INCREMENTAL HYBRID FINITE ELEMENT METHODS
FOR FINITE DEFORMATION PROBLEMS
(WITH SPECIAL EMPHASIS ON COMPLEMENTARY ENERGY PRINCIPLE)

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INCREMENTAL HYBRID FINITE ELEMENT METHODS

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SUMMARY

Finite element models, based on a complementary energy principle, for the analysis of finite deformation (large strain and rotation) problems of nonlinear compressible as well as incompressible elastic solids are developed. To this end, general (Hu-Washizu type) variational principles, in the total Lagrangean formulation, based on various measures of stresses and their conjugate strains are first studied. With these general principles as the basis, various special forms of variational principles are derived. Especially, the possibility of constructing a stationary complementary energy principle for the finite deformation problem of elastic solids is examined. It is concluded, through this study, that only the general principle based on the Jaumann stress measure can lead to rational and practical complementary energy principle which involves, unlike in the linear theory, both the unsymmetric Piola-Lagrange stress and the rotation tensor as variables. In such a complementary energy principle, the rotational equilibrium condition is enforced as an a posteriori condition through the stationarity condition of the functional corresponding to variations in the rotation tensor. Considering the feasibility for the practical application, the incremental form of variational principles leading to piecewise linear incremental solutions is derived. Further, introducing the concept of hybrid

finite element models, which allow for the a priori relaxations of the traction reciprocity condition and the displacement continuity condition at inter-element boundaries, incremental hybrid type variational principles are derived. Especially, the incremental hybrid complementary energy principles both in the total Lagrangean and updated Lagrangean formulations are proposed. These proposed principles are employed in the context of the finite element method, and incremental hybrid stress finite element models are developed to solve plane-stress finite deformation problems of compressible elastic solids. On the other hand, for the case of incompressible materials, the hydrostatic pressure is introduced as a Lagrange multiplier, and Hu-Washizu principles in which the incompressibility condition is relaxed a priori, are constructed. Then, following the same procedure as for compressible materials, a modified (hybrid) incremental complementary energy principle is derived. Based on this variational principle, an incremental hybrid stress finite element model for plane-stress finite deformation analysis of incompressible elastic solids is developed. Using these newly developed finite element models, example problems of finite strain plane-stress deformations of compressible as well as incompressible nonlinear elastic solids are solved. The results obtained by the present methods agree excellently with those, in literature, which were obtained by the compatible displacement finite element model. Through the results of the example problems, it is confirmed that the presently developed methods are powerful numerical tools to solve finite deformation problems of nonlinear elastic solids. Thoese methods are also more efficient than those in the literature based on potential energy principles.

LIST OF SYMBOLS

$\underline{x}$	Position Vector in Undeformed State
$\underline{y}$	Position Vector in Deformed State
$\underline{u}$	Displacement Vector
∇	Gradient Operator
$\underset{\sim}{F}$	Deformation Gradient
$\underset{\sim}{\alpha}$	Rotation Tensor
$\underset{\sim}{h}$	Symmetric Right Extensional Strain Tensor or Engineering Strain Tensor
$\underset{\sim}{e}$	Displacement Gradient
$\underset{\sim}{G}$	Deformation Tensor
$\underset{\sim}{g}$	Green-Lagrange Strain Tensor
$\underset{\sim}{\tau}$	True or Cauchy Stress Tensor
$\underset{\sim}{t}$	Piola-Lagrange Stress Tensor (In the Present) or "Unsymmetric Piola Stress Tensor" or "First Piola-Kirchhoff Stress Tensor"
$\underset{\sim}{s}$	Kirchhoff-Trefftz Stress Tensor (In the Present) or "Symmetric Kirchhoff Stress" or "Second Piola-Kirchhoff Stress"
$\underset{\sim}{r}$	Jaumann Stress Tensor
W	Strain Energy Density
ρ	Mass Density
$\underline{g}$	Body force per Unit Mass
J	Determinant of Deformation Gradient $\underset{\sim}{F}$
$\_$	(Under Symbol) Vector
$\sim$	(Under Symbol) Second Order Tensor

$\underline{a} \cdot \underline{b}$	Product of Two Vectors : $\underline{a} \cdot \underline{b} = a_i b_i$ (a_i and b_i are Rectangular Cartesian Components)
$\underset{\sim}{A} \cdot \underset{\sim}{B}$	Product of Two Tensors
$\underset{\sim}{A} : \underset{\sim}{B}$	Tensor Inner Product ; $\underset{\sim}{A} : \underset{\sim}{B} = \mathrm{trace}(\underset{\sim}{A}^T \cdot \underset{\sim}{B})$ $= A_{ij} B_{ij}$

CHAPTER I

INTRODUCTION

In the last two decades, the finite element method has been recognized as a powerful numerical solution technique for linear elastic problems, and several different models have been developed. Most of these finite element models are based on the well known minimum potential energy principle or minimum complementary energy principle. These two minimum principles, in the linear theory, provide upper and lower bounds for approximate numerical solutions. Also, mixed models based on Hellinger-Reissner principle are sometimes used. These variational principles* are summarized in a comprehensive work by Washizu [1]. In his work, it is shown that the above three variational principles can be systematically derived from the general principle, which is referred to as the "Hu-Washizu Principle".

Meanwhile, demands for solution techniques to analyze nonlinear behavior of structures have increased, and great efforts have been expended by many scientists and engineers to develop such numerical methods. The essential sources of nonlinearities are categorized in two parts. One is material

*From the literal meaning, it may be proper to use "variational theorem" instead of "variational principle". However, as often found in the literature, "principle" is used as an equivalent word to "theorem" in this thesis.

nonlinearity and the other is geometrical nonlinearity. The material nonlinearity alone does not bring significant change to the framework of the finite element scheme developed for linear elastic problems. However, if large deformation is considered, the geometrical nonlinearity brings several features which do not appear in linear theory. First of all, because of the large deformation, deformed and undeformed configurations must be clearly distinguished. Also, in the study of solid mechanics, in which we are interested in each material point in the solid body (Lagrangean Description) rather than a point in space (Eulerian Description), we need to introduce a reference configuration which will serve as a material co-ordinate system. The choice of this reference configuration is rather arbitrary. It can be an undeformed configuration, or it can be any intermediate known configuration. The first choice of the reference frame is often called Total or Stationary Lagrangean description. The second case is called Updated Lagrangean description, especially when it is used in incremental formulations.

Another feature of finite deformation analyses is the fact that several different stresses and their conjugate strains can be defined for finite deformation problems, namely, the unsymmetric Piola-Lagrange (First Piola-Kirchhoff) stress, symmetric Kirchhoff-Trefftz (Second Piola-Kirchhoff) stress, and the symmetric Jaumann stress, and their conjugate strains ; displacement gradient, Green-Lagrange strain, and right extensional strain tensor, respectively. These are discussed in chapter II. Therefore, because of the choice of the reference configuration and the definition of stress and strain tensors, quite a few different types of variational formulations are possible. Based on these variational principles, numerous number of finite element models have been reported. Comprehensive surveys of various aspects of the finite element methods for finite deformation problems were presented by Washizu [1], Nemat-Nasser and his co-workers [2, 3], Horrigmoe and Bergan [4], and Horrigmoe [5]. Most of these finite element models are based on the stationary potential energy principle or Hellinger-Reissner type principle. But hardly any stress model, strictly based on the complementary energy principle, can be found in the literature. The reason for the lack of stress models in literature is the controversy on the uniqueness of the inverse stress-strain relation, which is assumed in the complementary energy principle proposed by Levinson [6].

The main objective of the present thesis is to develop practical "stress finite element models" for finite deformation problems based on rational complementary energy principles, and to demonstrate their validity through proper numerical examples. The study of the complementary energy principle can be traced back to the work by Hellinger [7], which is considered as a landmark. This topic has attracted attentions of many researchers. Especially, in recent years, significant progress has been made, as seen from the recent works of Zubov [8], Fraeijs de Veubeke [9], Koiter [10, 11], Christoffersen [12], Dill [13] , and Atluri and Murakawa [14]. To begin with, these works are reviewed and the possibility of constructing rational and practical complementary energy principles is discussed in chapter III. For this purpose, general variational principles (Hu-Washizu principles) based on alternate definitions of stress and strain measures in total Lagrangean formulation are constructed. Then, by a priori satisfying some of the field equations and boundary conditions, these general principles are shown to be reduced to stationary potential energy principles, Hellinger-Reissner principles, or, if possible, complementary energy principles. However, as pointed out by Fraeijs de Veubeke [9], if the Kirchhoff-Trefftz stress is used in the variational formulations, the derived complementary energy principle involves both stress and displacement. And also, the a priori satisfaction of the translational equilibrium condition and the traction boundary condition, which are nonlinear in stress and displacement, is nearly impossible, in general. Thus the complementary energy principle based on the Kirchhoff-Trefftz stress fails to be a practically useful principle. On the other hand, if the Piola-Lagrange stress is used, the translational equilibrium condition and the traction boundary condition become linear equations involving stress alone. It is easy to satisfy these conditions a priori. Thus, if the complementary energy density in terms of the Piola-Lagrange stress exists, a complementary energy principle involving stress alone can be derived as shown by Levinson [5]. However, as pointed out by Truesdell and Noll [15] and recently by Dill [13], in general, the inverse of the stress-strain relation in terms of the Piola-Lagrange stress and the displacement gradient, which leads to the complementary energy density, is multi-valued. In the case of isotropic "semi-linear" materials, Zubov [8] attempts to establish unique inverse of the stress-strain relation. But, his arguments are refuted by Dill [13] and others who show that the inverse relation can be multi-valued. Meanwhile, Koiter, [11], proving the existence of the multi-valued inverse relation, proceeds to establish certain sufficient conditions for the validity of the minimum complementary energy principle using the complementary energy involving the Piola-Lagrange stress alone. Although, it can be used to solve simple problems in an analytical way, such a complementary energy principle involving the multi-valued inverse stress-strain relation can not be applied to a numerical method such as the finite element method. Moreover, there is an ambiguity on the satisfaction of the rotational equilibrium condition, which is nonlinear in Piola-Lagrange stress and displacement. These difficulties and ambiguities pointed out in the complementary energy principle based on the Kirchhoff-Trefftz stress or Piola-Lagrange stress can be avoided if the Jaumann stress is used. First of all, the inverse stress-strain relation in terms of the Jaumann stress and the right extensional strain tensor (engineering strain) is unique. Thus the complementary energy density function in terms of the Jaumann stress alone can be achieved. The Jaumann stress can be decomposed into the Piola-Lagrange stress and the rotation tensor. Further, the translational equilibrium condition and the traction boundary condition in terms of the Piola-Lagrange stress can be satisfied a priori. Thus, as discussed by Fraeijs de Veubeke [9] and Christoffersen [12], we can derive the most consistent and useful complementary energy principle involving both Piola-Lagrange stress and rotation tensor. In this type of complementary energy principle, the rotational equilibrium condition can be retained as an a posteriori condition through the stationarity condition of the functional with respect to the rotation. These complementary energy principles as well as other special variational principles derivable from the Hu-Washizu principles based on alternate stress and strain measures in total Lagrangean formulation, are summarized in chapter III.

The variational principles presented in chapter III can be applied to a finite element model. Such a model leads to a system of nonlinear algebraic equations in terms of unknown parameters, which are usually solved by Newton-Raphson method. However, depending on constitutive relations, the derived nonlinear equations, sometimes, become extremely complicated. To avoid this kind of algebraic complexity, incremental formulations, which lead to linear equations, are considered. For this purpose, incremental variational principles based on alternate stress and its conjugate strain measure, both in total Lagrangean and updated Lagrangean formulations, are derived in chapter IV.

In all these incremental variational principles including the incremental complementary energy principles, only functions, which satisfy required continuity conditions in the domain occupied by solid body, are allowed as admissible functions. For example, displacements must be continuous within the solid and the traction across any surface within the solid must be continuous (traction reciprocity). However, in the finite element formulations, the solid body is divided into a finite number of subdomains, which are called elements, and field variables are assumed in each element. In this situation, the required continuity conditions in the element can be easily satisfied by simply choosing continuous functions for these variables. But, in addition, these continuity conditions must be satisfied on interelement boundaries. In some cases, it is practically difficult to choose properly assumed functions which satisfy these interelement continuities. To deal with this difficult situation, the concept of "Hybrid Model" is introduced [16]. The hybrid finite element model is defined as a finite element model based on a modified (or hybrid) variational principle in which the constraints of displacement continuity and/or traction reciprocity condition at the interelement boundaries are relaxed a priori by using Lagrange multipliers. Thus, it leads to more versatility in choosing functions for displacement and/or stress in the element. The required continuity conditions are enforced a posteriori, at least in a weighted residual sense, through the stationarity condition of the modified functional with respect to Lagrange multipliers. Thus, functionals associated with various types of variational principles are further modified, and modified (hybrid) incremental functionals are constructed. Especially, incremental hybrid complementary energy principles which involve incremental Piola-Lagrange stress and rotation tensors , both in total Lagrangean and updated Lagrangean formulations, are proposed. Based on the proposed variational principle, an incremental hybrid stress finite element model in total Lagrangean formulation is developed. The detailed discussion of the finite element formulation for the analysis of finite deformation elastic problem is presented in chapter V. Using the newly developed method, an example problem of biaxial stretching of a thin sheet made of Blatz-Ko type [17] nonlinear elastic material is solved, and the numerical results are discussed.

It is known that, among the engineering materials which can deform in a large scale, many of them, such as rubbers, polymers, and solid-propellant rocket grains, are considered to be nearly or precisely incompressible. In the closed-form analysis, the incompressibility condition makes it easier to obtain solutions for certain simple problems

[46]. However, this is not the case for numerical methods based on energy type variational principles. An essential difference between compressible and incompressible materials is the fact that the stress can be determined by strain in the former case, whereas, in the latter case, the stress can not be fully determined by strain alone, and the hydrostatic pressure remains as an unknown. This implies that the complete stress-strain relation of the incompressible material can not be characterized by the usual strain energy density, as that for compressible materials, which is a function of strain alone. Moreover, in the case of incompressible materials the strain field must satisfy the incompressibility condition, which is, in general, nonlinear. The a priori satisfaction of this condition for the general case is practically impossible. Therefore, the variational principles derived for compressible materials are not valid for the incompressible case.

Some alternative approaches are suggested by Herrmann [19] and Key [20] , for linear elastic small deformation problems, and also by Oden [18] for the finite deformation problems. They introduce the hydrostatic pressure as a variable, and construct potential energy type or Hellinger-Reissner type variational principles, which are valid for nearly or precisely incompressible materials. In the present work, a complementary energy principle is used to solve finite elasticity problems of incompressible materials. First, by introducing the hydrostatic pressure as a Lagrange multiplier, functionals associated with Hu-Washizu principles in which the incompressibility condition is relaxed a priori through the Lagrange multiplier are constructed using alternate stress and strain measures. Then, from the Hu-Washizu principle based on the Jaumann stress, an incremental complementary energy principle and also its modified (hybrid) version are derived. Specifically, an incremental hybrid complementary energy principle is applied to the finite element method and a incremental hybrid stress model is developed. This proposed method is applied to solve finite strain plane stress problems for a nonlinear incompressible material of Mooney-Rivlin type [21]. Numerical results for biaxial stretching of a plane square sheet and a square sheet with centrally located circular hole are presented. The validity of the proposed method is demonstrated through comparison with the numerical results obtained by the displacement finite element model (Oden [18]).

The conclusions drawn from this study and recommendations for further study are given in chapter VII.

CHAPTER II

BASIC FORMULATIONS

Introduction

In the study of solid mechanics, we are interested in the state variables at material points of deformed solids, such as stress and strain. Thus, Lagrangean description is adopted to describe the behavior of solids. In this description, all the state variables are described as functions of material co-ordinates. In the case of linear theory, in which there is no distinction between deformed and undeformed configurations, usually, undeformed (equivalent to deformed) configuration is taken as a reference. The components of the position vector of the material point in the reference configuration are used as material co-ordinates to identify each material point. However, in the case of finite deformation problem, the undeformed and deformed configurations must be distinguished. Consequently, our choice of the reference becomes arbitrary. It can be the undeformed configuration, also it can be any intermediate known deformed configuration. If the undeformed configuration is chosen as a reference, it is called total or stationary Lagrangean description. If an intermediate deformed configuration is used, it is called updated Lagrangean description, especially when it is used in an incremental formulation. On the other hand, for the finite deformation problems, we can define stress and its conjugate strain in several different ways, so that the governing equations written in terms of these can be reduced to convenient mathematical forms.

As it is noticed, there are several different aspects involved in finite deformation problems. Also, notations are quite different from one author to another. Therefore, to avoid confusions due to notations, and to make the definitions consistent throughout the thesis, the definitions of alternate stress and strain measures are presented in this chapter. In connection with the definitions of these field variables, constitutive relations and the governing equations for finite deformation problems in terms of alternate stress and strain measures are also presented for both total and updated Lagrangean descriptions. Direct tensor notation, which is considered to be the most general way to describe the problem of solid mechanics is used for this purpose. The details of the direct tensor notation used in this thesis are given in the appendix A.

Total Lagrangean Description

Geometry of Deformed Solid

Consider a solid body in three-dimensional Euclidean space , as shown by Fig.1. The initial (stress free) configuration is denoted by C_0 and its volume and surface are denoted by V_0 and S_0. Similarly, the deformed (current) configuration is denoted by C, and V and S are its volume and surface. Since the initial configuration is taken as a reference, the material point P, which has a position P_0 in the initial configuration is identified by its position vector $\underline{x}$. The same material point moves to P in the deformed configuration through deformation of the body. Its position vector is denoted by vector $\underline{y}$. Thus, the displacement vector $\underline{u}$ is defined by,

$$\underline{u} = \underline{y} - \underline{x} = (y_i - x_i)\underline{e}_i \tag{2.1}$$

where y_i and x_i are rectangular Cartesian components, and $\underline{e}_i$ are unit base vectors. If $\underline{y}(\underline{x})$ is assumed to be differentiable with respect to $\underline{x}$, the deformation gradient $\underset{\sim}{F}$ is defined by,

$$\underset{\sim}{F} = (\nabla \underline{y})^T \tag{2.2}$$

or in rectangular Cartesian components,

$$F_{ij} = \frac{\partial y_i}{\partial x_j} \tag{2.3}$$

where the symbol ∇ denotes the gradient in the metric in C_0; and in the present notation, vectors and second order tensors are denoted by _ and ~ under symbols, respectively.

Definition of Strain Measures

The deformation gradient $\underset{\sim}{F}$ is not singular. It can be uniquely decomposed into right polar-decomposition,

$$\underset{\sim}{F} = \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \tag{2.4}$$

where $(\underset{\sim}{I} + \underset{\sim}{h})$ is a symmetric, positive definite tensor; $\underset{\sim}{I}$ is the identity tensor; and $\underset{\sim}{\alpha}$ is an orthogonal tensor, such that,

$$\underset{\sim}{\alpha}^T \cdot \underset{\sim}{\alpha} = \underset{\sim}{I} \tag{2.5}$$

Physically, the above decomposition means the separation of the deformation gradient into rigid body rotation and stretching. Thus, tensor $(\underset{\sim}{I}+\underset{\sim}{h})$ is called stretch tensor, and tensor $\underset{\sim}{h}$ is called right extensional strain tensor which provides one definition of strain. And $\underset{\sim}{\alpha}$ is called rotation tensor. Another strain measure is given by displacement gradient $\underset{\sim}{e}$, which is defined by,

$$\Delta \underset{\sim}{e} = (\nabla \underline{u})^T \tag{2.6}$$

A deformation tensor $\underset{\sim}{G}$ is defined by,

$$\underset{\sim}{G} = \underset{\sim}{F}^T \cdot \underset{\sim}{F} = (\underset{\sim}{h} + \underset{\sim}{I})^2 \tag{2.7}$$

Using deformation tensor $\underset{\sim}{G}$, the Green-Lagrange strain tensor $\underset{\sim}{g}$ is defined as,

$$\underset{\sim}{g} = 1/2 \; (\underset{\sim}{G} - \underset{\sim}{I}) = 1/2 \; \{ \nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u} \cdot \nabla \underline{u}^T \} \tag{2.8}$$

Thus, we defined three strain measures, namely, right extensional strain tensor, displacement gradient, and Green-Lagrange strain tensor. These strain tensors are related by,

$$\underset{\sim}{g} = 1/2 \; (\underset{\sim}{e} + \underset{\sim}{e}^T + \underset{\sim}{e}^T \cdot \underset{\sim}{e}) \tag{2.9}$$

$$\underset{\sim}{g} = 1/2\ (\ 2\underset{\sim}{h} + \underset{\sim}{h}\cdot\underset{\sim}{h}\ ) \tag{2.10}$$

Definition of Stress Measures

Following Truesdell and Noll [15], and Fraeijs de Veubeke [9], unsymmetric Piola-Lagrange stress tensor $\underset{\sim}{t}$, and symmetric Kirchhoff-Trefftz stress tensor $\underset{\sim}{s}$ are defined in terms of Cauchy or true stress $\underset{\sim}{\tau}$ in the deformed body, through the following relations,

$$\underset{\sim}{\tau} = (1/J)\ \underset{\sim}{F}\cdot\underset{\sim}{t} = (1/J)\ \underset{\sim}{F}\cdot\underset{\sim}{s}\cdot\underset{\sim}{F}^{T} \tag{2.11}$$

or inversely,

$$\underset{\sim}{t} = J\ \underset{\sim}{F}^{-1}\cdot\underset{\sim}{\tau} \tag{2.12}$$

$$\underset{\sim}{s} = J\ \underset{\sim}{F}^{-1}\cdot\underset{\sim}{\tau}\cdot(\underset{\sim}{F}^{-1})^{T} \tag{2.13}$$

and,

$$\underset{\sim}{t} = \underset{\sim}{s}\cdot\underset{\sim}{F}^{T} \tag{2.14}$$

where J is the determinant of $\underset{\sim}{F}$. Further, symmetric Jaumann stress tensor $\underset{\sim}{r}$ is defined by,

$$\underset{\sim}{r} = 1/2\ (\ \underset{\sim}{t}\cdot\underset{\sim}{a} + \underset{\sim}{a}^{T}\cdot\underset{\sim}{t}^{T}) \tag{2.15}$$

$$= 1/2\ \{\underset{\sim}{s}\cdot(\underset{\sim}{I}+\underset{\sim}{h}) + (\underset{\sim}{I}+\underset{\sim}{h})\cdot\underset{\sim}{s}\}$$

It is worth noting here that tensors $\underset{\sim}{s}$, $\underset{\sim}{g}$, and $\underset{\sim}{h}$ become coaxial for isotropic material. Thus, Eq.(2.15) is simplified and reduced to,

$$\underset{\sim}{r} = \underset{\sim}{t}\cdot\underset{\sim}{a} = \underset{\sim}{s}\cdot(\underset{\sim}{I}+\underset{\sim}{h}) \tag{2.16}$$

It is noted that the Piola-Lagrange stress $\underset{\sim}{t}$ defined by Eq.(2.11) corresponds to the transpose of that defined by

Truesdell [15]. Physical meanings of stress and strain measures defined in the above are presented in appendix B.

Constitutive Relations

Only an elastic material is considered in this section. If material is elastic, strain energy density function W, per unit undeformed volume, can be expressed as a function of Green-Lagrange strain $\underset{\sim}{g}$ alone. Further, it is assumed to be a symmetric function of $\underset{\sim}{g}$, so that the rotational equilibrium condition is embedded,

$$W(\underset{\sim}{g}) = W(\underset{\sim}{g}^T) \quad , \qquad \frac{\partial W}{\partial \underset{\sim}{g}^T} = \frac{\partial W}{\partial \underset{\sim}{g}} \tag{2.17}$$

Also, using Eq.(2.9), W can be expressed as a function of $\underset{\sim}{e}$:

$$W(\underset{\sim}{g}) = W[\underset{\sim}{g}(\underset{\sim}{e})] \tag{2.18}$$

Now, consider the variation of strain energy (virtual work) per unit undeformed volume, δW, which is given as,

$$\delta W = J\, \underset{\sim}{\tau} : \delta \underset{\sim}{F}^{*T} \tag{2.19}$$

where, $\delta \underset{\sim}{F}^* = \dfrac{\partial \delta y_i}{\partial y_j} \underline{e}_i \underline{e}_j = (\nabla \delta \underline{y})^T \cdot (\nabla \underline{y}^{-1})^T$

Using the definitions of stresses, Eqns.(2.12) and (2.13), it is reduced to,

$$\delta W = J(\nabla \underline{y}^{-1})^T \cdot \underset{\sim}{\tau} : (\nabla \delta \underline{y}) = \underset{\sim}{t} : \delta \underset{\sim}{e}^T = \underset{\sim}{s} : \delta \underset{\sim}{g} \tag{2.20}$$

On the other hand, from Eqns.(2.17) and (2.18),

$$\delta W = \frac{\partial W}{\partial \underset{\sim}{g}} : \delta \underset{\sim}{g} = \frac{\partial W}{\partial \underset{\sim}{e}} : \delta \underset{\sim}{e} \tag{2.21}$$

By comparing Eqns.(2.20) and (2.21), the following relations are obtained.

$$\frac{\partial W}{\partial \underset{\sim}{g}} = \underset{\sim}{S} \tag{2.22}$$

$$\frac{\partial W}{\partial \underset{\sim}{e}} = \underset{\sim}{t}^{T} \tag{2.23}$$

Further, using the relation,

$$\delta \underset{\sim}{g} = 1/2 \; [\; (\underset{\sim}{I} + \underset{\sim}{h}) \cdot \delta \underset{\sim}{h} + \delta \underset{\sim}{h} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \;] \tag{2.24}$$

Eq.(2.21) is rewritten as,

$$\delta W = \frac{1}{2} \{ \underset{\sim}{S} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) + (\underset{\sim}{I} + \underset{\sim}{h}) \cdot \underset{\sim}{S} \} : \delta \underset{\sim}{h} \tag{2.25}$$

Thus, the Jaumann stress $\underset{\sim}{r}$ is related to the strain energy density W by,

$$\frac{\partial W}{\partial \underset{\sim}{h}} = \underset{\sim}{r} \tag{2.26}$$

In fact, Fraeijs de Veubeke [9] defined Jaumann stress through the strain energy function as shown in the above.

For later use, we consider the inverse of the constitutive relations. As discussed by Fraeijs de Veubeke [9], the stress-strain relations given by Eqns.(2.22) and (2.26) are, in general, invertible, and the following contact transformations are achieved.

$$S(\underset{\sim}{s}) = \underset{\sim}{s} : \underset{\sim}{g}(\underset{\sim}{s}) - W[\underset{\sim}{g}(\underset{\sim}{s})] \tag{2.27}$$

$$R(\underset{\sim}{r}) = \underset{\sim}{r} : \underset{\sim}{h}(\underset{\sim}{r}) - W[\underset{\sim}{h}(\underset{\sim}{r})] \tag{2.28}$$

such that,

$$\frac{\partial S}{\partial \underset{\sim}{s}} = \underset{\sim}{g} \qquad (2.29)$$

$$\frac{\partial R}{\partial \underset{\sim}{r}} = \underset{\sim}{h} \qquad (2.30)$$

However, as noted by Truesdell and Noll [15] and more recently by Dill [13], unique inverse for Eq.(2.23) does not exist for general cases. In the case of semi-linear materials, Zubov [8] attempts to establish such an inverse relation. However, his arguments are refuted by Dill [13] and others who show that the inverse can be multi-valued. Following Dill [13], we closely investigate the inverse stress-strain relation in terms of $\underset{\sim}{t}$ and $\underset{\sim}{e}$. We assume that the material is isotropic and the Piola-Lagrange stress $\underset{\sim}{t}$ is given. Since material is isotropic, stress $\underset{\sim}{t}$ can be decomposed into the Jaumann stress $\underset{\sim}{r}$ and the rotation $\underset{\sim}{\alpha}$ as shown by Eq.(2.16).

$$\underset{\sim}{t} = \underset{\sim}{r} \cdot \underset{\sim}{\alpha}^T \qquad (2.16)^*$$

where $\underset{\sim}{r}$ is symmetric and $\underset{\sim}{\alpha}$ is orthogonal. Using Eq.(2.30), the Jaumann stress $\underset{\sim}{r}$ is uniquely related to the strain tensor $\underset{\sim}{h}$. Thus the strain $\underset{\sim}{h}$ is calculated and we can obtain the displacement gradient by,

$$\underset{\sim}{e} = \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) - \underset{\sim}{I} \qquad (2.31)$$

However, unlike in Eq.(2.4), the decomposition in Eq.(2.16)* is not unique because the tensor $\underset{\sim}{r}$ is only required to be symmetric. This can be seen from a simple example. We consider a semi-linear material, the strain energy density for which is given by,

$$W(\underset{\sim}{h}) = 1/2\ \lambda\,(\underset{\sim}{h} : \underset{\sim}{I})^2 + \mu(\underset{\sim}{h} : \underset{\sim}{h}) \tag{2.32}$$

such that,

$$\frac{\partial W}{\partial \underset{\sim}{h}} = \underset{\sim}{r} = \lambda(\underset{\sim}{h} : \underset{\sim}{I})\underset{\sim}{I} + 2\mu\underset{\sim}{h} \tag{2.33}$$

For simplicity, λ is assumed to be zero. Then Eq.(2.33) is reduced to,

$$\underset{\sim}{r} = 2\mu\underset{\sim}{h} \tag{2.34}$$

Suppose stress $\underset{\sim}{t}$ is given as,

$$\underset{\sim}{t} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \tag{2.35}$$

For the given $\underset{\sim}{t}$, the following decompositions are considered,

$$\underset{\sim}{t} = \underset{\sim}{r}_1 \cdot \underset{\sim}{a}_1^T = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.36}$$

$$\underset{\sim}{t} = \underset{\sim}{r}_2 \cdot \underset{\sim}{a}_2^T = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.37}$$

The strain tensors corresponding to the above Jaumann stresses are calculated to be,

$$\underset{\sim}{h}_1 = \frac{1}{2\mu}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \quad , \qquad \underset{\sim}{h}_2 = \frac{1}{2\mu}\begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \tag{2.38, 2.39}$$

Thus, it is seen that the inverse of Eq.(2.23) is multi-valued. Further, it is interesting to notice that the two strain fields obtained above satisfy the rotational equilibrium condition, which requires the symmetry of $\underset{\sim}{\tau}$ ($\underset{\sim}{\tau} = \frac{1}{J}\, \underset{\sim}{F} \cdot \underset{\sim}{t}$);

$$\underset{\sim}{F}_1 \cdot \underset{\sim}{t} = \underset{\sim}{a}_1 \cdot (\underset{\sim}{h}_1 + \underset{\sim}{I}) \cdot \underset{\sim}{t} = a \begin{bmatrix} \frac{a}{2\mu}+1 & 0 & 0 \\ 0 & \frac{a}{2\mu}+1 & 0 \\ 0 & 0 & \frac{a}{2\mu}+1 \end{bmatrix} = \text{symmetric} \tag{2.40}$$

$$\underset{\sim}{F}_2 \cdot \underset{\sim}{t} = \underset{\sim}{a}_2 \cdot (\underset{\sim}{h}_2 + \underset{\sim}{I}) \cdot \underset{\sim}{t} = a \begin{bmatrix} \frac{a}{2\mu}+1 & 0 & 0 \\ 0 & \frac{a}{2\mu}-1 & 0 \\ 0 & 0 & \frac{a}{2\mu}-1 \end{bmatrix} = \text{symmetric}$$

This example suggests that the rotational equilibrium condition alone is not enough to identify the strain field for the given stress $\underset{\sim}{t}$. As mentioned by Koiter [11], by considering the global deformation, it may be possible to select proper value among the multi values. However, it is practically impossible to select the correct inverse in the numerical solution process.

Field Equations and Boundary Conditions

Full mathematical description of the finite deformation problem of solid can be given by a complete set of field equations and proper boundary conditions, namely,

(a) translational equilibrium condition

(linear momentum balance)

(b) rotational equilibrium condition

(angular momentum balance)

(c) strain-displacement relation (kinematic relations)

(d) stress-strain relations (constitutive relations)

(e) displacement boundary conditions and/or traction boundary conditions and/or mixed boundary conditions.

The equilibrium conditions are essentially described in the deformed configuration in terms of true stress $\underset{\sim}{\tau}$. The translational equilibrium condition is expressed by,

$$\nabla^d \cdot \underset{\sim}{\tau} + \rho^d \underline{g} = 0 \tag{2.41}$$

where $\nabla^d\cdot$ represents divergence with respect to the metric in the deformed configuration; ρ^d is the mass density in the deformed configuration; and $\underline{g}$ is body force per unit mass. The rotational equilibrium condition is given as a symmetric property of tensor $\underset{\sim}{\tau}$,

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}^T \tag{2.42}$$

By using the geometrical relations and definitions of stresses, Eqns(2.41) and (2.42) are rewritten in terms of state variables defined in the undeformed configuration.

translational equilibrium conditions

$$\nabla \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^{T}) + \rho_0 \underline{g} = 0 \tag{2.43}$$

or $$\nabla \cdot \underset{\sim}{t} + \rho_0 \underline{g} = 0 \tag{2.44}$$

where ρ_0 is the mass density measured in the undeformed configuration.

rotational equilibrium conditions

$$\underset{\sim}{s}^{T} = \underset{\sim}{s} \tag{2.45}$$

$$\underset{\sim}{F} \cdot \underset{\sim}{t} = \underset{\sim}{t}^{T} \cdot \underset{\sim}{F}^{T} \tag{2.46}$$

$$(\underset{\sim}{h} + \underset{\sim}{I}) \cdot \underset{\sim}{t} \cdot \underset{\sim}{\alpha} = \text{symmetric} \tag{2.47}$$

Kinematic relations in terms of alternate strain measure are given by the following equations,

kinematic relations

$$\underset{\sim}{g} = 1/2 (\underset{\sim}{F}^{T} \cdot \underset{\sim}{F} - \underset{\sim}{I}) \tag{2.48}$$

$$\underset{\sim}{e} = (\nabla \underline{u})^{T} \tag{2.49}$$

$$\underset{\sim}{F} = \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \tag{2.50}$$

Assuming the existence of the strain energy density function W in terms of $\underset{\sim}{g}$, constitutive relations are expressed as,

constitutive relations

$$\underset{\sim}{s} = \frac{\partial W}{\partial \underset{\sim}{g}} , \tag{2.51}$$

$$\underset{\sim}{t}^{T} = \frac{\partial W}{\partial \underset{\sim}{e}} \quad (2.52) \qquad , \qquad \underset{\sim}{r} = \frac{\partial W}{\partial \underset{\sim}{h}} \quad (2.53)$$

Further, the inverse relations of Eqns.(2.51) and (2.53) can be obtained through the complementary energy density functions defined by Eqns.(2.27) and (2.28) as,

$$\underset{\sim}{g} = \frac{\partial S}{\partial \underset{\sim}{s}} \quad (2.54) \qquad , \qquad \underset{\sim}{h} = \frac{\partial R}{\partial \underset{\sim}{r}} \quad (2.55)$$

However, unique inverse of Eq.(2.52) does not exist for general cases.

<u>boundary conditions</u>

Stress boundary conditions and displacement boundary conditions are given by the following equations,

(a) $\bar{\underline{t}} = \underline{n} \cdot \underset{\sim}{t} = \underline{n} \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^{T})$ at S_{σ_0} (2.56)

where $\underline{n}$ is an unit normal to the surface S_{σ_0} where tractions are prescribed to be $\bar{\underline{t}}$.

(b) $\bar{\underline{u}} = \underline{u}$ at S_{u_0} (2.57)

where S_{u_0} is the undeformed surface where displacements are prescribed to be $\bar{\underline{u}}$.

<u>Updated Lagrangean Description</u>

The updated Lagrangean description seems somewhat unusual compared to the total Lagrangean description. However, it is widely employed in incremental formulations because of the fact that the formulations are greatly simplified by their use. In fact, sometimes, depending on the nature of problems and the stress or strain measures used, it is possible to use simple finite element computer programs developed for linear problem with minor modifications. Therefore, the updated Laglangean formulation will be discussed in the framework of the incremental formulation.

In the incremental formulation the external load, in general sense, is divided into a finite number of incremental loads. For given load increment, incremental equations are solved to obtain the next equilibrated state. With this equilibrated state as a current state, a new load increment is applied and the same procedure is repeated until the total load reaches the desired value. Now, we consider deformed configurations C_N and C_{N+1}, prior to and after the addition of the (N+1)th load increment as shown in Fig.2. The configuration C_N is considered to be an equilibrated known configulation, and C_{N+1} is an unknown state to be found. Thus, C_N is used as a reference instead of the initial configuration, to describe C_{N+1} state. If the C_{N+1} state is obtained, the reference will be updated and C_{N+1} will be a new reference. The name of updated Lagrangean description is given from this fact.

Since, our present reference is C_N state, all the state variables both in C_N and C_{N+1} states are referred to C_N configuration. The distinction between state variable in C_N and C_{N+1} are made by using superscripts N and N+1, respectively. Consider a material point P which has positions P_N and P_{N+1} in the (N)th and (N+1)th configurations. Position vectors of these points are denoted by $\underline{y}^N$ and $\underline{y}^{N+1}$. Since, in the Lagrangean description, material points are identified by their position vectors in the reference configuration, the components of vector $\underline{y}^N$ are taken as material coordinates for the present case. Thus, all the state variables in C_N and C_{N+1} states are considered as functions of $\underline{y}^N$. Symbolically, this statement is written as,

$$C_N = C_N(\underline{y}^N) \tag{2.58}$$

$$C_{N+1} = C_{N+1}(\underline{y}^N)$$

where C represents state variables in general. The displacement of a material point through the deformation from C_N to C_{N+1} is denoted by $\Delta\underline{u}$. It is written in terms of position vectors as,

$$\Delta\underline{u} = \underline{y}^{N+1}(\underline{y}^N) - \underline{y}^N \tag{2.59}$$

If $\underline{y}^{N+1}(\underline{y}^N)$ is differentiable with respect to the reference co-ordinate y_i^N, deformation gradient $\underset{\sim}{F}^{*N+1}$ in C_{N+1} with respect to $\underline{y}^N$ is defined by the following relation,

$$\underset{\sim}{F}^{*N+1} = (\nabla^* \underline{y}^{N+1})^T \quad , \qquad F^{*N+1}_{ij} = \frac{\partial y_i^{N+1}}{\partial y_j^N} \tag{2.60}$$

where ∇^* represents the gradient in the metric of C_N.

<u>Definition of Strain Measures</u>

The deformation gradient $\underset{\sim}{F}^{*N+1}$ is non-singular as is $\underset{\sim}{F}$ in the total Lagrangean description. It can be decomposed into polar-decomposition,

$$\underset{\sim}{F}^{*N+1} = \underset{\sim}{\alpha}^{*N+1} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^{*N+1}) \tag{2.61}$$

where $(\underset{\sim}{I} + \underset{\sim}{h}^{*N+1})$ is a symmetric, positive definite tensor and $\underset{\sim}{\alpha}^{*N+1}$ is an orthogonal tensor, such that,

$$(\underset{\sim}{\alpha}^{*N+1})^T \cdot (\underset{\sim}{\alpha}^{*N+1}) = \underset{\sim}{I} \tag{2.62}$$

and the superposed star implies state variables referred to C_N configuration. The physical interpretation of Eq.(2.61) is given in the analogous way as in the total Lagrangean description. The tensors $\underset{\sim}{\alpha}^{*N+1}$ and $(\underset{\sim}{I}+\underset{\sim}{h}^{*N+1})$ represent the rigid body rotation and the pure stretch of the infinitesimal material element through the deformation from C_N to C_{N+1}. Thus, $\underset{\sim}{h}^{*N+1}$ gives one strain measure. The displacement gradient $\underset{\sim}{e}^{*N+1}$ is defined by,

$$\underset{\sim}{e}^{*N+1} = (\nabla^* \underline{y}^{N+1} - \underset{\sim}{I})^T = (\nabla^* \Delta \underline{u})^T \tag{2.63}$$

Similarly, the deformation tensor $\underset{\sim}{G}^{*N+1}$ and the Green-Lagrange strain tensor $\underset{\sim}{g}^{*N+1}$ referred to the C_N configuration can be defined by the following equations,

$$\underset{\sim}{G}^{*N+1} = \underset{\sim}{F}^{*N+1^T} \cdot \underset{\sim}{F}^{*N+1} \tag{2.64}$$

$$\underset{\sim}{g}^{*N+1} = 1/2 \; (\underset{\sim}{G}^{*N+1} - \underset{\sim}{I}) \tag{2.65}$$

Thus, three strain measures $\underset{\sim}{h}^{*N+1}$, $\underset{\sim}{e}^{*N+1}$, and $\underset{\sim}{g}^{*N+1}$ are defined. These strain tensors are related to those defined in the total Lagrangean formulation through the following equations,

$$\underset{\sim}{e}^{N+1} = \underset{\sim}{e}*^{N+1} \cdot \underset{\sim}{F}^{N} + \underset{\sim}{e}^{N} \tag{2.66}$$

$$\underset{\sim}{g}^{N+1} = \underset{\sim}{F}^{N^T} \cdot \underset{\sim}{g}*^{N+1} \cdot \underset{\sim}{F}^{N} + \underset{\sim}{g}^{N} \tag{2.67}$$

where $\underset{\sim}{e}^{N+1}$, $\underset{\sim}{g}^{N+1}$, $\underset{\sim}{e}^{N}$, and $\underset{\sim}{g}^{N}$ are strain measures in C_{N+1} and C_N, but these are referred to the initial configuration ; and $\underset{\sim}{F}^{N} = (\nabla \underline{y}^{N})^{T}$. Further, the strains $\underset{\sim}{e}*^{N+1}$, $\underset{\sim}{g}*^{N+1}$, and $\underset{\sim}{h}*^{N+1}$ are related by the following equations.

$$\underset{\sim}{g}*^{N+1} = 1/2 \; \{ (\underset{\sim}{e}*^{N+1}) + (\underset{\sim}{e}*^{N+1})^{T} + (\underset{\sim}{e}*^{N+1})^{T} \cdot (\underset{\sim}{e}*^{N+1}) \} \tag{2.68}$$

$$\underset{\sim}{g}*^{N+1} = 1/2 \; \{ 2\underset{\sim}{h}*^{N+1} + \underset{\sim}{h}*^{N+1} \cdot \underset{\sim}{h}*^{N+1} \} \tag{2.69}$$

<u>Definition of Stress Measures</u>

In the updated Lagrangean formulation stress tensors are also referred to C_N configuration, instead of the initial configuration. Analogous to the case of the total Lagrangean description, Piola-Lagrange stress $\underset{\sim}{t}*^{N+1}$, Kirchhoff-Trefftz stress $\underset{\sim}{s}*^{N+1}$, and Jaumann stress $\underset{\sim}{r}*^{N+1}$ in the C_{N+1} state are defined through the following relations,

$$\underset{\sim}{\tau}^{N+1} = J^{N}/J^{N+1} \, \underset{\sim}{F}*^{N+1} \cdot \underset{\sim}{t}*^{N+1} = J^{N}/J^{N+1} \, \underset{\sim}{F}*^{N+1} \cdot \underset{\sim}{s}*^{N+1} \cdot (\underset{\sim}{F}*^{N+1})^{T} \tag{2.70}$$

or inversely,

$$\underset{\sim}{t}*^{N+1} = J^{N+1}/J^{N} \; (\underset{\sim}{F}*^{N+1})^{-1} \cdot \underset{\sim}{\tau}^{N+1} \tag{2.71}$$

$$\underset{\sim}{s}*^{N+1} = J^{N+1}/J^{N} \; (\underset{\sim}{F}*^{N+1})^{-1} \cdot \underset{\sim}{\tau}^{N+1} \cdot (\underset{\sim}{F}*^{N+1})^{-T} \tag{2.72}$$

and $$\underset{\sim}{t}*^{N+1} = \underset{\sim}{s}*^{N+1} \cdot (\underset{\sim}{F}*^{N+1})^{T} \tag{2.73}$$

where $\underset{\sim}{\tau}^{N+1}$ is the true stress in the C_{N+1} state and J^N and J^{N+1} are Jacobians which are defined by,

$$J^N = \det(\nabla \underline{y}^N) \quad \text{and} \quad J^{N+1} = \det(\nabla \underline{y}^{N+1}) \tag{2.74}$$

These stress tensors are related to those defined in the total Lagrangean description by the following equations,

$$\underset{\sim}{S}^{*N+1} = 1/J^N \,(\, \underset{\sim}{F}^N \cdot \underset{\sim}{s}^{N+1} \cdot \underset{\sim}{F}^{N^T}) \tag{2.75}$$

$$\underset{\sim}{t}^{*N+1} = 1/J^N \,(\, \underset{\sim}{F}^N \cdot \underset{\sim}{t}^{N+1}) \tag{2.76}$$

where $\underset{\sim}{s}^{N+1}$ and $\underset{\sim}{t}^{N+1}$ are Kirchhoff-Trefftz stress and Piola-Lagrange stress referred to the initial configuration. Further, the Jaumann stress $\underset{\sim}{r}^{*N+1}$ is defined by,

$$\begin{aligned} \underset{\sim}{r}^* &= (1/2)\{\underset{\sim}{t}^{*N+1} \cdot \underset{\sim}{\alpha}^{*N+1} + (\underset{\sim}{\alpha}^{*N+1})^T \cdot (\underset{\sim}{t}^{*N+1})^T\} \\ &= (1/2)\{\, \underset{\sim}{s}^{*N+1} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^{*N+1}) + (\underset{\sim}{I} + \underset{\sim}{h}^{*}) \cdot \underset{\sim}{s}^{*N+1}\} \end{aligned} \tag{2.77}$$

Constitutive Relations

We consider an elastic material discussed in the previous section. The existence of the strain energy density function W, which is measured in the undeformed configuration is assumed. We introduce strain energy density per unit volume in C_N configuration and denote it by W*. It is seen that W* is related to W by,

$$W^*(\underset{\sim}{g}^{*N+1}) = (1/J^N)\; W(\underset{\sim}{g}^{N+1}) \tag{2.78a}$$

with the additional conditions, that,

$$W^*(0) = (1/J^N)\ W(\underset{\sim}{g}^N)\ ;\quad \left.\frac{\partial W^*}{\partial \underset{\sim}{g}^{*N+1}}\right|_0 = \underset{\sim}{\tau}^N \qquad (2.78b,\ c)$$

The stress strain relations are shown to be derived through W* in the followings. First, we consider the variation of the strain energy (virtual work expended by virtual displacement) per unit volume in C_N,

$$\delta W^* = (1/J^N)\ \delta W \qquad (2.79)$$

The substitution of eqns.(2.21), (2.22), and (2.23) into Eq.(2.79) gives,

$$\delta W^* = (1/J^N)\ \underset{\sim}{s}^{N+1} : \delta \underset{\sim}{g}^{N+1} = (1/J^N)\ \underset{\sim}{t}^{N+1} : (\delta \underset{\sim}{e}^{N+1})^T \qquad (2.80)$$

Using the relations Eqns.(2.66, 67, 75, and 76), it is rewritten in terms of $\underset{\sim}{s}^{*N+1}$, $\underset{\sim}{t}^{*N+1}$, $\underset{\sim}{g}^{*N+1}$, and $\underset{\sim}{e}^{*N+1}$,

$$\delta W^* = \underset{\sim}{s}^{*N+1} : \delta \underset{\sim}{g}^{*N+1} = \underset{\sim}{t}^{*N+1} : (\delta \underset{\sim}{e}^{*N+1})^T \qquad (2.81)$$

On the other hand, W* is considered as a function of $\underset{\sim}{g}^{*N+1}$ or $\underset{\sim}{e}^{*N+1}$. Therefore , δW^* can also be written in the following forms,

$$\delta W^* = \frac{\partial W^*}{\partial \underset{\sim}{g}^{*N+1}} : \delta \underset{\sim}{g}^{*N+1} = \frac{\partial W^*}{\partial \underset{\sim}{e}^{*N+1}} : \delta \underset{\sim}{e}^{*N+1} \qquad (2.82)$$

By comparison between Eqns.(2.81) and (2.82), the stress and the strain are shown to be related by,

$$\frac{\partial W^*}{\partial \underset{\sim}{g}^{*N+1}} = \underset{\sim}{s}^{*N+1}\ ,\quad \frac{\partial W^*}{\partial \underset{\sim}{e}^{*N+1}} = \underset{\sim}{t}^{*N+1\,T} \qquad (2.83)$$

(2.84)

Similarly, considering W* as a function of $\underset{\sim}{h}^{*N+1}$, the following relation is derived.

$$\frac{\partial W^*}{\partial \underset{\sim}{h}^{*N+1}} = \underset{\sim}{r}^{*N+1} \tag{2.85}$$

Further, we consider the inverse of the above constitutive relations. As discussed earlier, the inverse stress-strain relation in terms of $\underset{\sim}{s}$ and $\underset{\sim}{g}$ is uniquely defined, and $\underset{\sim}{g}$ can be expressed as a function of $\underset{\sim}{s}$. Also $\underset{\sim}{g}^{N+1}$ and $\underset{\sim}{s}^{N+1}$ are linearly related to $\underset{\sim}{g}^{*N+1}$ and $\underset{\sim}{s}^{*N+1}$ through Eqns.(2.67) and (2.75). Therefore, $\underset{\sim}{g}^{*N+1}$ can be expressed in terms of $\underset{\sim}{s}^{*N+1}$. Thus, the contact transformation of W* in terms of $\underset{\sim}{s}^{*N+1}$ is achieved.

$$S^*(\underset{\sim}{s}^{*N+1}) = \underset{\sim}{s}^{*N+1} : \underset{\sim}{g}^{*N+1}(\underset{\sim}{s}^{*N+1}) - W^*[\underset{\sim}{g}^{*N+1}(\underset{\sim}{s}^{*N+1})] \tag{2.86}$$

such that,

$$\frac{\partial S^*}{\partial \underset{\sim}{s}^{*N+1}} = \underset{\sim}{g}^{*N+1} \tag{2.87}$$

Similarly, the contact transformation of W* in terms of $\underset{\sim}{r}^{*N+1}$, which is defined by the following equation, exists.

$$R^*(\underset{\sim}{r}^{*N+1}) = \underset{\sim}{r}^{*N+1} : \underset{\sim}{h}^{*N+1}(\underset{\sim}{r}^{*N+1}) - W^*[\underset{\sim}{h}^{*N+1}(\underset{\sim}{r}^{*N+1})] \tag{2.88}$$

such that,

$$\frac{\partial R^*}{\partial \underset{\sim}{r}^{*N+1}} = \underset{\sim}{h}^{*N+1} \tag{2.89}$$

However, as already shown, there is no unique inverse stress-strain relation in terms of $\underset{\sim}{t}^{N+1}$ and $\underset{\sim}{e}^{N+1}$, which are linearly related to $\underset{\sim}{t}^{*N+1}$ and $\underset{\sim}{e}^{*N+1}$ through Eqns.(2.66) and (2.76), respectively. Therefore, there is no unique inverse of Eq.(2.84). Thus, the contact transformation in terms of $\underset{\sim}{t}^{*N+1}$ can not be achieved.

Field Equations and Boundary Conditions

The field equations and the boundary conditions for the finite deformation elastic problems can be written in terms of alternate stress and their conjugate strain measures which are referred to C_N configuration. These equations are summarized in the following,

translational equilibrium conditions

$$\nabla^* \cdot [\, \underset{\sim}{s}^{*N+1} \cdot (\underset{\sim}{F}^{*N+1})^T \,] + \rho_N \underline{g}^{N+1} = 0 \tag{2.90}$$

$$\nabla^* \cdot \underset{\sim}{t}^{*N+1} + \rho_N \underline{g}^{N+1} = 0 \tag{2.91}$$

where ρ_N is the mass density per unit volume in C_N.

rotational equilibrium conditions

$$\underset{\sim}{s}^{*N+1} = (\, \underset{\sim}{s}^{*N+1})^T \tag{2.92}$$

$$\text{or} \quad \underset{\sim}{F}^{*N+1} \cdot \underset{\sim}{t}^{*N+1} = (\, \underset{\sim}{F}^{*N+1} \cdot \underset{\sim}{t}^{*N+1} \,)^T \tag{2.93}$$

$$\text{or} \quad (\underset{\sim}{h}^{*N+1} + \underset{\sim}{I}\,) \cdot \underset{\sim}{t}^{*N+1} \cdot \underset{\sim}{\alpha}^{*N+1} = \text{symmetric} \tag{2.94}$$

kinematic relations

$$\underset{\sim}{g}^{*N+1} = (1/2)\{ \overset{*}{\nabla}\Delta\underline{u} + \overset{*}{\nabla}\Delta\underline{u}^{T} + (\overset{*}{\nabla}\Delta\underline{u})\cdot(\overset{*}{\nabla}\Delta\underline{u})^{T} \} \tag{2.95}$$

$$\underset{\sim}{e}^{*N+1} = (\overset{*}{\nabla}\Delta\underline{u})^{T} \tag{2.96}$$

$$(\overset{*}{\nabla}\underline{y}^{N+1})^{T} = \underset{\sim}{a}^{*N+1}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^{*N+1}) \tag{2.97}$$

<u>constitutive relations</u>

$$\underset{\sim}{s}^{*N+1} = \frac{\partial W^*}{\partial \underset{\sim}{g}^{*N+1}} \tag{2.98}$$

$$\underset{\sim}{t}^{*N+1\,T} = \frac{\partial W^*}{\partial \underset{\sim}{e}^{*N+1}} \tag{2.99}$$

$$\underset{\sim}{r}^{*N+1} = \frac{\partial W^*}{\partial \underset{\sim}{h}^{*N+1}} \tag{2.100}$$

Further, through the complementary energy density functions defined by Eqns.(2.86) and (2.88), the inverse relations of Eqns.(2.98) and (2.100) are given by,

$$\underset{\sim}{g}^{*N+1} = \frac{\partial S^*}{\partial \underset{\sim}{s}^{*N+1}} \tag{2.101}$$

$$\underset{\sim}{h}^{*N+1} = \frac{\partial R^*}{\partial \underset{\sim}{r}^{*N+1}} \tag{2.102}$$

However, unique inverse of Eq.(2.99) does not exist for general cases.

<u>boundary conditions</u>

(a) $\quad \bar{\underline{t}}^{*N+1} = \underline{n}^{*}\cdot\underset{\sim}{t}^{*N+1} = \underline{n}^{*}\cdot\{ \underset{\sim}{s}^{*N+1}\cdot(\underset{\sim}{F}^{*N+1})^{T} \}$ at S_{σ_n} (2.103)

where $\underline{n}^*$ is the unit normal to the boundary S_{σ_n} in C_N where the traction is prescribed to be $\bar{\underline{t}}^{*N+1}$.

(b) $\quad \bar{\underline{u}}^{N+1} = \underline{u}^{N+1}$ at S_{u_n} (2.104) where Su_n is the boundary where displacements are prescribed to be $\bar{\underline{u}}^{N+1}$.

CHAPTER III

VARIATIONAL PRINCIPLES FOR FINITE DEFORMATION PROBLEMS (TOTAL LAGRANGEAN DESCRIPTION)

Introduction

As discussed in the preceding chapter the behavior of the deformed solid can be fully described by translational equilibrium equations, rotational equilibrium equations, kinematic relations, constitutive relations, and proper boundary conditions. In general, these equations are written in terms of displacement, strain, and stress. By eliminating some of these field variables, they are reduced to a set of partial differential equations and boundary conditions in terms of displacement or, if possible, stress alone. Usually the derived differential equations are nonlinear. Analytical solutions of these nonlinear equations for practically meaningful boundary conditions are very limited. Even for the small deformation problem in which governing equations are linear, an analytical solution is available only for ideal boundary conditions. Therefore, most of the practical works in solid mechanics are largely dependent on approximate numerical solution techniques. Among such numerical methos, finite element method has been widely used as a versatile tool.

The significant feature of fimite element methods is the fact that, in general, they have their strong theoretical bases on variational principles, such as stationary potential energy principle, stationary complementary energy principle, Hellinger-Reissner principle, etc. As it is seen from the works by Washizu [1], Nemat-Nasser and his co-workers [2, 3], Horrigmoe and Bergan [4], and Horrigmoe [5], variational principles have been playing an important role in the development of finite element models not only for small deformation problems but also for finite deformation problems. This implies that the development of a new finite element model can be made possible, if the corresponding variational formulation is derived. Since, the primary objective of this thesis is to develop assumed stress finite element models for finite deformation problems, rational complementary energy principles which lead to such models are sought. For this purpose, basic variational principles in total Lagrangean description are reviewed. Following Washizu [1], the general (Hu-Washizu) principles in terms of alternate stress and strain measures are constructed. With these general principles as bases, stationary potential energy principles, Hellinger-Reissner principles, and, if possible, stationary complementary energy principles are shown to be obtained as special cases. In this process the possibility of constructing a rational complementary energy principle is discussed in detail.

Hu-Washizu Variational Principles

A general variational principle was derived by Washizu

[1] and Hu [22] for linear elastic problems. In this principle, the functional is not subjected to any subsidiary (a priori) conditions. Its stationarity condition leads to all the field equations and boundary conditions, which fully describe the deformation of elastic body. Analogous general principles are constructed for the finite deformation problems in the following.

Based on $\underset{\sim}{s}$ and $\underset{\sim}{g}$

The Hu-Washizu functional in terms of displacement $\underline{u}$, Kirchhoff- Trefftz stress $\underset{\sim}{s}$, and Green-Lagrange strain $\underset{\sim}{g}$, for the finite-deformation case is derived, in a manner analogous to the original developments in [1], as,

$$\pi_{HW}(\underline{u}, \underset{\sim}{g}, \underset{\sim}{s}) = \int_{V_0} \Big\{ W(\underset{\sim}{g}) - \rho_0 \underline{g} \cdot \underline{u} + \frac{1}{2} \underset{\sim}{s} : \left[\nabla \underline{u} + (\nabla \underline{u})^T + (\nabla \underline{u}) \cdot (\nabla \underline{u})^T - 2\underset{\sim}{g} \right] \Big\} dv - \int_{S_{\sigma_0}} \bar{\underline{t}} \cdot \underline{u} \, ds - \int_{S_{u_0}} \underline{t} \cdot (\underline{u} - \bar{\underline{u}}) ds \tag{3.1}$$

where $\underline{t}$ is the traction on the boundary per unit undeformed area, which is defined by,

$$\underline{t} = \underline{n} \cdot \underset{\sim}{s} \cdot (\nabla \underline{y})$$

and $W(\underset{\sim}{g})$ is the strain energy density function (per unit initial volume) which is a symmetric function of $\underset{\sim}{g}$ as defined by Eq.(2.17); s_{σ_0} and s_{u_0} denote the portions of the boundary surface S_0, in the undeformed state, where the traction and the displacement are prescribed to be $\bar{t}$ and $\bar{u}$, respectively. The first variation of the above functional due to arbitrary variations $\delta\underline{u}$, $\delta\underset{\sim}{g}$, and $\delta\underset{\sim}{s}$ is obtained as,

$$\begin{aligned}\delta \pi_{HW} = &\int_{V_0}\Big\{\Big[\frac{\partial W}{\partial \underset{\sim}{g}} - \underset{\sim}{s}\Big] : \delta g + \underset{\sim}{s}\cdot\underset{\sim}{F}^T : (\delta \nabla \underline{u}) - \rho_0 \underline{g}\cdot\delta\underline{u} \\ &+ \Big[\frac{1}{2}\Big\{\nabla\underline{u} + \nabla\underline{u}^T + (\nabla\underline{u})\cdot(\nabla\underline{u})^T\Big\} - \underset{\sim}{g}\Big] : \delta\underset{\sim}{s}\Big\}dv \\ &- \int_{S_{\sigma_0}}\bar{\underline{t}}\cdot\delta\underline{u}\, ds - \int_{S_{u_0}}\Big\{\delta\underline{t}\cdot(\underline{u} - \bar{\underline{u}}) + \underline{t}\cdot\delta\underline{u}\Big\}ds\end{aligned} \tag{3.2}$$

If the stress $\underset{\sim}{s}$ and the displacement $\underline{u}$ are assumed to be differentiable with respect to $\underline{x}$, by using integration by parts, π_{HW} is rewritten in the following form,

$$\begin{aligned}\delta \pi_{HW} = &\int_{V_0}\Big\{\Big[\frac{\partial W}{\partial \underset{\sim}{g}} - \underset{\sim}{s}\Big] : \delta\underset{\sim}{g} - \Big[\nabla\cdot(\underset{\sim}{s}\cdot\underset{\sim}{F}^T) + \rho_0\underline{g}\Big]\cdot\delta\underline{u} \\ &+ \Big[\frac{1}{2}\Big\{\nabla\underline{u} + \nabla\underline{u}^T + (\nabla\underline{u})\cdot(\nabla\underline{u})^T\Big\} - \underset{\sim}{g}\Big] : \delta s\Big\}dv \\ &- \int_{S_{\sigma_0}}(\bar{\underline{t}} - \underline{n}\cdot\underline{t})\cdot\delta\underline{u}\, ds - \int_{S_{u_0}}\delta\underline{t}\cdot(\underline{u} - \bar{\underline{u}})\, ds\end{aligned} \tag{3.3}$$

Thus, it is readily seen that the stationarity condition leads to translational equilibrium condition Eq.(2.43), kinematic relation Eq.(2.48) , constitutive relation Eq(2.51), and boundary conditions Eqns.(2.56) and (2.57) as a posteriori conditions. In addition to these, from the symmetric property of $W(\underset{\sim}{g})$, the rotational equilibrium condition Eq.(2.45) is manifested as the condition of symmetry of $\underset{\sim}{s}$. Therefore, it is shown that the stationary condition of Eq.(3.1) is reduced to the full description of the finite deformation problem.

Based on $\underset{\sim}{t}$ and $\underset{\sim}{e}$

An analogous functional is derived in terms of displacement $\underline{u}$, Piola-Lagrange stress $\underset{\sim}{t}$, and displacement gradient $\underset{\sim}{e}$.

$$\pi_{HW}(\underline{u}, \underset{\sim}{e}, \underset{\sim}{t}) = \int_{V_0} \left\{ W(\underset{\sim}{e}) + \underset{\sim}{t}^T : (\nabla \underline{u}^T - \underset{\sim}{e}) - \rho_0 \underline{g} \cdot \underline{u} \right\} dv - \int_{S_{\sigma_0}} \bar{\underline{t}} \cdot \underline{u} \, ds - \int_{S_{u_0}} \underline{t} \cdot (\underline{u} - \bar{\underline{u}}) \, ds \tag{3.4}$$

where W is considered as a function of $\underset{\sim}{e}$ through $\underset{\sim}{g}$, as defined by Eq.(2.18). Its first variation is shown to be,

$$\delta \pi_{HW} = \int_{V_0} \left\{ \left[\frac{\partial W}{\partial \underset{\sim}{e}} - \underset{\sim}{t}^T \right] : \delta \underset{\sim}{e} + \left[\nabla \underline{u}^T - \underset{\sim}{e} \right] : \delta \underset{\sim}{t}^T - \left[\nabla \cdot \underset{\sim}{t} + \rho_0 \underline{g} \right] \cdot \delta \underline{u} \right\} dv - \int_{S_{\sigma_0}} (\bar{\underline{t}} - \underline{n} \cdot \underset{\sim}{t}) \cdot \delta \underline{u} \, ds - \int_{S_{u_0}} \delta \underline{t} \cdot (\underline{u} - \bar{\underline{u}}) ds \tag{3.5}$$

Thus, the stationarity condition of Eq.(3.4) leads to Eqns.(2.44), (2.46), (2.49), (2.52), (2.56), and (2.57). Again it is noticed that the rotational equilibrium condition is enforced through the symmetric structure of W from the following arguments. By the definition of W, the constitutive rulation can be expressed as,

$$\underset{\sim}{t} = \frac{\partial W}{\partial \underset{\sim}{e}^{T}} = \frac{\partial W}{\partial \underset{\sim}{g}} \cdot \underset{\sim}{F}^{T} = \underset{\sim}{s} \cdot \underset{\sim}{F}^{T} \tag{3.6}$$

where, $\underset{\sim}{s}^{T} = \underset{\sim}{s}$.

From the symmetry of $\underset{\sim}{s}$, which is embedded in W, the stress $\underset{\sim}{t}$, which is derived through Eq.(3.6), identically satisfies the rotational equilibrium condition given by Eq.(2.46).

Based on $\underset{\sim}{r}$ and $\underset{\sim}{h}$

We consider here that the strain energy density W is expressed as a symmetric function of right extensional strain tensor $\underset{\sim}{h}$. Then, the general principle is constructed based on the Jaumann stress $\underset{\sim}{r}$ and right extensional strain tensor $\underset{\sim}{h}$, as ,

$$\pi_{HW}(\underline{u}, \underset{\sim}{h}, \underset{\sim}{\alpha}, \underset{\sim}{t}) \tag{3.7}$$

$$= \int_{V_0} \Big\{ W(\underset{\sim}{h}) + \underset{\sim}{t}^{T} : \big[(\underset{\sim}{I} + \nabla \underline{u})^{T} - \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \big] - \rho_0 \underline{g} \cdot \underline{u} \Big\} dv$$

$$- \int_{S_{\sigma_0}} \bar{\underline{t}} \cdot \underline{u} \, ds - \int_{S_{u_0}} \underline{t} \cdot (\underline{u} - \bar{\underline{u}}) ds$$

Noting that the rotation tensor $\underset{\sim}{\alpha}$ is subjected to the orthogonality condition,

$$\underset{\sim}{\alpha}^{T} \cdot \underset{\sim}{\alpha} = \underset{\sim}{I} \tag{3.8}$$

the first variation of the functional Eq(3.7) is shown to be,

$$\delta \pi_{HW} = \int_{V_0} \Big\{ \Big[\frac{\partial W}{\partial \underset{\sim}{h}} - \frac{1}{2}(\underset{\sim}{t}\cdot\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T\cdot\underset{\sim}{t}^T) \Big] : \delta\underset{\sim}{h} \tag{3.9}$$

$$- \big[\nabla\cdot\underset{\sim}{t} + \rho_0 \underline{g} \big]\cdot \delta\underline{u} - \big[(\underset{\sim}{I} + \underset{\sim}{h})\cdot\underset{\sim}{t}\cdot\underset{\sim}{\alpha} \big] : (\underset{\sim}{\alpha}^T\cdot\delta\underset{\sim}{\alpha})^T$$

$$+ \big[(\underset{\sim}{I} + \nabla\underline{u})^T - \underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}) \big] : \delta\underset{\sim}{t}^T \Big\} dv$$

$$- \int_{S_{\sigma_0}} (\bar{\underline{t}} - \underline{n}\cdot\underset{\sim}{t})\cdot\delta\underline{u}\, ds - \int_{S_{u_0}} \delta\underline{t}\cdot(\underline{u} - \bar{\underline{u}}) ds$$

The constitutive relation Eq.(2.53), kinematic relation Eq.(2.50) translational equilibrium condition Eq.(2.44), and the boundary conditions Eqns.(2.56) and (2.57) are readily shown to be obtained from the stationarity condition of Eq.(3.7). We, now, consider the stationarity condition with respet to $\underset{\sim}{\alpha}$. From the orthogonality of $\underset{\sim}{\alpha}$, the variation must satisfy,

$$\underset{\sim}{\alpha}^T\cdot\delta\underset{\sim}{\alpha} + \delta\underset{\sim}{\alpha}^T\cdot\underset{\sim}{\alpha} = 0 \tag{3.10}$$

or

$$\underset{\sim}{\alpha}^T\cdot\delta\underset{\sim}{\alpha} = \text{skewsymmetric}$$

Thus, the condition of vanishing of the third term in Eq.(3.9) requires the symmetry of $(\underset{\sim}{I}+\underset{\sim}{h})\cdot\underset{\sim}{t}\cdot\underset{\sim}{\alpha}$, which is the exact statement of the rotational equilibrium condition as shown by Eq.(2.47). It is observeved, here, that rotational equilibrium condition is separated from the constitutive relation, and it is obtained directly from the stationarity condition of the functional as an a posteriori condition. As it will be discussed later, this feature holds the key to constructing a rational complementary energy principle.

Stationary Potential Energy Principles

Following Washizu [1], by a priori satisfying the kinematic relations, constitutive relations, and the displacement boundary conditions, the Hu-Washizu functionals given by Eqns.(3.1), (3.4), and (3.7) are reduced to the stationary potential energy principles. Since the potential energy functional involves only displacement $\underline{u}$, the reduced functionals become identical, and this functional is seen to be,

$$\pi_p(\underline{u}) = \int_{V_0}\left\{W(\underset{\sim}{g}) - \rho_0\underline{g}\cdot\underline{u}\right\}dv - \int_{S_{\sigma_0}} \underline{\bar{t}}\cdot\underline{u}\, ds \tag{3.11}$$

or equivalently

$$\pi_p(\underline{u}) = \int_{V_0}\left\{W(\underset{\sim}{e}) - \rho_0\underline{g}\cdot\underline{u}\right\}dv - \int_{S_{\sigma_0}} \underline{\bar{t}}\cdot\underline{u}\, ds \tag{3.12}$$

Its stationarity condition leads to the translational equilibrium condition Eq.(2.43 or 44) and the traction boundary condition Eq.(2.56). Moreover, when Eq.(3.12) is used, even though W is expressed as a function of $\underset{\sim}{e}$, the rotational equilibrium condition is inherently embedded in the structure of W as discussed earlier. This type of variational principles is commonly applied to the finite element method [18].

Principles of the "Hellinger-Reissner" Type

If the constitutive relation is invertible, the contact transformation of the strain energy density W exists. By using this transformation (to obtain the complementary energy density), strain tensor is eliminated from the Hu-Washizu functional and the Hellinger-Reissner functional can be derived.

As discussed by Fraeijs de Veubeke [9], the inverses of Eq.(2.22) and Eq.(2.26) exist, and the following contact transformations are achieved.

$$S(\underset{\sim}{s}) = \underset{\sim}{s} : \underset{\sim}{g}(\underset{\sim}{s}) - W[\underset{\sim}{g}(\underset{\sim}{s})] \tag{3.13}$$

$$R(\underset{\sim}{r}) = \underset{\sim}{r} : \underset{\sim}{h}(\underset{\sim}{r}) - W[\underset{\sim}{h}(\underset{\sim}{r})] \tag{3.14}$$

such that,

$$\frac{\partial S}{\partial \underset{\sim}{s}} = \underset{\sim}{g} \quad , \quad \frac{\partial R}{\partial \underset{\sim}{r}} = \underset{\sim}{h} \qquad (3.15),\ (3.16)$$

The substitution of these transformations into the Hu-Washizu functionals, Eq.(3.1) and Eq.(3.7), lead to the following two types of Hellinger-Reissner functionals.

Based on $\underset{\sim}{s}$ and $\underset{\sim}{g}$

(3.17)

$$\pi_{HR}(\underline{u}, \underset{\sim}{s})$$

$$= \int_{v_0} \left\{ -S(\underset{\sim}{s}) + \frac{1}{2}\underset{\sim}{s} : [\nabla\underline{u} + \nabla\underline{u}^T + (\nabla\underline{u})\cdot(\nabla\underline{u})^T] - \rho_0 \underline{g}\cdot\underline{u} \right\} dv$$

$$-\int_{S_{\sigma_0}} \bar{\underline{t}} \cdot \underline{u}\, ds \quad - \int_{S_{u_0}} \underline{t} \cdot (\underline{u} - \bar{\underline{u}})\, ds$$

The translational and rotational equilibrium conditions Eqns.(2.43) and (2.45) , compatibility condition Eq.(2.48), and the boundary conditions Eqns.(2.56) and (2.57) follow from the stationarity condition of the above functional. This form of variational principle is attributed to Hellinger [7] and Reissner [23]. Its applications to the finite element method are offten found in literature [3].

Based on $\underset{\sim}{r}$ and $\underset{\sim}{h}$

Likewise, based on Jaumann stress $\underset{\sim}{r}$, the following functional is derived.

$$\pi_{HR}(\underline{u}, \underset{\sim}{\alpha}, \underset{\sim}{t}) \tag{3.18}$$

$$= \int_{V_0} \{ -R(\underset{\sim}{r}) + \underset{\sim}{t}^T : [(\underset{\sim}{I} + \nabla \underline{u})^T - \underset{\sim}{\alpha}] - \rho_0 \underline{g} \cdot \underline{u} \} dv$$

$$-\int_{S_{\sigma_0}} \bar{\underline{t}} \cdot \underline{u}\, ds \quad - \int_{S_{u_0}} \underline{t} \cdot (\underline{u} - \bar{\underline{u}})\, ds$$

Its stationarity condition leads to Eqns.(2.44, 50, 56, and 57), and also rotational equilibrium condition Eq.(2.47).

Based on $\underset{\sim}{t}$ and $\underset{\sim}{e}$

As discussed by Novozhilov [24], Truesdell and Noll [15], and Dill [13], in general the inverse of the stress-strain relation, Eq.(2.52), is multivalued. Therefore, we can not derive practically useful Hellinger-Reissner principle based on $\underset{\sim}{t}$ and $\underset{\sim}{e}$ for general cases.

Stationary Complementary Energy Principles

In the linear theory, the Hellinger-Reissner principle is reduced to the minimum complementary energy principle, which involves stress alone, by a priori satisfying the translational equilibrium condition and the traction boundary condition. Analogous approach is adopted here to derive a complementary energy principle for finite deformation problems.

Based on $\underset{\sim}{s}$ and $\underset{\approx}{g}$

In the formulation based on Kirchhoff-Trefftz stress $\underset{\sim}{s}$, the translational equilibrium condition and the traction boundary condition are given by Eqns.(2.43) and (2.56), which are nonlinear and coupled partial defferential equations and boundary conditions involving both stress $\underset{\sim}{s}$ and displacement $\underline{u}$. The exact satisfaction of these nonlinear equations is considered to be impossible, in general. Moreover, as discussed by Fraeijs de Veubeke [9], even if they are satisfied somehow, the derived functional involves both $\underset{\sim}{s}$ and $\underline{u}$. It is formally shown by,

$$\pi_c(\underline{u}, \underset{\sim}{s}) = \int_{v_0} \{ S(\underset{\sim}{s}) + \frac{1}{2} \underset{\sim}{s} : [\nabla\underline{u} \cdot (\nabla\underline{u})^T] \} dv - \int_{S_{u_0}} \underline{t} \cdot \underline{\bar{u}} \, ds \qquad (3.19)$$

Noting the constraint on $\underset{\sim}{s}$ and $\underline{u}$, its stationarity condition leads to the kinematic relation Eq.(2.48) and displacement boundary condition Eq.(2.57).

Based on $\underset{\sim}{t}$ and $\underset{\sim}{e}$

The major obstacle for constructing complementary energy principle involving $\underset{\sim}{t}$ alone is found in the fact that there is no unique inverse relation for $\underset{\sim}{e}$ in terms of $\underset{\sim}{t}$, in general. However, assuming the existence of such a inverse relation and, consequently, the complementary energy density, further investigation is attempted here. The most atractive advantage in the formulation based on Piola-Lagrange stress $\underset{\sim}{t}$ is that the translational equilibrium condition and the traction boundary condition are linear in $\underset{\sim}{t}$ alone, and these can be easily satisfied a priori by the chosen stress field $\underset{\sim}{t}$. On the other hand, the rotational equilibrium condition becomes nonlinear in $\underset{\sim}{t}$ and $\underline{u}$ as shown by Eq.(2.46). To obtain a physically meaningful solution this condition must be satisfied either a priori through the structure of the complementary energy density or a posteriori through the variational principle. Although, its a priori satisfaction appears to be difficult, the study on the structure of the complementary energy density, which forces the rotational equilibrium condition, was made by Fraeijs de Veubeke [9]. Assuming the existence of the inverse stress-strain relation in terms of $\underset{\sim}{t}$ and $\underset{\sim}{e}$, he derived a set of nonlinear partial differential equations which characterize the structure of such complementary energy density so that it enforces the rotational equilibrium condition. However, because of the mathematical complexity, this approach does not appear worthwhile for practical applications. As it is seen in the above, there are ambiguities and difficulties involved in the formulation based on $\underset{\sim}{t}$. Thus, the complementary energy principle involving $\underset{\sim}{t}$ alone fails to be a rational and practical variational principle for general finite deformation problems.

<u>Based on $\underset{\sim}{r}$ and $\underset{\sim}{h}$</u>

It was shown that the Hellinger-Reissner principles based on Kirchhoff-Trefftz stress $\underset{\sim}{s}$ or Piola-Lagrange stress $\underset{\sim}{t}$ do not lead to a successful complementary energy principles, either because of the nonlinear equilibrium conditions or due to the multivalued inverse stress-strain relations. Now, we turn to the most successful formulation based on the Jaumann stress $\underset{\sim}{r}$. In this formulation, the translational equilibrium condition and the traction boundary condition are linear in $\underset{\sim}{t}$, and the constitutive relation in terms of $\underset{\sim}{r}$ and $\underset{\sim}{h}$ is invertible so that the complementary energy density exists. Moreover, the rotational equilibrium condition is directly satisfied through the stationarity condition of the functional. Thus, the ambiguity on its satisfaction can be avoided.

Assuming the a priori satisfaction of the translational equilibrium condition Eq.(2.44) and the traction boundary condition Eq.(2.56), the Hellinger-Reissner functional given by Eq.(3.18) is reduced to a complementary energy principle involving stress $\underset{\sim}{t}$ and rotation $\underset{\sim}{\alpha}$.

$$\Pi_c(\underset{\sim}{\alpha}, \underset{\sim}{t}) = \int_{V_0} \{ R(\underset{\sim}{r}) + \underset{\sim}{t}^T : [\underset{\sim}{\alpha} - \underset{\sim}{I}] \} dV \qquad (3.20)$$

$$- \int_{S_{u_0}} \underline{t} \cdot \underline{\bar{u}} \, ds$$

Noting the constraint condition on the variation of stress, i.e.,

$$\nabla \cdot \delta \underset{\sim}{t} = 0 \qquad \text{in } V_0 \qquad (3.21)$$

$$\delta \underline{t} = \underline{n} \cdot \delta \underset{\sim}{t} = 0 \qquad \text{at } S_{\sigma_0} \qquad (3.22)$$

and the orthogonality of the rotation tensor, i.e.,

$$\underset{\sim}{\alpha}^T \cdot \underset{\sim}{\alpha} = \underset{\sim}{I} \qquad (3.23)$$

the first variation of the functional Eq.(3.20) is obtained as,

$$\delta \Pi_c = \int_{V_0} \{ [\underset{\sim}{\alpha} \cdot \frac{\partial R}{\partial \underset{\sim}{r}} + \underset{\sim}{\alpha} - \underset{\sim}{I}] : \delta \underset{\sim}{t}^T \qquad (3.24)$$

$$+ \left(\frac{\partial R}{\partial \underset{\sim}{r}} + \underset{\sim}{I} \right) \cdot \underset{\sim}{t} : \delta \underset{\sim}{\alpha}^T \} dV$$

$$- \int_{S_{u_0}} \delta \underline{t} \cdot \underline{\bar{u}} \, ds$$

From the definition of R,

$$\frac{\partial R}{\partial \underset{\sim}{r}} = \underset{\sim}{h} \qquad (3.25)$$

By introducing the following identical equation,

$$\int_{V_o} (\nabla \underline{u})^T : \delta \underset{\sim}{t}^T \, dV = \int_{S_{u_o}} \delta \underline{t} \cdot \underline{u} \, ds \tag{3.26}$$

Eq.(3.24) is rewritten as,

$$\delta \pi_c = \int_{V_o} \Big\{ [\underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) - (\underset{\sim}{I} + \nabla \underline{u})^T] : \delta \underset{\sim}{t}^T$$
$$+ (\underset{\sim}{I} + \underset{\sim}{h}) \cdot \underset{\sim}{t} \cdot \underset{\sim}{\alpha} : (\underset{\sim}{\alpha}^T \cdot \delta \underset{\sim}{\alpha})^T \Big\} dV$$
$$+ \int_{S_{u_o}} \delta \underline{t} \cdot (\underline{u} - \underline{\bar{u}}) \, ds \tag{3.27}$$

Further, noting that,

$$\underset{\sim}{\alpha}^T \cdot \delta \underset{\sim}{\alpha} = \text{skewsymmetric}$$

from the orthogonality condition, the stationarity condition of the functional Eq.(3.20) leads to,

$$(\underset{\sim}{I} + \nabla \underline{u})^T = \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \qquad \text{in } V_o \tag{3.28}$$

$$(\underset{\sim}{I} + \underset{\sim}{h}) \cdot \underset{\sim}{t} \cdot \underset{\sim}{\alpha} = \text{symmetric} \qquad \text{in } V_o \tag{3.29}$$

$$\underline{u} = \underline{\bar{u}} \qquad \text{at } S_{u_o} \tag{3.30}$$

These equations are exact statements of the kinematic relation, rotational equilibrium condition, and the displacement boundary condition. Thus, the complementary energy principle as stated through Eq.(3.20) is the most rigorous, consistent and the most practically applicable version that has been derived to date.

CHAPTER IV

INCREMENTAL VARIATIONAL PRINCIPLES

Introduction

The various types of functionals summarised in the preceding chapter can be applied to the finite element models. In general, the derived finite element formulations lead to highly nonlinear algebraic equations in terms of undetermined parameters. Usually, these nonlinear equations are solved by using the imbedding techniques such as the Newton-Raphson method. Moreover, in the case of path-dependent inelastic materials, like elastic-plastic materials, the potentials W or its contact transformations S or R do not exist. Therefore, the variational principles governing the total deformation are not valid for these materials.

To deal with these difficulties, due to the algebraic complexity and the nature of the material, incremental formulations, which lead to piecewise linear incremental solutions, are considered. In the incremental formulations, the prescribed loads and/or displacements are considered to be applied in small but finite consecutive increments. We label the states (stress, strain, deformation, etc.) of the solid prior to and after the addition of the (N+1)th load increment as $\{C_N\}$ and $\{C_{N+1}\}$, respectively. Depending on whether the metric in C_0 (undeformed or initial configuration) or the metric in C_N is used to refer all the incremental state variables describing the transition from C_N to C_{N+1}, two types of incremental formulations are possible. These are generally referred to as the "total Lagrangean" and the "updated Lagrangean" formulations. The details are discussed for both formulations. Further, modified incremental variational principles, in which the continuity conditions at inter-element boundaries are relaxed a priori, are also presented in this chapter.

Total Lagrangean Formulation

Incremental Governing Equations

In the total Lagrangean description, the metric in C_0 is used to refer all the state variables in each of the subsequent states. Let state C_N be defined by the variables, $\{\tilde{s}^N, \tilde{t}^N, \tilde{r}^N, \tilde{g}^N, \tilde{e}^N, \tilde{h}^N, \tilde{a}^N, \underline{u}^N$ etc.$\}$, and a similar set of variables in C_{N+1} with the superscript (N+1). Let the incremental variables in passing from C_N to C_{N+1} be $\{\Delta\tilde{s}, \Delta\tilde{t}, \Delta\tilde{r}, \Delta\tilde{g}, \Delta\tilde{e}, \Delta\tilde{h}, \Delta\tilde{a}, \Delta\underline{u}$, etc.$\}$. These incremental state variables are symbolically denoted by ΔC. Thus, as a matter of formal symbolics, $C_{N+1}=C_N+\Delta C$. For later use, the definitions of all these incremental variables are listed as below,

$$\Delta\tilde{s} = \tilde{s}^{N+1} - \tilde{s}^N \quad ; \quad \Delta\tilde{t} = \tilde{t}^{N+1} - \tilde{t}^N \tag{4.1}$$

$$\Delta\tilde{r} = \tilde{r}^{N+1} - \tilde{r}^N \quad ; \quad \Delta\tilde{g} = \tilde{g}^{N+1} - \tilde{g}^N$$

$$\Delta\tilde{e} = \tilde{e}^{N+1} - \tilde{e}^N \quad ; \quad \Delta\tilde{h} = \tilde{h}^{N+1} - \tilde{h}^N$$

$$\Delta \underset{\sim}{\alpha} = \underset{\sim}{\alpha}^{N+1} - \underset{\sim}{\alpha}^{N} \quad ; \quad \Delta \underline{u} = \underline{u}^{N+1} - \underline{u}^{N}$$

The relations between the above incremental variables are shown by,

$$\Delta \underset{\sim}{t} = \Delta \underset{\sim}{s} \cdot \underset{\sim}{F}^{N^T} + \underset{\sim}{s}^{N} \cdot \nabla \Delta \underline{u} + \underline{\Delta \underset{\sim}{s} \cdot \nabla \Delta \underline{u}} \tag{4.2}$$

$$\Delta \underset{\sim}{r} = \frac{1}{2}\Big(\Delta \underset{\sim}{t} \cdot \underset{\sim}{\alpha}^{N} + \underset{\sim}{t}^{N} \cdot \Delta \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^{N^T} \cdot \Delta \underset{\sim}{t}^{T} + \Delta \underset{\sim}{\alpha}^{T} \cdot \underset{\sim}{t}^{N^T} + \underline{\Delta \underset{\sim}{t} \cdot \Delta \underset{\sim}{\alpha} + \Delta \underset{\sim}{\alpha}^{T} \cdot \Delta \underset{\sim}{t}^{T}}\Big) \tag{4.3}$$

$$\Delta \underset{\sim}{g} = (1/2)\left(\underset{\sim}{F}^{N^T} \cdot \Delta \underset{\sim}{e} + \Delta \underset{\sim}{e}^{T} \cdot \underset{\sim}{F}^{N} + \underline{\Delta \underset{\sim}{e}^{T} \cdot \Delta \underset{\sim}{e}} \right) \tag{4.4}$$

$$\Delta \underset{\sim}{e} = \Delta \underset{\sim}{\alpha} \cdot \left(\underset{\sim}{I} + \underset{\sim}{h}^{N} \right) + \underset{\sim}{\alpha}^{N} \cdot \Delta \underset{\sim}{h} + \underline{\Delta \underset{\sim}{\alpha} \cdot \Delta \underset{\sim}{h}} \tag{4.5}$$

$$\Delta \underset{\sim}{e} = (\nabla \Delta \underline{u})^{T} \tag{4.6}$$

Noting that ∇ is still the gradient operator in the metric C_0, the following incremental field equations and the boundary conditions governing the transition from C_N to C_{N+1} are derived.

translational equilibrium conditions

$$\nabla \cdot \left\{ \underset{\sim}{s}^{N} \cdot (\nabla \Delta \underline{u}) + \Delta \underset{\sim}{s} \cdot (\underset{\sim}{F}^{N^T} + \underline{\nabla \Delta \underline{u}}) \right\} + \rho_0 \Delta \underline{g} = 0 \tag{4.7}$$

or $$\nabla \cdot \Delta \underset{\sim}{t} + \rho_0 \Delta \underline{g} = 0 \tag{4.8}$$

where the underlined terms are nonlinear in the incremental variables, and these are neglected in the linearized formulations.

rotational equilibrium conditions

$$\Delta \underset{\sim}{s}^{T} = \Delta \underset{\sim}{s} \qquad (4.9)$$

$$(\underset{\sim}{F}^{N} + \underline{\nabla \Delta \underline{u}^{T}}) \cdot \Delta \underset{\sim}{t} + \nabla \Delta u^{T} \cdot \underset{\sim}{t}^{N} = \text{symmetric} \qquad (4.10)$$

$$\Delta \underset{\sim}{h} \cdot \underset{\sim}{t}^{N} \cdot \underset{\sim}{\alpha}^{N} + (\underset{\sim}{I} + \underset{\sim}{h}^{N}) \cdot (\Delta \underset{\sim}{t} \cdot \underset{\sim}{\alpha}^{N} + \underset{\sim}{t}^{N} \cdot \Delta \underset{\sim}{\alpha}) + \underline{(\underset{\sim}{h}^{N} + \underset{\sim}{I}) \cdot \Delta \underset{\sim}{t} \cdot \Delta \underset{\sim}{\alpha}} \qquad (4.11)$$

$$+ \underline{\Delta \underset{\sim}{h} \, (\Delta \underset{\sim}{t} \cdot \underset{\sim}{\alpha}^{N} + \underset{\sim}{t}^{N} \cdot \Delta \underset{\sim}{\alpha}) + \Delta \underset{\sim}{h} \cdot \Delta \underset{\sim}{t} \cdot \Delta \underset{\sim}{\alpha}} = \text{symmetric}$$

<u>kinematic relations</u>

$$\Delta \underset{\sim}{g} = (1/2)\{ \nabla \Delta \underline{u} + \nabla \Delta \underline{u} + (\nabla \Delta \underline{u}) \cdot (\nabla \underline{u}^{N})^{T} \qquad (4.12)$$

$$+ (\nabla \underline{u}^{N}) \cdot (\nabla \Delta \underline{u})^{T} + \underline{(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^{T}} \}$$

$$\Delta \underset{\sim}{e} = (\nabla \Delta \underline{u})^{T} \qquad (4.13)$$

$$(\nabla \Delta \underline{u})^{T} = \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^{N}) + \underset{\sim}{\alpha}^{N} \cdot \Delta \underset{\sim}{h} + \underline{\Delta \underset{\sim}{\alpha} \cdot \Delta \underset{\sim}{h}} \qquad (4.14)$$

<u>constitutive relations</u>

Assuming that the strain energy density W is a symmetric function of $\underset{\sim}{g}$ and that it can be expanded in Taylor series in terms of $\Delta \underset{\sim}{g}$, $\Delta \underset{\sim}{e}$, or $\Delta \underset{\sim}{h}$, the incremental potential function ΔW is defined by,

$$\Delta W(\Delta \underset{\sim}{g}) = \frac{1}{2} \left. \frac{\partial^2 W}{\partial \underset{\sim}{g}^2} \right|^{N} :: \Delta \underset{\sim}{g} \, \Delta \underset{\sim}{g} + H.O.T. \qquad (4.15)$$

Using rectangular Cartesian components, Eq.(4.15) is rewritten as,

$$\Delta W(\Delta g_{ij}) = \frac{1}{2} \frac{\partial^2 W}{\partial g_{ij} \partial g_{kl}} \Delta g_{ij} \, \Delta g_{kl} + \underline{H.O.T.}$$

Similarly,

$$\Delta W(\Delta \underset{\sim}{e}) = \frac{1}{2} \frac{\partial^2 W}{\partial \underset{\sim}{e}^2}\bigg|^N :: \Delta \underset{\sim}{e}\, \Delta \underset{\sim}{e} \; + \; \underline{H.O.T.} \tag{4.16}$$

$$\Delta W(\Delta \underset{\sim}{h}) = \frac{1}{2} \frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\bigg|^N :: \Delta \underset{\sim}{h}\, \Delta \underset{\sim}{h} \; + \; \underline{H.O.T.} \tag{4.17}$$

such that,

$$\frac{\partial \Delta W(\Delta \underset{\sim}{g})}{\partial \Delta \underset{\sim}{g}} = \Delta \underset{\sim}{S} \tag{4.18}$$

$$\frac{\partial \Delta W(\Delta \underset{\sim}{e})}{\partial \Delta \underset{\sim}{e}} = \Delta \underset{\sim}{t}^T \tag{4.19}$$

$$\frac{\partial \Delta W(\Delta \underset{\sim}{h})}{\partial \Delta \underset{\sim}{h}} = \Delta \underset{\sim}{r} \tag{4.20}$$

where $\frac{\partial^2 W}{\partial \underset{\sim}{g}^2}\Big|^N$, $\frac{\partial^2 W}{\partial \underset{\sim}{e}^2}\Big|^N$, and $\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\Big|^N$ are the second order derivatives of W, which are evaluated at C_N state. As discussed in chapter II, the contact transformations of W in terms of $\underset{\sim}{s}$ and $\underset{\sim}{r}$ exist, leading to complementary energy density functions S and R, respectively. Thus, the incremental complementary energy density ΔS and ΔR can be defined by,

$$\Delta S(\Delta \underset{\sim}{S}) = \frac{1}{2} \frac{\partial^2 S}{\partial \underset{\sim}{S}^2}\bigg|^N :: \Delta \underset{\sim}{S}\, \Delta \underset{\sim}{S} \; + \; \underline{H.O.T.} \tag{4.21}$$

$$\Delta R(\Delta \underset{\sim}{r}) = \frac{1}{2} \frac{\partial^2 R}{\partial \underset{\sim}{r}^2}\bigg|^N :: \Delta \underset{\sim}{r}\, \Delta \underset{\sim}{r} \; + \; \underline{H.O.T.} \tag{4.22}$$

such that,

$$\frac{\partial \Delta S}{\partial \Delta \underset{\sim}{S}} = \Delta \underset{\sim}{g} \tag{4.23}$$

$$\frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} = \Delta \underset{\sim}{h} \tag{4.24}$$

However, there is no unique complementary energy density in terms of $\underset{\sim}{t}$.

At this point, we look closely the incremental stress-strain relation in terms of $\Delta \underset{\sim}{t}$ and $\Delta \underset{\sim}{e}$. Using the relation between $\Delta \underset{\sim}{g}$ and $\Delta \underset{\sim}{e}$ given by Eq.(4.4), the incremental strain energy density defined by Eq.(4.16) can be rewritten in the following quadratic form.

$$\Delta W(\Delta \underset{\sim}{e}) = \frac{1}{2}\frac{\partial W}{\partial \underset{\sim}{g}}\Big|^{N} : (\Delta \underset{\sim}{e}^{T} \cdot \Delta \underset{\sim}{e}) + \frac{1}{2}\frac{\partial^2 W}{\partial \underset{\sim}{g}^2}\Big|^{N} :: \Delta \underset{\sim}{g}\, \Delta \underset{\sim}{g} \tag{4.25}$$

where, $\Delta \underset{\sim}{g} = (1/2)(\underset{\sim}{F}^{N^T} \cdot \Delta \underset{\sim}{e} + \Delta \underset{\sim}{e}^{T} \cdot \underset{\sim}{F}^{N})$

The incremental stress $\Delta \underset{\sim}{t}$ is obtained through Eq.(4.25) as,

$$\Delta \underset{\sim}{t} = \frac{\partial W}{\partial \underset{\sim}{g}}\Big|^{N} \cdot \Delta \underset{\sim}{e}^{T} + \left(\frac{\partial^2 W}{\partial g^2}\Big|^{N} : \Delta \underset{\sim}{g} \right) \cdot \underset{\sim}{F}^{N^T} \tag{4.26}$$

Eq.(4.26) can be rewritten by,

$$\Delta \underset{\sim}{t} = \underset{\sim}{s}^{N} \cdot \Delta \underset{\sim}{e}^{T} + \Delta \underset{\sim}{s} \cdot \underset{\sim}{F}^{N^T}$$

It is noticed that from the symmetry of $\Delta \underset{\sim}{s}$, which is embedded in W, stress increment $\Delta \underset{\sim}{t}$ obtained by Eq.(4.26) satisfies the linearized incremental rotational equilibrium, Eq.(4.10). For convenience, we introduce rectangular Cartesian components, and rewrite Eq.(4.26) as,

$$\begin{aligned} \Delta t_{ji} &= s^{N}_{jn}\, \Delta e_{in} + E^{N}_{mnoj}\, F^{N}_{io}\, F^{N}_{km}\, \Delta e_{kn} \\ &= \Delta e_{kn} \left[s^{N}_{jn}\, \delta_{ki} + E^{N}_{mnoj}\, F^{N}_{io}\, F^{N}_{km} \right] \\ &= \Delta e_{kn}\, E^{*}_{ijkn} \end{aligned} \tag{4.27}$$

In the above, the following notations are used,

$$E^{N}_{mnoj} = \frac{\partial^2 W}{\partial g_{mn} \partial g_{oj}}\Big|^{N} \tag{4.28}$$

$$E^{*}_{ijkn} = s^{N}_{jn}\,\delta_{ki} + E^{N}_{mnoj}\,F^{N}_{io}\,F^{N}_{km} \tag{4.29}$$

It is noted that, from definition, E^{N}_{mnoj} has the symmetry properties,

$$E^{N}_{mnoj} = E^{N}_{ojmn} = E^{N}_{nmoj} = E^{N}_{nmjo} \tag{4.30}$$

Whereas, E^{*}_{ijkn} has the only one symmetry property, such that,

$$E^{*}_{ijkn} = E^{*}_{knij} \tag{4.31}$$

Thus, if Eq.(4.27) is written in matrix notation,

$$\underset{9x1}{\{\Delta t\}} = \underset{9x9}{[E^{*}]}\ \underset{9x1}{\{\Delta e\}} \tag{4.32}$$

It is noticed that, in the first increment of the present piecewise linear incremental process, if the initial configuration C_0 is unstrained, it follows that $s^{o}_{jn} = 0$; $F^{o}_{io} = \delta_{io}$; hence $E^{*}_{ijkl} = E^{o}_{ijkl}$, and hence the (9x9) matrix in Eq.(4.32) cannot be inverted due to the property as in Eq.(4.30). However, in the second increment, one can set $\underset{\sim}{s}^{1} = \Delta \underset{\sim}{t}$ (of the first stress increment), $\underset{\sim}{F}^{1}$ is nonzero, and hence the (9x9) matrix in Eq.(4.32) may, in general, be inverted. Assuming that Eq.(4.32) is invertible, we obtain,

$$\Delta e_{ji} = E^{*-1}_{ijkl}\,\Delta t_{kl} \quad \text{or} \quad \{\Delta e\} = [E^{*^{-1}}]\{\Delta t\} \tag{4.33}$$

where, in general, $E^{*-1}_{ijkl} = E^{*-1}_{klij}$. Thus, using

Eq.(4.33), the contact transformation can be established to find ΔT such that,

$$\frac{\partial \Delta T}{\partial \Delta t_{ij}} = \Delta e_{ji} = E^{*-1}_{ijkl} \Delta t_{kl} \tag{4.34}$$

If the incremental rotational equilibrium condition is inherently built into the structure of ΔT, then the condition,

$$\Delta e_{ij} t^{N}_{jk} + F^{N}_{ij} \Delta t_{jk} = \text{symmetric} \tag{4.35}$$

must be identically satisfied when $\Delta \underset{\sim}{e}$ is expressed in terms of $\Delta \underset{\sim}{t}$. Doing so, it is found that the rotational equilibrium condition is expressed by the necessary condition that,

$$E^{*-1}_{jimn} \Delta t_{mn} t^{N}_{jk} + F^{N}_{ij} \Delta t_{jk} \quad \text{must be symmetric} \tag{4.36}$$

It is easily seen that neither of two terms in the above expression is by itself symmetric. The other possible ways in which the above sum of two terms can be symmetric are (a) Firstly, one term is a transpose of the other; however, it is easy to see that this is not the case. (b) Secondly, the first term can be expressed as the sum of a symmetric term and the transpose of the second term. However, E^{*-1}_{jimn} (with the only symmetry property, $E^{*-1}_{jimn} = E^{*-1}_{mnji}$) cannot be analytically derived. Thus it appears impossible, at present, to prove Eq.(4.36).

Even though the symmetry (or lack of it) of the term in Eq.(4.36) can be decided computationally, for a specific problem, it appears that, in general, we cannot expect the symmetry of the said term. Thus, it appears that even though the incremental contact transformation can be achieved to find ΔT in terms of $\Delta \underset{\sim}{t}$; since the rotational equilibrium conditions cannot be proved to be built into the structure of ΔT, the attendant complementary energy principle has little significance.

Finally, to complete the statement of the boundary value problem, we state the boundary conditions as follows.

boundary conditions

$$\text{(a)} \quad \underline{n} \cdot \{ \underset{\sim}{S}^{N} \cdot (\nabla \Delta \underline{u}) + \Delta \underset{\sim}{S} \cdot (\underset{\sim}{F}^{N^T} + \underline{\nabla \Delta \underline{u}}) \} = \Delta \underline{t} = \Delta \underline{\bar{t}} \tag{4.37}$$

$$\text{or} \quad \underline{n} \cdot \Delta \underset{\sim}{t} = \Delta \underline{\bar{t}} \qquad \text{at } S_{\sigma_0} \tag{4.38}$$

where $\Delta \underline{\bar{t}}$ is the prescribed incremental traction at S_{σ_0}.

$$\text{(b)} \quad \Delta \underline{u} = \Delta \underline{\bar{u}} \qquad \text{at } S_{u_0} \tag{4.39}$$

where $\Delta \underline{\bar{u}}$ is the prescribed incremental displacement at S_{u_0}.

General Procedure

The above incremental governing equations which describe the transition from C_N to C_{N+1} can be cast into equivalent variational statements based on the incremental Hu-Washizu type functionals. The general procedure to obtain such functionals can be illustrated as follows. First, we construct general functionals which govern the deformation of the solid in C_{N+1} state. Symbolically, it is denoted by

$\pi(C_{N+1})$. The state variables $\{C_{N+1}\}$ in π can be replaced by $\{C_N + \Delta C\}$. Thus, the functional is considered as a function of the incremental variables $\{\Delta C\}$, and it can be rearranged in the following form.

$$\begin{aligned}\pi(C_{N+1}) &= \pi(C_N + \Delta C) \\ &= \pi(C_N) + \text{constant} + \pi^1(\Delta C) + \pi^2(\Delta C) + \pi^3(\Delta C)\end{aligned} \tag{4.40}$$

where

$\pi(C_N)$: the value of the functional for C_N state and it is considered to be constant.

$\pi^1(\Delta C)$: first order terms of ΔC.

$\pi^2(\Delta C)$: second order terms of ΔC.

$\pi^3(\Delta C)$: third and higher order terms of ΔC.

It can be shown, in general, that the variation $\delta\pi^1$ vanishes if state C_N truly satisfies relevant field equations and boundary conditions. It can be also shown that $\delta(\pi^2+\pi^3)=0$ leads to the fully nonlinear incremental governing equations presented by Eqns.(4.7) through (4.39). However, if the increments are sufficiently small, the incremental governing equations can be linearized, and these, in general, can be shown to follow from $\delta\pi^2=0$ for a given variational principle. In the subsequent discussions π^3 is ignored so that the linearized functionals are derived. However, if the terms in π^3 were omitted in all increments prior to C_N, the state C_N may not truly satisfy the relevant field equations and boundary conditions. Thus, $\delta\pi^1$ may not vanish. Therefore, in practical applications, it is necessary to retain π^1 to generate iterative "correction procedures" so that the path of the piecewise linear incremental solutions is kept from straying away from the true solution as little as possible. Depending on their respective physical interpretations, these iterative corrections can be called as "equilibrium correction iteration", "compatibility mismatch iteration", etc. Such iterations, based on physical arguments as above, are entirely analogous to the mathematical procedures used in imbedding techniques for solving a system of nonlinear algebraic equations [18].

Incremental Hu-Washizu Principles

Following the general procedure discussed in the above, linearized incremental Hu-Washizu functionals, in which π^3 is ignored, are constructed for alternate incremental stress and strain measures. At the same time, the functionals π^1 which lead to the iterative correction procedures are derived in the following.

Based on $\Delta \underset{\sim}{S}$ and $\Delta \underset{\sim}{g}$. An incremental Hu-Washizu principle governing the transition from C_N to C_{N+1}, corresponding to Eq.(3.1), is,

$$\pi^2_{HW}(\Delta \underline{u}, \Delta \underset{\sim}{g}, \Delta \underset{\sim}{S}) \tag{4.41}$$

$$= \int_{V_0} \{ \Delta W(\Delta \underset{\sim}{g}) - \rho_0 \Delta \underset{\sim}{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T]$$

$$- \Delta \underset{\sim}{S} : [\Delta \underset{\sim}{g} - \frac{1}{2}(\nabla \Delta \underline{u} + \nabla \Delta \underline{u}^T + \nabla \underline{u}^N \cdot \nabla \Delta \underline{u}^T + \nabla \Delta \underline{u} \cdot \nabla \underline{u}^{N^T})]$$

$$- \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, ds \quad - \int_{S_{u_0}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) \, ds$$

where, $\Delta \underline{t} = \underline{n} \cdot (\Delta \underset{\sim}{S} \cdot \underset{\sim}{F}^{N^T} + \underset{\sim}{S}^N \cdot \nabla \Delta \underline{u})$

and

$$\Delta W(\Delta \underset{\sim}{g}) = \frac{1}{2} \frac{\partial^2 W}{\partial \underset{\sim}{g}^2}\Big|^N :: \Delta \underset{\sim}{g}\, \Delta \underset{\sim}{g}$$

such that,

$$\frac{\partial \Delta W(\Delta \underset{\sim}{g})}{\partial \Delta \underset{\sim}{g}} = \Delta \underset{\sim}{S}$$

It is noted that the incremental strain energy density ΔW is also linearized. The first variation of Eq.(4.41) is shown to be,

$$\delta \pi^2_{HW} = \int_{V_0} \Big\{ \Big[\frac{\partial \Delta W(\Delta \underset{\sim}{g})}{\partial \Delta \underset{\sim}{g}} - \Delta \underset{\sim}{S} \Big] : \delta \Delta \underset{\sim}{g} \qquad (4.42)$$

$$- \Big[\Delta \underset{\sim}{g} - \frac{1}{2} \big\{ \nabla \Delta \underline{u} + \nabla \Delta \underline{u}^T + \nabla \underline{u}^N \cdot \nabla \Delta \underline{u}^T + \nabla \Delta \underline{u} \cdot \nabla \underline{u}^{N^T} \big\} \Big] : \delta \Delta \underset{\sim}{S}$$

$$- \Big[\nabla \cdot (\Delta \underset{\sim}{S} \cdot \underset{\sim}{F}^{N^T} + \underset{\sim}{S}^N \cdot \nabla \Delta \underline{u}) + \rho_0 \Delta \underline{g} \Big] \cdot \delta \Delta \underline{u} \Big\} dV$$

$$- \int_{S_{\sigma_0}} (\Delta \bar{\underline{t}} - \Delta \underline{t}) \cdot \delta \Delta \underline{u}\, ds \; - \int_{S_{u_0}} \delta \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}})\, ds$$

Thus, the linearized form of governing equations, Eqns.(4.7, 9, 12, 18, 37, and 39), are obtained from the stationarity condition of Eq.(4.41). As mentioned before, the field variables that extremize the linearized functional do not truly satisfy the governing equations in nonlinear form. Therefore, the solution obtained through linearized functional is considered as a first guess. The correction to the first guess for the C_N state is provided through π^1.

For the present functional, π^1 is shown to be,

$$\pi'(\Delta\underline{u}, \Delta\underset{\sim}{g}, \Delta\underset{\sim}{S}) = \int_{V_0} \{ [\frac{\partial W}{\partial \underset{\sim}{g}} \Big|^N - \underset{\sim}{S}^N] : \Delta\underset{\sim}{g} \tag{4.43}$$

$$- [\underset{\sim}{g}^N - \frac{1}{2}(\nabla\underline{u}^N + \nabla\underline{u}^{N^T} + \nabla\underline{u}^N \cdot \nabla\underline{u}^{N^T})] : \Delta\underset{\sim}{S}$$

$$- \rho_0 \underline{g}^N \cdot \Delta\underline{u} + \underset{\sim}{S}^N \cdot \underset{\sim}{F}^{N^T} : (\nabla\Delta\underline{u}) \} dV$$

$$- \int_{S_{\sigma_0}} \underline{\bar{t}}^N \cdot \Delta\underline{u}\, ds - \int_{S_{u_0}} \{ \underline{t}^N \cdot \Delta\underline{u} + \Delta\underline{t} \cdot (\underline{u}^N - \underline{\bar{u}}^N) \} ds$$

The variation of π^1 is obtained as,

$$\delta\pi' = \int_{V_0} \{ [\frac{\partial W}{\partial \underset{\sim}{g}} \Big|^N - \underset{\sim}{S}^N] : \delta\Delta\underset{\sim}{g} \tag{4.44}$$

$$+ [\underset{\sim}{g}^N - \frac{1}{2}(\nabla\underline{u}^N + \nabla\underline{u}^{N^T} + \nabla\underline{u}^N \cdot \nabla\underline{u}^{N^T})] : \delta\Delta\underset{\sim}{S}$$

$$- [\nabla \cdot (\underset{\sim}{S}^N \cdot \underset{\sim}{F}^{N^T}) + \rho_0 \underline{g}^N] \cdot \delta\Delta\underline{u} \} dV$$

$$- \int_{S_{\sigma_0}} \{ \underline{\bar{t}}^N - \underline{n} \cdot (\underset{\sim}{S}^N \cdot \underset{\sim}{F}^{N^T}) \} \cdot \delta\Delta\underline{u}\, ds$$

$$- \int_{S_{u_0}} \delta\Delta\underline{t} \cdot (\underline{u}^N - \underline{\bar{u}}^N) ds$$

Thus, the condition of vanishing of $\delta\pi^1$ ensures the satisfaction of the governing equations in their total form at C_N state.

<u>Based on $\Delta\underset{\sim}{t}$ and $\Delta\underset{\sim}{e}$.</u> Likewise, the incremental form of the Hu-Washizu functional corresponding to Eq.(3.4) is,

$$\pi^2_{HW}(\Delta\underline{u}, \Delta\underset{\sim}{e}, \Delta\underset{\sim}{t}) \tag{4.45}$$

$$= \int_{V_0} \{ \Delta W(\Delta \underset{\sim}{e}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} + \Delta \underset{\sim}{t}^T : (\nabla \Delta \underline{u}^T - \Delta \underset{\sim}{e}) \} dV$$

$$- \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, ds - \int_{S_{u_0}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) \, ds$$

where,

$$\Delta W(\Delta \underset{\sim}{e}) = \frac{1}{2} \frac{\partial^2 W}{\partial \underset{\sim}{e}^2}\bigg|^N :: \Delta \underset{\sim}{e} \, \Delta \underset{\sim}{e}$$

such that,

$$\frac{\partial \Delta W(\Delta \underset{\sim}{e})}{\partial \Delta \underset{\sim}{e}} = \Delta \underset{\sim}{t}^T$$

The stationarity condition of the above functional leads to the incremental governing equations, Eqns.(4.8, 10, 13, 19, 38, and 39), in their linearized form. As discussed earlier, the rotational equilibrium condition is also retained through the functional Eq.(4.45). For the functional Eq.(4.45), π^1 is obtained as,

$$\pi^1(\Delta \underline{u}, \Delta \underset{\sim}{e}, \Delta \underset{\sim}{t}) \tag{4.46}$$

$$= \int_{V_0} \{ [\frac{\partial W}{\partial \underset{\sim}{e}}\bigg|^N - \underset{\sim}{t}^{N^T}] : \Delta \underset{\sim}{e} + [(\nabla \underline{u}^N)^T - \underset{\sim}{e}^N] : \Delta \underset{\sim}{t}^T$$

$$- \rho_0 \underline{g}^N \cdot \Delta \underline{u} + \underset{\sim}{t}^N : \nabla \Delta \underline{u} \} dV$$

$$- \int_{S_{\sigma_0}} \bar{\underline{t}}^N \cdot \Delta \underline{u} \, ds - \int_{S_{u_0}} \{ \underline{t}^N \cdot \Delta \underline{u} + \Delta \underline{t} \cdot (\underline{u}^N - \bar{\underline{u}}^N) \} ds$$

<u>Based on $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{h}$.</u> Similarly the incremental form of the Hu-Washizu principle corresponding to Eq.(3.7) is,

$$\pi^2_{HW}(\Delta \underline{u}, \Delta \underset{\sim}{h}, \Delta \underset{\sim}{\alpha}, \Delta \underset{\sim}{t}) \tag{4.47}$$

$$= \int_{V_0} \{ \Delta W(\Delta \underset{\sim}{h}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} + \Delta \underset{\sim}{t}^T : [\nabla \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{\alpha}^N \cdot \Delta \underset{\sim}{h}]$$

$$- \underset{\sim}{t}^{N^T} : \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{t}^{N^T} : \Delta \underset{\sim}{\alpha} \cdot \Delta \underset{\sim}{h} \} dV$$

$$- \int_{S_{\sigma_0}} \Delta \underline{\bar{t}} \cdot \Delta \underline{u} \, ds - \int_{S_{u_0}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \underline{\bar{u}}) \, ds$$

where,

$$\Delta W(\Delta \underset{\sim}{h}) = \frac{1}{2} \left. \frac{\partial^2 W}{\partial \underset{\sim}{h}^2} \right|^N :: \Delta \underset{\sim}{h} \Delta \underset{\sim}{h}$$

such that,

$$\frac{\partial \Delta W}{\partial \Delta \underset{\sim}{h}} = \Delta \underset{\sim}{\Gamma}$$

Noting that the incremental rotation tensor is subjected to the orthogonality condition*, i.e.,

$$(\underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha})^T \cdot (\underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha}) = \underset{\sim}{I} \tag{4.48}$$

and its variation satisfies,

$$(\underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha})^T \cdot \delta \Delta \underset{\sim}{\alpha} = \text{skewsymmetric} \tag{4.49}$$

the first variation of Π_{HW} is obtained as,

$$\delta \Pi^2_{HW} = \int_{V_0} \{ [\frac{\partial \Delta W}{\partial \Delta \underset{\sim}{h}} - \frac{1}{2} (\Delta \underset{\sim}{t} \cdot \underset{\sim}{\alpha}^N + \underset{\sim}{t}^N \cdot \Delta \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^{N^T} \cdot \Delta \underset{\sim}{t}^T + \Delta \underset{\sim}{\alpha}^T \cdot \underset{\sim}{t}^{N^T})] : \delta \Delta \underset{\sim}{h} \tag{4.50}$$

$$+ [\nabla \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{\alpha}^N \cdot \Delta \underset{\sim}{h}] : \delta \Delta \underset{\sim}{t}^T$$

$$- [(\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot (\underset{\sim}{t}^N + \Delta \underset{\sim}{t}) + \Delta \underset{\sim}{h} \cdot \underset{\sim}{t}^N] \cdot (\underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha}) : [(\underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha})^T \cdot \delta \Delta \underset{\sim}{\alpha}]^T$$

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*Due to this nonlinear constraint condition, the term $\underset{\sim}{t}^{N^T} : \Delta \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)$, which is linear in $\Delta \underset{\sim}{a}$, is retained in the incremental functional given by Eq.(4.47).

$$-[\nabla\cdot\Delta\underset{\sim}{t} + \rho_0\Delta\underline{g}]\cdot\delta\Delta\underline{u}\}dV$$

$$-\int_{S_{\sigma_0}}(\Delta\bar{\underline{t}} - \Delta\underline{t})\cdot\delta\Delta\underline{u}\,ds \quad -\int_{S_{u_0}}\delta\Delta\underline{t}\cdot(\Delta\underline{u} - \Delta\bar{\underline{u}})ds$$

It is readily shown that the stationarity condition of the functional Eq.(4.47) leads to the linearized form of Eqns.(4.8, 14, 20, 38, and 39). However, because of the nonlinear constraint on $\Delta\underset{\sim}{\alpha}$, the rotational equilibrium condition can not be noticed immediately. We examine the stationarity condition with respect to $\Delta\underset{\sim}{\alpha}$. It requires the symmetry of the terms, such that,

$$[(\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot(\underset{\sim}{t}^N + \Delta\underset{\sim}{t}) + \Delta\underset{\sim}{h}\cdot\underset{\sim}{t}^N]\cdot(\underset{\sim}{\alpha}^N + \Delta\underset{\sim}{\alpha}) = \text{symmetric} \tag{4.51}$$

or

$$\Delta\underset{\sim}{h}\cdot\underset{\sim}{t}^N\cdot\underset{\sim}{\alpha}^N + (\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot(\Delta\underset{\sim}{t}\cdot\underset{\sim}{\alpha}^N + \underset{\sim}{t}^N\cdot\Delta\underset{\sim}{\alpha}) + [(\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot\Delta\underset{\sim}{t} + \Delta\underset{\sim}{h}\cdot\underset{\sim}{t}^N]\cdot\Delta\underset{\sim}{\alpha} = \text{symmetric} \tag{4.52}$$

If the higher order terms are ignored, it is seen that the above equation is reduced to the linearized form of rotational equilibrium condition Eq.(4.11). However, if the constraint condition on $\Delta\underset{\sim}{\alpha}$ is assumed to be satisfied up to the linear term, the constrait condition is reduced to,

$$\underset{\sim}{\alpha}^{N^T}\cdot\Delta\underset{\sim}{\alpha} = -\Delta\underset{\sim}{\alpha}^T\cdot\underset{\sim}{\alpha}^N = \text{skewsymmetric} \tag{4.53}$$

and in its variational form,

$$\underset{\sim}{\alpha}^{N^T}\cdot\delta\Delta\underset{\sim}{\alpha} = -\delta\Delta\underset{\sim}{\alpha}^T\cdot\underset{\sim}{\alpha}^N = \text{skewsymmetric} \tag{4.54}$$

In this case the stationarity condition requires that,

$$[(\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot\Delta\underset{\sim}{t} + \Delta\underset{\sim}{h}\cdot\underset{\sim}{t}^N]\cdot\underset{\sim}{\alpha}^N = \text{symmetric}$$

This equation is different from the linearized form of Eq.(4.11) by the term $(\underset{\sim}{I}+\underset{\sim}{h}^N)\cdot\underset{\sim}{t}^N\cdot\Delta\underset{\sim}{\alpha}$, because of the linearization of the orthogonality condition. Even for this case, we can show that the exact rotational equilibrium condition can be retained through π^1 . For the functional given by Eq.(4.47), π^1 is obtained as,

$$\pi^1(\Delta\underline{u},\Delta\underset{\sim}{h},\Delta\underset{\sim}{\alpha},\Delta\underset{\sim}{t}) \tag{4.55}$$

$$= \int_{V_0}\Big\{\Big[\frac{\partial W}{\partial \underset{\sim}{h}}\Big|^N - \frac{1}{2}(\underset{\sim}{t}^N\cdot\underset{\sim}{\alpha}^N + \underset{\sim}{\alpha}^{N^T}\cdot\underset{\sim}{t}^{N^T})\Big]:\Delta\underset{\sim}{h}$$

$$+\big[(\underset{\sim}{I}+\nabla\underline{u}^N)^T - \underset{\sim}{\alpha}^N\cdot(\underset{\sim}{I}+\underset{\sim}{h}^N)\big]:\Delta\underset{\sim}{t}^T$$

$$-(\underset{\sim}{I}+\underset{\sim}{h}^N)\cdot\underset{\sim}{t}^N:\Delta\underset{\sim}{\alpha}^T - \rho_0\underline{g}^N\cdot\Delta\underline{u} + \underset{\sim}{t}^N:\nabla\Delta\underline{u}\Big\}dV$$

$$-\int_{S_{\sigma_0}}\bar{\underline{t}}^N\cdot\Delta\underline{u}\,ds \quad -\int_{S_{u_0}}\{\underset{\sim}{t}^N\cdot\Delta\underline{u} + \Delta\underset{\sim}{t}\cdot(\underline{u}^N-\bar{\underline{u}}^N)\}ds$$

Noting the constraint on $\Delta\underset{\sim}{\alpha}$, its variation is shown to be,

$$\delta\pi^1 = \int_{V_0}\Big\{\Big[\frac{\partial W}{\partial \underset{\sim}{h}}\Big|^N - \frac{1}{2}(\underset{\sim}{t}^N\cdot\underset{\sim}{\alpha}^N + \underset{\sim}{\alpha}^{N^T}\cdot\underset{\sim}{t}^{N^T})\Big]:\delta\Delta\underset{\sim}{h} \tag{4.56}$$

$$+\big[(\underset{\sim}{I}+\nabla\underline{u}^N)^T - \underset{\sim}{\alpha}^N\cdot(\underset{\sim}{I}+\underset{\sim}{h}^N)\big]:\delta\Delta\underset{\sim}{t}^T$$

$$-\big[(\underset{\sim}{I}+\underset{\sim}{h}^N)\cdot\underset{\sim}{t}^N\cdot\underset{\sim}{\alpha}^N\big]:\big[\underset{\sim}{\alpha}^{N^T}\cdot\delta\Delta\underset{\sim}{\alpha}\big]^T$$

$$-\big[\nabla\cdot\underset{\sim}{t}^N + \rho_0\underline{g}^N\big]\cdot\delta\Delta\underline{u}\Big\}dV$$

$$-\int_{S_{\sigma_0}}(\bar{\underline{t}}^N - \underline{t}^N)\cdot\delta\Delta\underline{u}\,ds \quad -\int_{S_{u_0}}\delta\Delta\underline{t}\cdot(\underline{u}^N-\bar{\underline{u}}^N)ds$$

Thus, all the governing equations including the rotational equilibrium condition are retained from the vanishing condition of $\delta\pi^1$.

As shown in the above, the incremental Hu-Washizu principles based on alternate incremental stress and strain measures are constructed. From these general incremental functionals, special types of incremental functionals and corresponding variational principles are derived in the following.

Incremental Potential Energy Principles

Based on $\Delta \underset{\sim}{s}$ and $\Delta \underset{\sim}{g}$. By a priori satisfying the incremental constitutive relation, kinematic relation and displacement boundary condition, Eqns.(4.12, 18, and 39), the functional in Eq.(4.41) is reduced to,

$$\pi_P^2(\Delta \underline{u}) = \int_{V_0} \left\{ \Delta W(\Delta \underset{\sim}{g}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T] \right\} dV - \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, dS \tag{4.57}$$

Its stationarity condition leads to incremental translational and rotational equilibrium conditions and also traction boundary condition, i.e. Eqns.(4.7), (4.9), and (4.37).

Based on $\Delta \underset{\sim}{t}$ and $\Delta \underset{\sim}{e}$. Analogous functionals based on $\Delta \underset{\sim}{t}$ can be derived, as,

$$\pi_P^2(\Delta \underline{u}) = \int_{V_0} \left\{ \Delta W(\Delta \underset{\sim}{e}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} \right\} dV - \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, dS \tag{4.58}$$

However, from the relation between $\Delta W(\Delta \underset{\sim}{e})$ and $\Delta W(\Delta \underset{\sim}{g})$ given by Eq.(4.25), the above functional can be shown to be identical to that given by Eq.(4.57).

Incremental Hellinger-Reissner Principles

Based on $\Delta \underset{\sim}{S}$ and $\Delta \underset{\sim}{g}$. By a priori satisfying the incremental constitutive relation Eq.(4.18) and introducing the incremental complementary energy density ΔS given by Eq.(4.21), the functional Eq(4.41) is reduced to the Hellinger-Reissner principle in its incremental form.

$$
\begin{aligned}
&\pi^2_{HR}(\Delta \underline{u}, \Delta \underset{\sim}{S}) \qquad (4.59)\\
&= \int_{V_0} \Big\{ -\Delta S(\Delta \underset{\sim}{S}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T] \\
&\quad + \frac{1}{2} \Delta \underset{\sim}{S} : [\nabla \Delta \underline{u} + \nabla \Delta \underline{u}^T + \nabla \underline{u}^N \cdot \nabla \Delta \underline{u}^T + \nabla \Delta \underline{u} \cdot \nabla \underline{u}^{N^T}] \Big\} dV \\
&\quad - \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, dS - \int_{S_{u_0}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) dS
\end{aligned}
$$

wherein,

$$\Delta S(\Delta \underset{\sim}{S}) = \frac{1}{2} \left. \frac{\partial^2 S}{\partial \underset{\sim}{S}^2} \right|^N :: \Delta \underset{\sim}{S} \, \Delta \underset{\sim}{S}$$

such that,

$$\frac{\partial \Delta S}{\partial \Delta \underset{\sim}{S}} = \Delta \underset{\sim}{g}$$

Its stationarity condition leads to the translational equilibrium condition Eq.(4.7), rotational equilibrium condition Eq.(4.9), kinematic relation Eq.(4.12), and boundary conditions Eqns.(4.37) and (4.39).

Based on $\Delta \underset{\sim}{t}$ and $\Delta \underset{\sim}{e}$. As shown earlier, if unique inverse stress-strain relation in terms of $\Delta \underset{\sim}{t}$ and $\Delta \underset{\sim}{e}$ is assumed, an incremental contact transformation can be achieved to express $-\Delta W(\Delta \underset{\sim}{e}) + \Delta \underset{\sim}{t}^T : \Delta \underset{\sim}{e} = \Delta T$ in terms of $\Delta \underset{\sim}{t}$ alone and thus formally derive an incremental Hellinger-Reissner functional from Eq.(4.45). However, as also shown earlier, since the incremental rotational equilibrium conditions are then not embedded in the structure of the thus derived incremental complementary energy density (in terms of $\Delta \underset{\sim}{t}$ alone), this formal Hellinger-Reissner Principle has no practical use. The same argument applies to the incremental complementary energy principle in terms of $\Delta \underset{\sim}{t}$ alone, which can be formally derived from Eq.(4.45) by using the contact transformation ΔT and satisfying the translational equilibrium condition and the traction boundary condition a priori.

<u>Based on $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{h}$ ($\Delta \underset{\sim}{t}$, $\Delta \underset{\sim}{\alpha}$, and $\Delta \underset{\sim}{h}$).</u> The substitution of ΔR, defined by Eq.(4.22), into the functional Eq.(4.47) leads to the incremental form of Hellinger-Reissner principle, the functional corresponding to which is,

$$\pi^2_{HR}(\Delta \underline{u}, \Delta \underset{\sim}{\alpha}, \Delta \underset{\sim}{t}) \qquad (4.60)$$

$$= \int_{V_0} \Big\{ -\Delta R(\Delta \underset{\sim}{r}) - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} - \underset{\sim}{t}^{N^T} : \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)$$

$$+ \Delta \underset{\sim}{t}^T : [\nabla \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)\Big\} dV$$

$$- \int_{S_{\sigma_0}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, ds - \int_{S_{u_0}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) ds$$

wherein,

$$\Delta R(\Delta \underset{\sim}{r}) = \frac{1}{2} \left. \frac{\partial^2 R}{\partial \underset{\sim}{r}^2} \right|^N :: \Delta \underset{\sim}{r} \, \Delta \underset{\sim}{r}$$

such that,

$$\frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} = \Delta \underset{\sim}{h}$$

and $\Delta \underset{\sim}{r} = (1/2)\{ \underset{\sim}{t}^N \cdot \Delta \underset{\sim}{\alpha} + \Delta \underset{\sim}{t} \cdot \underset{\sim}{\alpha}^N + \Delta \underset{\sim}{\alpha}^T \cdot \underset{\sim}{t}^{N^T} + \underset{\sim}{\alpha}^{N^T} \cdot \Delta \underset{\sim}{t}^T \}$

Its stationarity condition leads to Eqns.(4.8, 11, 14, 38, and 39).

Incremental Complementary Energy Principle

Based on $\Delta \underset{\sim}{s}$ and $\Delta \underline{g}$. By a priori satisfying, if possible, the translational equilibrium condition Eq.(4.7) and the traction boundary condition Eq.(4.37), functional defined by Eq.(4.59) is reduced to the incremental complementary energy principle,

$$\pi_c^2 (\Delta \underline{u}, \Delta \underset{\sim}{S}) = \int_{V_0} \{ \Delta S(\Delta \underset{\sim}{S}) + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T] \} dV - \int_{S_{u_0}} \Delta \underline{t} \cdot \Delta \underline{\bar{u}} \, ds \qquad (4.61)$$

Noting that $\Delta \underset{\sim}{s}$ and $\Delta \underline{u}$ are subjected to the constraint conditions, Eqns.(4.7) and (4.37), its stationarity condition leads to Eqns.(4.9, 12, and 39). However, as seen from Eq.(4.7), the incremental translational equilibrium condition is a set of nonlinear and coupled partial differential equations involving both $\Delta \underset{\sim}{s}$ and $\Delta \underline{u}$. It is impossible to choose admissible functions $\Delta \underset{\sim}{s}$ which exactly satisfy Eq.(4.7). In the practical application, the constraint condition, Eq.(4.7), may be linearized and it is reduced to,

$$\nabla \cdot \{ \underset{\sim}{s}^N \cdot (\nabla \Delta \underline{u}) + \Delta \underset{\sim}{s} \cdot \underset{\sim}{F}^{N^T} \} + \rho_0 \Delta \underline{g} = 0 \qquad (4.7)^*$$

Even in the above linearized equation, there is a strong coupling between $\underset{\sim}{S}^N$ and $\Delta\underline{u}$ on the one hand, and between $\Delta\underset{\sim}{S}$ and $\underset{\sim}{F}^N$ on the other. We notice that $\underset{\sim}{S}^N(\underline{x})$ and $\underset{\sim}{F}^N(\underline{x})$ are, in general, numerical solutions obtained up to the current stage and also they are functions of $\underline{x}$. Thus, the admissible stress field, to be used in a complementary energy principle, must represent a solution to the set of partial differential equations, Eq.(4.7)*, with variable coefficients. While, it may be mathematically possible to find such stress $\Delta\underset{\sim}{S}$, it defeats the very purpose of a variational principle forming the basis of a simple numerical method such as the finite element method. Thus, there does dot exist a practically useful incremental complementary energy principle, in the total Lagrangean form, when the Kirchhoff-Trefftz stress measure is used.

Even if $\Delta\underset{\sim}{S}$ were chosen somehow, so that Eq.(4.7)* is satisfied a priori, the associated functional in Eq.(4.61) involves both $\Delta\underset{\sim}{S}$ and $\Delta\underline{u}$. Moreover, Eq.(4.7)* is a linear approximation of the translational equilibrium condition. The correction to this approximation need to be retained in the iterative correction based on π^1. Including the correction terms to account for the prior linearization of the translational equilibrium condition, the functional π^1 is obtained as,

$$\pi^1(\Delta\underline{u},\Delta\underset{\sim}{S}) = \int_{V_0}\Big\{\Big[\frac{\partial S}{\partial \underset{\sim}{S}}\Big|^N - \underset{\sim}{g}^N\Big]:\Delta\underset{\sim}{S} + \rho_0\,\underline{g}^N\cdot\Delta\underline{u} \qquad (4.62)$$

$$- \underset{\sim}{S}^N\cdot\underset{\sim}{F}^{N^T}:(\nabla\Delta\underline{u})\Big\}\,dV$$

$$+\int_{S_{\sigma_0}} \bar{\underline{t}}^N \cdot \Delta\underline{u}\, ds \quad + \int_{S_{u_0}} \{ \underline{t}^N \cdot \Delta\underline{u} + \Delta\underline{t} \cdot (\underline{u}^N - \bar{\underline{u}}^N) \} ds$$

which is identical to that for incremental Hellinger-Reissner principle. For these two reasons, from a computational view point, it may be preferable and consistent to use the Hellinger-Reissner principle rather than a conplementary energy principle based on $\Delta\underset{\sim}{s}$.

<u>Based on $\Delta\underset{\sim}{r}$ and $\Delta\underset{\sim}{h}$.</u> The futility of deriving an incremental complementary energy principle in terms of $\Delta\underset{\sim}{t}$ has already been discussed above. We thus turn to the third alternative, a complementary energy principle based on $\Delta\underset{\sim}{r}$ and $\Delta\underset{\sim}{h}$, as below.

The difficulties detected for the functional Eq.(4.61) can be avoided if the incremental Hellinger-Reissner functional based on the incremental Jaumann stress is used. First of all, the incremental translational equilibrium condition Eq.(4.8) and the traction boundary condition Eq.(4.38) are linear equations, in $\Delta\underset{\sim}{t}$ alone, which can be easily satisfied. By the a priori satisfaction of the above equations, the incremental Hellinger-Reissner functional Eq.(4.60) is reduced to the incremental complementary energy functional π_c^2, involving only stress $\Delta\underset{\sim}{t}$ and rotation $\Delta\underset{\sim}{\alpha}$.

$$\pi_c^2 (\Delta\underset{\sim}{t}, \Delta\underset{\sim}{\alpha}) = \int_{V_0} \{ \Delta R(\Delta\underset{\sim}{r}) + (\underset{\sim}{t}^N + \Delta\underset{\sim}{t})^T : \Delta\underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) \} dV \quad (4.63)$$

$$-\int_{S_{u_0}} \Delta\underline{t} \cdot \Delta\bar{\underline{u}}\, ds$$

Moreover, as is already discussed for the general functional Eq.(4.47), the rotational equilibrium condition is shown to be enforced through the stationarity condition of the above functional with respect to the variation in $\Delta\underset{\sim}{\alpha}$. Noting the constraint conditions on $\Delta\underset{\sim}{t}$ and $\Delta\underset{\sim}{\alpha}$, i.e. Eqns.(4.8) and (4.48), its first variation is obtained as,

$$\delta\Pi_c^2 = \int_{V_0} \{ [(\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot(\underset{\sim}{t}^N + \Delta\underset{\sim}{t}) + \frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} \cdot \underset{\sim}{t}^N] : \delta\Delta\underset{\sim}{\alpha}^T \tag{4.64}$$

$$+ [\underset{\sim}{\alpha}^N \cdot \frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} + \Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^N)] : \delta\Delta\underset{\sim}{t}^T$$

$$- \int_{S_{u_0}} \delta\Delta\underline{t}\cdot\Delta\bar{\underline{u}}\, ds$$

Using an identical equation,

$$\int_{V_0} (\nabla\Delta\underline{u})^T : \delta\Delta\underset{\sim}{t}^T dV = \int_{S_{u_0}} \delta\Delta\underline{t}\cdot\Delta\underline{u}\, ds \tag{4.65}$$

Eq.(4.64) is rewritten as,

$$\delta\Pi_c^2 = \int_{V_0} \{ [(\underset{\sim}{I} + \underset{\sim}{h}^N)\cdot(\underset{\sim}{t}^N + \Delta\underset{\sim}{t}) + \frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} \cdot \underset{\sim}{t}^N] : \delta\Delta\underset{\sim}{\alpha}^T \tag{4.66}$$

$$- [(\nabla\Delta\underline{u})^T - \underset{\sim}{\alpha}^N \cdot \frac{\partial \Delta R}{\partial \Delta \underset{\sim}{r}} - \Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^N)] : \delta\Delta\underset{\sim}{t}^T \} dV$$

$$- \int_{S_{u_0}} \delta\Delta\underline{t}\cdot(\Delta\bar{\underline{u}} - \Delta\underline{u})\, ds$$

Thus, from an analogous argument as for incremental Hu-Washizu functional, Eq.(4.47), it is readily seen that the incremental kinematic relation Eq.(4.14) , the displacement boundary condition Eq.(4.39), and the rotational equilibrium condition Eq.(4.11), in their linear form, follow from the stationarity of the functional in Eq.(4.63). Also, even if the orthogonality condition, Eq.(4.48), is satisfied only up to linear terms, the exact rotational equilibrium condition can be retained through π^1, which is shown to be,

$$\pi'(\Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}) = \int_{V_0} \left\{ \left[\underset{\sim}{\alpha}^N \cdot \left(\underset{\sim}{I} + \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^N\right) - \underset{\sim}{I} \right] : \Delta\underset{\sim}{t}^T + \left[\left(\underset{\sim}{I} + \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^N\right) \cdot \underset{\sim}{t}^N \right] : \Delta\underset{\sim}{\alpha}^T \right\} dV - \int_{S_{u_0}} \Delta\underline{t} \cdot \underline{\bar{u}}^N ds \tag{4.67}$$

It is noted that, unlike for the functional Eq.(4.61), the correction of the translational equilibrium condition, which is exactly satisfied a priori, is not necessary.

Hybrid Type Incremental Variational Principles

Now, we turn to the application of the incremental variational principles discussed in the preceding sections to a finite element assembly . Let the continuum V_0 be divided into a finite set of nonoverlapping subdomains V_{0_m} (m=1, ...M) the boundaries of which are ∂V_{0_m} . It is easy to picture that in general, $\partial V_{0_m} = S_{\sigma_{0_m}} + S_{u_{0_m}} + \rho_{0_m}$; where $S_{\sigma_{0_m}}$ and $S_{u_{0_m}}$ are portions of ∂V_{0_m}, which coincide with the overall boundary of V_0, where tractions and displacements, respectively, are prescribed ; and ρ_{0_m} is the portion of an element boundary which is common to that of an adjoining element (inter-element boundary). It is easy to see that for an element which is completely surrounded by other elements, $\partial V_{0_m} = \rho_{0_m}$. Further, let us arbitrarily denote one side of ρ_{0_m}, as ρ_{0_m} is approached, by the superscript (+) and the other side of ρ_{0_m}, similarly , by a (-). Then it is seen that to obtain a physically meaningful solution certain continuity conditions must be satisfied at the inter-element boundary in addition to the field equations and the boundary conditions required for the continuum body. For the present case, the following conditions are required,

(a) displacement continuity condition

$$\Delta \underline{u}^{+} = \Delta \underline{u}^{-} \qquad \text{at } \rho_{0_m} \qquad (4.68)$$

(b) traction reciprocity

$$\Delta \underline{t}^{+} + \Delta \underline{t}^{-} = (\underline{n} \cdot \Delta \underset{\sim}{t})^{+} + (\underline{n} \cdot \Delta \underset{\sim}{t})^{-} = 0 \qquad \text{at } \rho_{0_m} \qquad (4.69)$$

If the ordinary incremental variational principles such as Eq.(4.41), in which only continuous functions are allowed as admissible functions, is directly applied to the finite element method, the assumed function defined in each element must satisfy the continuity conditions given by Eqns.(4.68) and (4.69). However, the choice of such functions is very limited. In some cases it is nearly impossible. Therefore, to preserve the wide choice of the assumed functions, the functionals are modified so that functions, which do not satisfy the continuity at inter-element boundaries, are allowed as admissible functions. Such modification can be achieved by introducing Lagrange multipliers, through which the continuity conditions are relaxed a priori. Then, these relaxed conditions are enforced, at least in the weighted residual sense, through the stationarity condition of the modified functional with respect to the Lagrange multipliers. These modified functionals lead to the incremental hybrid finite element models analogous to that first developed for linear problems by Pian [16].

There are two alternate ways to achieve such modifications, as discussed by Atluri [25]. These two versions of modified incremental functionals in terms of alternate stress and strain measures are derived in the following.

<u>Modified Incremental Hu-Washizu Principles</u>

<u>Based on $\Delta \underset{\sim}{S}$ and $\Delta \underset{\sim}{g}$.</u> The two versions of the modified functionals corresponding to the functional Eq.(4.41) are derived. They are shown together with π^1.

i) first version

$$\pi^2_{HWM1}\ (\Delta \underline{u}, \Delta \underset{\sim}{g}, \Delta \underset{\sim}{S}, \Delta \underline{\tilde{t}}_\rho) \tag{4.70}$$

$$= \sum_m \int_{V_{o_m}} \{ \Delta W(\Delta \underset{\sim}{g}) - \rho_o \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T]$$

$$- \Delta \underset{\sim}{S} : [\Delta \underset{\sim}{g} - \frac{1}{2} (\nabla \Delta \underline{u} + \nabla \Delta \underline{u}^T + \nabla \underline{u}^N \cdot \nabla \Delta \underline{u}^T + \nabla \Delta \underline{u} \cdot \nabla \underline{u}^{N^T})] \} dV$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u}\, ds \ - \sum_m{}' \int_{S_{u_{o_m}}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}})\, ds$$

$$- \sum_m \int_{\rho_{o_m}} \Delta \underline{\tilde{t}}_\rho \cdot \Delta \underline{u}\, ds$$

where $\Delta\tilde{\underline{t}}_\rho$ is a Lagrange multiplier (physically the traction at the inter-element boundary), whose magnitude is uniquely defined at the inter-element boundary but opposite signs are taken for two adjoinig elements, such that,

$$\Delta\tilde{\underline{t}}_\rho^{+} + \Delta\tilde{\underline{t}}_\rho^{-} = 0 \tag{4.71}$$

Also, π^1 corresponding to Eq.(4.70) is obtained as,

$$\pi'(\Delta\underline{u}, \Delta\underset{\sim}{g}, \Delta\underset{\sim}{S}, \Delta\tilde{\underline{t}}_\rho) \tag{4.72}$$

$$= \sum_m \int_{V_{o_m}} \{ [\frac{\partial W}{\partial \underset{\sim}{g}} \Big|^N - \underset{\sim}{S}^N] : \Delta\underset{\sim}{g} - \rho_o \underline{g}^N \cdot \Delta\underline{u} + \underset{\sim}{S}^N \cdot \underset{\sim}{F}^{NT} : (\nabla\Delta\underline{u})$$

$$- [\underset{\sim}{g}^N - \frac{1}{2} (\nabla\underline{u}^N + \nabla\underline{u}^{NT} + \nabla\underline{u}^N \cdot \nabla\underline{u}^{NT})] : \Delta\underset{\sim}{S} \} dV$$

$$- \sum_m \int_{S_{\sigma o_m}} \bar{\underline{t}}^N \cdot \Delta\underline{u}\, ds - \sum_m \int_{S_{u o_m}} \{ \underline{t}^N \cdot \Delta\underline{u} + \Delta\underline{t} \cdot (\underline{u}^N - \bar{\underline{u}}^N) \} ds$$

$$- \sum_m \int_{\rho_{o_m}} \{ \Delta\tilde{\underline{t}}_\rho \cdot \underline{u}^N + \tilde{\underline{t}}_\rho^N \cdot \Delta\underline{u} \} ds$$

The first variation of the functional Eq.(4.70) is shown to be,

$$\delta\pi^2_{HWM1} = \sum_m \int_{V_{o_m}} \{ [\frac{\partial \Delta W}{\partial \Delta\underset{\sim}{g}} - \Delta\underset{\sim}{S}] : \delta\Delta\underset{\sim}{g} \tag{4.73}$$

$$- [\Delta\underset{\sim}{g} - \frac{1}{2} (\nabla\Delta\underline{u} + \nabla\Delta\underline{u}^T + \nabla\underline{u}^N \cdot \nabla\Delta\underline{u}^T + \nabla\Delta\underline{u} \cdot \nabla\underline{u}^{NT})] : \delta\Delta\underset{\sim}{S}$$

$$- [\nabla \cdot (\Delta\underset{\sim}{S} \cdot \underset{\sim}{F}^{NT} + \underset{\sim}{S}^N \cdot \nabla\Delta\underline{u}) + \rho_o \Delta\underline{g}] \cdot \delta\Delta\underline{u} \} dV$$

$$-\sum_m \int_{S_{\sigma_{0_m}}} (\Delta\bar{\underline{t}} - \Delta\underline{t}) \cdot \delta\Delta\underline{u}\, ds \quad - \sum_m \int_{S_{u_{0_m}}} \delta\Delta\underline{t} \cdot (\Delta\underline{u} - \Delta\bar{\underline{u}})\, ds$$

$$-\sum_m \int_{\rho_{0_m}} \{ \delta\Delta\tilde{\underline{t}}_\rho \cdot \Delta\underline{u} + (\Delta\tilde{\underline{t}}_\rho - \Delta\underline{t}) \cdot \delta\Delta\underline{u} \} ds$$

Noting that the integral along the inter-element boundary is evaluated for element boundaries of two adjoining elements, the vanishing condition of the last integral in Eq.(4.73) requires that,

$$\Delta\underline{u}^+ = \Delta\underline{u}^- \qquad \text{at } \rho_{0_m} \qquad (4.74)$$

$$\text{and} \quad \Delta\underline{t}^\pm = \Delta\tilde{\underline{t}}_\rho^\pm \quad \text{or} \quad \Delta\underline{t}^+ + \Delta\underline{t}^- = 0 \qquad \text{at } \rho_{0_m} \qquad (4.75)$$

Thus, it is seen that the stationarity condition of the functional in Eq.(4.70) leads to the continuity conditions at inter-element boundaries and also all the field equations and the boundary conditions given by Eqns.(4.7, 9, 12, 18, 37, 39, 68, and 69).

ii) second version

$$\pi^2_{HWM2}(\Delta\underline{u}, \Delta\underset{\sim}{g}, \Delta\underset{\sim}{S}, \Delta\tilde{\underline{u}}_\rho, \Delta\underline{t}_\rho) \qquad (4.76)$$

$$= \{\text{first three terms are the same as in Eq.(4.70)}\}$$

$$-\sum_m \int_{\rho_{0_m}} \Delta\underline{t}_\rho \cdot (\Delta\underline{u} - \Delta\tilde{\underline{u}}_\rho)\, ds$$

where $\Delta\tilde{\underline{u}}_\rho$ is an incremental displacement vector uniquely defined at the inter-element boundary, whereas, $\Delta\underline{t}_\rho$ is a Lagrange multiplier (traction at ρ_{0_m} of an element) defined independently for each of two adjoining elements. Also, π^1 corresponding to Eq.(4.76) is obtained as,

$$\pi'(\Delta \underline{u}\ ,\ \Delta \underset{\sim}{g}\ ,\ \Delta \underset{\sim}{S}\ ,\ \Delta \underline{\tilde{u}}_\rho\ ,\ \Delta t_\rho\ ) \tag{4.77}$$

= {first three terms are the same as in Eq.(4.72)}

$$-\sum_m \int_{\rho_{o_m}} \{\ \underline{t}^N_\rho \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho) + \Delta \underline{t}_\rho \cdot (\underline{u}^N - \underline{\tilde{u}}^N_\rho)\} ds$$

The first variation of the functional Eq.(4.76) is obtained as,

$$\delta\, \pi^2_{HWM2} \tag{4.78}$$

= {first three terms are the same as in Eq.(4.73)}

$$-\sum_m \int_{\rho_{o_m}} \{\delta \Delta \underline{t}_\rho \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho) + (\Delta \underline{t}_\rho - \Delta \underline{t}) \cdot \delta \Delta \underline{u} - \Delta \underline{t}_\rho \cdot \delta \Delta \underline{\tilde{u}}_\rho \} ds$$

From the analogous argument as in the first version the vanishing condition of the last integral requires that,

$$\Delta \underline{u}^+ = \Delta \underline{\tilde{u}}_\rho = \Delta \underline{u}^- \tag{4.79}$$

$$\left.\begin{array}{l} \Delta \underline{t}^\pm = \Delta \underline{t}^\pm_\rho \\ \Delta \underline{t}^+_\rho + \Delta \underline{t}^-_\rho = 0 \end{array}\right\} \quad \Delta \underline{t}^+ + \Delta \underline{t}^- = 0 \tag{4.80}$$

However, as it is noticed, the Lagrange multiplier $\Delta \underline{t}_\rho$ is identified as the traction on the element boundary, defined by,

$$\Delta \underline{t}_\rho = \Delta \underline{t} = \underline{n} \cdot (\Delta \underset{\sim}{S} \cdot \underset{\sim}{F}^{N^T} + \underset{\sim}{S}^N \cdot \nabla \Delta \underline{u})$$

Thus, by a priori choosing Lagrange multiplier as $\Delta \underline{t}$, the variable $\Delta \underline{t}_\rho$ in Eq.(4.76) can be eliminated, and the functional is reduced to,

$$\pi^2_{HWM2}\,(\Delta\underline{u},\Delta\underset{\sim}{g},\Delta\underset{\sim}{S},\Delta\underline{\tilde{u}}_\rho) \tag{4.81}$$

= {first three terms are the same as in Eq.(4.76)}

$$-\sum_m\int_{\rho_{o_m}}\Delta\underline{t}\cdot(\Delta\underline{u}-\Delta\underline{\tilde{u}}_\rho)\,ds$$

<u>Based on $\Delta\underset{\sim}{t}$ and $\Delta\underset{\sim}{e}$.</u> Similarly, the modified functionals corresponding to Eq.(4.45) are derived as,

i) first version

$$\pi^2_{HWM1}\,(\Delta\underline{u},\Delta\underset{\sim}{e},\Delta\underset{\sim}{t},\Delta\underline{\tilde{t}}_\rho) \tag{4.82}$$

$$=\sum_m\int_{V_{o_m}}\{\Delta W(\Delta\underset{\sim}{e})-\rho_o\Delta\underline{g}\cdot\Delta\underline{u}+\Delta\underset{\sim}{t}^T:(\nabla\Delta\underline{u}^T-\Delta\underset{\sim}{e})\}dV$$

$$-\sum_m\int_{S_{\sigma_{o_m}}}\Delta\underline{\bar{t}}\cdot\Delta\underline{u}\,ds-\sum_m\int_{S_{u_{o_m}}}\Delta\underline{t}\cdot(\Delta\underline{u}-\Delta\underline{\bar{u}})\,ds$$

$$-\sum_m\int_{\rho_{o_m}}\Delta\underline{\tilde{t}}_\rho\cdot\Delta\underline{u}\,ds$$

$$\pi'(\Delta\underline{u},\Delta\underset{\sim}{e},\Delta\underset{\sim}{t},\Delta\underline{\tilde{t}}_\rho) \tag{4.83}$$

$$=\sum_m\int_{V_{o_m}}\{[\frac{\partial W}{\partial\underset{\sim}{e}}\Big|^N-\underset{\sim}{t}^{N^T}]:\Delta\underset{\sim}{e}+[(\nabla\underline{u}^N)^T-\underset{\sim}{e}^N]:\Delta\underset{\sim}{t}^T$$

$$-\rho_o\underline{g}^N\cdot\Delta\underline{u}+\underset{\sim}{t}^N:\nabla\Delta\underline{u}\}dV-\sum_m\int_{S_{\sigma_{o_m}}}\underline{\bar{t}}^N\cdot\Delta\underline{u}\,ds$$

$$-\sum_m\int_{S_{u_{o_m}}}\{\underline{t}^N\cdot\Delta\underline{u}+\Delta\underline{t}\cdot(\underline{u}^N-\underline{\bar{u}}^N)\,ds$$

$$-\sum_m\int_{\rho_{o_m}}\{\Delta\underline{\tilde{t}}_\rho\cdot\underline{u}^N+\underline{\tilde{t}}_\rho^N\cdot\Delta\underline{u}\}ds$$

ii) second version

$$\pi^{2}_{HWM2}\,(\Delta\underline{u}\,,\Delta\underset{\sim}{e}\,,\,\Delta\underset{\sim}{t}\,,\Delta\tilde{\underline{u}}_{\rho}\,) \tag{4.84}$$

= {first three terms are the same as in Eq.(4.82)}

$$-\sum_m \int_{\rho_{o_m}} \Delta\underline{t}\cdot(\Delta\underline{u}-\Delta\tilde{\underline{u}}_{\rho})\,ds$$

$$\pi'\,(\Delta\underline{u}\,,\Delta\underset{\sim}{e}\,,\Delta\underset{\sim}{t}\,,\,\Delta\tilde{\underline{u}}_{\rho}\,) \tag{4.85}$$

= {first three trems are the same as in Eq.(4.83)}

$$-\sum_m \int_{\rho_{o_m}} \{\,\underline{t}^{N}\cdot(\Delta\underline{u}-\Delta\tilde{\underline{u}}_{\rho})+\Delta\underline{t}\cdot(\,\underline{u}^{N}-\tilde{\underline{u}}^{N}_{\rho})\}\,ds$$

The stationarity conditions of the functionals given by Eqns.(4.82) and (4.84) lead to Eqns.(4.8, 10, 13, 19, 38, and 39) and also the continuity condition at the inter-element boundary, given by Eqns.(4.68) and (4.69).

<u>Based on $\Delta\underset{\sim}{r}$ and $\Delta\underset{\sim}{h}$.</u>

i) first version

$$\pi^{2}_{HWM1}\,(\Delta\underline{u}\,,\Delta\underset{\sim}{h}\,,\Delta\underset{\sim}{\alpha}\,,\,\Delta\underset{\sim}{t}\,,\Delta\tilde{\underline{t}}_{\rho}\,) \tag{4.85}$$

$$= \sum_m \int_{V_{o_m}} \{\Delta W(\Delta\underset{\sim}{h}) - \rho_o\Delta\underline{g}\cdot\Delta\underline{u} + \Delta\underset{\sim}{t}^{T}:[\nabla\Delta\underline{u}^{T}-\Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I}+\underset{\sim}{h}^{N})-\underset{\sim}{\alpha}^{N}\cdot\Delta\underset{\sim}{h}]$$

$$-\underset{\sim}{t}^{N^T}:\Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I}+\underset{\sim}{h}^{N}) - \underset{\sim}{t}^{N^T}:(\Delta\underset{\sim}{\alpha}\cdot\Delta\underset{\sim}{h})\}\,dV$$

$$-\sum_m \int_{S_{\sigma_{o_m}}} \Delta\bar{\underline{t}}\cdot\Delta\underline{u}\,ds \;-\sum_m \int_{S_{u_{o_m}}} \Delta\underset{\sim}{t}\cdot(\Delta\underline{u}-\Delta\bar{\underline{u}})\,ds -\sum_m \int_{\rho_{o_m}} \Delta\tilde{\underline{t}}_{\rho}\cdot\Delta\underline{u}\,ds$$

$$\pi'(\Delta\underline{u}\,,\Delta\underset{\sim}{h}\,,\Delta\underset{\sim}{\alpha}\,,\,\Delta\underset{\sim}{t}\,,\,\Delta\tilde{\underline{t}}_{\rho}\,) \tag{4.87}$$

$$= \sum_m \int_{V_{o_m}} \Big\{ \Big[\frac{\partial W}{\partial \underset{\sim}{h}} \Big|^N - \frac{1}{2} (\underset{\sim}{t}^N \cdot \underset{\sim}{\alpha}^N + \underset{\sim}{\alpha}^{N^T} \cdot \underset{\sim}{t}^{N^T}) \Big] : \Delta \underset{\sim}{h}$$

$$+ \big[(\underset{\sim}{I} + \nabla \underline{u}^N)^T - \underset{\sim}{\alpha}^N \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) \big] : \Delta \underset{\sim}{t}^T$$

$$- (\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot \underset{\sim}{t}^N : \Delta \underset{\sim}{\alpha}^T - \rho_o \underline{g}^N \cdot \Delta \underline{u} + \underset{\sim}{t}^N : (\nabla \Delta \underline{u}) \Big\} dv$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} \bar{\underline{t}}^N \cdot \Delta \underline{u} \, ds \quad - \sum_m \int_{S_{u_{o_m}}} \big\{ \underline{t}^N \cdot \Delta \underline{u} + \Delta \underline{t} \cdot (\underline{u}^N - \bar{\underline{u}}^N) \big\} ds$$

$$- \sum_m \int_{\rho_{o_m}} \big\{ \Delta \tilde{\underline{t}}_\rho \cdot \underline{u}^N + \tilde{\underline{t}}_\rho^N \cdot \Delta \underline{u} \big\} ds$$

ii) second version

$$\pi^2_{HWM2} \; (\Delta \underline{u} , \Delta \underset{\sim}{h} , \Delta \underset{\sim}{\alpha} , \Delta \underset{\sim}{t} , \Delta \tilde{\underline{u}}_\rho) \tag{4.88}$$

= {first three terms are the same as in Eq.(4.86)}

$$- \sum_m \int_{\rho_{o_m}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \tilde{\underline{u}}_\rho) \, ds$$

$$\pi' (\Delta \underline{u} , \Delta \underset{\sim}{h} , \Delta \underset{\sim}{\alpha} , \Delta \underset{\sim}{t} , \Delta \tilde{\underline{u}}_\rho) \tag{4.89}$$

= {first three terms are the same as in Eq.(4.87)}

$$- \sum_m \int_{\rho_{o_m}} \big\{ \underline{t}^N \cdot (\Delta \underline{u} - \Delta \tilde{\underline{u}}_\rho) + \Delta \underline{t} \cdot (\underline{u}^N - \tilde{\underline{u}}_\rho^N) \big\} ds$$

The stationarity conditions of the functional, Eqns.(4.86) and (4.88), lead to Eqns.(4.8, 11, 14, 20, 38, 39, 68, and 69).

Modified Incremental Potential Energy Principles

As it is shown, the incremental potential energy functionals based on $\Delta \underset{\sim}{s}$ and $\Delta \underset{\sim}{t}$ are identical. THe modifications of these functionals lead to the following modified incremental potential energy principles.

i) first version

$$\pi^2_{PM1}(\Delta \underline{u}, \Delta \underline{\tilde{t}}_\rho) = \sum_m \int_{V_{o_m}} \{\Delta W(\Delta \underset{\sim}{g}) \tag{4.90}$$

$$+ \frac{1}{2} \underset{\sim}{S}^N : [(\nabla\Delta\underline{u}) \cdot (\nabla\Delta\underline{u})^T] - \rho_o \Delta\underline{g} \cdot \Delta\underline{u}\} dv$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} \Delta\underline{\bar{t}} \cdot \Delta\underline{u}\, ds \; - \sum_m \int_{\rho_{o_m}} \Delta\underline{\tilde{t}}_\rho \cdot \Delta\underline{u}\, ds$$

or equivalently,

$$\pi^2_{PM1}(\Delta \underline{u}, \Delta \underline{\tilde{t}}_\rho) = \sum_m \int_{V_{o_m}} \{\Delta W(\Delta \underset{\sim}{e}) - \rho_o \Delta\underline{g} \cdot \Delta\underline{u}\} dv \tag{4.91}$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} \Delta\underline{\bar{t}} \cdot \Delta\underline{u}\, ds \; - \sum_m \int_{\rho_{o_m}} \Delta\underline{\tilde{t}}_\rho \cdot \Delta\underline{u}\, ds$$

ii) second version

$$\pi^2_{PM2}(\Delta \underline{u}, \Delta \underline{\tilde{u}}_\rho) \tag{4.92}$$

= {first two terms are the same as in EQ.(4.90) or Eq.(4.91)}

$$- \sum_m \int_{\rho_{o_m}} \Delta\underline{t} \cdot (\Delta\underline{u} - \Delta\underline{\tilde{u}}_\rho)\, ds$$

The stationarity conditions of the functionals, given by Eqns.(4.90) and (4.91), lead to Eqns.(4.7, 9, 37, 68, and 69) and (4.8, 10, 38, 68, and 69), respectively.

Based on these modified functionals so-called incremental hybrid displacement finite element models are derived [26]. Such finite element models are convenient to analyze plate bending or shell problems, for which the displacement continuity at the inter-element boundary is difficult to be achieve by conventional compatible models. Also, taking advantage of the freedom in choosing assumed functions, we can introduce dominant part of the analytical solution as assumed functions [27]. Thus more accurate solution can be obtained with less degree of freedom. This feature is common in all the hybrid finite element models.

Modified Incremental Hellinger-Reissner Principles

Based on $\Delta \underset{\sim}{S}$ and $\Delta \underset{\sim}{g}$. The incremental Hellinger-Reissner functional given by Eq.(4.59) is modified and the following two versions of functionals are derived.

i) first version

$$\pi^2_{HRM1}(\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \underline{\tilde{t}}_\rho) \tag{4.93}$$

$$= \sum_m \int_{V_{o_m}} \Big\{ -\Delta S(\Delta \underset{\sim}{S}) - \rho_o \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T]$$

$$+ \frac{1}{2} \Delta \underset{\sim}{S} : [\nabla \Delta \underline{u} + (\nabla \Delta \underline{u})^T + \nabla \underline{u}^N \cdot (\nabla \Delta \underline{u})^T + \nabla \Delta \underline{u} \cdot (\nabla \underline{u}^N)^T \Big\} dV$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u}\, ds - \sum_m \int_{S_{u_{o_m}}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}})\, ds - \sum_m \int_{\rho_{o_m}} \Delta \tilde{\underline{t}}_\rho \cdot \Delta \underline{u}\, ds$$

ii) second version

$$\pi^2_{HRM2}(\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \tilde{\underline{u}}_\rho) \tag{4.94}$$

= {first three terms are the same as in Eq.(4.93)}

$$- \sum_m \int_{\rho_{o_m}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \tilde{\underline{u}}_\rho)\, ds$$

The stationarity conditions of the above functionals lead to Eqns.(4.7, 9, 12, 37, 39, 68, and 69). The functionals given by Eqns.(4.93) and (4.94) can be applied to the finite element method, and they lead to the incremental hybrid mixed model finite element models [28].

Based on $\Delta\underset{\sim}{r}$ and $\Delta\underset{\sim}{h}$. Similarly, the modified functionals corresponding to Eq.(4.60) are derived as,

i) first version

$$\pi^2_{HRM1}(\Delta\underline{u}, \Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\tilde{\underline{t}}_\rho) \tag{4.95}$$

$$= \sum_m \int_{V_{0m}} \{ -\Delta R(\Delta\underset{\sim}{r}) - \rho_0 \Delta\underline{g}\cdot\Delta\underline{u} + \Delta\underset{\sim}{t}^T : [\nabla\Delta\underline{u}^T - \Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^N)]$$

$$- \underset{\sim}{t}^{N^T} : \Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^N)\} dV \quad - \sum_m \int_{S_{\sigma_{0m}}} \Delta\bar{\underline{t}}\cdot\Delta\underline{u}\, ds$$

$$- \sum_m \int_{S_{u_{0m}}} \Delta\underline{t}\cdot(\Delta\underline{u} - \Delta\bar{\underline{u}})\, ds \quad - \sum_m \int_{\rho_{0m}} \Delta\tilde{\underline{t}}_\rho\cdot\Delta\underline{u}\, ds$$

ii) second version

$$\pi^2_{HRM2}(\Delta\underline{u}, \Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\tilde{\underline{u}}_\rho) \tag{4.96}$$

= {first three terms are the same as in Eq.(4.95)}

$$- \sum_m \int_{\rho_{0m}} \Delta\underline{t}\cdot(\Delta\underline{u} - \Delta\tilde{\underline{u}}_\rho)\, ds$$

The stationarity conditions of the functionals given by Eqns.(4.95) and (4.96) lead to Eqns.(4.8, 11, 14, 38, 39, 68, and 69).

The application of the above functionals to the finite element method is possible. However, as it is seen, too many independent variables are involved. In this aspect, it is not preferable to use such finite element models for practical purposes.

Modified Incremental Complementary Energy Principles

Based on $\Delta \underset{\sim}{S}$ and $\Delta \underset{\sim}{g}$. Assuming that the incremental translational equilibrium condition Eq.(4.7) and the traction boundary condition Eq.(4.37) are satisfied up to linear terms, the modified functionals given by Eqns.(4.93) and (4.94) are reduced to the following functionals which correspond to Eq.(4.61).

i) first version

$$\pi^2_{CM1}(\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \tilde{\underline{t}}_\rho) \tag{4.97}$$

$$= \sum_m \int_{V_{o_m}} \left\{ \Delta S(\Delta \underset{\sim}{S}) + \frac{1}{2} \underset{\sim}{S}^N : [(\nabla \Delta \underline{u}) \cdot (\nabla \Delta \underline{u})^T] \right\} dV$$

$$- \sum_m \int_{S_{u_{o_m}}} \Delta \underline{t} \cdot \Delta \underline{\bar{u}} \, dS - \sum_m \int_{\rho_{o_m}} (\Delta \underline{t} - \Delta \tilde{\underline{t}}_\rho) \cdot \Delta \underline{u} \, dS$$

Since the translational equilibrium condition and the traction boundary condition are a priori satisfied only approximately, the correction to these conditions must be retained. Thus π^1 corresponding to Eq.(4.97) is obtained as,

$$\pi^1(\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \tilde{\underline{t}}_\rho) \tag{4.98}$$

$$= \sum_m \int_{V_{o_m}} \left\{ \left[\frac{\partial S}{\partial \underset{\sim}{S}} \Big|^N - \frac{1}{2} (\nabla \underline{u}^N + \nabla \underline{u}^{N^T} + \nabla \underline{u}^N \cdot \nabla \underline{u}^{N^T}) \right] : \Delta \underset{\sim}{S} \right.$$

$$\left. + \rho_o \underline{g}^N \cdot \Delta \underline{u} - \underset{\sim}{S}^N \cdot \underset{\sim}{F}^{N^T} : (\nabla \Delta \underline{u}) \right\} dV$$

$$+ \sum_m \int_{S_{\sigma_{o_m}}} \underline{\bar{t}}^N \cdot \Delta \underline{u} \, dS + \sum_m \int_{S_{u_{o_m}}} \{ \underline{t}^N \cdot \Delta \underline{u} + \Delta \underline{t} \cdot (\underline{u}^N - \underline{\bar{u}}^N) \} dS$$

$$+\sum_m \int_{\rho_{o_m}} \{ \Delta \tilde{\underline{t}}_\rho \cdot \underline{u}^N + \tilde{\underline{t}}_\rho^N \cdot \Delta \underline{u} \} dS$$

ii) second version

$$\pi^2_{CM2} (\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \tilde{\underline{u}}_\rho) \tag{4.99}$$

= {first two terms are the same as in Eq.(4.97)}

$$- \sum_m \int_{\rho_{o_m}} \Delta \underline{t} \cdot \Delta \tilde{\underline{u}}_\rho \, dS$$

From analogous argument as in the case of the first version, π' corresponding to Eq.(4.99) is obtained as,

$$\pi' (\Delta \underline{u}, \Delta \underset{\sim}{S}, \Delta \tilde{\underline{u}}_\rho) \tag{4.100}$$

= {first three terms are the same as in Eq.(4.98)}

$$+ \sum_m \int_{\rho_{o_m}} \{ \underline{t}^N \cdot (\Delta \underline{u} - \Delta \tilde{\underline{u}}_\rho) + \Delta \underline{t} \cdot (\underline{u}^N - \tilde{\underline{u}}^N_\rho) \} dS$$

The stationarity conditions of the above functionals lead to Eqns.(4.10, 12, 39, 68, and 69).

The functionals given by Eqns.(4.97) and (4.99) can be applied to finite element models [29]. However, if we consider their failure to satisfy the translational equilibrium condition and the traction boundary condition and also the fact that the functionals involve displacements as variables, no significant advantage is found in this type of finite element models as compared to the hybrid mixed model based on Eq.(4.93) or Eq.(4.94).

Based on $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{h}$. Finally, the incremental complementary energy principle in the most consistent form, which is given by Eq.(4.63), is modified following the same procedure. After modifications, two versions of modified incremental complementary energy functionals are derived.

i) first version

$$\Pi^2_{CM1}(\Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\underline{\tilde{t}}_\rho, \Delta\underline{u}_\rho) \tag{4.101}$$

$$= \sum_m \int_{V_{o_m}} \{\Delta R(\Delta\underset{\sim}{r}) + (\underset{\sim}{t}^N + \Delta\underset{\sim}{t})^T : \Delta\underset{\sim}{\alpha}\cdot(\underset{\sim}{I} + \underset{\sim}{h}^N)\} dV$$

$$- \sum_m \int_{S_{u_{o_m}}} \Delta\underline{t}\cdot\Delta\underline{\bar{u}}\, ds - \sum_m \int_{\rho_{o_m}} (\Delta\underline{t} - \Delta\underline{\tilde{t}}_\rho)\cdot\Delta\underline{u}_\rho\, ds$$

$$\Pi'(\Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\underline{\tilde{t}}_\rho, \Delta\underline{u}_\rho) \tag{4.102}$$

$$= \sum_m \int_{V_{o_m}} \{[\underset{\sim}{\alpha}^N\cdot\frac{\partial R}{\partial \underset{\sim}{r}}\Big|^N + \underset{\sim}{\alpha}^N - \underset{\sim}{I}\,] : \Delta\underset{\sim}{t}^T$$

$$+ [(\underset{\sim}{I} + \frac{\partial R}{\partial\underset{\sim}{r}}\Big|^N)\cdot\underset{\sim}{t}^N] : \Delta\underset{\sim}{\alpha}^T\} dV - \sum_m \int_{S_{u_{o_m}}} \Delta\underline{t}\cdot\underline{\bar{u}}^N ds$$

$$- \sum_m \int_{\rho_{o_m}} \{(\Delta\underline{t} - \Delta\underline{\tilde{t}}_\rho)\cdot\underline{u}^N_\rho + (\underline{t}^N - \underline{\tilde{t}}^N_\rho)\cdot\Delta\underline{u}_\rho\} ds$$

where $\Delta\underline{u}_\rho$ is the displacement vector at inter-element boundaries which is independently defined for the adjoining elements.

ii) second version

$$\Pi^2_{CM2}(\Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\underline{\tilde{u}}_\rho) \tag{4.103}$$

= {first two terms are the same as in Eq.(4.101)}

$$- \sum_m \int_{\rho_{o_m}} \Delta\underline{t}\cdot\Delta\underline{\tilde{u}}_\rho\, ds$$

$$\pi'(\Delta\underset{\sim}{\alpha}, \Delta\underset{\sim}{t}, \Delta\underline{\tilde{u}}_\rho) \tag{4.104}$$

= {first two terms are the same as in Eq.(4.102)}

$$-\sum_m \int_{\rho_{o_m}} \{ \underline{t}^N \cdot \Delta\underline{\tilde{u}}_\rho + \Delta\underline{t} \cdot \underline{\tilde{u}}_\rho^N \} ds$$

The stationarity conditions of the functionals given by Eqns.(4.101) and (4.103) lead to Eqns.(4.11, 14, 39, 68, and 69).

It is emphasized again that, in the above functionals, the translational equilibrium condition and the traction boundary condition are exactly satisfied a priori and the rotational equilibrium condition is enforced through the stationarity conditions of the respective functionals. Moreover, the continuity conditions at inter-element boundaries are relaxed a priori so that arbitrary functions which do not satisfy these conditions are also allowed as admissible functions. Following the general procedure in the derivation of finite element models, these functionals are discretized and they lead to the incremental hybrid stress finite element models, in which the above advantageous features are preserved in their discretized forms. Thus, the derived incremental hybrid stress finite element models are considered to be consistent and also versatile in practical applications.

Updated Lagrangean Formulation

Incremental Governing Equations

In the updated Lagrangean formulation, state variables in both C_N state and C_{N+1} state are referred to the metric in C_N state. Also, the transition from C_N to C_{N+1} is described using the same metric. Let state variables in C_N and C_{N+1} be symbolically denoted by $\{C^*_N\}$ and $\{C^*_{N+1}\}$. The superscript star is used to distinguish variables referred to C_N from those referred to the initial configuration C_0. Let the incremental state variables in passing from C_N to C_{N+1} be denoted by $\{\Delta C^*\}$. Thus, in general, it is shown that,

$$\{C^*_{N+1}\} = \{C^*_N\}+\{\Delta C^*\} \tag{4.105}$$

Now we consider closely the state variables both in C_N and C_{N+1}. The C_{N+1} state is described in terms of alternate stress and strain measures referred to C_N, namely ; Piola-Lagrange stress $\underset{\sim}{t}^{*N+1}$, Kirchhoff-Trefftz stress $\underset{\sim}{s}^{*N+1}$, and Jaumann stress $\underset{\sim}{r}^{*N+1}$ and also their conjugate strains : displacement gradient $\underset{\sim}{e}^{*N+1}$, Green-Lagrange strain $\underset{\sim}{g}^{*N+1}$, and stretch tensor $\underset{\sim}{h}^{*N+1}$. The definitions of these stresses and strains are given by Eqns.(2.61) through (2.77). On the other hand, state variables in C_N are also referred to the metric of C_N. It is noticed that this way of describing the state variables is equivalent to that of the Eulerian. Thus, the differences among various definitions of stresses disappear, and all the stresses become identical with the true stress $\underset{\sim}{\tau}^N$ in C_N.

$$\underset{\sim}{s}^{*N} = \underset{\sim}{t}^{*N} = \underset{\sim}{r}^{*N} = \underset{\sim}{\tau}^N \tag{4.106}$$

Similarly, various definitions of strains in C_N become identical, and they are shown to be zero from their definitions,

$$\underset{\sim}{g}^{*N} = \underset{\sim}{e}^{*N} = \underset{\sim}{h}^{*N} = 0 \tag{4.107}$$

And also, the deformation gradient $\underset{\sim}{F}^{*N}$ and the rotation tensor $\underset{\sim}{\alpha}^{*N}$ are,

$$\underset{\sim}{F}^{*N} = \underset{\sim}{I} \quad \text{and} \quad \underset{\sim}{\alpha}^{*N} = \underset{\sim}{I} \tag{4.108}$$

Following the general definition, Eq.(4.105), incremental state variables are defined as follows,

$$\begin{aligned}
\Delta \underline{u} &= \underline{u}^{N+1} - \underline{u}^{N} \\
\Delta \underset{\sim}{s}^{*} &= \underset{\sim}{s}^{*N+1} - \underset{\sim}{\tau}^{N} \\
\Delta \underset{\sim}{t}^{*} &= \underset{\sim}{t}^{*N+1} - \underset{\sim}{\tau}^{N} \\
\Delta \underset{\sim}{r}^{*} &= \underset{\sim}{r}^{*N+1} - \underset{\sim}{\tau}^{N} \\
\Delta \underset{\sim}{g}^{*} &= \underset{\sim}{g}^{*N+1} = (1/2)\{\overset{*}{\nabla}\Delta\underline{u} + (\overset{*}{\nabla}\Delta\underline{u})^{T} + \underline{(\overset{*}{\nabla}\Delta\underline{u})\cdot(\overset{*}{\nabla}\Delta\underline{u})^{T}}\} \\
\Delta \underset{\sim}{e}^{*} &= \underset{\sim}{e}^{*N+1} = (\overset{*}{\nabla}\Delta\underline{u})^{T} \\
\Delta \underset{\sim}{h}^{*} &= \underset{\sim}{h}^{*N+1} \\
\Delta \underset{\sim}{\alpha}^{*} &= \underset{\sim}{\alpha}^{*N+1} - \underset{\sim}{I}
\end{aligned} \tag{4.109}$$

where ∇^{*} represents the gradient operator in the metric of C_N. Further , these incremental variables are related through the following equations,

$$\Delta \underset{\sim}{t}^{*} = \Delta \underset{\sim}{s}^{*} + \underset{\sim}{\tau}^{N}\cdot\overset{*}{\nabla}\Delta\underline{u} + \underline{\Delta \underset{\sim}{s}^{*}\cdot\overset{*}{\nabla}\Delta\underline{u}} \tag{4.110}$$

$$\begin{aligned}
\Delta \underset{\sim}{r}^{*} = (1/2)\{&\Delta \underset{\sim}{t}^{*} + \underset{\sim}{\tau}^{N}\cdot\Delta\underset{\sim}{\alpha}^{*} + \Delta\underset{\sim}{t}^{*T} + \Delta\underset{\sim}{\alpha}^{*T}\cdot\underset{\sim}{\tau}^{NT} \\
&+ \underline{\Delta\underset{\sim}{t}^{*}\cdot\Delta\underset{\sim}{\alpha}^{*} + \Delta\underset{\sim}{\alpha}^{*T}\cdot\Delta\underset{\sim}{t}^{*T}}\}
\end{aligned} \tag{4.111}$$

$$\Delta \underset{\sim}{g}^{*} = (1/2)\{\Delta\underset{\sim}{e}^{*} + \Delta\underset{\sim}{e}^{*T} + \underline{\Delta\underset{\sim}{e}^{*T}\cdot\Delta\underset{\sim}{e}^{*}}\} \tag{4.112}$$

$$= (1/2)[2\Delta \underset{\sim}{h}^* + \underline{\Delta \underset{\sim}{h}^* \cdot \Delta \underset{\sim}{h}^*}]$$

$$\Delta \underset{\sim}{e}^* = \Delta \underset{\sim}{a}^* + \Delta \underset{\sim}{h}^* + \underline{\Delta \underset{\sim}{a}^* \cdot \Delta \underset{\sim}{h}^*} \tag{4.113}$$

In the above equations as well as the subsequent equations, nonlinear terms in incremental variables are indicated by underlines. And these are neglected in the following piecewise linear incremental formulation.

Using the incremental variables defined in the above, the field equations and the boundary conditions given by Eqns.(2.90) through (2.104) are reduced to the incremental equations which govern the transition from C_N to C_{N+1}, as shown in the following.

<u>translational equilibrium conditions</u>

$$\nabla^* \cdot (\Delta \underset{\sim}{S}^* + \underset{\sim}{\tau}^N \cdot \nabla^* \Delta \underline{u} + \underline{\Delta \underset{\sim}{S}^* \cdot \nabla^* \Delta \underline{u}}) + \rho_N \Delta \underline{g} = 0 \tag{4.114}$$

$$\nabla^* \cdot (\Delta \underset{\sim}{t}^*) + \rho_N \Delta \underline{g} = 0 \tag{4.115}$$

where ρ_N is the mass density per unit volume in C_N.

<u>rotational equilibrium conditions</u>

$$\Delta \underset{\sim}{s}^{*T} = \Delta \underset{\sim}{s}^* \tag{4.116}$$

or $$(\nabla^* \Delta \underline{u})^T \cdot \underset{\sim}{\tau}^N + \Delta \underset{\sim}{t}^* + \underline{(\nabla^* \Delta \underline{u})^T \cdot \Delta \underset{\sim}{t}^*} = \text{symmetric} \tag{4.117}$$

or $$\Delta \underset{\sim}{h}^* \cdot \underset{\sim}{\tau}^N + \Delta \underset{\sim}{t}^* + \underset{\sim}{\tau}^N \cdot \Delta \underset{\sim}{a}^* + \underline{\Delta \underset{\sim}{h}^* \cdot \Delta \underset{\sim}{t}^* + \Delta \underset{\sim}{t}^* \cdot \Delta \underset{\sim}{a}^*}$$
$$\underline{+ \Delta \underset{\sim}{h}^* \cdot \underset{\sim}{\tau}^N \cdot \Delta \underset{\sim}{a}^* + \Delta \underset{\sim}{h}^* \cdot \Delta \underset{\sim}{t}^* \cdot \Delta \underset{\sim}{a}^*} = \text{symmetric} \tag{4.118}$$

<u>kinematic relations</u>

$$\Delta \underset{\sim}{g}^* = (1/2)\{\nabla^*\Delta \underline{u} + (\nabla^*\Delta \underline{u})^T + \underline{(\nabla^*\Delta \underline{u})\cdot(\nabla^*\Delta \underline{u})^T}\} \tag{4.119}$$

$$\Delta \underset{\sim}{e}^* = (\nabla^*\Delta \underline{u})^T \tag{4.120}$$

$$(\nabla^*\Delta u)^T = \Delta \underset{\sim}{a}^* + \Delta \underset{\sim}{h}^* + \underline{\Delta \underset{\sim}{a}^*\cdot \Delta \underset{\sim}{h}^*} \tag{4.121}$$

<u>constitutive relations</u>

Assuming that the change in the field variables from C_N to C_{N+1} is sufficiently small, the strain energy density W^* defined by Eq.(2.78) is expanded in Taylor series in terms of $\Delta \underset{\sim}{g}^*$, $\Delta \underset{\sim}{e}^*$, and $\Delta \underset{\sim}{h}^*$, and the following incremental strain energy density functions are derived.

$$\Delta W^*(\Delta \underset{\sim}{g}^*) = \frac{1}{2}\left.\frac{\partial^2 W^*}{\partial \underset{\sim}{g}^{*2}}\right|^N :: \Delta \underset{\sim}{g}^* \Delta \underset{\sim}{g}^* + \underline{H.O.T.} \tag{4.122}$$

$$\Delta W^*(\Delta \underset{\sim}{e}^*) = \frac{1}{2}\left.\frac{\partial^2 W^*}{\partial \underset{\sim}{e}^{*2}}\right|^N :: \Delta \underset{\sim}{e}^* \Delta \underset{\sim}{e}^* + \underline{H.O.T.} \tag{4.123}$$

$$\Delta W^*(\Delta \underset{\sim}{h}^*) = \frac{1}{2}\left.\frac{\partial^2 W^*}{\partial \underset{\sim}{h}^{*2}}\right|^N :: \Delta \underset{\sim}{h}^* \Delta \underset{\sim}{h}^* + \underline{H.O.T.} \tag{4.124}$$

such that,

$$\frac{\partial \Delta W^*(\Delta \underset{\sim}{g}^*)}{\partial \Delta \underset{\sim}{g}^*} = \Delta \underset{\sim}{S}^*, \quad \frac{\partial \Delta W^*(\Delta \underset{\sim}{e}^*)}{\partial \Delta \underset{\sim}{e}^*} = \Delta \underset{\sim}{t}^{*T}, \quad \frac{\partial \Delta W^*(\Delta \underset{\sim}{h}^*)}{\partial \Delta \underset{\sim}{h}^*} = \Delta \underset{\sim}{r}^* \qquad \begin{matrix}(4.125)\\(4.126)\\(4.127)\end{matrix}$$

Further, the contact transformation of W^* in terms of $\underset{\sim}{s}^*$ and $\underset{\sim}{r}^*$ exists, as defined by Eqns.(2.86) and (2.88), and they are denoted by S^* and R^*, respectively. Thus, the incremental potential functions ΔS^* and ΔR^* can be defined by,

$$\Delta S^*(\Delta \underset{\sim}{S}^*) = \frac{1}{2}\left.\frac{\partial^2 S^*}{\partial \underset{\sim}{S}^{*2}}\right|^N :: \Delta \underset{\sim}{S}^* \Delta \underset{\sim}{S}^* + \underline{H.O.T.} \tag{4.128}$$

$$\Delta R^*(\Delta \underset{\sim}{r}^*) = \frac{1}{2}\left.\frac{\partial^2 R^*}{\partial \underset{\sim}{r}^{*2}}\right|^N :: \Delta \underset{\sim}{r}^* \Delta \underset{\sim}{r}^* + \underline{H.O.T.} \tag{4.129}$$

such that,

$$\frac{\partial \Delta S^*}{\partial \Delta \underset{\sim}{S}^*} = \Delta \underset{\sim}{g}^* \quad , \quad \frac{\partial \Delta R^*}{\partial \Delta \underset{\sim}{r}^*} = \Delta \underset{\sim}{h}^* \qquad (4.130), (4.131)$$

<u>boundary conditions</u>

$$\text{(a)} \quad \underline{n}^* \cdot \{\Delta \underset{\sim}{S}^* + \underset{\sim}{\tau}^N \cdot \nabla^* \Delta \underline{u} + \underline{\Delta \underset{\approx}{S}^* \cdot \nabla^* \Delta \underline{u}}\} = \Delta \underline{t}^* = \Delta \bar{\underline{t}}^* \qquad (4.132)$$

$$\text{or} \quad \underline{n}^* \cdot \Delta \underset{\sim}{t}^* = \Delta \bar{\underline{t}}^* \qquad \text{at } S_{\sigma_n} \qquad (4.133)$$

where $\underline{n}^*$ is a unit outward normal to the surface in C_N, and S_{σ_n} is a portion of the surface of the body in C_N at which the incremental traction is prescribed to be $\Delta \bar{\underline{t}}^*$.

$$\text{(b)} \quad \Delta \underline{u} = \Delta \bar{\underline{u}} \qquad \text{at } S_{u_n} \qquad (4.134)$$

where S_{u_n} is a portion of the surface in C_N at which the incremental displacement is prescribed to be $\Delta \bar{\underline{u}}$.

<u>Hu-Washizu Principles</u>

Following the general procedure as in the incremental total Lagrangean description, the incremental governing equations derived in the above can be cast into the variational principles in the most general form.

<u>Based on $\Delta \underset{\approx}{S}^*$ and $\Delta \underset{\approx}{g}^*$.</u> The incremental Hu-Washizu principle governing the transition from C_N to C_{N+1} is derived as,

$$\pi_{HW}^{*2}(\Delta \underline{u}, \Delta \underset{\sim}{g}^*, \Delta \underset{\sim}{S}^*) \qquad (4.135)$$

$$= \int_{V_n} \{ \Delta W^*(\Delta \underset{\sim}{g}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{\tau}^N : [(\nabla^* \Delta \underline{u}) \cdot (\nabla^* \Delta \underline{u})^T]$$

$$-\Delta \underset{\sim}{S}^{*}:[\Delta \underset{\sim}{g}^{*} - \tfrac{1}{2}\{(\nabla^{*}\Delta \underline{u}) + (\nabla^{*}\Delta \underline{u})^{T}\}]\Big\}\, dV$$

$$-\int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^{*}\cdot \Delta \underline{u}\, ds \;\; -\int_{S_{u_n}} \Delta \underline{t}^{*}\cdot (\Delta \underline{u} - \Delta \bar{\underline{u}})\, ds$$

where,

$$\Delta W^{*}(\Delta \underset{\sim}{g}^{*}) = \frac{1}{2}\, \frac{\partial^2 W^{*}}{\partial \underset{\sim}{g}^{*2}}\Big|^{N} :: \Delta \underset{\sim}{g}^{*} \Delta \underset{\sim}{g}^{*}$$

such that,

$$\frac{\partial \Delta W^{*}(\Delta \underset{\sim}{g}^{*})}{\partial \Delta \underset{\sim}{g}^{*}} = \Delta \underset{\sim}{S}^{*}$$

and, $\Delta \underline{t}^{*} = \underline{n}^{*}\cdot(\Delta \underset{\sim}{S}^{*} + \underset{\sim}{\tau}^{N}\cdot \nabla^{*}\Delta \underline{u})$.

Its stationarity condition leads to the complete set of incremental governing equations in their linearized form, namely; translational equilibrium condition Eq.(4.114), rotational equilibrium condition Eq.(4.116), kinematic relation Eq.(4.119), constitutive relation Eq.(4.125), and boundary conditions Eqns.(4.132) and (4.134). The iterative procedure to correct the linearized solution obtained through the above functional is provided by π', which is given as,

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{g}^{*}, \Delta \underset{\sim}{S}^{*}) = \int_{V_n} \Big\{ [\frac{\partial W^{*}}{\partial \underset{\sim}{g}^{*}}\Big|^{N+1} - \underset{\sim}{S}^{*N+1}] : \Delta \underset{\sim}{g}^{*} \qquad (4.136)$$

$$-[\underset{\sim}{g}^{*N+1} - \tfrac{1}{2}\{(\underset{\sim}{F}^{*N+1})^{T}\cdot(\underset{\sim}{F}^{*N+1}) - \underset{\sim}{I}\}] : \Delta \underset{\sim}{S}^{*}$$

$$-\rho_N\, \underline{g}^{N+1}\cdot \Delta \underline{u} + \underset{\sim}{S}^{*N+1}\cdot(\underset{\sim}{F}^{*N+1})^{T} : (\nabla^{*}\Delta \underline{u})\Big\}\, dV$$

$$-\int_{S_{\sigma_n}} \bar{\underline{t}}^{*N+1}\cdot \Delta \underline{u}\, ds \;\; -\int_{S_{u_n}} \{\underline{t}^{*N+1}\cdot \Delta \underline{u} + \Delta \underline{t}^{*}\cdot(\underline{u}^{N+1} - \bar{\underline{u}}^{N+1})\}\, ds$$

where, $\underset{\sim}{F}^{*N+1} = (\nabla^{*}\underline{y}^{N+1})^{T}$.

<u>Based on $\Delta \underset{\sim}{t}^*$ and $\Delta \underset{\sim}{e}^*$.</u> Likewise, the incremental Hu-Washizu functional in terms of $\Delta \underset{\sim}{t}^*$ and $\Delta \underset{\sim}{e}^*$ is obtained as,

$$\pi_{HW}^{*2}(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*) \tag{4.137}$$

$$= \int_{V_n} \{\Delta W^*(\Delta \underset{\sim}{e}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} + \Delta \underset{\sim}{t}^{*T} : (\nabla^* \Delta \underline{u}^T - \Delta \underset{\sim}{e}^*)\} dV$$

$$- \int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^* \cdot \Delta \underline{u} \, ds - \int_{S_{u_n}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) \, ds$$

where,

$$\Delta W^*(\Delta \underset{\sim}{e}^*) = \frac{1}{2} \frac{\partial^2 W^*}{\partial \underset{\sim}{e}^{*2}}\bigg|^N :: \Delta \underset{\sim}{e}^* \Delta \underset{\sim}{e}^*$$

such that,

$$\frac{\partial \Delta W^*(\Delta \underset{\sim}{e}^*)}{\partial \Delta \underset{\sim}{e}^*} = \Delta \underset{\sim}{t}^{*T}$$

Its stationarity condition leads to Eqns.(4.115, 117, 120, 126, 133, and 134) in their linearized form. In addition, the incremental rotational equilibrium condition is retained through the structure of $\Delta W^*(\Delta \underset{\sim}{e}^*)$ from the analogous argument as shown for the total Lagrangean description. π' corresponding to Eq.(4.137) is,

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*) \tag{4.138}$$

$$= \int_{V_n} \Big\{ \Big[\frac{\partial W^*}{\partial \underset{\sim}{e}^*}\bigg|^{N+1} - \underset{\sim}{t}^{*N+1^T} \Big] : \Delta \underset{\sim}{e}^* + [(\nabla^* \underline{y}^{N+1})^T - (\underset{\sim}{e}^{*N+1} + \underset{\sim}{I})] : \Delta \underset{\sim}{t}^{*T}$$

$$- \rho_N \underline{g}^{N+1} \cdot \Delta \underline{u} + (\underset{\sim}{t}^{*N+1}) : \nabla^* \Delta \underline{u} \Big\} dV$$

$$- \int_{S_{\sigma_n}} \bar{\underline{t}}^{*N+1} \cdot \Delta \underline{u} \, ds - \int_{S_{u_n}} \{ \underline{t}^{*N+1} \cdot \Delta \underline{u} + \Delta \underline{t}^* \cdot (\underline{u}^{N+1} - \bar{\underline{u}}^{N+1}) \} ds$$

Based on $\Delta \underset{\sim}{r}^*$ and $\Delta \underset{\sim}{h}^*$. Similarly the following incremental Hu-Washizu functional in terms of $\Delta \underset{\sim}{r}^*$ and $\Delta \underset{\sim}{h}^*$ is derived.

$$\Pi_{HW}^{*2} (\Delta \underline{u}, \Delta \underset{\sim}{h}^*, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*) \tag{4.139}$$

$$= \int_{V_n} \{ \Delta W^*(\Delta \underset{\sim}{h}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} + \Delta \underset{\sim}{t}^{*T} : [\nabla^* \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha}^* - \Delta \underset{\sim}{h}^*]$$

$$- \underset{\sim}{\tau}^{NT} : \Delta \underset{\sim}{\alpha}^* - \underset{\sim}{\tau}^{NT} : \Delta \underset{\sim}{\alpha}^* \cdot \Delta \underset{\sim}{h}^* \} dV$$

$$- \int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^* \cdot \Delta \underline{u} \, ds \quad - \int_{S_{u_n}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) \, ds$$

where,

$$\Delta W^*(\Delta \underset{\sim}{h}^*) = \frac{1}{2} \left. \frac{\partial^2 W^*}{\partial \underset{\sim}{h}^{*2}} \right|^N :: \Delta \underset{\sim}{h}^* \Delta \underset{\sim}{h}^*$$

such that,

$$\frac{\partial \Delta W^*(\Delta \underset{\sim}{h}^*)}{\partial \Delta \underset{\sim}{h}^*} = \Delta \underset{\sim}{r}^*$$

Close investigation is made specially for the present functional to obtain its stationarity condition. Considering the orthogonality condition on the rotation tensor, i.e.,

$$(\underset{\sim}{I} + \Delta \underset{\sim}{\alpha}^*)^T \cdot (\underset{\sim}{I} + \Delta \underset{\sim}{\alpha}^*) = \underset{\sim}{I} \tag{4.140}$$

and its variational form,

$$(\underset{\sim}{I} + \Delta \underset{\sim}{\alpha}^*)^T \cdot \delta \Delta \underset{\sim}{\alpha}^* = \text{skewsymmetric} \tag{4.141}$$

the first variation of the functional Eq.(4.139) is shown to be,

$$\delta \Pi_{HW}^{*2} = \int_{V_n} \{ [\frac{\partial \Delta W^*}{\partial \Delta h^*} - \frac{1}{2}(\Delta t^* + \tau^N \cdot \Delta \alpha^* + \Delta t^{*T} + \Delta \alpha^{*T} \cdot \tau^{NT})] : \delta \Delta h^* \tag{4.142}$$

$$+ [(\nabla^* \Delta u)^T - \Delta \alpha^* - \Delta h^*] : \delta \Delta t^{*T}$$

$$- [(\tau^N + \Delta h^* \cdot \tau^N + \Delta t^*) \cdot (I + \Delta \alpha^*)] : [(I + \Delta \alpha^*)^T \cdot \delta \Delta \alpha^*]^T$$

$$- [\nabla^* \cdot \Delta t^* + \rho_N \Delta g] \cdot \delta \Delta u \} dV$$

$$- \int_{S_{\sigma_n}} (\Delta \bar{t}^* - \Delta t^*) \cdot \delta \Delta u \, dS - \int_{S_{u_n}} \delta \Delta t^* \cdot (\Delta u - \Delta \bar{u}) dS$$

Thus, it is shown that the stationarity condition of the functional Eq.(4.139) leads to Eqns.(4.115, 118, 121, 127, 133, and 134). However, in practical applications the nonlinear constraint condition, Eq.(4.140), can be satisfied only in the linear fashion,

$$\Delta \alpha^* + \Delta \alpha^{*T} = 0 \tag{4.143}$$

Even for this case, exact rotational equilibrium condition is retained through π', which is given by,

$$\Pi' (\Delta u, \Delta h^*, \Delta \alpha^*, \Delta t^*) \tag{4.144}$$

$$= \int_{V_n} \{ [\frac{\partial W^*}{\partial h^*} \Big|^{N+1} - \frac{1}{2} \{ t^{*N+1} \cdot \alpha^{*N+1} + (\alpha^{*N+1})^T \cdot (t^{*N+1})^T \}] : \Delta h^*$$

$$+ [(\nabla^* y^{N+1})^T - \alpha^{*N+1} \cdot (I + h^{*N+1})] : \Delta t^{*T}$$

$$- (I + h^{*N+1}) \cdot t^{*N+1} : \Delta \alpha^{*T}$$

$$- \rho_N g^{N+1} \cdot \Delta u + t^{*N+1} : (\nabla^* \Delta u) \} dV$$

$$-\int_{S_{\sigma_n}} \bar{\underline{t}}^{*N+1} \cdot \Delta \underline{u}\, ds \quad -\int_{S_{u_n}} \{ \underline{t}^{*N+1} \cdot \Delta \underline{u} \; + \Delta \underline{t}^{*} \cdot (\underline{u}^{N+1} - \bar{\underline{u}}^{N+1}) \} \, ds$$

In the iterative correction process, the incremental orthogonality condition Eq.(4.143) is replaced by,

$$(\underset{\sim}{a}^{*N+1})^{T} \cdot \Delta \underset{\sim}{a}^{*} + \Delta \underset{\sim}{a}^{*T} \cdot \underset{\sim}{a}^{*N+1} = 0 \tag{4.145}$$

and in its variational form,

$$(\underset{\sim}{a}^{*N+1})^{T} \cdot \delta \Delta \underset{\sim}{a}^{*} = \text{skewsymmetric} \tag{4.146}$$

Thus, from the analogous argument as shown for the total Lagrangean formulation, the vanishing condition of $\delta\pi'$ reduces to the exact governing equations for C_{N+1} state.

Incremental Potential Energy Principles

By a priori satisfying the incremental constitutive relation Eq.(4.125) , kinematic relation Eq.(4.119), and displacement boundary condition Eq.(4.134), the functional in Eq.(4.135) is reduced to the following incremental stationary potential energy functional.

$$\Pi_P^{*2}(\Delta \underline{u}) = \int_{V_n} \{ \Delta W^{*}(\Delta \underset{\sim}{g}^{*}) \; - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} \tag{4.147}$$

$$+ \frac{1}{2} \underset{\sim}{\tau}^{N} : [(\nabla^{*} \Delta \underline{u}) \cdot (\nabla^{*} \Delta \underline{u})^{T}] \} \, dV \; - \int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^{*} \cdot \Delta \underline{u} \, ds$$

Similarly, by assuming a priori satisfaction of Eqns.(4.120, 126, and 134), the functional in Eq.(4.137) is reduced to,

$$\Pi_P^{*2}(\Delta \underline{u}) = \int_{V_n} \{ \Delta W^{*}(\Delta \underset{\sim}{e}^{*}) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} \} \, dV \tag{4.148}$$

$$-\int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^{*} \cdot \Delta \underline{u}\, ds$$

However, the incremental strain energy density functions $\Delta W^*(\Delta \underset{\sim}{g}^*)$ and $\Delta W^*(\Delta \underset{\sim}{e}^*)$ are related by,

$$\Delta W^*(\Delta \underset{\sim}{e}^*) = \Delta W^*(\Delta \underset{\sim}{g}^*) + \frac{1}{2} \underset{\sim}{\tau}^N : [(\nabla^*\Delta \underline{u}) \cdot (\nabla^*\Delta \underline{u})^T] \tag{4.149}$$

Thus, it is seen that the functionals given by Eqns.(4.147) and (4.148) are identical. The stationarity conditions of these functionals lead to Eqns.(4.114, 116, and 132) and (4.115, 117, and 133), respectively.

Incremental Hellinger-Reissner Principles

As discussed earlier, the contact transformations of ΔW^* in terms of $\Delta \underset{\sim}{S}^*$ and $\Delta \underset{\sim}{r}^*$ exist. By introducing these transformations, functionals given by Eqns.(4.135) and (4.139) can be reduced to incremental Hellinger-Reissner principles.

Based on $\Delta \underset{\sim}{S}^*$ and $\Delta \underset{\sim}{g}^*$.

$$\pi_{HR}^{*2}(\Delta \underline{u}, \Delta \underset{\sim}{S}^*) = \int_{V_n} \Big\{ -\Delta S^*(\Delta \underset{\sim}{S}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} \tag{4.150}$$

$$+ \frac{1}{2} \underset{\sim}{\tau}^N : [(\nabla^*\Delta \underline{u}) \cdot (\nabla^*\Delta \underline{u})^T] + \frac{1}{2} \Delta \underset{\sim}{S}^* : [(\nabla^*\Delta \underline{u}) + (\nabla^*\Delta \underline{u})^T] \Big\} dv$$

$$-\int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^{*} \cdot \Delta \underline{u}\, ds \; - \int_{S_{u_n}} \Delta \underline{t}^{*} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}})\, ds$$

where,

$$\Delta S^*(\Delta \underset{\sim}{S}^*) = \frac{1}{2} \left. \frac{\partial^2 S^*}{\partial \underset{\sim}{S}^{*2}} \right|^N :: \Delta \underset{\sim}{S}^* \Delta \underset{\sim}{S}^*$$

such that,

$$\frac{\partial \Delta S^*}{\partial \Delta \underset{\sim}{s}^*} = \Delta \underset{\sim}{g}^*$$

The stationarity condition of the above functional leads to Eqns.(4.114, 116, 119, 132, and 134).

Based on $\Delta \underset{\sim}{r}^*$ and $\Delta \underset{\sim}{h}^*$.

$$\pi_{HR}^{*2}(\Delta \underline{u}, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*) = \int_{V_n} \{ -\Delta R^*(\Delta \underset{\sim}{r}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} \tag{4.151}$$

$$+ \Delta \underset{\sim}{t}^{*T} : [\nabla^* \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha}^*] - \underset{\sim}{\tau}^{N^T} : \Delta \underset{\sim}{\alpha}^* \} dV$$

$$- \int_{S_{\sigma_n}} \Delta \bar{\underline{t}}^* \cdot \Delta \underline{u} \, ds - \int_{S_{u_n}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) \, ds$$

where,

$$\Delta R^*(\Delta \underset{\sim}{r}^*) = \frac{1}{2} \left. \frac{\partial^2 R^*}{\partial \underset{\sim}{r}^{*2}} \right|^N :: \Delta \underset{\sim}{r}^* \Delta \underset{\sim}{r}^*$$

such that,

$$\frac{\partial \Delta R^*}{\partial \Delta \underset{\sim}{r}^*} = \Delta \underset{\sim}{h}^*$$

Its stationarity condition leads to Eqns.(4.115, 118, 121, 133, and 134).

Based on $\Delta \underset{\sim}{t}^*$ and $\Delta \underset{\sim}{e}^*$. Eventhough one can use a formal contact transformation to express $-\Delta W^* + \Delta \underset{\sim}{t}^* : \Delta \underset{\sim}{e}^{*T} = \Delta T^*$ in terms of $\Delta \underset{\sim}{t}^*$ alone and thus formally derive a Hellinger-Reissner type principle from Eq.(4.137). The rotational equilibrium conditions can be seen, as in the total Lagrangean incremental (rate) formulation, to be not embedded in the structure of ΔT^*. For this reason, the above formally derived Hellinger-Reissner (updated Lagrangean) incremental principle has no practical use. The same comment applies to the incremental complementary energy principle in terms of $\Delta \underset{\sim}{t}^*$ alone, which can again be formally derived from Eq.(4.137).

Incremental Complementary Energy Principle

Based on $\Delta \underset{\sim}{s}^*$ and Δg^*. By a priori satisfaction of the translational equilibrium condition Eq.(4.114) and the traction boundary condition Eq.(4.132), the functional given by Eq.(4.150) is reduced to,

$$\pi_c^{*2}(\Delta \underline{u}, \Delta \underset{\sim}{S}) = \int_{V_n} \{ \Delta S^*(\Delta \underset{\sim}{S}) \tag{4.152}$$

$$+ \frac{1}{2} \underset{\sim}{\tau}^N : [(\nabla^* \Delta \underline{u}) \cdot (\nabla^* \Delta \underline{u})^T] \} dv - \int_{S_{u_n}} \Delta \underline{t}^* \cdot \Delta \bar{\underline{u}} \, ds$$

However, it is noticed that the incremental translational equilibrium condition given by Eq.(4.114) is a set of nonlinear and coupled partial differential equations involving both stress $\Delta \underset{\sim}{s}^*$ and displacement $\Delta \underline{u}$. It is impossible to satisfy this condition in its nonlinear form. In practical applications, Eq.(4.114) may be linearized to,

$$\nabla^* \cdot (\Delta \underset{\sim}{s}^* + \underset{\sim}{\tau}^N \cdot \nabla^* \Delta \underline{u}) + \rho_N \Delta \underline{g} = 0 \tag{4.153a}$$

The above linearized equation is used as a constraint condition on $\Delta \underset{\sim}{s}^*$ instead of Eq.(4.114). Unlike in the total Lagrangean formulation, the stress $\Delta \underset{\sim}{s}^*$ is not coupled with deformation gradient in Eq.(4.153a). Thus, as shown by

Atluri [14, 32], it becomes somewhat easier to choose $\Delta\underset{\sim}{s}^*$ which satisfies the linearized translational equilibrium condition, Eq.(4.153a), a priori. By introducing the symmetric Maxwell-Morera-Beltrami second order stress function $\underset{\sim}{A}$, such stress for the general three dimensional case can be assumed by,

$$\Delta\underset{\sim}{s}^* = \text{curl}(\,\text{curl}\ \underset{\sim}{A}\,)^T + \Delta\underset{\sim}{s}^{*P} \qquad (4.153b)$$

Using the rectangular Cartesian components, Eq.(4.153b) is rewritten as,

$$\Delta s^*_{ij} = e_{imn}\, e_{jpq}\, A_{nq,mp} + \Delta s^{*P}_{ij} \qquad (4.153c)$$

where $\Delta\underset{\sim}{s}^{*P}$ is any symmetric particular solution, such that,

$$\nabla^*\cdot\Delta\underset{\sim}{s}^{*P} = -\rho_N\Delta\underset{\sim}{g} - \nabla^*\cdot(\underset{\sim}{\tau}^N\cdot\nabla^*\Delta\underline{u}) \qquad (4.153d)$$

Then, the error due to the linearization of the translational equilibrium condition can be corrected through the iterative correction based on π^1, which is given by,

$$\pi^1(\Delta\underline{u}, \Delta\underset{\sim}{s}^*) \qquad (4.153e)$$

$$= \int_{V_n}\Big\{\Big[\frac{\partial S^*}{\partial\underset{\sim}{s}^*}\Big|^{N+1} - \frac{1}{2}\{(\underset{\sim}{F}^{*N+1})^T\cdot(\underset{\sim}{F}^{*N+1}) - \underset{\sim}{I}\}\Big]:\Delta\underset{\sim}{s}^*$$

$$+\rho_N\,\underline{g}^{N+1}\cdot\Delta\underline{u} - \underset{\sim}{s}^{*N+1}\cdot(\underset{\sim}{F}^{*N+1})^T:(\nabla^*\Delta\underline{u})\Big\}dV$$

$$+\int_{S_{\sigma_n}}\bar{\underline{t}}^{*N+1}\cdot\Delta\underline{u}\,dS + \int_{S_{u_n}}\{\underline{t}^{*N+1}\cdot\Delta\underline{u} + \Delta\underline{t}^*\cdot(\underline{u}^{N+1} - \bar{\underline{u}}^{N+1})\}dS$$

One simple way of satisfying Eq.(4.153a) may be to assume particular solution for the direct stress Δs^{*P}_{ii} (no sum on i : i=1, 2, 3) only. However, then, the question of completeness remains to be answered. Also, because of the term, $\nabla^{*}\cdot(\underset{\sim}{\tau}^{N}\cdot\nabla^{*}\Delta\underline{u})$, in Eq.(4.153d), it is still difficult to choose the particular solution $\Delta\underset{\sim}{s}^{*P}$. Alternative way to avoid the above difficulty may be to define the particular solution as,

$$\nabla^{*}\cdot\Delta\underset{\sim}{s}^{*P} = -\rho_{N}\Delta\underline{g} \tag{4.153f}$$

Such $\Delta\underset{\sim}{s}^{*P}$ can be easily found. Then, $\Delta\underset{\sim}{s}^{*}$ is expressed as,

$$\Delta\underset{\sim}{s}^{*} = \mathrm{curl}(\,\mathrm{curl}\,\underset{\sim}{A}\,)^{T} - \underset{\sim}{\tau}^{N}\cdot(\nabla^{*}\Delta\underline{u}) + \Delta\underset{\sim}{s}^{*P} \tag{4.153g}$$

However, in this case the assumed stress $\Delta\underset{\sim}{s}^{*}$ ceases to be symmetric, and it violates the rotational equilibrium condition. Thus, the rotational equilibrium condition must be introduced as a constraint condition into the associated complementary energy functional of the type given in Eq.(4.152) through additional Lagrange multipliers.

Considering the fact that the complementary energy functional, Eq.(4.152), involves both stress $\Delta\underset{\sim}{s}^{*}$ and displacement $\Delta\underline{u}$ as variables and the difficulty associated with choosing symmetric $\Delta\underset{\sim}{s}^{*}$ which satisfies Eq.(4.153a), it appears that a incremental complementary energy principle based on $\Delta\underset{\sim}{s}^{*}$ does not provide a basis of a practically useful finite element model.

<u>Based on $\Delta\underset{\sim}{r}^{*}$ and $\Delta\underset{\sim}{h}^{*}$.</u> Incremental complementary energy principle, which is considered to be consistent and

suitable for practical applications, can be derived from the functional given by Eq.(4.151) based on incremental Jaumann stress. By a priori satisfying Eqns.(4.115) and (4.133), which are linear and uncoupled in $\Delta \underset{\sim}{t}^*$ alone, the Hellinger-Reissner type functional, Eq.(4.151), is reduced to the incremental complementary energy functional π_c^{*2}, which involves stress $\Delta \underset{\sim}{t}^*$ and rotation $\Delta \underset{\sim}{\alpha}^*$.

$$\pi_c^{*2}(\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*) \tag{4.154}$$
$$= \int_{V_n} \left\{ \Delta R^*(\Delta \underset{\sim}{r}^*) + \underset{\sim}{\tau}^{N^T} : \Delta \underset{\sim}{\alpha}^* + \Delta \underset{\sim}{t}^{*^T} : \Delta \underset{\sim}{\alpha}^* \right\} dV$$
$$- \int_{S_{u_n}} \Delta \underline{t}^* \cdot \Delta \bar{\underline{u}} \, dS$$

Following the same argument as for the incremental Hu-Washizu principle given by Eq.(4.139), the stationarity condition of the above functional is shown to lead to Eqns.(4.116, 119, and 134), in their linearized form. The correction procedure to the linearized solution is derived through π^1 which is given by,

$$\pi^1(\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*) \tag{4.155}$$

$$= \int_{V_n} \left\{ \left[\underset{\sim}{\alpha}^{*N+1} \cdot \left(\underset{\sim}{I} + \frac{\partial R^*}{\partial \underset{\sim}{r}^*}\Big|^{N+1} \right) - \underset{\sim}{I} \right] : \Delta \underset{\sim}{t}^{*^T} \right.$$

$$\left. + \left[\left(\underset{\sim}{I} + \frac{\partial R^*}{\partial \underset{\sim}{r}^*}\Big|^{N+1} \right) \cdot \underset{\sim}{t}^{*N+1} \right] : \Delta \underset{\sim}{\alpha}^{*^T} \right\} dV$$

$$- \int_{S_{u_n}} \Delta \underline{t}^* \cdot \bar{\underline{u}}^{N+1} \, dS$$

Hybrid Type Incremental Variational Principles

The incremental variational principles in the updated Lagrangean formulation are summarized in the preceding sections. These functionals can be directly applied to the finite element method, provided that the interpolation (or assumed) functions are chosen so that the continuity conditions at inter-element boundaries are satisfied a priori. However, as discussed in the total Lagrangean formulation, such direct applications are limited. Therefore, these functionals are also modified and hybrid type functionals, considered to be more versatile in their applications, are derived.

In the present updated Lagrangean formulation, the configuration in C_N state is taken as reference. Thus, the deformed body V_N in C_N state is divided into a finite number of subdomains V_{n_m} (m=1...M). The portion of the element boundary which coincides with that of the overall boundary of the body V_N, where the displacement or the traction is prescribed, is denoted by Su_{n_m} or $S\sigma_{n_m}$, respectively. Also, inter-element boundary is denoted by ρ_{n_m}. Then, the displacement continuity condition and the traction reciprocity condition at ρ_{n_m}, in their incremental forms, are given by,

$$\Delta \underline{t}^{*+} + \Delta \underline{t}^{*-} = 0 \qquad \text{at} \quad \rho_{n_m} \qquad (4.156)$$

$$\Delta \underline{u}^{+} = \Delta \underline{u}^{-} \qquad \text{at} \quad \rho_{n_m} \qquad (4.157)$$

Following the same procedure as discussed in detail for total Lagrangean formulation, the various functionals given by Eqns.(4.135, 137, etc.) are modified , and the following hybrid type incremental functionals are derived.

Modified Incremental Hu-Washizu Principles

Based on $\Delta \underset{\sim}{S}^*$ and $\Delta \underset{\sim}{g}^*$. The two versions of modified functionals are derived from the incremental Hu-Washizu functional given by Eq.(4.135)

i) first version

$$\pi^{*2}_{HWM1}(\Delta \underline{u}, \Delta \underset{\sim}{g}^*, \Delta \underset{\sim}{S}^*, \Delta \tilde{\underline{t}}^*_\rho) \tag{4.158}$$

$$= \sum_m \int_{V_{n_m}} \{ \Delta W^*(\Delta \underset{\sim}{g}^*) - \rho_N \Delta \underline{g} \cdot \Delta \underline{u} + \frac{1}{2} \underset{\sim}{\tau}^N : [(\nabla^* \Delta \underline{u}) \cdot (\nabla^* \Delta \underline{u})^T]$$

$$- \Delta \underset{\sim}{S}^* : [\Delta \underset{\sim}{g}^* - \frac{1}{2}\{(\nabla^* \Delta \underline{u}) + (\nabla^* \Delta \underline{u})^T\}]\} dV$$

$$- \sum_m \int_{S\sigma_{n_m}} \Delta \bar{\underline{t}}^* \cdot \Delta \underline{u}\, dS \quad - \sum_m \int_{Su_{n_m}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) dS$$

$$- \sum_m \int_{\rho_{n_m}} \Delta \tilde{\underline{t}}^*_\rho \cdot \Delta \underline{u}\, dS$$

where, as in the total Lagrangean formulation, $\Delta \tilde{\underline{t}}^*_\rho$ is the traction (per unit area in C_N) at the inter-element boundary, whose magnitude is uniquely defined at the inter-element boundary but opposite sign is taken for each of the two adjoining elements, such that,

$$\Delta \tilde{\underline{t}}^{*+}_\rho + \Delta \tilde{\underline{t}}^{*-}_\rho = 0 \tag{4.159}$$

π^1 corresponding to Eq.(4.158) is shown to be,

$$\Pi'(\Delta\underline{u}, \Delta\underset{\sim}{g}^*, \Delta\underset{\sim}{S}^*, \Delta\tilde{\underline{t}}_\rho^*) \tag{4.160}$$

$$= \sum_m \int_{V_{n_m}} \{ [\frac{\partial W^*}{\partial \underset{\sim}{g}^*} |^{N+1} - \underset{\sim}{S}^{*N+1}] : \Delta\underset{\sim}{g}^*$$

$$- [\underset{\sim}{g}^{*N+1} - \frac{1}{2}\{(\underset{\sim}{F}^{*N+1})^T \cdot (\underset{\sim}{F}^{*N+1}) - \underset{\sim}{I}\}] : \Delta\underset{\sim}{S}^*$$

$$- \rho_N \underline{g}^{N+1} \cdot \Delta\underline{u} + \underset{\sim}{S}^{*N+1} \cdot (\underset{\sim}{F}^{*N+1})^T : (\nabla^* \Delta\underline{u}) \} dV$$

$$- \sum_m \int_{S_{\sigma_{n_m}}} \bar{\underline{t}}^{*N+1} \cdot \Delta\underline{u}\, ds \quad - \sum_m \int_{S_{u_{n_m}}} \{ \underline{t}^{*N+1} \cdot \Delta\underline{u} + \Delta\underline{t}^* \cdot (\underline{u}^{N+1} - \bar{\underline{u}}^{N+1}) \} ds$$

$$- \sum_m \int_{\rho_{n_m}} \{ \Delta\tilde{\underline{t}}_\rho^* \cdot \underline{u}^{N+1} + \tilde{\underline{t}}_\rho^{*N+1} \cdot \Delta\underline{u} \} ds$$

ii) second version

$$\Pi^{*2}_{HWM2}(\Delta\underline{u}, \Delta\underset{\sim}{g}^*, \Delta\underset{\sim}{S}^*, \Delta\tilde{\underline{u}}_\rho) \tag{4.161}$$

= {first three terms are the same as in Eq.(4.158)}

$$- \sum_m \int_{\rho_{n_m}} \Delta\underline{t}^* \cdot (\Delta\underline{u} - \Delta\tilde{\underline{u}}_\rho)\, ds$$

where $\Delta\tilde{\underline{u}}_\rho$ is an incremental displacement vector uniquely defined at the inter-element boundary.

$$\Pi'(\Delta\underline{u}, \Delta\underset{\sim}{g}^*, \Delta\underset{\sim}{S}^*, \Delta\tilde{\underline{u}}_\rho) \tag{4.162}$$

= {first three terms are the same as in Eq.(4.160)}

$$- \sum_m \int_{\rho_{n_m}} \{ \underline{t}^{*N+1} \cdot (\Delta\underline{u} - \Delta\tilde{\underline{u}}_\rho) + \Delta\underline{t}^* \cdot (\underline{u}^{N+1} - \tilde{\underline{u}}_\rho^{N+1}) \} ds$$

The stationarity conditions of the functionals, Eqns.(4.158) and (4.161), lead to Eqns.(4.114, 116, 119, 125, 132, and

134) and also the continuity conditions at inter-element boundaries , which is given by Eqns.(4.156) and (4.157).

Based on $\Delta \underset{\sim}{t}^*$ and $\Delta \underset{\sim}{e}^*$. Similarly, the modified functionals corresponding to Eq.(4.137) are shown to be,

i) first version

$$\pi^{*2}_{HWM1}(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*, \Delta \tilde{\underline{t}}^*_\rho) \tag{4.163}$$

= {first three terms are the same as in Eq.(4.137) except that the integrals over the volume and the surface are replaced by the sum of those for each element} $- \sum_m \int_{\rho_{n_m}} \Delta \tilde{\underline{t}}^*_\rho \cdot \Delta \underline{u} \, ds$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*, \Delta \tilde{\underline{t}}^*_\rho) \tag{4.164}$$

= {first three terms are the same as in Eq.(4.138) except that the integrals over the volume and the surface are replaced by the sum of those for each element} $- \sum_m \int_{\rho_{n_m}} \{ \Delta \tilde{\underline{t}}^*_\rho \cdot \underline{u}^{N+1} + \tilde{\underline{t}}^{*N+1}_\rho \cdot \Delta \underline{u} \} \, ds$

ii) second version

$$\pi^{*2}_{HWM2}(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*, \Delta \tilde{\underline{u}}_\rho) \tag{4.165}$$

= {first three terms are the same as in Eq.(4.163)}

$$- \sum_m \int_{\rho_{n_m}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \tilde{\underline{u}}_\rho) \, ds$$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{e}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.166}$$

= {first three terms are the same as in Eq.(4.164)}

$$-\sum_m \int_{\rho_{n_m}} \left\{ \underline{t}^{*N+1} \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho) + \Delta \underline{t}^* \cdot (\underline{u}^{N+1} - \underline{\tilde{u}}^{N+1}) \right\} ds$$

The stationarity conditions of the functionals given by Eqns.(4.163 and 165) lead to Eqns.(4.115, 117, 120, 126, 133, 134, 156, and 157) as a posteriori conditions.

<u>Based on $\Delta \underset{\sim}{r}^*$ and $\Delta \underset{\sim}{h}^*$.</u> Likewise, analogous modifications can be done to Eq.(4.139). These Modifications lead to,

i) first version

$$\pi^{*2}_{HWM1}(\Delta \underline{u}, \Delta \underset{\sim}{h}^*, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{t}}^*_\rho) \tag{4.167}$$

= {first three terms are the same as in Eq.(4.139) except that the integrals over the volume and the surface are replaced by the sum of those for each element} $- \sum_m \int_{\rho_{n_m}} \Delta \underline{\tilde{t}}^*_\rho \cdot \Delta \underline{u} \, ds$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{h}^*, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{t}}^*_\rho) \tag{4.168}$$

= {first three terms are the same as in Eq.(4.144) except that the integrals over the volume and the surface are replaced by the sum of those for each element} $- \sum_m \int_{\rho_{n_m}} \left\{ \Delta \underline{\tilde{t}}^*_\rho \cdot \underline{u}^{N+1} + \underline{\tilde{t}}^{*\,N+1}_\rho \cdot \Delta \underline{u} \right\} ds$

ii) second version

$$\pi^{*2}_{HWM2}(\Delta \underline{u}, \Delta \underset{\sim}{h}^*, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.169}$$

= {first three terms are the same as in Eq.(4.167)}

$$-\sum_m \int_{\rho_{nm}} \Delta \underline{t}^* \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho)\, ds$$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{h}^*, \Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.170}$$

= {first three terms are the same as in Eq.(4.168)}

$$-\sum_m \int_{\rho_{nm}} \{ \underline{t}^{*N+1} \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho) + \Delta \underline{t}^* \cdot (\underline{u}^{N+1} - \underline{\tilde{u}}_\rho^{N+1}) \}\, ds$$

The incremental governing equations, Eqns.(4.115, 118, 121, 127, 133, and 134) and the continuity conditions at inter-element boundaries, Eq.(4.156 and 157), follow from the stationarity conditions of the functionals given by Eqns.(4.167 and 169).

Modified Incremental Potential Energy Principles

The two forms of incremental potential energy principles given by Eqns.(4.147 and 148), which are shown to be identical, lead to the following modified functionals.

i) first version

$$\pi^{*2}_{PM1}(\Delta \underline{u}, \Delta \underline{\tilde{t}}_\rho^*) \tag{4.171}$$

= {first two terms are the same as in Eq.(4.147) or Eq.(4.148), except that the integrals are replaced by the sum of those for each element}

$$-\sum_m \int_{\rho_{nm}} \Delta \underline{\tilde{t}}_\rho^* \cdot \Delta \underline{u}\, ds$$

ii) second version

$$\pi^{*2}_{PM2}\,(\Delta\underline{u}\,,\,\Delta\underline{\tilde{u}}_{\rho}) \tag{4.172}$$

$= \{$first two terms are the same as in Eq.(4.171)$\}$

$$-\sum_{m}\int_{\rho_{nm}} \Delta\underline{t}^{*}\cdot(\Delta\underline{u}-\Delta\underline{\tilde{u}}_{\rho})\,ds$$

The stationarity conditions of the functionals, Eqns.(4.171 and 172), lead to Eqns.(4.114, 116, 132, 156, and 157) and (4.115, 117, 133, 156, and 157), respectively.

Based on the above functionals incremental hybrid displacement finite element models are derived [30]. This type of finite element models are also practically as useful as those in the total Lagrengean formulation.

Modified Incremental Hellinger-Reissner Principles

The functionals given by Eqns.(4.150 and 151) are modified, and the following modified incremental Hellinger-Reissner functionals are derived.

Based on $\Delta\underset{\sim}{S}^{*}$ and $\Delta\underset{\sim}{g}^{*}$.

i) first version

$$\pi^{*2}_{HRM1}\,(\Delta\underline{u}\,,\,\Delta\underset{\sim}{S}^{*},\,\Delta\underline{\tilde{t}}^{*}_{\rho}) \tag{4.173}$$

$= \{$first three terms are the same as in Eq.(4.150) except that the integrals are replaced by the sum of those for each element$\}$ $-\sum_{m}\int_{\rho_{nm}}\Delta\underline{\tilde{t}}^{*}_{\rho}\cdot\Delta\underline{u}\,ds$

ii) second version

$$\pi^{*2}_{HRM2}\,(\Delta\underline{u},\,\Delta\underset{\sim}{S}^{*},\,\Delta\tilde{\underline{u}}_{\rho}) \tag{4.174}$$

= {first three terms are the same as in Eq.(4.173)}

$$-\sum_{m}\int_{\rho_{n_m}} \Delta\underline{t}^{*}\cdot(\Delta\underline{u}-\Delta\tilde{\underline{u}}_{\rho})\,ds$$

The stationarity conditions of the functionals given by Eqns.(4.173 and 174) lead to Eqns.(4.114, 116, 119, 132, 134, 156, and 157).

Based on the above modified functionals, incremental hybrid mixed finite element models analogous to those in the updated Lagrangean formulation can be derived [31].

Based on $\Delta\underset{\sim}{r}^{*}$ and $\Delta\underset{\sim}{h}^{*}$.

i) first version

$$\pi^{*2}_{HRM1}\,(\Delta\underline{u},\,\Delta\underset{\sim}{d}^{*},\,\Delta\underset{\sim}{t}^{*},\,\Delta\tilde{\underline{t}}^{*}_{\rho}) \tag{4.175}$$

= {first three terms are the same as in Eq.(4.151), except that the integrals are replaced by the sum of those for each element}

$$-\sum_{m}\int_{\rho_{n_m}} \Delta\tilde{\underline{t}}^{*}_{\rho}\cdot\Delta\underline{u}\,ds$$

ii) second version

$$\pi^{*2}_{HRM2}\,(\Delta\underline{u},\,\Delta\underset{\sim}{d}^{*},\,\Delta\underset{\sim}{t}^{*},\,\Delta\tilde{\underline{u}}_{\rho}) \tag{4.176}$$

= {first three terms are the same as in Eq.(4.175)}

$$-\sum_{m}\int_{\rho_{n_m}} \Delta\underline{t}^{*}\cdot(\Delta\underline{u}-\Delta\tilde{\underline{u}}_{\rho})\,ds$$

The stationarity conditions of the functionals, Eqns.(4.175

and 176), lead to Eqns.(4.115, 118, 121, 133, 134, 156, and 157) as a posteriori conditions.

Modified Incremental Complementary Energy Principle

Based on $\Delta \underset{\sim}{S}^*$ and $\Delta \underset{\sim}{g}^*$. The incremental complementary energy principle given by Eq.(4.152) is modified and the following modified functionals are obtained. However, the translational equilibrium condition and the traction boundary condition, a priori satisfied only approximately, are retained in π'.

i) first version

$$\pi^{*2}_{CM1}\,(\Delta \underline{u}, \Delta \underset{\sim}{S}^*, \Delta \tilde{\underline{t}}^*_\rho) \tag{4.177}$$

= {first two terms are the same as in Eq.(4.152), except that the integrals are replaced by the sum of those for each element}

$$-\sum_m \int_{\rho_{nm}} (\Delta \underline{t}^* - \Delta \tilde{\underline{t}}^*_\rho) \cdot \Delta \underline{u}\, ds$$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{S}^*, \Delta \tilde{\underline{t}}^*_\rho) \tag{4.178}$$

= {first two terms are the same as in Eq.(4.153), except that the integrals are replaced by the sum of those for each element}

$$+\sum_m \int_{\rho_{nm}} \{\Delta \tilde{\underline{t}}^*_\rho \cdot \underline{u}^{N+1} + \tilde{\underline{t}}^{*\,N+1}_\rho \cdot \Delta \underline{u}\}\, ds$$

ii) second version

$$\pi^{*2}_{CM2}\,(\Delta \underline{u}, \Delta \underset{\sim}{S}^*, \Delta \tilde{\underline{u}}_\rho) \tag{4.179}$$

= {first two terma are the same as in Eq.(4.177)}

$$- \sum_m \int_{\rho_{n_m}} \Delta \underline{t}^* \cdot \Delta \underline{\tilde{u}}_\rho \, ds$$

$$\pi'(\Delta \underline{u}, \Delta \underset{\sim}{S}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.180}$$

= {first two terms are the same as in Eq.(4.178)}

$$+ \sum_m \int_{\rho_{n_m}} \left\{ \underline{t}^{*N+1} \cdot (\Delta \underline{u} - \Delta \underline{\tilde{u}}_\rho) + \Delta \underline{t}^* \cdot (\underline{u}^{N+1} - \underline{\tilde{u}}_\rho^{N+1}) \right\} ds$$

The stationarity conditions of the above functionals lead to Eqns.(4.116, 119, 134, 156, and 157) as a posteriori conditions.

The modified functionals given by Eqns.(4.177 and 178) can be applied to finite element models [32]. However, as discussed for the total Lagrangean formulations, no significant advantage is founf in this type of finite element models as compared to the hybrid type mixed finite element models based on Eqns.(4.173 and 174).

Based on $\Delta \underset{\sim}{r}^*$ and $\Delta \underset{\sim}{h}^*$. The functional given by Eq.(4.154), which is considered to be a basis for the most consistent incremental complementary energy principle, as with the analogous principle in the total Lagrangean formulation, is modified further to yield the following modified functionals, which form the basis of hybrid stress finite element incremental models.

i) first version

$$\pi^{*2}_{CM1}(\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{t}}^*_\rho, \Delta \underline{u}_\rho) \tag{4.181}$$

$$= \sum_m \int_{V_{n_m}} \{ \Delta R^*(\Delta \underset{\sim}{r}^*) + (\underset{\sim}{\tau}^N + \Delta \underset{\sim}{t}^*)^T : \Delta \underset{\sim}{\alpha}^* \} dV$$

$$- \sum_m \int_{S_{u_{n_m}}} \Delta \underline{t}^* \cdot \Delta \underline{\bar{u}} \, ds - \sum_m \int_{\rho_{n_m}} (\Delta \underline{t}^* - \Delta \underline{\tilde{t}}_\rho^*) \cdot \Delta \underline{u}_\rho \, ds$$

$$\pi'(\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{t}}_\rho^*, \Delta \underline{u}_\rho) \tag{4.182}$$

$$= \sum_m \int_{V_{n_m}} \{ [\underset{\sim}{\alpha}^{*\,N+1} \cdot (\underset{\sim}{I} + \frac{\partial R^*}{\partial \underset{\sim}{r}^*} \Big|^{N+1}) - \underset{\sim}{I}] : \Delta \underset{\sim}{t}^{*T}$$

$$+ [(\underset{\sim}{I} + \frac{\partial R^*}{\partial \underset{\sim}{r}^*} \Big|^{N+1}) \cdot \underset{\sim}{t}^{*\,N+1}] : \Delta \underset{\sim}{\alpha}^{*T} \} dV$$

$$- \sum_m \int_{S_{u_{n_m}}} \Delta \underline{t}^* \cdot \underline{\bar{u}}^{N+1} ds$$

$$- \sum_m \int_{\rho_{n_m}} \{ (\Delta \underline{t}^* - \Delta \underline{\tilde{t}}_\rho) \cdot \underline{u}_\rho^{N+1} + (\underline{t}^{*\,N+1} - \underline{\tilde{t}}_\rho^{*\,N+1}) \cdot \Delta \underline{u}_\rho \} ds$$

where $\Delta \underline{u}_\rho$ is the displacement vector at inter-element boundaries which is independently defined for the adjoining elements.

ii) second version

$$\pi_{CM2}^{*2} (\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.183}$$

= {first two terms are the same as in Eq.(4.181)}

$$- \sum_m \int_{\rho_{n_m}} \Delta \underline{t}^* \cdot \Delta \underline{\tilde{u}}_\rho \, ds$$

$$\pi'(\Delta \underset{\sim}{\alpha}^*, \Delta \underset{\sim}{t}^*, \Delta \underline{\tilde{u}}_\rho) \tag{4.184}$$

= {first two terms are the same as in Eq.(4.182)}

$$- \sum_m \int_{\rho_{n_m}} \{ \underline{t}^{*\,N+1} \cdot \Delta \underline{\tilde{u}}_\rho + \Delta \underline{t}^* \cdot \underline{\tilde{u}}_\rho^{N+1} \} ds$$

The stationarity conditions of the above functionals lead to Eqns.(4.118, 121, 134, 156, and 157) as a posteriori conditions. Based on these functionals, incremental hybrid stress finite element models which involve undetermined parameters for stress $\Delta \underset{\sim}{t}^{*}$, rotation $\Delta \underset{\sim}{a}^{*}$, and inter-element boundary traction and/or displacement. These types of finite element models also have the same advantageous features as discussed for the analogous models in the total Lagrangran formulation. Thus, they will provide consistent and versatile numerical tools to analyze finite deformation problems of solids.

CHAPTER V

FINITE DEFORMATION PROBLEMS OF NONLINEAR COMPRESSIBLE ELASTIC SOLID

Introduction

In this chapter, an incremental hybrid stress finite element model, based on the modified incremental complementary energy principle is derived, and its application to the finite deformation problems of nonlinear compressible elastic solids is discussed.

Two versions of a modified incremental complementary energy principles, using both the total and updated Lagrangean formulations, are proposed in the preceding chapter. Consequently, four different types of assumed stress finite element models can be derived. The choice among these four models largely depends on the nature of the problems to be solved. For the present nonlinear elasticity problems, in which the strain energy density is defined in the initial configuration, the total Lagrangean formulation in which the initial configuration is used for reference is preferable. Also, if the two versions of modified functionals are compared, the second version, involving only inter-element boundary displacement as an additional variable, appears more convenient than the first version. Thus, based on the functional given by Eq.(4.103), an incremental hybrid stress finite element model using the total Lagrangean formulation is derived. The detailed finite element formulations are presented for general three-dimensional problems. Further, the general formulation is reduced for two-dimensional problems, and the plane stress four-noded quadrilateral element is developed. Using the newly developed finite element model, the example problem of the prescribed stretching of a thin elastic sheet is solved. In this example, the material is considered to be a Blatz-Ko type nonlinear elastic compressible material [17]. The numerical results for this example problem are discussed.

Finite Element Formulation of Incremental Hybrid Stress Model

A finite element formulation is considered, in general, as a discretized equivalent of the corresponding variational principle. For the incremental hybrid stress model, the development of which is one of the objectives of the present work, the modified incremental complementary energy principle stated by Eq.(4.103) is taken as its basis. If the application of the functional Eq.(4.103) to the finite element model is considered, the traction boundary condition, which is assumed to be satisfied a priori in Eq.(4.103), is difficult to be achieved by the assumed functions. Therefore, this condition is also relaxed a priori in the same manner as shown for the Hu-Washizu principle. On the other hand, the displacement boundary condition can be directly enforced by selecting element boundary displacement $\Delta\tilde{\underline{u}}_\rho$ such that,

$$\Delta \tilde{\underline{u}}_\rho = \Delta \bar{\underline{u}} \qquad \text{on } S_{u_{0m}} \tag{5.1}$$

Since, $\Delta \tilde{\underline{u}}_\rho$ is taken as independent variable, this condition can be introduced at the stage when the final algebraic equations for the whole system are solved. Thus, in the discretizing process the displacement boundary condition is removed from the functional, and the functional Eq.(4.103) is rewritten in the following form,

$$\begin{aligned} &\pi^2_{HS2}(\Delta \underset{\sim}{a}, \Delta \underset{\sim}{t}, \Delta \tilde{\underline{u}}_\rho) \\ &= \sum_m \int_{V_{0m}} \left\{ \Delta R(\Delta \underset{\sim}{r}) + (\underset{\sim}{t}^N + \Delta \underset{\sim}{t})^T : [\, \Delta \underset{\sim}{a} \cdot (\underset{\sim}{h}^N + \underset{\sim}{I}) \,] \right\} dv \\ &+ \sum_m \int_{S_{\sigma_{0m}}} \Delta \bar{\underline{t}} \cdot \tilde{\underline{u}}_\rho \, ds - \sum_m \int_{\partial V_{0m}} \Delta \underline{t} \cdot \Delta \tilde{\underline{u}}_\rho \, ds \end{aligned} \tag{5.2}$$

The corresponding iterative correction procedure is obtained by retaining the following functional,

$$\begin{aligned} &\pi^1(\Delta \underset{\sim}{a}, \Delta \underset{\sim}{t}, \Delta \tilde{\underline{u}}_\rho) \\ &= \sum_m \int_{V_{0m}} \left\{ [\, \underset{\sim}{a}^N \cdot \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^N + \underset{\sim}{a}^N - \underset{\sim}{I} \,] : \Delta \underset{\sim}{t}^T \right. \\ &\qquad \left. + [\, (\underset{\sim}{I} + \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^N) \cdot \underset{\sim}{t}^N] : \Delta \underset{\sim}{a}^T \right\} dv \\ &+ \sum_m \int_{S_{\sigma_{0m}}} \bar{\underline{t}}^N \cdot \Delta \tilde{\underline{u}}_\rho \, ds - \sum_m \int_{\partial V_{0m}} \left\{ \underline{t}^N \cdot \Delta \tilde{\underline{u}}_\rho + \Delta \underline{t} \cdot \tilde{\underline{u}}_\rho^N \right\} ds \end{aligned} \tag{5.3}$$

In the functional given by Eq.(5.2), the translational equilibrium condition Eq.(4.8) and constitutive relation Eq.(4.20) are satisfied a priori. Keeping this in mind, Eq.(5.2) and Eq.(5.3) are discretized in the following manner. The first step is the discretization of the variables in the functionals. The stress field, which satisfies the translational equilibrium condition, viz.,

$$\nabla \cdot \underset{\sim}{t} + \rho_0 \underline{g} = 0 \tag{5.4}$$

and in its incremental form,

$$\nabla \cdot \Delta \underset{\sim}{t} + \rho_0 \Delta \underline{g} = 0 \tag{5.5}$$

can be assumed by introducing a first-order stress functions $\underset{\sim}{\psi}$, such that,

$$\underset{\sim}{t} = \nabla \times \underset{\sim}{\psi} + \underset{\sim}{t}^P \tag{5.6}$$

or in its incremental form,

$$\Delta \underset{\sim}{t} = \nabla \times \Delta \underset{\sim}{\psi} + \Delta \underset{\sim}{t}^P \tag{5.7}$$

where $\underset{\sim}{t}^P$ is a particular solution which satisfies,

$$\nabla \cdot \underset{\sim}{t}^P = - \rho_0 \underline{g} \tag{5.8}$$

For convenience, vectors and tensors are decomposed into the rectangular Cartesian components, and equations are presented by using index notations in the subsequent development. Then, Eq.(5.6) is rewritten in components as,

$$t_{ij} = e_{imn} \psi_{nj,m} + t^P_{ij} \tag{5.9}$$

where e_{imn} is the permutation symbol, and a comma followed by subscript implies the derivative with respect to the corresponding Cartesian coordinate. Now, we assume that ψ_{nj} are defined by linear combinations of a finite number of linearly independent functions f_q (q=1...J) with undetermined parametres β_{njq}, such that,

$$\psi_{nj} = f_q \beta_{njq} \qquad (5.10)$$

The substitution of Eq.(5.10) into Eq.(5.9) yields,

$$t_{ij} = e_{imn} f_{q,m} \beta_{njq} + t^p_{ij} \qquad (5.11)$$

Further, the matrix notation is introduced, and Eq.(5.11) is rewritten as,

$$\{t_{ij}\} = \{e_{imn} f_{q,m} \beta_{njq} + t^p_{ij}\} \qquad (5.11)^*$$

$$= \underset{9xa}{[A]} \underset{axl}{\{\beta_{njq}\}} + \underset{9xl}{\{t^p_{ij}\}}$$

where a (a=9xJ) is the number of undetermined stress parameters, and column and row vectors are denoted by {} and ⌊⌋, respectively : and matrices are denoted by [ ]. Since Eq.(5.11)* is a linear equation, the stress increments Δt_{ij} are also defined in the same way.

$$\{\Delta t_{ij}\} = [A]\{\Delta\beta_{njq}\} + \{\Delta t^p_{ij}\} \qquad (5.12)$$

Using this asuumed function, the tractions at the element boundary t_j can be expressed by,

$$\{t_j\} = \{n_i t_{ij}\} = [A^*]\{\beta_{njq}\} + \{T^p{}_j\} \qquad (5.13)$$

And its incremental form is,

$$\{\Delta t_j\} = [A^*]\{\Delta\beta_{njq}\} + \{\Delta T^p_j\} \tag{5.14}$$

On the other hand, the rotation tensor in the three-dimensional case is subjected to the orthogonality condition, which is nonlinear,

$$\underset{\sim}{a}^T \cdot \underset{\sim}{a} = \underset{\sim}{I} \tag{5.15}$$

Therefore rotation can be uniquely defined by three independent parameters θ_i (i=1, 2, 3), such as the Euler angles. In general, each component of the rotation tensor is a nonlinear function of θ_i. Assuming the distribution of θ_i as a linear combination of linearly independent functions w_j with undetermined rotation parametres μ_{ij} (i=1, 2, 3 and j=1...K) such that,

$$\theta_i = w_j \mu_{ij} \tag{5.16}$$

the components of the rotation tensor can be expressed as nonlinear functions of μ_{ij}. Following the definition Eq.(4.1), their increments are obtained by,

$$\begin{aligned} \Delta a_{ij} &= a^{N+1}_{ij}(\mu^{N+1}_{mn}) - a^{N}_{ij}(\mu^{N}_{mn}) \\ &= a^{N+1}_{ij}(\mu^{N}_{mn} + \Delta\mu_{mn}) - a^{N}_{ij}(\mu^{N}_{mn}) \end{aligned} \tag{5.17}$$

which are nonlinear functions of $\Delta\mu_{mn}$. Retaining up to the second order terms* of $\Delta\mu_{mn}$, discretized incremental rotation is defined by,

* As it will be seen in Eq.(5.27), the second order terms in $\Delta\mu_{mn}$ has contributions to the discretized incremental functional through the term, $\int_{V_{0m}} \underset{\sim}{t}^{N^T} : \Delta\underset{\sim}{a} \cdot (\underset{\sim}{h}^N + \underset{\sim}{I})\, dV$.

$$\{\Delta a_{ij}\} = [B]\{\Delta\mu_{mn}\} + \{\Delta\mu^2\} \tag{5.18}$$
$$9x1 \qquad 9xb \quad bx1 \qquad\qquad 9x1$$

where $[B]\{\Delta\mu_{mn}\}$ represents the linear terms in $\Delta\mu_{mn}$, and the second order terms are symbolically denoted by $\{\Delta\mu^2\}$; and b (b=3XK) is the number of undetermined rotation parameters.

The element boundary displacements $\tilde{u}_{\rho i}$ are uniquely interpolated using nodal displacements (displacements at nodes of an element) as undetermined parameters, such that,

$$\tilde{u}_{\rho i} = L_j \, q_{ij} \tag{5.19}$$

where L_j (j=1..N; N is the number of nodes) are interpolation functions and q_{ij} (i=1, 2, 3 and j=1, ..N) are nodal displacements. Eq.(5.19) is rewritten in the matrix form,

$$\{\tilde{u}_{\rho i}\} = [L]\{q_{ij}\} \tag{5.20}$$
$$3x1 \qquad\qquad 3xc \quad cx1$$

where c (c=3XN) is the number of displacement parameters. Similarly, the displacement increments are defined by,

$$\{\Delta\tilde{u}_{\rho i}\} = [L]\{\Delta q_{ij}\} \tag{5.21}$$

From the definition, Eq.(4.3), the incremental Jaumann stress $\Delta\underset{\sim}{r}$ is expressed in terms of $\Delta\underset{\sim}{t}$ and $\Delta\underset{\sim}{a}$ by,

$$\Delta\underset{\sim}{r} = \frac{1}{2}\left(\Delta\underset{\sim}{t}\cdot\underset{\sim}{a}^N + \underset{\sim}{t}^N\cdot\Delta\underset{\sim}{a} + \underset{\sim}{a}^{NT}\cdot\Delta\underset{\sim}{t}^T + \Delta\underset{\sim}{a}^T\cdot\underset{\sim}{t}^{NT}\right) \tag{5.22}$$

At this point, it is noted that the state variables in the C_N state, such as $\underset{\sim}{t}^N$, $\underset{\sim}{a}^N$, and $\underset{\sim}{h}^N$, are known quantities. Thus, using the discretized variables defined by Eqns.(5.12) and (5.18), the incremental Jaumann stress $\Delta\underset{\sim}{r}$ is expressed in discretized form.

$$\{\Delta r_{ij}\} = [D_1]\{\Delta\beta_{mnq}\} + [D_2]\{\Delta\mu_{mq}\} + \{\Delta r^p_{ij}\} \qquad (5.23)$$

where $\{\Delta r^p_{ij}\}$ are the contributions from the particular solutions, and the definitions of $[D_1]$ and $[D_2]$ in terms of $[A]$, $[B]$, $\underset{\sim}{t}^N$, and $\underset{\sim}{a}^N$ immediately follow from Eqns.(5.12, 18, and 22). Also, for later use, the tnsor $\Delta\underset{\sim}{a}\cdot(\underset{\sim}{h}^N+\underset{\sim}{I})$ is rewritten in the matrix form. Noting that $\Delta\underset{\sim}{a}$ involves the second order terms in $\Delta\mu_{ij}$, it is expressed in the following form,

$$\{\Delta a_{im}(h^N_{mj} + \delta_{mj})\} = [R_1]\{\Delta\mu\} + \{R_2(\Delta\mu^2)\} \qquad (5.24)$$

where $\{R_2(\Delta\mu^2)\}$ is a vector, the components of which are of second order in $\Delta\mu_{ij}$.

Now, we consider the discretization of Eq.(5.2). For simplicity, the subscripts are omitted in the matrix notation in the following equations. As defined by Eq.(4.22), the complementary energy density ΔR is a quadratic function of $\Delta\underset{\sim}{r}$. Using matrix notation, it can be written as,

$$\Delta R = \frac{1}{2}\lfloor\Delta r\rfloor[C]\{\Delta r\} \qquad (5.25)$$

By substituting Eqns.(5.23) and (5.25), the first term in Eq.(5.2) is reduced to the following discretized form.

$$\int_{V_{o_m}} \Delta R \, dv = \frac{1}{2}\int_{V_{o_m}} \lfloor\Delta r\rfloor[C]\{\Delta r\} \, dv \qquad (5.26)$$

$$= \frac{1}{2} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}^T \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix} + \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}^T \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 \end{Bmatrix}$$

where

$$[H_{11}] = \int_{v_{o_m}} [D_1]^T [C] [D_1] \, dv$$

$$[H_{12}] = [H_{21}]^T = \int_{v_{o_m}} [D_1]^T [C] [D_2] \, dv$$

$$[H_{22}] = \int_{v_{o_m}} [D_2]^T [C] [D_2] \, dv$$

$$\{\Delta Q_1\} = \int_{v_{o_m}} [D_1]^T [C] \{\Delta r^P\} \, dv$$

$$\{\Delta Q_2\} = \int_{v_{o_m}} [D_2]^T [C] \{\Delta r^P\} \, dv$$

similarly, other terms in Eq.(5.2) are discretized.

$$\int_{v_{o_m}} \underset{\sim}{t}^{N^T} : [\Delta \underset{\sim}{a} \cdot (\underset{\sim}{h}^N + \underset{\sim}{I})] \, dv = \int_{v_{o_m}} \lfloor t^N \rfloor ([R_1] \{\Delta\mu\} + \{R_2(\Delta\mu^2)\}) dv \quad (5.27)$$

$$= \lfloor \Delta\mu \rfloor \{Q_r\} + \frac{1}{2} \lfloor \Delta\mu \rfloor [H^*] \{\Delta\mu\}$$

$$\int_{v_{o_m}} \Delta \underset{\sim}{t}^T : [\Delta \underset{\sim}{a} \cdot (\underset{\sim}{h}^N + \underset{\sim}{I})] \, dv \quad (5.28)$$

$$= \int_{v_{o_m}} (\lfloor \Delta\beta \rfloor [A]^T + \lfloor \Delta t^P \rfloor) ([R_1] \{\Delta\mu\} + \{R_2(\Delta\mu^2)\}) \, dv$$

$$= \lfloor \Delta\beta \rfloor [P] \{\Delta\mu\} + \frac{1}{2} \lfloor \Delta\mu \rfloor [S] \{\Delta\mu\} + \lfloor \Delta\mu \rfloor \{\Delta Q_3\}$$

$$+ \text{ higher order terms}$$

$$\int_{\partial v_{o_m}} \Delta \underline{t} \cdot \Delta \tilde{\underline{u}}_\rho \, ds = \int_{\partial v_{o_m}} (\lfloor \Delta\beta \rfloor [A^*]^T + \lfloor \Delta T^P \rfloor) [L] \{\Delta q\} \, ds \quad (5.29)$$

$$= \lfloor \Delta\beta \rfloor [G] \{\Delta q\} + \lfloor \Delta q \rfloor \{\Delta Q_4\}$$

$$\int_{s\sigma_{o_m}} \Delta \underline{\bar{t}} \cdot \Delta \underline{\tilde{u}}_{\rho} \, ds = \int_{s\sigma_{o_m}} \lfloor \Delta \bar{t} \rfloor [L] \{\Delta q\} \, ds = \lfloor \Delta q \rfloor \{\Delta Q_5\} \tag{5.30}$$

The substitution of Eqns.(5.26) through (5.30) into the functional, Eq.(5.2), leads to the discretized functional:

$$\pi^2_{HS2}(\Delta\beta, \Delta\mu, \Delta q) \tag{5.31}$$

$$= \sum_m \frac{1}{2} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}^T \begin{bmatrix} H_{11} & (H_{12} + P) \\ (H_{12} + P)^T & (H_{22} + H^* + S) \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}$$

$$+ \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}^T \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 + \Delta Q_3 \end{Bmatrix} - \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix}^T \begin{bmatrix} G \\ 0 \end{bmatrix} \{\Delta q\}$$

$$+ \sum_m \{-\Delta Q_4 + \Delta Q_5\}^T \{\Delta q\}$$

where the term $[\Delta\mu]\{Q_r\}$, which corresoonds to the rotational equilibrium check, is removed, and it will be retained in π^1. The stress parameters $\Delta\beta$ and the rotation parameters $\Delta\mu$ are independent for each element, whereas nodal displacements Δq are common to a set of adjoining elements. Thus, the stationarity condition of Eq.(5.31) with respect to arbitrary variation of $\Delta\beta$ and $\Delta\mu$ gives,

$$[H] \begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix} \{\Delta q\} - \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 + \Delta Q_3 \end{Bmatrix} \tag{5.32}$$

for individual elements, where,

$$\left[H \right] = \begin{bmatrix} H_{11} & (H_{12} + P) \\ (H_{12} + P)^T & (H_{22} + H^* + S) \end{bmatrix} \tag{5.33}$$

As discussed by Fraeijs de Veubeke [33] for the linear elastic case, the matrix $[H_{11}]$ cannot by itself be inverted, due to the fact that certain combinations of stress parameters $\Delta\beta$ produce zero stress energy state; thus, there exists a non-zero vector $\Delta\beta$ for which the stress energy is zero;however, the entire matrix $[H]$ in Eq.(5.32) can be inverted. Therefore, Eq.(5.32) can be solved for $[\Delta\beta, \Delta\mu]$, and we obtain,

$$\begin{Bmatrix} \Delta\beta \\ \Delta\mu \end{Bmatrix} = \left[H \right]^{-1} \left(\begin{bmatrix} G \\ 0 \end{bmatrix} \left\{ \Delta q \right\} - \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 + \Delta Q_3 \end{Bmatrix} \right) \tag{5.34}$$

By substituting back this equation into Eq.(5.31), the functional is expressed in terms of only Δq.

$$\pi^2_{HS2}(\Delta q) = -\frac{1}{2} \sum_m \lfloor \Delta q \rfloor_m \left[K_m \right] \left\{ \Delta q \right\}_m + \lfloor \Delta q \rfloor_m \left\{ \Delta Q \right\}_m \tag{5.35}$$

where

$$\left[K_m \right] = \begin{bmatrix} G \\ 0 \end{bmatrix}^T \left[H \right]^{-1} \begin{bmatrix} G \\ 0 \end{bmatrix} \tag{5.36}$$

$$\left\{ \Delta Q \right\}_m = \begin{bmatrix} G \\ 0 \end{bmatrix}^T \left[H \right]^{-1} \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 + \Delta Q_3 \end{Bmatrix} + \left\{ -\Delta Q_4 + \Delta Q_5 \right\} \tag{5.37}$$

The stationarity condition of the above functional with respect to Δq is obtained as,

$$\sum_m \left[K_m \right] \left\{ \Delta q \right\}_m = \sum_m \left\{ \Delta Q \right\}_m \tag{5.38}$$

where $[K_m]$ are incremental (or tangent) stiffness matrix of elements, and $\{\Delta Q\}_m$ are equivalent nodal forces. After the summation over the elements is properly carried out, so that the connectivity of nodes is maintained, Eq.(5.38) leads to a system of algebraic equations, from which the incremental nodal displacements are obtained.

$$\left[K_G \right] \left\{ \Delta q_G \right\} = \left\{ \Delta Q_G \right\} \tag{5.39}$$

where $[K_G]$ is a global tangent stiffness matrix: $\{\Delta q_G\}$ and $\{\Delta Q_G\}$ are global nodal displacements and nodal forces. This equation can be solved numerically by using the digital computer, and the incremental nodal displacements $\{\Delta q_G\}$ are thus determined. Once $\{\Delta q_G\}$ is known, $\{\Delta\beta\}$ and $\{\Delta\mu\}$ can be calculated through Eq.(5.34). From these values, $\Delta \underset{\sim}{t}$ and $\Delta \underset{\sim}{\alpha}$ can be found by Eqns.(5.12) and (5.18). However, $\Delta \tilde{\underline{u}}_\rho$, $\Delta \underset{\sim}{t}$, and $\Delta \underset{\sim}{\alpha}$ obtained here are linear approximations. Thus, values $\tilde{\underline{u}}_\rho^{N+1}$, $\underset{\sim}{t}^{N+1}$, and $\underset{\sim}{\alpha}^{N+1}$ estimated by,

$$\tilde{\underline{u}}_\rho^{N+1} = \tilde{\underline{u}}_\rho^{N} + \Delta \tilde{\underline{u}}_\rho, \quad \underset{\sim}{t}^{N+1} = \underset{\sim}{t}^{N} + \Delta \underset{\sim}{t}, \quad \underset{\sim}{\alpha}^{N+1} = \underset{\sim}{\alpha}^{N} + \Delta \underset{\sim}{\alpha} \tag{5.40}$$

are also approximate values. The correction to these approximatios can be carried out by the iterative procedure based on π_1 given by Eq.(5.3). The idea of such iteration is similar to that of the well-known Newton-Raphson iteration methods in solving nonlinear algebraic equations.

Following the same procedure as shown for π^2_{HS2}, π^1 given by Eq.(5.3), which corresponds to the correction for C_N state, can be discretized. However, for convenience in the later discussion, π^1 for C_{N+1} state is considered. Its discretized form is obtained as,

$$\pi^1(\Delta\beta, \Delta\mu, \Delta q) = \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta q \end{Bmatrix}^T \begin{Bmatrix} Q_k \\ Q_r \\ Q_t \end{Bmatrix} \tag{5.41}$$

where

$$\lfloor \Delta\beta \rfloor \{Q_k\} = \int_{V_{o_m}} \Big[\underset{\sim}{a}^{N+1} \cdot \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^{N+1} + \underset{\sim}{a}^{N+1} - \underset{\sim}{I} \Big] : \Delta \underset{\sim}{t}^T \, dv - \int_{\partial V_{o_m}} \Delta \underline{t} \cdot \tilde{\underline{u}}_\rho^{N+1} \, ds \tag{5.42}$$

$$\lfloor \Delta\mu \rfloor \{Q_r\} = \int_{V_{o_m}} \Big[\Big(\underset{\sim}{I} + \frac{\partial R}{\partial \underset{\sim}{r}}\Big|^{N+1} \Big) \cdot \underset{\sim}{t}^{N+1} \Big] : \Delta \underset{\sim}{a}^T \, dv \tag{5.43}$$

$$\lfloor \Delta q \rfloor \{Q_t\} = \int_{S_{\sigma_{o_m}}} \bar{\underline{t}}^{N+1} \cdot \Delta \tilde{\underline{u}}_\rho \, ds - \int_{\partial V_{o_m}} \underline{t}^{N+1} \cdot \Delta \tilde{\underline{u}}_\rho \, ds \tag{5.44}$$

The values $\underset{\sim}{a}^{N+1}$, $\underset{\sim}{t}^{N+1}$, $\tilde{\underline{u}}_\rho^{N+1}$, and $\frac{\partial R}{\partial \underset{\sim}{r}}\Big|^{N+1}$ in Eqns.(5.42, 43, and 44) are considered as approximate values obtained either by Eq.(5.39) or after some iterations. Thus, Q_k, Q_r, and Q_t are interpreted as discretized errors in the compatibility condition (kinematic relation), rotational equilibrium condition, and both the traction reciprocity at inter-element boundary and traction boundary condition, respectively. These errors are added on the right hand side of Eq.(5.39) as residual forces in a general sense. Then we obtain,

$$\sum_m \left[K_m\right] \{q_c\}_m = \sum_m \{Q_c\}_m \tag{5.45}$$

where $\{q_c\}_m$ is a correction vector and $\{Q_c\}_m$ is a residual force vector which is defined by,

$$\{Q_c\}_m = \begin{bmatrix} G \\ 0 \end{bmatrix}^T \left[H\right]^{-1} \begin{Bmatrix} Q_k \\ Q_r \end{Bmatrix} + \left\{Q_t\right\} \tag{5.46}$$

After the summation over the elements, Eq.(5.45) is reduced to,

$$\left[K_G\right]\left\{q_{cG}\right\} = \left\{Q_{cG}\right\} \tag{5.47}$$

where $\{q_{cG}\}$ and $\{Q_{cG}\}$ are global correction vector for nodal displacements and global residual forces at nodes. By solving Eq.(5.47) for $\{q_{cG}\}$, the correction vector is obtained. The correction vectors for stress and rotaion are calculated by,

$$\begin{Bmatrix} \beta_c \\ \mu_c \end{Bmatrix} = \left[H^{-1}\right]\left(\begin{bmatrix} G \\ 0 \end{bmatrix} \{q_c\} - \begin{Bmatrix} Q_k \\ Q_r \end{Bmatrix}\right) \tag{5.48}$$

which is analogous to Eq.(5.34). Using these correction vectors, new values of $\underline{\tilde{u}}_{\rho}^{N+1}$, $\underset{\sim}{a}^{N+1}$, and $\underset{\sim}{t}^{N+1}$ are calculated. Again these values are substituted into Eq.(5.41), and the errors are estimated. If the errors are not smaller than desired values, the above corrections are repeated until they are reduced to a prescribed tolerence level. At this point, it is noted that $[K_G]$ in Eq.(5.47) is a tangent stiffness matrix at C_N state. If $[K_G]$ is kept same througnout the iterations , the manner of the convergence is illustrated by Fig.3 which corresponds to a modified Newton-Raphson method. However, as shown by Eqns.(5.22 through 5.47), matrices $[K_m]$ and $[H]$, and equivalently , $[K_G]$ and $\{Q_{cG}\}$ involve $\underset{\sim}{t}^N$, $\underset{\sim}{\alpha}^N$, and $\tilde{\underline{u}}_\rho^N$. If these matrices are evaluated for the new approximate values $\underset{\sim}{t}^{N+1}$, $\underset{\sim}{\alpha}^{N+1}$, and $\tilde{\underline{u}}_\rho^{N+1}$, at each iteration, in other words, the tangent stiffness matrices are replaced by that for new approximate C_{N+1} state, the convergence of the iteration can be improved, as shown by Fig.4, which corresponds to the Newton-Raphson method.

Plane Stress Problem

The general theoretical developments presented in the preceding section are now specialized to the case of plane-stress problems, and the incremental hybrid stress finite element model using four-noded rectangular element is developed. The problem of the prescribed stretching of a thin elastic sheet made of Blatz-Ko type [17] nonlinear elastic material is solved as an example.

For the plane-stress case the stress field is assumed to be constrained by the condition,

$$t_{13} = t_{31} = t_{23} = t_{32} = t_{33} = 0 \tag{5.49}$$

where x_1 and x_2 are inplane coordinates and x_3 is a coordinate in the thickness direction. Similarly, rotation field is subjected to the conditions,

$$\alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = 0, \quad \alpha_{33} = 1 \tag{5.50}$$

Using these assumptions, the Jaumann stress defined by Eq.(2.15) is reduced to,

$$[\underset{\sim}{r}] = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.51}$$

For simplicity, the body forces are assumed to be zero. Then, following the general procedure discussed in the preceding section, stress field is assumed using stress functions ψ_1 and ψ_2, as,

$$t_{11} = \psi_{1,2} \quad , \quad t_{12} = \psi_{2,2} \quad , \quad t_{21} = -\psi_{1,1} \quad , \quad t_{22} = -\psi_{2,1} \tag{5.52}$$

where ψ_1 and ψ_2 are chosen to be complete polynomials of inplane coordinates x_1 and x_2, i.e.,

$$\psi_1 = x_1 \beta_1^1 + x_2 \beta_2^1 + x_1^2 \beta_3^1 + x_1 x_2 \beta_4^1 + x_2^2 \beta_5^1 + \cdots\cdots \tag{5.53}$$

$$\psi_2 = x_1 \beta_1^2 + x_2 \beta_2^2 + x_1^2 \beta_3^2 + x_1 x_2 \beta_4^2 + x_2^2 \beta_5^2 + \cdots\cdots$$

The incremental stress field is also assumed in the same manner. It is noted that the special case, when $\beta_1^1 = \beta_2^2$ and other β's are zero, corresponds to a zero stress energy state in the first increment. This is responsible for the need to invert the matrix [H] as a whole in Eq.(5.32).

The two dimensional rotation field is assumed as,

$$[\ a_{11},\ a_{12}\ ,\ a_{21}\ ,\ a_{22}\ ] = [\mathrm{Cos}\theta,\ \mathrm{Sin}\theta,\ -\mathrm{Sin}\theta,\ \mathrm{Cos}\theta] \quad (5.54)$$

where the rotation parameter corresponds to the rotation angle around x_3 direction. It is readily shown that the above rotation field satisfies the orthogonality condition. Then, retaining the second order terms in $\Delta\theta$, the incremental rotation field defined by Eq.(4.1) is assumed as,

$$[\ \Delta a_{11},\ \Delta a_{12}\ ,\ \Delta a_{21}\ ,\ \Delta a_{22}\ ] = [-\mathrm{Sin}\,\theta^N,\ \mathrm{Cos}\,\theta^N,\ -\mathrm{Cos}\,\theta^N,\ -\mathrm{Sin}\,\theta^N]\Delta\theta$$
$$-\tfrac{1}{2}[\mathrm{Cos}\,\theta^N,\ \mathrm{Sin}\,\theta^N,\ -\mathrm{Sin}\,\theta^N,\ \mathrm{Cos}\,\theta^N]\Delta\theta^2 \quad (5.55)$$

Now, we examine the orthogonality condition of $(\ \underset{\sim}{a}^N + \Delta\underset{\sim}{a}\ )$ which is given by,

$$(\underset{\sim}{a}^N + \Delta\underset{\sim}{a})^T \cdot (\underset{\sim}{a}^N + \Delta\underset{\sim}{a}) = \underset{\sim}{I}$$

By substituting the incremental rotation tensor assumed as in Eq.(5.55) into the above equation, we obtain,

$$(\underset{\sim}{a}^N + \Delta\underset{\sim}{a})^T \cdot (\underset{\sim}{a}^N + \Delta\underset{\sim}{a}) = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}\Delta\theta^4 \quad (5.56)$$

Thus, the assumed incremental rotation tensor satisfies the orthogonality condition up to the third order in $\Delta\theta$. Moreover, if only linear part in $\Delta\underset{\sim}{a}$ is considered, the incremental rotation field can be shown to satisfy the required linearized orthgonality condition, viz.,

$$\underset{\sim}{a}^{N^T} \cdot \Delta\underset{\sim}{a} = \text{skewsymmetric} \quad (5.57)$$

Similarly, the variation $\delta \Delta \underset{\sim}{a}$ satisfies,

$$\underset{\sim}{a}^{N^T} \cdot \delta \Delta \underset{\sim}{a} = \text{skewsymmetric} \tag{5.58}$$

which is required in the iteration process as shown by Eq.(4.54). Thus, the rotational equilibrium condition is guaranteed by iterative corrections. The distribution of the rotation angle θ in the element is also assumed by complete polynomials,

$$\theta = \mu_1 + x_1 \mu_2 + x_2 \mu_3 + \cdots\cdots \tag{5.59}$$

Similarly the increment of the rotation angle $\Delta\theta$ is assumed in the same manner.

For the present problem, four-noded isoparametric element is considered. The geometry of an element in the undeformed state C_0 is shown by Fig.5. For convenience, the four nodes are numbered in the anticlockwise direction. The coordinates of node n $(n=1, \ldots 4)$ are denoted by (x_1^n, x_2^n). An arbitrary shaped quadrilateral geometry of the element can be transformed into a square in (r, s) plane by the isoparametric mapping which is given by,

$$\begin{aligned} x_i = \; & (1/4)\{(1-r)(1-s)x_i^1 + (1+r)(1-s)x_i^2 \\ & + (1+r)(1+s)x_i^3 + (1-r)(1+s)x_i^4\} \quad (i=1,\ 2) \end{aligned} \tag{5.60}$$

Further, denoting the nodal displacements at node n by (q_1^n, q_2^n), displacement fields on the element boundary are assumed by,

$$\tilde{u}_{\rho i} = (1/4)\{(1-r)(1-s)q_i^1 + (1+r)(1-s)q_i^2 \tag{5.61}$$

$$+(1+r)(1+s)q_i^3 \ +(1-r)(1+s)q_i^4 \ \} \qquad (i=1,\ 2)$$

where $(1-r^2)(1-s^2)=0$ and $|r|,\ |s|<1$

The incremental displacements at element boundary $\Delta\tilde{\underline{u}}_\rho$ are also assumed in the same way.

Now, we consider the constitutive relations. The material is assumed to be Blatz-Ko type nonlinear elastic material [17]. Its mechanical properties are characterized by the strain energy density W per unit initial volume, which is given by,

$$W = (\mu f/2)[J_1 - 3 + \frac{2}{\alpha}(J_3^{-\alpha} - 1)] + \frac{\mu(1-f)}{2}[J_2 - 3 + \frac{2}{\alpha}(J_3^{\alpha} - 1)] \tag{5.62}$$

where J_1, J_2, and J_3 are defined by,

$$J_1 = I_1 \ ; \ J_2 = I_2/I_3 \ ; \ \text{and } J_3 = \sqrt{I_3} \tag{5.63}$$

where I_i are the principal invariants of the deformation tensor $\underset{\sim}{G}$. In Eq.(5.63), μ represents the shear modulus, and α is related to the Poisson's ratio ν through,

$$\alpha = 2\nu/(1-2\nu) \tag{5.64}$$

Also, f is a material constant.

The incremental constitutive relations can be obtained through the incremental strain energy density function ΔW which is defined by,

$$\Delta W(\Delta\underset{\sim}{h}) = \frac{1}{2}\left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^N :: \Delta\underset{\sim}{h}\,\Delta\underset{\sim}{h} \tag{5.65}$$

where

$$\frac{\partial \Delta W}{\partial \Delta \underset{\sim}{h}} = \Delta \underset{\sim}{r}$$

The derivation of ΔW is rather lengthy but it can be obtained in a straightforward manner. Let the invariants of $(\underset{\sim}{h}+\underset{\sim}{I})$ be denoted by h_i. For the present plane-stress case, in which,

$$h_{13} = h_{31} = h_{23} = h_{32} = 0$$

they are defined by,

$$h_1 = h_{11} + h_{22} + h_{33} \tag{5.66}$$

$$h_2 = h_{11}\, h_{22} - h_{12}\, h_{21} + h_{22}\, h_{33} + h_{33}\, h_{11}$$

$$h_3 = (h_{11}\, h_{22} - h_{12}\, h_{21})\, h_{33}$$

From the relation between $\underset{\sim}{G}$ and $\underset{\sim}{h}$, Eq.(2.7), it is shown that I_i and h_i are related by,

$$\begin{aligned} I_1 &= h_1^2 - 2h_2 \\ I_2 &= h_2^2 - 2h_3h_1 \\ I_3 &= h_3^2 \end{aligned} \tag{5.67}$$

Using Eqns.(5.63, 66, and 67), W can be expressed in terms of $\underset{\sim}{h}$. Similarly, if we consider the strain tensor $\underset{\sim}{h}^{N+1}$ in C_{N+1}

state, which can be written in terms of $\underset{\sim}{h}^N$ and $\Delta \underset{\sim}{h}$ as,

$$\underset{\sim}{h}^{N+1} = \underset{\sim}{h}^N + \Delta \underset{\sim}{h} \tag{5.68}$$

the strain energy density W^{N+1} in C_{N+1} state can be expressed as a function of the incremental strain $\Delta \underset{\sim}{h}$. Then, the incremental strain energy density ΔW is obtained as the sum of the second order terms of $\Delta \underset{\sim}{h}$ in W^{N+1}. It is shown to be,

$$\begin{aligned}
\Delta W = \frac{\mu f}{2} \Big[& \Big\{ \left(\Delta h_1\right)^2 - 2\Delta^2 h_2 \Big\} + \frac{2}{\alpha} \Big\{ - \alpha \left(J_3^N\right)^{-\alpha-1} \Delta^2 h_3 + \frac{1}{2}\alpha(\alpha+1)\left(J_3^N\right)^{-\alpha-2}\left(\Delta h_3\right)^2 \Big\} \Big] \\
& + \frac{\mu(1-f)}{2} \Big[\frac{1}{I_3^N} \Big\{ \left(\Delta h_2\right)^2 + 2h_2^N \Delta^2 h_2 - 2h_1^N \Delta^2 h_3 - 2\Delta h_3 \Delta h_1 \Big\} \\
& + I_2 \Big\langle - \frac{1}{(I_3^N)^2} \Big\{ \left(\Delta h_3\right)^2 + 2h_3^N \Delta^2 h_3 \Big\} + \frac{1}{(I_3^N)^3} \Big\{ 2h_3^N \Delta h_3 \Big\}^2 \Big\rangle \\
& - \frac{2}{(I_3^N)^2} h_3^N \Big\{ 2h_2^N \Delta h_2 - 2h_3^N \Delta h_1 - 2h_1^N \Delta h_3 \Big\} \Delta h_3 \\
& + \frac{2}{\alpha} \Big\{ \alpha \left(J_3^N\right)^{\alpha-1} \Delta^2 h_3 + \frac{1}{2}\alpha(\alpha-1)\left(J_3^N\right)^{\alpha-2}\left(\Delta h_3\right)^2 \Big\} \Big]
\end{aligned} \tag{5.69}$$

where Δh_1, Δh_2, and Δh_3 are linear parts of the principal invariants h_i^{N+1}, whereas, $\Delta^2 h_2$ and $\Delta^2 h_3$ are second order trems of $\Delta \underset{\sim}{h}$ in h_2^{N+1} and h_3^{N+1}, respectively. They are given by,

$$\Delta h_1 = \Delta h_{11} + \Delta h_{22} + \Delta h_{33} \tag{5.70}$$

$$\begin{aligned}
\Delta h_2 = & (2 + h_{22} + h_{33})\Delta h_{11} - h_{21}\Delta h_{12} - h_{12}\Delta h_{21} \\
& + (2 + h_{11} + h_{33})\Delta h_{22} + (2 + h_{11} + h_{22})\Delta h_{33}
\end{aligned}$$

$$\Delta h_3 = (1 + h_{22})(1 + h_{33})\Delta h_{11} - (1 + h_{33})h_{21}\Delta h_{12}$$

$$- (1 + h_{33})h_{12}\Delta h_{21} + [(1 + h_{11})(1 + h_{22}) - h_{12}h_{21}]\Delta h_{33}$$

$$+ (1 + h_{33})(1 + h_{11})\Delta h_{22}$$

$$\Delta^2 h_2 = \Delta h_{11}\Delta h_{22} - \Delta h_{12}\Delta h_{21} + \Delta h_{11}\Delta h_{33} + \Delta h_{22}\Delta h_{33}$$

$$\Delta^2 h_3 = (1 + h_{33})\{\Delta h_{11}\Delta h_{22} - \Delta h_{12}\Delta h_{21}\}$$

$$+ \Delta h_{33}\{(1 + h_{11})\Delta h_{22} + (1 + h_{22})\Delta h_{11} - h_{12}\Delta h_{21} - h_{21}\Delta h_{12}\}$$

In the above, h_{ij} are known quantities h_{ij}^N at the Nth stage and the super-script N has been omitted for convenience. Using the matrix notation, ΔW can be rewritten as,

$$\Delta W = \frac{1}{2}\{\Delta h\}^T [E]\{\Delta h\} \tag{5.71}$$

where

$$\{\Delta h\}^T = \lfloor \Delta h_{11}, \Delta h_{12}, \Delta h_{21}, \Delta h_{22}, \Delta h_{33} \rfloor$$

From the definition of ΔW, the incremental Jaumann stress is obtained as,

$$\{\Delta r\} = [E]\{\Delta h\} \tag{5.72}$$

where

$$\{\Delta r\}^T = \lfloor \Delta r_{11}, \Delta r_{12}, \Delta r_{21}, \Delta r_{22}, \Delta r_{33} \rfloor$$

Noting the symmetric properties of $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{h}$, viz.,

$$\Delta r_{12} = \Delta r_{21} \quad , \quad \Delta h_{12} = \Delta h_{21}$$

the above relation can be inverted, and we obtain,

$$\{\Delta h\} = [C]\{\Delta r\} \tag{5.73}$$

Further, incremental complementary energy density is obtained as,

$$\Delta R = \frac{1}{2}\{\Delta r\}^T [C]\{\Delta r\} \tag{5.74}$$

Considering the fact that $\Delta r_{33} = 0$ for plane-stress case, ΔR is finally written in the following form.

$$\Delta R(\Delta \underset{\sim}{r}) = \frac{1}{2} \begin{Bmatrix} \Delta r_{11} \\ \Delta r_{12} \\ \Delta r_{21} \\ \Delta r_{22} \end{Bmatrix}^T \begin{bmatrix} C \end{bmatrix} \begin{Bmatrix} \Delta r_{11} \\ \Delta r_{12} \\ \Delta r_{21} \\ \Delta r_{22} \end{Bmatrix} \tag{5.75}$$

where

$$\Delta r_{ij} = \frac{1}{2} (t^N_{im} \Delta \alpha_{mj} + \Delta t_{im} \alpha^N_{mj} + t^N_{jm} \Delta \alpha_{mi} + \Delta t_{jm} \alpha^N_{mi}) \tag{5.76}$$
$$(i,j,m = 1,2)$$

Using the assumptions Eqns.(5.49 and 50) and the incremental complementary energy density given by Eq.(5.75), the functional derived for three-dimensional case, Eq.(5.2), is reduced to,

$$\pi^2_{HS2}(\Delta t_{pq}, \Delta \alpha_{pq}, \Delta \tilde{u}_{\rho_p}) \tag{5.77}$$

$$= \sum_m \int_{V_{o_m}} \Big\{ \Delta R(\Delta r_{ij}) + \Delta t_{ij} \Delta \alpha_{jk}'(h^N_{ki} + \delta_{ki})$$

$$+ t^N_{ij} \Delta\alpha_{jk} \left(h^N_{ki} + \delta_{ki} \right) \Big\} dv$$

$$- \sum_m \int_{\partial v_{o_m}} \Delta t_i \Delta \tilde{u}_{\rho i} \, ds + \sum_m \int_{s_{\sigma_{om}}} \Delta \bar{t}_i \Delta \tilde{u}_{\rho i} \, ds$$

$$(i,j,k,p,q = 1,2)$$

The volume integrals and the surface integrals in the above equation are reduced to integrals over the area of the element and line integrals along its boundary. Further, by using the isoparametric transformation, these integrals defined in the (x_1, x_2) plane are mapped into those in the (r, s) plane. Thus, they are reduced to area integrals over the square region ($|r|, |s|<1$) and line integrals along the line parallel to the coordinate lines, which can be easily evaluated by numerical quadrature, such as Gaussian integration.

By substituting the discretized assumed functions defined by Eqns.(5.52, 54, and 61) into the functional, Eq.(5.77), and carrying out the integration, it is reduced to the discretized functional analogous to Eq.(5.31). Following the general procedures discussed for three-dimensional case, incremental hybrid stress finite element model for plane-stress problems is developed.

Before applying the newly developed finite element model to boundary value problems, the properties of a single element are studied. According to the assumptions Eq.(5.53 and 59), the stress function and the rotation angle can be assumed as complete polynomials of any order. However, as shown by Fraeijs de Veubeke [33] for linear elastic case, certain conditions must be satisfied by the numbers of undetermined parameters for stress field "a", rotation field "b", and displacement field "c", so that the matrix [H] defined by Eq.(5.32) can be inverted, and further, the element stiffness matrix $[K_m]$ defined by Eq.(5.36) does not involve any kinematic deformation modes other than the three rigid body modes, namely, translations in x_1 and x_2 direction and rigid body rotation. By a close investigation of the mathematical properties of these matrices, we may possibly obtain such conditions in an analytical way. But these mathematical arguments are left for future studies, and a numerical approach is employed. The behavior of the element is characterized by the eigen-values and corresponding eigen-vectors of the stiffness matrix. Physically, the eigen values are proportional to the amount of strain energy stored in the element through the deformations which have the same pattern as the respective eigen vectors. Therefore, these eigen-values must be non-negative for all materials. Further, as shown by Bathe [34] for assumed displacement finite elements in linear theory, a properly formulated element must have three zero eigen-values and these correspond to the rigid body modes. Therefore, by checking the eigen-values and eigen-vectors of the stiffness matrix, we can tell whether the behavior of the element is physically proper or not. Since, the stiffness matrix is changing with the deformation in the finite deformation problem, it is impossible to predict its behavior in the arbitrary deformed state. But by examining the stiffness matrix at the undeformed state, the essential information can be obtained. Then, the eigen-values and eigen-modes of the stiffness matrix of the newly developed finite element are calculated for various combinations of number of stress parameters "a" and number of rotation parameters "b" (number of displacement parameters c=8). The geometry of the element is square and material constants f, μ, and α are chosen so that they are equivalent to the case of E=1 psi and ν=0.3 (E:Young's modulus, ν:Poisson's ratio). The calculated eigen-values are presented in Table 1, and the eigen-modes for the case (a, b, c)=(10, 1, 8) are shown by Fig.7. It is observed from these results that if number of stress parameters "a" is 10 or more, the element behaves properly. However, it is noticed that if the number of rotation parameters, which are considered to constrain the stress field through the rotational equilibrium condition, is taken large compared to that of stress parameters, matrix [H] may become singular as in the case of the combination (4, 3, 8).

Numerical Examples

Now, we turn to the numerical application of the incremental hybrid stress finite element model to a boundary value problem. The example problem considered is the prescribed stretching of a thin elastic sheet (8"X8"X.05"). The sheet is clamped at the loading edges x_1 =±4. and it is stretched to twice its original length in x_1 direction as shown by the inset in Fig.8. Thus the boundary conditions can be stated as,

$$t_1 = t_2 = 0 \qquad \text{at} \quad x_2 = \pm 4. \tag{5.78}$$

$$u_1 = \pm 4(\lambda - 1), \quad u_2 = 0 \qquad \text{at } x_1 = \pm 4. \tag{5.79}$$

where λ is an extension ratio $(1 < \lambda < 2)$. The material is assumed to be a Blatz-Ko type material [17] such as foamed rubber. The specific material constants μ, f, and α in Eq.(5.62) are chosen to be,

$$\mu = 40 \text{ psi} \quad ; \quad f = 0 \quad ; \quad \alpha = 1 \quad (\nu = 0.25)$$

Considering the symmetry of the problem, a quarter of the sheet is analyzed using a 6X6 non-uniform mesh finite element assembly as shown by the inset in Fig.8. The four-noded element which has 10 stress parameters (linear distribution of stress) and 3 rotation parameters (also linear distribution), is used for the present example. The considered total stretch (λ=2) prescribed on the edge $x_1 = \pm 4$ is imposed in 20 increments $(\Delta\lambda = 0.05)$. The Newton-Raphson type iterations based on Eq.(5.3) are carried out at each increment. During the iterations the tangent stiffness matrix is kept unchanged as illustrated by Fig.3. In general, the errors involved in the solution is estimated by the residual load $\{Q_{CG}\}$ in Eq.(5.44). If the norm of the residual load vector and the total load vector, denoted by $\|Q_C\|$ and $\|Q\|$, respectively, are calculated, the measure of the errors $\mathcal{E}$ is estimated by,

$$\mathcal{E} = \|Q_c\| / \|Q\| \tag{5.80}$$

The iterations are repeated until $\mathcal{E}$ becomes less than 1%. For the present example the desired convergence is achieved after 2 iterations, on an average. Fig.8 shows the total axial load necessary to achieve various ratios of stretch ($1< \lambda <2$). For the purpose of comparison, the results obtained by using finer increments ($\Delta\lambda=0.025$) but without iterations are also plotted in Fig.8. However, the results without iterations were not noticeably different from those with twice large increments and with iterations. Hence, no distinction is made between these in Fig.8. From these results, it appears that if the increments are taken small enough, practically reasonable solutions can be obtained without iterations. However it is recommended to check the errors by iterations at least every few increments. The reduction ratio of the width of the sheet at the center line (lateral cntraction ratio) is plotted in Fig.9. The deformed configurations of the sheet at $\lambda=1.5$ and $\lambda=2.0$, along with the initial configuration are shown in Fig.10. The contours of computed rotation angle θ at the final stage ($\lambda=2.0$) are plotted in Fig.11. The rotation field shown by the figure is consistent with the displacement pattern shown by Fig.10, and the maximum rotation occurs at the corner of the loading edge. The contours of the axial (x_1 direction) component τ_{11} of the true or Cauchy stress at $\lambda=2.0$ are plotted on the deformed configuration in Fig.12. Similarly, the distributions of τ_{11}, τ_{12}, and τ_{22} at $\lambda=1.5$ are presented by Figs.13, 14, and 15. It is noted here that all the stress components are considered in the rectangular Cartesian coordinates. As shown by Figs.13 and 14, although number of data points is not sufficiently large due to the coarse finite element mesh, maximum values of τ_{11} and τ_{12} also appear to be found at the corner of the loading edge. From the comparison between Figs.12 and 13 , it is observed that the distribution of τ_{11} at $\lambda=2.0$ is notably different from that at $\lambda=1.5$. This difference is explained by the fact that the material almost reaches the maximum strength after $\lambda=1.8$, and the stress distribution becomes close to being uniform. For comparison, the axial components t_{11} and s_{11} of the Piola-Lagrange stress and Kirchhoff-Trefftz stress, respectively, at $\lambda=1.5$, are plotted on the undeformed configuration in Figs.16 and 17. If the Figs.13, 16, and 17 are compared it is noticed that the stress τ_{11} measured per unit deformed area, which is smaller than that of undeformed state, has the largest value compared to others, whereas, that of s_{11} measured per unit undeformed area and decomposed with respect to the base vectors in deformed configuration is the smallest. This result is consistent with the definitions of stresses given by Eqns.(2.12 and 13). Moreover, because of the fact that the Piola-Lagrange stress $\underset{\sim}{t}$ and true stress $\underset{\sim}{\tau}$ are subjected to the same form of differential equations, viz.,

$$\nabla \cdot \underset{\sim}{t} + \rho_0 \underline{g} = 0 \tag{5.81}$$

$$\nabla^* \cdot \underset{\sim}{\tau} + \rho^* \underline{g} = 0 \tag{5.82}$$

where ∇ and ∇^* are gradient operators in the undeformed and current deformed configurations, respectively; ρ_0 and ρ^* are mass density per unit undeformed and deformed volume, respectively, very close distribution patterns of t_{11} and τ_{11} are observed. On the other hand, s_{11} which is subjected to,

$$\nabla \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T) + \rho_0 \underline{g} = 0 \tag{5.83}$$

has a significantly different distribution. Similar comparison is made among t_{21} , s_{12} , and τ_{12} shown by Figs.18, 19, and 14. Unlike in the case of the axial components, the effect of the difference in the boundary condtions along the free edge is relatively large, also since, the Cauchy stress is symmetric, whereas the Piola-Lagrange stress is unsymmetric. Therefore, a slight difference is observed between Figs.14 and 18.

For incmpressible Mooney type material, similar problems are solved by Oden [18] and Becker [35] using displacement finite element model. Although, direct comparison is not possible, the results obtained by the proposed method show good qualitative agreement with those obtained by Oden [18] and Becker [35]. From the above discussions, it is seen that the numerical results obtained by the proposed method are consistent from both the mathematical and physical points of view.

CHAPTER VI

FINITE DEFORMATION PROBLEMS OF INCOMPRESSIBL ELASTIC SOLIDS

Introduction

Various types of incremental variational principles and their modified versions both in the total Lagrangean and updated Lagrangean formulations, based on alternate stress and strain measures, are discussed in chapter IV. Especially, using the modified incremental complementary energy principle, incremental hybrid stress finite element models for finite deformation problems of a solid are proposed. These variational principles and finite element models are valid for general compressible materials. However, there are many engineering materials, such as rubber, solid propellant rocket grains, and polymers, which are effectively incompressible. With the increasing use of such materials in practical engineering developments, the demands for the theoretical analysis of their behavior have increased in recent years. To analyze such problems, an incremental hybrid stress finite element model which is valid for incompressible materials is developed in this chapter.

The kinematic constraint of precise incompressiblity in finite deformation problems is stated by,

$$I_3 = 1 \quad \text{or} \quad h_3 = 1 \tag{6.1}$$

where I_3 and h_3 are the third principal invariants of the deformation tensor $\underset{\sim}{G}$ and stretch tensor $(\underset{\sim}{I}+\underset{\sim}{h})$, respectively. It is noticed here that these conditions are third order nonlinear equations. The assumption of incompressibility makes it easier to obtain analytical (exact) solutions to certain problems in finite elasticity. However, in general, reverse is the case with numerical solutions such as energy-based finite element methods.

In the case of linear elastic infinitesimal deformation problems, the incompressibility condition is reduced to,

$$\nabla \cdot \underline{u} = 0 \tag{6.2}$$

which is linear. The stain energy density for isotropic linear elastic material can be expressed by,

$$W(\underset{\sim}{\epsilon}) = \frac{1}{2}\lambda(\underset{\sim}{\epsilon}:\underset{\sim}{I})^2 + \mu\, \underset{\sim}{\epsilon}:\underset{\sim}{\epsilon} \tag{6.3}$$

where,

$$\underset{\sim}{\epsilon} = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

The stress is obtained by,

$$\begin{aligned} \underset{\sim}{\sigma} = \frac{\partial W}{\partial \underset{\sim}{\epsilon}} &= \lambda(\underset{\sim}{\epsilon}:\underset{\sim}{I})\underset{\sim}{I} + 2\mu\underset{\sim}{\epsilon} \\ &= (\lambda + \tfrac{2}{3}\mu)(\underset{\sim}{\epsilon}:\underset{\sim}{I})\underset{\sim}{I} + 2\mu\underset{\sim}{\epsilon}' \end{aligned} \tag{6.4}$$

where $\underset{\sim}{\epsilon}'$ is the deviatric strain defined by,

$$\epsilon' = \underset{\sim}{\epsilon} - \frac{1}{3} (\underset{\sim}{\epsilon} : \underset{\sim}{I}) \underset{\sim}{I} \qquad (6.5)$$

If the material is incompressible, λ becomes infinite. Consequently, the strain energy density becomes singular, thus the finite element based on the strain energy is not valid for this case. Also, as noticed from Eq.(6.4), only the deviatoric part of stress can be determined from the strain. Its hydrostatic part can not be determined by strain. It can be determined from the boundary conditions.

Alternative approaches for analyzing nearly or precisely incompressible linear elastic materials are proposed by Herrmann [19] and Key [20]. They construct mixed type variational principles involving both displacement and hydrostatic pressure as variables. In their functionals the incompressibility condition is relaxed by introducing the hydrostatic pressure as a Lagrange multiplier. The incompressibility condition is then preserved a posteriori through the stationarity condition of the functionals. This approach can be extended to the finite deformation problems. Oden [18], based on a modified (not hybrid) stationary potential energy principle, which involves both displacement and hydrostatic pressure, derived a finite element model for finite deformation problems of incompressible elastic solids. If the problem is a plane-stress problem, as discussed by Oden [18], the hydrostatic presure can be expressed in terms of displacements due to the assumption of the vanishing normal stress in the thickness direction. Thus the functional can be expressed in terms of displacements alone. However, as shown by Oden [18], the reduced functional becomes highly nonlinear. Consequently, it leads to a set of nonlinear algebraic equations in the finite element formulation, which can be numerically solved by Newton-Raphson method. An analogous functional is constructed by Becker [35], using simplified incompressibility condition. The solution is obtained by direct minimization of the functional, instead of deriving nonlinear equations.

In this chapter, a modified (hybrid type) incremental complementary energy principle for incompressible material is derived startiong from the Hu-Washizu principle involving the hydrostatic pressure as an additional variable. Based on the derived incremental variational principle, a incremental hybrid stress finite element model for plane stress problem is developed. Two types of plane-stress problems of Mooney-Rivlin type material are solved as examples by using the proposed method.

Hu-Washizu Variational Principles

As briefly mentioned for the linear elastic case, the stress can not be fully determined by strains (equivalently strain energy density), and the mean pressure remains undetermined for incompressible nonlinear elastic materials. The mean pressure (hydrostatic pressure) can be determined only by considering the boundary conditions imposed on the solid. If the material is isotropic, the strain energy density is a function of three principal invariants of the deformation tensor $\underset{\sim}{G}$. In view of Eq.(6.1), strain energy density for an incompressible material is considered as a function of only I_1 and I_2,

$$W = W(I_1, I_2) \tag{6.6}$$

By introducing the hydrostatic pressure p into the strain energy density, we can define a modified strain energy density, whose derivative with respect to strain gives stress. Depending on which strain measure is used, there are three alternative ways to express such a modified strain energy density $\hat{W}$ per unit undeformed volume.

<u>based on $\underset{\sim}{g}$</u>

$$\hat{W}(\underset{\sim}{g}) = W(I_1(\underset{\sim}{g}), I_2(\underset{\sim}{g})) + \frac{p}{2}(I_3 - 1) \tag{6.7}$$

where W is a symmetric function of $\underset{\sim}{g}$.

<u>based on $\underset{\sim}{e}$</u>

$$\hat{W}(\underset{\sim}{e}) = W(I_1(\underset{\sim}{e}), I_2(\underset{\sim}{e})) + p(J - 1) \tag{6.8}$$

where W is considered as a function of $\underset{\sim}{e}$, and J is the third invariant of $\underset{\sim}{F}$.

<u>based on $\underset{\sim}{h}$</u>

$$\hat{W}(\underset{\sim}{h}) = W(I_1(\underset{\sim}{h}), I_2(\underset{\sim}{h})) + p(h_3 - 1) \tag{6.9}$$

where W is considered as a function of $\underset{\sim}{h}$, and h_3 is the third invariant of $(\underset{\sim}{h}+\underset{\sim}{I})$. From these strain energy density functions, the stresses are obtained as,

$$\underset{\sim}{s} = \frac{\partial \hat{W}(\underset{\sim}{g})}{\partial \underset{\sim}{g}} = \frac{\partial W}{\partial \underset{\sim}{g}} + pI_3 \underset{\sim}{G}^{-1} \tag{6.10}$$

$$\underset{\sim}{t}^{T} = \frac{\partial \hat{W}(\underset{\sim}{e})}{\partial \underset{\sim}{e}} = \frac{\partial W}{\partial \underset{\sim}{e}} + pJ\underset{\sim}{F}^{-1} \tag{6.11}$$

$$\underset{\sim}{r} = \frac{\partial \hat{W}(\underset{\sim}{h})}{\partial \underset{\sim}{h}} = \frac{\partial W}{\partial \underset{\sim}{h}} + ph_3 (\underset{\sim}{I} + \underset{\sim}{h})^{-1} \tag{6.12}$$

where, according to the definition presented in the appendix A, the inverse of the unsymmetric tensor $\underset{\sim}{F}$ is defined by $\underset{\sim}{F}^{-1}\cdot\underset{\sim}{F}=\underset{\sim}{I}$. Noting that $I_3 = J = h_3 = 1$ for incompressible case, it is seen that the stress obtained through Eqns.(6.10, 11, and 12) are consistent with the definition of stress given by Eqns.(2.11, 12, and 13).

By introducing the strain energy density function $\hat{W}$ and treating the hydrostatic pressure as a variable, the following Hu-Washizu principles in the total Lagrangean formulation corresponding to Eqns.(3.1, 4, and 7) are derived,

$$\pi_{HW}(\underline{u}, \underset{\sim}{g}, \underset{\sim}{s}, p) \tag{6.13}$$

$$= \int_{V_0} \Big\{ W(\underset{\sim}{g}) + \frac{p}{2}(I_3 - 1) - \rho_0 \underline{g}\cdot\underline{u} + \frac{1}{2}\underset{\sim}{s}:\big[\nabla\underline{u} + (\nabla\underline{u})^T + \nabla\underline{u}\cdot(\nabla\underline{u})^T - 2\underset{\sim}{g}\big]\Big\}\, dv$$

$$- \int_{s_{\sigma_o}} \underline{\bar{t}} \cdot \underline{u} \, ds - \int_{s_{u_o}} \underline{t} \cdot (\underline{u} - \underline{\bar{u}}) ds$$

$$\pi_{HW}(\underline{u}, \underset{\sim}{e}, \underset{\sim}{t}, p) \tag{6.14}$$

$$= \int_{v_o} \Big\{ W(\underset{\sim}{e}) + p(J - 1) - \rho_o \underline{g} \cdot \underline{u}$$

$$+ \underset{\sim}{t}^{T} : [(\nabla \underline{u})^{T} - \underset{\sim}{e}] \Big\} dv$$

$$- \int_{s_{\sigma_o}} \underline{\bar{t}} \cdot \underline{u} \, ds - \int_{s_{u_o}} \underline{t} \cdot (\underline{u} - \underline{\bar{u}}) ds$$

$$\pi_{HW}(\underline{u}, \underset{\sim}{h}, \underset{\sim}{a}, \underset{\sim}{t}, p) \tag{6.15}$$

$$= \int_{v_o} \Big\{ W(\underset{\sim}{h}) + p(h_3 - 1) - \rho_o \underline{g} \cdot \underline{u}$$

$$+ \underset{\sim}{t}^{T} : [(\underset{\sim}{I} + \nabla \underline{u})^{T} - \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h})] \Big\} dv$$

$$- \int_{s_{\sigma_o}} \underline{\bar{t}} \cdot \underline{u} \, ds - \int_{s_{u_o}} \underline{t} \cdot (\underline{u} - \underline{\bar{u}}) ds$$

If, for example, Eq.(6.15) is considered, its first variation can be shown as,

$$\delta \pi_{HW} = \int_{v_o} \Big\{ \Big[\frac{\partial W}{\partial \underset{\sim}{h}} + p \frac{\partial h_3}{\partial \underset{\sim}{h}} - \frac{1}{2} (\underset{\sim}{t} \cdot \underset{\sim}{a} + \underset{\sim}{a}^{T} \cdot \underset{\sim}{t}^{T}) \Big] : \delta \underset{\sim}{h} - [(\underset{\sim}{I} + \underset{\sim}{h}) \cdot \underset{\sim}{t}] : \delta \underset{\sim}{a}^{T} \tag{6.16}$$

$$+ \delta p (h_3 - 1) + \delta \underset{\sim}{t}^{T} : [(\underset{\sim}{I} + \nabla \underline{u})^{T} - \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h})] - [\nabla \cdot \underset{\sim}{t} + \rho_o \underline{g}] \cdot \delta \underline{u} \Big\} dv$$

$$- \int_{s_{\sigma_o}} (\underline{\bar{t}} - \underline{t}) \cdot \delta \underline{u} \, ds - \int_{s_{u_o}} \delta \underline{t} \cdot (\underline{u} - \underline{\bar{u}}) ds$$

Thus, the stationarity condition of Eq.(6.15) yields to all the field equations and boundary conditions and also the incompressibility condition.

Since, the objective here is to derive a modified incremental complementary energy principle for incompressible material, the discussion is focused on the functional given by Eq.(6.15).

Incremental Governing Equations

The incremental governing equations for incompressible materials are the same as those for compressible materials except for the incremental constitutive relation and the additional incompressibility condition. To derive the incremental constitutive relation, the constitutive relation, given by Eq.(6.9), for C_{N+1} state is considered:

$$\underset{\sim}{r}^{N+1} = \frac{\partial W^{N+1}}{\partial \underset{\sim}{h}^{N+1}} + p^{N+1} \frac{\partial h_3^{N+1}}{\partial \underset{\sim}{h}^{N+1}} \tag{6.17}$$

Since the strain $\underset{\sim}{h}_{N+1}$ can be expressed as,

$$\underset{\sim}{h}^{N+1} = \underset{\sim}{h}^{N} + \Delta \underset{\sim}{h} \tag{6.18}$$

Eq.(6.17) is considered to be expressed in terms of $\Delta\underset{\sim}{h}$. Thus, it can be expanded in the Taylor series,

$$\underset{\sim}{r}^{N+1} = \underset{\sim}{r}^{N} + \Delta \underset{\sim}{r} = \left.\frac{\partial W}{\partial \underset{\sim}{h}}\right|^{N} + \left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^{N} : \Delta \underset{\sim}{h} + (p^N + \Delta p)\left\{\left.\frac{\partial h_3}{\partial \underset{\sim}{h}}\right|^{N} + \left.\frac{\partial^2 h_3}{\partial \underset{\sim}{h}^2}\right|^{N} : \Delta \underset{\sim}{h}\right\} + \text{higher order terms} \tag{6.19}$$

Noting that for C_N state,

$$\underset{\sim}{r}^{N} = \left.\frac{\partial W}{\partial \underset{\sim}{h}}\right|^{N} + p^N \left.\frac{\partial h_3}{\partial \underset{\sim}{h}}\right|^{N} \tag{6.20}$$

by ignoring the higher order terms in $\Delta \underset{\sim}{h}$, the incremental constitutive relation is obtained as,

$$\Delta \underset{\sim}{r} = \left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^N : \Delta \underset{\sim}{h} + p^N \left.\frac{\partial^2 h_3}{\partial \underset{\sim}{h}^2}\right|^N : \Delta \underset{\sim}{h} + \Delta p \left.\frac{\partial h_3}{\partial \underset{\sim}{h}}\right|^N \tag{6.21}$$

Similarly the incompressibility condition:

$$h_3(\underset{\sim}{h}^{N+1}) = h_3(\underset{\sim}{h}^N + \Delta \underset{\sim}{h}) = 1 \tag{6.22}$$

is also expanded in Taylor series. And ignoring the higher order terms, the incremental incompressibility condition is obtained as,

$$\left.\frac{\partial h_3}{\partial \underset{\sim}{h}}\right|^N : \Delta \underset{\sim}{h} = 0 \tag{6.23}$$

Incremental Hu-Washizu Principle

Following the general procedure discussed in chapter IV, the linearized incremental functional and the corresponding π^1 which provides the correction procedure are obtained as,

$$\begin{aligned}\pi^2_{HW}(\Delta \underline{u}, \Delta \underset{\sim}{h}, \Delta \underset{\sim}{\alpha}, \Delta \underset{\sim}{t}, \Delta p) = \int_{V_O} \Big\{ & \Delta \hat{W}(\Delta \underset{\sim}{h}) + \Delta p \left.\frac{\partial h_3}{\partial \underset{\sim}{h}}\right|^N : \Delta \underset{\sim}{h} \\ & - \rho_0 \Delta \underline{g} \cdot \Delta \underline{u} + \Delta \underset{\sim}{t}^T : \left[\nabla \Delta \underline{u}^T - \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{\alpha}^N \cdot \Delta \underset{\sim}{h}\right] - \underset{\sim}{t}^{N^T} : \Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N + \Delta \underset{\sim}{h}) \Big\} dv \\ & - \int_{S_{\sigma_O}} \Delta \bar{\underline{t}} \cdot \Delta \underline{u} \, ds - \int_{S_{u_O}} \Delta \underline{t} \cdot (\Delta \underline{u} - \Delta \bar{\underline{u}}) ds \end{aligned} \tag{6.24}$$

where, $\Delta \hat{W}(\Delta \underset{\sim}{h}) = \frac{1}{2} \left(\left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^N + p^N \left.\frac{\partial^2 h_3}{\partial \underset{\sim}{h}^2}\right|^N \right) :: \Delta \underset{\sim}{h} \, \Delta \underset{\sim}{h}$

(6.25)

$$\pi'(\Delta\underline{u}, \Delta\underset{\sim}{h}, \Delta\underset{\sim}{a}, \Delta\underset{\sim}{t}, \Delta p)$$

$$= \int_{V_0} \Big\{ \Big[\frac{\partial \hat{W}}{\partial \underset{\sim}{h}}\Big|^{N} - \frac{1}{2}(\underset{\sim}{t}^{N}\cdot\underset{\sim}{a}^{N} + \underset{\sim}{a}^{N^T}\cdot\underset{\sim}{t}^{N^T})\Big] : \Delta\underset{\sim}{h} - (\underset{\sim}{I}+\underset{\sim}{h}^{N})\cdot\underset{\sim}{t}^{N} : \Delta\underset{\sim}{a}^{T} + \Delta p(h_3^{N}-1)$$

$$- \Delta\underset{\sim}{t}^{T} : \Big[(\underset{\sim}{I}+\nabla\underline{u}^{N})^{T} - \underset{\sim}{a}^{N}\cdot(\underset{\sim}{I}+\underset{\sim}{h}^{N})\Big] + \underset{\sim}{t}^{N^T} : \nabla\Delta\underline{u}^{T} - \rho_0\,\underline{g}^{N}\cdot\Delta\underline{u} \Big\}\, dv$$

$$- \int_{S_{\sigma_0}} \bar{\underline{t}}^{N}\cdot\Delta\underline{u}\, ds - \int_{S_{u_0}} \Big\{ \underline{t}^{N}\cdot\Delta\underline{u} + \Delta\underline{t}\cdot(\underline{u}^{N} - \bar{\underline{u}}^{N}) \Big\} ds$$

The stationarity condition of the functional Eq.(6.24) leads to all the incremental governing equations, i.e., Eqns.(4.8, 11, 14, 38, 39, and 6.21) and, in addition, the incremental incompressibility condition given by Eq.(6.23).

Incremental Complementary Energy Principle

To construct the complementary energy principle, the incremental strain $\Delta\underset{\sim}{h}$ in Eq.(6.24) must be eliminated. Now, we group the terms involving $\Delta\underset{\sim}{h}$ in Eq.(6.24), and designate this group as A.

$$A = \Delta\hat{W}(\Delta\underset{\sim}{h}) - \Big(\Delta\underset{\sim}{t}\cdot\underset{\sim}{a}^{N} + \underset{\sim}{t}^{N}\cdot\Delta\underset{\sim}{a} - \Delta p\,\frac{\partial h_3}{\partial\underset{\sim}{h}}\Big|^{N}\Big) : \Delta\underset{\sim}{h} \qquad (6.26)$$

By introducing the stress increment $\Delta\hat{\underset{\sim}{r}}$ defined by,

$$\Delta\hat{\underset{\sim}{r}} = \frac{1}{2}(\Delta\underset{\sim}{t}\cdot\underset{\sim}{a}^{N} + \underset{\sim}{t}^{N}\cdot\Delta\underset{\sim}{a} + \underset{\sim}{a}^{N^T}\cdot\Delta\underset{\sim}{t}^{T} + \Delta\underset{\sim}{a}^{T}\cdot\underset{\sim}{t}^{N^T}) - \Delta p\,\frac{\partial h_3}{\partial\underset{\sim}{h}}\Big|^{N} \qquad (6.27)$$

Eq.(6.26) can be rewritten as,

$$A = \Delta \hat{W}(\Delta \underset{\sim}{h}) - \Delta \hat{\underset{\sim}{r}}:\Delta \underset{\sim}{h} \tag{6.28}$$

Also, the incremental constitutive relation, Eq.(6.21) is rewritten as,

$$\Delta \hat{\underset{\sim}{r}} = \left(\frac{\partial^2 W}{\partial h^2}\bigg|^N + p^N \frac{\partial^2 h_3}{\partial h^2}\bigg|^N \right):\Delta \underset{\sim}{h} \tag{6.29}$$

Assuming the a priori satisfaction of Eq.(6.29) and taking its inverse, strain $\Delta \underset{\sim}{h}$ can be expressed in terms of $\Delta \hat{\underset{\sim}{r}}$. Thus the following contact transformation is achieved.

$$\Delta \hat{R}(\Delta \hat{\underset{\sim}{r}}) = \Delta \hat{\underset{\sim}{r}}:\Delta \underset{\sim}{h} - \Delta \hat{W}(\Delta \underset{\sim}{h}) \tag{6.30}$$

such that, $\quad \dfrac{\partial \Delta \hat{R}(\Delta \hat{\underset{\sim}{r}})}{\partial \Delta \hat{\underset{\sim}{r}}} = \Delta \underset{\sim}{h}$

By introducing $\Delta \hat{R}$ defined by Eq.(6.30) and assuming a priori satisfaction of the translational equilibrium condition Eq.(4.8) and the traction boundary condition Eq.(4.38), the functional given by Eq.(6.24) is reduced to a incremental complementary energy principle:

$$\pi_c^2(\Delta \underset{\sim}{a}, \Delta \underset{\sim}{t}, \Delta p) = \int_{V_0} \Big\{ \Delta \hat{R}(\Delta \hat{\underset{\sim}{r}}) \tag{6.31}$$

$$+ (\Delta \underset{\sim}{t} + \underset{\sim}{t}^N)^T : \Delta \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) \Big\} dv - \int_{s_{u_0}} \Delta \underline{t} \cdot \Delta \underline{\bar{u}} \, ds$$

Also, the corresponding π^1 is given by,

$$\pi^1(\Delta\underset{\sim}{a}, \Delta\underset{\sim}{t}, \Delta p) = \int_{v_o}\Big\{ \Delta\underset{\sim}{t}^T : \big[\underset{\sim}{a}^N \cdot (\underset{\sim}{I}+\underset{\sim}{h}^N) - \underset{\sim}{I}\big] - (h_3^N - 1)\Delta p + (\underset{\sim}{I}+\underset{\sim}{h}^N)\cdot \underset{\sim}{t}^N : \Delta\underset{\sim}{a}^T\Big\}dv - \int_{su_o}\Delta\underline{t}\cdot\underline{\bar{u}}^N ds \tag{6.32}$$

<u>Modified Incremental Complementary Energy Principle</u>

Further, by relaxing the inter-element boundary continuity conditions, given by Eqns.(4.68 and 69), in the same manner as discussed in chapter IV, two versions of modified incremental complementary energy principles are derived. For example, the second version is shown to be,

$$\pi^2_{CM2}(\Delta\underset{\sim}{a}, \Delta\underset{\sim}{t}, \Delta p, \Delta\underline{\tilde{u}}_\rho) = \sum_m \int_{v_{o_m}}\Big\{\Delta\hat{R}(\Delta\hat{r}) + (\Delta\underset{\sim}{t} + \underset{\sim}{t}^N)^T : \Delta\underset{\sim}{a}\cdot(\underset{\sim}{I}+\underset{\sim}{h}^N)\Big\}dv - \sum_m \int_{su_{o_m}}\Delta\underline{t}\cdot\Delta\underline{\bar{u}}\, ds - \sum_m \int_{\rho_{o_m}}\Delta\underline{t}\cdot\Delta\underline{\tilde{u}}_\rho ds \tag{6.33}$$

And respective π^1 is given by,

$$\pi^1(\Delta\underset{\sim}{a}, \Delta\underset{\sim}{t}, \Delta p, \Delta\underline{\tilde{u}}_\rho) = \sum_m \int_{v_{o_m}}\Big\{ \Delta\underset{\sim}{t}^T : \big[\underset{\sim}{a}^N \cdot (\underset{\sim}{I}+\underset{\sim}{h}^N) - \underset{\sim}{I}\big] - (h_3^N - 1)\Delta p + (\underset{\sim}{I}+\underset{\sim}{h}^N)\cdot \underset{\sim}{t}^N : \Delta\underset{\sim}{a}^T\Big\}dv - \sum_m \int_{su_{o_m}}\Delta\underline{t}\cdot\underline{\bar{u}}^N ds \tag{6.34}$$

$$- \sum_m \int_{\rho_{o_m}} (\underline{t}^N \cdot \Delta \tilde{\underline{u}} + \Delta \underline{t} \cdot \tilde{\underline{u}}^N) ds$$

The stationarity condition of the functional in Eq.(6.33) yields to Eqns.(4.11, 14, 39, 68, 69, and 6.23).

Finite Element Formulation

The finite element formulation for the general three-dimensional case is discussed in this section, in the same manner as in chapter V. With a slight modification, Eq.(6.33) leads to the functional analogous to that given by Eq.(5.2), which has the suitable form for the finite element formulation.

$$\pi^2_{HS2}(\Delta \underset{\sim}{a}, \ \Delta \underset{\sim}{t}, \ \Delta p, \Delta \tilde{\underline{u}}_\rho) \tag{6.35}$$

$$= \sum_m \int_{v_{o_m}} \left\{ \Delta \hat{R}(\Delta \hat{\underset{\sim}{r}}) + (\underset{\sim}{t}^N + \Delta \underset{\sim}{t})^T : \Delta \underset{\sim}{a} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) \right\} dv$$

$$+ \sum_m \int_{s_{\sigma o_m}} \Delta \bar{\underline{t}} \cdot \Delta \tilde{\underline{u}}_\rho ds - \sum_m \int_{\partial v_{o_m}} \Delta \underline{t} \cdot \Delta \tilde{\underline{u}}_\rho ds$$

and the corresponding π^1,

$$\pi^1(\Delta \underset{\sim}{a}, \Delta \underset{\sim}{t}, \Delta p, \Delta \tilde{\underline{u}}_\rho) = \sum_m \int_{v_{o_m}} \left\{ \Delta \underset{\sim}{t}^T : \left[\underset{\sim}{a}^N \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{I} \right] \right. \tag{6.36}$$

$$\left. - (h_3^N - 1) \Delta p + (\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot \underset{\sim}{t}^N : \Delta \underset{\sim}{a}^T \right\} dv$$

$$+ \sum_m \int_{s_{\sigma o_m}} \bar{\underline{t}}^N \cdot \Delta \tilde{\underline{u}}_\rho ds - \sum_m \int_{\partial v_{o_m}} (\underline{t}^N \cdot \Delta \tilde{\underline{u}}_\rho + \Delta \underline{t} \cdot \tilde{\underline{u}}_\rho^N) ds$$

We take the first variation of Eq.(6.35) to show its stationarity conditions

$$\delta \pi^2_{HS2} = \sum_m \int_{V_{o_m}} \Big\{ \Big[\Delta \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + \underset{\sim}{\alpha}^N \cdot \frac{\partial \Delta \hat{R}}{\partial \Delta \hat{\underset{\sim}{r}}} - (\nabla \Delta \underline{u})^T \Big] : \delta \Delta \underset{\sim}{t}^T \tag{6.37}$$

$$- \Big[(\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot (\underset{\sim}{t}^N + \Delta \underset{\sim}{t}) + \frac{\partial \Delta \hat{R}}{\partial \Delta \hat{\underset{\sim}{r}}} \cdot \underset{\sim}{t}^N \Big] : \delta \Delta \underset{\sim}{\alpha}^T + \frac{\partial h_3}{\partial \underset{\sim}{h}} \Big|^N : \frac{\partial \Delta \hat{R}}{\partial \Delta \hat{\underset{\sim}{r}}} \, \delta \Delta p \Big\} dv$$

$$- \sum_m \int_{S_{\sigma_{o_m}}} (\Delta \underline{t} - \Delta \bar{\underline{t}}) \cdot \delta \Delta \tilde{\underline{u}}_\rho \, ds$$

$$- \sum_m \int_{\partial V_{o_m}} \Big\{ \delta \Delta \underline{t} \cdot (\Delta \tilde{\underline{u}}_\rho - \Delta \underline{u}) + \Delta \underline{t} \cdot \delta \Delta \tilde{\underline{u}}_\rho \Big\} ds$$

Thus, it is shown that the stationarity condition of the functional leads to Eqns.(4.11, 14, 38, 68, 69, and 6.23) as a posteriori conditions.

If the functional, Eq.(6.35), is compared with the corresponding functional for compressible materials Eq.(5.2), it is noticed that only one term ΔR in Eq.(5.2) is replaced by $\Delta \hat{R}$ in Eq.(6.35). Similarly, π^1 given by Eq.(6.36) has only one additional term, $(h_3 - 1)\Delta p$, compared to Eq.(5.3). Therefore, most of the equations derived in chapter V can be used for the present case, and the detailed formulations are limited to these two terms in the following discussins.

There are four field variables involved in the functional in Eq.(6.35). The stress, rotation, and the element boundary displacement field are assumed in the same way as for compressible materials. Thus these variables are assumed as in Eqns.(5.12, 18, and 21). The new variable, hydrostatic pressure p, is assumed as a linear combination of linearly independent functions M_i with undetermined pressure parameters p_i as coefficients, such that,

$$p = M_i \; P_i \; = \lfloor M_i \rfloor \{p_i\} \tag{6.38}$$

Similarly, its increment is assumed as,

$$\Delta p = \lfloor M_i \rfloor \{\Delta p_i\} \tag{6.39}$$

The incremental stress $\Delta \hat{\underset{\sim}{r}}$, defined by,

$$\Delta \hat{\underset{\sim}{r}} = \frac{1}{2} \left(\Delta \underset{\sim}{t} \cdot \underset{\sim}{a}^N + \underset{\sim}{t}^N \cdot \Delta \underset{\sim}{a} + \underset{\sim}{a}^{N^T} \cdot \Delta \underset{\sim}{t}^T + \Delta \underset{\sim}{a}^T \cdot \underset{\sim}{t}^{N^T} \right) - \left. \frac{\partial h_3}{\partial \underset{\sim}{h}} \right|^N \Delta P \tag{6.40}$$

can be expressed in terms of the undetermined parameters $\Delta\beta$, $\Delta\mu$, and Δp, using Eqns.(5.12, 18, and 6.39),

$$\{\Delta \hat{r}\} = [\hat{D}_1]\{\Delta\beta\} + [\hat{D}_2]\{\Delta\mu\} + [\hat{D}_3]\{\Delta p\} + \{\Delta \hat{r}^P\} \tag{6.41}$$

The incremental complementary energy density $\Delta\hat{R}$, defined by Eq.(6.30), is expressed in the matrix notation as,

$$\Delta \hat{R} = \frac{1}{2} \{\Delta \hat{\underset{\sim}{r}}\}^T [\, \hat{C} \,] \{\Delta \hat{\underset{\sim}{r}}\} \tag{6.42}$$

Its integration over the element after substituting Eq.(6.41) yields to,

$$\int_{v_{o_m}} \Delta \hat{R} \, dv = \frac{1}{2} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta P \end{Bmatrix}^T \begin{bmatrix} \hat{H}_{11}, & \hat{H}_{12}, & \hat{H}_{13} \\ \hat{H}_{21}, & \hat{H}_{22}, & \hat{H}_{23} \\ \hat{H}_{31}, & \hat{H}_{32}, & \hat{H}_{33} \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta P \end{Bmatrix} + \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta p \end{Bmatrix}^T \begin{Bmatrix} \Delta\hat{Q}_1 \\ \Delta\hat{Q}_2 \\ \Delta\hat{Q}_6 \end{Bmatrix} \tag{6.43}$$

where,

(6.44)

$$[\hat{H}_{11}] = \int_{v_{o_m}} [\hat{D}_1]^T [\hat{C}][\hat{D}_1] \, dv$$

$$[\hat{H}_{12}] = [\hat{H}_{21}]^T = \int_{v_{o_m}} [\hat{D}_1]^T [\hat{C}][\hat{D}_2] \, dv$$

$$[\hat{H}_{13}] = [\hat{H}_{31}]^T = \int_{v_{o_m}} [\hat{D}_1]^T [\hat{C}][\hat{D}_3] \, dv$$

$$[\hat{H}_{22}] = \int_{v_{o_m}} [\hat{D}_2]^T [\hat{C}][\hat{D}_2] \, dv$$

$$[\hat{H}_{23}] = [\hat{H}_{32}] = \int_{v_{o_m}} [\hat{D}_2]^T [\hat{C}][\hat{D}_3] \, dv$$

$$[\hat{H}_{33}] = \int_{v_{o_m}} [\hat{D}_3]^T [\hat{C}][\hat{D}_3] \, dv$$

$$\lfloor \Delta\hat{Q}_1 \rfloor = \int_{v_{o_m}} [\hat{D}_1]^T [\hat{C}] \{\Delta r^P\} \, dv$$

$$\lfloor \Delta\hat{Q}_2 \rfloor = \int_{v_{o_m}} [\hat{D}_2]^T [\hat{C}] \{\Delta r^P\} \, dv$$

$$\lfloor \Delta\hat{Q}_6 \rfloor = \int_{v_{o_m}} [\hat{D}_3]^T [\hat{C}] \{\Delta r^P\} \, dv$$

Simply by replacing the terms corresponding to $\int_{V_{o_m}} \Delta R \, dV$ in Eq.(5.31) by Eq.(6.43), the discretized form of the functional for incompressible material, Eq.(6.35), is obtained.

$$\pi^2_{HS2}(\Delta\beta, \Delta\mu, \Delta P, \Delta q) \tag{6.45}$$

$$= \frac{1}{2} \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta P \end{Bmatrix}^T [\hat{H}] \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta p \end{Bmatrix} - \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta p \end{Bmatrix}^T \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix} \{\Delta q\}$$

$$+ \sum_m \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta p \end{Bmatrix}^T \begin{Bmatrix} \Delta\hat{Q}_1 \\ \Delta\hat{Q}_2 + \Delta\hat{Q}_3 \\ \Delta\hat{Q}_6 \end{Bmatrix} + \sum_m \left\{ -\Delta Q_4 + \Delta Q_5 \right\}^T \left\{ \Delta q \right\}$$

where G, ΔQ_3, ΔQ_4, and ΔQ_5 are defined by Eqns.(5.26, 28, 29, and 30); and $\hat{H}$ is defined by,

$$\left[\hat{H} \right] = \begin{bmatrix} \hat{H}_{11} & , \hat{H}_{12} + P & , \hat{H}_{13} \\ (\hat{H}_{12} + P)^T, & \hat{H}_{22} + H^* + S, & \hat{H}_{23} \\ \hat{H}_{31} & , \hat{H}_{32} & , \hat{H}_{33} \end{bmatrix} \tag{6.46}$$

where $[P]$, $[S]$, and $[H^*]$ are implicitly defined by Eq.(5.26 and 28).

It is noted here that, unlike for the compressible material, the matrix $[\hat{H}]$ cannot be inverted as a whole for general three-dimensional case. This implies that although the parameters for stress, rotation , and hydrostatic pressure are independently assumed in each element, they can not be eliminated at the same time. Thus, the hydrostatic pressure, through which the incompressibility condition is inposed, must be kept as unknowns, as discussed by Key [20] and Herrmann [19] for linear case.

However, in the case of plane-stress problem, the material area on the inplane surface, (x_1, x_2) plane, can change whithout changing the volume, but it is not the case for plane-strain case. This implies that in the plane-stress case, the stiffness (or resistance) against the change of the area is finite, whereas, in the plane-strain case, it becomes infinite. Thus, the matrix $[\hat{H}]$ can be inverted in the plane-stress case, and we can obtain stiffness matrix $[K_m]$ with finite values.

In the same manner as in the derivation of Eq.(5.41), the discretized form of π^1, given by Eq.(6.36), is obtained by adding the contribution from the term $(h_3^{N+1} - 1)\Delta p$ to Eq.(5.41). Thus, π^1 is obtained as,

$$\pi^1(\Delta\beta, \Delta\mu, \Delta p, \Delta q) = \begin{Bmatrix} \Delta\beta \\ \Delta\mu \\ \Delta p \\ \Delta q \end{Bmatrix}^T \begin{Bmatrix} Q_k \\ Q_r \\ Q_p \\ Q_t \end{Bmatrix} \tag{6.47}$$

where $\{Q_k\}$, $\{Q_r\}$, and $\{Q_t\}$ are defined by Eqns.(5.42, 43, and 44); and $\{Q_p\}$ is defined by,

$$\lfloor \Delta p \rfloor \{Q_p\} = -\int_{Vo_m} (h_3^{N+1} - 1)\,\Delta p \; dv \tag{6.48}$$

Plane Stress Problem

Now, we consider the plane-stress problem as a special case. Following the same procedure as shown in chapter V, incremental hybrid stress finite element models using four-noded and also eight-noded elements are derived based on the discretized functional given by Eq.(6.45).

In the present plane-stress problem, the stress and the rotation fields are subjected to the same constraint conditions as for compressible materials, viz.,

$$t_{13} = t_{31} = t_{23} = t_{32} = t_{33} = 0 \tag{6.49}$$

$$\alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = 0, \qquad \alpha_{33} = 1 \tag{6.50}$$

Then the Jaumann stress defined by Eq.(2.15) is reduced to,

$$\left[\underset{\sim}{r} \right] = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.51}$$

However, the stress increment $\Delta \underset{\sim}{\hat{r}}$ defined by Eq.(6.27) has nonzero component $\Delta \hat{r}_{33}$.

$$\left[\Delta \underset{\sim}{\hat{r}} \right] = \begin{bmatrix} \Delta r_{11} - \Delta p \left.\dfrac{\partial h_3}{\partial h_{11}}\right|^N, & \Delta r_{12} , & 0 \\ \Delta r_{21} , & \Delta r_{22} - \Delta p \left.\dfrac{\partial h_3}{\partial h_{22}}\right|^N, & 0 \\ 0 , & 0 , & -\Delta p \left.\dfrac{\partial h_3}{\partial h_{33}}\right|^N \end{bmatrix} \tag{6.52}$$

The incremental stress and rotation are assumed by the same functions given by Eqns.(5.52 and 55). The additional hydrostatic pressure is also assumed in terms of complete polynomials, as,

$$\Delta p = \Delta p_1 + x_1 \Delta p_2 + x_2 \Delta p_3 + x_1^2 \Delta p_4 + x_1 x_2 \Delta p_5 + \cdots\cdots \tag{6.53}$$

The incremental element boundary displacements $\Delta \tilde{u}_{\rho_1}$ and $\Delta \tilde{u}_{\rho_2}$

for four-noded element are assumed by Eq.(5.61). Similarly, those for eight-noded element are assumed by,

$$\begin{aligned}\Delta \tilde{u}_{\rho_i} = & \frac{1}{4}(1-r)(1-s)(-r-s-1)\Delta q_i^1 + \frac{1}{2}(1-r^2)(1-s)\Delta q_i^2 \\ & + \frac{1}{4}(1+r)(1-s)(r-s-1)\Delta q_i^3 + \frac{1}{2}(1+r)(1-s^2)\Delta q_i^4 \\ & + \frac{1}{4}(1+r)(1+s)(r+s-1)\Delta q_i^5 + \frac{1}{2}(1-r^2)(1+s)\Delta q_i^6 \\ & + \frac{1}{4}(1-r)(1-s)(-r+s-1)\Delta q_i^7 + \frac{1}{2}(1-r)(1-s^2)\Delta q_i^8\end{aligned} \tag{6.54}$$

where Δq_i^j are the i th (i=1, 2) component of the nodal displacement at j th (j=1, ...8) node.

The material considered here is a Mooney-Rivlin type incompressible material, whose mechanical properties are characterized by W.

$$W = C_1(I_1-3) + C_2(I_2-3) \tag{6.55}$$

where C_1 and C_2 are material constants; and I_1 and I_2 are the first and the second invariants of deformation tensor $\underset{\sim}{G}$, respectively. Then the incremental strain energy density is defined by,

$$\Delta \hat{W} = \frac{1}{2}\left\{ \left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^N + p^N \left.\frac{\partial^2 h_3}{\partial \underset{\sim}{h}^2}\right|^N \right\} :: \Delta \underset{\sim}{h}\, \Delta \underset{\sim}{h} \tag{6.56}$$

such that,

$$\frac{\partial \Delta \hat{W}}{\partial \underset{\sim}{h}} = \Delta \hat{\underset{\sim}{r}} \tag{6.57}$$

The first term in Eq.(6.56) can be obtained in tne same way as for ΔW given by Eq.(5.69). Thus, we obtain,

$$\frac{1}{2}\left.\frac{\partial^2 W}{\partial \underset{\sim}{h}^2}\right|^N :: \Delta \underset{\sim}{h}\, \Delta \underset{\sim}{h} = C_1\left\{(\Delta h_1)^2 - 2\, \Delta^2 h_2\right\} + C_2\left\{(\Delta h_2)^2 + 2h_2\, \Delta^2 h_2 - 2(h_1\, \Delta^2 h_3 + \Delta h_1 \Delta h_3)\right\} \tag{6.58}$$

where h_1 , h_2 , h_3 , Δh_1 , Δh_2 , Δh_3 , $\Delta^2 h_2$, and $\Delta^2 h_3$ are defined by Eq.(5.70). The second term in Eq.(6.56), which is independent of the material, is obtained for the plane-stress case as,

$$\frac{1}{2}\, p^N \left.\frac{\partial^2 h_3}{\partial \underset{\sim}{h}^2}\right|^N :: \Delta \underset{\sim}{h}\, \Delta \underset{\sim}{h} = \Delta h_{11} \Delta h_{22}\, h_{33}^N + \Delta h_{22} \Delta h_{33} h_{11}^N + \Delta h_{33} \Delta h_{11} h_{22}^N - \Delta h_{12} \Delta h_{21} h_{33}^N - \Delta h_{21} \Delta h_{33} h_{12}^N - \Delta h_{33} \Delta h_{12} h_{21}^N \tag{6.59}$$

Then, $\Delta \hat{W}$ in Eq.(6.56) is expressed in the matrix notation by,

$$\Delta \hat{W} = \frac{1}{2}\{\Delta h\}^T [\, \hat{E}\,]\{\Delta h\} \tag{6.60}$$

where

$$\{\Delta h\}^T = \lfloor\, \Delta h_{11}, \Delta h_{12}, \Delta h_{21}, \Delta h_{22}, \Delta h_{33} \rfloor$$

Following the same procedure as in obtaining ΔR in Eq.(5.75), the the contact transformation of $\Delta \hat{W}$ in terms of $\Delta \underset{\sim}{\hat{r}}$ is achieved.

$$\Delta \hat{R} = \frac{1}{2}\{\Delta \underset{\sim}{\hat{r}}\}^T [\, \hat{C}\,]\{\Delta \hat{r}\} \tag{6.61}$$

where,

$$\left\{\Delta\hat{r}\right\}^T = \left\lfloor \Delta\hat{r}_{11}, \Delta\hat{r}_{12}, \Delta\hat{r}_{21}, \Delta\hat{r}_{22}, \Delta\hat{r}_{33} \right\rfloor$$

It is noted here that, from the definition of $\Delta\hat{\underset{\sim}{r}}$ given by Eq.(6.27), $\Delta\hat{r}_{33}$ is not zero, and it has contributions to $\Delta\hat{R}$.

Noting the constraint condition for the plane-stress problem, by substituting the assumed functions given by Eqns.(5.52, 55, and 6.53) and (5.61 or 6.54) and $\Delta\hat{R}$ given by Eq.(6.61) into Eq.(6.35), we obtain a discretized functional in terms of $\Delta\beta$, $\Delta\mu$, Δp, and Δq, wich is the reduced form of Eq.(6.45) for plane-stress problems. Since the matrix $[\hat{H}]$ in Eq.(6.45) is invertible for plane-stress case, the parameters for stress, rotation, and hydrostatic pressure are all eliminated at the element level. Finally, we obtain the discretized functional analogous to Eq.(5.35), which involves the element boundary displacement alone. Based on this functional, incremental hybrid stress finite element models using four-noded and eight-noded isoparametric elements are derived. Further, by introducing the iterative procedure based on π', shown by Eq.(6.36), complete numerical scheme to solve plane-stress problem of incompressible solids is developed.

Before applying the newly developed finite element model to specific problems, eigen-values and eigen-vectors of an element stiffness matrix are calculated for various combinations of numbers of stress parameters "a", rotation parameters "b", hydrostatic pressure parameters "d", and element boundary displacement parameters "c". The element considered for this purpose is a square element, and the elastic constants C_1 and C_2 in Eq.(6.55) are chosen to be, C_1=24.0 psi and C_2=1.5 psi. The results for four-noded element (c=8) and eight-noded element (c=16) are presented in Table 2 and Table 3, respectively. The eigen-modes for the case (a, b, d, c)= (10, 3, 6, 8) and (28, 3, 10, 16) are shown in Figs.20 and 21.

It is interesting to notice that, if the number of hydrostatic pressure parameter is taken small, such as in the case of (a, b, d, c)= (10, 1, 1, 8), (10, 3, 1, 8), or (18, 3, 1, 8), the lowest eigen-values become unusually small. Even, they becomenegative in the case of (a, b, d, c)= (18, 1, 1, 16), (18, 1, 3, 16), or (28, 3, 3, 16). These results imply that the number of hydrostatic pressure parameters must be taken sufficiently large, so that the incompressibility condition is imposed on the element in a strong manner, otherwise the element does not behave properly. Moreover, it is noticed that these eigen-values calculated for initial state cannot tell the behavior of the element after large deformation. In general, the nonlinearity of the incompressibility condition becomes much stronger under large deformation. Therefore, to ensure such highly nonlinear condition, at least one order higher hydrostatic pressure field than that required from Tables 2 and 3 is recommended from the weighted residual point of view. Also it is observed from Table 3 that a smaller number of stress parameters "a", such as (18, 3, 6, 16), results in a stiffness matrix which has more than three kinematic modes (zero eigen-values), whereas a properly behaving stiffness matrix has three kinematic modes corresponding to rigid body motions.

If the four-noded element and the eight-noded element are compared, the lowest possible total degree of freedom of eight-noded element is 51, whereas, that for four-noded element is 20. Therefore, four-noded element appears to be more convenient for practical applications.

Numerical Examples

For comparison, two types of plane-stress problems, same as those solved by Oden [18] using finite element model based on the stationary potential energy principle, are chosen as example problems.

Prescribed Stretching of an Elastic Sheet

The first example problem considered is the problem of prescribed stretching of a thin elastic sheet (8"X 8"X 0.05") to twice its original length. Thus the boundary conditions imposed on the sheet are described by,

$$t_1 = t_2 = 0 \qquad \text{on} \quad x_2 = \pm 4.0 \qquad (6.62)$$

$$u_1 = \pm 4(\lambda - 1), \quad u_2 = 0 \qquad \text{on} \quad x_1 = \pm 4.0$$

where λ is a extension ratio ($1 < \lambda < 2$). As noted in [18], this problem corresponds to the biaxial strip test used to characterize the ultimate properties of synthetic rubber, etc., and no exact solution is available for this problem. The material of the sheet is assumed to be a Mooney-Rivlin material of the type given by Eq.(6.55). The material constants C_1 and C_2 in Eq.(6.55) are $C_1=24.0$ psi and $C_2=1.5$ psi. This problem is solved by using the proposed four-noded element with the combination of numbers of parameters (a, b, d, c)=(10, 3, 6, 8). From the symmetry of the problem, a quarter of the sheet is simulated by a 6X6 finite element mesh shown by the inset in Fig.22. The prescribed displacements at $x_1=\pm4.0$ are applied in 20 steps with the increment $\Delta\lambda=0.05$. At each increment, iterations are carried out so that the error, defined by Eq.(5.80), is kept less than 1%.

The net horizontal boundary force F, required to produce various ratios of stretch, $1<\lambda<2$, are plotted in Fig.22. However, because of the solution method, only the value of F at $\lambda=2.0$ for various finite element meshes, are given in [18], where it is found that F is approximately 36.0 lb. The present result for F at $\lambda=2.0$, F=36.4 lb, is in close agreement with [18]. The computer-plotted deformed profiles of a quarter of a sheet at various values of λ are presented in Fig.23. The contours of the components of the true or Cauchy stress τ_{11}, τ_{12}, and τ_{22} at $\lambda=2.0$ are hand-plotted on the deformed configuration in Figs.24, 25, and 26. Also, the distribution of the axial components of Piola-Lagrange and Kirchhoff-Trefftz stress, t_{11} and s_{11}, respectively, at $\lambda=2.0$ are presented in Figs.27 and 28. As discussed for the example problem of compressible material, similarity between the distributions of τ_{11} and t_{11} is observed. The contour lines of the rotation angle θ and the extension ratio in the thickness direction, $(h_{33}+1)$, at $\lambda=2.0$ are given in Figs.29 and 30. The distributions of τ_{11} and τ_{22} at $\lambda=1.5$ are shown in Figs.31 and 32, respectively. These results of the distributions of stress or strain are not given in [18], hence, no further comparison is attempted. However, Oden [18] gives the results for τ_{11} and τ_{22} at $\lambda=1.5$, obtained by Becker [35] for same type of Mooney-Rivlin material, but with different material constants, $C_1=8$ psi and $C_2=1.0$ psi. Although the materials are different, there is an excellent qualitative agreement between the present results and those of [35]. As noted in [18], the present results for stress distribution, as well as those in [18], differ significantly from those predicted by the infinitesimal theory of incompressible solids [36]. Further, it is noticed that Becker's results are obtained by using 400 four-noded elements, whereas the present results are obtained by using 37 four-noded elements. This may perhaps confirm the commonly held notion that an accurate stress distribution can be obtained more efficiently using a stress finite element model based on complementary energy principle as in the present work.

Uniaxial Stretching of a Sheet with a Circular Hole

The second example problem is that of the uniaxial stretching of a square sheet (6.5"X 6.5"X 0.079") with a circular hole of 0.5" diameter. The boundary conditions imposed on the sheet are,

$$u_2 = 0 \ , \quad t_1 = 0 \qquad \text{on} \quad x_2 = \pm 3.25 \tag{6.63}$$
$$u_1 = 3.25(\lambda - 1) \ , \quad u_2 = 0 \qquad \text{on} \quad x_1 = \pm 3.25$$

where $(1 < \lambda < 3)$ is the axial extension ratio. The material is assumed to be a Mooney-Rivlin material of the type given by Eq.(6.55) with material constants, $C_1 = 27.02$ psi and $C_2 = 1.42$ psi. Oden [18] solved this problem incrementally by using three-noded triangular displacement finite element. The number of elements used in his analysis is 192 per quarter sheet. The same problem is solved by using the presently developed four-noded hybrid stress model finite element, with the combination of numbers of parameters, (a, b, d, c)=(10, 3, 6, 8). The finite element mesh used is a 6X6 mesh as shown by the inset in Fig.33. The prescribed displacements are applied in 80 steps with the increment $\Delta\lambda = 0.025$. Because of the stress concentration around the hole, the increment is taken smaller than in the first problem.

The present result for the required total edge force as a function of λ is shown in Fig.33 along with comparison result of [18]. The two setsof results shown in Fig.33 are seen to correlate well. Eventhough the rigorous mathematical discussion of the convergence of the numerical solution based on variational principle in the finite deformation problem is beyond the scope of the present work, it may be surmised from Fig.33 that the exact solution may exist in the neighbourhood of these two approximate solutions. The deformed configurations at various extension ratios are presented in Fig.34. Also the deformed profiles of the circular hole at various values of λ are shown in Fig.35 along with a comparison result available from [18], which, however, gives the deformed profile for the edge load of 64 lb, corresponding to $\lambda=2.175$. Once again, the correlation is found to be excellent. The contours of the components of true stress τ_{11}, τ_{12}, and τ_{22} at $\lambda=3.0$ are shown in Figs.36, 37, and 38. As in the infinitesimal deformation theory, the maximum axial stress $\tau_{11\,MAX}$ is observed at the minor axis location of the hole ($x_1=0$, $x_2=0.25$). The stress concentration factor, defined as the ratio of the maximum stress $\tau_{11\,MAX}$ to the average Cauchy stress $\overline{\tau}_{11}$ at the edge of the sheet ($x_1=+3.25$) is shown in Fig.39 as a function of λ. According to the infinitesimal theory, the stress concentration factor for the same problem, but for an infinite plate is 2.5 [37]. Although the finite element mesh is relatively coarse (the smallest element size is 2/5 of the radius of the hole), the calculated stress concentration factor in the range of small deformation is very close to that predicted by infinitesimal theory. Further, it is interesting to notice that, unlike in the infinitesimal theory, the stress concentration factor increases with the stretching. The distribution of the rotation angle θ and the extension ratio in the thickness direction is presented in Figs.40 and 41. The rotation field shown by Fig.40 is consistent with the deformation pattren shown by Fig.34. For comparison, cotours of t_{11} and s_{11} at λ=3.0 are plotted on the undeformed configurations in Figs.42 and 43.

The results of the two examples discussed in the above would appear to indicate that the proposed incremental hybrid stress finite element model based on the complementary energy principle is a viable numerical tool for the analysis of the finite deformation problems of incompressible solids. Also, it is seen that the accurate solution for stresses can be obtained relatively efficiently by the present method compared with those based on the potential energy principle.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In the present Dissertation, various types of modified (hybrid type) incremental (rate) variational principles, governing the finite deformation (large strain and rotation) problems, based on alternate stress and its conjugate strain measures and in both total Lagrangean and updated Lagrangean formulations are presented. Especially, modified incremental complementary energy principles, involving as variables the incremental Piola-Lagrange stress and the rotation tensors, which are considered to be most rational and suitable for applications through finite element methods, are proposed. Based on these variational principles, an incremental hybrid stress finite element model, in the total Lagrangean formulation, is derived. The above developments are extended to the problem of finite deformation of incompressible elastic solids, and a hybrid type incremental complementary energy principle, in which the incompressibility condition is relaxed a priori through the introduction of the hydrostatic pressure as a Lagrange multiplier, is derived. This type of variational principle is also applied to the finite element method, and the incremental hybrid stress model is derived.

The above hybrid stress finite element models are used to solve finite strain plane-stress problems of compressible as well as incompressible nonlinear elastic solids. Several examples of such problems are solved. The validity and the feasibility of the proposed methods are demonstrated through the numerical examples.

The conclusions of the present work are enumerated as follows.

A. Variational Principles and Finite Element Formulations

1. A complementary energy principle can be formally derived using the Piola-Lagrange stress alone. However, due to the multi-valued inverse stress-strain relation and the ambiguity on the satisfaction of the rotational equilibrium condition, it can not be applied in the solution of practical problems, in general.

2. Because of the fact that the translational equilibrium condition and the traction boundary condition are nonlinear equations in terms of Kirchhoff-Trefftz stress and displacement, the exact satisfaction of which is impossible, a complementary energy principle based on the Kirchhoff-Trefftz stress does not lead to a successful finite element model.

3. The difficulties pointed out in (1) and (2) remain even in the incremental formulations.

4. The complementary energy principle based on the Jaumann stress, which involves Piola-Lagrange stress and rotation, is considered to be the most rational and suitable for application to the finite element method. In such a complementary energy principle, the inverse stress-strain relation is uniquely defined, and the translational equilibrium condition and the traction boundary condition are linear in terms of the Piola-Lagrange stress. Moreover, the rotational equilibrium condition is retained unambiguously as an a posteriori condition.

5. The proposed incremental hybrid stress finite element models are essentially based on the complementary energy principle described in (4). Thus, they are considered to be the most consistent assumed stress finite element models, for the analysis of problems involving geometrical as well as material nonlinearities, developed to date.

6. The hybrid formulation of the present model allows for the a priori relaxation of the continuity conditions at inter-element boundaries. Thus, the wide choice of the assumed functions for stress and/or displacement is preserved.

7. The incremental formulation leads to linear algebraic equations, which are much easier to solve compared to nonlinear equations. In addition, in the present method,

iterative corrections are embedded in the solution scheme. Thus, the piecewise linear solution can be kept from straying away from the correct solution path.

8. If the material is incompressible, the hydrostatic pressure is introduced as a Lagrange multiplier, and the modified incremental complementary energy principle is derived. This variational principle leads to an incremental hybrid stress finite element model which also has the same features as stated in (6) anfd (7).

B. Numerical Examples

1. From the study of the eigen-values of the element stiffness matrix , it is observed that if the number of the stress parameters is not sufficiently large compared to that of the boundary displacement, it results in an improper stiffness matrix which has more than three zero eigen-values (kinematic modes). In the incompressible case, a small number of the hydrostatic pressure parameters results in an unusually small or even negative eigenvalues, which are also physically improper.

2. The numerical results of the example problems for both compressible and incompressible elastic solids by the present methods are qualitatively consistent from both the physical and mathematical points of view.

3. The results for the finite deformation problem of incompressible elastic sheets obtained by the present method agree excellently with those obtained by a compatible displacement model (Oden [18]); but the number of degrees of freedom used presently is substantially smaller than that in [18].

4. Through example problems, the validity of the proposed method is established.

5. As demonstrated by the numerical result for the stress concentration factor in a sheet with a circular hole, accurate solution for stress can be obtained by the present stress model more efficiently compared to a displacement model.

Recommendations

In the present Dissertation, only compressible or incompressible nonlinear elastic materials are considered. However, metals, such as mild steel, are also capable of large scale, but plastic, deformations. This property is used to form metals. In metal forming processes, such as metal extrusion, plastic strains of order unity occur. Depending on the manufacturing condition, metal forming processes cause the internal or surface cracks or undesirable residual stresses. In order to assess the onset of these material forming defects, it is necessary to develop a method of analysis which is able to solve large strain elastic-plastic deformation problems.

The theoretical basis for large strain elastic-plastic deformation problem is found in work of Hill [38]. Hill has discussed the general framework for the classical rate-constitutive relations for elastic-plastic solid with smooth yield surfaces at finite strain. A special form of rate-constitutive relation using the corotational rate of Kirchhoff stress is proposed by Budiansky [39] as a generalization of the J_2 flow theory in the small deformation problem. Based on this rate constitutive relation, several finite element models have been developed. In general, these models are categorized in two types. One is the total Lagrangean incremental formulation and the other is the updated Lagrangean incremental formulation. The former approach is adopted by Hutchinson [40], Needlman [41], and Tvergaard [42]. The latter approach is taken by McMeeking and Rice [43], Lee, Mallett and Yang [44], and Yamada [45]. All these methods are based on the virtual work theorem.

Although, in most of these finite element models, the rate-constitutive relation in terms of the corotational rate of Kirchhoff stress is used, we may introduce alternate stress rates in the analogous manner as discussed in chapter II. Such attempt is made by Yamada [45]. However, his models are based on the virtual work theorem. Thus, there is no significant difference between the use of stress rates of different deffinitions. If we, further, consider

Hellinger-Reissner or complementary energy type variational principles we may derive various finite element models based on alternate stress rates.

As discussed in the present thesis, the stress finite element model, in which stress is directly taken as an independent variable, is more efficient to obtain an accurate solution for stress compared to displacement models, in which stress is indirectly obtained by taking the derivatives of displacements. Therefore, the present incremental hybrid stress finite element model may be extended, and a numerical solution technique for the analysis of large strain elastic-plastic deformation problems of solids can be developed.

APPENDIX A

DIRECT TENSOR NOTATIONS

In the study of solid mechanics, index notation, in which tensor equations are described in terms of components referred to some co-ordinate system, is commonly used. However, depending on the choice of the reference co-ordinate system, equations describing the same physical phenomenon change their forms. This nature of the index notation is at times inconvenient when we are trying to describe the physical phenomenon in the general mathematical form. On the other hand, in the direct tensor notation, equations are expressed in terms of vectors and tensors themselves instead of their components. It is known that all the vectors and tensors encountered in the study of solid mechanics, such as the displacement vector and stress tensor, are physical quantities which do not depend on the co-ordinate system chosen as a reference. Thus, if the direct tensor notation is employed, the mathematical representation of the problem of solid mechanics in the general form, which does not depend on the co-ordinate system, can be achieved. Also, by using the direct tensor notation, equations are largely simplified, and this offers convenience in book-keeping.

In the mathematical description of solid mechanics, various tensors of different order are involved. For example, the strain energy density, complementary energy density, mass density, and also, functionals corresponding to various variational principles are zeroth order tensors (or scalars). The displacement vector, body force, and traction at boundaries are first order tensors (or vectors). And the various measures of stress and strain are second order tensors. Further, the so-called elasticity tensor and compliance tensor, which characterize the mechanical property of the material, are considered to be fourth order tensors. The governing equations and the functionals are described in terms of these tensors of different order. Moreover, such equations themselves have the property of tensors. Thus, they are called as tensor equations. In these tensor equations, several tensor operations among different order tensors are involved. The general discussions of the tensor operations are available in textbooks, such as [47]. However, the tensor operations involved in solid mechanics are limited to certain types. For these tensor operations, the details of definitions are presented in the following.

For convenience, a rectangular Cartesian co-ordinate system (x_1, x_2, x_3) with unit base vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is introduced to define the tensor operations used in this thesis. In the following as well as in the text, a scalar is represented by a simple Roman or Greek letter. A vector and a second order tensor are indicated, respectively, by $\_$ and $\sim$ under symbols.

Let $\underline{a}$ and $\underset{\sim}{c}$ be a vector and a second order tensor, respectively. They are decomposed into the Cartesian components a_i and c_{ij} .

$$\underline{a} = a_i \underline{e}_i \tag{A.1}$$

$$\underset{\sim}{c} = c_{ij} \underline{e}_i \underline{e}_j \tag{A.2}$$

The transpose, or conjugate tensor of the second order tensor $\underset{\sim}{c}$ is denoted by $\underset{\sim}{c}^T$, and its definition is given by,

$$\underset{\sim}{c}^T = c_{ij} \underline{e}_j \underline{e}_i \tag{A.3}$$

If the second order tensor $\underset{\sim}{c}$ has a property, such that,

$$\underset{\sim}{c}^T = \underset{\sim}{c} \tag{A.4}$$

it is said to be self-conjugate or symmetric. Further, a unit second order tensor (or identity tensor) $\underset{\sim}{I}$ is defined by,

$$\underset{\sim}{I} = \delta_{ij} \underline{e}_i \underline{e}_j \tag{A.5}$$

where,

$$\delta_{ij} = 1 \quad \text{if} \quad i = j$$

$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j$$

Now, we consider certain operations among vectors and tensors. The product of two vectors is defined in the usual way, such that,

$$\underline{a} \cdot \underline{b} = a_i \ b_i \tag{A.6}$$

Similarly, the vector product of two vectors is defined by,

$$\underline{a} \times \underline{b} = e_{ijk}\, a_j b_k \underline{e}_i \tag{A.7}$$

where e_{ijk} is a component of permutation tensor, such that $e_{ijk}=1$ if i,j,k take on values 1,2,3 in cyclic order, $e_{ijk}=-1$ if i,j,k take on 3,2,1 in cyclic order, otherwise $e_{ijk}=0$. The product between a vector and a second order tensor is defined by,

$$\underline{a} \cdot \underset{\sim}{c} = \underset{\sim}{c}^T \cdot \underline{a} = a_i c_{ij} \underline{e}_j \tag{A.8}$$

Following the definitions, Eqns.(A.6 and 8), the operation among two vectros and one second order tensor can be defined by,

$$\underline{a} \cdot \underset{\sim}{c} \cdot \underline{b} = \underline{b} \cdot \underset{\sim}{c}^T \cdot \underline{a} = a_i c_{ij} b_j \tag{a.9}$$

The product of two second order tensors is defined by,

$$\underset{\sim}{c} \cdot \underset{\sim}{d} = c_{ij} d_{jk} \underline{e}_i \underline{e}_k \tag{A.10}$$

If $\underset{\sim}{c} \cdot \underset{\sim}{d} = \underset{\sim}{I}$, tensor $\underset{\sim}{c}$ is said to be an inverse of the second order tensor $\underset{\sim}{d}$. Denoting inverse of $\underset{\sim}{d}$ as $\underset{\sim}{d}^{-1}$, its definition is wrtten as,

$$\underset{\sim}{d}^{-1} \cdot \underset{\sim}{d} = \underset{\sim}{I} \tag{A.11}$$

It is noted, here, that $\underset{\sim}{d} \cdot \underset{\sim}{d}^{-1} \neq \underset{\sim}{I}$, unless $\underset{\sim}{d}$ is symmetric.

The trace of a second order tensor is defined by,

$$\mathrm{trace}(\underset{\sim}{c}) = c_{ii} \tag{A.12}$$

We define a tensor inner product, $\underset{\sim}{d} : \underset{\sim}{e}$, as,

$$\underset{\sim}{d}:\underset{\sim}{e} = \mathrm{trace}(\underset{\sim}{d}^T \cdot \underset{\sim}{e}) = d_{ij}\, e_{ij} \tag{A.13}$$

It is shown from the definition that,

$$\underset{\sim}{d}:\underset{\sim}{e} = \underset{\sim}{e}:\underset{\sim}{d} = \underset{\sim}{d}^T:\underset{\sim}{e}^T \tag{A.14}$$

Also, it is shown for a symmetric tensor $\underset{\sim}{d}$ that,

$$\underset{\sim}{d}:\underset{\sim}{e} = \underset{\sim}{d}:\underset{\sim}{e}^T \tag{A.15}$$

Combining the above operations, more complicated operations can be described in a simple manner,

$$(\underset{\sim}{c}\cdot\underset{\sim}{d}):\underset{\sim}{e} = \mathrm{trace}[(\underset{\sim}{c}\cdot\underset{\sim}{d})^T\cdot\underset{\sim}{e}] = c_{ik}\, d_{kj}\, e_{ij} \tag{A.16}$$

It is noted that Eq.(A.16) can be rewritten in several ways,

$$(\underset{\sim}{c}\cdot\underset{\sim}{d}):\underset{\sim}{e} = (\underset{\sim}{e}\cdot\underset{\sim}{d}^T):\underset{\sim}{c} = (\underset{\sim}{c}^T\cdot\underset{\sim}{e}):\underset{\sim}{d} \tag{A.17}$$

This property is very convenient in constructing variational principles.

As shown by Eq.(2.17), the strain energy density is considered as a function of the strain tensor itself, and its derivative with respect to the strain tensor gives stress tensor. However, for the validity of this statement, the concept of the derivative with respect to tensor must be clearly defined. This can be generarilized from the usual mathematical concept of derivative. Let the strain energy density W be a function of the Cartesian components of the Green-Lagrange strain tensor, g_{ij} . Then the total derivative of W can be expressed as,

$$dW(g_{ij}) = \frac{\partial W}{\partial g_{ij}}\, dg_{ij} \tag{A.18}$$

This can be equivalently rewritten by,

$$dW(g_{ij}) = \left(\frac{\partial W}{\partial g_{ij}}\, \underline{e}_i\, \underline{e}_j\right):\left(dg_{kl}\, \underline{e}_k\, \underline{e}_l\right) \tag{A.19}$$

Then, the total derivative dW can be expressed in tensor form,

$$dW = \frac{\partial W}{\partial \underset{\sim}{g}} : d\underset{\sim}{g} \tag{A.20}$$

where,

$$\frac{\partial W}{\partial \underset{\sim}{g}} = \frac{\partial W}{\partial g_{ij}}\, \underline{e}_i\, \underline{e}_j$$

and

$$d\underset{\sim}{g} = dg_{kl}\, \underline{e}_k\, \underline{e}_l$$

Thus, the derivative of scalar with respect to tensor is defined. Similarly, the derivative of tensor $\underset{\sim}{s}$ with respect to tensor $\underset{\sim}{g}$ is defined by,

$$\frac{\partial \underset{\sim}{s}}{\partial \underset{\sim}{g}} = \frac{\partial s_{ij}}{\partial g_{kl}}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k\, \underline{e}_l \tag{A.21}$$

Further, by replacing $\underset{\sim}{s}$ by $\frac{\partial W}{\partial \underset{\sim}{g}}$, we can define a second order derivative of W with respect to $\underset{\sim}{g}$ as,

$$\frac{\partial^2 W}{\partial \underset{\sim}{g}^2} = \frac{\partial^2 W}{\partial g_{ij} \partial g_{kl}}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k\, \underline{e}_l \tag{A.22}$$

which is a fourth order tensor. Using the above notations, the Taylor expansion of $W(\underset{\sim}{g}^N + \Delta\underset{\sim}{g})$ in terms of $\Delta\underset{\sim}{g}$ can be expressed by,

$$W(\underset{\sim}{g}^N+\Delta\underset{\sim}{g}) = W(\underset{\sim}{g}^N) + \frac{\partial W}{\partial \underset{\sim}{g}}\bigg|^N : \Delta\underset{\sim}{g} + \frac{1}{2}\frac{\partial^2 W}{\partial \underset{\sim}{g}^2}\bigg|^N :: \Delta\underset{\sim}{g}\,\Delta\underset{\sim}{g} + \cdots \tag{A.23}$$

where the derivatives are evaluated for $\underset{\sim}{g}^N$.

The gradient operator is also a tensor of first order, which is defined by,

$$\nabla = \underline{e}_i \frac{\partial}{\partial x_i} \tag{A.24}$$

Thus, the gradient of a scalar p and a vector $\underline{a}$ are defined by,

$$\nabla p = \frac{\partial p}{\partial x_i}\,\underline{e}_i \tag{A.25}$$

$$\nabla \underline{a} = \frac{\partial a_j}{\partial x_i}\,\underline{e}_i\,\underline{e}_j \tag{A.26}$$

Considering the gradient operator as a vector, the following operations are defined,

$$\nabla\cdot\underline{a} = (\nabla\underline{a}):\underset{\sim}{I} = \frac{\partial a_i}{\partial x_i} \tag{A.27}$$

$$\nabla\cdot\underset{\sim}{c} = \frac{\partial c_{ij}}{\partial x_i}\,\underline{e}_j \tag{A.28}$$

Although it is not presented here, the tensor operations defined in the above can be decomposed into components of any convenient co-ordinate system, such as polar co-ordinates and cylindrical co-ordinates.

APPENDIX B

PHYSICAL MEANING OF STRESS AND STRAIN MEASURES

Stress Measures

In this section, the hpysical meanings of stress measures in the total Lagrangean description are explored. To this end, we start from geometrical relations between nudeformed and deformed configurations of a solid. For simplicity, a fixed rectangular Cartesian co-ordinate system with unit base vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is employed to describe both undeformed and deformed state. We consider a material point, the positions of which in undeformed and deformed configurations are P and P' as shown in Fig.B-1. Co-ordinates of P and P' are x_i and y_i, respectively. Then, co-ordinates x_i represent material (or Lagrangean) co-ordinates, and y_i represent spacial (or Eulerian) co-ordinates. It is noted here that the material co-ordinates x_i (Cartesian) in the undeformed configuration become curvilinear co-ordinates x_i in the deformed configuration. For later use, we introduce base vectors $\underline{g}_i$ for the convected co-ordinate system (x_1, x_2, x_3) in the deformed configuration, which are defined as,

$$\underline{g}_i = \frac{\partial y_j}{\partial x_i} \underline{e}_j \qquad (B.1)$$

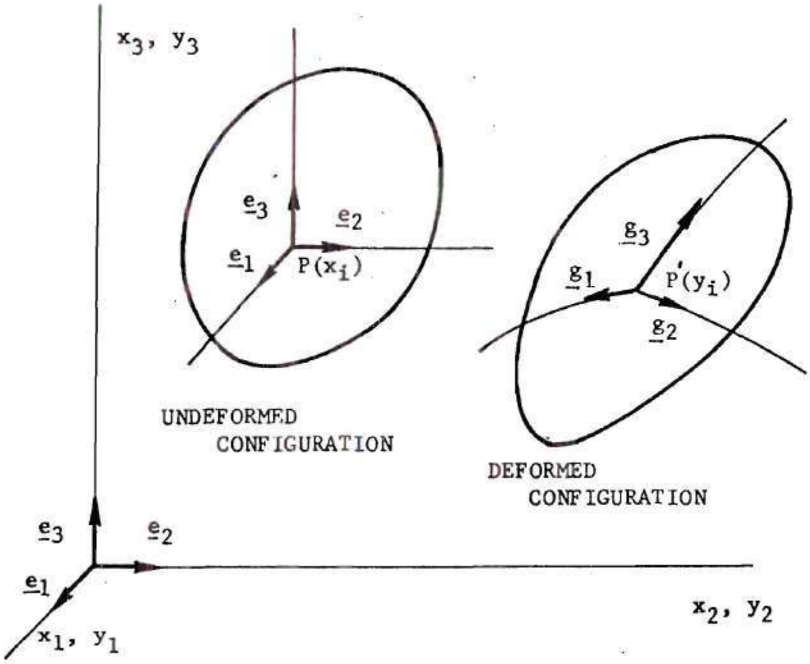


Fig.B-1 Co-ordinate System

Also, we need to know the change of volume of a infinitesimal material element and the change of oriented area of a material surface element through deformation of a body.

Let dV_0 and dV be volumes of a material element in the undeformed and deformed configurations, respectively. These volumes are related by,

$$J \, dV_o = dV \tag{B.2}$$

where,

$$J = \det \left| \frac{\partial y_i}{\partial x_j} \right|$$

Let ds_0 and ds be areas of a infinitesimal material surface element in undeformed and deformed configurations : $\underline{n}$ and $\underline{\nu}$ be unit outward normals to ds_0 and ds, respectively. Then, oriented areas $\underline{n}ds_0$ and $\underline{\nu}ds$ are related by,

$$\underline{n}\, ds_o = \frac{1}{J}\ (\nabla \underline{y}) \cdot \underline{\nu}\, ds \tag{B.3}$$

or in components,

$$n_i\ ds = \frac{1}{J}\ \frac{\partial y_j}{\partial x_i}\ \nu_j\ ds \tag{B.4}$$

Using the above relations, the physical meanings of Cauchy stress $\underset{\sim}{\tau}$, Piola-Lagrange stress $\underset{\sim}{t}$, and Kirchhoff-Trefftz stress $\underset{\sim}{s}$ can be shown in the following.

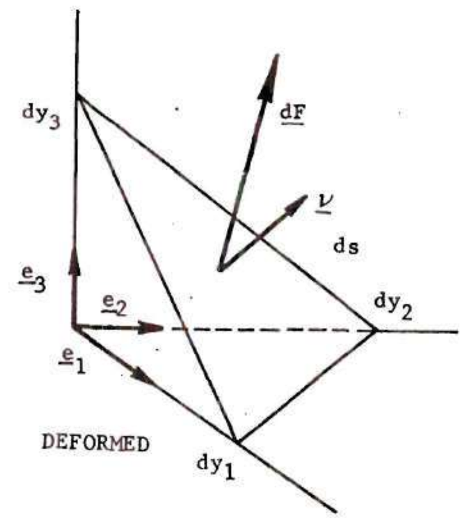


Fig.B-2 Physical Meaning of Cauchy Stress

Consider an infinitesimal surface element ds in the deformed state as shown in Fig.B-2. Unit outward normal to ds is denoted by $\underline{\nu}$. Let a force vector acting on ds be $d\underline{F}$. Then, stress vector $\underline{T}$ per unit area in the deformed configuration can be defined as,

$$\underline{T} = \frac{d\underline{F}}{ds} \tag{B.5}$$

or in components, $$T_i\ \underline{e}_i = \frac{dF_i}{ds}\ \underline{e}_i \tag{B.6}$$

Further, Cartesian components of Cauchy stress τ_{ij} are defined through the following relation.

$$\nu_i \tau_{ij} = T_j = \frac{dF_j}{ds} \tag{B.7}$$

or,

$$\underline{dF} = (\nu_i \tau_{ij}\, ds)\, \underline{e}_j \tag{B.8}$$

Thus, τ_{ij} is a stress per unit area in deformed configuration.

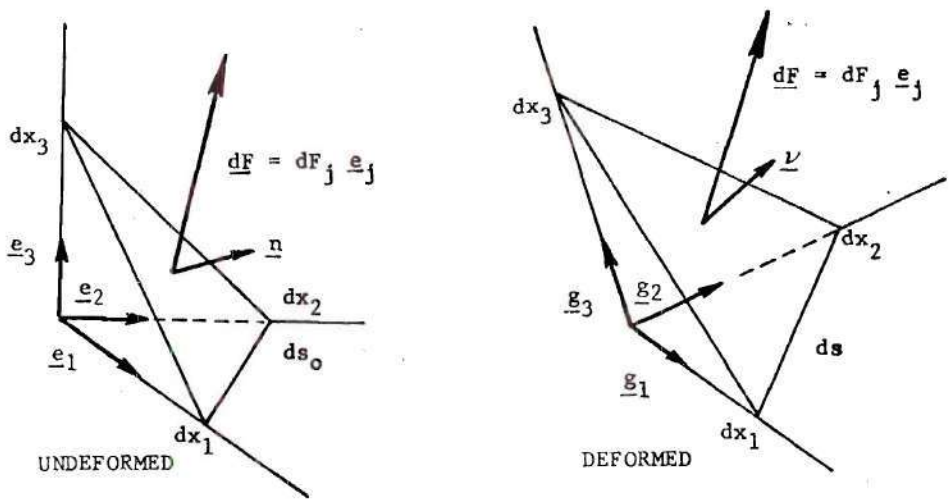


Fig.B-3 Physical Meaning of Piola-Lagrange Stress

On the other hand, if we use undeformed area to measure stress vector, Piola-Lagrange stress is defined in the following manner. First we translate the force vector $\underline{dF}$ acting on the deformed area ds to the undeformed area ds_0 as shown by Fig.B-3. Then, the stress vector per unit undeformed area $\underline{t}$ is defined by,

$$\underline{t} = t_i \, \underline{e}_i = \frac{dF_i}{ds_o} \, \underline{e}_i \tag{B.9}$$

Through the stress vector $\underline{t}$, Cartesian components of the Piola-Lagrange stress t_{ij} are defined by,

$$n_i \, t_{ij} = t_j = \frac{dF_j}{ds_o} \tag{B.10}$$

where n_i are components of unit normal to the undeformed surface ds_0. The force vector $\underline{dF}$ is expressed in terms of t_{ij} as,

$$\underline{dF} = (n_i \, t_{ij} \, ds_o) \, \underline{e}_j \tag{B.11}$$

Thus, t_{ij} is a stress per unit area in undeformed configuration.

Similarly, the physical meaning of the Kirchhoff-Trefftz stress can be shown. Unlike the case of the Piola-Lagrange stress, the force vector $\underline{dF}$ is decomposed with respect to the convected base vector $\underline{g}_i$ before translation.

$$\underline{dF} = d\hat{F}_i \, \underline{g}_i \tag{B.12}$$

Then we define an alternative force vector $\underline{d\hat{F}}$ whose Cartesian components are $d\hat{F}_i$, such that,

$$\underline{d\hat{F}} = d\hat{F}_i \, \underline{e}_i \tag{B.13}$$

This force vector is translated to the undeformed area ds_o as shown in Fig.B-4.

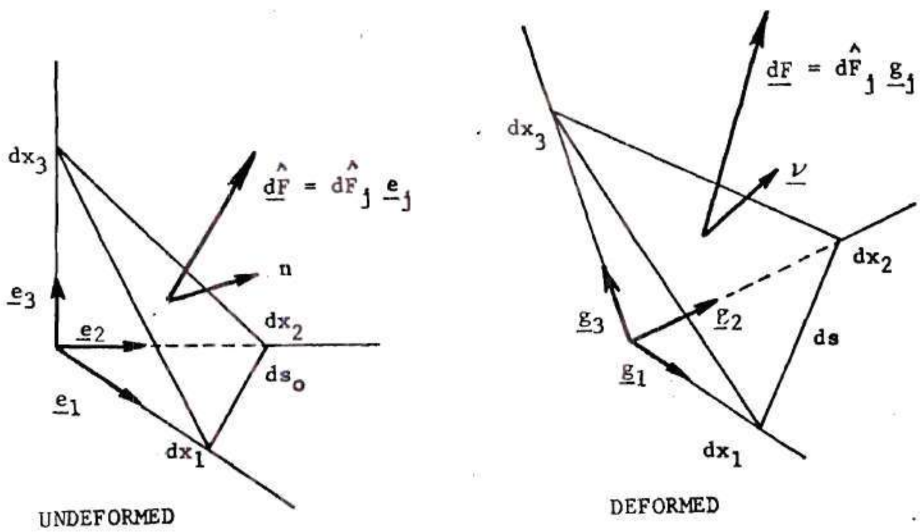


Fig.B-4 Physical Meaning of Kirchhoff-Trefftz Stress

Then, the stress vector $\underline{\hat{t}}$ per unit area in the undeformed configuration is defined.

$$\underline{\hat{t}} = \hat{t}_i \, \underline{e}_i = \frac{d\hat{F}_i}{ds_o} \underline{e}_i \tag{B.14}$$

Through stress vector $\underline{\hat{t}}$, Cartesian components of the Kirchhoff-Trefftz stress s_{ij} are defined by the following relation.

$$n_i \, s_{ij} = \hat{t}_j = \frac{d\hat{F}_j}{ds_o} \tag{B.15}$$

$$\underline{dF} = d\hat{F}_j\, \underline{g}_j = (n_i\, s_{ij}\, ds_o)\underline{g}_j \tag{B.16}$$

Thus s_{ij} is a stress per unit area in the undeformed configuration, but stress vector is defined by Eq.(B.14).

The stress measures defied above can be related through the force vector $\underline{dF}$.

$$\underline{dF} = (\nu_i\, \tau_{ij}\, ds)\, \underline{e}_j \tag{B.17}$$

$$= (n_i\, t_{ij}\, ds_o)\, \underline{e}_j$$

$$= (n_i\, s_{ij}\, ds_o)\, \underline{g}_j$$

Using Eqns.(B.1 and 4), the above relations are reduced to,

$$\tau_{ij} = \frac{1}{J}\frac{\partial y_i}{\partial x_k}\, t_{kj} = \frac{1}{J}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_l}\, s_{kl} \tag{B.18}$$

which are used as definitions of stress measures in the text.

Strain Measures

The details of the physical meaning of Green-Lagrange strain are available in textbooks such as Novozhilov [24]. Therefore, discussion in this section is focused on the strain measure $\underset{\sim}{h}$.

We consider two infinitely close material points M and N. Let $\underline{dx}$ and $\underline{dy}$ be vectors connecting the material points M and N in undeformed and deformed configurations. These are related through deformation gradient $\underset{\sim}{F}$ by,

$$\underline{dy} = \underline{dx}\cdot\nabla\underline{y} = \underset{\sim}{F}\cdot\underline{dx} \tag{B.19}$$

Let ds_0 and ds be distances of the two points in undeformed and deformed configurations, respectively. Then, squares of ds_0 and ds are obtained as,

$$(ds_o)^2 = d\underline{x} \cdot d\underline{x} \tag{B.20}$$

$$(ds)^2 = d\underline{x} \cdot (\nabla\underline{y} \cdot \nabla\underline{y}^T) \cdot d\underline{x} = d\underline{x} \cdot \underset{\sim}{G} \cdot d\underline{x} \tag{B.21}$$

where $\underset{\sim}{G}$ is a deformation tensor. Since $\underset{\sim}{G}$ is related to Green-Lagrange strain $\underset{\sim}{g}$ and right extensional strain $\underset{\sim}{h}$ through,

$$\underset{\sim}{G} = (2\underset{\sim}{g} + \underset{\sim}{I}) = (\underset{\sim}{I} + \underset{\sim}{h}) \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \tag{B.22}$$

Eq.(B.21) can be rewritten in terms of $\underset{\sim}{g}$ and $\underset{\sim}{h}$.

$$(ds)^2 = d\underline{x} \cdot (2\underset{\sim}{g} + \underset{\sim}{I}) \cdot d\underline{x} = d\underline{x}\ (\underset{\sim}{I} + \underset{\sim}{h}) \cdot (\underset{\sim}{I} + \underset{\sim}{h}) \cdot d\underline{x} \tag{B.23}$$

Using the above relations, the relative elongation of the infinitesimal segment MN, defined as,

$$E_{MN} = (ds - ds_o)/ds_o \tag{B.24}$$

can be expressed in terms of $\underset{\sim}{g}$ and $\underset{\sim}{h}$. For convenience, we introduce a rectangular Cartesian co-ordinate system whose co-ordinate lines are parallel to the principal direction of the deformation tensor $\underset{\sim}{G}$. Denoting unit base vectors in this co-ordinate system as $\hat{\underline{e}}_i$, the deformation tensor $\underset{\sim}{G}$ can be decomposed into,

$$\underset{\sim}{G} = G_i\ \hat{\underline{e}}_i\ \hat{\underline{e}}_i \tag{B.25}$$

where G_i are principal values of $\underset{\sim}{G}$. Since tensors $\underset{\sim}{G}$, $\underset{\sim}{g}$, and $\underset{\sim}{h}$ are coaxial, $\underset{\sim}{g}$ and $\underset{\sim}{h}$ can be also decomposed in the same manner.

$$\underset{\sim}{g} = g_i \, \hat{\underline{e}}_i \, \hat{\underline{e}}_i \tag{B.26}$$

$$\underset{\sim}{h} = h_i \, \hat{\underline{e}}_i \, \hat{\underline{e}}_i \tag{B.27}$$

where g_i and h_i are principal values of $\underset{\sim}{g}$ and $\underset{\sim}{h}$, respectively.

Now, we choose material points located on a line parallel to $\hat{\underline{e}}_1$ in undeformed configuration. Then vector $d\underline{x}$ is reduced to,

$$d\underline{x} = d\hat{x}_1 \, \hat{\underline{e}}_1 \tag{B.28}$$

From Eqns.(B.20 and 23), squares of the length of the material line in undeformed and deformed configurations are obtained as,

$$(ds_o^1)^2 = (d\hat{x}_1)^2 \tag{B.29}$$

$$(ds^1)^2 = (2g_1 + 1)(d\hat{x}_1)^2 = (1 + h_1)^2(d\hat{x}_1)^2 \tag{B.30}$$

Using Eqns.(B.29 and 30), relative elongation of the infinitesimal material line can be expressed in terms of g_1 and h_1.

$$(ds^1 - ds_o^1)/ds_o^1 = \sqrt{2g_1 + 1} - 1 = h_1 \tag{B.31}$$

As seen in the above equation, principal value of right extensional strain h_1 corresponds to the relative elongation of a material line which is parallel to the principal direction in the undeformed configuration. For this reason, $\underset{\sim}{h}$ is sometimes called as engineering strain.

APPENDIX C

ILLUSTRATIONS

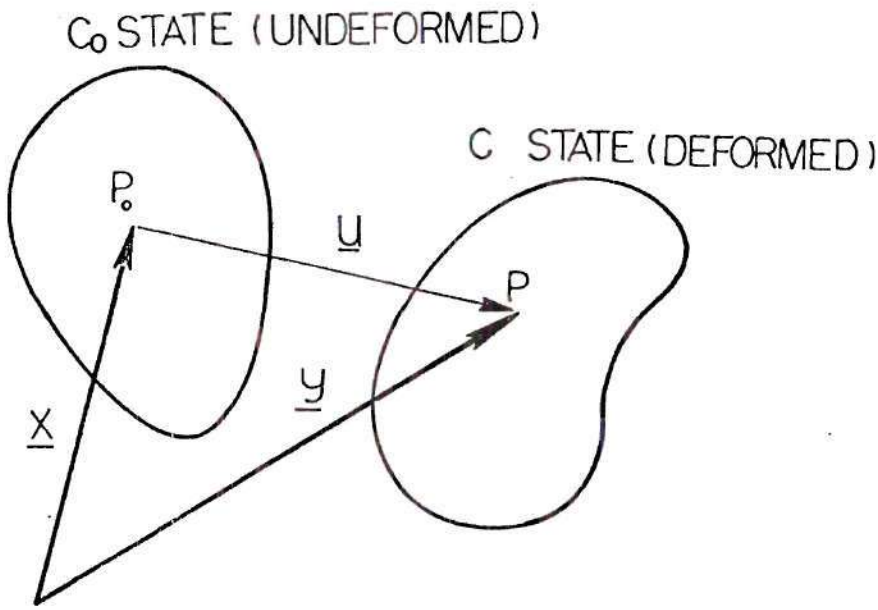


Fig. 1 Description of a Deformed Solid
(Total Lagrangean Description)

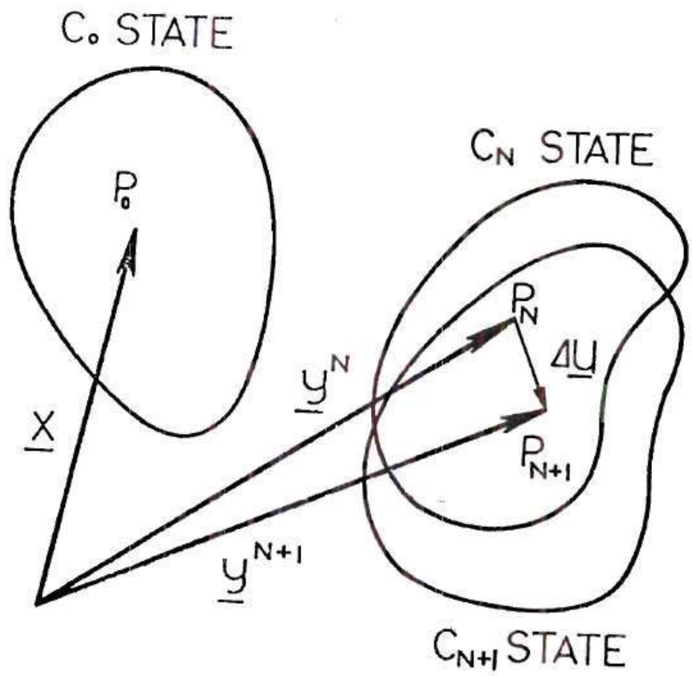


Fig. 2 Description of a Deformed Solid
(Updated Lagrangean Description)

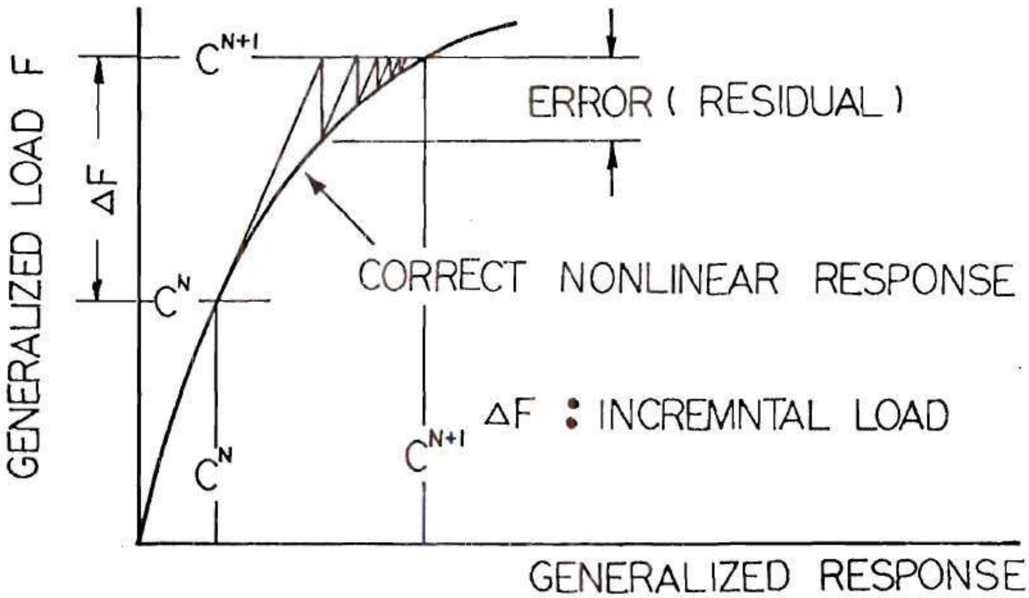


Fig. 3 Iterative Correction Procedure
(Modified Newton-Raphson)

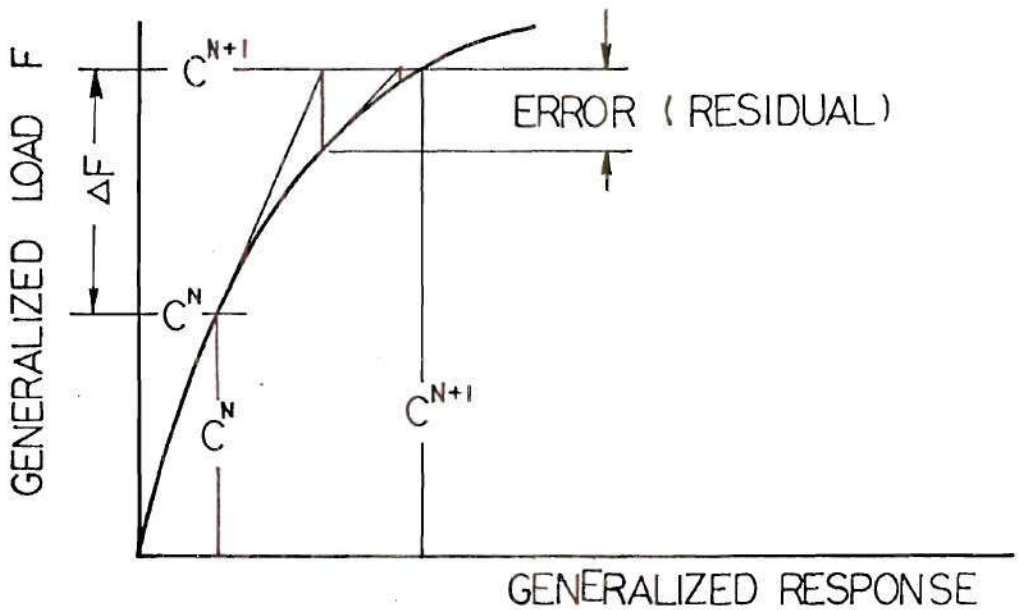


Fig. 4 Iterative Correction Procedure
(Newton-Raphson)

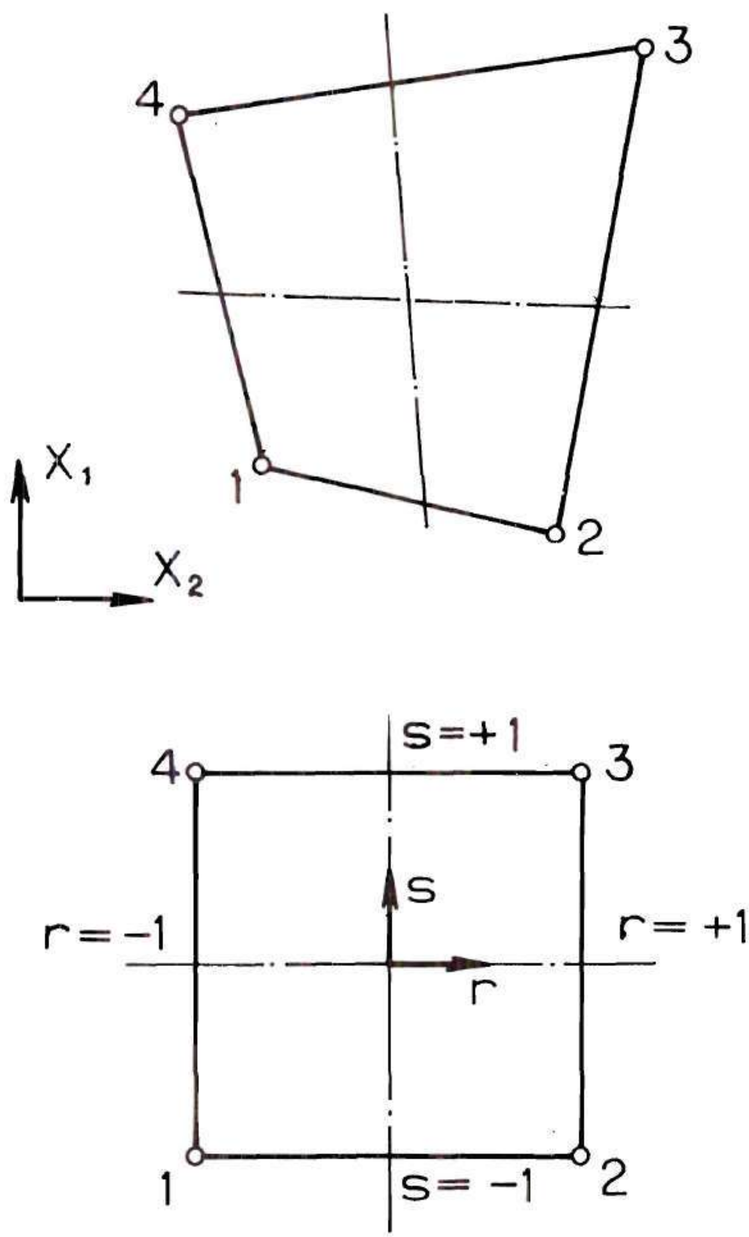


Fig. 5 Four-Noded Isoparametric Element

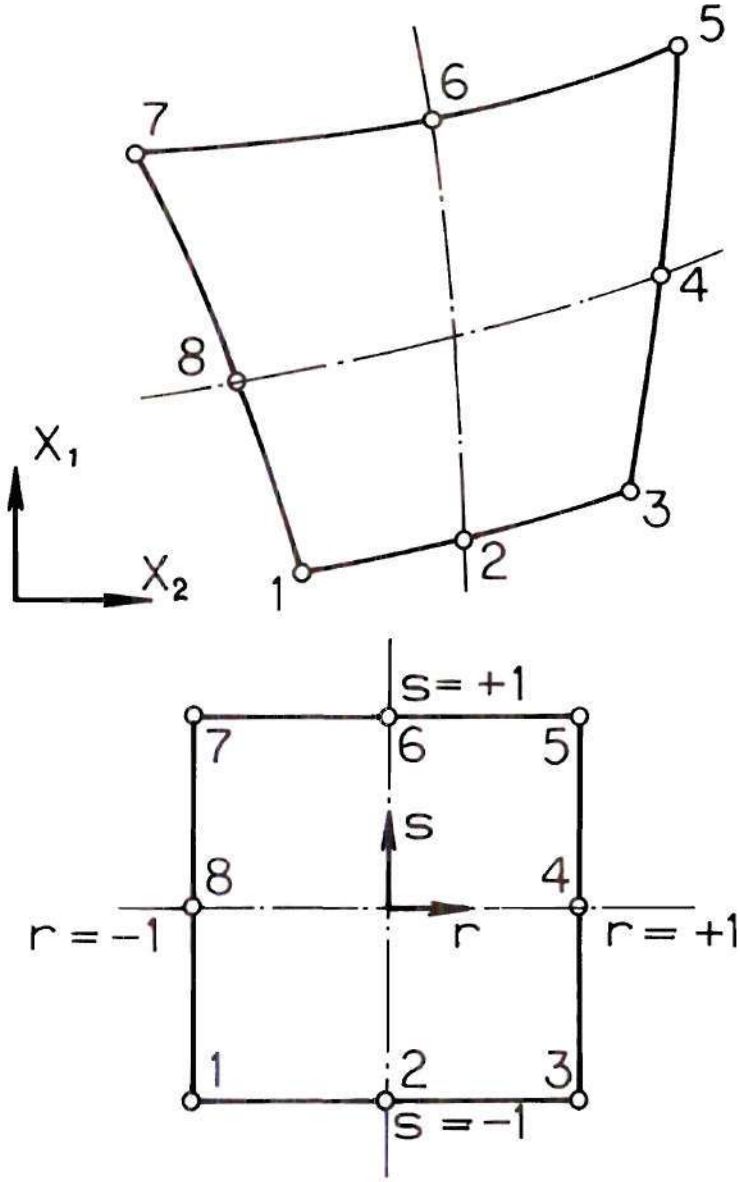


Fig. 6 Eight-Noded Isoparametric Element

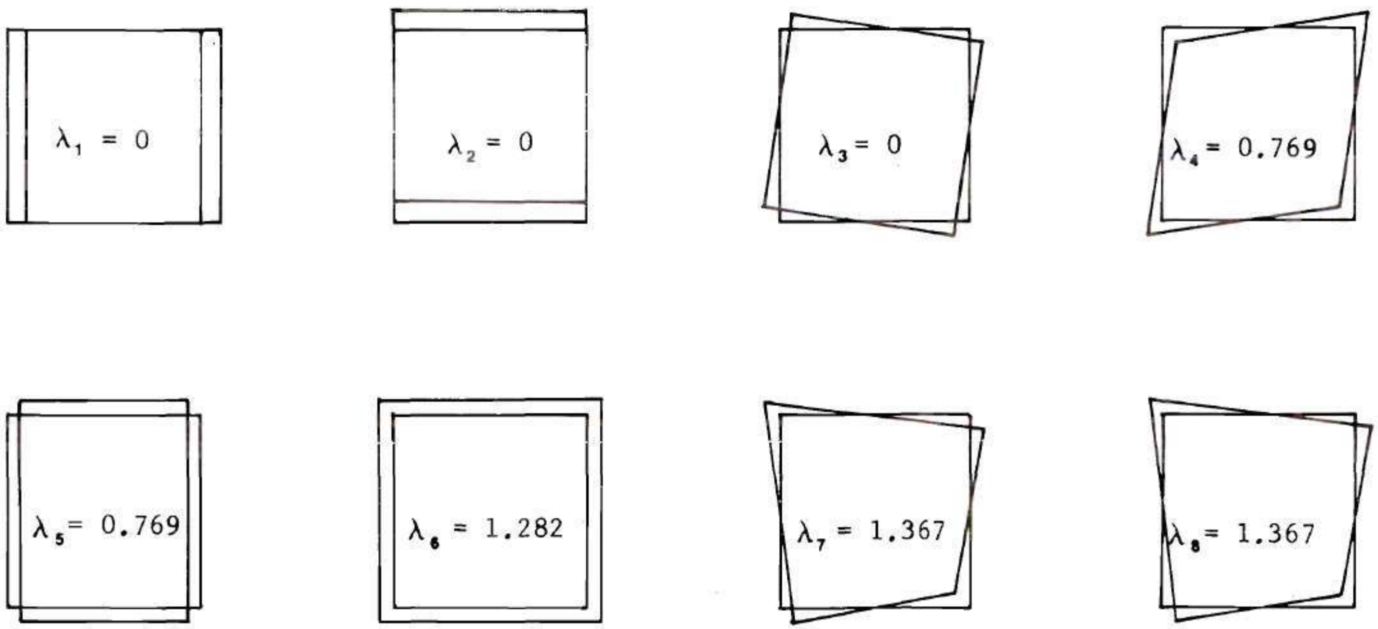


Fig.7 Eigen-Modes of the Stiffness Matrix of Four-Noded Element

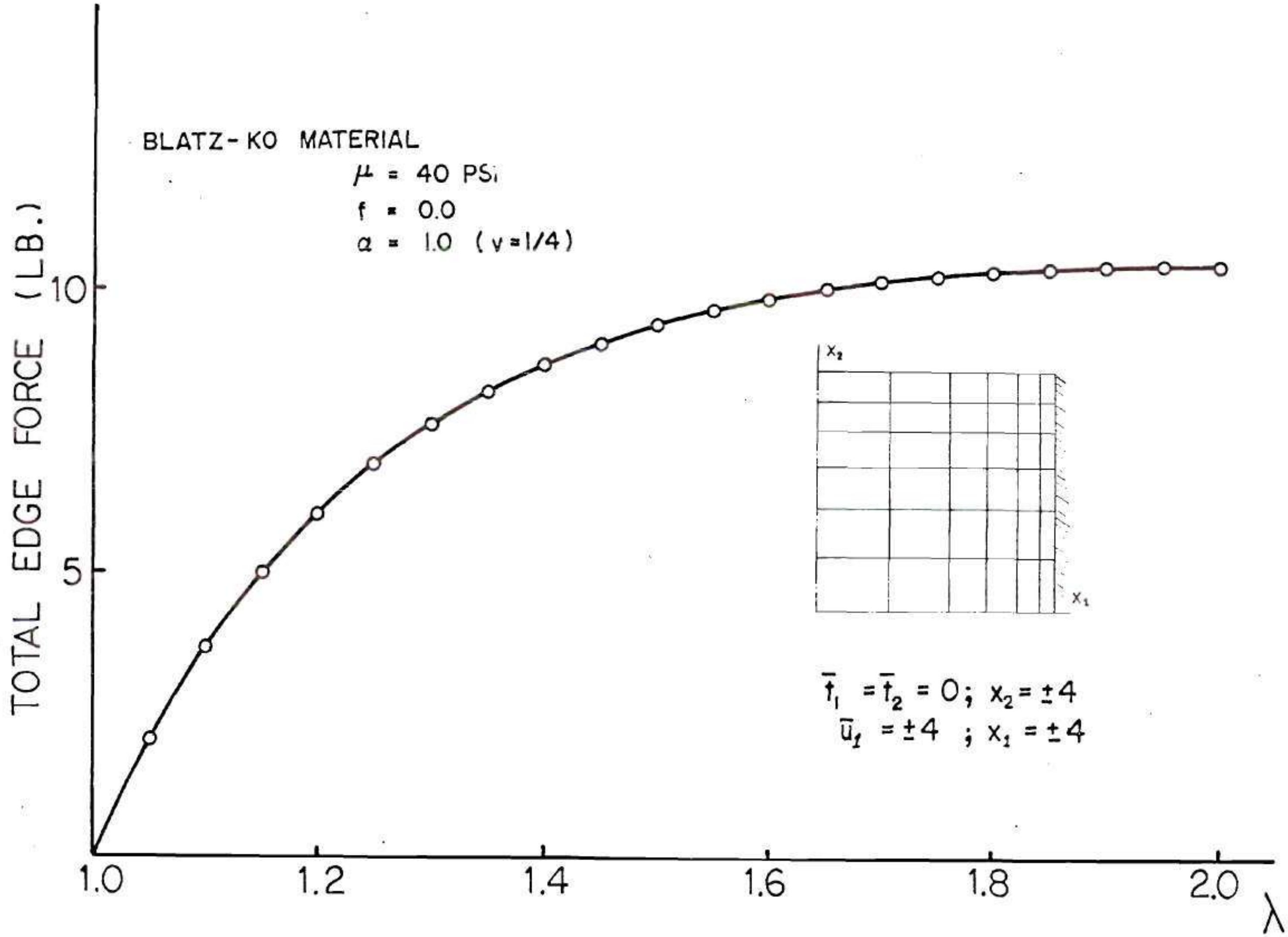


Fig. 8 Total Edge Force Versus Axial Extension Ratio

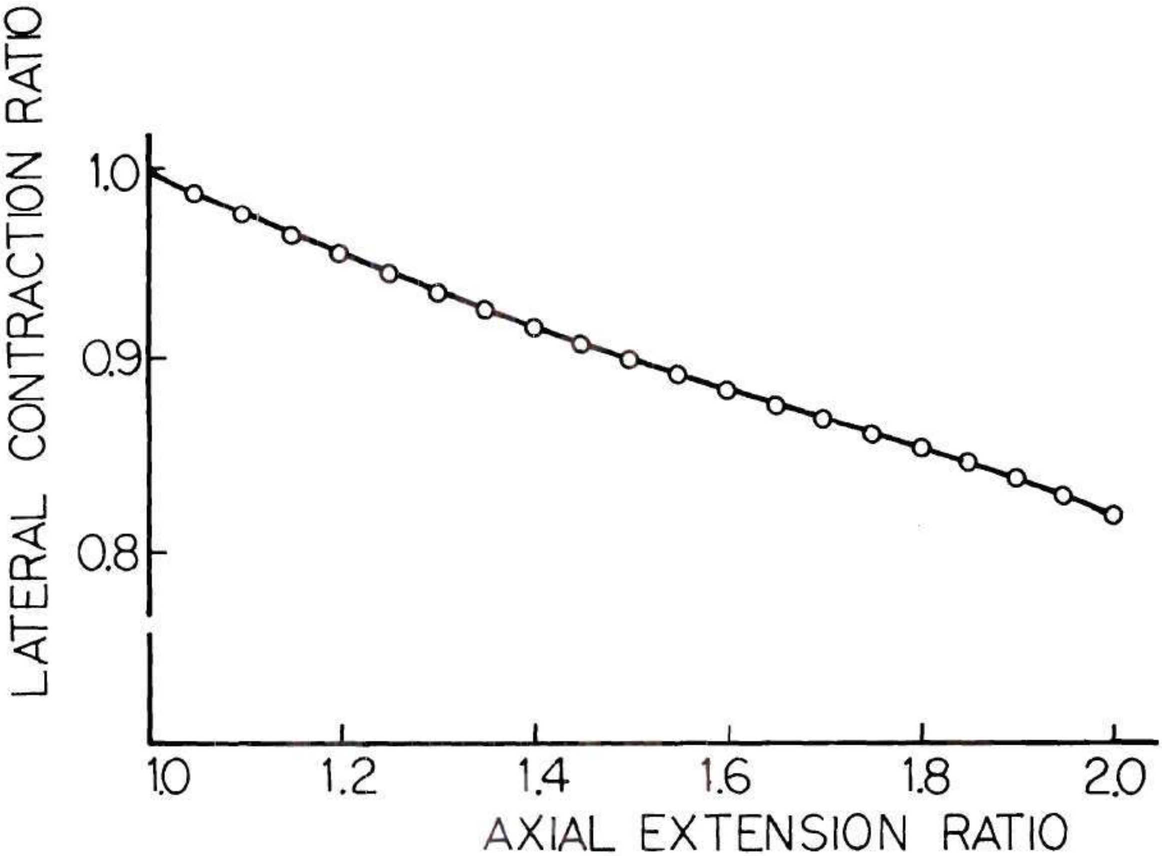
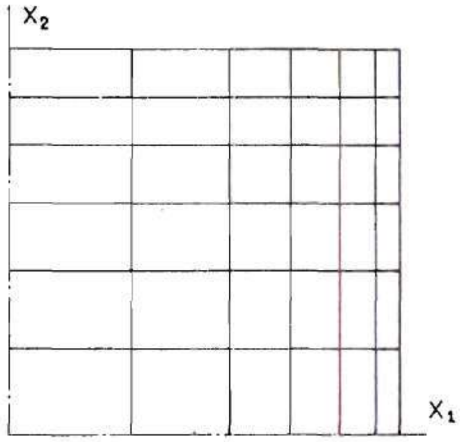
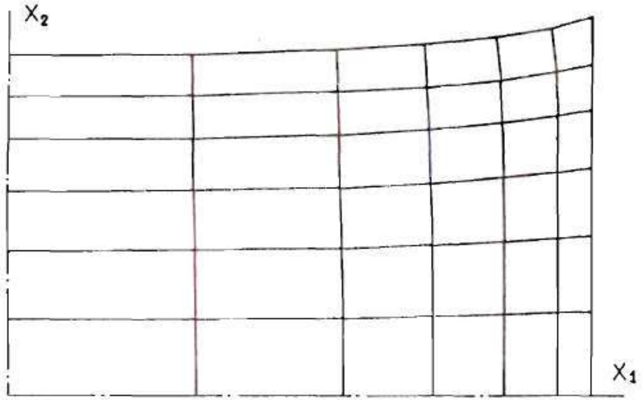


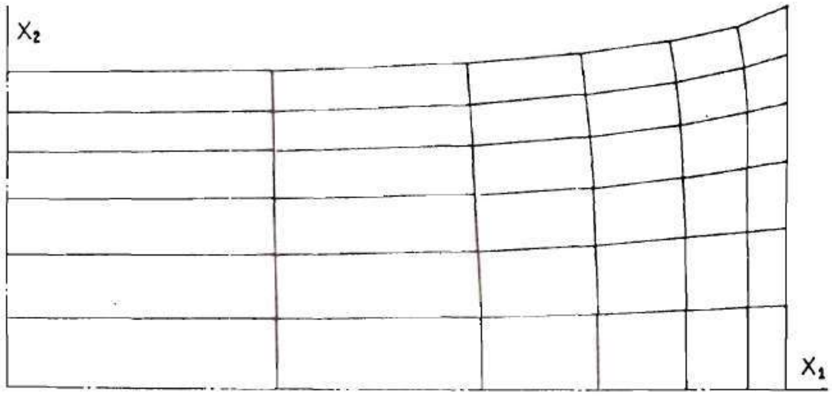
Fig. 9 Lateral Contraction Ratio Versus Axial Extension Ratio



(Initial Configuration)



(Deformed Configuration at $\lambda = 1.5$)



(Deformed Configuration at $\lambda = 2.0$)

Fig. 10 Deformed Configurations of a Square Sheet

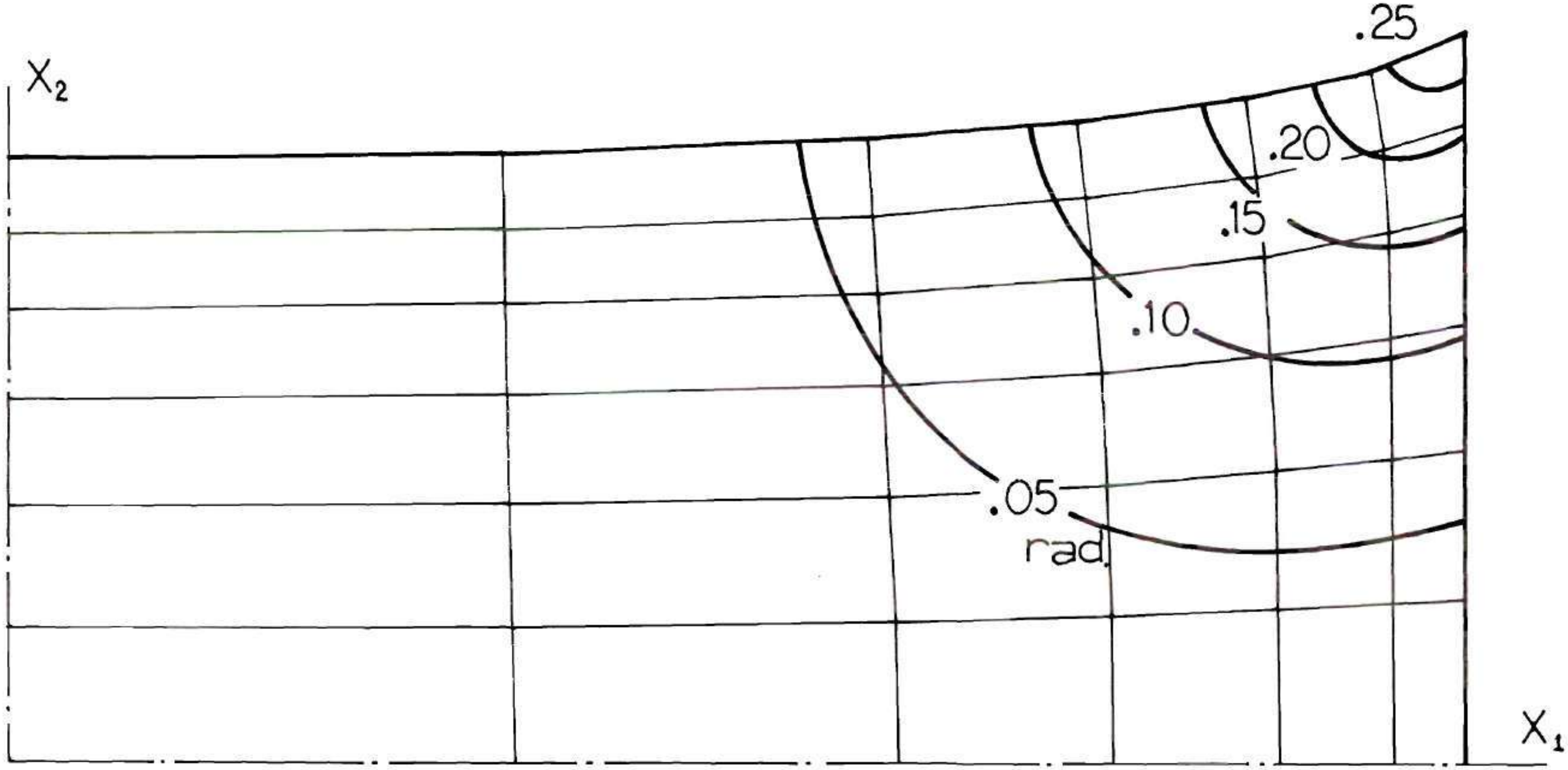


Fig. 11 Contours of Rotation Angle at Final Deformed Configuration

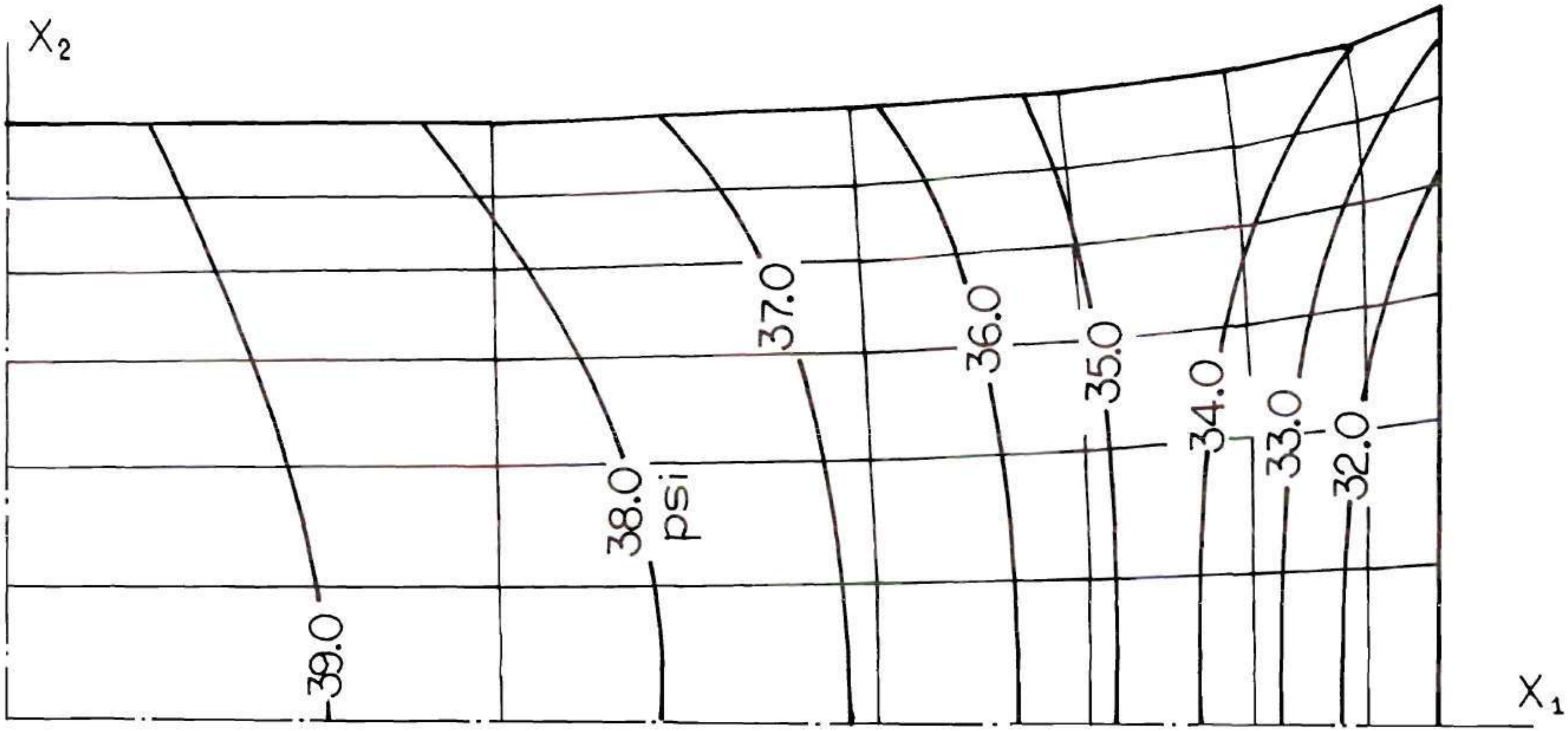


Fig. 12 Contours of Axial Component of Cauchy Stress t_{11} at $\lambda = 2.0$

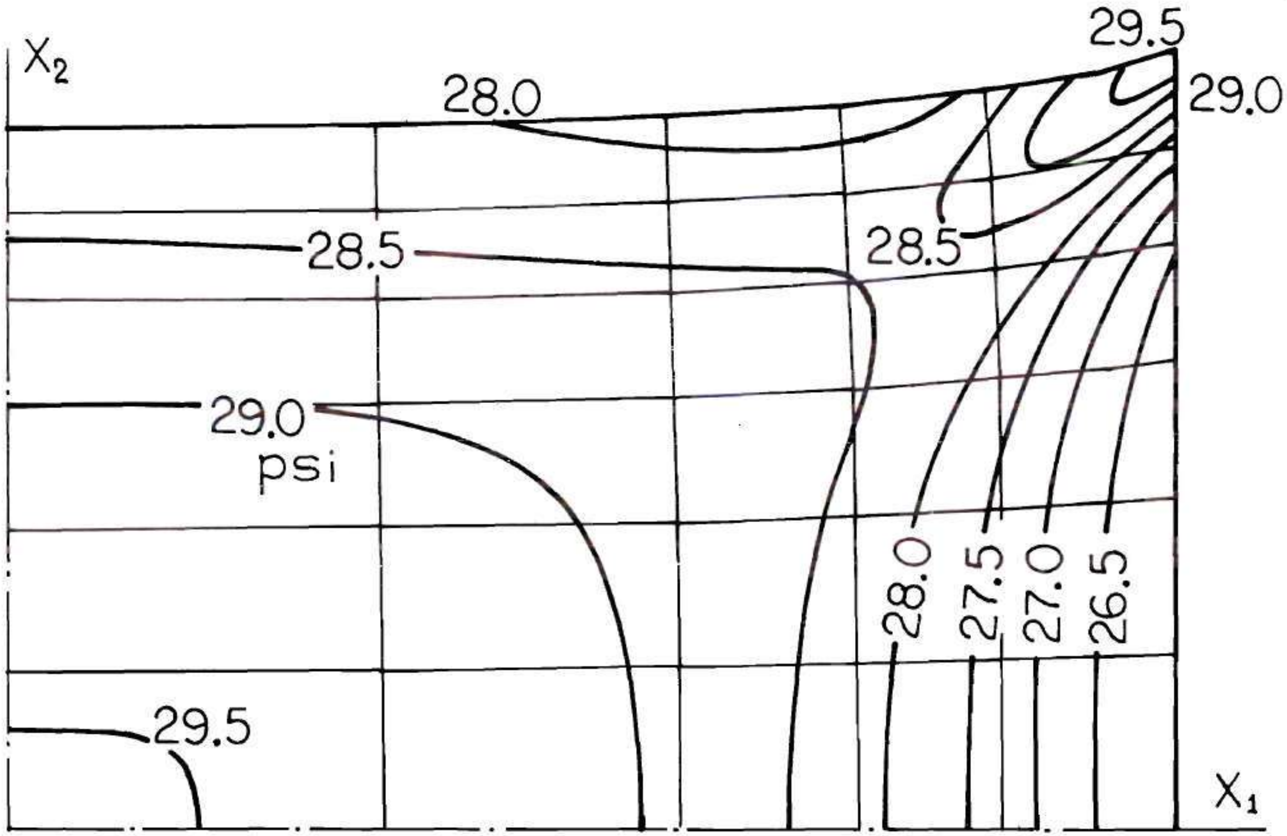


Fig. 13 Contours of Axial Component of Cauchy Stress τ_{11} at $\lambda = 1.5$

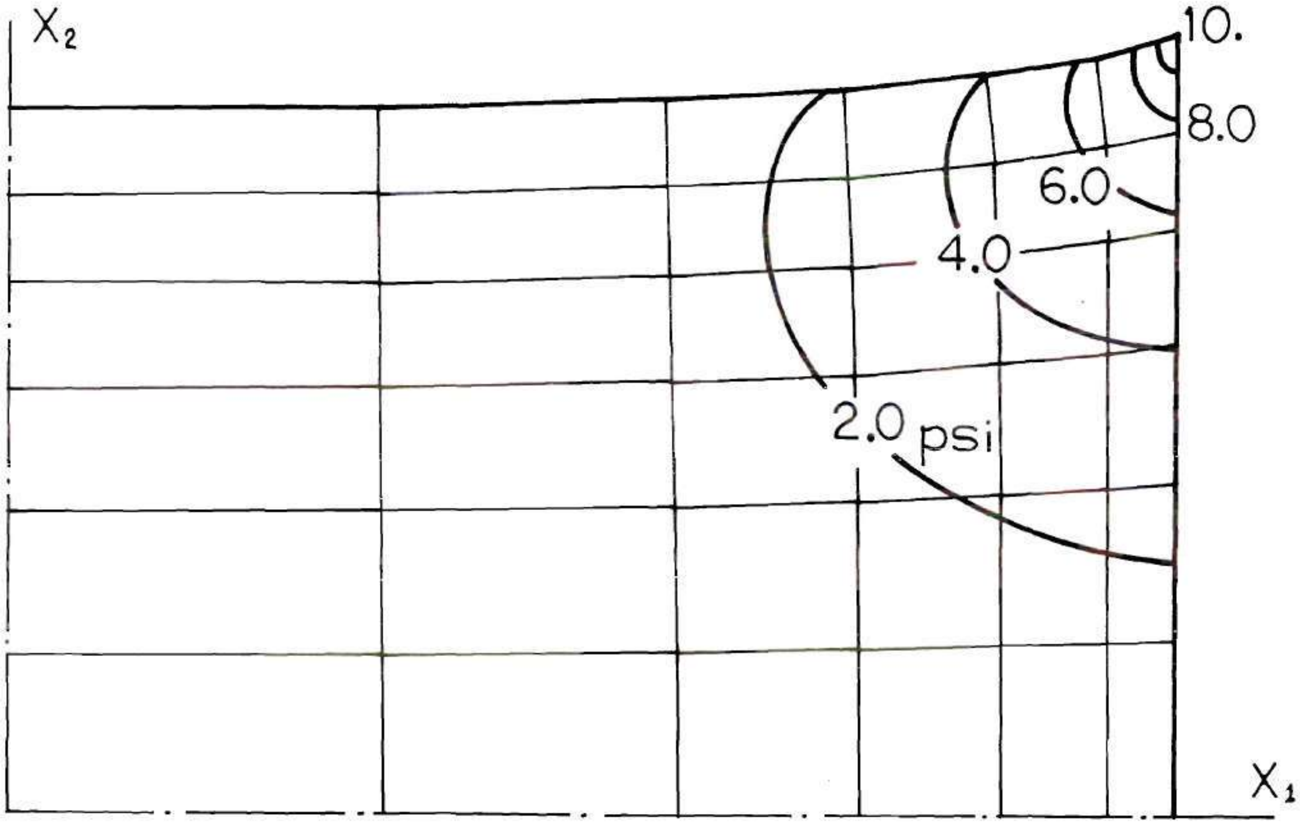


Fig. 14 Contours of Shear Component of Cauchy Stress τ_{12} at $\lambda = 1.5$

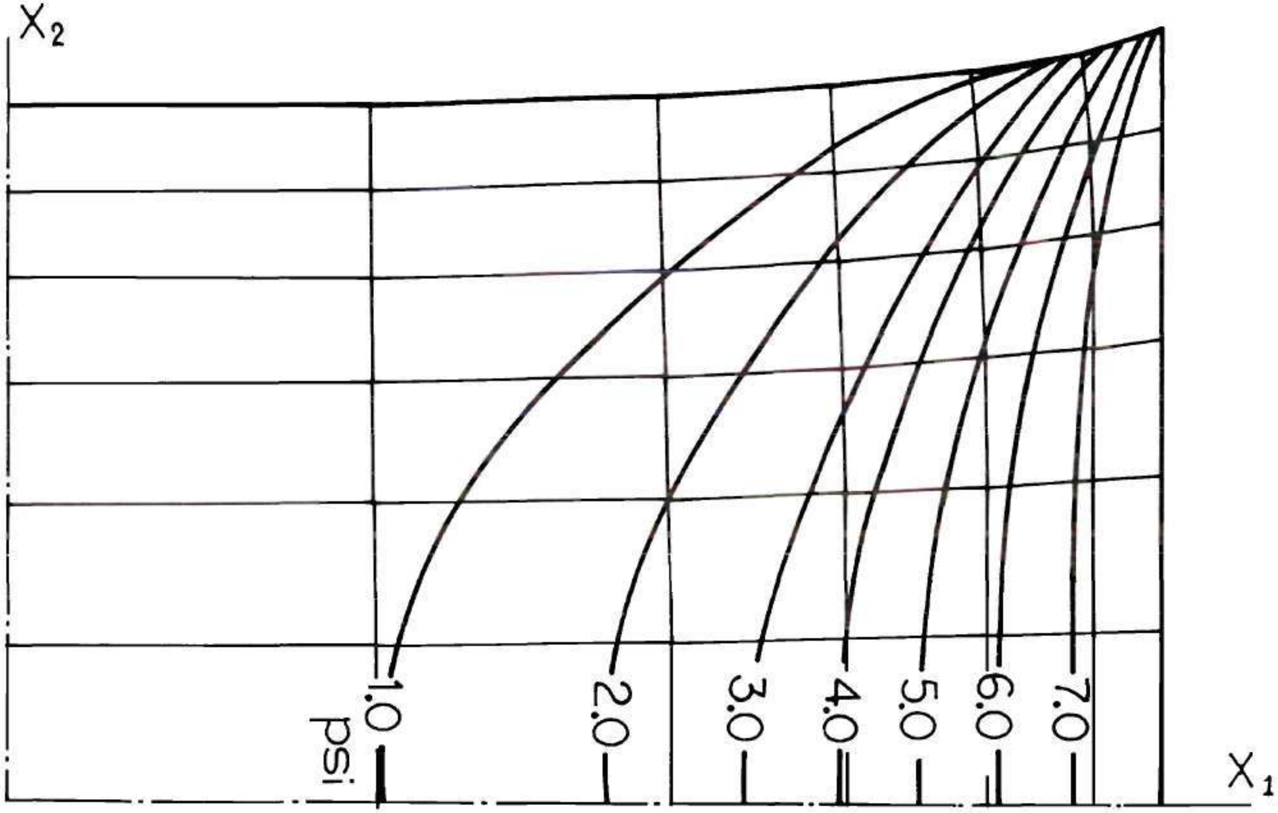


Fig.15 Contours of Lateral Component of Cauchy Stress τ_{22} at $\lambda = 1.5$

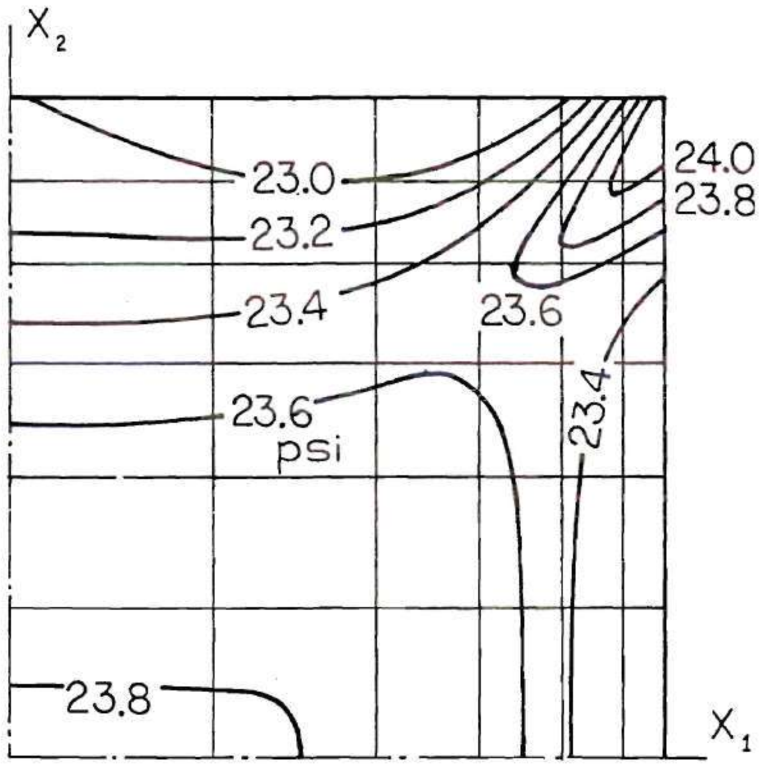


Fig. 16 Contours of Axial Component of Piola-Lagrange Stress t_{11} at $\lambda = 1.5$

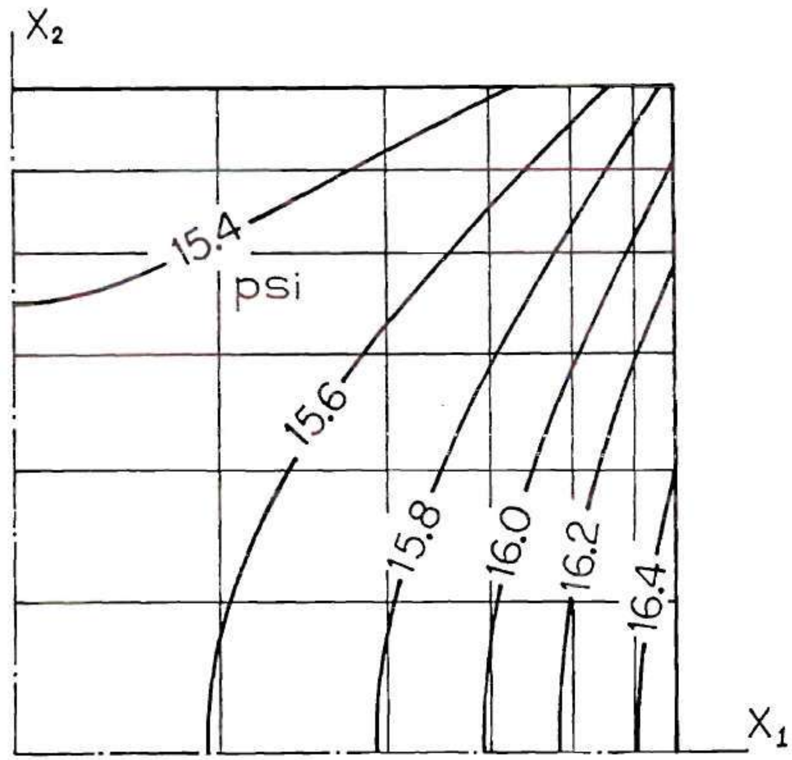


Fig. 17 Contours of Axial Component of Kirchhoff-Trefftz Stress s_{11} at $\lambda = 1.5$

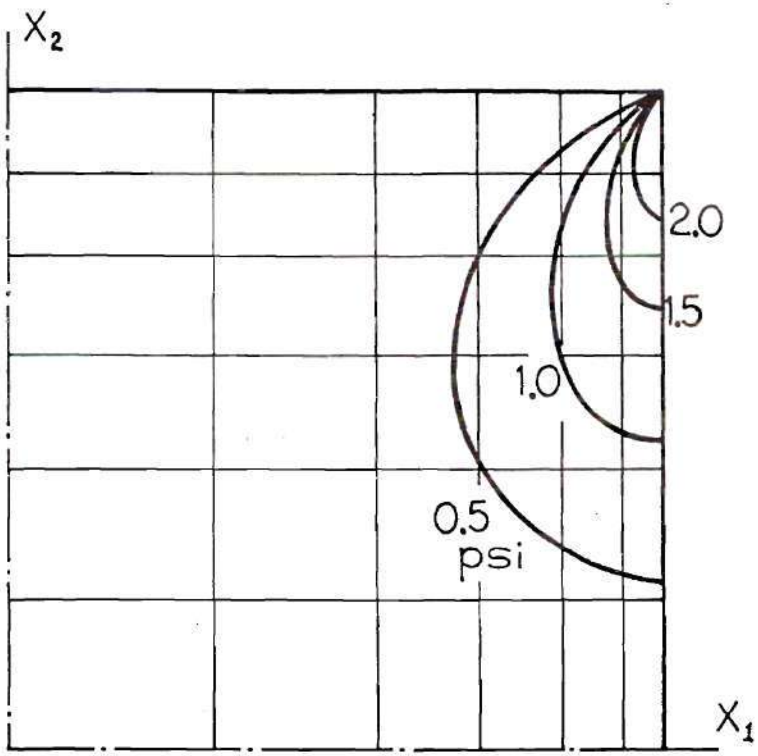


Fig. 18 Contours of Shear Component of Piola-Lagrange Stress t_{21} at $\lambda = 1.5$

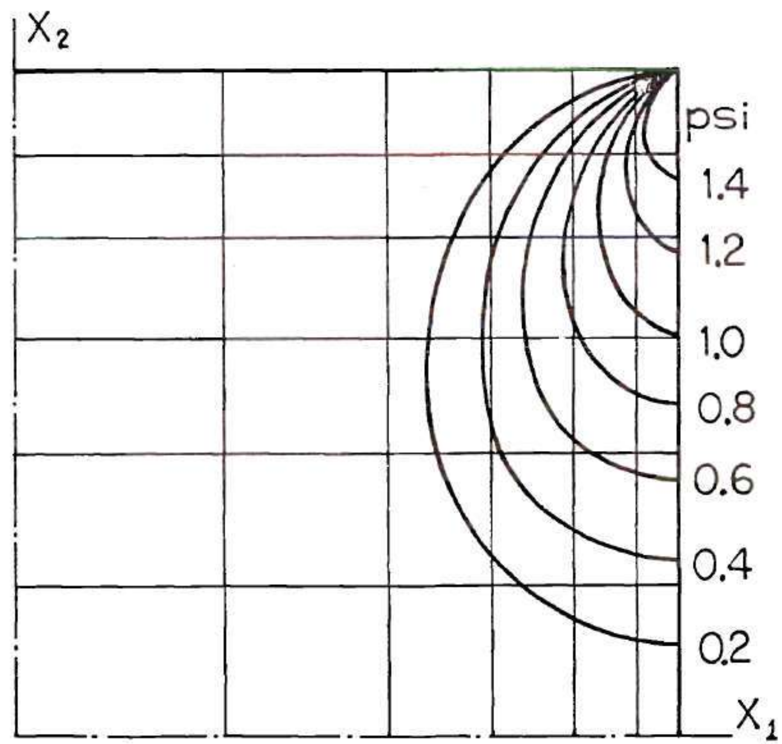


Fig. 19 Contours of Shear Component of Kirchhoff-Trefftz Stress S_{12} at $\lambda = 1.5$

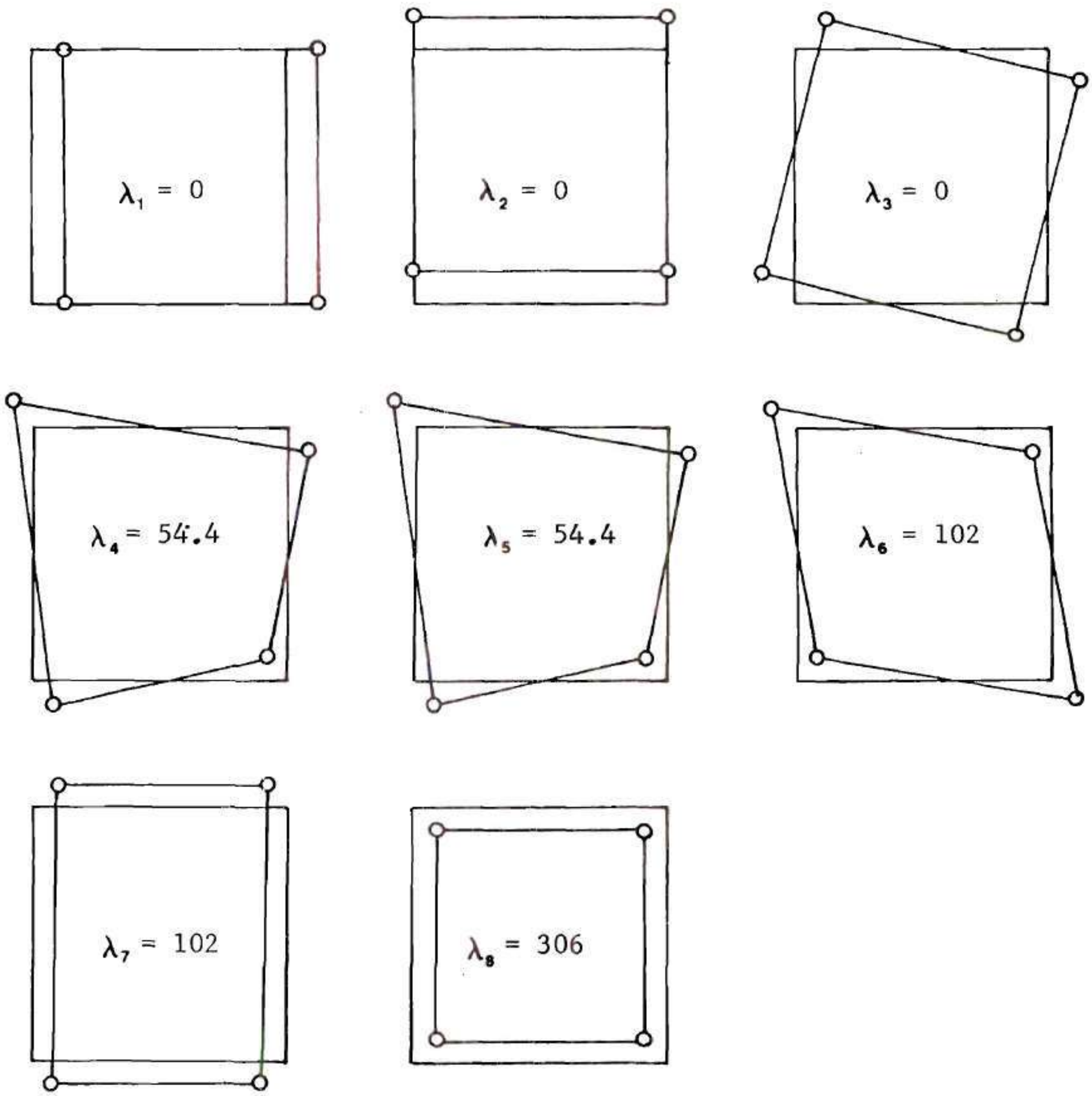


Fig.20 Eigen-Modes of the Stiffness Matrix of Four-Noded Element (Incompressible, Plane-Stress)

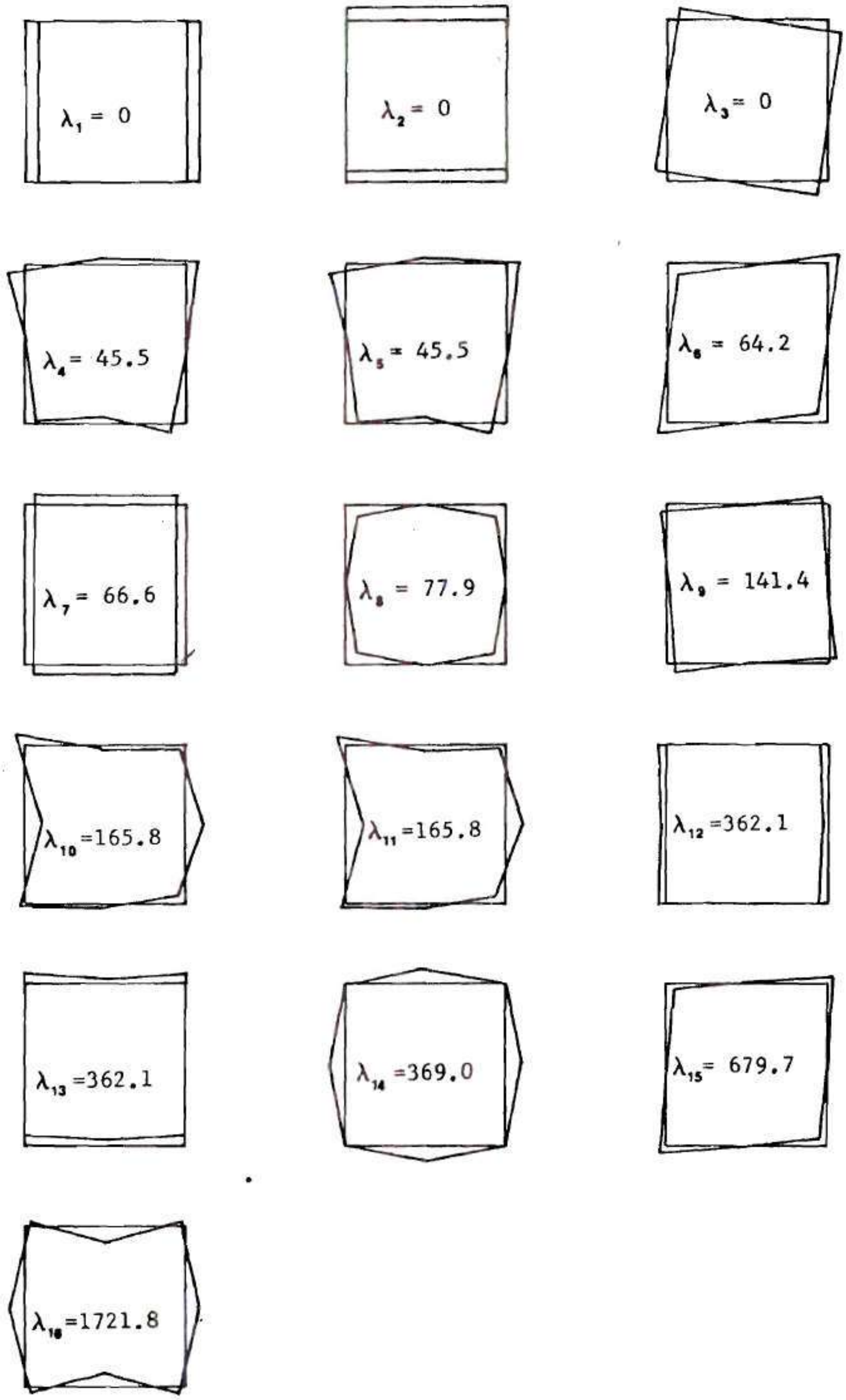


Fig.21 Eigen-Modes of the Stiffness Matrix of Eight-Noded Element (Incompressible, Plane-Stress)

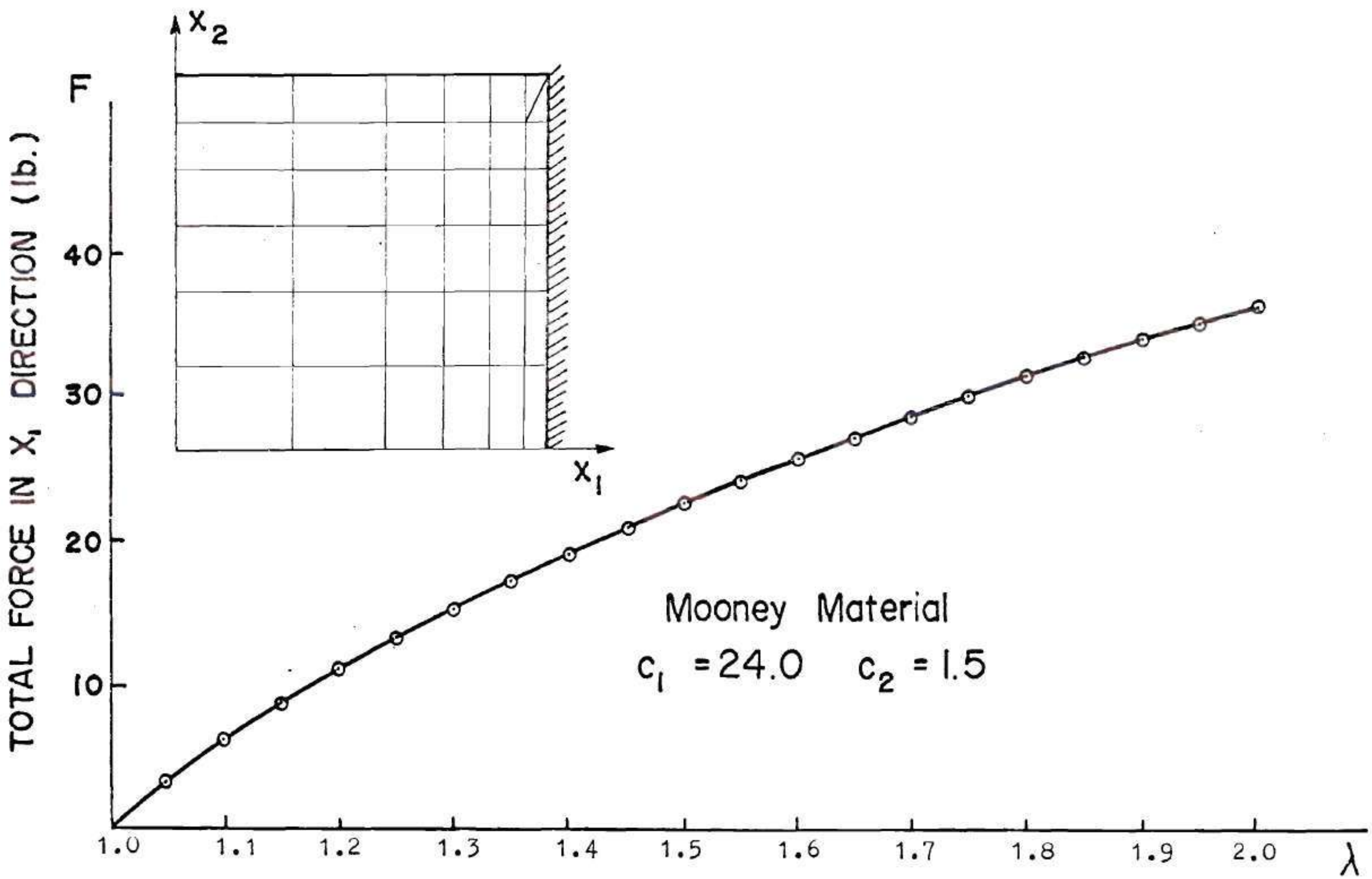
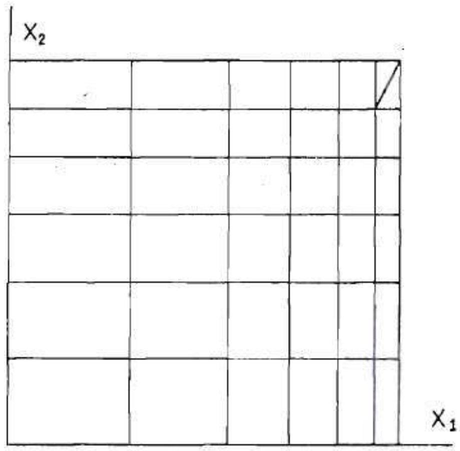
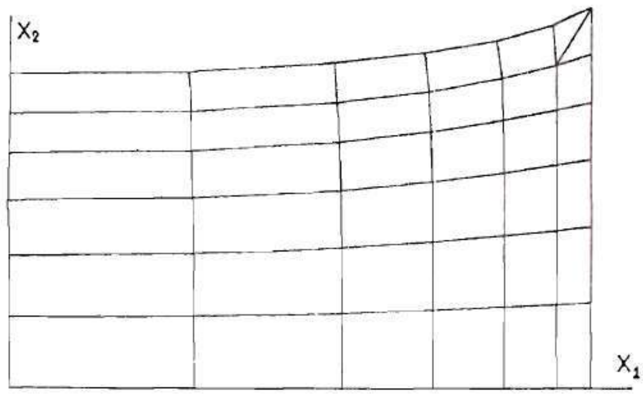


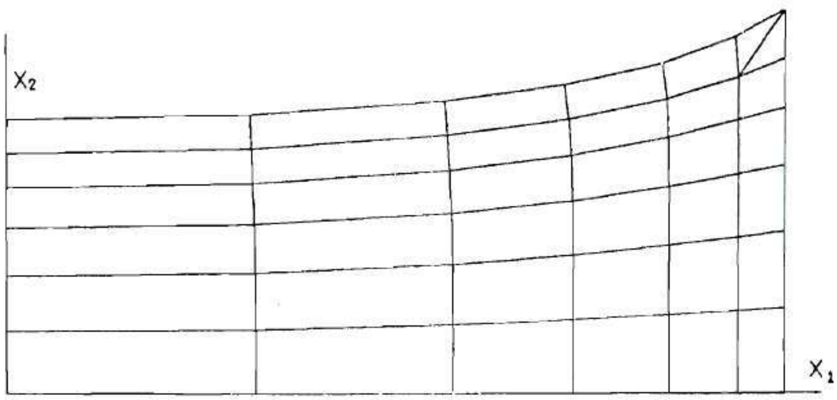
Fig.22 Total Edge Load Versus Axial Extension Ratio



(Initial Configuration)



(Deformed Configuration at λ = 1.5)



(Deformed Configuration at λ = 2.0)

Fig.23 Deformed Configurations of a Square Sheet

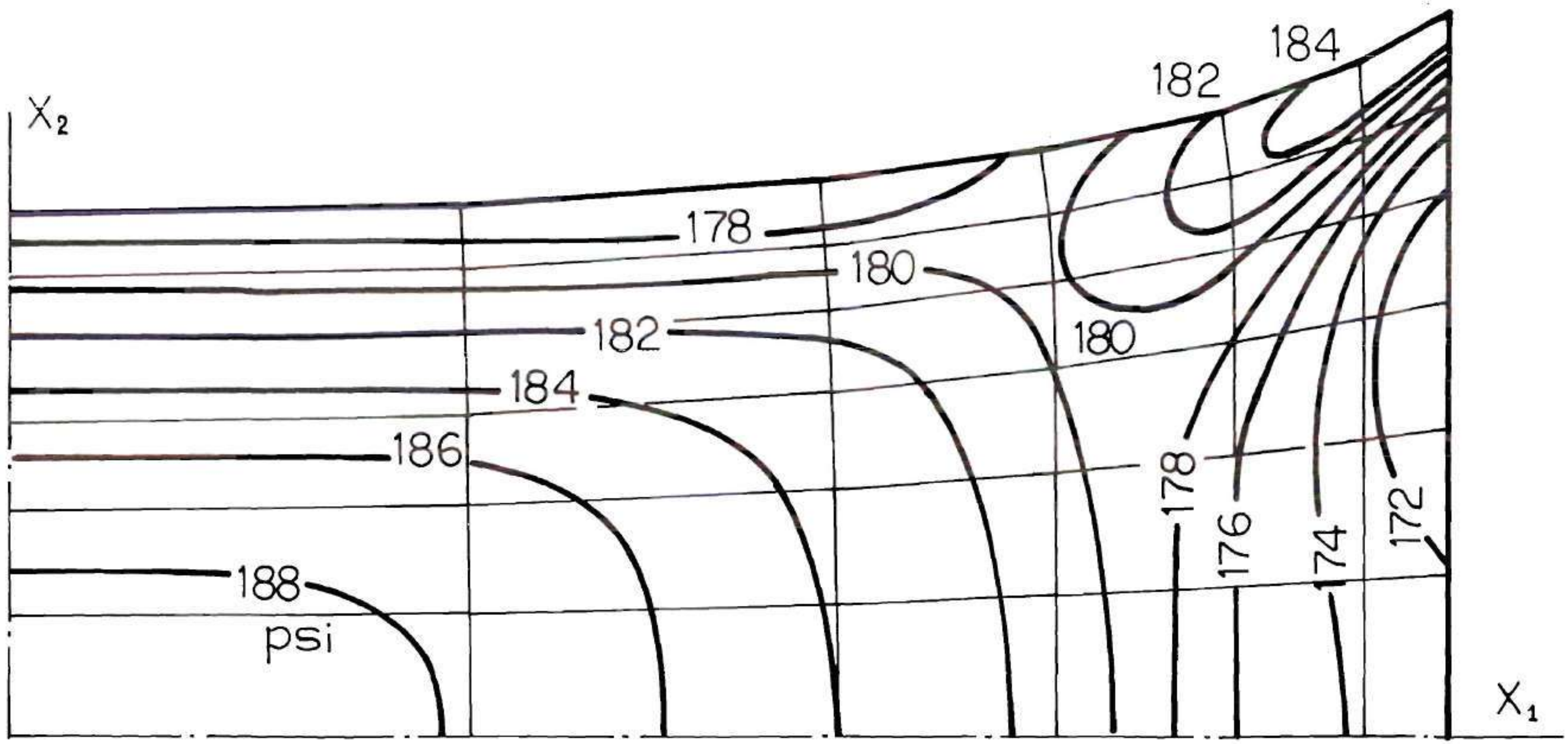


Fig.24 Contours of Axial Component of Cauchy Stress τ_{11} at $\lambda = 2.0$

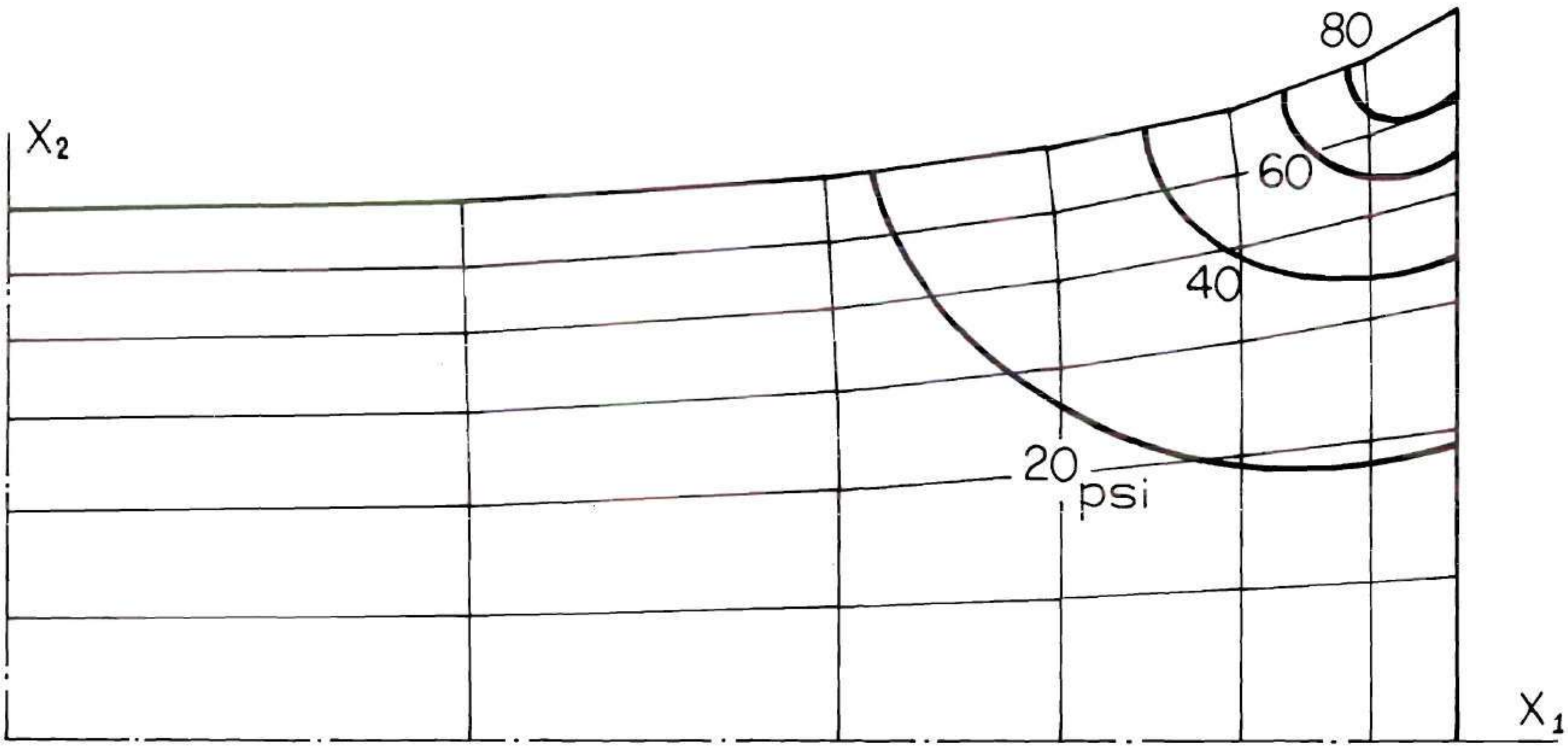


Fig.25 Contours of Shear Component of Cauchy Stress τ_{12} at λ=2.0

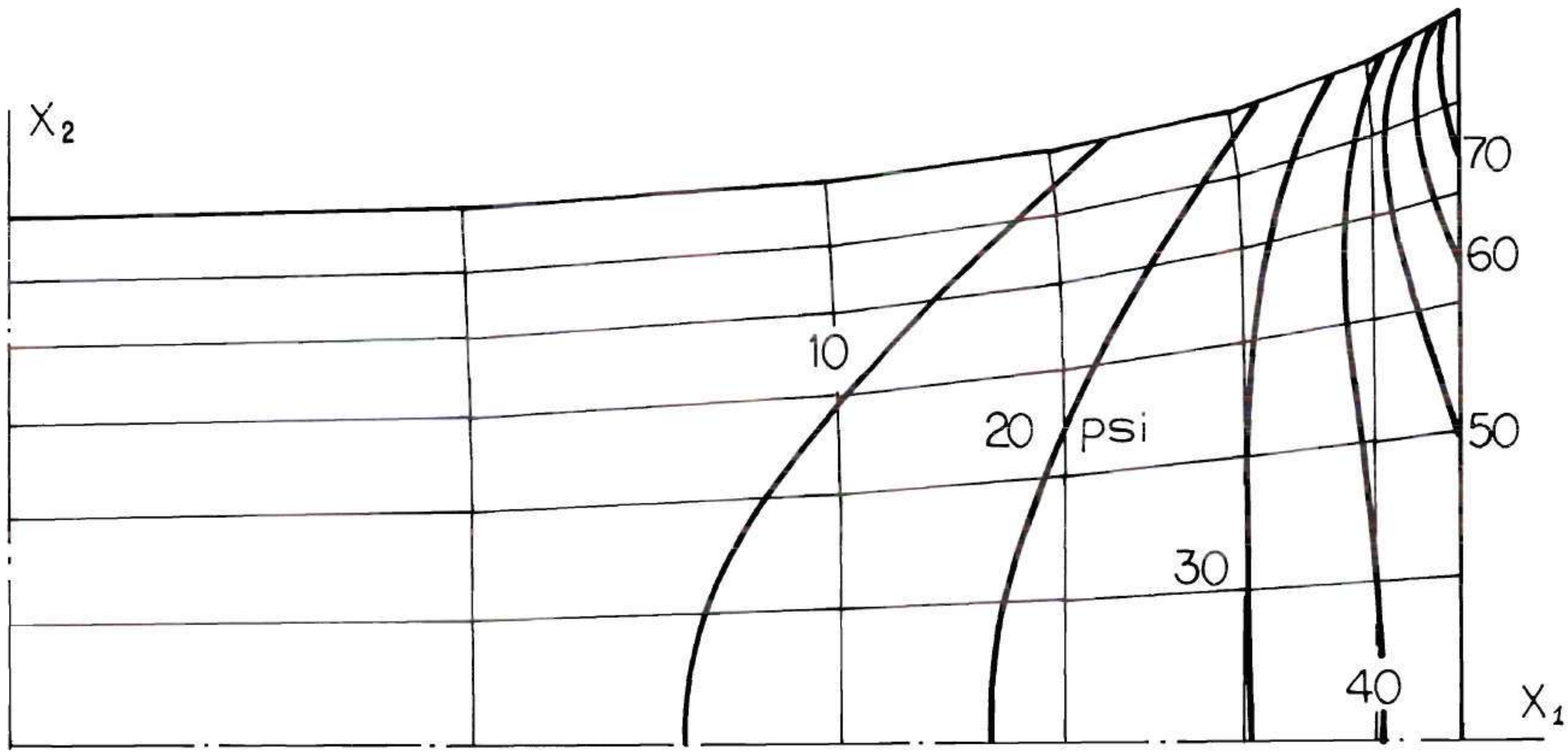


Fig.26 Contours of Lateral Component of Cauchy Stress τ_{22} at $\lambda = 2.0$

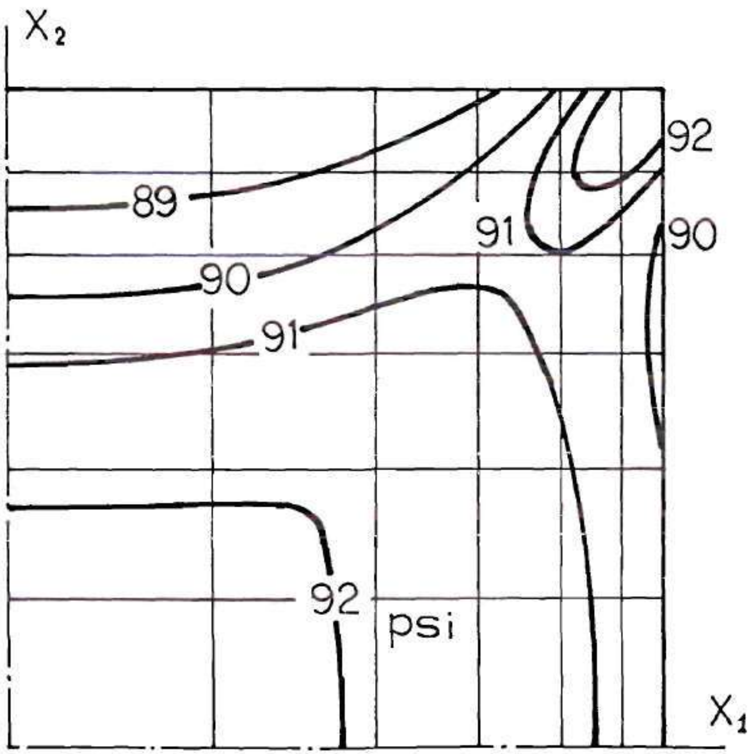


Fig.27 Contours of Axial Component of Piola-Lagrange Stress t_{11} at $\lambda = 2.0$

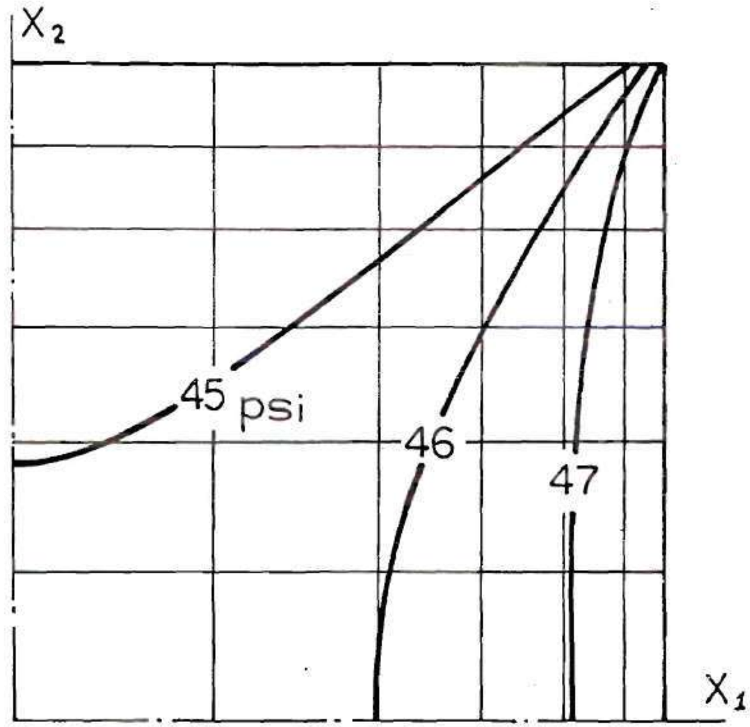


Fig.28 Contours of Axial Component of Kirchhoff-Trefftz Stress S_{11} at $\lambda = 2.0$

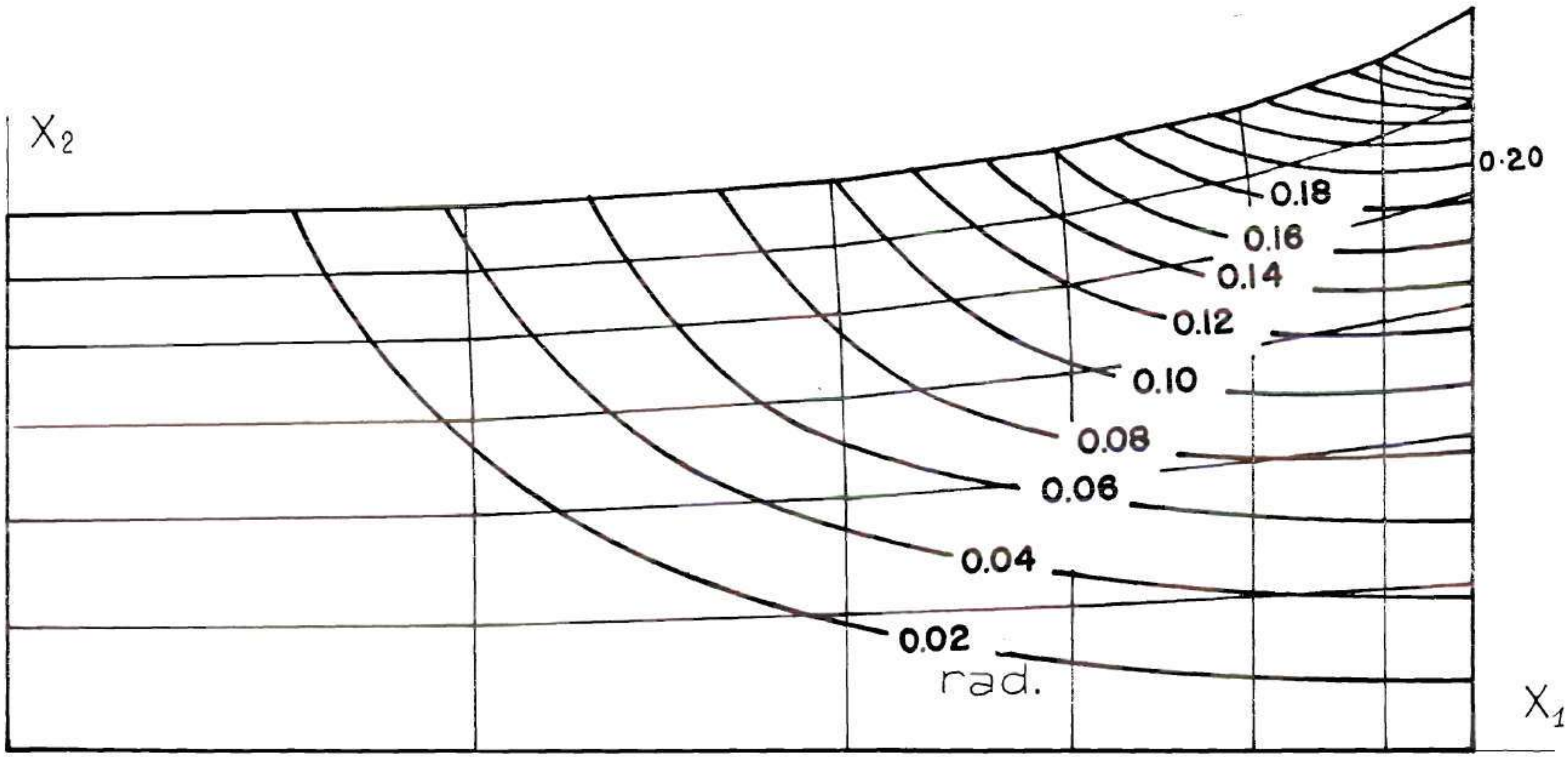


Fig.29 Contours of Rotation Angle θ at $\lambda = 2.0$

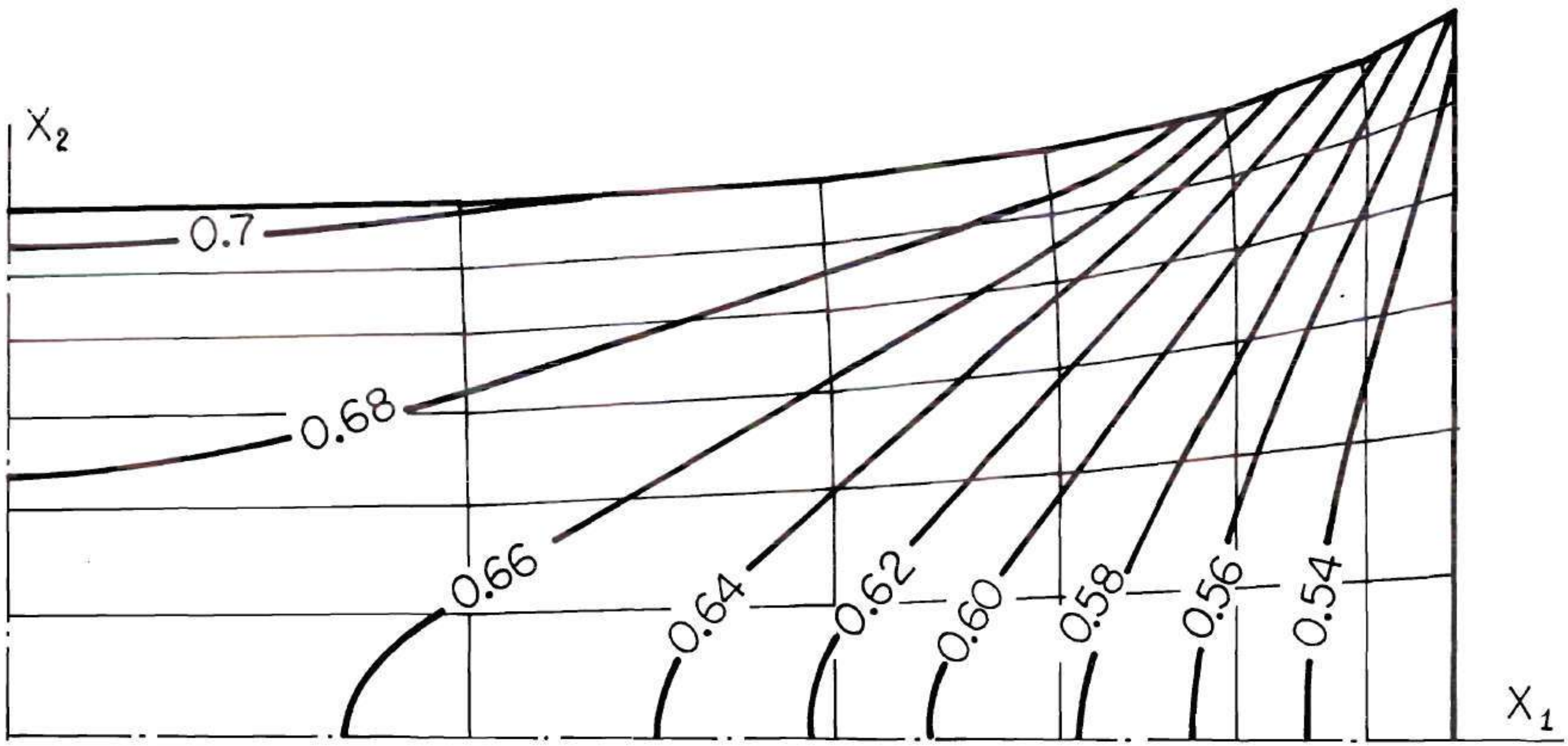


Fig.30 Contours of the Extension Ratio in the Thickness Direction, $(1+h_{33})$, at $\lambda = 2.0$

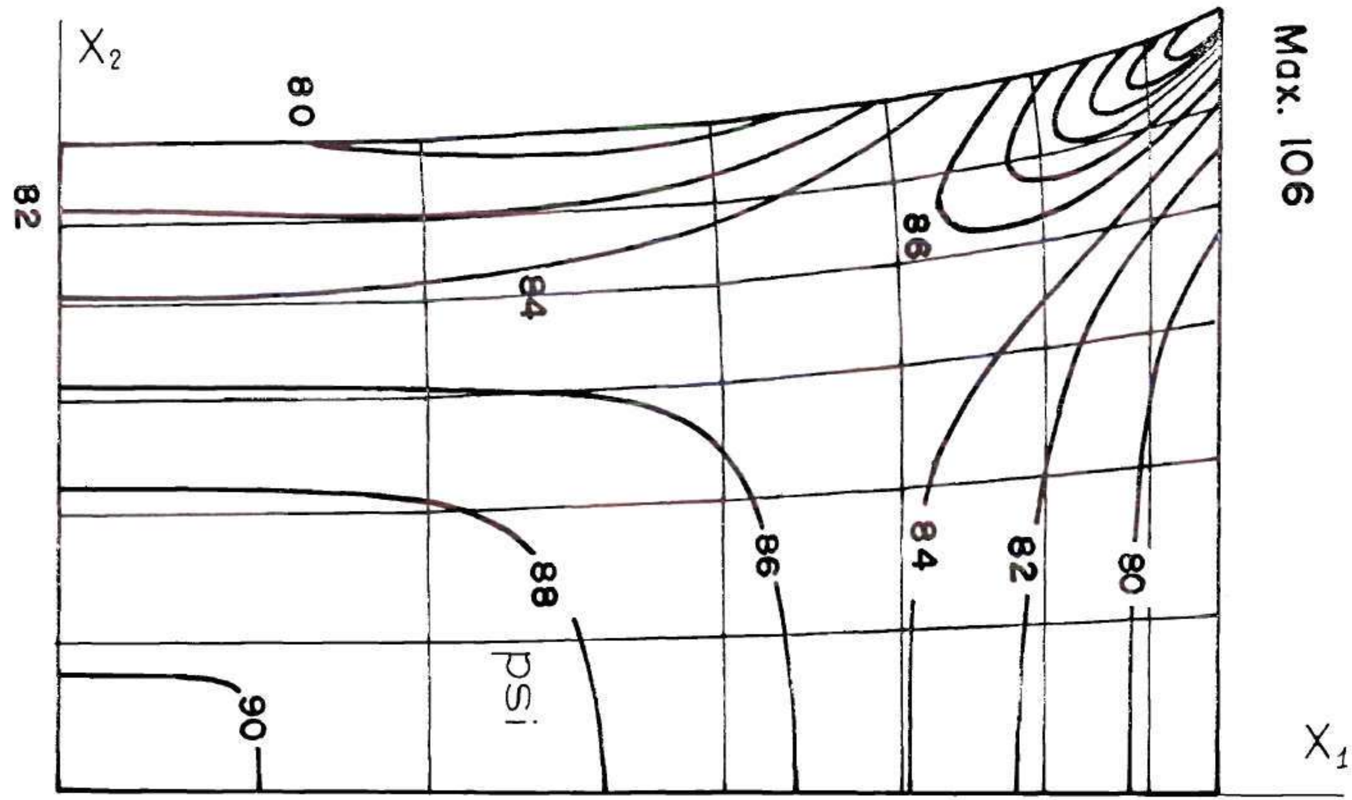


Fig.31 Contours of Axial Component of Cauchy Stress τ_{11} at $\lambda = 1.5$

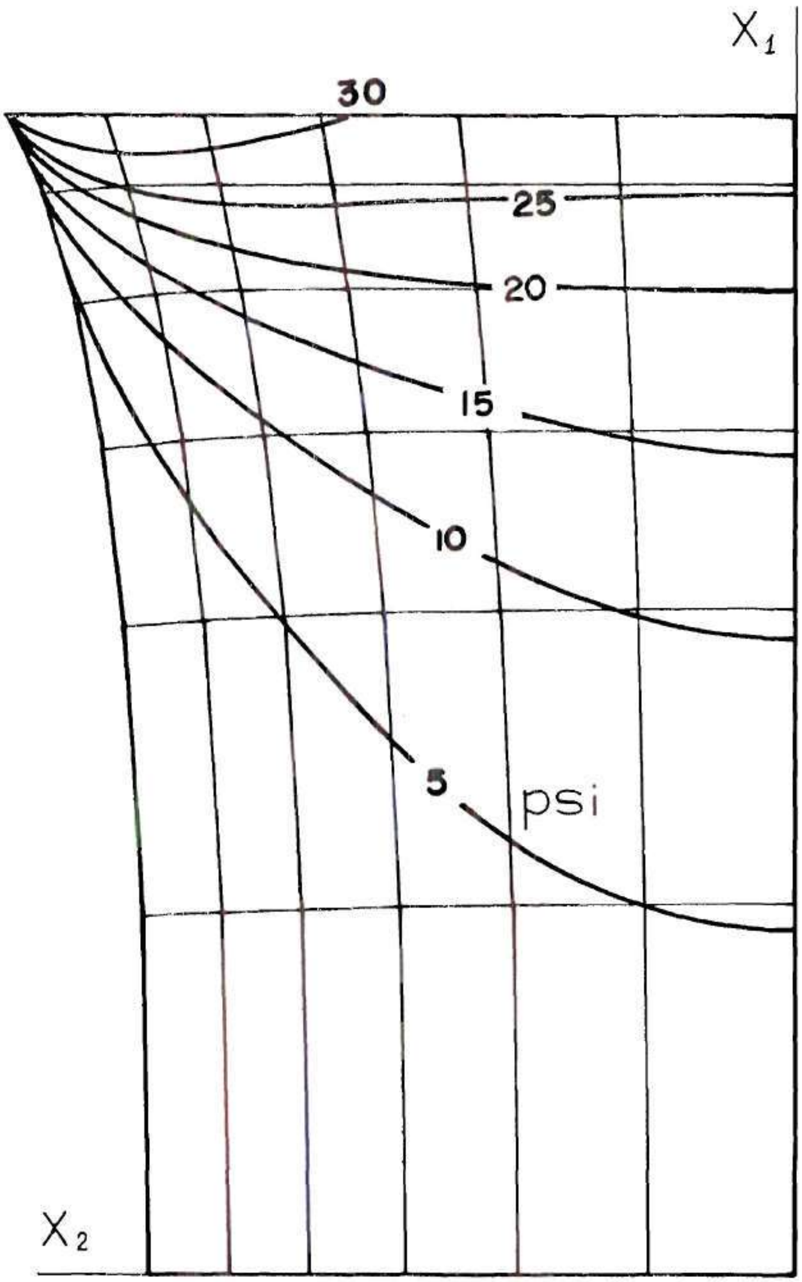


Fig.32 Contours of Lateral Component of Cauchy Stress τ_{22} at $\lambda = 1.5$

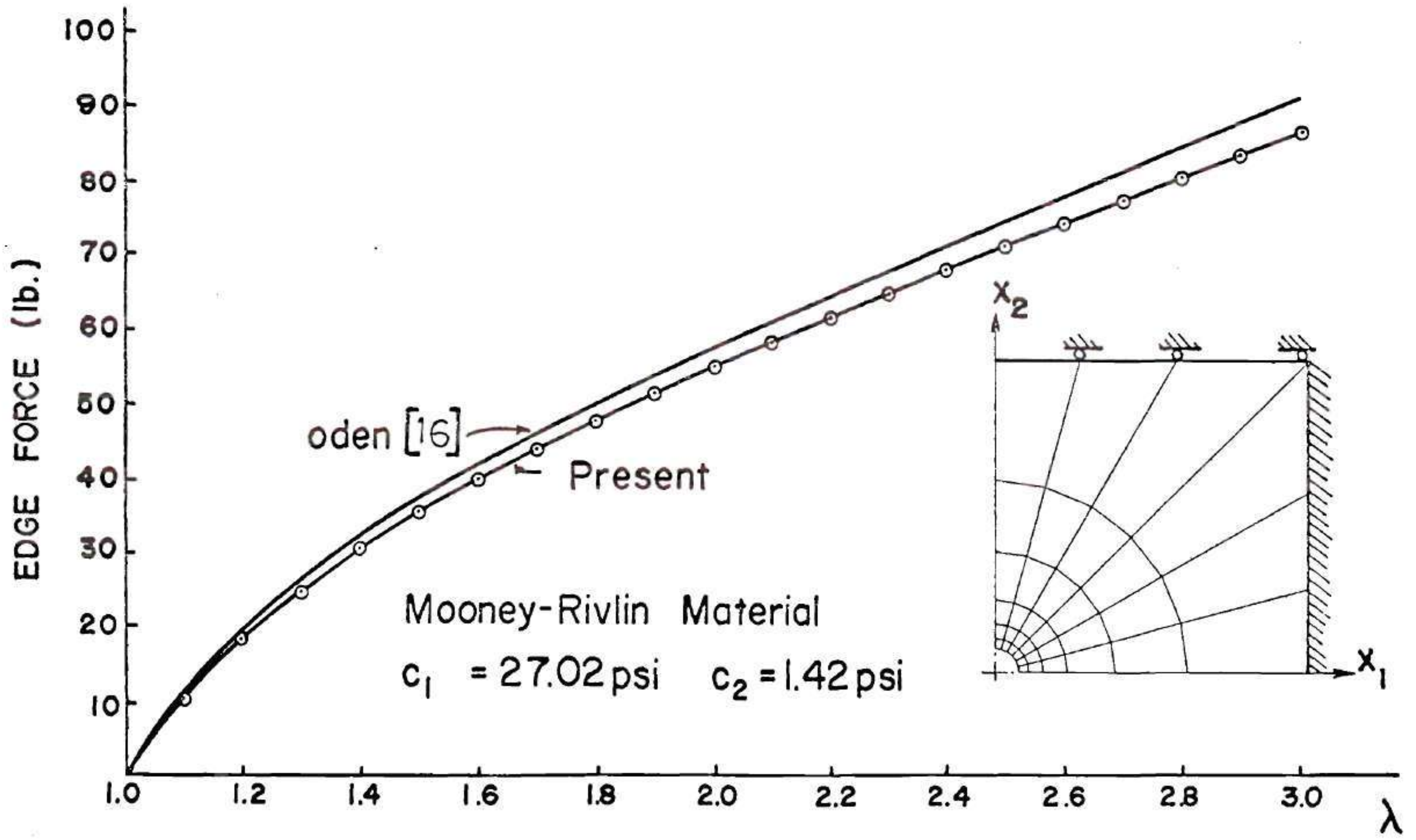
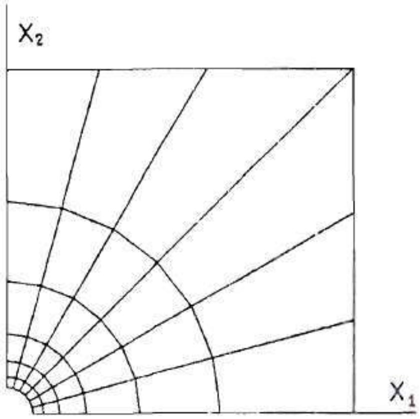
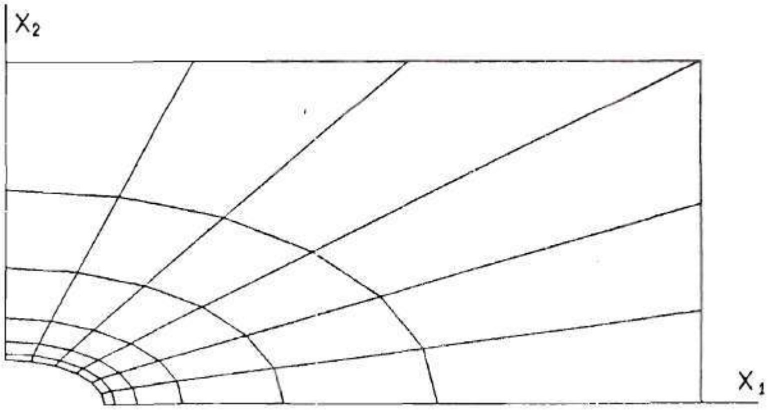


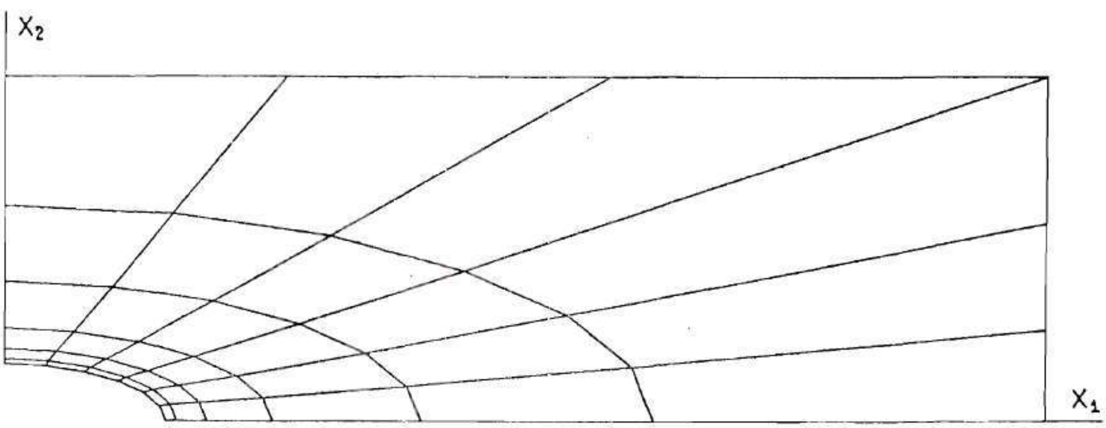
Fig.33 Total Edge Force Versus Axial Extension Ratio



(Initial Configuration)



(Deformed Configuration at $\lambda = 2.0$)



(Deformed Configuration at $\lambda = 3.0$)

Fig.34 Deformed Configurations of a Square Sheet with a Circular Hole

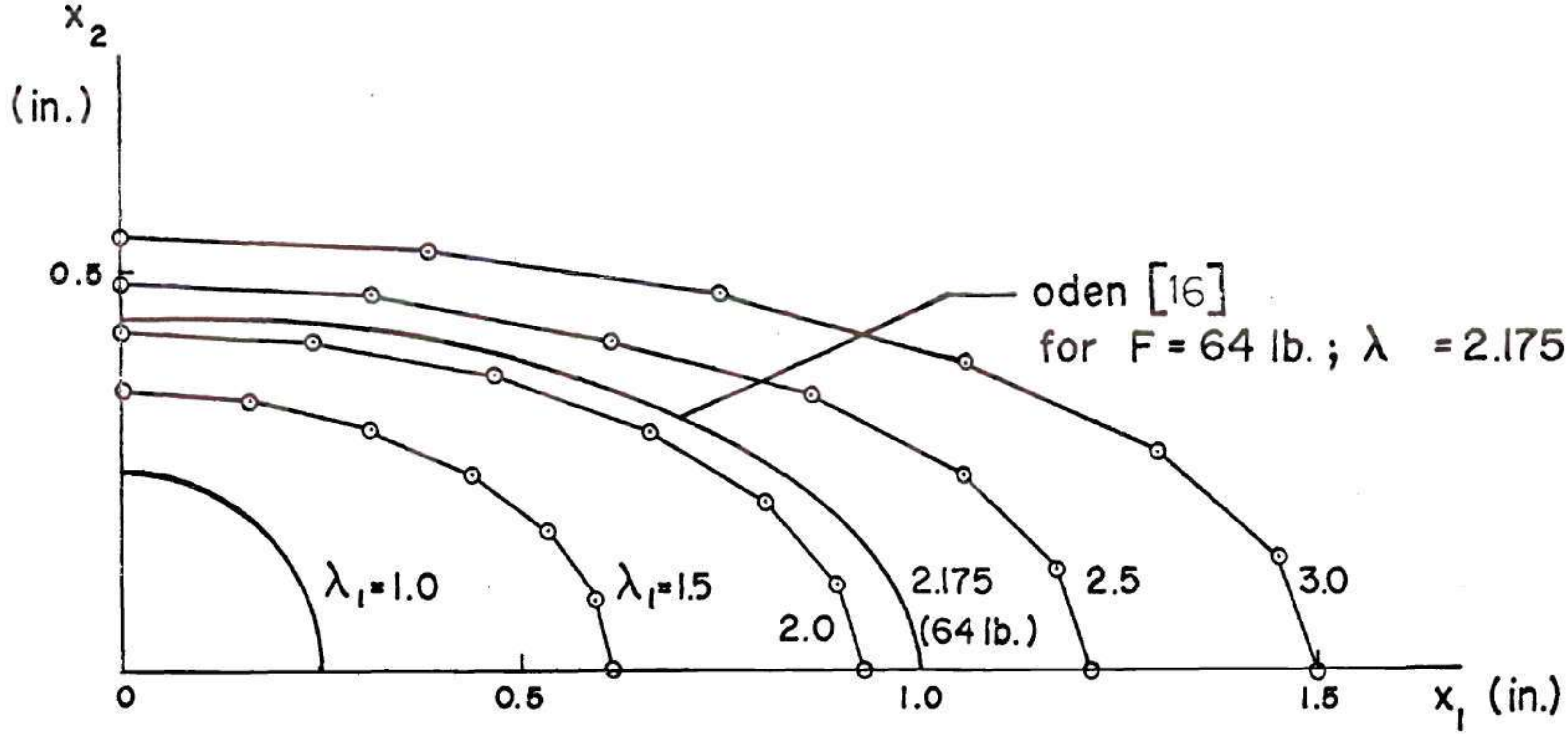


Fig.35 Deformed Profiles of the Circular Hole

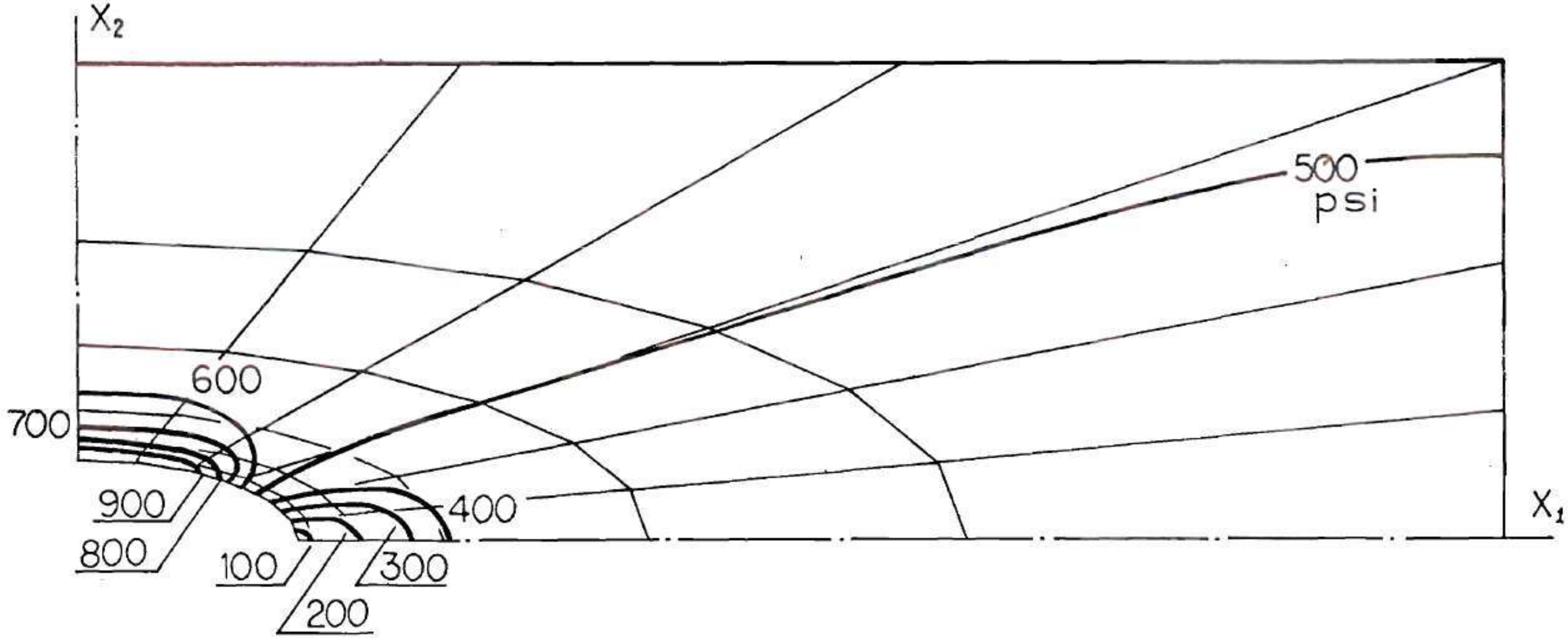


Fig.36 Contours of Axial Component of Cauchy Stress τ_{11} at λ = 3.0

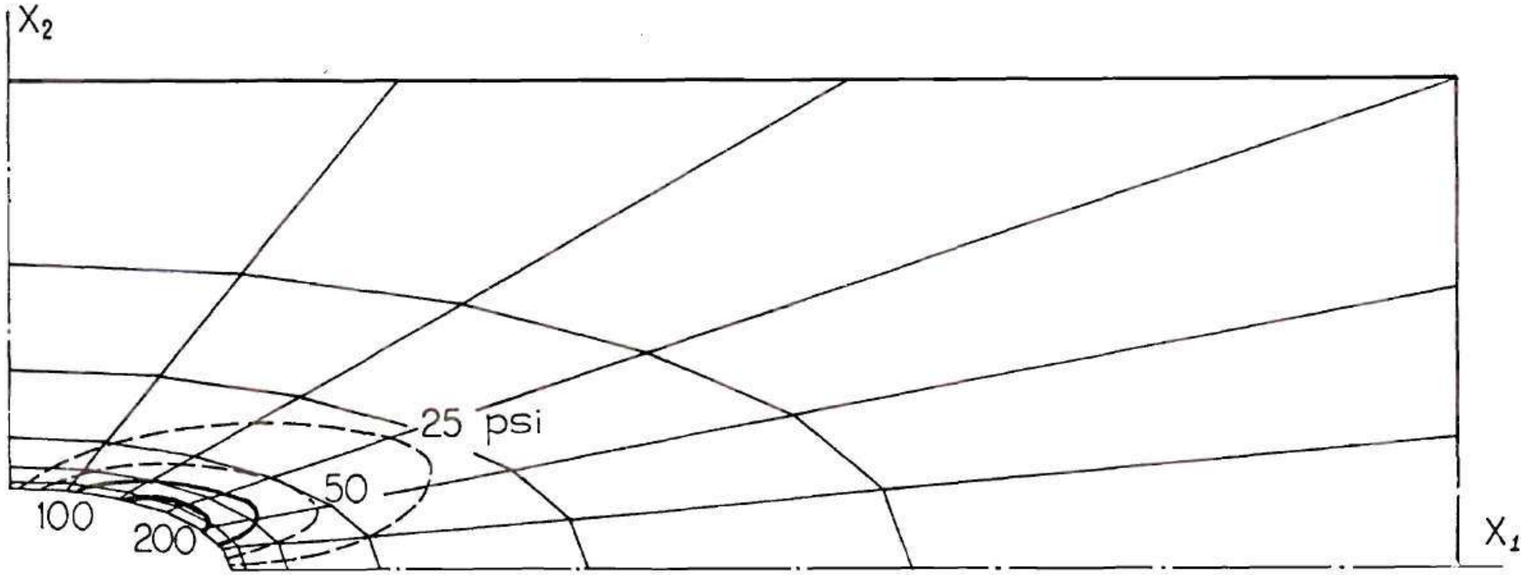


Fig.37 Contours of Shear Component of Cauchy Stress τ_{12} at $\lambda = 3.0$

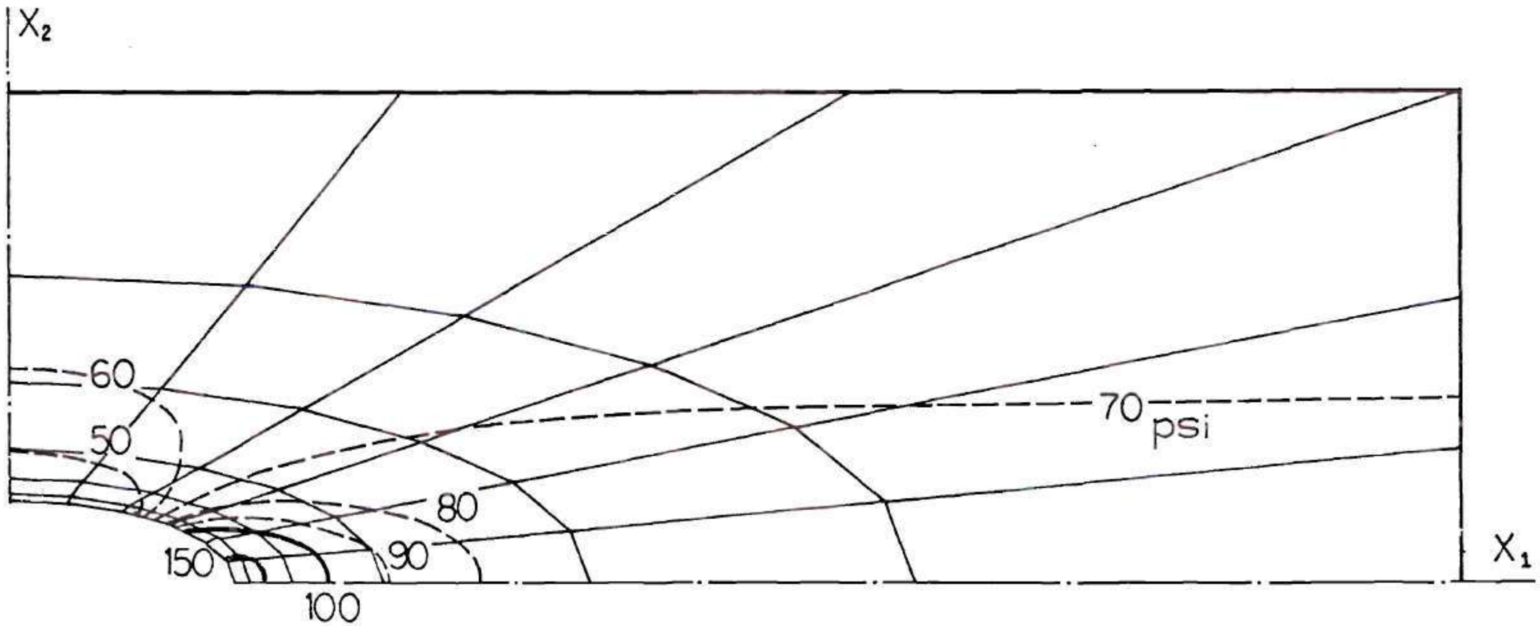


Fig.38 Contours of Lateral Component of Cauchy Stress τ_{22} at $\lambda = 3.0$.

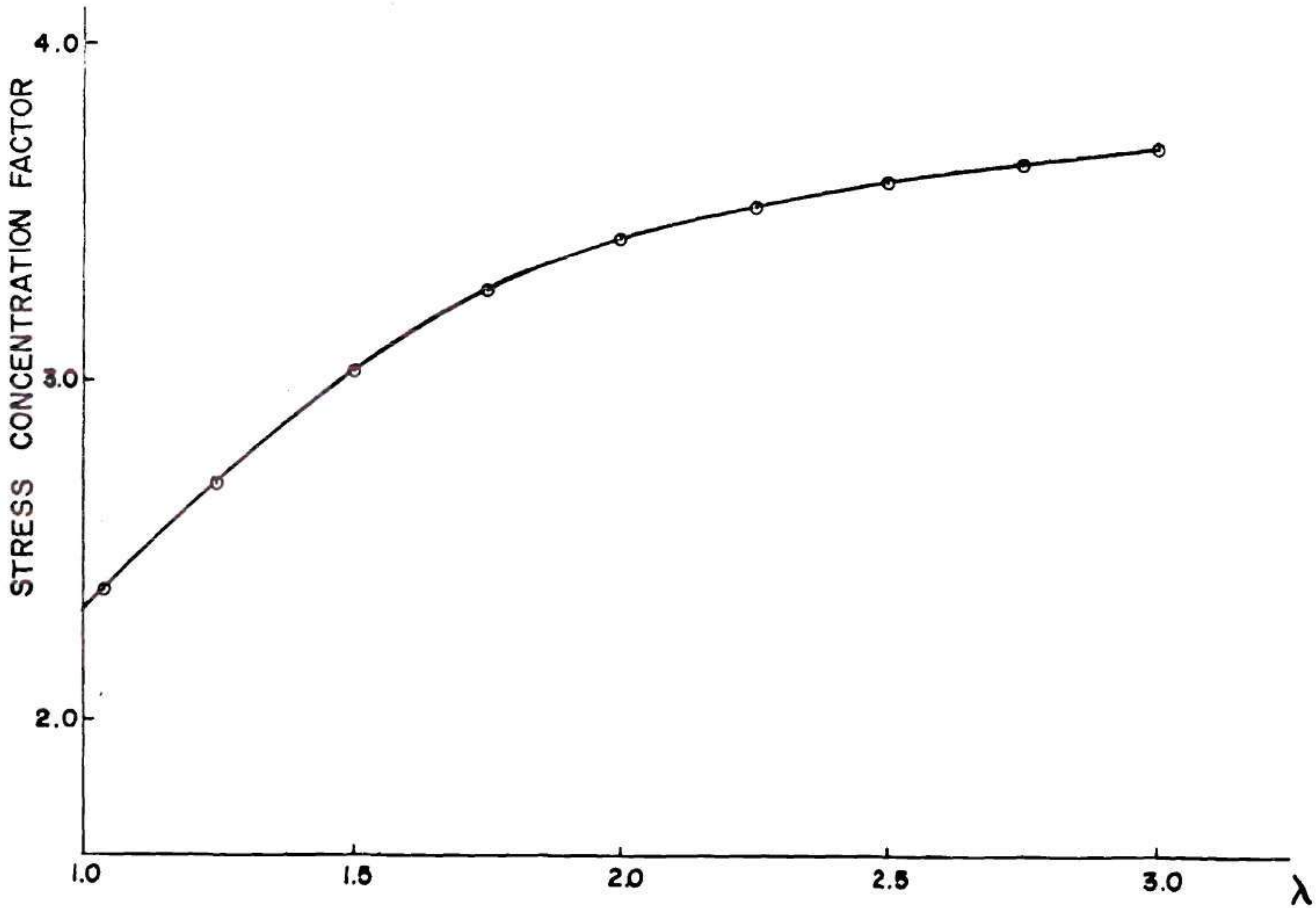


Fig.39 Stress Concentration in a Square Sheet with a Circular Hole

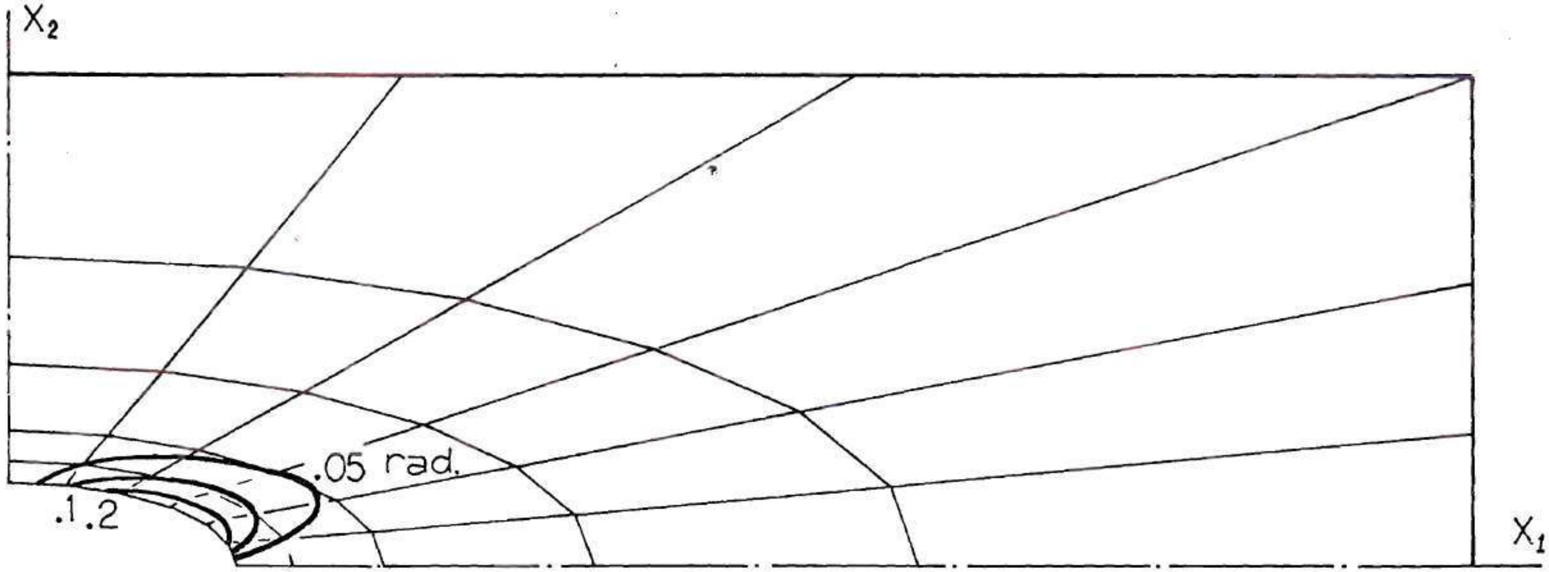


Fig. 40 Contours of Rotation Angle θ at $\lambda = 3.0$

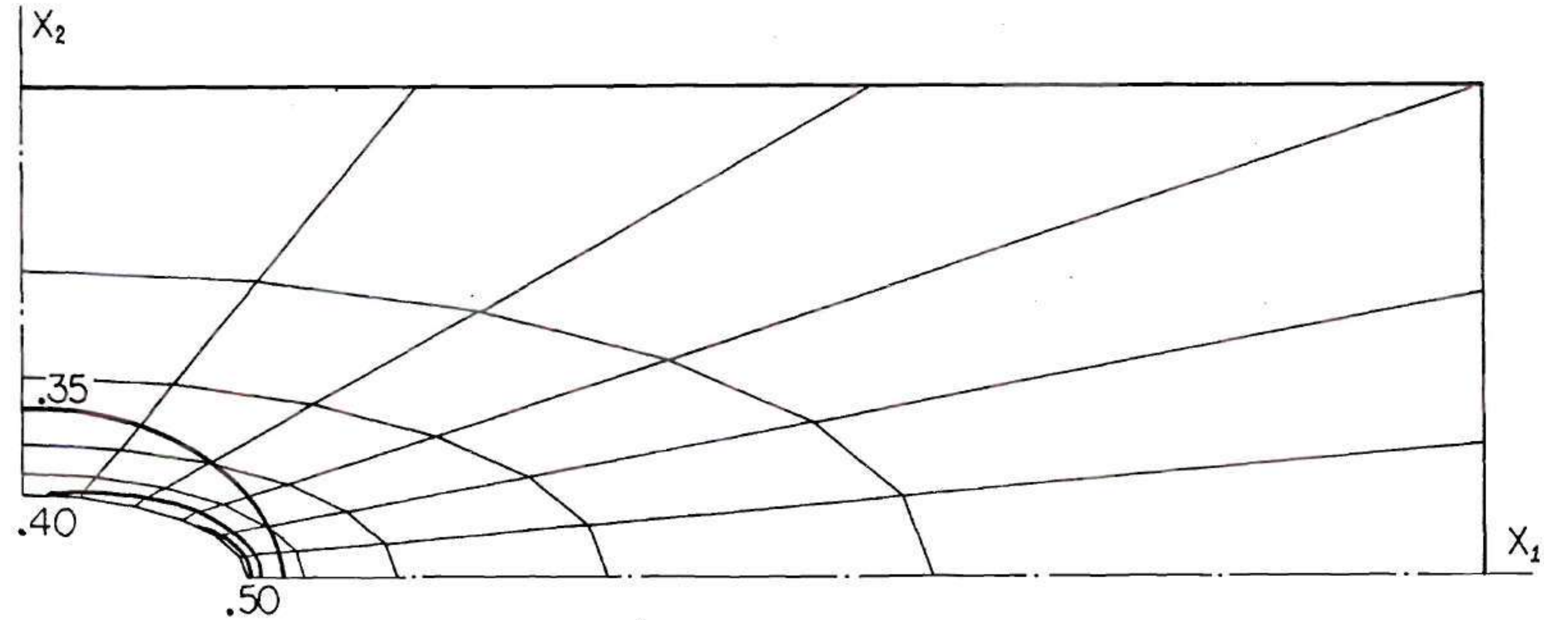


Fig.41 Contours of the Extension Ratio in the Thickness Direction, $(1+h_{33})$, at $\lambda = 3.0$

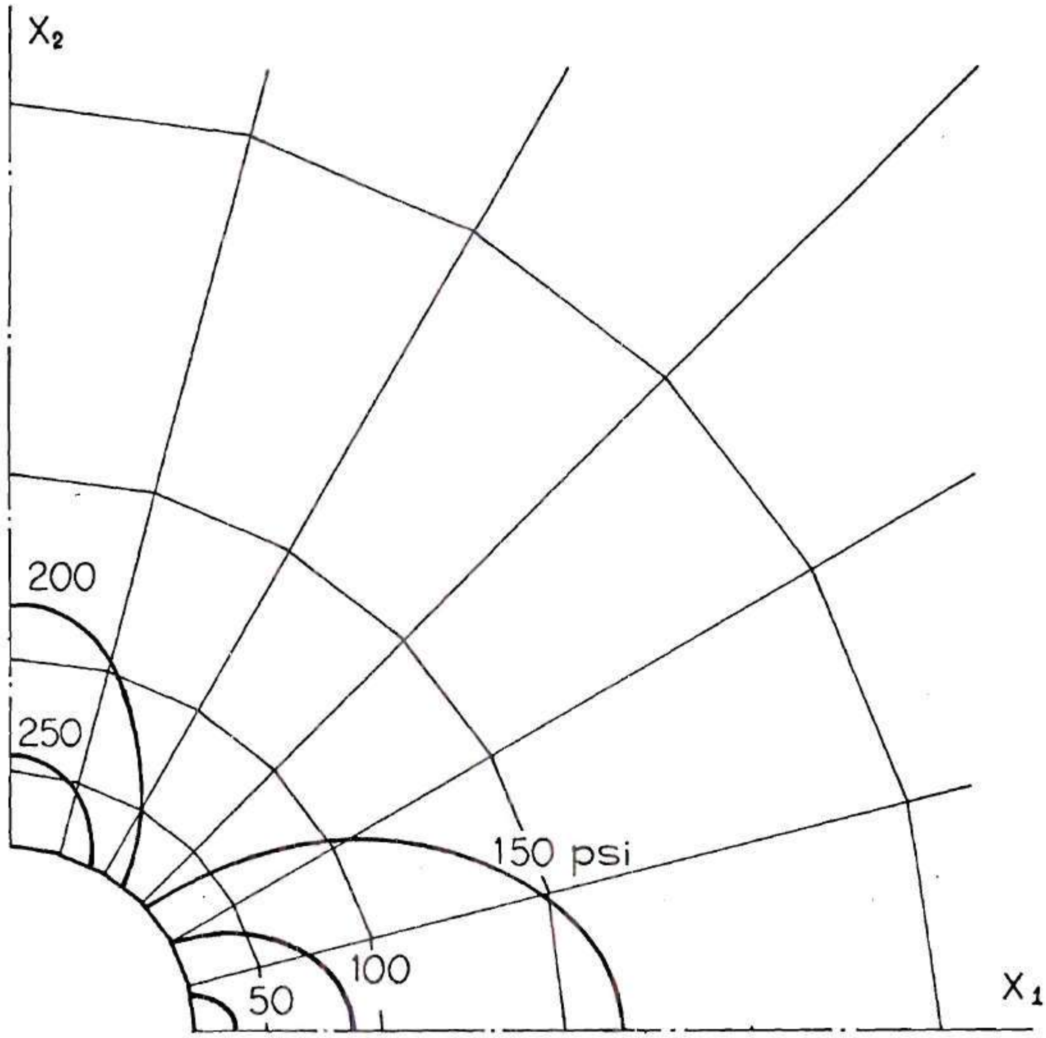


Fig.42 Contours of Axial Component of Piola-Lagrange Stress t_{11} at $\lambda = 3.0$

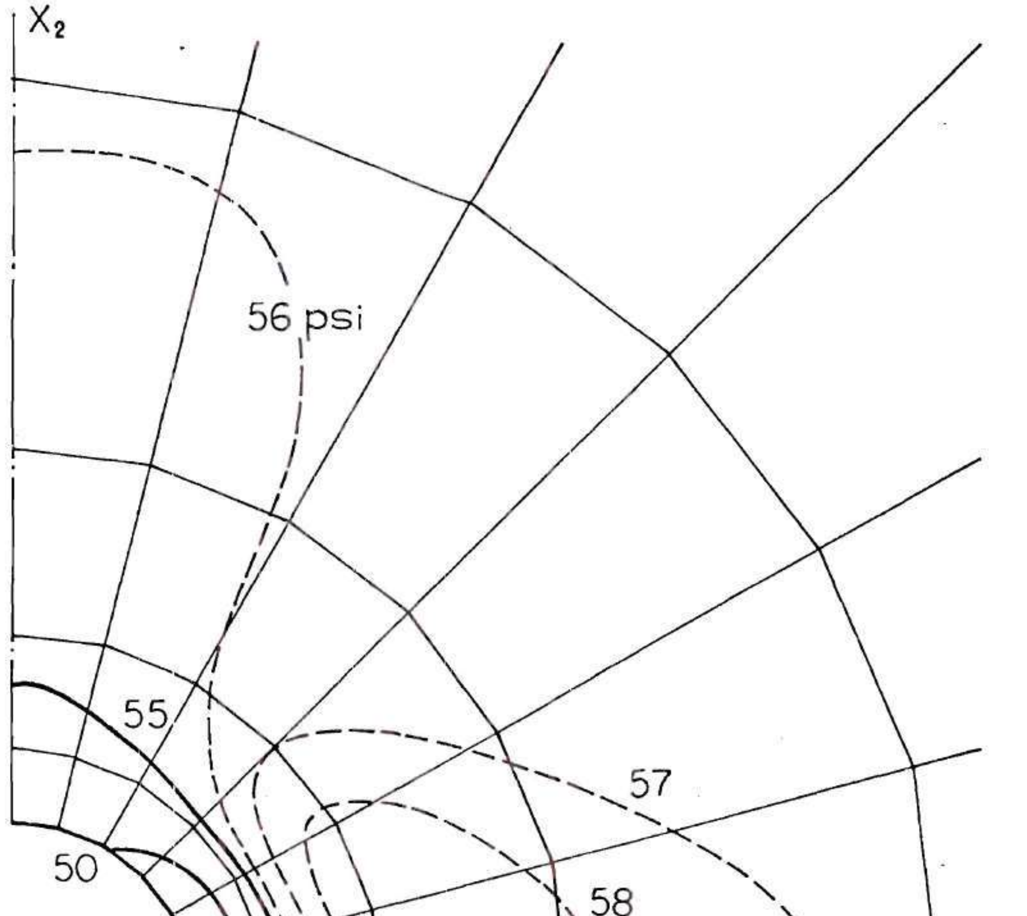


Fig.43 Contours of Axial Component of Kirchhoff-Treffts Stress S_{11} at $\lambda = 3.0$

APPENDIX D

TABLES

Table 1. Eigen-Values of Stiffness Matrices of the Four-Noded Plane-Stress Element (Compressible)

(a,b,c)	λ_4	λ_5	λ_6	λ_7	λ_8
(4,1,8)	0*	0*	.769	.769	1.28
(4,3,8)	H	becomes singular			
(10,1,8)	.769	.769	1.28	1.37	1.37
(10,3,8)	.373	.373	.769	.769	1.28
(18,1,8)	.769	.769	1.28	1.37	1.37
(18,3,8)	.373	.373	.769	.769	1.28

(* : physically improper eigen-values)

Table 2. Eigen-Values of Stiffness Matrices of the Four-Noded Plane-Stress Element (Incompressible)

(a,b,d,c)	λ_4	λ_5	λ_6	λ_7	λ_8
(10,1,1,8)	15.1*	15.1*	102	102	306
(10,1,3,8)	102	102	272	272	306
(10,1,6,8)	102	102	272	272	306
(10,3,1,8)	12.4*	12.4*	102	102	306
(10,3,3,8)	54.4	54.4	102	102	306
(10,3,6,8)	54.4	54.4	102	102	306
(18,1,1,8)	15.1*	15.1*	102	102	306
(18,1,3,8)	102	102	272	272	306
(18,1,6,8)	102	102	272	272	306
(18,3,1,8)	12.4*	12.4*	102	102	306
(18,3,3,8)	54.4	54.4	102	102	306
(18,3,6,8)	54.4	54.4	102	102	306

(* : physically improper eigen-values)

Table 3. Eigen-Values of Stiffness Matrices of the Eight-Noded Plane-Stress Element (Incompressible)

(a,b, d, c)	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}
(18,1, 1,16)	0*	0*	-70.4*	-70.4*	76.6	79.1	-79.6*	83.7	-140*	141	146	146	367
(18,1, 3,16)	0*	0*	76.6	79.1	-79.6*	83.7	103	103	-140*	141	367	1257	1257
(18,1, 6,16)	0*	0*	64.2	66.6	77.9	103	103	141	369	680	1257	1257	1722
(18,3, 1,16)	0*	0*	76.6	79.1	-79.6*	-81.9*	-81.9*	83.7	-140*	141	144	144	367
(18,3, 3,16)	0*	0*	50.0	50.0	76.6	79.1	-79.6*	83.7	-140*	141	222	222	367
(18,3, 6,16)	0*	0*	50.0	50.0	64.2	66.6	77.9	141	222	222	369	680	1722
(28,1, 1,16)	0.33*	0.33*	-70.2*	-70.2*	76.6	79.1	-79.6*	83.7	-140*	141	146	146	367
(28,1, 3,16)	0.33*	0.33*	76.6	79.1	-79.6*	83.7	103	103	-140*	141	367	1257	1257
(28,1, 6,16)	0.33*	0.33*	64.2	66.6	77.9	103	103	141	369	680	1257	1257	1722
(28,1,10,16)	64.2	66.6	77.9	102	102	141	309	309	369	680	1574	1574	1722
(28,3, 1,16)	0.33*	0.33*	76.6	79.1	-79.6*	-81.8	-81.8*	83.7	-140*	141	144	144	367
(28,3, 3,16)	0.33*	0.33*	50.0	50.0	76.6	79.1	-79.6*	83.7	-140*	141	222	222	367
(28,3, 6,16)	0.33*	0.33*	50.0	50.0	64.2	66.6	77.9	141	222	222	369	680	1722
(28,3,10,16)	45.5	45.5	64.2	66.6	77.9	141	166	166	362	362	369	680	1722

(* : physically improper eigen-values)

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