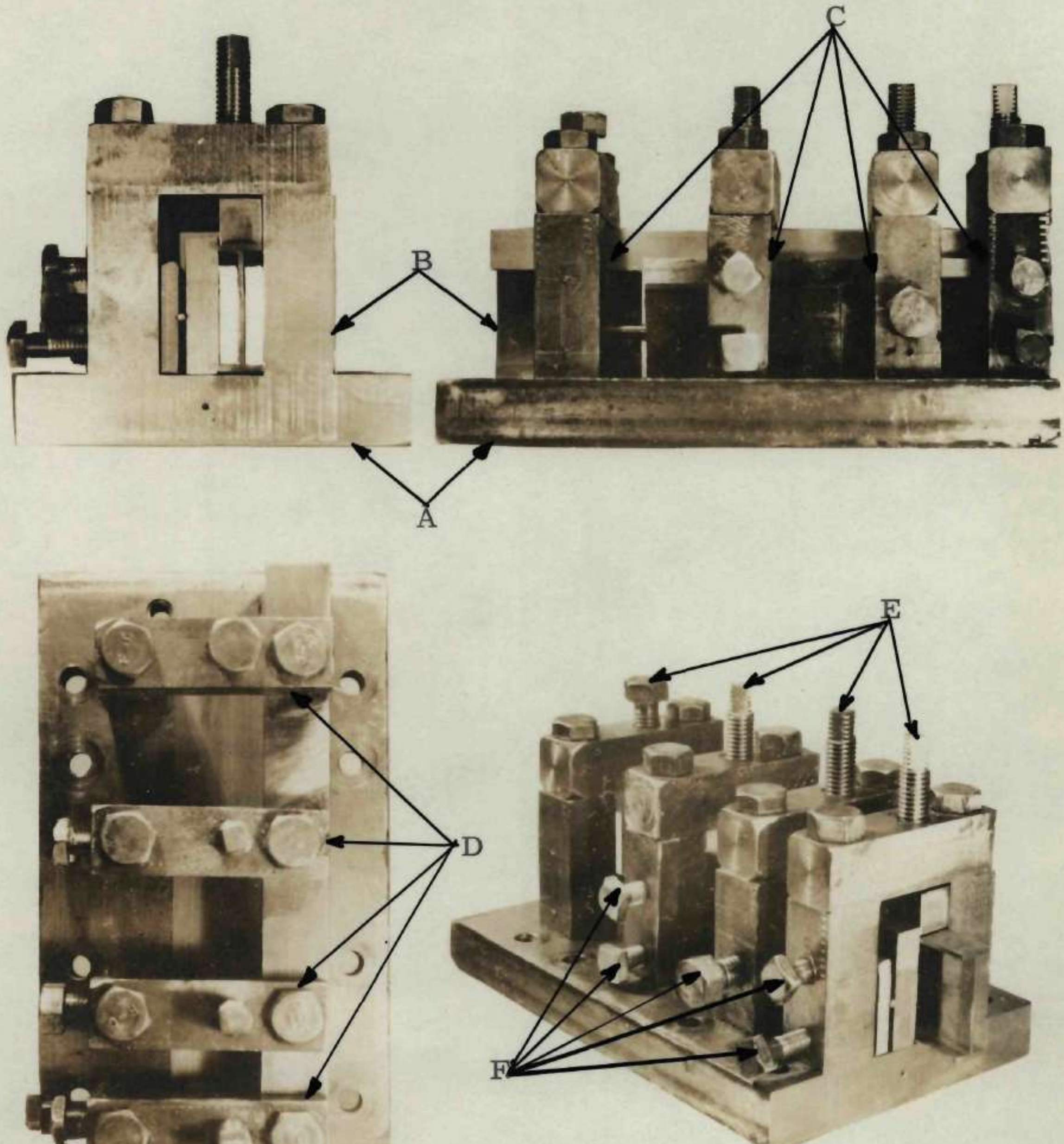
Clamp Base Platform and Clamp, --The clamp base platform (C, Fig. 3) was made from two 4" x 4" x 16" H-beams, carefully milled on both top



Clamp Base and Clamp Base Platform

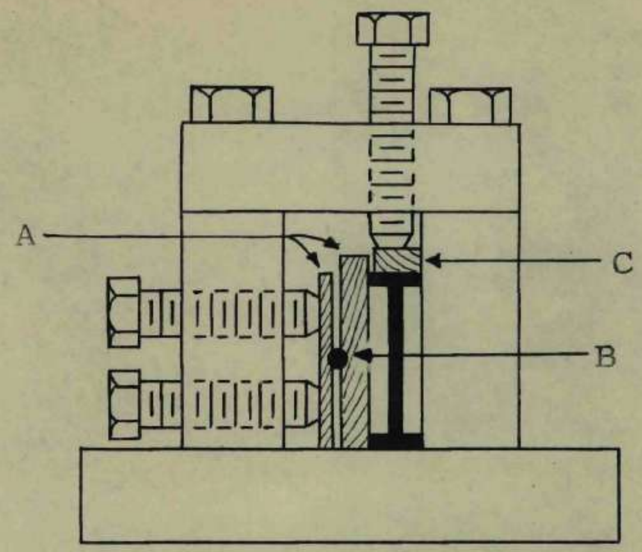
Fig. 3

and bottom surfaces. These beams were placed side by side, one and one-half inches apart, and clamped at one end perpendicular to the base. The clamp (Fig. 4) was then attached to the clamp base platform with eight $\frac{3}{8}$" x 2" bolts. This clamp was made from a flat plate of steel with the dimensions of 1" x 5.6" x 9.2". On this flat plate (A, Fig. 4) was mounted another flat plate (B, Fig. 4)perpendicular to the bottom plate. This second flat plate was cut to the dimensions of 1" x 2.5" x 9.0". Four square bars (C, Fig. 4) with the dimensions of 1" x 1" x 2.5" were then mounted vertically. Four more square bars (D, Fig. 4) with the dimensions of 1" x 1" x 3.4" were then secured across the top of each vertical square bar and the vertical flat plate. Adjusting screws (E, Fig. 4) were set into the four horizontal square bars for clamping the I-beam into place vertically. Five adjusting screws (F, Fig. 4) were set into three of the four vertical square bars for the purpose of giving lateral clamping to the I-beam. The front face, the bottom surface, and the inside clamping surfaces of the clamp were then milled to present square flat surfaces of contact. Sufficient width was maintained in the construction of this clamp so that a method of insuring that there would be the same pressure against the edges of the top and the bottom flanges of the model I-beam could be employed. This device consisted of two distribution plates (A, Fig. 5) separated by a longitudinal rod (B, Fig. 5). The rod was laid into milled grooves in the distribution plates on the midline between the top and



Clamp for I-Beam

Fig. 4



A - Vertical Distribution Plates

B - Top Distribution Plate

C - Longitudinal Rod

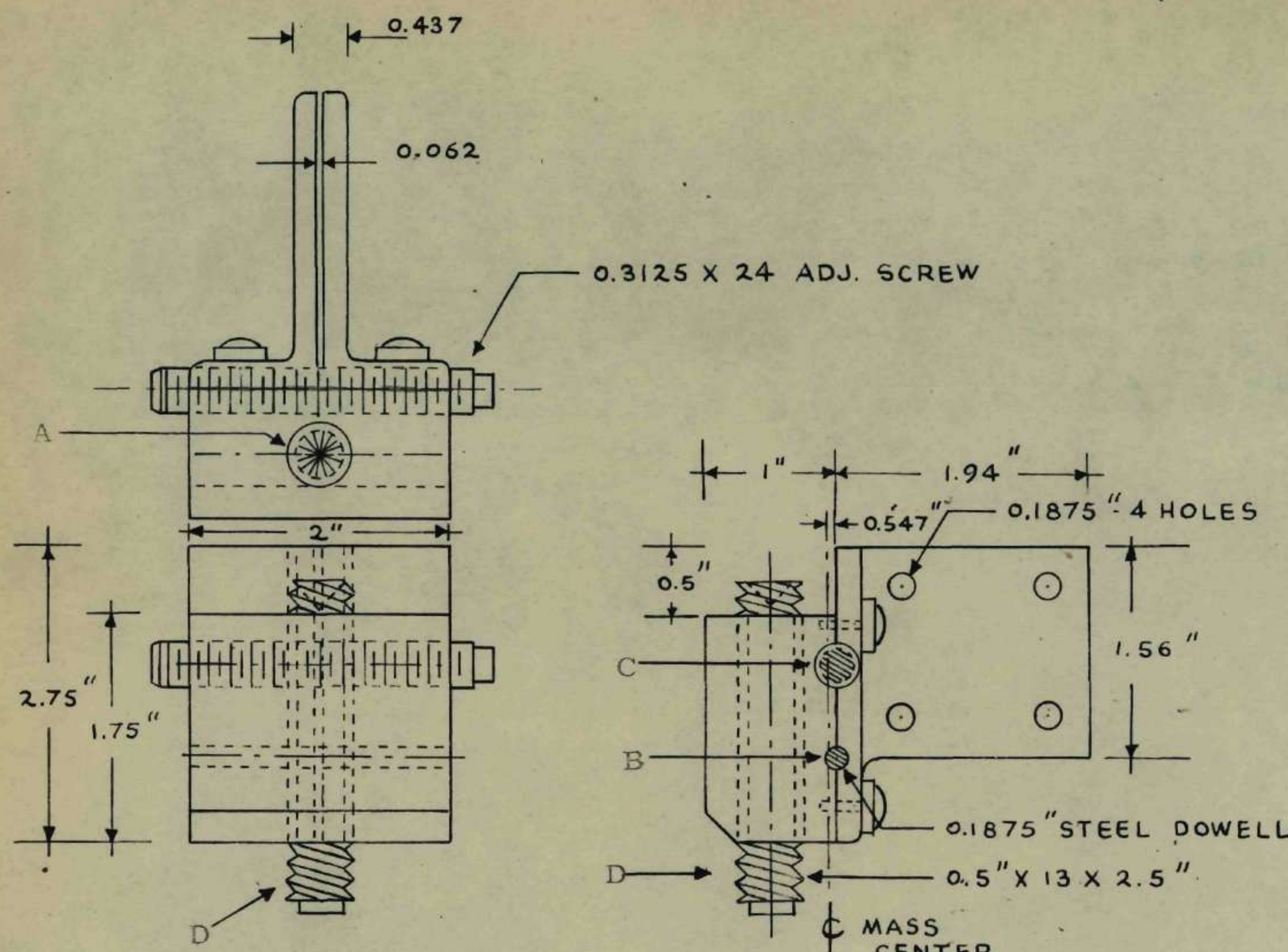
End View of Clamp

Fig. 5

the bottom flanges. The pressure of the adjusting screws upon the distribution plates held the rod in its groove. Adequate vertical clearance was also provided so as to distribute the pressure of the adjusting screws on the top of the clamp along the distribution plate (C, Fig. 5) and onto the top flange of the model I-beam.

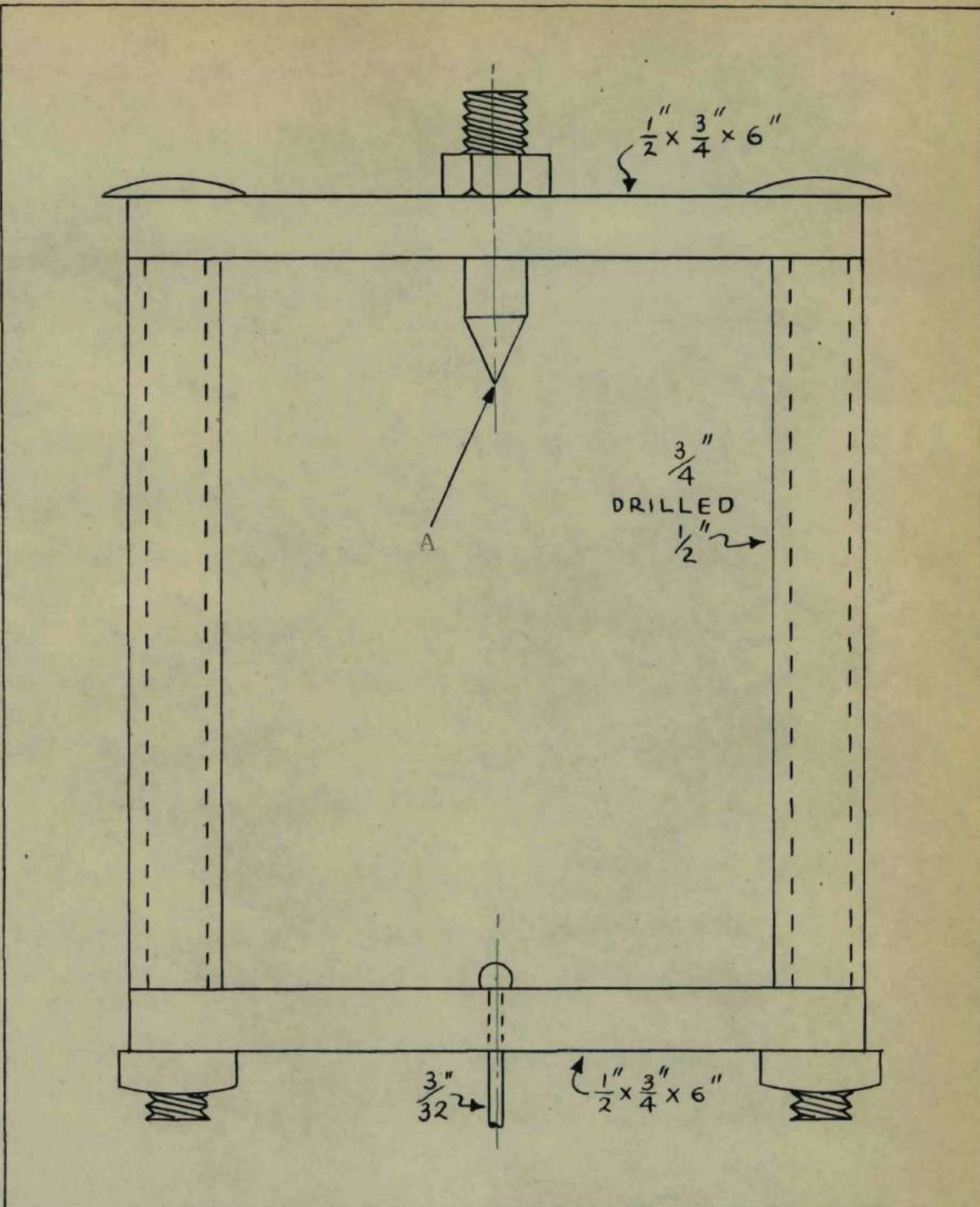
Loading Head, -- The loading head (Figs. 6 and 20) was designed and used as a device to transmit the load from the application point of the load swing support (A, Fig. 7) to a definite point in the loading head (A, Fig. 6) and to a particular point in the end cross section of the I-beam. It was necessary to provide a small lateral adjustment when clamped onto the model I-beam and to be able to adjust the application point of the load from a position above the top of the top flange to a position below the lottom of the lower flange. In this experiment, only that position which was directly at the centroid of the cross section was used. The lateral adjustment was obtained by a dowell and screw device (B and C, Fig. 6) and the vertical adjustment was obtained by a vertical screw (D, Fig. 6). This loading head was securely fastened to the end of the model I-beam with four bolts (D, Fig. 20). The weight of the loading head was computed to be 1.54 pounds and was included in all measurements of the load.

The lateral adjustment of the loading head with respect to the model I-beam was to allow for any irregularity in the beam itself, i. e., provide a means to always load the beam through the apparent centroid.



Loading Head

Fig. 6



Load Swing Support

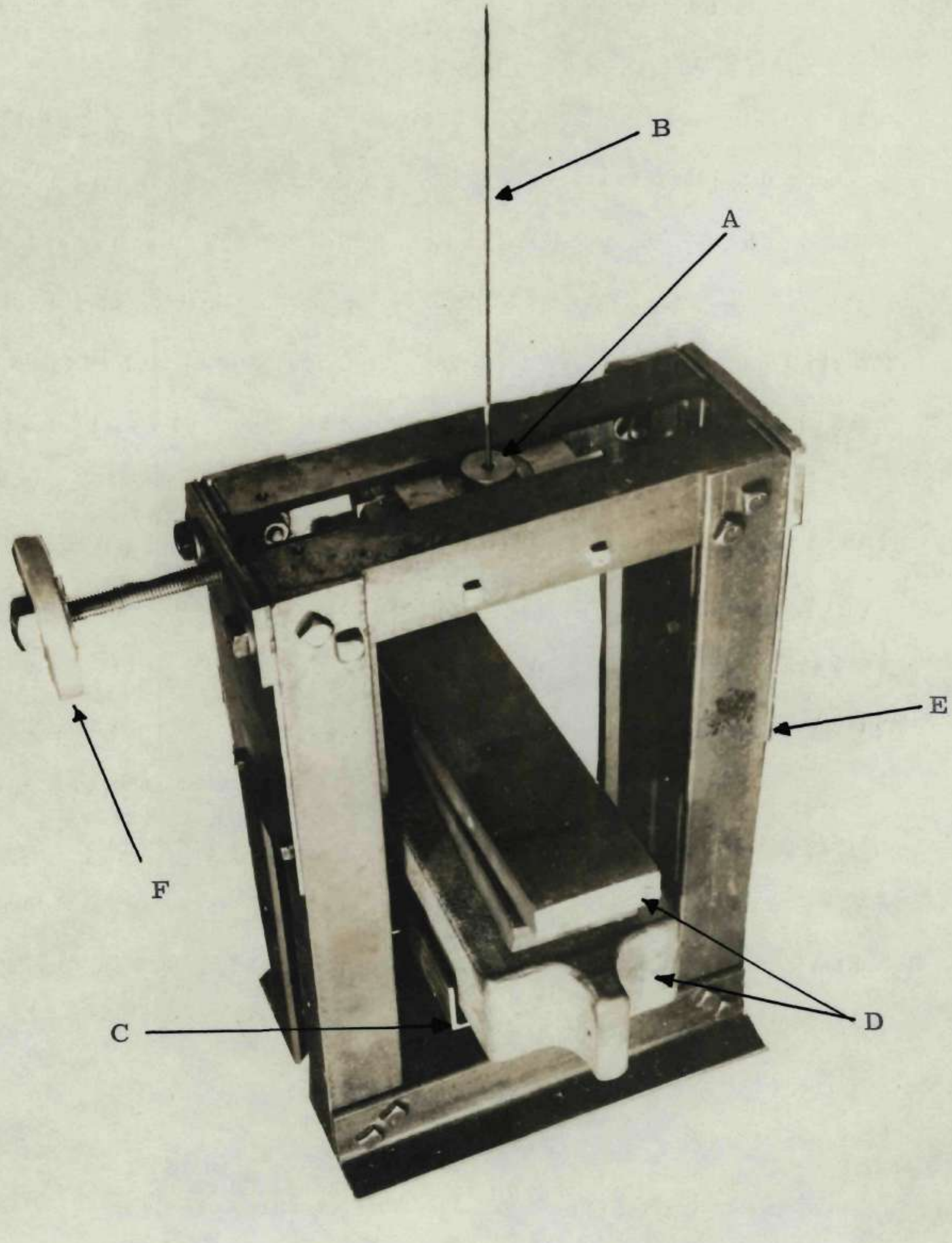
Fig. 7

If, due to non-homogeneity in the beam, the shear center during loading shifted from the centroid of the cross section, slight lateral adjustment could be made with the head adjustment so as to assure the application of the load exactly through the apparent shear center. This particular piece of equipment was constructed for use at some future date as well as for the purpose of this experiment. It was thought that an investigation of lateral buckling in a beam loaded with vertical and lateral eccentricity should be made by someone in the future.

Load Swing, --This swing was designed to give great sensitivity under all conditions of loading. The swing was constructed in two parts, namely, the support (A, Fig. 9) and the seat (C, Fig. 8). The load swing assembly weighed 6.50 pounds. These two parts were connected by the support wire (B, Fig. 8).

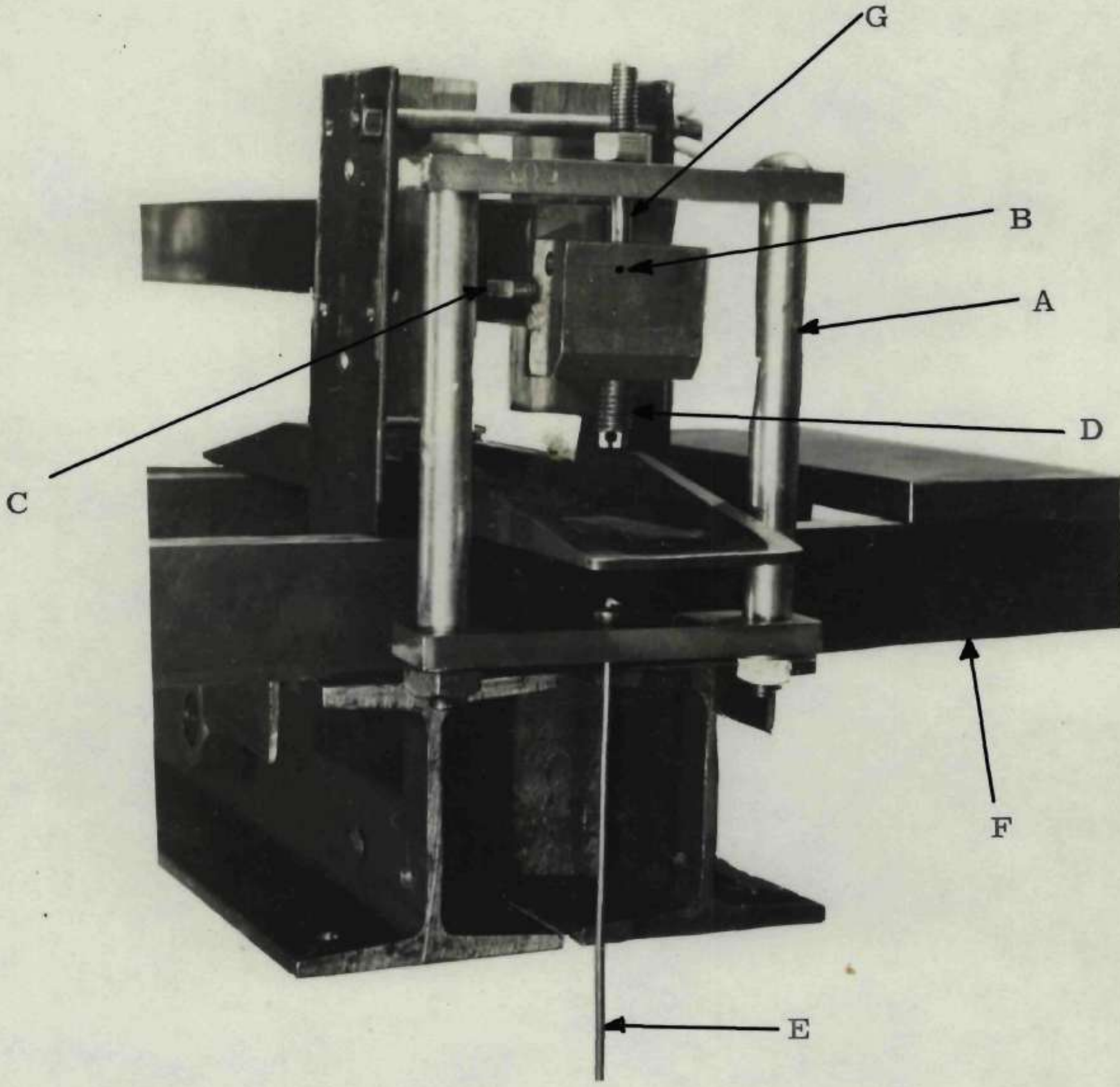
The support was constructed so that loads could be applied through a sharp pointed applicator (G, Fig. 9) to the desired point in the loading head (B, Fig. 9). The support wire was securely fastened to the support.

At the other end of the support wire, there was attached the seat of the load swing. The seat (C, Fig. 8) was merely a framework in which could be placed selected weights (D, Fig. 8). With the combination of the weight of the loading head, the load swing, and the selected weights which were placed on the seat, very accurate loads



Load Swing Seat and Load Restrainer

Fig. 8



Load Swing Support and Loading Head

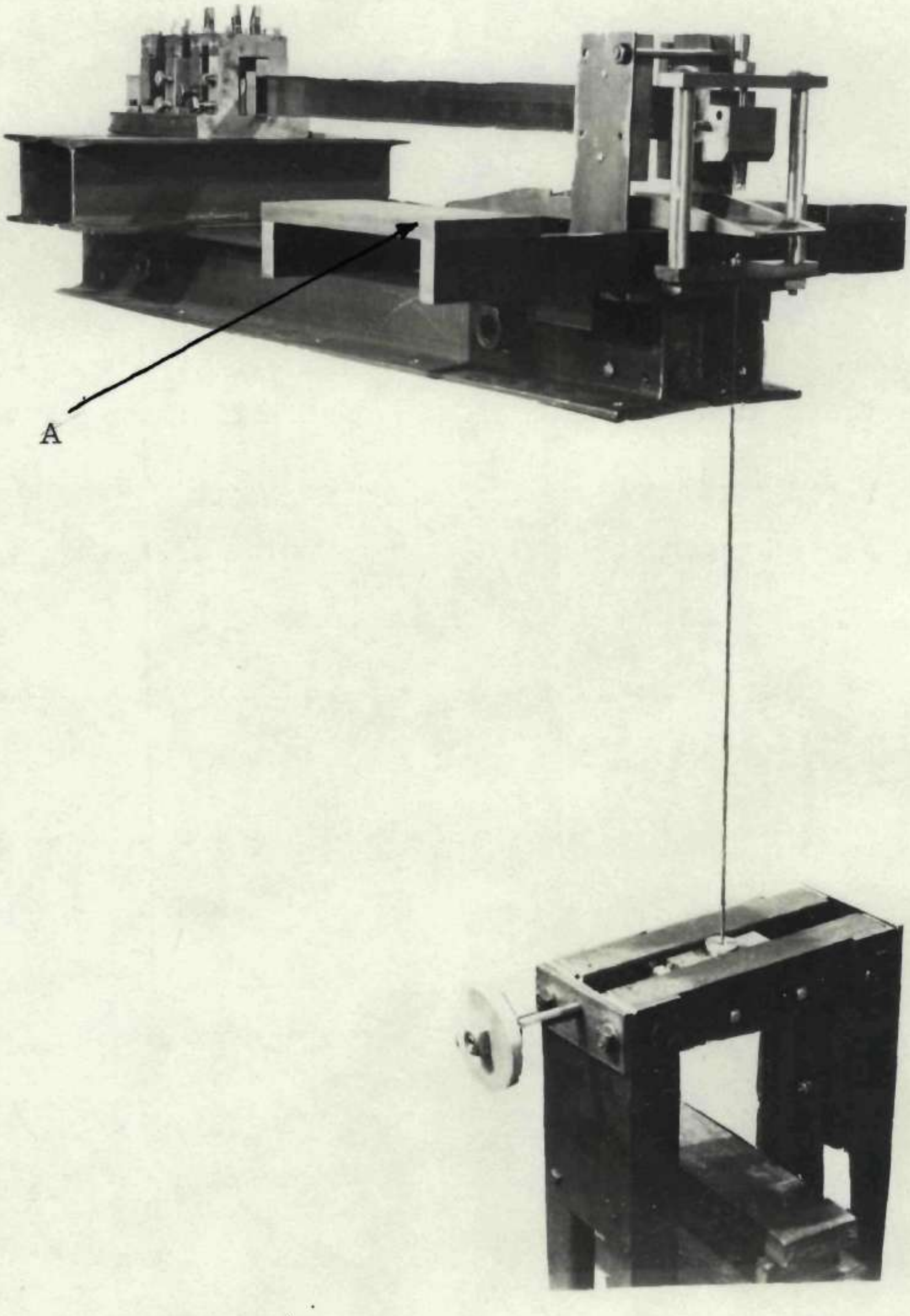
Fig. 9

could be applied to the model I-beam.

Measuring Devices

Lateral Load Restrainer, --Although this framework (E, Fig. 8) is not truly a measuring device, it is included in this section due to the fact that it was constantly used in taking the measurements of the load. The frame was built entirely of unequal angle sections $1'' \times 1\frac{1}{2}'' \times \frac{1}{4}''$. This frame was bolted onto the milled surface of the base, opposite to the clamp base platform and clamp. The restrainer made it possible to prevent rapid lateral movement of the end of the model I-beam by prohibiting excessive lateral movement of the load swing support wire. When the end of the I-beam gradually shifted laterally, the horizontal positioning screw (F, Fig. 8) was adjusted so that no lateral restraint was exerted upon the supporting wire (B, Fig. 8) of the load swing seat. This whole device was used for the purpose of determining when P_{cr_e} had been reached. As long as the horizontal positioning screw was able to move the plug (A, Fig. 8) laterally so that the supporting wire did not come in contact with the sides of the hole in the plug, P_{cr_e} had not been reached. A photograph of the entire apparatus may be seen in Fig. 10.

Measuring Table, --The measuring table was constructed as indicated in A, Fig. 10. After it was completed, 0.015 inch was milled from



Entire Apparatus

Fig. 10

both the bottom of the legs and the top measuring surface. This machining was employed in order to give a smooth flat surface to rest upon the base and also to give a smooth flat surface from which all measurements could be made. This further insured that the measuring surface would be as nearly parallel to the base as possible. The measuring table itself rested, without being attached, upon the base and was held in place there by its own weight and conformation.

CHAPTER III

PROCEDURE

Preparation

Theoretical, --The theoretical preparation consisted of carefully perusing theoretical results already submitted, following through the various derivations of other writers, and in particular, those derivations of Timoshenko which apply to a cantilever I-beam loaded with a single concentrated load at the free end. Dr. Timoshenko's latest book[6] presents a more complete derivation of the formula than any of his previous articles. The translation of this derivation from German may be found in the appendix.

Mathematical, --When determining the value for P_{cr} for any I-beam is found to be necessary, the first step is to determine the values of the various constants E, I_1, I_2, B_1, B_2, h, L, C, G, etc.[7] Sample calculations may be found in the appendix. Fig. 1 may be consulted for the values used for the model I-beam used in this experiment. Having de-

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[6]Stephen P. Timoshenko, The Collected Papers, trans by Kibbey M. Horne, Capt., USA, New York, N. Y., McGraw Hill, 1952, p. 34

[7]*Supra*, p. xii

termined the values for these preliminary constants, a table[8] was used to determine values of "m" to be used in the theoretical equation $P_{cr} = \frac{m\sqrt{B_1 C}}{L^2}$. For values of $\frac{L^2}{a^2}$ less than 40, this table provided the value for m. For values of $\frac{L^2}{a^2}$ greater than 40, an approximate "m" was determined from the equation $m = \frac{4.01}{(1 - \frac{a}{L})^2}$.

Curves were then drawn in which P_{cr} versus length for both thin rectangular beams and I-beams were plotted on standard graph paper (Figs. 11 and 12) and on logarithmic graph paper (Figs. 16 and 17). A table of values was also prepared which gave all necessary values of constants for determining P_{cr} for Prandtl's thin rectangular beams and for Timoshenko's I-beams. With these mathematical tools, it was easily apparent for each length of beam to be tested, what load could be expected to produce P_{cr}.

Test Data, --In order that accurate values of the modulus of elasticity (E) of the model I-beam could be determined, and also an accurate value for the elastic limit of the material be available, the test specimens which had been cut from the same bar of steel and had been subjected to the same heat treating process were tested by several means. These specimens were placed in a Tinius Olson Plastiversal Testing Machine, and a tension test conducted four times. The first time, readings of strain were made by using a Tuckerman Optical Strain

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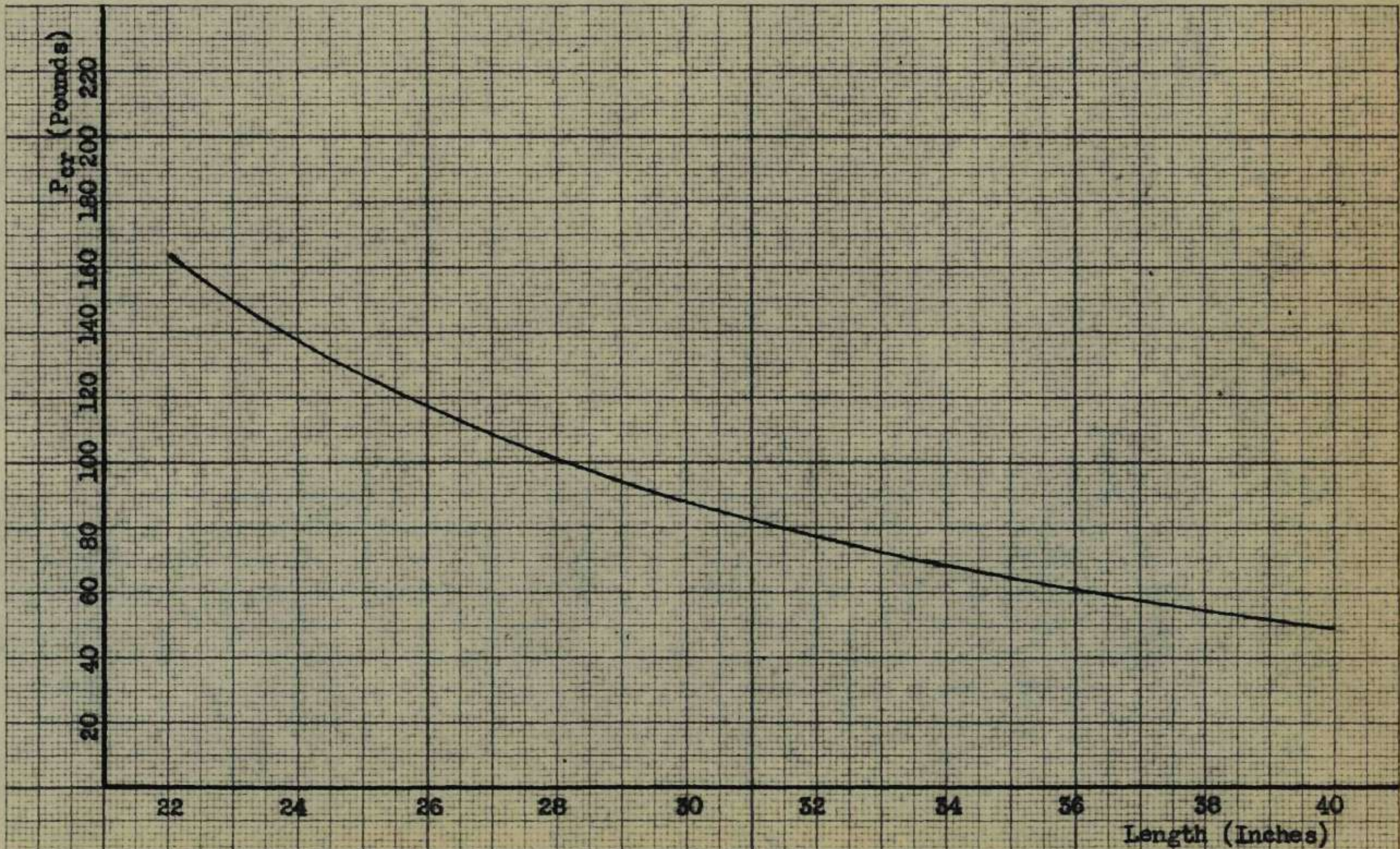
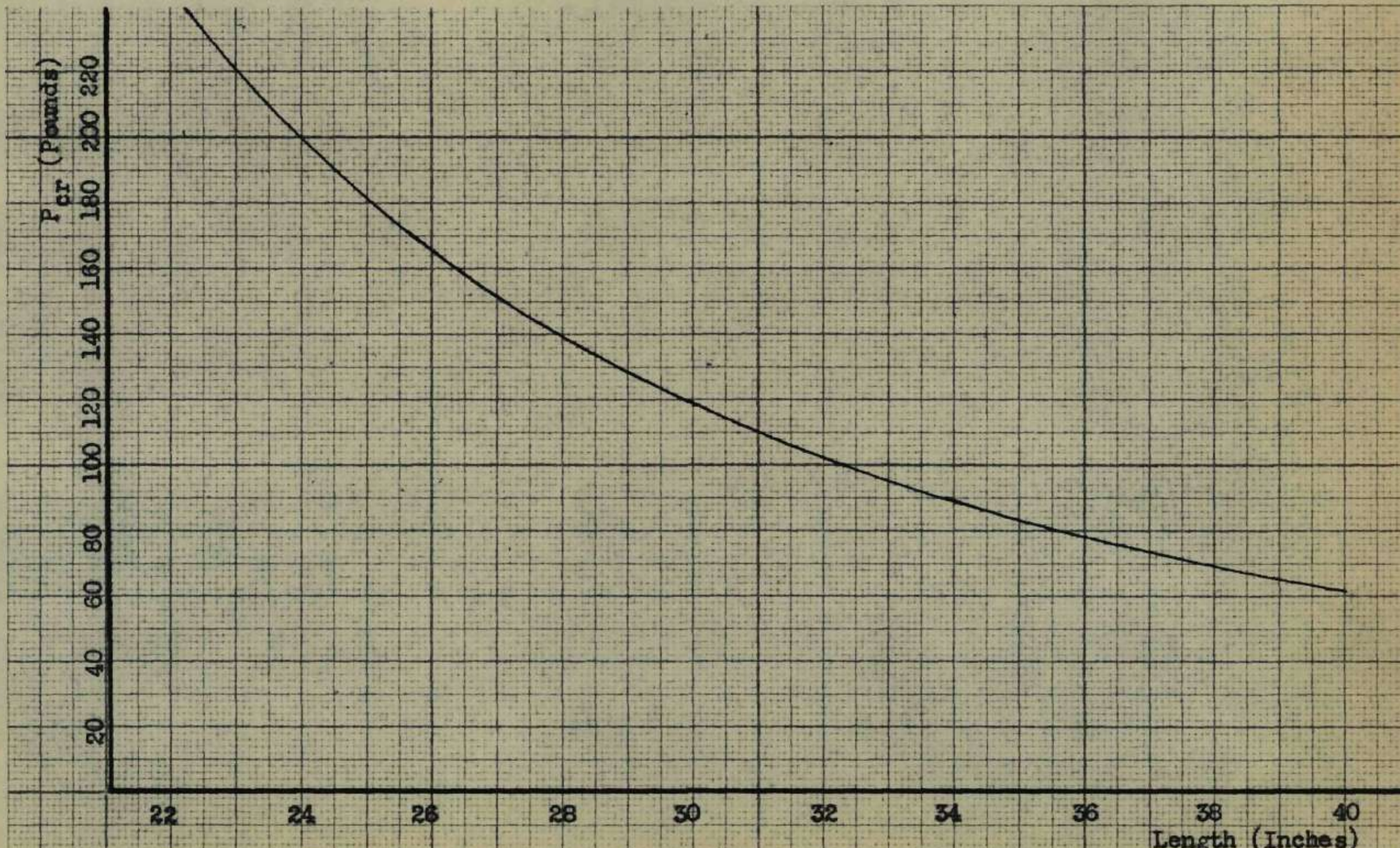
[8]Stephen P. Timoshenko, "Theory of Elastic Stability," New York, N. Y., McGraw Hill, 1936, p. 264



P_{cr} vs. Length for Thin Rectangular Cross Sections

Fig. 11



P_{cr} vs. Length for I-Beams

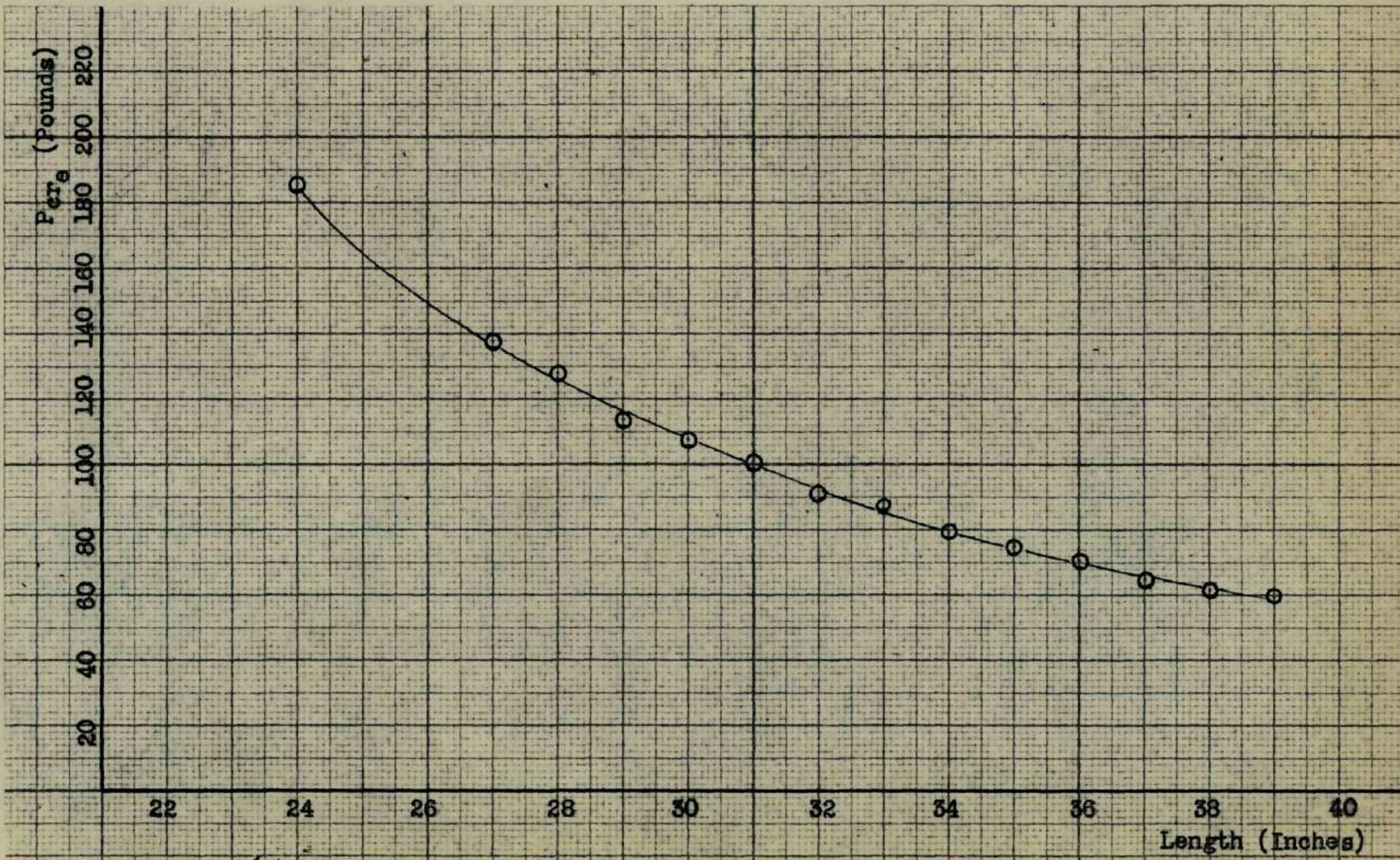
Fig. 12

Gage and Auto Collimator, while loading the specimen from zero to five thousand pounds. The second run was made on the down load of the first, still using the same strain gage. The third run was conducted in a similar manner; however, an SR-4 strain gage was employed in conjunction with a Baldwin Portable Strain Gage Indicator, Type M. The fourth run was made with a Baldwin Universal Testing Machine and a mechanical extensometer. The results of all of these tests proved to be quite alike. Sample graphs were made for determining the modulus of elasticity and for determining the elastic limit (Figs. 21 and 22 respectively). Tables 4, 5, 6, 7, and 8 in the appendix may be seen to find the recordings of the test runs. Values of the modulus of elasticity, determined from Fig. 21, were found to be 30×10^6 pounds per square inch. The elastic limit, as determined from Fig. 22, was computed to be 38, 500 pounds per square inch.

Experimental

The model I-beam was clamped into position at its most extreme length of 39. 25 inches. At this length, successive weights were added to the load swing seat. Vertical deflections were measured at several values of P for each length. As the load was increased, a spirit level, attached to the top of the top flange, was constantly checked for any beginning twist. At any time a twist was noted, the lateral positioning screw of the loading head was adjusted sufficiently for the beam to regain its untwisted state. The position of the wire which supports the seat of the load swing was constantly checked also to assure that the plug was not providing any lateral restraint on the wire. This lateral restraint would prevent the model I-beam from becoming laterally unstable at the true value of P_{cr_e}. With the constant adjustment being made at these two points (the loading head and the lateral load restrainer, the beam was loaded to that point at which no amount of further adjustment could hold it in a level and stable state. Adjustment of the lateral positioning screw of the loading head would untwist the beam, only to have it flop to the opposite extreme twisted position. Any further adjustment of the plug in the lateral load restrainer in a lateral direction was closely followed by the swing wire itself as the swing constantly sought lateral support. This maximum load that it was possible to add onto the end of the beam was considered to be P_{cr_e}.

This same procedure was followed for each of the successive lengths indicated in Tab. 3. These values were then plotted and a smooth curve drawn between these points. This is indicated on standard graph paper in Fig. 13, and on logarithmic graph paper in Fig. 18.



P_{cr_e} vs. Length

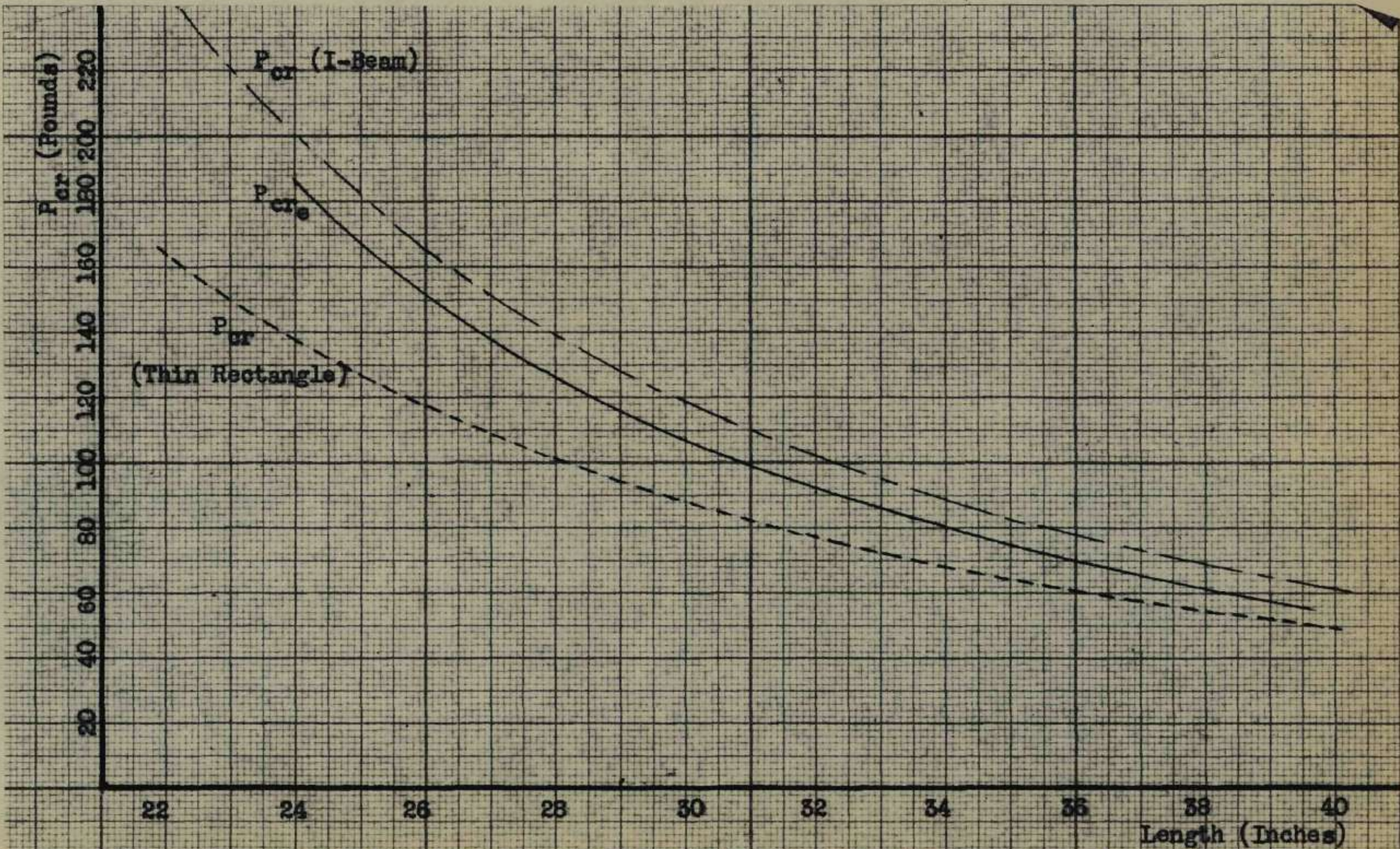
Fig. 13

CHAPTER IV

DISCUSSION OF RESULTS

The readings taken on the model I-beam used in this experiment formed a smooth curve when plotted on standard graph paper (Fig. 13). For purposes of comparison, the theoretical values for this research were tabulated and graphed on standard graph paper Tab 2 and Figs. 11 and 12). In order that a proper comparison could be made, P_{cr} for Prandtl's thin rectangular beams and Timoshenko's I-beams, and the experimental values, P_{cr_e}, were all plotted on the same sheet of paper (Fig. 14). As may be seen from this graph, for any given length of beam, the theoretical values of P_{cr} for I-beams were greater than those for thin rectangles. The experimental results fell between the two curves, but were within approximately 10 per cent of the theoretical values of P_{cr} for I-beams. The values of P_{cr} for I-beams slowly approached those values for thin rectangles as the length increased. This, of course, was due to the fact that as the length increased, the added resistance to bending in the planes of the flanges was minimized.

It had been suggested that interesting results might be obtained from plotting P_{cr} versus length on logarithmic graph paper. To this



P_{cr} (Thin Rectangle), P_{cr} (I-Beam), and P_{cr_e} vs. Length

Fig. 14

end, the theoretical values derived from a very careful series of substitutions into Timoshenko's equation for P_{cr} were plotted against length on a sheet of logarithmic graph paper. When this graph proved to be a straight line, similar graphs were drawn for Prandtl's thin rectangular beams, for the experimental values of P_{cr_e}, and for a composite graph of all three curves (Figs. 16, 17, 18, and 19). This use of logarithmic plotting made it possible to make a much better comparison between the three curves (Fig. 19) due to the fact that they did define straight lines. Fig. 19 also shows that the curve plotted from the values of P_{cr} for I-beams and the curve plotted from the values of P_{cr_e} are exactly parallel. In addition, it was possible to derive an equation by graphical analysis for the curve plotted from the values of P_{cr_e}. This equation was found to be $P_{cr_e} = \frac{315,800}{L^{2.345}}$.

The relatively small number of points on the experimental graph were few due to the fact that having once put permanent set into the beam at the wall, the beam was of no further use for evaluating P_{cr_e} for this length or of longer length. After each reading was made, the beam was moved inward into the clamp so that that portion of the beam, in which the elastic limit of the material had been exceeded, never entered into the tests for P_{cr_e} for subsequent lengths. It had been determined by previous calculation that moving the beam inward a distance of one inch insured that the remaining length of the beam had not been stressed beyond the elastic limit of the material.

CHAPTER V

CONCLUSIONS

Within the accuracy and completeness of this investigation, the following conclusions seem justified. They are based upon work carried out on a model I-beam loaded as a cantilever and carrying a concentrated load at its free end.

1. It is reasonable to produce a model I-beam for the pur- of analyzing experimentally the value of P_{cr_e} as it applies to lateral stability, instead of using standard structural sizes.
2. The test platform, clamp, and method of loading which has been established in this test run gave very satisfactory results.
3. Although it does not seem possible to plot one curve on logarithmic paper in order to predict values of P_{cr} for any beam of any length, it is found that it is only necessary to plot two points, found from Timoshenko's equation, for P_{cr} versus length in order to have a complete range of values for any one beam size.
4. Plotted curves of the theoretical values and the experimental values of P_{cr} both form straight parallel lines on logarithmic graph paper.

5. Timoshenko's equation for P_{cr} for a cantilever beam loaded with a single concentrated load at the free end is verified.

CHAPTER VI

RECOMMENDATIONS

As this experimental test run has indicated, there are many possible future investigations. It is hoped that there may be some more graphical analysis with logarithmic graph paper, which would enable the practicing engineer to merely look at a graph for a value of P_{cr} and correct it to his own needs and building codes. It is felt that a simpler relationship between length and P_{cr} can be found.

It is also recommended that similar tests be run on other cantilever beams with the view to testing predicted values in the plastic range. Tests should also be made in which the load is varied eccentrically from the centroid of the cross section laterally and vertically, and sample curves drawn therefrom. There are a variety of other type load conditions, i.e., simply supported, fixed ended, continuous, etc., which would bear similar investigations.

APPENDIX

DERIVATION OF P_{cr}

The following derivation of the theoretical value of P_{cr} has been made by Dr. Timoshenko. The location of this particular derivation may be found in his latest book, "The Collected Papers." In this book, it was necessary to have the work translated from German.

> Par. 15. A beam, fixed at one end, is loaded at the free end with a single force.
>
> Let the point of application of the bending force coincide with the centroid of the cross section. The force acts perpendicular to the longitudinal axis of the beam. Place the origin of coordinates, X, Y, and Z, on the point of application of the load, letting the X axis lie along the straight axis of the beam and directing the Z axis according to the force P. By gradually increasing the force P, one can reach the limit at which the straight state of equilibrium ceases to be stable, and the beam will twist. Further, let us assume a flexible coordinate system, ξ, η, ζ ; then, for any desired beam cross section with small curvatures, the following relationships exist:
>
> $$M_\xi = P\left(y - x\frac{\partial y}{\partial x}\right) \qquad M_\eta = -Px \qquad M_\zeta = Px\,\varphi$$
>
> Substituting in the basic equation, this results in:
>
> $$(11)\quad P\left(x\frac{\partial y}{\partial x} - y\right) = C\varphi' + \frac{Dh^2}{2}\varphi''', \quad B_2\frac{\partial^2 y}{\partial x^2} = Px\,\varphi, \quad B_1\frac{\partial^2 z}{\partial x^2} = Px$$
>
> and we obtain therefrom the equation determining :
>
> $$(12)\quad \frac{P^2x^2}{B_2}\varphi = -C\varphi'' + \frac{Dh^2}{2}\varphi^{iv}$$
>
> Substituting D = 0 in this equation, that is, disregarding the bending of the flange, we arrive at the bending investigated by Professor Prandtl of a beam of thin rectangular cross section. If Equation (12) is divided by the coefficients of φ^{iv}, we arrive at:
>
> $$(12')\quad \varphi^{iv} - \frac{1}{a^2}\varphi'' - \frac{x^2}{b^6}\varphi = 0$$

where

(13) $\frac{1}{a^2} = \frac{2C}{Dh^2}$ $\frac{1}{b^6} = \frac{2P^2}{DB_2h^2}$

We will carry through the solution of Equation (12) in the form of an infinite series

$$\varphi = A_0 + A_1 x + A_2 x^2 + \cdots$$

Substituting this in Equation (12'), we can find the relation between the coefficients A_0, A_1, A_2, ... and represent the general form of Equation (12') in the following form:

(14) $\varphi = \varphi_0 [M] + \varphi_0' x [N] + \frac{1}{2} \varphi_0'' x^2 [Q] + \frac{1}{6} \varphi_0''' x^3 [P]$

where φ_0 is the angle of twist and the values of its derivatives for $X = 0$, (M), (N), ... are infinite series arranged according to increasing powers of X^2. The first differential of the term (14) with respect to X is:

$$\varphi_0' = \frac{1}{x} \varphi_0 [R] + \varphi_0' [S] + \frac{1}{2} \varphi_0'' x [U] + \frac{1}{6} \varphi_0''' x^2 [T]$$

The arbitrary constants φ_0, and φ_0' are determined by the end conditions of the beam. Since there is no moment at the right end of the beam $(X = 0)$, it follows that:

(1) $\varphi_0'' = 0$ (2) $-C\varphi_0' + \frac{Dh^2}{2} \varphi_0''' = 0$

For the fixed end of the beam, $(X = L)$, the following conditions exist:

(3) $\varphi_L' = 0$ (4) $\varphi_L = 0$

Considering (1) and (2) above, these conditions may now be written thus:

$$(15) \begin{cases} \varphi_0 [M]_L + L\varphi_0' \left\{ [N]_L + \frac{1}{6} \frac{L^2}{a^2} [P]_L \right\} = 0 \\ \varphi_0 [R]_L + L\varphi_0' \left\{ [S]_L + \frac{1}{6} \frac{L^2}{a^2} [T]_L \right\} = 0 \end{cases}$$

The equations above are satisfied when $\varphi_0 = \varphi_0' = 0$ is substituted, then, however, $\varphi_0 = 0$ along the entire length of the beam and we have, therefore, the straight state of equilibrium. In order for a bent state of equilibrium to be possible, the determinants of the Equations (15) must be equal to zero, that is, it must be

$$(16)\quad [M]_L \left\{ [S]_L + \frac{1}{6}\frac{L^2}{a^2}[T]_L \right\} - R_L \left\{ [N]_L + \frac{1}{6}\frac{L^2}{a^2}[P]_L \right\} = 0$$

It should be noted that the infinite series M_L, N_L ... are made up only of different powers of the values $\frac{L^2}{a^2}$ and $\frac{L^6}{a^6}$. From relation (13) the value of $\frac{L^2}{a^2}$ for a given beam can be calculated in advance. Then the values of $\frac{L^6}{a^6}$ (which depend on the bending force P) and consequently also the critical value P_{cr} can be determined by solving the transcendental equation (16).

If one designates $\left(\frac{L^6}{b^6}\right) : \left(\frac{L^2}{a^2}\right) = K^2$

then, considering (13), it follows that:

$$P_{cr} = \frac{K\sqrt{B_2 C}}{L^2}$$

Table 1. Dimensions of Model I-Beam at Various Stations

DIMENSIONS	-1-	-2-	-3-	-4-	-5-
A	1.757	1.756	1.757	1.757	1.757
B	0.273	0.273	0.273	0.273	0.273
C	0.273	0.273	0.273	0.273	0.281
D	0.273	0.273	0.273	0.273	0.273
E	o.273	0.273	0.266	0.273	0.266
F	0.061	0.061	0.062	0.063	0.063
G	0.082	0.078	0.081	0.079	0.079
H	0.082	0.085	0.087	0.085	0.084
I	0.084	0.087	0.085	0.085	0.084
J	0.080	0.078	0.076	0.076	0.075
*K	0.002	0.000	0.003	0.002	0.004
*L	0.002	0.004	0.001	0.002	0.000
*M	0.004	0.000	0.003	0.002	0.005
*N	0.000	0.003	0.001	0.002	0.000
O	0.608	0.609	0.607	0.607	0.596
P	0.609	0.608	0.608	0.605	0.604

*This symbol indicates that this measurement was the variation from a straight edge.

Table 2. P_{cr} for Thin Rectangular Beams and for I-Beams

L (Inches)	P_{cr}* (Pounds)	$\frac{L^2}{a^2}$	m	$\frac{m}{L^2}$	P_{cr}** (Pounds)
22	163.8	28.40	6.015	0.01243	245.6
24	137.7	33.70	5.820	0.01010	199.6
26	117.3	39.60	5.650	0.00836	165.2
28	101.1	45.90	5.520	0.00704	139.1
30	88.1	52.70	5.390	0.00599	118.3
32	77.4	60.00	5.290	0.00517	102.1
34	68.6	67.70	5.200	0.00450	88.9
36	61.2	75.90	5.120	0.00395	78.1
38	54.9	84.60	5.050	0.00350	69.1
40	49.6	93.70	4.990	0.00312	61.6

*P_{cr} for thin rectangular beams.

**P_{cr} for I-beams.

Table 3. Recorded Loads and Some Vertical Deflections

LENGTH (Inches)	Δ_v (Inches)	P (Pounds)	LENGTH (Inches)	Δ_v (Inches)	P (Pounds)
39	2.145	8.04	34	1.451	1.54
		44.54		1.207	44.54
		53.82		1.029	71.59
	1.714	54.93		0.976	*79.40
		*59.37		1.438	1.54
38	2.344	1.54	33	1.478	1.54
	2.063	44.54		1.233	44.54
	1.969	53.31		1.084	71.59
		57.75			*87.00
		*61.93		1.442	1.54
37	1.484	1.54	32	1.463	1.54
	1.172	44.54		1.257	44.54
	1.156	56.14			71.59
		*64.91			*91.00
	1.469	1.54		1.434	1.54
36	1.468	1.54	31	1.464	1.54
	1.188	44.54		1.271	44.54
	1.109	55.04		1.103	80.84
	1.047	62.81			*100.50
	1.046	63.81			
		*70.81	30	1.391	1.54
				1.216	44.54
35	1.469	1.54		1.064	80.84
	1.203	44.54			106.34
	1.078	62.31			*107.45
		71.59			
	1.016	73.81	29	1.398	1.54
	0.984	*74.92		1.243	44.54
	1.443	1.54		1.098	80.84

Table 3. (Continued)

LENGTH (Inches)	Δ_v (Inches)	P (Pounds)	LENGTH (Inches)	Δ_v (Inches)	P (Pounds)
29		106.34	27	1.404	1.54
		*114.00		1.222	62.11
	1.376	1.54		1.091	98.61
				1.023	118.39
28	1.414	1.54			*137.94
	1.265	44.54		1.349	1.54
	1.139	80.84			
		106.30	24	1.359	1.54
		*128.00		1.217	62.11
	1.376	1.54		1.131	99.31
					150.05
					*185.39

*This indicates P_{cr_e}

Table 4. Tensile Test of Specimen*

LOAD (Pounds)	STRESS (psi)	STRAIN GAGE READING	Δ GAGE READING	CUMULATIVE Δ GAGE READING
0	0	11, 250	0	0
1, 015	5, 075	11, 417	167	167
2, 000	10, 000	11, 583	166	333
3, 000	15, 000	11, 748	165	498
4, 000	20, 000	11, 917	169	667
5, 000	25, 000	12, 082	165	832
6, 000	30, 000	12, 253	171	1, 003
6, 520	32, 600	12, 343	90	1, 093
6, 860	34, 300	12, 573	230	1, 323

*This test was accomplished with a Tinius Olson Plastiversal Testing Machine. Measuring devices used were one (1) SR-4 Strain Gage and a Baldwin Portable Strain Gage Indicator (Type K).

Table 5. Tensile Test of Specimen*

LOAD (Pounds)	STRESS (psi)	TUCKERMAN FRONT	TUCKERMAN REAR	AVERAGE Δ TUCKERMAN
UP LOADING				
1,000	5,000	16.79	6.65	0.00
1,500	7,500	17.77	7.41	0.87
2,000	10,000	18.77	8.10	0.85
2,500	12,500	19.75	8.77	0.83
3,000	15,000	20.72	9.43	0.82
3,500	17,500	21.69	10.11	0.83
4,000	20,000	22.64	10.80	0.82
4,500	22,500	23.55	11.50	0.80
4,900	24,500	24.31	12.10	0.68

*This test was accomplished with a Tinius Olson Plastiversal Testing Machine. Measuring devices used were two (2) Tuckerman Optical Strain Gages and Auto Collimator.

Table 6. Tensile Test of Specimen*

DOWN LOADING				
LOAD (Pounds)	STRESS (psi)	TUCKERMAN FRONT	TUCKERMAN REAR	AVERAGE Δ TUCKERMAN
4, 900	24, 500	22. 37	12. 10	0. 00
4, 500	22, 500	21. 58	11. 51	0. 69
4, 000	20, 000	20. 64	10. 80	0. 83
3, 500	17, 500	19. 69	10. 09	0. 83
3, 000	15, 000	18. 73	9. 40	0. 83
2, 500	12, 500	17. 76	8. 71	0. 83
2, 000	10, 000	16. 77	8. 04	0. 83
1, 500	7, 500	15. 77	7. 40	0. 82
1, 000	5, 000	14. 80	6. 72	0. 83

*This test was accomplished with a Tinius Olson Plastiversal Testing Machine. Measuring devices used were two (2) Tuckerman Optical Strain Gages and Auto Cillimator.

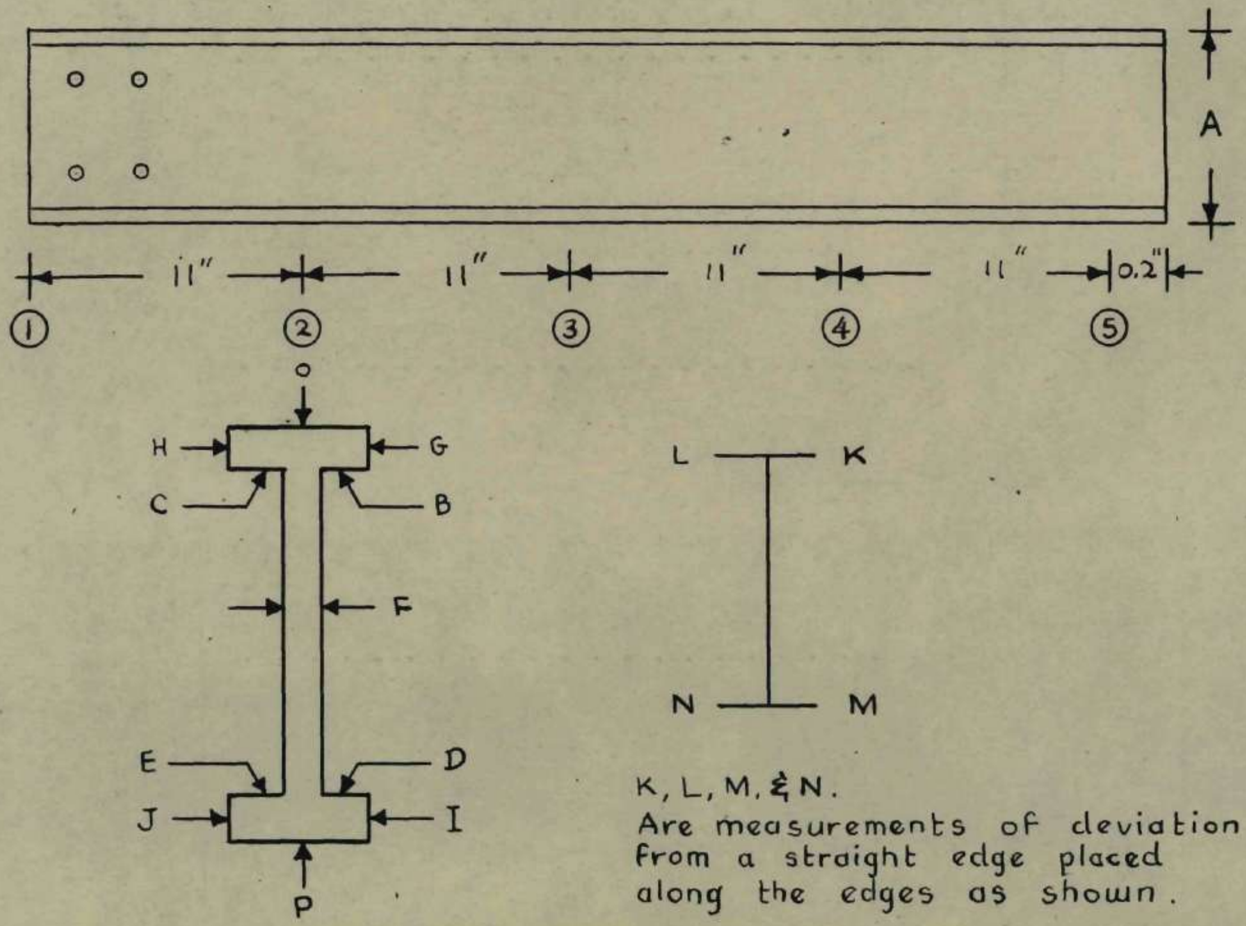
Table 7. Averaged Tuckerman Results of Tensile Test Specimen

STRESS (psi)	STRAIN UP LOAD (Microinches per Inch) Δ	CUMULATIVE Δ	STRAIN DOWN LOAD (Microinches per Inch) Δ	CUMULATIVE Δ
5, 000	00. 0	00. 0	00. 0	00. 0
7, 500	86. 9	86. 9	82. 9	82. 9
10, 000	84. 9	171. 8	81. 9	164. 8
12, 500	82. 9	254. 7	82. 9	247. 7
15, 000	81. 9	336. 6	82. 9	330. 6
17, 500	82. 9	419. 5	82. 9	413. 5
20, 000	81. 9	501. 4	82. 9	496. 4
22, 500	79. 9	581. 3	82. 9	579. 3
24, 000	67. 9	649. 2	68. 9	648. 2

Table 8. Tensile Test of Specimen*

LOAD (Pounds)	STRESS (psi)	EXTENSOMETER READING	STRAIN (Microinches) (per Inch)
0	0	0.00	0.00
500	2, 500	1.78	0.89
1, 000	5, 000	3.44	1.72
1, 500	7, 500	5.18	2.59
2, 000	10, 000	6.64	3.32
2, 500	12, 500	8.38	4.19
3, 000	15, 000	10.08	5.04
3, 500	17, 500	11.78	5.89
4, 000	20, 000	13.30	6.65
4, 500	22, 500	15.00	7.50
5, 000	25, 000	16.70	8.35
5, 500	27, 500	18.46	9.23
6, 000	30, 000	19.88	9.94
6, 500	32, 500	21.72	10.86
7, 000	35, 000	23.48	11.74
7, 500	37, 500	25.00	12.50
7, 600	38, 000	25.38	12.69
7, 800	39, 000	25.88	12.94
8, 000	40, 000	26.70	13.35
8, 200	41, 000	27.62	13.81
8, 400	42, 000	28.80	14.40
8, 600	43, 000	29.88	14.94
8, 800	44, 000	31.52	15.76

*This test was accomplished with a Tinius Olson Universal Testing Machine. Measuring device was one (1) mechanical Extensometer.



Stations of Model I-Beam

Fig. 15

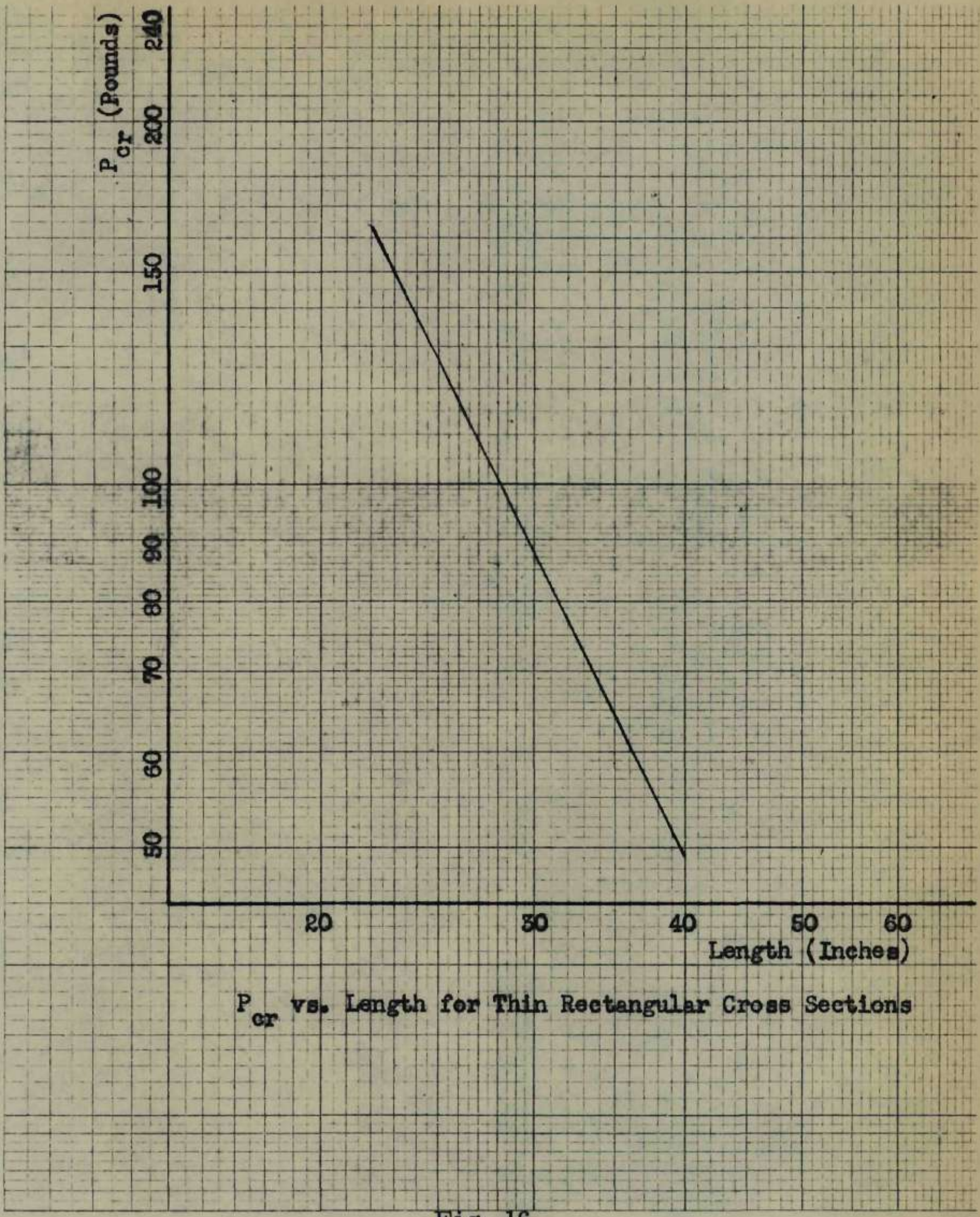
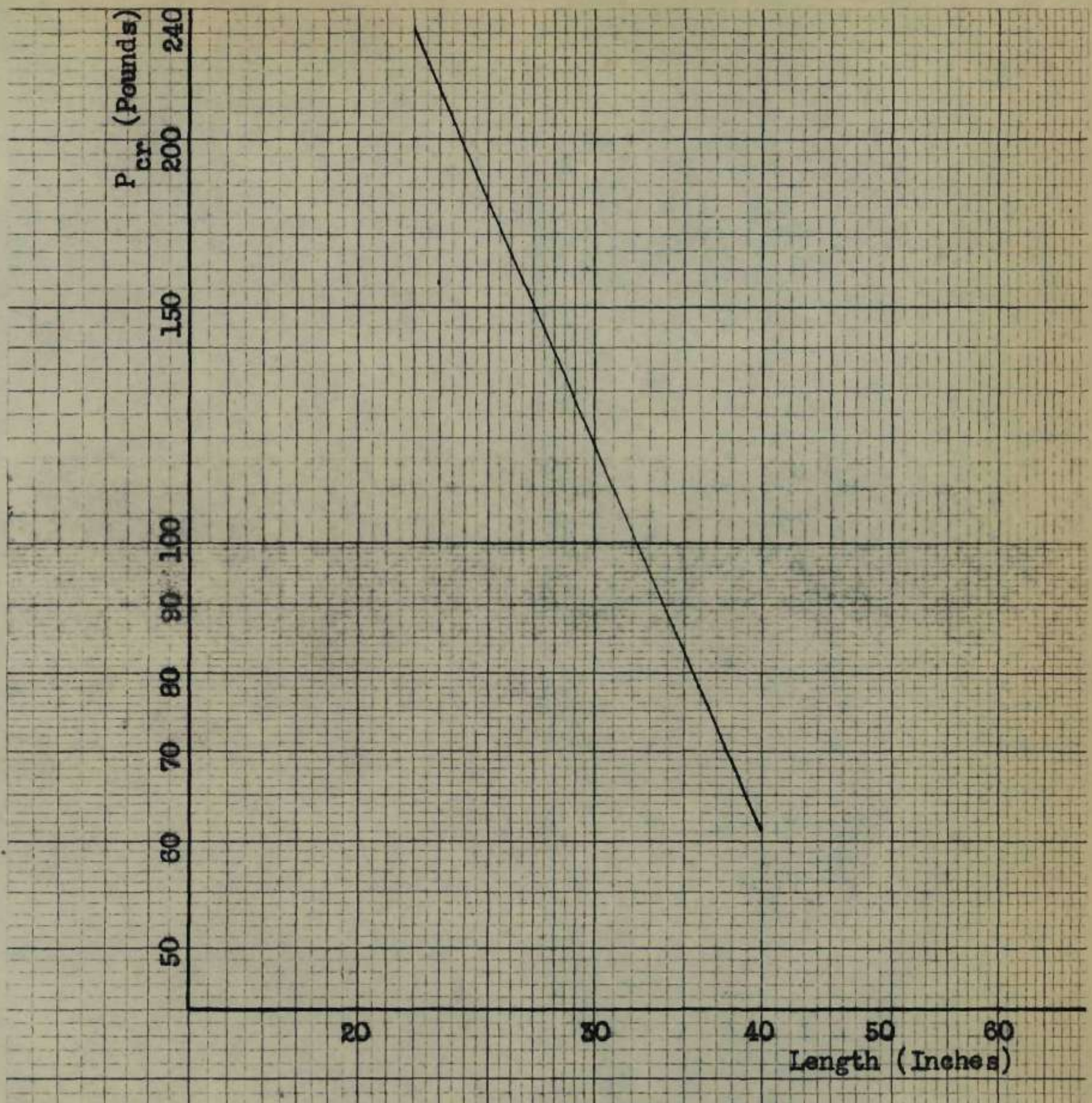
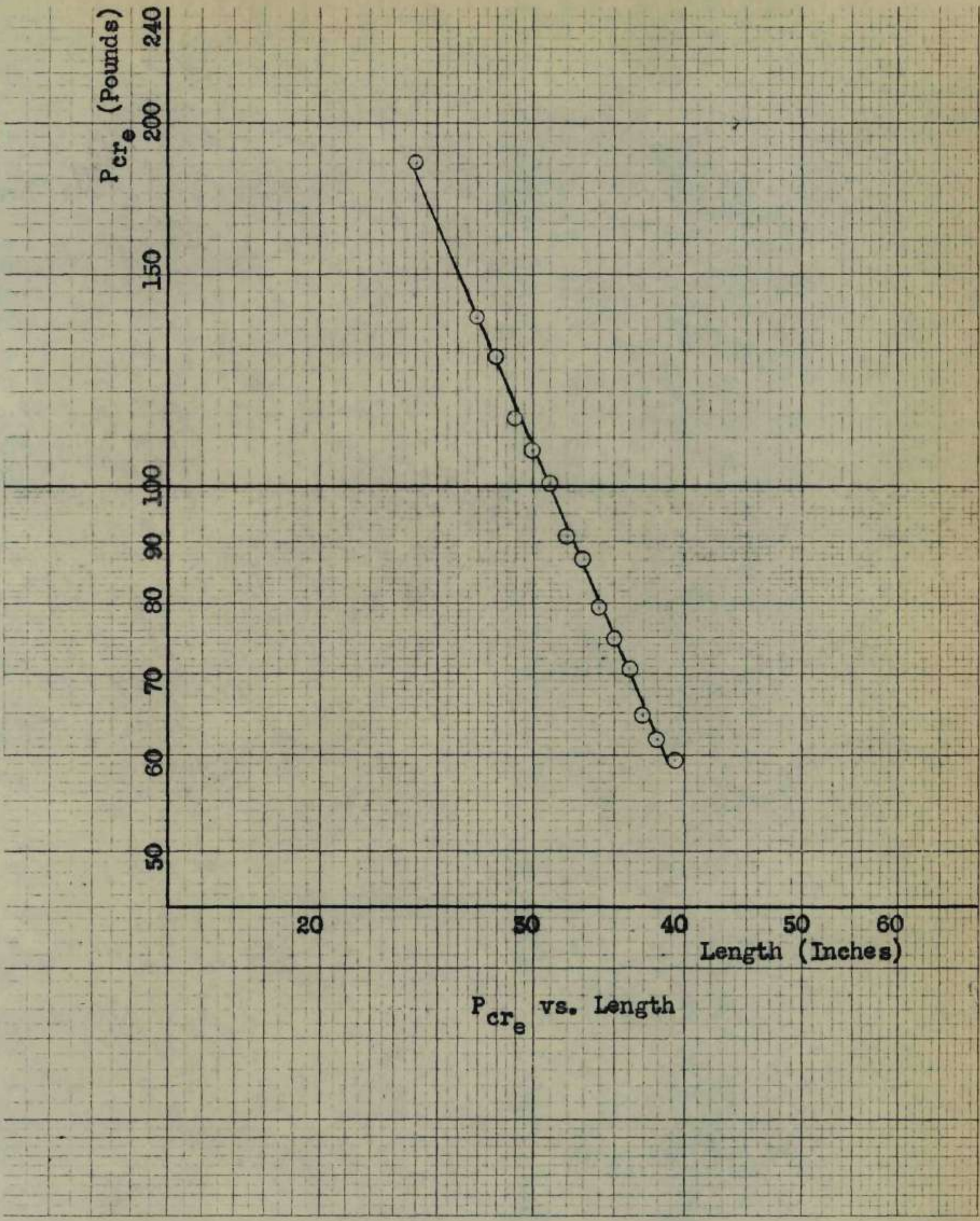

P_{cr} (Pounds)
240
200
150
100
90
80
70
60
50
20
30
40
50
60
Length (Inches)
P_{cr} vs. Length for Thin Rectangular Cross Sections

Fig. 16



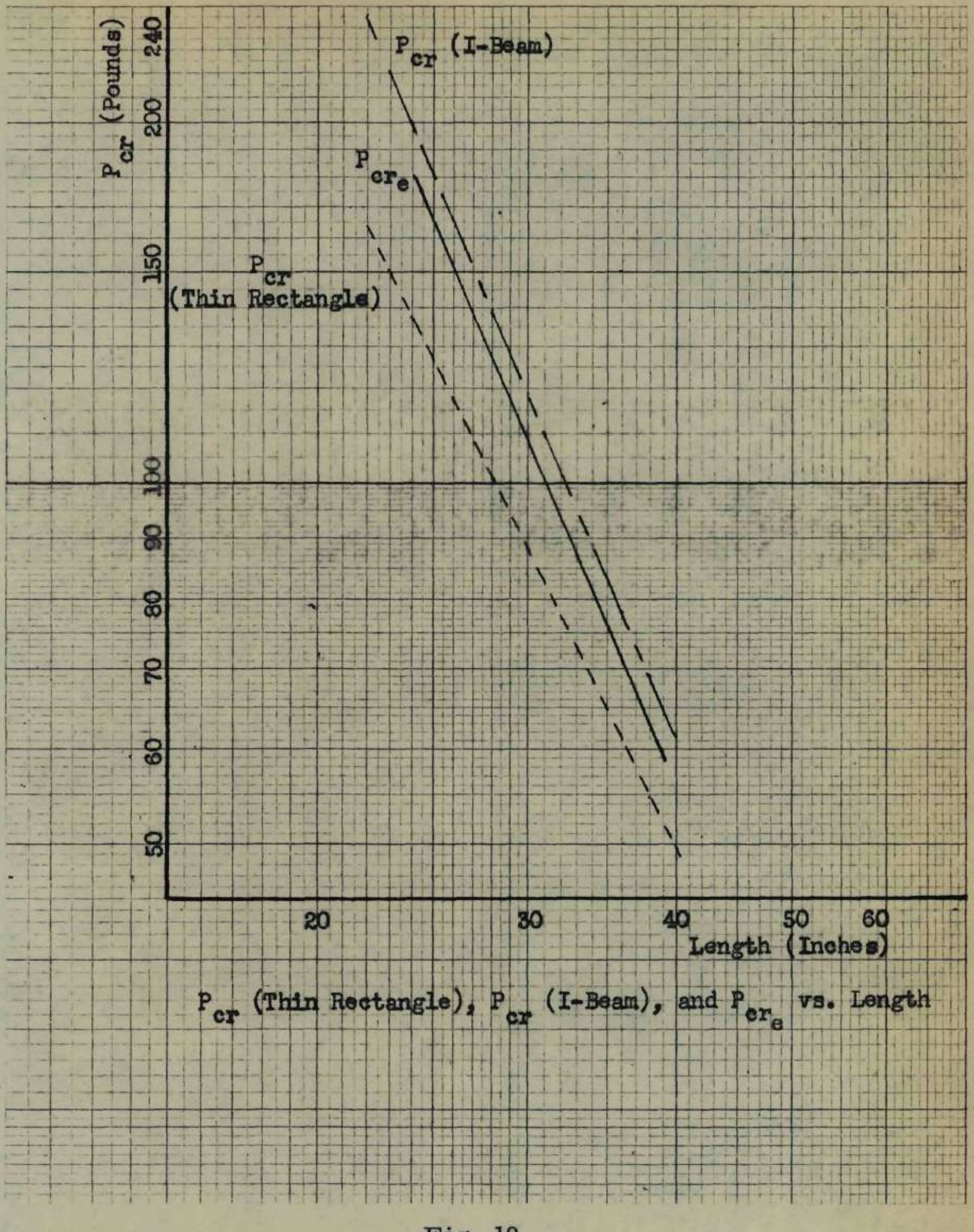
P_{cr} vs. Length for I-Beams

Fig. 17



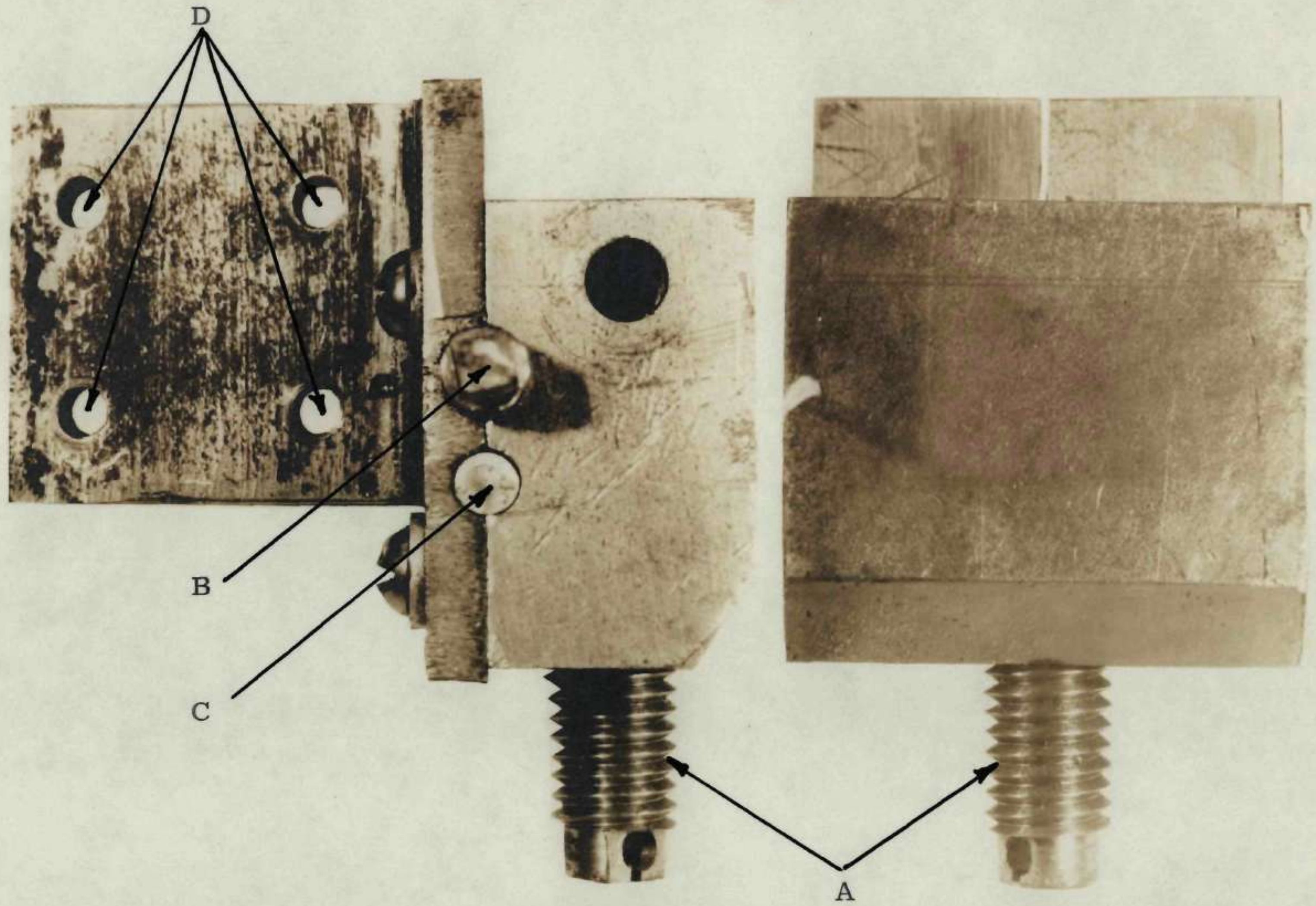
P_{cr_e} (Pounds)
240
200
150
100
90
80
70
60
50
20
30
40
50
60
Length (Inches)
P_{cr_e} vs. Length

Fig. 18



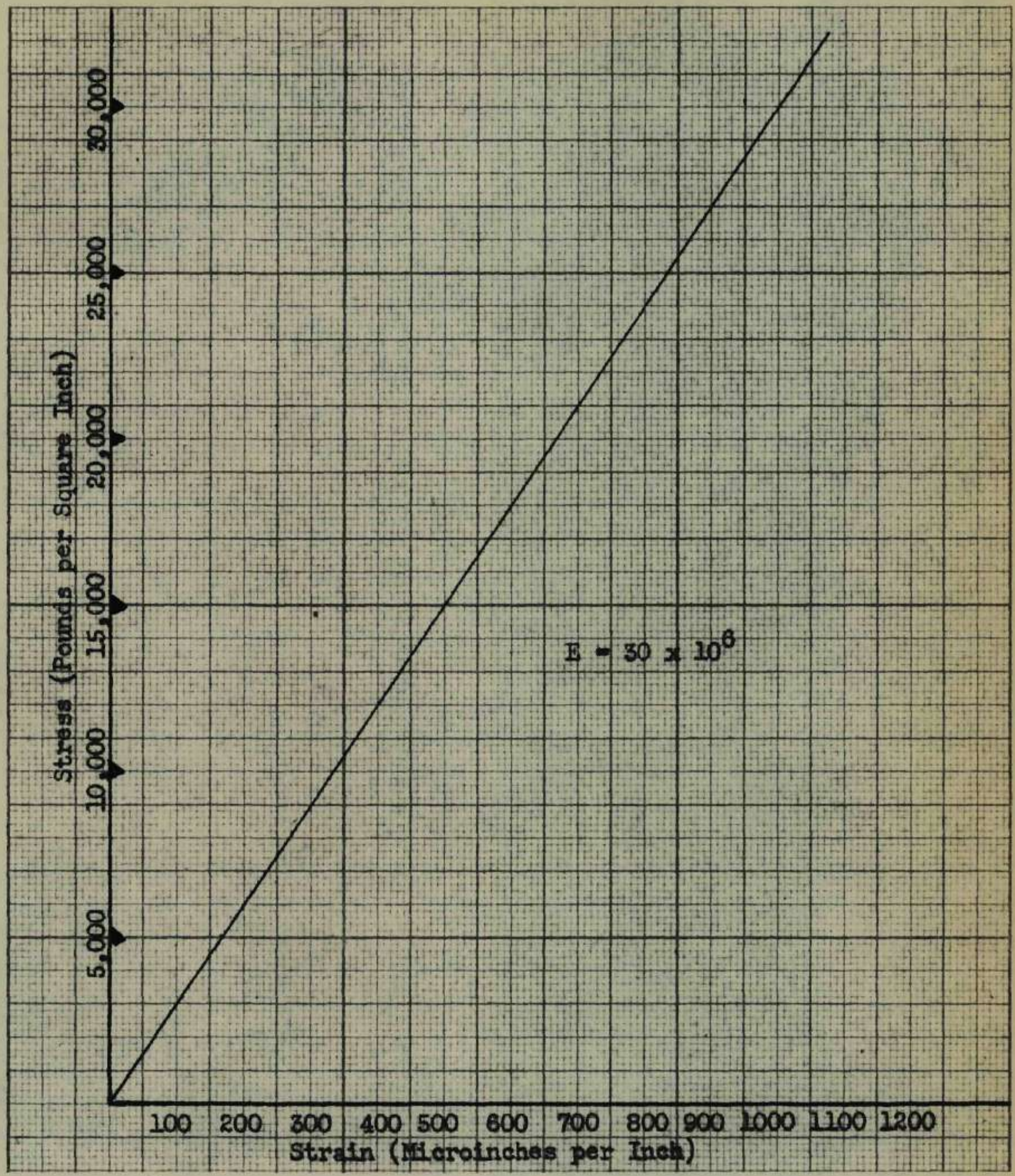
P cr (Pounds)
240
200
150
100
90
80
70
60
50
P cr (I-Beam)
P cr e
P cr
(Thin Rectangle)
20
30
40
50
60
Length (Inches)
P cr (Thin Rectangle), P cr (I-Beam), and P cr e vs. Length

Fig. 19



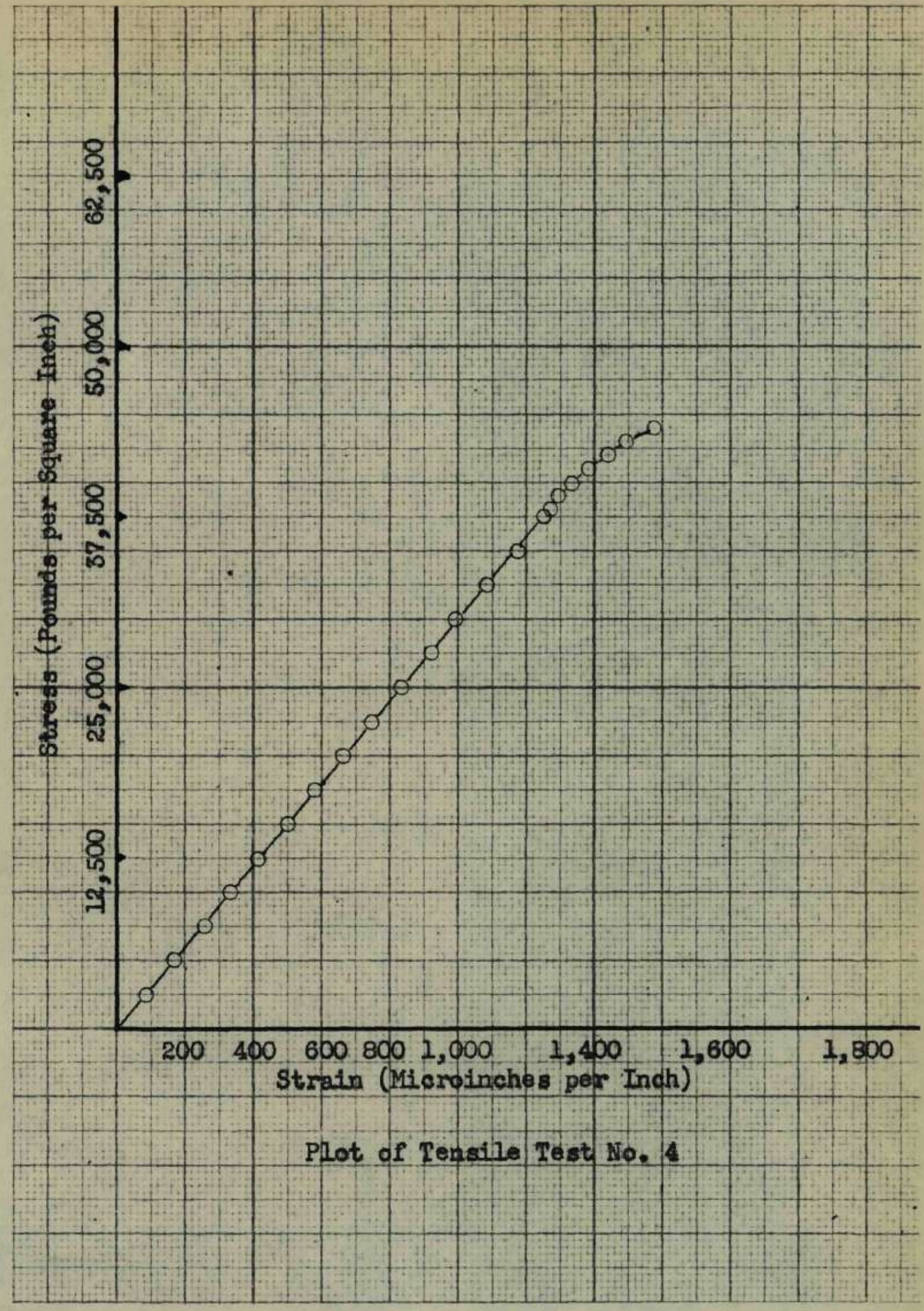
Loading Head

Fig. 20



Stress Versus Strain for Computation of Young's Modulus (E).

Fig. 21



Stress Versus Strain for Computation of Elastic Limit

Fig. 22

SAMPLE CALCULATIONS

1. $$I_1 = \left[\frac{1}{12}(1.593)(0.062)^3\right] + 2\left[\frac{1}{12}(0.082)(0.607)^3\right]$$
$$= 3,088 \times 10^{-6} \text{ in.}^4$$

2. $$I_2 = \left[\frac{1}{12}(0.607)(1.757)^3\right] - 2\left[\frac{1}{12}(0.2725)(1.593)^3\right]$$
$$= 90,800 \times 10^{-6} \text{ in.}^4$$

3. $$C = G\left[\frac{2}{3}(0.607)(0.082)^3 + \frac{1}{3}(1.757)(0.062)^3\right] \text{(where } G = 11.58 \times 10^6 \text{psi.)}$$
$$= 4,201 \text{ lb.-in.}^2$$

4. $$E = 30 \times 10^6 \text{ psi. (See Fig. 21)}$$

5. $$B_1 = (E)(I_1) = (30 \times 10^6)(3,088 \times 10^{-6})$$
$$= 92,950 \text{ lb.-in.}^2$$

6. $$B_2 = (E)(I_2) = (30 \times 10^6)(90,800 \times 10^{-6})$$
$$= 2,733,100 \text{ lb.-in.}^2$$

7. $$D = \frac{1}{2}B_1 = \frac{1}{2}(92,950)$$
$$= 46,475 \text{ lb.-in.}^2$$

8. $$a^2 = \frac{Dh^2}{2C} = \frac{(46,475)(1.757)^2}{(2)(4,201)}$$
$$= 17.07 \text{ in.}^2$$

9. $$m = \frac{4.01}{(1 - \frac{a}{L})^2} = \frac{4.01}{(1 - \frac{4.132}{30})^2}$$
$$= 5.39$$

10. $$P_{cr} = \frac{m\sqrt{B_1 C}}{L^2} \qquad \frac{5.39\sqrt{(92,950)(4,201)}}{(30)^2}$$
$$= 118.3 \text{ lbs.}$$

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