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SOLUTIONS OF SOME COUNTABLE SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A THESIS

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SOLUTIONS OF SOME COUNTABLE SYSTEMS OF

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SUMMARY

The major goal of the investigation is the use of recursively generated orthogonal polynomials for constructing solutions to initial-value problems of certain countable systems of ordinary differential equations. The infinite systems considered have the form $y' = My$ or $y'' = My$, where M is an infinite tri-diagonal matrix of real constants. For any positive integer N, an N^{th}-order tri-diagonal matrix M_N can be formed from the infinite tri-diagonal matrix M simply by deleting from M all but its first N rows and columns. The analysis concentrates on the two natural pairs

(i) $y' = My$ with its truncation $y_N' = M_N y_N$,

(ii) $y'' = My$ with its truncation $y_N'' = M_N y_N$,

where y_N denotes a column vector of N functions.

The elements of the infinite matrix M are used to determine the coefficients in a three-term recurrence of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \; n \geq 0$$

(where $P_{-1} \equiv 0$ and $P_o \equiv 1$), which then generates a sequence $\{P_n(x)\}$ of polynomials; a simple (necessary and sufficient) condition on these elements insures that the corresponding polynomials P_n are orthogonal. For each of the two pairs of systems (i) and (ii) a solution y of the infinite system and the solution y_N of the finite truncation are given in terms of the orthogonal polynomials P_n (when they exist). Estimates of the difference between the n^{th} components of y and y_N (for any $n \leq N$)

are deduced, so that the initial N-segment of y can be used as an approximation to y_N. The procedure for solving a given infinite system (or for approximating the solution of a given finite system with the furnished solution of an infinite system for which the finite system is a truncation) is illustrated in two detailed examples.

Of necessity a considerable part of the discussion is devoted to orthogonality of the recursively generated polynomials P_n. The known cases of the classical Sturm-Liouville polynomials are mentioned and, in addition, a one-parameter family of non-classical polynomials is introduced; it is shown that the latter are orthogonal over the interval $[-1,1]$ with weight $w(x) = |x|^\alpha$ (where $\alpha > -1$ and $\alpha \neq 0$).

The countable systems of differential equations considered may be used as models for various physical systems -- for example, coupled harmonic oscillators or ladder networks. A number of these physical systems, accompanied by their mathematical models with solutions, are catalogued in an appendix.

CHAPTER I

INTRODUCTION

The pages to follow recount the results of an investigation which is closely related to previous work by J. W. Jayne [13] and F. L. Cook [4]. As might be inferred from the title, the principal object is to exhibit solutions of some countable systems of linear ordinary differential equations with constant coefficients. A number of the problems considered are, in a suitable sense, mathematical models of readily conceivable physical systems; two of these models have been investigated in some detail by H. W. Gatzke [6].

For denumerable systems of ordinary differential equations with constant coefficients [9], theorems on existence, uniqueness and properties of solutions are known in numerous cases [11,12,15-20]. In the present study such questions are not considered. No existence theorems are proved; no conditions are stated which are sufficient to guarantee the uniqueness of a solution; and there is no listing of properties of solutions. The goal is simply to construct solutions -- in the elementary sense of finding sequences of sufficiently differentiable functions which reduce all the differential equations to identities and satisfy prescribed initial conditions. Whenever the goal is attained, an existence theorem has clearly been proved.

Various properties of orthogonal polynomials and several ideas associated with the classical moment problem* are frequently used. The explanation begins with these concepts, since orthogonal polynomials generated by suitable three-term recurrence relations play a crucial role in the method presented for solving the infinite differential systems discussed.

A three-term recurrence

$$P_0(x) = 1 ,$$

$$P_1(x) = A_0 x + B_0 ,$$

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \; n \geq 1 , \qquad (1)$$

where A_n $(n \geq 0)$, B_n $(n \geq 0)$ and C_n $(n \geq 1)$ are real numbers for which $A_n C_{n+1} \neq 0$ $(n = 0,1,2,\ldots)$, generates a sequence $\{P_n\}$ of polynomials in which P_n is of degree exactly n. Some (but not all) sequences so generated consist of orthogonal polynomials associated with a distribution $d\alpha(x)$ over some interval $[a,b]$ of the real line; that is, they are polynomials for which there exists an integrator $\alpha(x)$ such that in the Stieltjes sense

$$\int_a^b P_i(x)P_j(x)d\alpha(x) = 0, \quad i \neq j , \qquad (2)$$

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*The classical moment problem may be stated as follows: For a given sequence $\{\mu_n\}$ $(n \geq 0)$ with $\mu_0 = 1$, what properties of the sequence will insure the existence of an integrator $\alpha(x)$ of a prescribed type [3] over some interval $[a,b]$ so that $\mu_n = \int_a^b x^n \, d\alpha(x)$ for $n = 0,1,2,\ldots$? When such an integrator exists, the μ_n $(n \geq 0)$ are called the <u>moments of $d\alpha(x)$ over $[a,b]$</u>.

where $\alpha(x)$ is bounded, is non-decreasing and assumes infinitely many different values over $[a,b]$. It is demonstrated that a necessary and sufficient condition for such orthogonality of the polynomials P_n is that the coefficients in the recurrence (1) satisfy the relation

$$\frac{C_n}{A_n A_{n-1}} > 0, \quad n = 1,2,3,\ldots . \tag{3}$$

Whenever the distribution $d\alpha(x)$ has the property that $d\alpha(x) = \omega(x)dx$, where $\omega(x)$ is non-negative and Riemann integrable (perhaps improperly) on $[a,b]$, the Stieltjes integral (2) reduces to a Riemann integral and the polynomials P_n are orthogonal polynomials with weight $\omega(x)$ over the interval $[a,b]$. The weight and interval can be determined, for example, if the coefficients in the recurrence (1) are such that the polynomials P_n constitute a Sturm-Liouville sequence [10]. The general problem of finding a practicable construction for the weight and interval (when they exist) in terms of the coefficients in (1) is as yet, however, unsolved.*

When (3) holds, the polynomials P_n can be used to construct solutions to countable systems of ordinary differential equations in which the (constant) coefficients are intimately related to the coefficients in the recurrence (1). The explanation of this remark comprises a large

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*A related problem of some interest is that of specifying reasonable conditions on the coefficients in the recursion (1) which are sufficient to insure that the polynomials P_n are orthogonal in the sense (2) on a <u>finite</u> interval. Some necessary and sufficient conditions for the existence of certain types of weights over finite intervals are known in terms of prospective moments of a weight [2], but it appears difficult to reformulate these conditions in a practicable way in terms of the coefficients in the recurrence (1).

part of the body of the thesis. However, some indication of the connection between the orthogonal polynomials and the differential equations is given in the following paragraph.

The countably infinite systems considered are initial-value problems for differential equations of the form $y' = My$ or $y'' = My$, where M is an infinite tri-diagonal matrix of constants. For any positive integer N, an N^{th}-order tri-diagonal matrix M_N can be formed from the infinite tri-diagonal matrix M simply by deleting from M all but its first N rows and columns. The analysis to follow concentrates on the two natural pairs:

(i) $y' = My$ with its truncation $y'_N = M_N y_N$,

(ii) $y'' = My$ with its truncation $y''_N = M_N y_N$,

where y_N denotes a column vector of N functions. The elements of the infinite matrix M are used to construct a recurrence of the form (1), which then yields a sequence $\{P_n\}$ of polynomials that are orthogonal if (3) is satisfied. For each of the two pairs of systems (i) and (ii) a solution of the infinite system and the solution of the finite truncation are given in terms of the orthogonal polynomials P_n. In each case the solution of the truncation involves the zeros of P_N, but the solution furnished for the infinite system has the attractive feature that no zeros of the P_n need be known. The initial N-segment of the given solution y of the infinite system may be used as an approximation to the solution y_N of the corresponding truncation. Estimates of the error incurred by doing so (that is, estimates of the difference between the n^{th} components of y and y_N for any $n \leq N$) are deduced.

The remarks of Chapter I are primarily for orientation. In Chapter II a detailed study of the recurrence polynomials P_n is undertaken. A device, apparently introduced by J. Favard [5], is used to generate recursively a sequence $\{v_n\}$ of real numbers in terms of the coefficients in the recurrence (1). It is shown that $\{v_n\}$ is a moment sequence for a distribution $d\alpha(x)$ over an interval $[a,b]$ if and only if the polynomials P_n of the recurrence (1) are orthogonal polynomials associated with $d\alpha(x)$ over $[a,b]$. Conditions (3) imply, however, that $\{v_n\}$ <u>is</u> a moment sequence for some distribution; hence the recurrence polynomials are orthogonal whenever (3) holds. The converse of this last statement is also proved -- namely, that conditions (3) <u>follow</u> from orthogonality of the polynomials P_n. Thus conditions (3) are necessary and sufficient for orthogonality of the P_n (with respect to a distribution over an interval) -- which is probably the most important point in Chapter II.

The two classes of countable systems of ordinary differential equations are introduced in Chapter III. Their solutions (when conditions (3) hold) are given in terms of orthogonal polynomials P_n, along with the error estimates mentioned previously. It is also noted that for any $n \leq N$ the n^{th} component of the given solution y of the infinite system (say $y^{(n)}$) and the n^{th} component of the solution y_N of the truncation (say $y_N^{(n)}$) have an interesting connection: (a) the finite sum $y_N^{(n)}$ is a Riemann-Stieltjes sum for the integral $y^{(n)}$; (b) when the interval of orthogonality is finite, $y_N^{(n)}(t) \to y^{(n)}(t)$ as $N \to \infty$.

Chapter IV consists of a study of a one-parameter family of polynomials which are orthogonal over the interval $[-1,1]$ with weight

$\omega(x) = |x|^{\alpha}$ (where $\alpha > -1$). These polynomials are prescribed by a recurrence of the form (1) and the proof of their orthogonality given here illustrates the techniques discussed in Chapter II. It is also shown with the aid of [10] that these polynomials are Sturm-Liouville polynomials if and only if $\alpha = 0$.

Three examples are examined, in some detail, in Chapter V. The solutions presented for these three systems illustrate the techniques developed in preceding chapters; they also serve to indicate related but unanswered questions in the study of infinite systems of differential equations.

The countable systems introduced in Chapter III may be used as models for various physical systems -- for example, coupled linear harmonic oscillators or lossless transmission lines (see, also, [7] and [23]). A number of these physical systems, accompanied by their mathematical models with solutions, are catalogued in the Appendix for reference purposes.

CHAPTER II

RECURSIVELY GENERATED (ORTHOGONAL) POLYNOMIALS

Polynomials Determined by a Three-Term Recurrence

The principal goals of this chapter are to deduce necessary and sufficient conditions for orthogonality of polynomials generated by a recurrence (1) and to infer certain properties of the polynomials so generated. The chapter consists of a series of definitions and lemmas which culminate in Theorem 2.

In trying to decide whether the polynomials P_n generated by (1) are orthogonal it is convenient to consider a sequence of polynomials φ_n which are simply scalar multiples of the P_n; certainly any orthogonality property of either sequence is inherited by the other. To this end, let

$$b_n = \frac{B_n}{A_n}, \quad n \geq 0,$$

$$c_n = \frac{C_n}{A_n A_{n-1}}, \quad n \geq 1, \tag{4}$$

$$\varphi_o = 1$$

and

$$\varphi_{n+1} = \frac{1}{A_0 A_1 A_2 \ldots A_n} P_{n+1}, \; n \geq 0 .$$

It is readily seen from (1) and (4) that

$$\varphi_o(x) = 1,$$

$$\varphi_1(x) = x + b_o$$

and $$\varphi_{n+1}(x) = (x + b_n)\varphi_n(x) - c_n\varphi_{n-1}(x), \; n \geq 1 , \tag{5}$$

where $$c_n \neq 0 \quad \text{for} \quad n = 1,2,3,\ldots \quad .$$

Let $\{v_n\}$ be the sequence of real numbers determined iteratively from recurrence (5) as follows:

if $\varphi_n(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_o$, then

$$v_o = 1 , \tag{6}$$

$$v_{2n-1} = -[a_{n-1}v_{2n-2} + a_{n-2}v_{2n-3} + \ldots + a_1 v_n + a_o v_{n-1}], \; n \geq 1,$$

and $$v_{2n-2} = -[a_{n-1}v_{2n-3} + a_{n-2}v_{2n-4} + \ldots + a_1 v_{n-1} + a_o v_{n-2}], \; n \geq 2.$$

Such a sequence $\{v_n\}$ will be called the <u>sequence of quasi-moments generated by the φ_n</u>.* Further, let L be the linear operator which maps all real polynomials onto the real numbers in accordance with the rule

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*Notice that <u>if</u> $\int_a^b \varphi_i(x)\,\varphi_j(x)d\alpha(x) = 0, \quad i \neq j,$ then

$$\int_a^b \varphi_n(x)x^{n-1}d\alpha(x) = 0 \text{ for } n \geq 1$$

and $$\int_a^b \varphi_n(x)x^{n-2}\,d\alpha(x) = 0 \quad \text{for } n \geq 2 .$$

If these two conditions are written in the form (6), then the quasi-moments v_k which they generate are in fact the moments of $d\alpha(x)$ over $[a,b]$.

$$L(d_n x^n + d_{n-1}x^{n-1} + \dots + d_1 x + d_o) = d_n v_n + d_{n-1}v_{n-1} + \dots + d_1 v_1 + d_o v_o . \quad (7)$$

Lemma 1. For the polynomials φ_n given by the recurrence (5),

$$L(\varphi_i(x)\varphi_j(x)) = 0 \quad \text{whenever } i \neq j$$

if and only if $L(\varphi_n(x)x^{n-1}) = 0$ for $n \geq 1$ and $L(\varphi_n(x)x^{n-2}) = 0$ for $n \geq 2$.

Proof. (a) If $L(\varphi_i(x)\varphi_j(x)) = 0$ whenever $i \neq j$, then $L(\varphi_n(x)x^{n-1}) = 0$ for $n \geq 1$, since x^{n-1} can be written as a linear combination of φ_o, φ_1, $\varphi_2,\dots,\varphi_{n-1}$ and L is a linear operator. Similarly $L(\varphi_n(x)x^{n-2}) = 0$ for $n \geq 2$.

(b) Suppose $L(\varphi_n(x)x^{n-1}) = 0$ for $n \geq 1$ and $L(\varphi_n(x)x^{n-2}) = 0$ for $n \geq 2$. For each integer $k \geq 1$ let T_k be the statement

"for each $m = 1,2,3,\dots,k$, $L(\varphi_n(x)x^{n-m}) = 0$ for all $n \geq m$."

It will first be shown, by an induction argument, that T_k is true for each $k = 1,2,3,\dots$. By hypothesis T_1 and T_2 are true. To show that T_k implies T_{k+1} it is sufficient to demonstrate that $L(\varphi_n(x)x^{n-[k+1]}) = 0$ for all $n \geq (k + 1)$ since, by the induction hypothesis, $L(\varphi_n(x)x^{n-m}) = 0$ (whenever $n \geq m$) for each $m = 1,2,3,\dots,k$. For any $n \geq (k + 1)$, multiplication throughout the recurrence (5) by $x^{n-[k+1]}$, followed by one application of the operator L, yields

$$L(\varphi_n(x)x^{n-[k+1]}) = L(\varphi_{n-1}(x)x^{n-k}) + b_{n-1}L(\varphi_{n-1}(x)x^{n-[k+1]}) - c_{n-1}L(\varphi_{n-2}(x)x^{n-[k+1]}) .$$

[i] $L(\varphi_{n-1}(x)x^{n-k}) = L(\varphi_{n-1}(x)x^{[n-1]-m})$, where $m = k - 1$. By the induction hypothesis, the right side here vanishes for

all $(n-1) \geq m$ since $m < k$; that is, it vanishes for all $n \geq k$ and hence for all $n \geq (k+1)$.

[ii] As in part [i], $L(\varphi_{n-1}(x)x^{n-[k+1]}) = L(\varphi_{n-1}(x)x^{[n-1]-k}) = 0$ for all $(n-1) \geq k$; that is, for all $n \geq (k+1)$.

[iii] $L(\varphi_{n-2}(x)x^{n-[k+1]}) = L(\varphi_{n-2}(x)x^{[n-2]-m})$ where $m = k - 1$ and, thus, $m < k$. As in the preceding parts, the right side is zero for all $(n-2) \geq m$ -- i.e., for all $n \geq (k+1)$.

Hence the induction is completed and T_k is true for all $k \geq 1$. Now, for any integer $j \geq 1$, the statement T_j yields in particular that $L(\varphi_j(x)x^{j-m}) = 0$ for $m = 1,2,3,\ldots,j$; equivalently, $L(\varphi_j(x)x^m) = 0$ for $m = 0,1,2,\ldots,j-1$. But, since L is linear, $L(\varphi_j(x)\varphi_i(x)) = 0$ for any $i < j$, which completes the proof.

Observe that (6) describes the conditions $L(\varphi_n(x)x^{n-1}) = 0$ for $n \geq 1$ and $L(\varphi_n(x)x^{n-2}) = 0$ for $n \geq 2$. Consequently the recurrence polynomials φ_n have the property

$$L(\varphi_i(x)\varphi_j(x)) = 0 \quad (\text{whenever } i \neq j) \; . \tag{8}$$

As will be shown in Lemma 3, the operator L and the quasi-moments v_n have significant interpretations if the recurrence polynomials φ_n (or P_n) are orthogonal. Prior to Lemma 3, another notable property of the φ_n (or of the P_n) is deduced as Lemma 2; and then, with the aid of Lemma 3, it is shown that this property also has a significant interpretation (see (11)) if the polynomials φ_n are orthogonal.

<u>Lemma 2</u>. For $n \geq 0$, let

$$\begin{aligned} \gamma_n &= L(\varphi_n^{\,2}(x)) \\ \text{and} \quad \zeta_n &= L(P_n^{\,2}(x)) \; . \end{aligned} \tag{9}$$

Then

$$\text{(a)}\quad \gamma_n = c_0 c_1 c_2 \ldots c_n \quad \text{for } n \geq 0 \tag{10}$$

and

$$\text{(b)}\quad \zeta_n = \frac{A_0}{A_n}\, C_0 C_1 C_2 \ldots C_n \quad \text{for } n \geq 0,$$

where $c_0 \equiv C_0 \equiv 1$.

Proof. (a) Since x^n can be written as a linear combination of $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n$ and $L(\varphi_k(x)\varphi_n(x)) = 0$ if $k < n$, it is easy to see that $L(\varphi_n(x)x^n) = L(\varphi_n^2(x))$ for $n \geq 0$. Multiplication throughout the recurrence (5) by x^{n-1}, followed by one use of the operator L, shows that $0 = L(\varphi_n(x)x^n) - c_n L(\varphi_{n-1}(x)x^{n-1})$ for $n \geq 1$, whence $\gamma_n = c_n\gamma_{n-1}$ for $n \geq 1$. But, from the definition of L, $\gamma_0 = 1$; and a simple induction yields that $\gamma_n = c_1 c_2 c_3 \ldots c_n$ for $n \geq 1$ -- which is precisely assertion (a).

(b) From (4),

$$\begin{aligned} L(P_n^2(x)) &= [A_0 A_1 A_2 \ldots A_{n-1}]^2\, L(\varphi_n^2(x)) \\ &= [A_0 A_1 A_2 \ldots A_{n-1}]^2\, c_1 c_2 \ldots c_n \\ &= [A_0 A_1 A_2 \ldots A_{n-1}]^2 \\ &\quad \cdot \left\{ \left[\frac{C_1}{A_1 A_0}\right]\left[\frac{C_2}{A_2 A_1}\right]\left[\frac{C_3}{A_3 A_2}\right] \cdots \left[\frac{C_n}{A_n A_{n-1}}\right] \right\} \\ &= \frac{A_0}{A_n}\, C_1 C_2 C_3 \ldots C_n \quad \text{for } n \geq 1 , \end{aligned}$$

and assertion (b) follows at once.

Definition 1. Let $\alpha(x)$ be a function which is bounded and non-decreasing and assumes infinitely many different values over an interval $[a,b]$ on which it is defined (here $-\infty \leq a < b \leq +\infty$). Further, let $\int_a^b d\alpha(x) = 1$

($\int_a^b d\alpha(x) = \alpha(b) - \alpha(a) > 0$ and $\alpha(x)$ can be scaled). Then $d\alpha(x)$ will be called a <u>distribution over the interval [a,b]</u>. The <u>moments</u> of a distribution $d\alpha(x)$ over an interval [a,b] are the numbers μ_n given by:

$$\mu_n = \int_a^b x^n d\alpha(x), \quad n = 0,1,2,\dots .$$

<u>Definition 2</u>. Let $\{Q_n\}$ be a sequence of polynomials in which Q_n has degree exactly n for $n = 0,1,2,\dots$. If there exists a distribution $d\alpha(x)$ over an interval [a,b] for which $\int_a^b Q_i(x)Q_j(x)d\alpha(x) = 0$ (whenever $i \neq j$), then the polynomials Q_n, $n \geq 0$, will be called <u>orthogonal polynomials associated with the distribution $d\alpha(x)$ over the interval [a,b]</u>. If, in addition, the orthogonal polynomials Q_n are normalized, that is, $\int_a^b Q_n^2(x)d\alpha(x) = 1$ for $n \geq 0$, then they will be called <u>orthonormal polynomials associated with the distribution $d\alpha(x)$ over the interval [a,b]</u>.* Finally, if there exists a function $\omega(x)$ such that $d\alpha(x) = \omega(x)dx$ for $a < x < b$ then orthogonal (orthonormal) polynomials associated with the distribution $d\alpha(x)$ over the interval [a,b] are called orthogonal (orthonormal) polynomials <u>with weight $\omega(x)$ over the interval [a,b]</u>.

<u>Lemma 3</u>. Let $d\alpha(x)$ be a distribution over an interval [a,b]. The following three statements are equivalent.

(a) The quasi-moments ν_n are the moments of $d\alpha(x)$ over [a,b].

(b) The operator L has the form

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*Such orthonormal polynomials are also called <u>the orthogonal polynomials associated with the distribution $d\alpha(x)$</u> when they are standardized so that the coefficient of the highest power of x is positive [22].

$$L(p(x)) = \int_a^b p(x)d\alpha(x)$$

for any real polynomial $p(x)$.

(c) The recurrence polynomials φ_n are orthogonal polynomials associated with $d\alpha(x)$ over $[a,b]$.

<u>Proof</u>. [i] ((a) $\Rightarrow$ (b)). Suppose $\nu_n = \int_a^b x^n d\alpha(x)$ for $n = 0,1,2,\ldots$. Then, from (7),

$$\begin{aligned} L(d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_o) &= d_n\nu_n + d_{n-1}\nu_{n-1} + \ldots + d_1\nu_1 + d_o\nu_o \\ &= d_n \int_a^b x^n d\alpha(x) + d_{n-1}\int_a^b x^{n-1}d\alpha(x) + \ldots + \\ &\quad d_1\int_a^b x d\alpha(x) + d_o\int_a^b d\alpha(x) \\ &= \int_a^b \left\{d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_o\right\} d\alpha(x) \end{aligned}$$

for any real polynomial $d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_o$.

[ii] ((b) $\Rightarrow$ (c)). Suppose $L(p(x)) = \int_a^b p(x)d\alpha(x)$ for any real polynomial $p(x)$. Then, from (8), $\int_a^b \varphi_i(x)\varphi_j(x)d\alpha(x) = 0$ whenever $i \neq j$ -- that is, the φ_n, $n \geq 0$, are orthogonal polynomials associated with $d\alpha(x)$ over $[a,b]$.

[iii] ((c) $\Rightarrow$ (a)). Suppose the recurrence polynomials φ_n are orthogonal polynomials associated with the distribution $d\alpha(x)$ over the interval $[a,b]$. Then $\int_a^b \varphi_n(x)x^{n-1}d\alpha(x) = 0$ for $n \geq 1$ and $\int_a^b \varphi_n(x)x^{n-2}d\alpha(x) = 0$ for $n \geq 2$; in other words, the moments μ_n of $d\alpha(x)$ over $[a,b]$ satisfy (6). But the quasi-moments ν_n generated by the φ_n

also satisfy (6) and a simple induction argument, using (6), verifies that $\nu_n = \mu_n$ for $n = 0,1,2,\ldots$. This completes the proof of Lemma 3.

In case the polynomials φ_n given by recurrence (5) <u>are</u> orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$, the operator L is integration and Lemma 2 takes a very useful form:

$$\zeta_n \overset{d}{=} \int_a^b P_n^2(x)d\alpha(x) = \frac{A_o}{A_n}\; C_oC_1C_2 \ldots C_n \quad \text{for } n \geq 0 . \qquad (11)$$

That is, the square of the norm of P_n can be computed directly from the coefficients in the three-term recurrence satisfied by the P_n.

It is demonstrated in the next section that if $c_n > 0$ for $n \geq 1$, then the sequence $\{\nu_n\}$ of quasi-moments generated by the recurrence polynomials φ_n is the sequence of <u>moments</u> of some distribution $d\alpha(x)$ over some interval $[a,b]$. Hence, by Lemma 3, if $c_n > 0$ for $n \geq 1$ then the polynomials φ_n generated by (5) are orthogonal polynomials associated with some distribution $d\alpha(x)$ over some interval $[a,b]$. The converse of this last assertion is also proved, which then gives the important equivalent of orthogonality described in Lemma 5.

<u>Conditions for Orthogonality of the Recurrence Polynomials</u>

Let φ_n, $n \geq 0$, be the polynomials given by the recurrence (5) and let $\{\nu_n\}$ be the sequence of quasi-moments generated by the φ_n. Let $\Delta_o \equiv 1$ and, for $n \geq 0$, let Δ_{n+1} be the <u>$(n+1)^{st}$ Hankel determinant for the sequence $\{\nu_n\}$</u>; that is,

$$\Delta_{n+1} = \begin{vmatrix} \nu_o & \nu_1 & \nu_2 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \nu_3 & \cdots & \nu_{n+1} \\ \nu_2 & \nu_3 & \nu_4 & \cdots & \nu_{n+2} \\ & \cdots & & & \\ \nu_n & \nu_{n+1} & \nu_{n+2} & \cdots & \nu_{2n} \end{vmatrix} \quad \text{for } n \geq 0 . \tag{12}$$

<u>Lemma 4</u>. Suppose that, in (5), $c_n > 0$ for $n \geq 1$. Then

$$\Delta_{n+1} = c_o^{n+1} c_1^{n} c_2^{n-1} \cdots c_{n-2}^{3} c_{n-1}^{2} c_n, \quad n \geq 0 . \tag{13}$$

<u>Proof</u>. It is first shown that $L(p^2(x))$ is positive for any non-zero real polynomial $p(x)$; this fact is used to show that $\Delta_n \neq 0$, which implies that $\Delta_{n+1} = (c_o c_1 c_2 \ldots c_n)\Delta_n$ for $n = 0,1,2,\ldots$. An induction argument with this last relation then verifies the assertion of the Lemma. Let $p(x)$ be any non-zero real polynomial of degree $n \geq 0$. There exist constants k_o, k_1, $k_2, \ldots, k_n$, not all zero, such that $p = k_o\varphi_o + k_1\varphi_1 + k_2\varphi_2 + \ldots + k_n\varphi_n$ and hence

$$\begin{aligned} L(p^2(x)) &= L\left(\left[\sum_{j=o}^{n} k_j\varphi_j(x)\right]\left[\sum_{i=o}^{n} k_i\varphi_i(x)\right]\right) \\ &= \sum_{j=o}^{n}\left[\sum_{i=o}^{n} k_i k_j \, L(\varphi_i(x)\varphi_j(x))\right] \\ &= \sum_{j=o}^{n} k_j^2 L(\varphi_j^2(x)) \\ &= \sum_{j=o}^{n} [k_j^2 (c_o c_1 c_2 \ldots c_j)] , \end{aligned}$$

where (8) and (10) have been used. Thus, $L(p^2(x))$ is a sum of non-negative terms of which at least one is positive; so $L(p^2(x)) > 0$. Further, let $p(x) = d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_o$; then

$$0 < L(p^2(x)) = L\left(\left[\sum_{i=o}^{n} d_i x^i\right]\left[\sum_{j=o}^{n} d_j x^j\right]\right)$$

$$= \sum_{i=o}^{n}\sum_{j=o}^{n} d_i d_j L(x^{i+j})$$

$$= \sum_{i=o}^{n}\sum_{j=o}^{n} d_i d_j \, v_{i+j}$$

where (7) has been used. That is, the quadratic form $\sum_{i=o}^{n}\sum_{j=o}^{n}(v_{i+j})d_i d_j$ in the n+1 variables $d_o, d_1, d_2, \ldots, d_n$ is positive definite; hence the determinant of the form, namely Δ_{n+1}, is positive. Since $\Delta_1 = v_o = c_o = 1$, the relation $\Delta_{n+1} = (c_o c_1 c_2 \ldots c_n)\Delta_n$ holds when $n = 0$. For $n > 0$ let $\varphi_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_o$, where φ_n is given by (5) and $a_n \equiv 1$. Now for $i = 0,1,2,\ldots,n-1$

$$L(\varphi_n(x)x^i) = \sum_{j=o}^{n} a_j v_{i+j} = 0 \ ,$$

and for $i = n$

$$L(\varphi_n(x)x^i) = \sum_{j=o}^{n} a_j v_{n+j} = c_o c_1 c_2 \ldots c_n = \gamma_n \ ,$$

since $L(\varphi_n(x)x^n) = L(\varphi_n^2(x))$.

These n+1 relations together can be written

$$\begin{bmatrix} v_o & v_1 & \cdots & v_n \\ v_1 & v_2 & \cdots & v_{n+1} \\ & \cdots & & \\ v_{n-1} & v_n & \cdots & v_{2n-1} \\ v_n & v_{n+1} & \cdots & v_{2n} \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma_n \end{bmatrix},$$

where the coefficient matrix is non-singular since $\Delta_{n+1} > 0$. One use of Cramer's Rule for the last component in the solution followed by a simple Laplace expansion gives $1 = \frac{\gamma_n \Delta_n}{\Delta_{n+1}}$ or $\Delta_{n+1} = (c_o c_1 c_2 \ldots c_n)\Delta_n$ for $n = 1,2,3,\ldots$. A direct induction argument with this last relation completes the proof.

Since each Hankel determinant Δ_{n+1} for the sequence $\{v_n\}$ is positive (when $c_n > 0$ for $n \geq 1$), it follows [8] that there exists a distribution $d\alpha(x)$ over some interval $[a,b]$ such that the v_n are the moments of $d\alpha(x)$ over $[a,b]$. This fact, combined with Lemma 3, yields an important result, due to J. Favard [5], which may be stated as

Theorem 1. Let φ_n, $n \geq 0$, be the polynomials generated by the recurrence (5) and suppose that $c_n > 0$ for $n = 1,2,3,\ldots$. Then there exists a distribution $d\alpha(x)$ over some interval $[a,b]$ such that the polynomials φ_n are orthogonal polynomials associated with $d\alpha(x)$ over $[a,b]$.

A converse to Theorem 1 holds: when polynomials P_n are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$, they [22] satisfy a three-term recurrence of the type (1) and hence the multiples φ_n of these polynomials satisfy a recurrence (5). It follows that each c_n is necessarily positive. For suppose that the polynomials φ_n given by (5) are orthogonal polynomials associated with some distribution $d\alpha(x)$ over some interval $[a,b]$. Multiplication throughout the recurrence by $\varphi_{n-1}(x)$, followed by integration and use of the orthogonality, shows that

$$0 = \int_a^b x\varphi_{n-1}(x)\varphi_n(x)d\alpha(x) - c_n \int_a^b \varphi_{n-1}^2(x)d\alpha(x)$$

or

$$\int_a^b \varphi_n^2(x)d\alpha(x) = c_n \int_a^b \varphi_{n-1}^2(x)d\alpha(x) \quad \text{for } n = 1,2,3,\ldots .$$

But $\alpha(x)$ has an infinite number of points of increase over $[a,b]$ which implies [22, p. 43] that $\int_a^b \varphi_j^2(x)d\alpha(x) > 0$ for $j \geq 0$. Consequently c_n is a quotient of positive numbers and hence is positive for $n \geq 1$.

This last result combined with Theorem 1 is stated in the notation of recurrence (1) to give

Lemma 5. A necessary and sufficient condition for the polynomials P_n (given by recurrence (1)) to be orthogonal polynomials associated with some distribution over some interval is that $C_n/(A_n A_{n-1})$ be positive for all $n \geq 1$.

Lemma 5 can be invoked, incidentally, to illuminate a point raised by H. L. Krall and O. Frink [14, p. 114]. The Bessel polynomials satisfy a recurrence of the form (1) in which $B_0 = 1$, $B_n = 0$ for $n \geq 1$, $A_n = 2n+1$ for $n \geq 0$ and $C_n = -1$ for $n \geq 1$. It is also known [14] that they are orthogonal polynomials with weight $\omega(x) = \exp(-2/x)$ around the unit circle in the complex plane. Krall and Frink observe that the Bessel polynomials are not orthogonal polynomials associated with a (non-negative valued) weight over an interval. In point of fact more can be said: Lemma 5 shows that the Bessel polynomials are not even orthogonal polynomials associated with a <u>distribution</u> over an interval.

Formulation of Results

A number of the properties of the recursively generated polynomials P_n are used in the subsequent analysis. Some of these results are described in the preceding material but are stated for the polynomials φ_n. They are included in Theorem 2 in terms of the coefficients A_n, B_n and C_n of the recurrence (1).

Suppose conditions (3) hold so that the recurrence polynomials P_n are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$. The polynomials $Q_n = \frac{1}{\sqrt{\zeta_n}} P_n$, $n \geq 0$, are orthonormal polynomials associated with $d\alpha(x)$ over $[a,b]$, and the zeros of P_n are precisely those of Q_n. It is well known [22] that for each $n > 1$ the zeros of Q_n are real and distinct and lie in (a,b); in addition, the zeros of Q_n separate the zeros of Q_{n+1}. These facts together with Lemma 5 and relation (11) give

<u>Theorem 2</u>. Let P_n, for $n \geq 0$, be the polynomials generated by the recurrence (1) and suppose that $C_n/(A_n A_{n-1})$ is positive for $n = 1,2,3,\ldots$. Then (a) there exists a distribution $d\alpha(x)$ over some interval $[a,b]$ such that

$$\int_a^b P_i(x)P_j(x)d\alpha(x) = 0 \quad \text{if } i \neq j;$$

(b) for each $n \geq 0$

$$\zeta_n = \int_a^b P_n^2(x)d\alpha(x) = \frac{A_o}{A_n} C_oC_1C_2 \ldots C_n \text{ (where } C_o \equiv 1);$$

(c) for each $n > 1$ the zeros of P_n are real and distinct and lie in (a,b). Furthermore, no zero of P_n can be a zero of P_{n+1}.

CHAPTER III

TRI-DIAGONAL COUNTABLE SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

The investigation in this chapter covers the solutions of certain countable systems of first- and second-order linear ordinary differential equations with (constant) coefficient matrices that are tri-diagonal. The entries in a coefficient matrix are used for the coefficients in a recurrence (1) and, under the hypothesis (3) on these elements, the orthogonal polynomials P_n are employed for constructing solutions to both the infinite system and a finite truncation of the system. The initial segment of the solution to the infinite system can be conveniently taken as an approximation to the more cumbersome solution of the finite system since error estimates for such an approximation are included in the development.

The well known [22] relation between a Stieltjes integral and one of its Riemann-Stieltjes sums in which the summand has Christoffel numbers for its coefficients is an important one for the following analysis. It is introduced in the next section, reformulated in terms of the notation (1) and summarized as Theorem 3 (The Quadrature Formula).

The Quadrature Formula

Suppose conditions (3) hold so that the recurrence polynomials P_n are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$. Let $N > 2$ be an integer and let $z_1, z_2, \ldots, z_N$ be the zeros of P_N. For $j = 1,2,\ldots,N$ let $\{\lambda_j\}$ be the Christoffel numbers associated with the zeros $\{z_j\}$; i.e., using the orthonormal polynomials $Q_n = \frac{1}{\sqrt{\gamma_n}} \varphi_n$,

$$\lambda_j = \frac{-\sqrt{\frac{\gamma_N}{\gamma_{N+1}}}}{Q_{N+1}(z_j)Q_N'(z_j)} \quad \text{for } j = 1,2,\ldots,N . \tag{14}$$

Let $f(x)$ be any function which is continuous over $[a,b]$ and for which $f^{(2N)}(x)$ is continuous over $[a,b]$; then [22, p. 369] there exists a ξ in $[a,b]$ such that

$$\int_a^b f(x)d\alpha(x) = \sum_{j=1}^{N} \lambda_j f(z_j) + \frac{f^{(2N)}(\xi)}{(2N)!} \gamma_N . \tag{15}$$

Substituting from (10) and (4) into (14) and (15) leads to

Theorem 3 (The Quadrature Formula). Suppose conditions (3) hold so that the polynomials P_n are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$. Let $N > 2$ be an integer and let $z_1, z_2, \ldots, z_N$ be the zeros of P_N. For $j = 1,2,\ldots,N$ let

$$\lambda_j = - \frac{A_0 C_1 C_2 \cdots C_N}{P_{N+1}(z_j)P_N'(z_j)} . \tag{16}$$

Let $f(x)$ be continuous and have a continuous $2N^{th}$-order derivative over $[a,b]$. Then there exists a ξ in $[a,b]$ such that

$$\int_a^b f(x)d\alpha(x) = \sum_{j=1}^{N} \lambda_j f(z_j) + E , \tag{17}$$

where

$$E = \frac{C_1 C_2 \cdots C_N}{A_o A_1^2 A_2^2 \cdots A_{N-1}^2 A_N} \frac{f^{(2N)}(\xi)}{(2N)!} \quad . \tag{18}$$

The Quadrature Formula has an interesting interpretation: the Tchebicheff-Markhoff-Stieltjes Separation Theorem [22, p. 49] shows that $\sum_{j=1}^{N} \lambda_j f(z_j)$ has the character of a Riemann-Stieltjes sum for $\int_a^b f(x)d\alpha(x)$.

Solutions of Initial-Value Problems for First- and Second-Order Countable Systems

The discussion in this section concentrates first on a pair of countable second-order systems and then on a pair of first-order systems. It will be assumed throughout that the coefficients $A_n (n \geq 0)$ and $C_n (n \geq 1)$ satisfy conditions (3) so that the polynomials P_n given by (1) are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$ and thus the Quadrature Formula (17) holds.

Let k be a prescribed non-negative integer which is less than the integer $N > 2$, and let α_k and β_k be specified constants. Consider the two systems

$$\begin{aligned}
A_o \ddot{y}_o &= B_o y_o - y_1 \\
A_1 \ddot{y}_1 &= -C_1 y_o + B_1 y_1 - y_2 \\
A_2 \ddot{y}_2 &= -C_2 y_1 + B_2 y_2 - y_3 \\
&\quad \cdots \\
A_n \ddot{y}_n &= -C_n y_{n-1} + B_n y_n - y_{n+1} \quad (n=1,2,\ldots,N-2) \\
&\quad \cdots \\
A_{N-1} \ddot{y}_{N-1} &= -C_{N-1} y_{N-2} + B_{N-1} y_{N-1}
\end{aligned} \tag{19}$$

and

$$
\begin{aligned}
A_o\ddot{y}_o &= \qquad\qquad B_o y_o - y_1 \\
A_1\ddot{y}_1 &= -C_1 y_o + B_1 y_1 - y_2 \\
A_2\ddot{y}_2 &= -C_2 y_1 + B_2 y_2 - y_3 \\
&\quad \cdots \\
A_n\ddot{y}_n &= -C_n y_{n-1} + B_n y_n - y_{n+1}, (n \geq 1) ,
\end{aligned} \tag{20}
$$

both subject to the initial conditions

$$
\begin{aligned}
&y_n(0) = \dot{y}_n(0) = 0 \quad \text{for } n \neq k , \\
&y_k(0) = \alpha_k \text{ and } \dot{y}_k(0) = \beta_k .
\end{aligned} \tag{21}
$$

For each non-negative integer n, for $t \geq 0$ and for any x let

$$f_n(x,t) = P_n(x)F(x,t) ,$$

where

$$
F(x,t) = \begin{cases} \dfrac{1}{\zeta_k} P_k(x) \left[\alpha_k \cosh (t\sqrt{-x}) + \dfrac{\beta_k}{\sqrt{-x}} \sinh (t\sqrt{-x})\right], & x < 0 \\[2ex] \dfrac{1}{\zeta_k} P_k(x) \left[\alpha_k \cos (t\sqrt{x}) + \dfrac{\beta_k}{\sqrt{x}} \sin (t\sqrt{x})\right], & x \geq 0 . \end{cases} \tag{22}
$$

<u>Theorem 4</u>. Suppose that, in system (20), $\dfrac{C_n}{A_n A_{n-1}} > 0$ for $n = 1,2,3,\ldots$. Let P_n $(n \geq 0)$ be the polynomials prescribed by recurrence (1) -- thus these polynomials are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$. For $n = 0,1,2,\ldots N$ let

$$x_n(t) = \sum_{j=1}^{N} \lambda_j f_n(z_j, t) \tag{23}$$

and for $n \geq 0$ let

$$X_n(t) = \int_a^b f_n(x,t)d\alpha(x) \; . \tag{24}$$

Suppose that for each $t \geq 0$

$$\dot{X}_n(t) = \int_a^b \frac{\partial}{\partial t} f_n(x,t)d\alpha(x) \quad \text{and} \quad \ddot{X}_n(t) = \int_a^b \frac{\partial^2}{\partial t^2} f_n(x,t)d\alpha(x), \tag{25}$$

$$(n \geq 0) \, .$$

Then: (a) the $X_n(t)$ $(n \geq 0)$ constitute a solution of the infinite initial-value problem (20)-(21) for $t \geq 0$;

(b) the $x_n(t)$ $(n = 0,1,2,\ldots,N-1)$ constitute the solution of the finite initial-value problem (19) and (21) for $t \geq 0$;

(c) for any $n = 0,1,2,\ldots,N-1$ and any fixed $t \geq 0$ there exists a ξ in $[a,b]$ such that

$$X_n(t) = x_n(t) + E_x \, ,$$

where

$$E_x = \frac{C_1 C_2 \ldots C_N}{[(2N)!] A_0 A_1^2 A_2^2 \ldots A_{N-1}^2 A_N} \frac{\partial^{2N}}{\partial x^{2N}} f_n(x,t) \bigg|_{(x,t)=(\xi,t)} \; . \tag{26}$$

<u>Proof</u>. It is evident from the orthogonality of the polynomials P_n (and the first of hypotheses (25)) that the functions $X_n(t)$, $n \geq 0$, satisfy the initial conditions (21). Also, for each $n = 0,1,2,\ldots,N-1$

$$x_n(0) = \sum_{j=1}^{N} \lambda_j \tilde{f}_n(z_j), \text{ where } \tilde{f}_n(x) = \frac{\alpha_k}{\zeta_k} P_n(x)P_k(x) \text{ is a polynomial of}$$

degree at most 2N-2, and hence from the Quadrature Formula

$$X_n(0) = \int_a^b \tilde{f}_n(x)d\alpha(x) = \sum_{j=1}^{N} \lambda_j \tilde{f}_n(z_j) = x_n(0) .$$

Similarly $\dot{X}_n(0) = \mathring{x}_n(0)$ so that the functions $x_n(t)$ satisfy the same initial conditions as the functions $X_n(t)$ $(n = 0,1,2,\ldots,N-1)$ -- namely those in (21). Clearly assertion (c) is a restatement of the Quadrature Formula for the functions $f_n(x,t)$. Hence it remains to show that $\{X_n(t)\}$ satisfies the system (20) and $\{x_n(t)\}$ satisfies (19). From (22), $\frac{\partial^2}{\partial t^2} f_n(x,t) = -xf_n(x,t)$ for $n \geq 0$, any $t \geq 0$ and any x; for convenience of notation in the remainder of the proof, let $X_{-1}(t) \equiv x_{-1}(t) \equiv 0$.

[i] For each $t \geq 0$,

$$-A_n\ddot{X}_n(t) - C_nX_{n-1}(t) + B_nX_n(t) - X_{n+1}(t)$$

$$= -A_n \int_a^b [-xP_n(x)F(x,t)]d\alpha(x) - C_n \int_a^b P_{n-1}(x)F(x,t)d\alpha(x)$$

$$+ B_n \int_a^b P_n(x)F(x,t)d\alpha(x) - \int_a^b P_{n+1}(x)F(x,t)d\alpha(x)$$

$$= \int_a^b [(A_nx + B_n)P_n(x) - C_nP_{n-1}(x) - P_{n+1}(x)]F(x,t)d\alpha(x)$$

$$= 0 \quad \text{for } n \geq 0 .$$

This completes the proof of assertion (a).

[ii] Similarly, for each $t \geq 0$,

$$-A_n\dot{x}_n(t) - C_n x_{n-1}(t) + B_n x_n(t) - x_{n+1}(t)$$

$$= \sum_{j=1}^{N} \lambda_j[(A_n z_j + B_n)P_n(z_j) - C_n P_{n-1}(z_j) - P_{n+1}(z_j)]F(z_j,t)$$

$$= 0 \quad \text{for } n = 0,1,2,\ldots,N-1$$

(note that the last equation of (19) is satisfied since $x_N(t) \equiv 0$); this gives assertion (b) and hence the proof of the theorem is complete.

The next theorem treats the case of a corresponding pair of countable first-order systems. Those details of proof which parallel portions of the proof for Theorem 4 are omitted but other major parts are included.

Let k be a prescribed non-negative integer which is less than the integer $N > 2$ and let ε_k be a specified constant. Consider the two systems

$$\begin{aligned}
A_o\dot{y}_o &= B_o y_o - y_1 \\
A_1\dot{y}_1 &= -C_1 y_o + B_1 y_1 - y_2 \\
A_2\dot{y}_2 &= -C_2 y_1 + B_2 y_2 - y_3 \\
&\cdots \\
A_n\dot{y}_n &= -C_n y_{n-1} + B_n y_n - y_{n+1} \quad (n=1,2,\ldots,N-2) \\
&\cdots \\
A_{N-1}\dot{y}_{N-1} &= -C_{N-1} y_{N-2} + B_{N-1} y_{N-1}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
A_o\dot{y}_o &= B_o y_o - y_1 \\
A_1\dot{y}_1 &= -C_1 y_o + B_1 y_1 - y_2 \\
A_2\dot{y}_2 &= -C_2 y_1 + B_2 y_2 - y_3 \\
&\cdots \\
A_n\dot{y}_n &= -C_n y_{n-1} + B_n y_n - y_{n+1} \quad (n \geq 1),
\end{aligned} \tag{28}$$

both subject to the initial conditions

$$y_n(0) = 0 \quad \text{for } n \neq k$$
$$\text{and} \quad y_k(0) = \varepsilon_k . \tag{29}$$

For each non-negative integer n, for $t \geq 0$ and for any x let

$$g_n(x,t) = P_n(x)G(x,t) ,$$
where
$$G(x,t) = \frac{\varepsilon_k}{\zeta_k} P_k(x)e^{-tx} . \tag{30}$$

<u>Theorem 5</u>. Suppose that, in system (28), $\frac{C_n}{A_n A_{n-1}} > 0$ for $n=1,2,3,\ldots$. Let P_n, $n \geq 0$, be the polynomials prescribed by recurrence (1) -- thus these polynomials are orthogonal polynomials associated with a distribution $d\alpha(x)$ over an interval $[a,b]$. For $n = 0,1,2,\ldots,N$ let

$$v_n(t) = \sum_{j=1}^{N} \lambda_j g_n(z_j, t) \tag{31}$$

and for $n \geq 0$ let

$$V_n(t) = \int_a^b g_n(x,t)d\alpha(x) . \tag{32}$$

Suppose that for each $t \geq 0$

$$\dot{V}_n(t) = \int_a^b \frac{\partial}{\partial t} g_n(x,t)d\alpha(x) \quad (n \geq 0). \tag{33}$$

Then: (a) the $V_n(t)$ $(n \geq 0)$ constitute a solution of the infinite initial-value problem (28) - (29) for $t \geq 0$;

(b) the $v_n(t)$ $(n = 0,1,2,\ldots,N-1)$ constitute the solution of the finite initial-value problem (27) and (29);

(c) for any $n = 0,1,2,\ldots,N-1$ and any fixed $t \geq 0$ there exists a ξ in $[a,b]$ such that

$$V_n(t) = v_n(t) + E_v ,$$

where
$$E_v = \frac{C_1 C_2 \cdots C_N}{[(2N)!] A_0 A_1^2 A_2^2 \cdots A_{N-1}^2 A_N} \frac{\partial^{2N}}{\partial x^{2N}} g_n(x,t)\Bigg|_{(x,t)=(\xi,t)} . \quad (34)$$

Proof. The orthogonality of the polynomials P_n implies that the functions $V_n(t)$ $(n = 0,1,2,\ldots)$ meet the initial conditions (29). For each $n = 0,1,2,\ldots,N-1$ the Quadrature Formula shows that $v_n(0) = V_n(0)$; hence the functions $v_n(t)$ also satisfy (29). Assertion (34) is a reformulation of the Quadrature Formula for the functions $g_n(x,t)$. For convenience of notation let $V_{-1}(t) \equiv v_{-1}(t) \equiv 0$. Since $\frac{\partial}{\partial t} g_n(x,t) = -xP_n(x)G(x,t)$ for $n \geq 0$, any $t \geq 0$ and any x, there follows:

[i] for each $t \geq 0$, $-A_n\dot{V}_n(t) - C_nV_{n-1}(t) + B_nV_n(t) - V_{n+1}(t) =$

$$\int_a^b [(A_nx + B_n)P_n(x) - C_nP_{n-1}(x) - P_{n+1}(x)]G(x,t)d\alpha(x) = 0$$

for $n \geq 1$

and [ii] similarly, for each $t \geq 0$, $-A_n\dot{v}_n(t) - C_nv_{n-1}(t) + B_nv_n(t) - v_{n+1}(t) = 0$ for $n = 0,1,2,\ldots,N-1$, where $v_N(t) \equiv 0$,

which completes the proof of the Theorem.

If the interval of orthogonality $[a,b]$ is finite, conclusions additional to the ones in Theorems 4 and 5 can be reached. In particular, hypotheses (25) and (33) are satisfied while $E_x \to 0$ and $E_v \to 0$ as $N \to \infty$; these statements are justified in Lemmas 6 and 7 below.

Lemma 6. Suppose the interval of orthogonality $[a,b]$ is finite. Then hypothesis (25) is satisfied; furthermore, for any fixed integer $n \geq 0$ and any fixed $t > 0$, $x_n(t) \rightarrow X_n(t)$ as $N \rightarrow \infty$; that is, in (26), $E_x \rightarrow 0$ as $N \rightarrow \infty$.

Proof. $\alpha(x)$ is a bounded non-decreasing function over the finite interval $a \leq x \leq b$ and thus is of bounded variation there. Let T be an arbitrary positive number. It can be easily verified that for $n \geq 0$ $f_n(x,t)$, $\frac{\partial}{\partial t} f_n(x,t)$ and $\frac{\partial^2}{\partial t^2} f_n(x,t)$ are all continuous at each point (x,t) of the plane region $a \leq x \leq b$, $-T \leq t \leq T$. Consequently a standard theorem* can be invoked twice to show that (25) holds. The result $E_x \rightarrow 0$ as $N \rightarrow \infty$ is well known [22, p. 342].

Since $g_n(x,t)$ and $\frac{\partial}{\partial t} g_n(x,t)$ are both continuous over the plane region $a \leq x \leq b$, $-T \leq t \leq T$ (for arbitrary $T > 0$), a completely analogous proof to the preceding one establishes

Lemma 7. Suppose the interval of orthogonality $[a,b]$ is finite. Then hypothesis (33) is satisfied; furthermore, for any fixed integer $n \geq 0$ and any fixed $t > 0$, $v_n(t) \rightarrow V_n(t)$ as $N \rightarrow \infty$.

It might be noted at this juncture that finiteness of the interval of orthogonality is a sufficient condition for hypotheses (25) and (33) to be satisfied but is by no means necessary. For example, the Laguerre polynomials $L_n(x)$ are orthogonal with weight $\omega(x) = e^{-x}$ over the interval $[0, \infty)$; using the analog of (33) say, the Weierstrass M-Test insures absolute and uniform convergence of the integrals in question and hence

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*See, for example, T. M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Co. (1960), p. 219.

$$\frac{d}{dt}\int_0^\infty L_n(x)L_k(x)e^{-tx}\,e^{-x}dx = \int_0^\infty \frac{\partial}{\partial t}\left[L_n(x)L_k(x)e^{-tx}e^{-x}\right]dx \text{ for } t \geq 0.$$

CHAPTER IV

A NON-CLASSICAL FAMILY OF POLYNOMIALS ORTHOGONAL OVER A FINITE INTERVAL

Since many of the polynomial sequences determined by a recurrence (1) are composed of familiar Sturm-Liouville polynomials [10], it seems desirable to have available sequences of polynomials which are not of this classical type. Just such a one-parameter family is described below in (35). It is demonstrated that these polynomials are orthogonal polynomials with weight $\omega(x) = |x|^{\alpha}$ ($\alpha > -1$ a parameter) over the interval $[-1,1]$ but are not Sturm-Liouville polynomials if $\alpha \neq 0$. The proof of their orthogonality given here illustrates a number of points raised in Chapter II and, as indicated in Lemma 9, also depends on the evaluation of an interesting determinant D_n.

The Non-Classical $|x|^{\alpha}$ Polynomials

Let $\alpha > -1$ be a parameter and let $S_n^{(\alpha)}$ $(n \geq 0)$ be the polynomials determined by:

$$\begin{aligned} S_o^{(\alpha)}(x) &= 1 \\ S_1^{(\alpha)}(x) &= x \\ S_{n+1}^{(\alpha)}(x) &= x\, S_n^{(\alpha)}(x) - e_n S_{n-1}^{(\alpha)}(x), \quad n \geq 1, \end{aligned} \tag{35}$$

where

$$e_n = \frac{\left\{n + \alpha \sin^2\left(\frac{n\pi}{2}\right)\right\}^2}{(2n + \alpha - 1)(2n + \alpha + 1)} \quad \text{for } n \geq 1 .$$

Note that (35) has the form of recurrence (5) in which $b_n = 0$ for $n \geq 0$ and $c_n = e_n$ for $n \geq 1$. Now let

$$\Delta = (2b_2 - b_1 - b_o)[(b_1 - b_o)^2 + 4(c_1 + c_2)] + 9c_2(b_o - b_2) ,$$

$$g_1(n) = [(n+1)b_{n+1} + (1-n)b_n - b_1 - b_o][(b_1 - b_o)^2 + 4(c_1 + c_2)]/3c_2$$
$$+ [(-2n-1)b_{n+1} + (2n-3)b_n + b_1 + 3b_o] , \quad n \geq 1 ,$$

and

$$g_2(n) = [(n+1)b_n b_{n+1} - nb_n^2 - b_o b_1 + c_1 - (2n+1)c_{n+1} + (2n-3)c_n][(b_1 - b_o)^2 + 4(c_1 + c_2)]/3c_2$$
$$+ [(-2n-1)b_n b_{n+1} + (2n-1)b_n^2 + b_o b_1 + b_o^2 + 4nc_{n+1} + (-4n+8)c_n], \quad n \geq 1.$$

To prove that the polynomials $S_n^{(\alpha)}(x)$ $(n \geq 0)$ are not Sturm-Liouville polynomials if $\alpha \neq 0$, it suffices to show [10] that $\Delta = 0$ but that it is not the case that $g_1(n) = g_2(n) = 0$ for every positive integer n. It is readily seen here that $\Delta=0$, $g_1(n) = 0$ for $n \geq 1$ and

$$g_2(n) = [e_1 - (2n+1)e_{n+1} + (2n-3)e_n][4(e_1+e_2)]/3e_2 + 4ne_{n+1} + (-4n+8)e_n, n \geq 1.$$

The substitution for e_1, e_2, e_n and e_{n+1} then yields that

$$g_2(n) = \frac{(1+\alpha)(3+\alpha)}{3} + \frac{(\alpha^2+6\alpha+3)[\alpha\{\sin^2(\frac{n\pi}{2}) - \sin^2(\frac{n\pi+\pi}{2})\} - 1]n^3 + h_1 n^2 + h_2 n + h_3}{3n^3 + h_4 n^2 + h_5 n + h_6}, \quad (36)$$

where h_1, $h_2, \ldots, h_6$ are independent of n. To show that the condition

$$g_2(n) = 0 \quad \text{for} \quad n \geq 1 \qquad (37)$$

is not satisfied when $\alpha \neq 0$, it is sufficient to show that $\lim_{n\to\infty} g_2(n)$

either does not exist or is not zero -- this result will be demonstrated through an examination of g_2 at odd and even integers.

It can be easily computed from (36) that

$$\lim_{m\to\infty} g_2(2m+1) = \frac{\alpha^3 + 6\alpha^2 + \alpha}{3}$$

and

$$\lim_{m\to\infty} g_2(2m) = \frac{-\alpha^3 - 6\alpha^2 - 5\alpha}{3} .$$

Now if $\lim\limits_{m\to\infty} g_2(2m+1) \neq \lim\limits_{m\to\infty} g_2(2m)$, $\lim\limits_{n\to\infty} g_2(n)$ does not exist. So the only values of α for which $\lim\limits_{n\to\infty} g_2(n)$ does exist are those for which $(\alpha^3 + 6\alpha^2 + \alpha)/3 = (-\alpha^3 - 6\alpha^2 - 5\alpha)/3$ or $\alpha(\alpha + 3 + \sqrt{6})(\alpha + 3 - \sqrt{6}) = 0$; that is, $\alpha = 0$, $-3 - \sqrt{6}$ or $-3 + \sqrt{6}$. But the first two values are excluded, since $\alpha \neq 0$ and $\alpha > -1$; and for $\alpha = -3 + \sqrt{6}$,

$$\lim_{n\to\infty} g_2(n) = \lim_{m\to\infty} g_2(2m+1) = \frac{\alpha^3 + 6\alpha^2 + \alpha}{3} = \frac{6 - 2\sqrt{6}}{3} \neq 0 .$$

Consequently condition (37) is not satisfied and thus the polynomials $S_n^{(\alpha)}$ are not Sturm-Liouville polynomials if $\alpha \neq 0$.*

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*If $\alpha = 0$, then $\Delta = 0$ and $g_1(n) = g_2(n) = 0$ for $n \geq 1$, which imply [10] that the polynomials $S_n^{(0)}$ are Sturm-Liouville polynomials. But the Legendre polynomials P_n satisfy a recurrence (1) where $A_n = (2n+1)/(n+1)$ for $n \geq 0$, $B_n = 0$ for $n \geq 0$ and $C_n = n/(n+1)$ for $n \geq 1$, so that the corresponding polynomials $\varphi_n = [A_0 A_1 A_2 \ldots A_{n-1}]^{-1} P_n$ satisfy recurrence (5). However, (5) here is precisely the recurrence (35) with $\alpha = 0$ and hence $S_n^{(0)} = \varphi_n$; in other words

$$S_n^{(0)}(x) = \frac{n 2^{n-1}[(n-1)!]^2}{(2n-1)!} P_n(x), \quad n \geq 1 ,$$

where the P_n $(n = 0,1,2,\ldots)$ are the Legendre polynomials.

Orthogonality and Other Properties of the $|x|^\alpha$ Polynomials

It has been shown that the polynomials $S_n^{(\alpha)}$, $n \geq 0$, are not Sturm-Liouville polynomials if $\alpha \neq 0$. Some other interesting properties of these polynomials (particularly their orthogonality over a finite interval) appear in the following development. The first property given (concerning the evenness and oddness of $S_n^{(\alpha)}$ and the vanishing of the corresponding quasi-moments) is a basic one for the subsequent discussion.

An elementary induction argument with (35) shows that $S_n^{(\alpha)}$ is an even or odd function according as n is an even or odd integer.* Now let $\{\tau_n\}$ be the sequence of quasi-moments generated by the $S_n^{(\alpha)}$. It is readily seen, using the even-odd property of $\left\{S_n^{(\alpha)}\right\}$, that $\tau_{2j+1} = 0$ for $j = 0,1,2,\ldots$.

In preparation for the basic lemma used in proving orthogonality, let $\{\nabla_n\}$ be the sequence of Hankel determinants for $\{\tau_n\}$; that is, let

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*In fact, a direct but tedious induction argument using the recurrence (35) verifies that

$$S_n^{(\alpha)}(x) = x^n + \sum_{k=1}^{[n/2]} \left(-\frac{1}{2}\right)^k \binom{[n/2]}{k} \cdot$$

$$\frac{\{2n+2\alpha+(-1)^{n+1}-1\}\{2n+2\alpha+(-1)^{n+1}-5\}\ldots\{2n+2\alpha+(-1)^{n+1}-4k+3\}}{(2n+\alpha-1)(2n+\alpha-3)\ldots(2n+\alpha-2k+1)} x^{n-2k}$$

for $n \geq 0$, where $[r]$ denotes the greatest integer which is less than or equal to r.

$$\nabla_o = 1$$

and
$$\nabla_{n+1} = \begin{vmatrix} \tau_o & \tau_1 & \tau_2 & \cdots & \tau_n \\ \tau_1 & \tau_2 & \tau_3 & \cdots & \tau_{n+1} \\ \tau_2 & \tau_3 & \tau_4 & \cdots & \tau_{n+2} \\ & & \cdots & & \\ \tau_n & \tau_{n+1} & \tau_{n+2} & \cdots & \tau_{2n} \end{vmatrix} \quad \text{for } n = 0,1,2,\ldots \,. \tag{38}$$

Lemma 8. Let $\{y_n\}$, $n \geq 0$, be a sequence of real numbers for which $y_o = 1$ and $y_{2j+1} = 0$ for $j = 0,1,2,\ldots$, and let

$$H_o = 1$$

and
$$H_{n+1} = \begin{vmatrix} y_o & y_1 & y_2 & \cdots & y_n \\ y_1 & y_2 & y_3 & \cdots & y_{n+1} \\ y_2 & y_3 & y_4 & \cdots & y_{n+2} \\ & \cdots & & & \\ y_n & y_{n+1} & y_{n+2} & \cdots & y_{2n} \end{vmatrix}$$

for $n \geq 0$. If $H_n = \nabla_n$ for $n \geq 0$, then $y_k = \tau_k$ for all $k \geq 0$.

Proof. It is clear that $y_o = \tau_o$ and $y_1 = \tau_1$. It will be shown that if $y_j = \tau_j$ for $j = 0,1,2,\ldots,2m$ then $y_j = \tau_j$ for $j = 0,1,2,\ldots,(2m+2)$. But $y_{2m+1} = 0 = \tau_{2m+1}$ -- hence it need only be shown that $y_{2m+2} = \tau_{2m+2}$. In view of Lemma 4, (38) and (35) yield that

$$\nabla_{m+2} = e_1^{m+1} e_2^m e_3^{m-1} \ldots e_m^2 e_{m+1}$$

and Laplace expansion of ∇_{m+2} by its last column thus gives

$$\tau_{2m+2}\nabla_{m+1} + f(\tau_o,\tau_1,\tau_2,\ldots,\tau_{2m+1}) = e_1^{m+1}e_2^m e_3^{m-1}\ldots e_m^2 e_{m+1} , \quad (39)$$

where $f(\tau_o,\tau_1,\tau_2,\ldots,\tau_{2m+1}) = \sum_{i=o}^{2m+1} \tau_i A_i^*$ and A_i^* denotes the cofactor of τ_i for $i = 0,1,2,\ldots,(2m+1)$. By the induction hypothesis $\nabla_{m+1} = H_{m+1}$ and, since $y_{2m+1} = \tau_{2m+1}$, $f(\tau_o,\tau_1,\tau_2,\ldots,\tau_{m+1}) = f(y_o,y_1,y_2,\ldots,y_{m+1})$. On the other hand, Laplace expansion of H_{m+2} then gives

$$y_{2m+2}\nabla_{m+1} + f(\tau_o,\tau_1,\tau_2,\ldots,\tau_{2m+1}) = e_1^{m+1}e_2^m e_3^{m-1}\ldots e_m^2 e_{m+1} . \quad (40)$$

But $\nabla_{m+1} \neq 0$ (by Lemma 4) and a comparison of (39) and (40) shows that $y_{2m+2} = \tau_{2m+2}$ which completes the proof of the Lemma.

The goal of the remaining discussion is the verification that the polynomials $S_n^{(\alpha)}$ are orthogonal polynomials associated with the weight $\omega(x) = \frac{\alpha+1}{2} |x|^\alpha$ over the interval $[-1,1]$.

For $n \geq 0$, let

$$\mu_n = \frac{\alpha+1}{2}\int_{-1}^{1} x^n |x|^\alpha dx ; \quad (41)$$

that is,

$$\mu_o = 1 ,$$
$$\mu_{2j+1} = 0 \quad \text{for } j = 0,1,2,\ldots \quad (42)$$

and
$$\mu_{2j} = \frac{\alpha+1}{2j+\alpha+1} \text{ for } j = 0,1,2,\ldots .$$

Further, let

$$D_o = 1$$

and
$$D_{n+1} = \begin{vmatrix} \mu_o & \mu_1 & \mu_2 & \cdots \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots \mu_{n+1} \\ \mu_2 & \mu_3 & \mu_4 & \cdots \mu_{n+2} \\ & \cdots & & \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots \mu_{2n} \end{vmatrix} \text{ for } n \geq 0 . \qquad (43)$$

Observe that D_{n+1} is positive (for $n \geq 0$) since it is the determinant of the quadratic form described by

$$\sum_{i=o}^{n} \sum_{j=o}^{n} d_i d_j \mu_{i+j} = \frac{\alpha+1}{2} \int_{-1}^{1} (d_n x^n + d_{n-1} x^{n-1} + \ldots + d_1 x + d_o)^2 \, |x|^\alpha dx$$

and the right side is positive for any real d_o, $d_1, \ldots, d_n$ (not all zero). It will be shown, by evaluating D_{n+1}, that $D_{n+1} = \nabla_{n+1}$ $(n = 0, 1, 2, \ldots)$ and hence from Lemma 8 that $\tau_n = \mu_n$ for $n \geq 0$. But then $\{\tau_n\}$ is the moment sequence for $\omega(x) = \frac{\alpha+1}{2} |x|^\alpha$, which implies (from Lemma 3) orthogonality of the polynomials $S_n^{(\alpha)}$.

Lemma 9. For any $n \geq 1$,

$$D_{n+1} = e_1^n e_2^{n-1} e_3^{n-2} \ldots e_{n-1}^2 e_n ,$$

where e_n is given in (35).

Proof. It will be shown that $D_{n+1} = (e_1 e_2 \ldots e_n) D_n$ or, what is simpler here, that $D_{n+1} = (e_1 e_2 \ldots e_n)[e_1 e_2 \ldots e_{n-1} D_{n-1}]$ $(n \geq 2)$, from which the assertion follows by a direct induction. The proof falls naturally into two cases according as n is an even or odd integer. In each case the verification is somewhat detailed; hence merely a recipe for the proof is given.

Case 1: let $n > 3$ be an even integer. Then

$$D_{n+1} = \begin{vmatrix} \frac{\alpha+1}{\alpha+1} & 0 & \frac{\alpha+1}{\alpha+3} & 0 & \cdots & \frac{\alpha+1}{\alpha+n-1} & 0 & \frac{\alpha+1}{\alpha+n+1} \\ 0 & \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & \cdots & 0 & \frac{\alpha+1}{\alpha+n+1} & 0 \\ \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & 0 & \cdots & \frac{\alpha+1}{\alpha+n+1} & 0 & \frac{\alpha+1}{\alpha+n+3} \\ & & & \cdots & & & & \\ \frac{\alpha+1}{\alpha+n+1} & 0 & \frac{\alpha+1}{\alpha+n+3} & 0 & \cdots & \frac{\alpha+1}{\alpha+2n-1} & 0 & \frac{\alpha+1}{\alpha+2n+1} \end{vmatrix} .$$

[i] Multiply each of the n+1 rows by $\frac{1}{\alpha+1}$.

[ii] Multiply rows 1 and 2 by $(\alpha+n+1)$, rows 3 and 4 by $(\alpha+n+3),\ldots,$ rows $(n-1)$ and n by $(\alpha+2n-1)$, and row $(n+1)$ by $(\alpha+2n+1)$.

[iii] Subtract column $(n+1)$ from column 1, from column 3,..., from column $(n-3)$, and from column $(n-1)$. Subtract column n from column 2, from column 4,..., from column $(n-4)$, and from column $(n-2)$.

[iv] Multiply column 1 by $\frac{1}{n}$, columns 2 and 3 by $\frac{1}{n-2}$, columns 4 and 5 by $\frac{1}{n-4}$,..., and columns $(n-2)$ and $(n-1)$ by $\frac{1}{2}$.

[v] Multiply columns 1 and 2 by $(\alpha + n + 1)$, columns 3 and 4 by $(\alpha + n + 3),\ldots,$ columns $(n-3)$ and $(n-2)$ by $(\alpha + 2n - 3)$, and column $(n-1)$ by $(\alpha + 2n - 1)$.

[vi] Subtract row $(n+1)$ from row 1, from row 3,..., and from row $(n-1)$. Subtract row n from row 2, from row 4,..., and from

row (n-2).

[vii] Multiply row 1 by $\frac{1}{n}$, rows 2 and 3 by $\frac{1}{n-2}$, rows 4 and 5 by $\frac{1}{n-4}$,..., and rows (n-2) and (n-1) by $\frac{1}{2}$.

[viii] Multiply each of the first n-1 rows by $(\alpha+1)$.

Then,

$$D_{n+1} = Y_n \begin{vmatrix} \frac{\alpha+1}{\alpha+1} & 0 & \frac{\alpha+1}{\alpha+3} & 0 & \cdots & 0 & \frac{\alpha+1}{\alpha+n-1} & 0 & 0 \\ 0 & \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & \cdots & \frac{\alpha+1}{\alpha+n-1} & 0 & 0 & 0 \\ \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & 0 & \cdots & 0 & \frac{\alpha+1}{\alpha+n+1} & 0 & 0 \\ & & & & \cdots & & & & \\ \frac{\alpha+1}{\alpha+n-1} & 0 & \frac{\alpha+1}{\alpha+n+1} & 0 & \cdots & 0 & \frac{\alpha+1}{\alpha+2n-3} & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 \end{vmatrix} \tag{44}$$

$$\text{where } Y_n = \frac{(\alpha+1)^2\,[2^2 4^2 6^2 \cdots (n-2)^2 n]^2}{[(\alpha+n+1)^2(\alpha+n+3)^2\cdots(\alpha+2n-3)^2]^2\,(\alpha+2n-1)^3(\alpha+2n+1)} \,. \tag{45}$$

In (44), two Laplace expansions (by a last column) give immediately that $D_{n+1} = Y_n D_{n-1}$. It can be verified directly that $e_1^2 e_2^2 e_3^2 \cdots e_{n-1}^2 e_n = Y_n$, and thus the proof for Case 1 is complete.

Case 2: let $n > 2$ be an odd integer. Then

$$D_{n+1} = \begin{vmatrix} \frac{\alpha+1}{\alpha+1} & 0 & \frac{\alpha+1}{\alpha+3} & 0 & \cdots & \frac{\alpha+1}{\alpha+n} & 0 \\ 0 & \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & \cdots & 0 & \frac{\alpha+1}{\alpha+n+2} \\ \frac{\alpha+1}{\alpha+3} & 0 & \frac{\alpha+1}{\alpha+5} & 0 & \cdots & \frac{\alpha+1}{\alpha+n+2} & 0 \\ & & \cdots & & & & \\ 0 & \frac{\alpha+1}{\alpha+n+2} & 0 & \frac{\alpha+1}{\alpha+n+4} & \cdots & 0 & \frac{\alpha+1}{\alpha+2n+1} \end{vmatrix} .$$

[i] Multiply each of the n+1 rows by $\frac{1}{\alpha+1}$.

[ii] Multiply row 1 by $(\alpha+n)$, rows 2 and 3 by $(\alpha+n+2)$, rows 4 and 5 by $(\alpha+n+4), \ldots$, rows $(n-1)$ and n by $(\alpha+2n-1)$, and row $(n+1)$ by $(\alpha+2n+1)$.

[iii] Subtract column $(n+1)$ from column 2, from column $4, \ldots$, and from column $(n-1)$. Subtract column n from column 1, from column $3, \ldots$, and from column $(n-2)$.

[iv] Multiply columns 1 and 2 by $\frac{1}{n-1}$, columns 3 and 4 by $\frac{1}{n-3}, \ldots$, and columns $(n-2)$ and $(n-1)$ by $\frac{1}{2}$.

[v] Multiply column 1 by $(\alpha+n)$, columns 2 and 3 by $(\alpha+n+2)$, columns 4 and 5 by $(\alpha+n+4), \ldots$, columns $(n-3)$ and $(n-2)$ by $(\alpha+2n-3)$, and column $(n-1)$ by $(\alpha+2n-1)$.

[vi] Subtract row $(n+1)$ from row 2, from row $4, \ldots$, and from row $(n-1)$. Subtract row n from row 1, row $3, \ldots$, and from row $(n-2)$.

[vii] Multiply rows 1 and 2 by $\frac{1}{n-1}$, rows 3 and 4 by $\frac{1}{n-3}, \ldots$, and rows $(n-2)$ and $(n-1)$ by $\frac{1}{2}$.

[viii] Multiply each of the first n-1 rows by $(\alpha+1)$.

As in Case 1, it now follows that

$$D_{n+1} = Z_n D_{n-1} ,$$

where
$$Z_n = \frac{(\alpha+1)^2[2^2 4^2 6^2 \ldots (n-1)^2]^2}{[(\alpha+n)^2(\alpha+n+2)^2 \ldots (\alpha+2n-3)^2]^2 (\alpha+2n-1)^3(\alpha+2n+1)}$$

$$= e_1^2 e_2^2 e_3^2 \ldots e_{n-1}^2 e_n ,$$

which completes the proof of the Lemma.

By Lemma 4, $\nabla_{n+1} = e_1^n e_2^{n-1} e_3^{n-2} \ldots e_{n-1}^2 e_n$; consequently $\nabla_{n+1} = D_{n+1}$ for $n \geq 0$. Hence, by Lemma 8, $\tau_n = \mu_n$ for $n \geq 0$; that is, the quasi-moment sequence $\{\tau_n\}$ is the moment sequence for $\omega(x) = \frac{\alpha+1}{2} |x|^\alpha$ over $[-1,1]$. But then Lemma 3 implies that the corresponding polynomials $S_n^{(\alpha)}$ $(n = 0,1,2,\ldots)$ are orthogonal polynomials with weight $\omega(x) = \frac{\alpha+1}{2} |x|^\alpha$ over the interval $[-1,1]$.

CHAPTER V

ILLUSTRATIVE EXAMPLES AND COMMENTS

The presentation in this chapter is framed around three examples of countable systems of ordinary differential equations. The solutions presented for these systems illustrate many of the techniques developed in preceding chapters; in addition, they serve to indicate related but unanswered questions in the study of infinite differential systems. In each case the example is preceded by a discussion of some of the properties to be illustrated.

Example 1

The example below, which illustrates many of the methods developed in Chapters II and III, also shows that the solution furnished for the infinite system may, in some cases, be represented in a (simpler) form which is quite different in appearance from the original one. The differential system considered here may be interpreted as a model for an infinite vertical stack of flat plates, sliding horizontally with respect to each other, with viscous friction between adjacent plates (see the Appendix).

Consider the infinite system

$$\frac{m}{\rho}\,\mathring{y}_o = -y_o + y_1$$

$$\frac{m}{\rho}\,\mathring{y}_1 = y_o - 2y_1 + y_2 \qquad (46)$$

$$\frac{m}{\rho}\dot{y}_2 = y_1 - 2y_2 + y_3$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{m}{\rho}\dot{y}_n = y_{n-1} - 2y_n + y_{n+1} \quad (n \geq 1),$$

subject to the initial conditions

$$y_n(0) = 0 \quad \text{for} \quad n \neq k \tag{47}$$

and

$$y_k(0) = \varepsilon_k ,$$

where m and ρ are positive constants and k is a fixed non-negative integer.

When (46) is expressed in the form of the infinite system (28) and the coefficients are identified, there results

$$A_n = -\frac{m}{\rho} \quad \text{for} \quad n \geq 0 ,$$

$$B_o = 1 \quad \text{and} \quad B_n = 2 \quad \text{for } n \geq 1 \tag{48}$$

and

$$C_n = 1 \quad \text{for } n \geq 1 .$$

Let P_n $(n = 0,1,2,\ldots)$ be the polynomials generated by the recurrence (1) in which the coefficients A_n, B_n and C_n have the values given in (48). According to (4) and (5), the polynomials

$$\varphi_o = 1$$

and

$$\varphi_n = \frac{(-1)^n \rho^n}{m^n} P_n \quad \text{for} \quad n \geq 1 \tag{49}$$

satisfy the recurrence

$$\varphi_o(x) = 1 ,$$

$$\varphi_1(x) = x + b_o \tag{50}$$

and
$$\varphi_{n+1}(x) = (x+b_n)\varphi_n(x) - c_n\varphi_{n-1}(x), \; n \geq 1 ,$$

where
$$b_o = -\frac{\rho}{m} , \; b_n = -\frac{2\rho}{m} \quad \text{for } n \geq 1 \tag{51}$$

and
$$c_n = \frac{\rho^2}{m^2} \quad \text{for} \quad n \geq 1 .$$

From [10], the polynomials $\varphi_n(x)$ are (apart from a linear change of the independent variable and multiplicative factors which may depend on n but not on x) Jacobi polynomials satisfying the differential equation

$$[-\frac{1}{2} x^2 + \frac{2\rho}{m} x]\varphi_n''(x) + [-x + \frac{\rho}{m}]\varphi_n'(x) + \frac{n(n+1)}{2} \varphi_n(x) = 0 \tag{52}$$

for n = 0,1,2,... . Let $x = \frac{2\rho}{m} t + \frac{2\rho}{m}$ and $\psi_n(t) = \varphi_n(\frac{2\rho}{m} t + \frac{2\rho}{m})$; then (52) takes the form

$$[-t^2 + 1]\psi_n''(t) + [-2t - 1]\psi_n'(t) + n(n+1)\psi_n(t) = 0 \tag{53}$$

for n = 0,1,2,... . Identification of (53) with the Jacobi differential equation

$$[-t^2+1]y'' + [\beta - \alpha - (\alpha+\beta+2)t]y' + n(n+\alpha+\beta+1)y = 0$$

shows that $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ and the polynomials $\psi_n(t)$ are thus the Jacobi polynomials $P_n^{(1/2,\,-1/2)}(t)$ (apart from multiplicative factors which may depend on n but not on t). These Jacobi polynomials satisfy the recurrence

$$P_o^{(1/2,\ -1/2)}(t) = 1 ,$$

$$P_1^{(1/2,\ -1/2)}(t) = t + \frac{1}{2} \tag{54}$$

and $$P_{n+1}^{(1/2,\ -1/2)}(t) = \frac{2n+1}{n+1}\, t\, P_n^{(1/2,\ -1/2)}(t) - \frac{(2n-1)(2n+1)}{2n(2n+2)}\, P_{n-1}^{(1/2,-1/2)}(t),$$

$$n \geq 1 .$$

The Jacobi polynomials $P_n^{(1/2,\ -1/2)}(t)$ are orthogonal over the interval $-1 \leq t \leq 1$ with weight $W(t) = \dfrac{\sqrt{1-t}}{\pi\sqrt{1+t}}$ where $\int_{-1}^{1} W(t)dt = 1$. But $t = \frac{mx}{2\rho} - 1$ and hence the polynomials P_n satisfy

$$P_n(x) = q_n P_n^{(1/2,\ -1/2)}\left(\frac{mx}{2\rho} - 1\right) \quad (n = 0,1,2,\ldots) \tag{55}$$

for some constants $q_o, q_1, q_2, \ldots$ and are orthogonal over the interval $0 \leq x \leq \frac{4\rho}{m}$ with weight

$$\omega(x) = \frac{m\sqrt{4\rho - mx}}{2\rho\pi\sqrt{mx}} ,$$

where $\int_o^{\frac{4\rho}{m}} \omega(x)dx = 1$. The multiplicative terms q_n can be determined as follows: let R_n $(n \geq 0)$ be the polynomials described by

$$R_n(x) = P_n^{(1/2,\ -1/2)}\left(\frac{mx}{2\rho} - 1\right) . \tag{56}$$

Then (54) yields

$$R_o(x) = 1 ,$$

$$R_1(x) = \frac{m}{2\rho}\, x - \frac{1}{2} \tag{57}$$

and $$R_{n+1}(x) = \left[\frac{m(2n+1)}{2\rho(n+1)}\, x - \frac{2n+1}{n+1}\right] R_n(x) - \frac{(2n-1)(2n+1)}{2n(2n+2)} R_{n-1}(x),\ n \geq 1. \quad (57)$$

From recurrence (1) (where the coefficients are given in (48)), the coefficient of the highest power of x in $P_n(x)$ is

$$A_o A_1 A_2 \ldots A_{n-1} = \frac{(-1)^n m^n}{\rho^n}, \quad \text{for } n \geq 1 . \quad (58)$$

Similarly, from (57), the coefficient of the highest power of x in $R_n(x)$ is

$$\frac{2m^n[(2n-1)!]}{n\rho^n 4^n[(n-1)!]^2} \text{ for } n \geq 1 . \quad (59)$$

Consequently relations (55) through (59) show that $q_o = 1$ and

$$(-1)^n \frac{m^n}{\rho^n} = q_n \frac{2m^n[(2n-1)!]}{n\rho^n 4^n[(n-1)!]^2} \quad \text{for } n \geq 1 .$$

Hence,

$$q_o = 1$$

and $$q_n = \frac{n(-1)^n 4^n[(n-1)!]^2}{2[(2n-1)!]}, \ n \geq 1 . \quad (60)$$

Now, $$\zeta_o \overset{d}{=} \int_o^{\frac{4\rho}{m}} \omega(x)dx = 1$$

and, from (11),

$$\zeta_n \overset{d}{=} \int_o^{\frac{4\rho}{m}} P_n^2(x)\omega(x)dx = 1 \quad \text{for } n \geq 1 .$$

Theorem 5 shows that a solution of the initial-value problem (46)-(47) is thus given by

$$V_n(t) = \varepsilon_k \int_0^{\frac{4\rho}{m}} P_n(x)P_k(x)\omega(x)e^{-tx}\,dx$$

$$= \frac{\varepsilon_k q_n q_k m}{2\rho\pi} \int_0^{\frac{4\rho}{m}} P_n^{(1/2,-1/2)}(\tfrac{mx}{2\rho} - 1)P_k^{(1/2,-1/2)}(\tfrac{mx}{2\rho} - 1) \cdot \frac{\sqrt{4\rho - mx}}{\sqrt{mx}}\, e^{-tx}\,dx$$

$$= \frac{\varepsilon_k q_n q_k}{\pi} \int_0^{\pi} P_n^{(1/2,-1/2)}(-\cos\theta)P_k^{(1/2,-1/2)}(-\cos\theta) \cdot \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \sin\theta\, e^{\frac{2\rho}{m}(-1+\cos\theta)t}\, d\theta, n \geq 0.$$

But, from [22, p. 59], $P_j^{(1/2,-1/2)}(-\cos\theta) = \dfrac{\cos\left\{(2j+1)\frac{\theta}{2}\right\}}{q_j \cos(\frac{\theta}{2})}$ for $j \geq 0$;

and thus it is true that

$$V_n(t) = \frac{\varepsilon_k}{\pi} \int_0^{\pi} \cos\left\{(2n+1)\frac{\theta}{2}\right\} \cos\left\{(2k+1)\frac{\theta}{2}\right\}\left[\frac{\sqrt{1+\cos\theta}\,\sin\theta}{\sqrt{1-\cos\theta}\,\cos^2(\frac{\theta}{2})}\right] e^{\frac{2\rho}{m}(-1+\cos\theta)t}\, d\theta$$

$$= \frac{\varepsilon_k}{\pi} \int_0^{\pi} 2\cos\left\{(2n+1)\frac{\theta}{2}\right\} \cos\left\{(2k+1)\frac{\theta}{2}\right\} e^{\frac{2\rho}{m}(-1+\cos\theta)t}\, d\theta$$

$$= \frac{\varepsilon_k}{\pi} \int_0^{\pi} [\cos\{(n+k+1)\theta\} + \cos\{(n-k)\theta\}]e^{\frac{2\rho}{m}(-1+\cos\theta)t}\, d\theta$$

$$= \varepsilon_k \int_0^{1} [\cos\{(n+k+1)\pi x\} + \cos\{(n-k)\pi x\}]e^{\frac{2\rho}{m}(-1+\cos\pi x)t}\, dx,$$

for $n = 0,1,2,...$. This last expression has a very different appearance from the form first given for $V_n(t)$.

Example 2

In this section, after some preliminary remarks, an example is given in which the furnished solution of an infinite system is a good approximation to the solution of its finite truncation over a considerable range of the independent variable.

It is indicated in Chapter III that the solution given for the infinite system may be conveniently used to approximate the solution of a finite truncation, although there is no assurance of a "close" approximation. Error estimates for such approximations are supplied in Theorems 4 and 5; and in Lemmas 6 and 7 it is shown that if the interval of orthogonality is finite, the errors approach zero with increasing N (where N is the order of the finite system). However, the problem of obtaining practicable upper bounds for these errors for *fixed* N appears to be a difficult one which has not, as yet, been subjected to a detailed investigation.

The furnished solution for the infinite system has a computationally simple form, whereas the solution of the truncation is cumbersome and requires that the zeros of a certain polynomial (of degree the order of the system) be known. Consequently a solution of the infinite system is always an attractive approximation; it is also a useful one whenever the error incurred can be shown to be small.

The approximation $X_n(t) \sim x_n(t)$ (or $V_n(t) \sim v_n(t)$) is, in many cases, an accurate approximation for small values of t. However, there may be large values of t at which the error incurred by using the approximation is no longer small; the reader is referred to [6] for an interesting illustration in a physical setting.

A finite first-order differential system is considered below as an illustration. The first component of the solution is approximated by the first component of a solution to an infinite system for which the given system is a truncation. The error estimate for the pair of systems is also examined. This particular example was deliberately chosen to show that there exist non-trivial problems in which a solution of the infinite system is a good approximation to the solution of the finite system even when the order of the system is relatively small and t is reasonably large.

Consider the system

$$\begin{aligned} A_o\dot{y}_o &= B_oy_o - y_1 \\ A_1\dot{y}_1 &= -C_1y_o + B_1y_1 - y_2 \\ &\cdots \\ A_n\dot{y}_n &= -C_ny_{n-1} + B_ny_n - y_{n+1} \quad (n=1,2,3,\ldots,99) \\ &\cdots \\ A_{100}\dot{y}_{100} &= -C_{100}y_{99} + B_{100}y_{100} \quad , \end{aligned} \tag{61}$$

with initial conditions

$$y_o(0) = \varepsilon_o$$

and

$$y_n(0) = 0 \quad \text{for } n = 1,2,3,\ldots,100 \ , \tag{62}$$

where

$$A_n = \frac{2(2n+3)}{n+2} \quad \text{for } n \geq 0 \,,$$

$$B_n = \frac{-4(n+1)^2}{(n+2)(2n+1)} \quad \text{for } n \geq 0 \tag{63}$$

and

$$C_n = \frac{n(2n+3)}{(n+2)(2n+1)} \quad \text{for } n \geq 1 \,.$$

Let P_n $(n \geq 0)$ be the polynomials generated by the recurrence (1) in which the coefficients A_n, B_n and C_n have the values given in (63). It can be shown* that the polynomials P_n are (apart from a linear change of the independent variable and multiplicative factors which may depend on n but not on x) the Jacobi polynomials $P_n^{(0,1)}$. It then follows that the polynomials $P_n(x)$ are orthogonal over the interval $0 \leq x \leq 1$ with weight $\omega(x) = 2x$. Theorems 2 and 5 show that the first component of the solution to the initial-value problem (61) - (62) may be written

$$v_o(t) = \varepsilon_o[A_o C_1 C_2 C_3 \ldots C_{101}] \sum_{j=1}^{101} \frac{-e^{-tz_j}}{P_{102}(z_j)P'_{101}(z_j)} \,, \tag{64}$$

where $z_1, z_2, \ldots, z_{101}$ are the zeros of P_{101}. However, the first component of the corresponding solution to the infinite problem has the simple form

$$\begin{aligned} V_o(t) &= \varepsilon_o \int_o^1 e^{-tx}\,(2x)dx \\ &= 2\varepsilon_o \left[\frac{-xe^{-tx}}{t} - \frac{e^{-tx}}{t^2} \right]_o^1 \\ &= \frac{2\varepsilon_o(1 - e^{-t} - te^{-t})}{t^2} \,. \end{aligned} \tag{65}$$

---

*See the preceding example for a detailed illustration of the procedure followed here.

Furthermore, at any $t \geq 0$

$$E_v \overset{d}{=} V_o(t) - v_o(t)$$

$$= \frac{\varepsilon_o(C_1C_2C_3 \,\ldots\, C_{101})t^{202}\, e^{-t\xi}}{[(202)!]\; A_oA_1^2A_2^2A_3^2 \ldots A_{100}^2A_{101}}\text{, for some } \xi \text{ in } [0,1]\ ,$$

$$= \frac{\varepsilon_o(102)^2[(101)!]^4 t^{202} e^{-t\xi}}{(203)^2[(202)!]^3}\ .$$

A direct computation* shows that for any $t \geq 0$,

$$E_v \leq \varepsilon_o(10)^{-500}\ (6.0702)t^{202}e^{-t\xi}$$

$$\leq \varepsilon_o(10)^{-500}\ (6.0702)t^{202}\ .$$

Thus, for example, if $0 \leq t \leq 250$ then

$$E_v \leq (1.47)(10)^{-15}\varepsilon_o\ .$$

Example 3

The infinite linear differential system introduced below involves both first and second derivatives and hence is not of a type considered in Theorems 4 and 5. Consequently none of the techniques described earlier will supply a solution.

The class of systems having both first- and second-derivative terms is a natural area to consider for extending the methods of Chapter III. The example presented here suggests that further investigation

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*See M. Alliaume, Tables Jusqu'à n = 1200 des Factorielles n!, Louvain, Librairie Universitaire (1928).

of this area may be rewarding.

The infinite system discussed can be considered as a model for an infinite system of coupled harmonic oscillators in which the spring constants k_n $(n \geq 0)$, the masses m_n $(n \geq 0)$ and the damping coefficients d_n $(n \geq 0)$ have the form

$$
\begin{aligned}
k_n &= k_o r^n \quad \text{for } n \geq 0\,, \\
m_n &= m_o r^n \quad \text{for } n \geq 0 \\
\text{and} \qquad d_n &= d_o r^n \quad \text{for } n \geq 0\,,
\end{aligned}
\tag{66}
$$

where k_o, m_o, d_o and r are positive constants (see Figure 1).

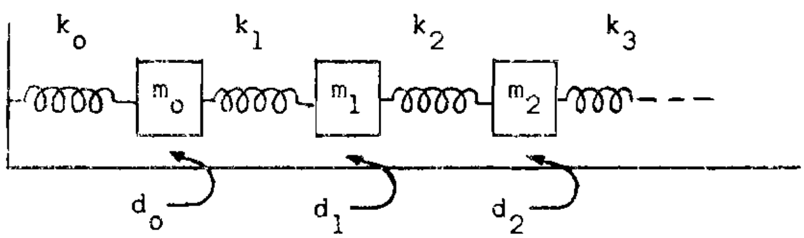

Figure 1. An Infinite System of Coupled Harmonic Oscillators with Viscous Damping.

With $\alpha \equiv \dfrac{d_o}{m_o}$ and $\beta \equiv \dfrac{k_o}{m_o}$, the equations of motion are

$$
\begin{aligned}
\ddot{y}_o &= - \alpha\dot{y}_o - \beta(1+r)y_o + \beta r y_1 \\
\ddot{y}_1 &= \beta y_o - \alpha\dot{y}_1 - \beta(1+r)y_1 + \beta r y_2 \\
\ddot{y}_2 &= \beta y_1 - \alpha\dot{y}_2 - \beta(1+r)y_2 + \beta r y_3 \\
&\quad\cdots \\
\ddot{y}_n &= \beta y_{n-1} - \alpha\dot{y}_n - \beta(1+r)y_n + \beta r y_{n+1}\,, \quad n \geq 1\,.
\end{aligned}
\tag{67}
$$

Let prescribed initial conditions be

$$\begin{aligned} & y_n(0) = \dot{y}_n(0) = 0 \quad \text{for} \quad n \neq k , \\ & y_k(0) = \alpha_k \quad \text{and} \quad \dot{y}_k(0) = \beta_k \quad (\text{for some fixed } k \geq 0). \end{aligned} \tag{68}$$

Then a direct induction argument and a simple check for the initial conditions (68) show that a solution of the infinite initial-value problem (67)-(68) is

$$X_n(t) = \left(\frac{1}{\sqrt{r}}\right)^{n-k} e^{-\frac{t\alpha}{2}} \int_0^1 \frac{\sin[(n+1)\pi x]\sin[(k+1)\pi x]}{v(x)} \Big\{(2\beta_k + \alpha_k \alpha)\sinh[tv(x)] + 2\alpha_k v(x)\cosh[tv(x)]\Big\} dx , \quad \text{for } n = 0,1,2,\ldots,$$

where

$$v(x) = \sqrt{\frac{\alpha^2}{4} - \beta(1+r) + 2\beta\sqrt{r}\cos \pi x} \, .$$

Note that for each x in $[0,1]$, $v(x)$ is either real and non-negative or pure imaginary. It is of some interest that $\{X_n(t)\}$ is a solution of (67) - (68) for <u>any</u> constants α and β (not necessarily positive).

APPENDIX

A number of physical counterparts of the infinite second- and first-order systems discussed in Theorems 4 and 5 are presented in the subsequent pages. Both a physical and electrical analog are given for each system, along with a tabulation of solutions in a number of cases.

The following conventions are retained throughout the remaining material:

[i] $P_n^{(\alpha,\beta)}$, $n \geq 0$, are the Jacobi polynomials described in Szegö [22];

[ii] $L_n^{(\alpha)}$, $n \geq 0$, are the Laguerre polynomials described in Szegö [22];

[iii] $S_n^{(\alpha)}$, $n \geq 0$, are the polynomials described in Chapter IV.

FINITE PROTOTYPES

Coupled Harmonic Oscillators

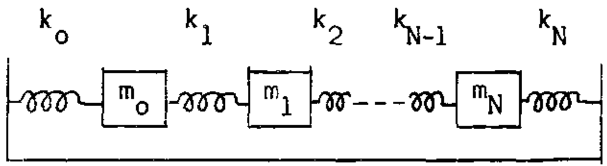

Figure 2(a). A Finite System of Coupled Harmonic Oscillators.

INFINITE PROTOTYPES

Coupled Harmonic Oscillators

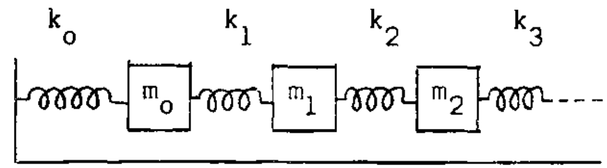

Figure 2(b). An Infinite System of Coupled Harmonic Oscillators.

(For $n \geq 0$, let k_n be the spring constant for the $(n+1)^{st}$ spring and let m_n be the $(n+1)^{st}$ mass.)

For $n = 0,1,2,\ldots,N-1$, let $x_n(t)$ denote the displacement of mass m_n at time t, measured positively to the right from the position which m_n occupies when all the springs are unstressed.

For $n = 0,1,2,\ldots$, let $X_n(t)$ denote the displacement of mass m_n at time t, measured positively to the right from the position which m_n occupies when all the springs are unstressed.

Ladder Network

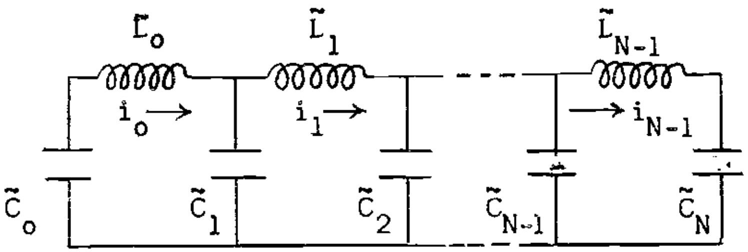

Figure 3(a). A Finite Ladder Network.

Ladder Network

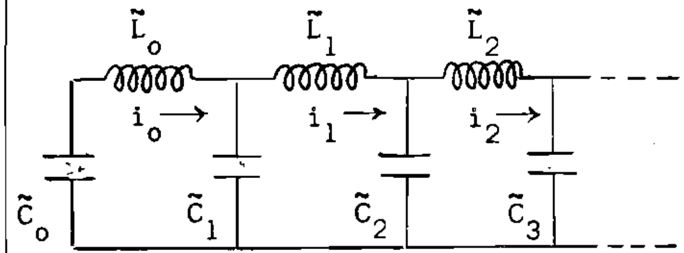

Figure 3(b). An Infinite Ladder Network.

(For $n \geq 0$, let $\tilde{L}_n$ be the inductance of the $(n+1)^{st}$ inductor and $\tilde{C}_n$ the capacitance of the $(n+1)^{st}$ capacitor.)

For $n = 0,1,2,\ldots,N-1$, let $x_n(t)$ denote the quantity $q_n(t)$, where

$$q_n(t) = \int_o^t i_n(\tau)d\tau + q_n(0^+)$$

and $q_{-1}(t) \equiv q_N(t) \equiv 0$.

[Note: The charge $Q_n(t)$ on the capacitor $\tilde{C}_n$ $(0 \leq n \leq N)$ is $q_{n-1}(t) - q_n(t)$.]

For $n = 0,1,2,\ldots,$ let $X_n(t)$ denote the quantity $q_n(t)$, where

$$q_n(t) = \int_o^t i_n(\tau)d\tau + q_n(0^+)$$

and $q_{-1}(t) \equiv 0$.

[Note: The charge $Q_n(t)$ on the capacitor $\tilde{C}_n$ $(n \geq 0)$ is $q_{n-1}(t) - q_n(t)$.]

FINITE MODEL (corresponding to the harmonic-oscillator prototype)

System

$$A_o\ddot{x}_o = B_o x_o - x_1$$
$$A_1\ddot{x}_1 = -C_1 x_o + B_1 x_1 - x_2$$
$$A_2\ddot{x}_2 = -C_2 x_1 + B_2 x_2 - x_3$$
$$\cdots$$
$$A_n\ddot{x}_n = -C_n x_{n-1} + B_n x_n - x_{n+1} \quad (n = 1,2,\ldots,N-2)$$
$$\cdots$$
$$A_{N-1}\ddot{x}_{N-1} = -C_{N-1}x_{N-2} + B_{N-1}x_{N-1} .$$

Initial Conditions

$$x_n(0) = \dot{x}_n(0) = 0 \quad \text{for } n \neq k,$$
$$x_k(0) = \alpha_k \quad \text{and} \quad \dot{x}_k(0) = \beta_k .$$

Solution

$$x_n(t) = \sum_{j=1}^{N} \lambda_j P_n(z_j)F(z_j,t) \quad \text{for } n=0,1,2,\ldots,N-1 .$$

INFINITE MODEL (corresponding to the harmonic-oscillator prototype)

System

$$A_o\ddot{X}_o = B_o X_o - X_1$$
$$A_1\ddot{X}_1 = -C_1 X_o + B_1 X_1 - X_2$$
$$A_2\ddot{X}_2 = -C_2 X_1 + B_2 X_2 - X_3$$
$$\cdots$$
$$A_n\ddot{X}_n = -C_n X_{n-1} + B_n X_n - X_{n+1} \quad (n = 1,2,3,\ldots).$$

Initial Conditions

$$X_n(0) = \dot{X}_n(0) = 0 \quad \text{for } n \neq k,$$
$$X_k(0) = \alpha_k \quad \text{and} \quad \dot{X}_k(0) = \beta_k .$$

Solution

$$X_n(t) = \int_a^b P_n(x)F(x,t)\omega(x)dx \quad \text{for } n=0,1,2,\ldots .$$

(For $F(x,t)$, see equation (22), p. 24.)

For the harmonic oscillators, $A_n = -m_n/k_{n+1}$ (for $n \geq 0$), $B_n = 1 + k_n/k_{n+1}$ (for $n \geq 0$) and $C_n = k_n/k_{n+1}$ (for $n \geq 1$). The replacement of k_n by $(\tilde{C}_n)^{-1}$ and m_n by $\tilde{L}_n$ here and in Table 1 yields mathematical models and various solutions for the ladder network.

Table 1. Solutions for Systems of Harmonic Oscillators*

	k_n	m_n	ζ_n	$P_n(x)$	$[a,b]$	$\omega(x)$
1.	$k_o r^n$, $n \geq 0$ (r constant)	$m_o r^n$ (r constant)	$\frac{1}{r^n}$	$q_n P_n^{(1/2,1/2)}\left(\frac{m_o}{2k_o\sqrt{r}}x - \frac{1+r}{2\sqrt{r}}\right)$, $q_n = \frac{(-1)^n 4^n [n!][(n+1)!]}{(\sqrt{r})^n[(2n+1)!]}$	$[\frac{k_o}{m_o}(1-\sqrt{r})^2, \frac{k_o}{m_o}(1+\sqrt{r})^2]$	$\frac{m_o\sqrt{2k_o m_o(1+r)x - m_o^2 x^2 - k_o^2(1-r)^2}}{2\pi k_o^2 r}$
2.	$k_o = 0$, $k_n = k$, $n \geq 1$ (k constant)	m (m constant)	1	$q_n P_n^{(1/2,-1/2)}(\frac{mx}{2k} - 1)$, $q_n = \frac{(-1)^n n 4^n[(n-1)!]^2}{2[(2n-1)!]}$	$[0, \frac{4k}{m}]$	$\frac{m\sqrt{4k - mx}}{2k\pi\sqrt{mx}}$
3.	$k_o = 0$, $k_n = nk$, $n \geq 1$ (k constant)	$(2n+1)m$ (m constant)	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}(\frac{m}{k}x - 1)$	$[0, \frac{2k}{m}]$	$\frac{m}{2k}$
4.	$k_o = 0$, $k_n = nk$, $n \geq 1$ (k constant)	m (m constant)	1	$L_n^{(0)}(\frac{m}{k}x)$	$[0, \infty)$	$\frac{m}{k}e^{-\frac{m}{k}x}$
5.	$k_n = k$, $n \geq 0$ (k constant)	$\frac{m}{n+1}$ (m constant)	$n+1$	$L_n^{(1)}(\frac{m}{k}x)$	$[0, \infty)$	$x\frac{m^2}{k^2}e^{-\frac{m^2}{k^2}x}$
6.	$k_o = 0$, $k_n = (n+\alpha\sin^2(\frac{n\pi}{2}))k$ for $n \geq 1$ (k constant)	$(2n+1+\alpha)m$ (m constant)	$\frac{1+\alpha}{2n+1+\alpha}$	$q_n S_n^{(\alpha)}(\frac{m}{k}x - 1)$, $q_n = \frac{(-1)^n(\alpha+1)(\alpha+3)\dots(\alpha+2n-1)}{(1+\alpha)(2)(3+\alpha)(4)\dots(n+\alpha\sin^2[\frac{n\pi}{2}])}$	$[0, \frac{2k}{m}]$	$\frac{m(\alpha+1)}{2k}\lvert\frac{m}{k}x - 1\rvert^{\alpha}$

*F. L. Cook has shown [4] that it is impossible to choose an infinite sequence of values for the spring constants and an infinite sequence of values for the masses in such a way that every truncation of the corresponding infinite chain of harmonic oscillators has a secular polynomial $P_N(x)$ which is (apart from a linear change of the independent variable and multiplicative factors which may depend on n but not on x) an Hermite polynomial.

FINITE PROTOTYPES

Sliding Plates

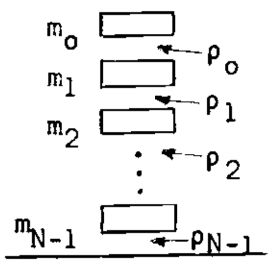

Figure 4(a). A Finite Stack of Sliding Plates.

INFINITE PROTOTYPES

Sliding Plates

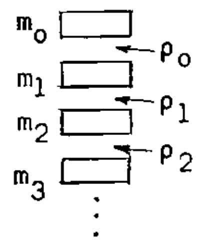

Figure 4(b). An Infinite Stack of Sliding Plates.

(For $n \geq 0$, let m_n be the mass of the $(n+1)^{st}$ plate and ρ_n the coefficient of viscous friction at the bottom of the $(n+1)^{st}$ plate.)

For $n = 0,1,2,\ldots,N-1$, let $v_n(t)$ be the velocity of the $(n+1)^{st}$ plate at time t.

For $n = 0,1,2,\ldots$, let $V_n(t)$ be the velocity of the $(n+1)^{st}$ plate at time t.

Ladder Network

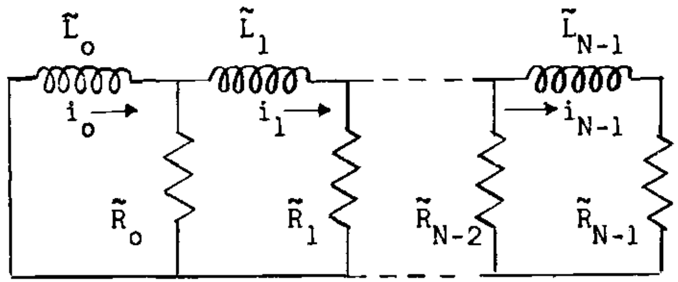

Figure 5(a). A Finite Ladder Network.

Ladder Network

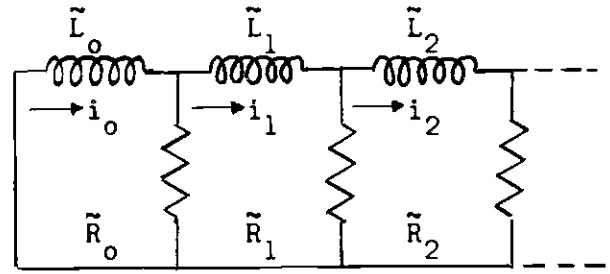

Figure 5(b). An Infinite Ladder Network.

For $n \geq 0$, let $\tilde{L}_n$ be the inductance of the $(n+1)^{st}$ inductor and $\tilde{R}_n$ the resistance of the $(n+1)^{st}$ resistor.

For $n = 0,1,2,\ldots,N-1$, let $v_n(t)$ be the current through the $(n+1)^{st}$ resistor at time t.

For $n = 0,1,2,\ldots$, let $V_n(t)$ be the current through the $(n+1)^{st}$ resistor at time t.

FINITE MODEL (corresponding to the sliding-plate prototype)

System

$$A_o\dot{v}_o = B_o v_o - v_1$$

$$A_1\dot{v}_1 = -C_1 v_o + B_1 v_1 - v_2$$

$$A_2\dot{v}_2 = -C_2 v_1 + B_2 v_2 - v_3$$

$$\cdots$$

$$A_n\dot{v}_n = -C_n v_{n-1} + B_n v_n - v_{n+1} \quad (n=1,2,3,\ldots,N-2)$$

$$\cdots$$

$$A_{N-1}\dot{v}_{N-1} = -C_{N-1}v_{N-2} + B_{N-1}v_{N-1} .$$

Initial Conditions

$$v_n(0) = 0 \text{ for } n \neq k$$

and $v_k(0) = \varepsilon_k$.

Solution

$$v_n(t) = \sum_{j=1}^{N} \lambda_j P_n(z_j) G(z_j,t) \text{ for } n=0,1,2,\ldots,N-1 .$$

INFINITE MODEL (corresponding to the sliding-plate prototype)

System

$$A_o\dot{V}_o = B_o V_o - V_1$$

$$A_1\dot{V}_1 = -C_1 V_o + B_1 V_1 - V_2$$

$$A_2\dot{V}_2 = -C_2 V_1 + B_2 V_2 - V_3$$

$$\cdots$$

$$A_n\dot{V}_n = -C_n V_{n-1} + B_n V_n - V_{n+1} \quad (n=1,2,3,\ldots) .$$

Initial Conditions

$$V_n(0) = 0 \text{ for } n \neq k$$

and $V_k(0) = \varepsilon_k$.

Solution

$$V_n(t) = \int_a^b P_n(x) G(x,t) \omega(x) dx \text{ for } n=0,1,2,\ldots .$$

(For $G(x,t)$, see equation 30, p. 28.)

For the sliding plates, $A_n = -m_n/\rho_n$ (for $n \geq 0$), $B_o = 1$, $B_n = 1 + \rho_{n-1}/\rho_n$ (for $n \geq 1$) and $C_n = \rho_{n-1}/\rho_n$ (for $n \geq 1$). The replacement of m_n by $\tilde{L}_n$ and ρ_n by $\tilde{R}_n$ here and in Table 2 yields mathematical models and various solutions for the ladder network.

Table 2. Solutions for Stacks of Sliding Plates.

	ρ_n	m_n	ζ_n	$P_n(x)$	$[a,b]$	$\omega(x)$
1.	ρ (ρ constant)	m (m constant)	1	$q_n P_n^{(1/2,-1/2)}(\frac{mx}{2\rho} - 1)$, $q_n = \frac{n(-1)^n 4^n[(n-1)!]^2}{2[(2n-1)!]}$	$[0, \frac{4\rho}{m}]$	$\frac{m\sqrt{4\rho - mx}}{2\rho\pi\sqrt{mx}}$
2.	$(n+1)\rho$ (ρ constant)	$(2n+1)m$ (m constant)	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}(\frac{m}{\rho}x - 1)$	$[0, \frac{2\rho}{m}]$	$\frac{m}{2\rho}$
3.	$(n+1+\alpha \sin^2(n+1)\frac{\pi}{2})\rho$ (ρ constant)	$(2n+1+\alpha)m$ (m constant)	$\frac{1+\alpha}{2n+1+\alpha}$	$q_n S_n^{(\alpha)}(\frac{m}{\rho}x - 1)$, $q_n = \frac{(-1)^n(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1)}{(1+\alpha)(2)(3+\alpha)(4)\ldots(n+\alpha\sin^2[\frac{n\pi}{2}])}$	$[0, \frac{2\rho}{m}]$	$\frac{m(\alpha+1)}{2\rho}\lvert\frac{m}{\rho}x-1\rvert^{\alpha}$
4.	$(n+1)\rho$ (ρ constant)	m (m constant)	1	$L_n^{(0)}(\frac{m}{\rho}x)$	$[0,\infty)$	$\frac{m}{\rho}e^{-\frac{m}{\rho}x}$

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Alan Greenwell Law was born August 21, 1936, in Seaham, Co. Durham, England. In 1948 his family migrated to Nanaimo, B. C., Canada, where he subsequently received the remainder of his secondary education. He earned the B.A. (1958) and M.A. (1961) degrees in mathematics at the University of British Columbia. From 1957 to 1961 he served as an Assistant at the Computing Centre of the University of British Columbia.

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