ADVANCED HIGH ORDER THEORIES AND ELASTICITY SOLUTIONS FOR CURVED SANDWICH COMPOSITE PANELS

A Thesis
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Aerospace Engineering

Georgia Institute of Technology

August 2017

ADVANCED HIGH ORDER THEORIES AND ELASTICITY SOLUTIONS FOR CURVED SANDWICH COMPOSITE PANELS

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To my mother, Numthip, this is for you.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my advisor, Dr. George Kardomateas. Through out my time from the very first moment to last day of PhD studies, his support, guidance, motivation, immense knowledge are tremendously propel to completion of PhD dissertation. Not only he has flawlessly fulfill the academic role, his kindness concerns toward student life and health are exemplary

Next, I would like to thank, Dr. Yehoshua Frostig, Technion-Israel, his expertise on the subject and guidance are invaluable. During his visits at Georgia Tech, his sincere willingness to help has shown as we had several in depth and detail discussions, after each visit, I always have papers filling with his hand written.

I would also like to thank committees, Dr. Satyanaraya Hanagud, Dr. Massimo Ruzzene, and Dr. Julian Rimoli for their comments, suggestions, and approvals. Their classes, during my early year at the institute, unquestionably have build a strong foundation to pursue my research.

I could not imagine my success without family support and their unconditional love, my mother who sacrifices her career twice, to stay with me during hard times due to medical issues for the recent two years, my father, an engineer, who is my inspiration and role model, my dearly sister, a medical doctor, I am very proud of you.

Living while pursuing the degree would not be possible without financial support, I would like to express my deepest gratitude to Dr. Yapa Rajapakse, Office of Naval Research, for an uninterrupted research grants and Dr. Jachiel Jagoda for his constant trust in my Teaching Assistant position.

These acknowledgments would not be complete without mentorship and friendship from Dr. Pisit Jarumaneeroj. And I am thankful to all individuals who fulfill my life as a foreign abroad, they are, Thai Student Organization, Badminton club at Georgia Tech, office lab colleagues, and Georgia Tech Cycling.

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SUMMARY

A new one-dimensional Extended High order Sandwich Panel Theory (EHSAPT) for curved panels is presented. The theory accounts for the sandwich core compressibility in the radial direction as well as the core circumferential rigidity. Two distinct core displacement fields are proposed and investigated. One is a logarithmic (it includes terms that are linear, inverse, and logarithmic functions of the radial coordinate). The other is a polynomial (it consists of second and third order polynomials of the radial coordinate) and it is an extension of the corresponding field for the flat panel. In both formulations the two thin curved face sheets are assumed to be perfectly bonded to the core and follow the classical Euler-Bernoulli beam assumptions. The new theory is formulated by Principle of Minimum Total Potential Energy for static and Hamilton's principle for free vibration analysis. Then, the linear elasticity displacement formulation and solutions for a generally asymmetric simply support sandwich curved beam/panel consisting of orthotropic core and face sheets are presented. Closed-form analytical solutions are derived for the curved sandwich subjected to a top face distributed static transverse loading; and the method of Frobenius series is applied in free vibration analysis. Next, due to the curvature, the first order shear deformation (FOSD) theory for curved sandwich panels is not a direct extension of the corresponding one for flat panels and thus, it is formulated accordingly, and its unique features, such as the reference curve, are discussed. Three versions of the FOSD theory are formulated: the one based on direct variational formulation based on the assumed through-thickness displacement field (termed "basic"), one based on the definition of an equivalent shear modulus for the section (termed "G_{eq}") and one based on derivation of a shear correction factor, which is considered in conjunction with the equivalent shear modulus. In addition, the classical theory for curved sandwich panels which does not include transverse shear is also presented. The results from following: the new proposed EHSAPT, the existing high order sandwich panel theory HSAPT (from literature), three variants FOSD theory, and Classical theory are compared with Elasticity which serves as a benchmark in assessing the accuracy of the various sandwich panel theories. The case examined are transverse static loads and free vibration of simply supported curved sandwich panels, for which a closed form elasticity solution is formulated. It is shown that the new EHSAPT is the most accurate among other presented theories with the logarithmic formulation is more accurate than the polynomial.

CHAPTER 1
INTRODUCTION

In aerospace or naval construction, structural shapes are often not flat (e.g. ship hulls or airplane fuselages). Thus, although the majority of the structural theories are formulated and studied on flat panels, there is a definite need to formulate such theories for the geometry of curved panels and properly address the effect of curvature. When it comes to sandwich structures, which consist of two thin high-stiffness face sheets, usually metallic or laminated composites, bonded to a core made of low-density and low-stiffness materials such as honeycomb, polymer foam or balsa wood, most of the research has been done on flat panels, see book by Carlsson and Kardomateas (2011) [1].

The majority of the research on this topic has been done for flat panels. For static problems involving laminated composite or sandwich flat panels, a few closed form elasticity solutions exist, namely by Vlasov (1957) [2] for isotropic plates, by Pagano [3] (1969,1970),[4] for a beam and plate configuration respectively, both under restrictive assumptions, extended by Kardomateas (2009) [5] and Kardomateas and Phan (2011) [6] for general sandwich plates and beams, respectively. Regarding the dynamic case, an elasticity solution for the free vibration of homogeneous and laminated plates was presented by Srinivas et al [7] (1970) and an elasto-dynamic solution for a sandwich beam/wide plate under blast loading was developed by Kardomateas et al (1992) [8]. The latter was later extended to a sandwich plate of arbitrary aspect ratio by Kardomateas et al (2015) [9].

In sandwich panels the core is supposed to provide the shear resistance/stiffness and including transverse shear has been long recognized as a necessary characteristic of sandwich analysis [1] . Thus, the simplest sandwich structural theories assume that the core is incompressible in the transverse direction and with negligible in-plane rigidity in the longitudinal direction. The most popular theory that includes the transverse shear effect is the First Order Shear Deformation Theory that replaces the layered panel with an equivalent single layer and assumes that the core is incompressible, see for example the books by Allen (1969) [10] or Carlsson and Kardomateas (2011) [1]. However, the first order shear alone does not account for parabolic shear stress distribution unless augmentation of shear correction factor. Reddy (1984) [11] the simple high order shear theory accounts for parabolic distribution and zero shear stress on surfaces while retain the same number of unknown displacement functions as in the first order theory; thus, the theory is categorized as equivalent single layer. Numerous refined theories for sandwich panel analyis were reported [12], [13]. According to Carrera and Brischetto (2009) [12], Such theories were classified into: equivalent single layered model, layerwise model base on layered dependency of unknown variables and these consequently were characterized into classical, first order, higher order, zigzag, layer wise, and mixed theories. Carrera (2003) [14] historically summarized theories for multilayered structures that include zigzag effect which describing a through thickness piecewise displacement field and fulfill interlaminar traction conditions.

Experimental studies of sandwich foam cores subject to impact blast loading reported large core compressive deformations and core cracking failures [15], [16], [17], [18], [19], [20]. Recent advanced approach that include core compressibilty effect is based on high-order models, see for Frostig et al (1992) [21], and Hohe and Librescu (2003) [22]. Despite both are describing thickness wise 2nd and 3rd order of sandwich core transverse and tangential displacement respectively, the expressions are difference. Frostig et al (1992) [21] core displacement field is derived from the elasticity equations in a closed-form by assuming no in-plane core rigidity which is very accurate for very flexible foam core. In suchs model, the overall response is a combination of the responses of the face sheets and the core through equilibrium and compatibility.

The most recent advanced high order theory is the Extended High Order Sandwich Panel Theory which includes the effect of the in-plane core rigidity. The theory hereditarily derive from Frostig et al (1992) [21]; and has been formulated for flat beams/wide panels, Phan et al (2012) [23] for the static, Phan et al (2013) [24] for the dynamic case, and Siddiqui (2015) [25] for plates. This theory was shown to be very close to the elasticity prediction in both the static and the dynamic cases. In addition, Phan et al (2012) [26] applied the theory to study instabilities of honeycomb sandwich panels, the results shown to be in a good agreement with the experiments. Finite element method based on the theory was also formulated, see Yuan et al (2015) [27], to study geometric nonlinearity effects, see Yuan et al (2016) [28].

The literature is rather limited when it comes to sandwich curved panels. Unlike the flat plate geometry, the curvature introduces strains that are not a linear function of the distance from the reference axis, but have a dependence on this distance in the denominator, as will be seen in the analysis section. Noor et al (1996, 1997) [29],[30] outlined models that assume that the core is incompressible following the First-Order Shear Deformable (FOSD) Theory and Vaswani et el. (1988) [31] studied the vibration and damping analysis of curved sandwich beams by using the Flugge shell theory while assuming that the face sheets are membranes only and the core is incompressible. Yin-Jiang (1989) [32] studied the stability of shallow cylindrical sandwich panels with orthotropic layers by use of the FOSD theory but with membrane and flexural rigidities of the face sheets. Furthermore, Di Sciuva(1987) [33] developed a model that takes into account the shear deformation but assumes that the core is incompressible and linear. The use of the FOSD in the literature has been mostly done by neglecting the distance from reference axis, which is valid for very thin curved panels. For example, Qatu (1993) [34] made this assumption in the study of natural frequencies for laminated composite curved beams. Another consequence of non-linearity of the strains due to the curvature, is that the reference axis definition is not, in general, at the middle of the section, and needs to be defined accordingly Timoshenko (1940) [35]. Again, the use of the FOSD in the literature has been mostly done by considering the reference axis to be at mid-thickness. For example, Tseng et al (2000) [36] made this assumption, along with the assumption of a parabolic distribution of shear, by direct extension of the one from flat panels. The study was done for laminated composites and this assumption also led to the calculation of a shear correcting function, which, again, is the one from the homogeneous flat panel analysis. In this report, FOSD will be formulated with inclusion of reference axis and the proper shear correction factor will be derived for curved panels geometry.

The compressibility of the core was included in the High Order Sandwich Panel Theory introduced by Frostig et al (1992) [21] and was adopted for the curved sandwich panel configuration by Frostig (1999) [37] for the linear and Bozhevolnaya and Frostig (1997) [38] for the nonlinear case. In addition, Bozhevolnaya and Frostig (2001) [39] studied the free vibration and Frostig and Thomsen (2009) [40] the thermal effects that induce deformation as well as degradation of properties in curved sandwich panels. As already mentioned, the effect of the in-plane core rigidity has been included in the extended high order sandwich panel theory (EHSAPT), already formulated for flat panels in Phan et al (2012) [23]. The formulation is extended to the configuration of a curved sandwich panel. The theory will be developed based on the following assumptions: The face sheets have in-plane (circumferential) and bending rigidities with negligible shear deformations, see Brush and Almroth (1975) [24] and Simitses (1976) [29]. The core is considered as a 2D linear elastic continuum obeying small deformation kinematic relations and where the core height may change (compressible core) and the section plane does not remain plane after deformation. The core is assumed to possess shear, radial normal and circumferential stiffness; full bonding between the face sheets and the core is assumed and the interfacial layers can resist shear as well as radial normal stresses. Two variant core displacement field base on logarithmic function whose essence from Frostig (1999) [37] and high order polynomial whose directly extend from straight flat case Phan et al (2012) [23] will be studied.

Elasticity closed-form static solutions for curved panels (or beams) have been derived based on a stress function approach in Timoshenko and Goodier (1970) [41] for the case of an isotropic curved bar under pure bending or a curved bar loaded by a force at the end. In both cases the displacement field included a $\log r$ term. The case of an anisotropic curved bar, again under pure bending or loaded by a force at the end, was studied again by the stress function approach in Lekhnitskii (1981) [42]. The same approach was used by Ren (1987) [43] to extend Lekhnitskii's solution to the case of laminated cylindrical shells. In this report, the closed-form elasticity solution shall be derived for a sandwich curved panel under distributed load on the top bounding surface and a different approach shall be followed, i.e. a displacement-based rather than a stress function-based approach. The problem under consideration has full dependence on r (the radial coordinate) and θ (the circumferential coordinate). It should be noted that for the much simpler axisymmetric problem of a cylindrical shell (only dependence on r) rather than the curved panel configuration, Kardomateas (2001) [44] used Lekhnitskii's approach to derive a closed form elasticity solution for the axisymmetric problem of a sandwich cylindrical shell under external pressure, internal pressure and axial load.

Three dimensional elasto-dynamics of cylinders and cylindrical shells were reviewed in Soldatos (1994) [45]. Numerous free vibration studies reported in Soldatos (1994) [45] revealed that solving isotropic material problems often in the scope of well known Bessel functions. The complexities arise when considering orthotropic material because additional elastic constants are introduced [46], [47]. A more general solution method namely, the method of Frobenius series are generally applied; and solving the ordinary differential equations using the method is very detail. Moreover, the series convergence and computational resource becomes an issue in order to obtain higher frequency range. Mirsky (1964) [46] derived a closed form solution for axisymmetric of orthotropic cylinder using a method of Frobenius series with restrictive solution forms. Ding and Chen (1993) [48] studied the same problem and spherical shells, they solved the problems in a matrix form and carefully included various expressions of solution; however, no numerical results were produced. A full three dimensional flexural waves in anisotropic bars (solid cylindrical shape) was studied by Ohnabe and Nowinski (1971) [49]; solutions were approximate and limited to trasnverly isotropic flexural waves. And orthotropic cylinders was studied in Chou et al (1972) [47]; numerical results were produced for infinitely long solid cylinders. Armenakas and Reitz (1973) [50] studied free vibration cylindrical hollow shell, various solution expressions were addressed; and simplified cases, for instance, axisymmtric non-torsional, torsional results were numerically generated. Most mentioned studies were considered cylindrical geometry, thus, complete circular ring. Curved panels (incomplete ring full dependence on r and θ) were studied in Sharma (2001) [51] for transversely isotropic; potential functions whose derivatives are displacement were applied to help facilitate solution and obtained as modified Bessel functions. Sandwich curved panels are predominantly made from orthotropic material; thus, the Frobenius method will be applied to solve the free vibration problem and various associated solution forms will be derived.

A focus on this thesis is to develop effective tools to analyze curved sandwich beams and wide panels, in particular, they are subjected to static loads and free vibration analysis. Firstly, the new Extended high order theory will be developed for curved sandwich panel as it had been proven in flat sandwich beams and plates [23], [24], and [25]. Next, elasticity closed form analytical solutions will be derived for curved sandwich panels with simply supported on both ends; the purpose is to serve as an accuracy benchmark against other theories. Lastly, the concept of equivalent shear modulus will be incorporated to the First order shear deformation theory, unifying layerwise properties to an equivalent single layer; and appropriate shear correction factor derivation for curved sandwich configuration will be shown. In addition, classical theory was formulated as a self contain; and the high order theory [37] numerical results will be included to compare.

CHAPTER 2

CURVED SANDWICH COMPOSITES BEAMS/WIDE PANELS THEORIES

In the following, we consider a curved sandwich panel of a unit width, and consisting of two thin top/bottom face sheets of thickness f_t and f_b, respectively, separated by a thick core of thickness $2c$. Both face sheets and core have cylindrical orthotropy materials. The panel is bounded in the plane by two concentric circles of radii R_1 and R_2 and two radial segments forming an arbitrary angle α (figure 2.1). The mean radii for the top face, core and bottom face are denoted by R_t, R_c and R_b, respectively. In addition, we denote by r_{tc} = $R_c + c$, the radius at the top face/core interface and by r_{bc} = $R_c - c$, the corresponding radius at the bottom face/core interface. A polar coordinate system (r, θ) is used to describe the sandwich geometry and kinematics. The polar angle θ is measured from the left end. Furthermore, local normal-tangential coordinate systems are defined at mid top/bottom face sheets and mid core to help simplify the formulation; which are (z_t, θ),(z_c, θ) and (z_b, θ). Radial and circumferential displacements are represented by w and u respectively, and the superscripts t, b, c denote top face sheet, bottom face sheet, and core.

2.1 Extended High Order Theory for Curved Sandwich Beams/Panels

A new one-dimensional high order sandwich panel theory for curved panels is presented. The theory accounts for the sandwich core compressibility in the radial direction as well as the core circumferential rigidity. Two distinct core displacement fields are proposed and investigated. One is a logarithmic (it includes terms that are linear, inverse, and logarithmic functions of the radial coordinate). The other is a polynomial (it consists of second and third order polynomials of the radial coordinate) and it is an extension of the corresponding field for the flat panel. In both formulations the two thin curved face sheets are assumed to be perfectly bonded to the core and follow the classical Euler-Bernoulli beam assumptions.

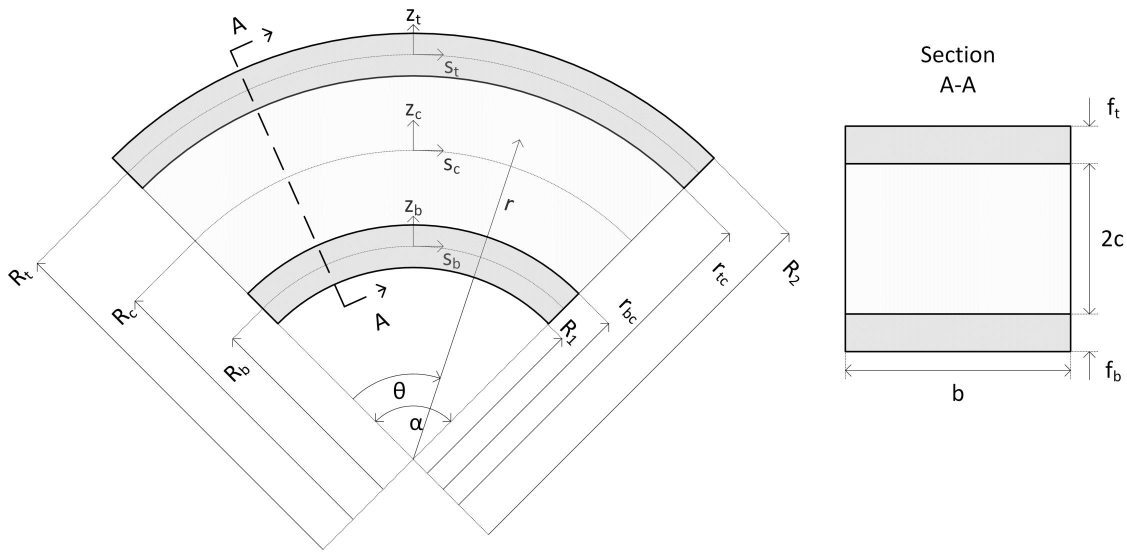

Figure 2.1: Geometry of a curved sandwich panel

2.1.1 Kinematic descriptions

The two face sheets are assumed to follow the assumptions of the Euler-Bernoulli beam theory. Therefore, the displacement field for the top face sheet, $-f_t/2 \leq z_t \leq f_t/2$, is:

$$w^t(z_t, \theta) = w_0^t(\theta) \; ; \quad u^t(z_t, \theta) = u_0^t(\theta) + \left[u_0^t(\theta) - {w_0^t}'(\theta)\right] \frac{z_t}{R_t} \; , \tag{2.1a}$$

and for the bottom face sheet, $-f_b/2 \leq z_b \leq f_b/2$, is:

$$w^b(z_b, \theta) = w_0^b(\theta) \; ; \quad u^b(z_b, \theta) = u_0^b(\theta) + \left[u_0^b(\theta) - {w_0^b}'(\theta)\right] \frac{z_b}{R_b} \; . \tag{2.1b}$$

The only non-zero corresponding linear strain is:

$$\epsilon_{\theta\theta}^{t,b} = \epsilon_{\theta\theta 0}^{t,b} + z_{t,b} \kappa_{\theta\theta 0}^{t,b} \; , \tag{2.1c}$$

where

$$\epsilon_{\theta\theta 0}^{t,b}(\theta) = \frac{1}{R_{t,b}} \left[u_0^{t,b\,\prime}(\theta) + w_0^{t,b}(\theta) \right] \ ; \quad \kappa_{\theta\theta 0}^{t,b}(\theta) = \frac{1}{R_{t,b}} \left[u_0^{t,b\,\prime}(\theta) + w_0^{t,b}(\theta) \right] \ . \tag{2.1d}$$

The Extended High Order Sandwich Panel Theory allows for core compressibility and circumferential rigidity effects. Two distinct presumed core displacement fields are proposed.

One is a closed form displacement function involving a logarithmic term. The rational for adopting this displacement profile is as follows: In general, the core in sandwich structures has axial rigidity significantly less than that of the faces. Accordingly, if we neglect the core circumferential stress, i.e. we assume that $\sigma_{\theta\theta} = 0$, then the elasticity equilibrium equations become:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr}}{r} = 0 \ ; \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \ . \tag{2.2a}$$

These can be re-written as:

$$\frac{\partial (r\sigma_{rr})}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} = 0 \ ; \quad \frac{\partial (r^2 \tau_{r\theta})}{\partial r} = 0 \ . \tag{2.2b}$$

Integrating the second of (2.2b) for $\tau_{r\theta}$ and then substituting in the first of (2.2b) and integrating for σ_{rr}, results in:

$$\tau_{r\theta} = \frac{f_1(\theta)}{r^2} \ ; \quad \sigma_{rr} = \frac{f_1'(\theta)}{r^2} + \frac{f_2(\theta)}{r} \ . \tag{2.2c}$$

Using the constitutive and strain-displacement relations gives:

$$\sigma_{rr} = E_c \epsilon_{rr} = E_c \frac{\partial w}{\partial r} = \frac{f_1'(\theta)}{r^2} + \frac{f_2(\theta)}{r} \ , \tag{2.2d}$$

which would integrate to

$$w(r,\theta) = \frac{1}{E_c}\left[f_2(\theta)\ln r - \frac{f_1'(\theta)}{r} + f_3(\theta)\right] . \tag{2.2e}$$

Similarly, the shear stress relation becomes:

$$\tau_{r\theta} = G_c\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial r} - \frac{u}{r}\right) = G_c r\left[\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right] = \frac{f_1(\theta)}{r^2} . \tag{2.2f}$$

Substituting w from (2.2e) and integrating gives:

$$u(r,\theta) = -\frac{f_1(\theta)}{2G_c r} - \frac{1}{E_c}\left[f_2'(\theta)(1+\ln r) + f_3' - \frac{f_1''}{2r}\right] + r f_4(\theta) . \tag{2.2g}$$

Therefore, it can be seen that in this case the displacements would contain terms $\ln r$, r and $1/r$.

Although our high order theory includes the axial rigidity of the core, it is reasonable to assume, based on the above discussion, that the curvature of the panel would induce a logarithmic dependence, thus the first displacement profile, termed "logarithmic", is as follows:

$$u_c(r,\theta) = u_0^c(\theta) + r u_1^c(\theta) + \frac{u_2^c(\theta)}{r} + u_3^c(\theta)\ln(r) , \tag{2.3a}$$

$$w_c(r,\theta) = w_0^c(\theta) + \frac{w_1^c(\theta)}{r} + w_2^c(\theta)\ln(r) . \tag{2.3b}$$

Assuming perfect bonding of the two face sheets with the core, displacement continuity is imposed at the two interfaces resulting in four compatibility equations:

$$w^t(z_t = -\frac{f_t}{2},\theta) = w^c(r = r_{tc},\theta) ; \quad u^t(z_t = -\frac{f_t}{2},\theta) = u^c(r = r_{tc},\theta) , \tag{2.4a}$$

$$w^b(z_b = +\frac{f_b}{2}, \theta) = w^c(r = r_{bc}, \theta) \; ; \quad u^b(z_b = +\frac{f_b}{2}, \theta) = u^c(r = r_{bc}, \theta) \; . \tag{2.4b}$$

The four compatibility equations are then solved for the four dependent variables $u_2^c(\theta)$, $u_3^c(\theta)$, $w_1^c(\theta)$, and $w_2^c(\theta)$. These are obtained in terms of

$$D = r_{tc} \ln r_{tc} - r_{bc} \ln r_{bc} \; , \tag{2.5a}$$

as follows:

$$u_2^c(\theta) = \frac{r_{tc} r_{bc}}{D} \Big\{ u_0^b(\theta) \ln r_{tc} - u_0^t(\theta) \ln r_{bc} + u_0^c(\theta)(\ln r_{bc} - \ln r_{tc}) + u_1^c(\theta)(r_{tc} \ln r_{bc} - r_{bc} \ln r_{tc}) + \\ + \frac{(f_b/2) \ln r_{tc}}{r_b} [u_0^b(\theta) - w_0^b{}'(\theta)] + \frac{(f_t/2) \ln r_{bc}}{r_t} [u_0^t(\theta) - w_0^t{}'(\theta)] \Big\} \; , \tag{2.5b}$$

$$u_3^c(\theta) = \frac{1}{D} \Big[r_{tc} u_0^t(\theta) - r_{bc} u_0^b(\theta) + (r_{bc} - r_{tc}) u_0^c(\theta) + (r_{bc}^2 - r_{tc}^2) u_1^c(\theta) - \\ - \frac{(f_b/2) r_{bc}}{r_b} \left[u_0^b(\theta) - w_0^b{}'(\theta) \right] - \frac{(f_t/2) r_{tc}}{r_t} \left[u_0^t(\theta) - w_0^t{}'(\theta) \right] \Big] \; , \tag{2.5c}$$

$$w_1^c(\theta) = \frac{r_{tc} r_{bc}}{D} \left[w_0^b(\theta) \ln r_{tc} - w_0^t(\theta) \ln r_{bc} + w_0^c(\theta) \ln \frac{r_{bc}}{r_{tc}} \right] , \tag{2.5d}$$

$$w_2^c(\theta) = \frac{1}{D} \left[r_{tc} w_0^t(\theta) - r_{bc} w_0^b(\theta) + (r_{bc} - r_{tc}) w_0^c(\theta) \right] , \tag{2.5e}$$

The corresponding linear strains in polar coordinates are:

$$\epsilon_{rr}^c(r, \theta) = \frac{\partial w_c(r, \theta)}{\partial r} \; , \tag{2.6a}$$

$$\epsilon_{\theta\theta}^c(r, \theta) = \frac{1}{r} \frac{\partial u_c(r, \theta)}{\partial \theta} + \frac{w_c(r, \theta)}{r} \; , \tag{2.6b}$$

$$\gamma_{r\theta}^{c}(r,\theta) = \frac{\partial u_c(r,\theta)}{\partial r} + \frac{1}{r}\frac{\partial w_c(r,\theta)}{\partial \theta} - \frac{u_c(r,\theta)}{r} \ . \tag{2.6c}$$

The other presumed core displacement field takes the form of a high-order polynomial and it is a direct extension of the displacement field for sandwich flat panels [23]. The polynomial core kinematics description is simple when expressed in the local coordinate (z_c, θ) as follows:

$$\begin{aligned} u^c(z_c,\theta) = \left(1-\frac{z_c^2}{c^2}\right)u_0^c(\theta) + z_c\left(1-\frac{z_c^2}{c^2}\right)u_1^c(\theta) + \frac{[(f_b/2)+R_b]z_c^2}{2c^2R_b}\left(1-\frac{z_c}{c}\right)u_0^b(\theta)+ \\ + \frac{[(-f_t/2)+R_t]z_c^2}{2c^2R_t}\left(1+\frac{z_c}{c}\right)u_0^t(\theta) + \frac{f_b z_c^2}{4c^2R_b}\left(-1+\frac{z_c}{c}\right){w_0^b}'(\theta)+ \\ \frac{f_t z_c^2}{4c^2R_t}\left(1+\frac{z_c}{c}\right){w_0^t}'(\theta) \ , \end{aligned} \tag{2.7a}$$

$$w_c(z_c,\theta) = \left(\frac{-z_c}{2c}+\frac{z_c^2}{2c^2}\right)w_0^b(\theta) + \left(1-\frac{z_c^2}{2c^2}\right)w_0^c(\theta) + \left(\frac{z_c}{2c}+\frac{z_c^2}{2c^2}\right)w_0^t(\theta) \ . \tag{2.7b}$$

The polynomial displacement field in (2.7) has been defined in such a way that satisfies the four interfacial displacement compatibility conditions, (2.4). Thus, in contrast with (2.3), the polynomial functions do not contain $u_2^c(\theta)$, $u_3^c(\theta)$, $w_1^c(\theta)$,and $w_2^c(\theta)$ that need to be determined through interfacial compatibilities.

The corresponding linear-strain in the local coordinate (z_c, ϕ) are the same as in eqs (2.6), with interchanging variable, $r = z_c + R_c$

Thus, the extended high order theory formulation for sandwich curved panels is in terms of seven dependent variables as a function of θ: two for the top face sheet, w_0^t, u_0^t, two for the bottom face sheet, w_0^b, u_0^b, and three for the core, w_0^c, u_0^c,and u_1^c.

2.1.2 Constitutive relations

In the following, $c_{ij}^{t,b,c}$ denote the orthotropic material stiffness constants where $i, j = 1, 3, 5$ and $1 \equiv \theta$, $3 \equiv r$, and $5 \equiv r\theta$. The orthotropic stress-strain relations for the core read:

$$\begin{bmatrix} \sigma_{\theta\theta}^c \\ \sigma_{rr}^c \\ \tau_{r\theta}^c \end{bmatrix} = \begin{bmatrix} c_{11}^c & c_{13}^c & 0 \\ c_{13}^c & c_{33}^c & 0 \\ 0 & 0 & c_{55}^c \end{bmatrix} \begin{bmatrix} \epsilon_{\theta\theta}^c \\ \epsilon_{rr}^c \\ \gamma_{r\theta}^c \end{bmatrix} , \tag{2.8a}$$

The stiffness matrix coefficient c_{ij}^c are derived from an inverse of following cylindrical orthotropic compliance matrix :

$$A = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{31}}{E_3} & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{1}{E_3} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} , \tag{2.8b}$$

hence, core stiffness coefficients are :

$$c_{11}^c = \frac{E_1^c}{1 - \nu_{13}^c \nu_{31}^c} \; ; \quad c_{33}^c = \frac{E_3^c}{1 - \nu_{13}^c \nu_{31}^c} \; ; \quad c_{13}^c = \frac{E_1^c \nu_{31}^c}{1 - \nu_{13}^c \nu_{31}^c} \; ; \quad c_{55}^c = G^c \, . \tag{2.8c}$$

For the face sheets, from the kinematic assumptions (2.1), the only non-zero strain is the $\epsilon_{\theta\theta}$, and as a consequence,

the associated non-zero resulting stress are:

$$\sigma_{\theta\theta}^{t,b} = c_{11}^{t,b} \epsilon_{\theta\theta}^{t,b} \; ; \quad \text{where} \, , c_{11}^{t,b} = E_1^{t,b} \tag{2.8d}$$

2.1.3 Principle of Minimum Total Potential Energy

Governing equations and associated boundary conditions are derived from the Principle of Minimum Total Potential energy:

$$\delta(U+V)=0\;. \tag{2.9a}$$

where U is the strain energy of the sandwich panel and V is the external potential due to the applied loads.

The first variation of the strain energy of the sandwich beam is:

$$\delta U=\int\limits_{0}^{\alpha}\Bigg[\int\limits_{-f_b/2}^{f_b/2}\sigma_{\theta\theta}^{b}\delta\epsilon_{\theta\theta}^{b}R_b dz_b+\int\limits_{r_{bc}}^{r_{tc}}(\sigma_{rr}^{c}\delta\epsilon_{rr}^{c}+\sigma_{\theta\theta}^{c}\delta\epsilon_{\theta\theta}^{c}+\tau_{r\theta}^{c}\delta\gamma_{r\theta}^{c})r dr+\int\limits_{-f_t/2}^{f_t/2}\sigma_{\theta\theta}^{t}\delta\epsilon_{\theta\theta}^{t}R_t dz_t\Bigg]b d\theta\;. \tag{2.9b}$$

Recall that, in this paper, two distinct core displacement fields are presented: one is described in polar coordinates (r,θ), the other is described in the local tangential coordinates (z_c,θ). In eqn(2.9b), the core strain-energy integral, denoted by the superscript ^{c}, is in the polar coordinates. If the core strain-energy is expressed in the local tangential coordinates (z_c,θ), the integral can be easily converted to the local coordinate as $r=z_c+R_c$, $dr=dz_c$, and the integration limits is changed from $\int\limits_{r_{bc}}^{r_{tc}}$ to $\int\limits_{-c}^{c}$. Also, please notice that due to the small thickness of the faces, in the first integral in (2.9b) we carry the integration with $R_b dz_b$ instead of $(R_b+z_b)dz_b$; same with the last integral in (2.9b).

The sandwich panel is subjected to various loadings (see figure 2.2) on both face sheets, and the first variation of the external potential is:

$$\begin{aligned}\delta V=-\int\limits_{0}^{\alpha}\Big\{&[n_{\theta\theta}^{t}(\theta)\delta u_0^t(\theta)+q^t(\theta)\delta w_0^t(\theta)+m^t(\theta)\delta\beta^t(\theta)]R_t\\&+[n_{\theta\theta}^{b}(\theta)\delta u_0^b(\theta)+q^b(\theta)\delta w_0^b(\theta)+m^b(\theta)\delta\beta^b(\theta)]R_b\Big\}d\theta\\&-N_t\delta u_0^t(\theta_e)-P_t\delta w_0^t(\theta_e)+M_t\delta\beta^t(\theta_e)-N_b\delta u_0^b(\theta_e)-P_b\delta w_0^b(\theta_e)+M_b\delta\beta^b(\theta_e)\;.\end{aligned} \tag{2.9c}$$

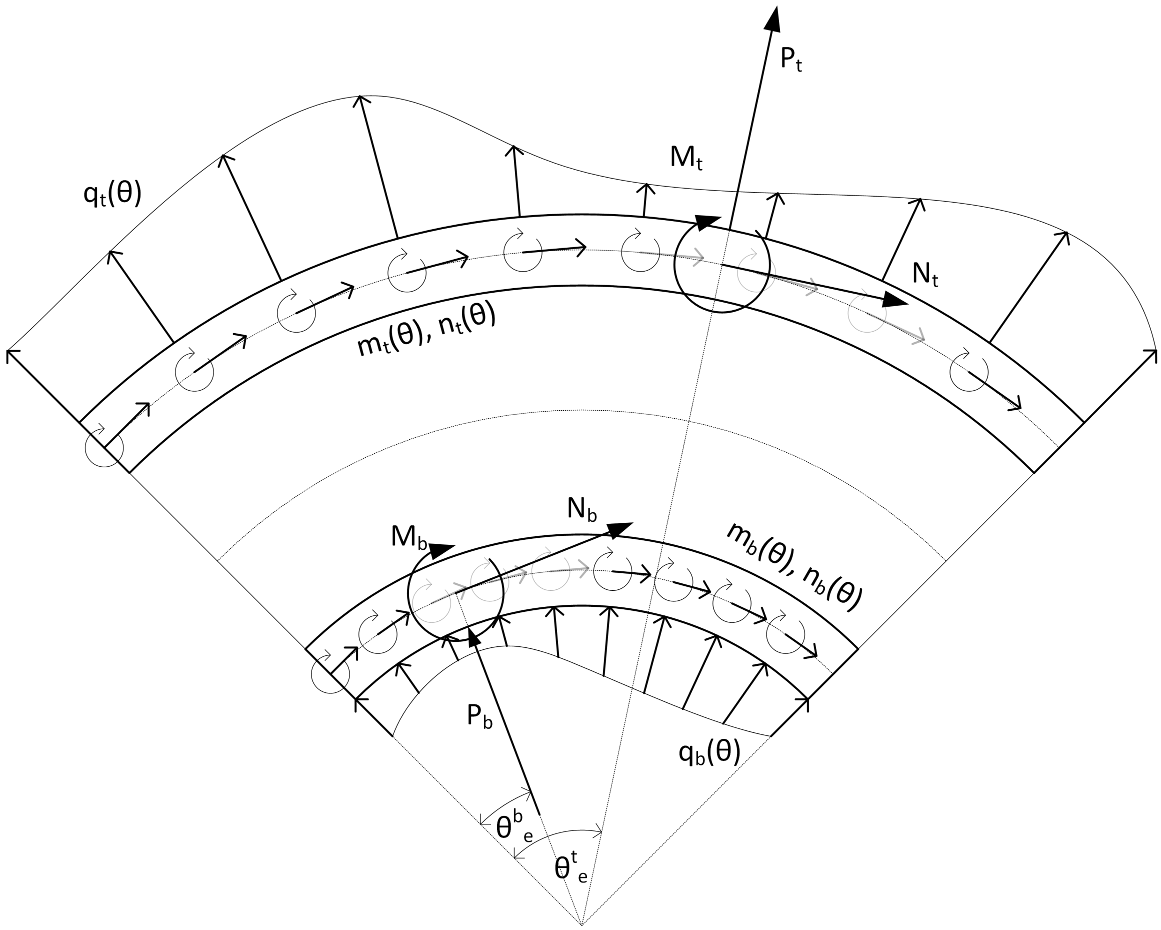

Figure 2.2: External forces and moments subject to a curve sandwich panel

where, face sheet rotation is $\beta^{t,b} = u^{t,b} - w^{t,b\prime}(\theta)/R_{t,b}$, and $q^{t,b}(\theta)$ is the distributed normal force (along the radius), $n^{t,b}(\theta)$ is the distributed tangential force (along θ) and $m^{t,b}(\theta)$ is the distributed moment on the top and bottom face sheets, respectively. In addition, $P_{t,b}$ is the concentrated normal (along the radius) force, $N_{t,b}$ is the concentrated tangential (along θ) force and $M_{t,b}$ is the concentrated moment applied at the end θ_e on the top and bottom face sheets, respectively.

In the foregoing equation we denote by θ_e the boundary points, commonly $\theta_e = 0$ or $\theta_e = \alpha$. The procedure below will make this assumption. If concentrated external forces/moments are applied at a θ_e between 0 and α, then the boundary conditions can be treated in separate ranges i.e. $0 \leq \theta \leq \theta_e$ and $\theta_e \leq \theta \leq \alpha$, with continuity conditions applied at $\theta = \theta_e$.

Then the governing equations are obtained by substituting the stress-strain relations, (2.8), into the strain energy, (2.9b); this way the expressions are in terms of strains. Next, by substituting the strain-displacement relations, (2.6), into the strain energy(2.9b), the expression are written in terms of displacements. Once all the terms in the variational expressions are in terms of displacements, integration by parts is carried in order to obtain the governing differential equations and the associated boundary conditions. As a result, we obtain seven linear ordinary differential equations in terms of the seven generalized coordinates: w_0^t, u_0^t, w_0^b, u_0^b, w_0^c, u_0^c and ϕ_0^c.

The resulting governing equations for $0 \leq \theta \leq \alpha$ and associated boundary conditions are as follows:

2.1.4 Governing differential equations and associated boundary conditions

Corresponding seven differential equations are as follows:

Top Face Sheet

δw_0^t :

$$\left[A_3^c+\frac{c_{11}^t b f_t}{R_t}+(A_3^a-A_7^b+A_9^c)\frac{\partial^2}{\partial\theta^2}+(A_9^a+\frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^4}{\partial\theta^4}\right]w_0^t+\left[A_1^c+(A_1^a-A_5^b+A_8^c)\frac{\partial^2}{\partial\phi^2}+A_8^a\frac{\partial^4}{\partial\theta^4}\right]w_0^b$$
$$+\left[A_2^c+(A_2^a-A_6^b)\frac{\partial^2}{\partial\theta^2}\right]w_0^c+\left[(A_6^c-A_3^b+\frac{c_{11}^t b d_t}{R_t})\frac{\partial}{\partial\theta}+(A_6^a+\frac{c_{11}^t b d_t^3}{12R_t^3})\frac{\partial^3}{\partial\theta^3}\right]u_0^t$$
$$+\left[(A_4^c-A_1^b)\frac{\partial}{\partial\theta}+A_4^a\frac{\partial^3}{\partial\theta^3}\right]u_0^b+\left[(A_5^c-A_2^b)\frac{\partial}{\partial\theta}+A_5^a\frac{\partial^3}{\partial\theta^3}\right]u_0^c+\left[(A_7^c-A_4^b)\frac{\partial}{\partial\theta}+A_7^a\frac{\partial^3}{\partial\theta^3}\right]u_1^c$$
$$=R_t q_t+\frac{\partial}{\partial\theta}m_t\ , \quad (2.10a)$$

δu_0^t :

$$\left[(D_7^c-D_3^b-\frac{c_{11}^t b d_t}{R_t})\frac{\partial}{\partial\theta}+(\frac{c_{11}^t b d_t^3}{12R_t^3}-D_9^b)\frac{\partial^3}{\partial\theta^3}\right]w_0^t+\left[(D_5^c-D_1^b)\frac{\partial}{\partial\theta}-D_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b$$
$$+(D_6^c-D_2^b)\frac{\partial}{\partial\theta}w_0^c+\left[D_3^c-(D_6^b+\frac{c_{11}^t b d_t}{R_t}+\frac{c_{11}^t b d_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]u_0^t$$

$$+ (D_1^c - D_4^b \frac{\partial^2}{\partial\theta^2}) u_0^b + (D_2^c - D_5^b \frac{\partial^2}{\partial\theta^2}) u_0^c + (D_4^c - D_7^b \frac{\partial^2}{\partial\theta^2}) u_1^c = R_t n_{\theta\theta}^t + m_t \ . \quad (2.10b)$$

Bottom Face Sheet

δw_0^b :

$$\left[B_3^c + (B_3^a - B_7^b + B_9^c)\frac{\partial^2}{\partial\theta^2} + B_9^a \frac{\partial^4}{\partial\theta^4}\right] w_0^t + \left[B_1^c + \frac{c_{11}^b b d_b}{R_b} + (B_1^a - B_5^b + B_8^c)\frac{\partial^2}{\partial\theta^2} + (B_8^a + \frac{c_{11}^b b d_b^3}{12 R_b^3})\frac{\partial^4}{\partial\theta^4}\right] w_0^b$$
$$+ \left[B_2^c + (B_2^a - B_6^b)\frac{\partial^2}{\partial\theta^2}\right] w_0^c + \left[(B_6^c - B_3^b)\frac{\partial}{\partial\theta} + B_6^a \frac{\partial^3}{\partial\theta^3}\right] u_0^t$$
$$+ \left[(B_4^c - B_1^b + \frac{c_{11}^b b d_b}{R_b})\frac{\partial}{\partial\theta} + (B_4^a - \frac{c_{11}^b b d_b^3}{12 R_b^3})\frac{\partial^3}{\partial\theta^3}\right] u_0^b + \left[(-B_2^b + B_5^c)\frac{\partial}{\partial\theta} + B_5^a \frac{\partial^3}{\partial\theta^3}\right] u_0^c$$
$$+ \left[(-B_4^b + B_7^c)\frac{\partial}{\partial\theta} + B_7^a \frac{\partial^3}{\partial\theta^3}\right] u_1^c = -R_b q_b - \frac{\partial}{\partial\theta} m_b \ , \quad (2.10c)$$

δu_0^b :

$$\left[(E_7^c - E_3^b)\frac{\partial}{\partial\theta} - E_9^b \frac{\partial^3}{\partial\theta^3}\right] w_0^t + \left[(E_5^c - E_1^b - \frac{c_{11}^b b d_b}{R_b})\frac{\partial}{\partial\theta} + (\frac{c_{11}^b b d_b^3}{12 R_b^3} - E_8^b)\frac{\partial^3}{\partial\theta^3}\right] w_0^b$$
$$+ (E_6^c - E_2^b)\frac{\partial}{\partial\theta} w_0^c + (E_3^c - E_6^b \frac{\partial^2}{\partial\theta^2}) u_0^t + \left[E_1^c - (E_4^b + \frac{c_{11}^b b d_b}{R_b} + \frac{c_{11}^b b d_b^3}{12 R_b^3})\frac{\partial^2}{\partial\theta^2}\right] u_0^b$$
$$+ (E_2^c - E_5^b \frac{\partial^2}{\partial\theta^2}) u_0^c + (E_4^c - E_7^b \frac{\partial^2}{\partial\theta^2}) u_1^c = -R_b n_{ss,b} - m_b \ . \quad (2.10d)$$

Core

δw_0^c :

$$\left[C_3^c + (C_9^c - C_7^b)\frac{\partial^2}{\partial\theta^2}\right] w_0^t + \left[C_1^c + (C_8^c - C_5^b)\frac{\partial^2}{\partial\theta^2}\right] w_0^b + \left(C_2^c - C_6^b \frac{\partial^2}{\partial\theta^2}\right) w_0^c$$
$$+ (C_6^c - C_3^b)\frac{\partial}{\partial\theta} u_0^t + (C_4^c - C_1^b)\frac{\partial}{\partial\theta} u_0^b + (C_5^c - C_2^b)\frac{\partial}{\partial\theta} u_0^c + (C_7^c - C_4^b)\frac{\partial}{\partial\theta} u_1^c = 0 \ ,$$
$$(2.10e)$$

δu_0^c :

$$\left[(F_7^c - F_3^b)\frac{\partial}{\partial\theta} - F_9^b\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(F_5^c - F_1^b)\frac{\partial}{\partial\theta} - F_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (F_6^c - F_2^b)\frac{\partial}{\partial\theta}w_0^c$$
$$+ (F_3^c - F_6^b\frac{\partial^2}{\partial\theta^2})u_0^t + (F_1^c - F_4^b\frac{\partial^2}{\partial\theta^2})u_0^b + (F_2^c - F_5^b\frac{\partial^2}{\partial\theta^2})u_0^c + (F_4^c - F_7^b\frac{\partial^2}{\partial\theta^2})u_1^c = 0\ , \tag{2.10f}$$

δu_1^c :

$$\left[(G_7^c - G_3^b)\frac{\partial}{\partial\theta} - G_9^b\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(G_5^c - G_1^b)\frac{\partial}{\partial\theta} - G_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (G_6^c - G_2^b)\frac{\partial}{\partial\theta}w_0^c$$
$$+ (G_3^c - G_6^b\frac{\partial^2}{\partial\theta^2})u_0^t + (G_1^c - G_4^b\frac{\partial^2}{\partial\theta^2})u_0^b + (G_2^c - G_5^b\frac{\partial^2}{\partial\theta^2})u_0^c + (G_4^c - G_7^b\frac{\partial^2}{\partial\theta^2})u_1^c = 0\ . \tag{2.10g}$$

The corresponding boundary conditions are at $\theta = 0$ and $\theta = \alpha$, read as follows (at each end there are nine boundary conditions, three for each of the two face sheets and three for the core):

Top Face Sheet

Either $\delta w_0^t = 0$ *or*,

$$\left[(A_7^b - A_3^a)\frac{\partial}{\partial\theta} - (A_9^a + \frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(A_5^b - A_1^a)\frac{\partial}{\partial\theta} - A_8^a\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (A_6^b - A_2^a)\frac{\partial}{\partial\theta}w_0^c$$
$$+\left[A_3^b + (\frac{c_{11}^t b f_t^3}{12R_t^3} - A_6^a)\frac{\partial^2}{\partial\theta^2}\right]u_0^t + \left(A_1^b - A_4^a\frac{\partial^2}{\partial\theta^2}\right)u_0^b + \left(A_2^b - A_5^a\frac{\partial^2}{\partial\theta^2}\right)u_0^c + \left(A_4^b - A_7^a\frac{\partial^2}{\partial\theta^2}\right)u_1^c$$
$$= P_t - m_t\ , \tag{2.11a}$$

Either $\delta w_0^{t\,\prime} = 0$ *or*,

$$\left[A_3^a + (A_9^a + \frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^t + \left(A_1^a + A_8^a\frac{\partial^2}{\partial\theta^2}\right)w_0^b + A_2^a w_0^c + \left[A_6^a\frac{\partial}{\partial\theta} - \frac{c_{11}^t b f_t^3}{12R_t^3}\frac{\partial^3}{\partial\theta^3}\right]u_0^t$$
$$+ A_4^a\frac{\partial}{\partial\theta}u_0^b + A_5^a\frac{\partial}{\partial\theta}u_0^c + A_7^a\frac{\partial}{\partial\theta}u_1^c = -\frac{M_t}{R_t}\ , \tag{2.11b}$$

Either $\delta u_0^t = 0$ *or,*

$$\left[D_3^b + \frac{c_{11}^t b f_t}{R_t} + (D_9^b - \frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^t + \left(D_1^b + D_8^b\frac{\partial^2}{\partial\theta^2}\right)w_0^b + D_2^b w_0^c$$
$$+ \left[D_6^b + \frac{c_{11}^t b f_t^3}{12R_t^3} + \frac{c_{11}^t b f_t}{R_t}\right]\frac{\partial}{\partial\theta}u_0^t + D_4^b\frac{\partial}{\partial\theta}u_0^b + D_5^b\frac{\partial}{\partial\theta}u_0^c + D_7^b\frac{\partial}{\partial\theta}u_1^c = N_t + \frac{M_t}{R_t}\ , \tag{2.11c}$$

<u>*Bottom Face Sheet*</u>

Either $\delta w_0^b = 0$ *or,*

$$\left[(B_7^b - B_3^a)\frac{\partial}{\partial\theta} - B_9^a\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(B_5^b - B_1^a)\frac{\partial}{\partial\theta} - (\frac{c_{11}^b b f_b^3}{12R_b^3} + B_8^a)\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (B_6^b - B_2^a)\frac{\partial}{\partial\theta}w_0^c$$
$$+\left(B_3^b - B_6^a\frac{\partial^2}{\partial\theta^2}\right)u_0^t + \left[B_1^b + (\frac{c_{11}^b b f_b^3}{12R_b^3} - B_4^a)\frac{\partial^2}{\partial\theta^2}\right]u_0^b + \left(B_2^b - B_5^a\frac{\partial^2}{\partial\theta^2}\right)u_0^c + \left(B_4^b - B_7^a\frac{\partial^2}{\partial\theta^2}\right)u_1^c$$
$$= P_t + m_b\ , \tag{2.11d}$$

Either $\delta w_0^{b\,\prime} = 0$ *or,*

$$\left(B_3^a + B_9^a\frac{\partial^2}{\partial\theta^2}\right)w_0^t + \left[B_1^a + (B_8^a + \frac{c_{11}^b b f_b^3}{12R_b^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^b + B_2^a w_0^c + B_4^a\frac{\partial}{\partial\theta}u_0^t$$
$$+ \left(B_4^a\frac{\partial}{\partial\theta} - \frac{c_{11}^b b f_b^3}{12R_b^3}\frac{\partial^3}{\partial\theta^3}\right)u_0^b + B_5^a\frac{\partial}{\partial\theta}u_0^c + B_7^a\frac{\partial}{\partial\theta}u_1^c = -\frac{M_b}{R_b}\ , \tag{2.11e}$$

Either $\delta u_0^b = 0$ *or,*

$$\left(E_3^b + E_9^b\frac{\partial^2}{\partial\theta^2}\right)w_0^t + \left(E_1^b + \frac{c_{11}^b b f_b}{R_b} + (E_8^b - \frac{c_{11}^b b f_b^3}{12R_b^3})\frac{\partial^2}{\partial\theta^2}\right)w_0^b + E_2^b w_0^c + E_5^b\frac{\partial}{\partial\theta}u_0^t$$
$$\left(E_4^b + \frac{c_{11}^b b f_b^3}{12R_b^3} + \frac{c_{11}^b b f_b}{R_b}\right)\frac{\partial}{\partial\theta}u_0^b + E_5^b\frac{\partial}{\partial\theta}u_0^c + E_7^b\frac{\partial}{\partial\theta}u_1^c = N_b + \frac{M_b}{R_b}\ , \tag{2.11f}$$

<u>*Core*</u>

Either $\delta w_0^c = 0$ *or,*

$$C_7^b\frac{\partial}{\partial\theta}w_0^t + C_5^b\frac{\partial}{\partial\theta}w_0^b + C_6^b\frac{\partial}{\partial\theta}w_0^c + C_3^b u_t^c + C_1^b u_0^b + C_2^b u_0^c + C_4^b u_1^c = 0 \;, \tag{2.11g}$$

Either $\delta u_0^c = 0$ *or,*

$$\left(F_3^b+F_9^b\frac{\partial^2}{\partial\theta^2}\right)w_0^t+\left(F_1^b+F_8^b\frac{\partial^2}{\partial\theta^2}\right)w_0^b+F_2^b w_0^c+F_6^b\frac{\partial}{\partial\theta}u_0^t+F_4^b\frac{\partial}{\partial\theta}u_0^b+F_5^b\frac{\partial}{\partial\theta}u_0^c+F_7^b\frac{\partial}{\partial\theta}u_1^c=0\,, \tag{2.11h}$$

Either $\delta u_1^c = 0$ *or,*

$$\left(G_3^b+G_9^b\frac{\partial^2}{\partial\theta^2}\right)w_0^t+\left(G_1^b+G_8^b\frac{\partial^2}{\partial\theta^2}\right)w_0^b+G_2^b w_0^c+G_6^b\frac{\partial}{\partial\theta}u_0^t+G_4^b\frac{\partial}{\partial\theta}u_0^b+G_5^b\frac{\partial}{\partial\theta}u_0^c+G_7^b\frac{\partial}{\partial\theta}u_1^c=0\,, \tag{2.11i}$$

where $A_i^{a,b,c}$, $B_i^{a,b,c}$, $C_i^{b,c}$, $D_i^{b,c}$, $E_i^{b,c}$, $F_i^{b,c}$, $G_i^{b,c}$ are constants which include both geometric and material properties and are defined in Appendix A and Appendix B for logarithmic and polynomial variant, respectively.

The governing equations and boundary conditions shown above are expressed in terms of displacement functions. Alternatively, logarithmic variant i.e. (2.3), defining axial stress resultants as following:

$$N_{ss}^t = \int_{-f_t/2}^{f_t/2} \sigma_{\theta\theta}^t b dz_t \quad ; \quad N_{ss}^b = \int_{-f_b/2}^{f_b/2} \sigma_{\theta\theta}^b b dz_b \;,$$

$$N_{ss}^c = \int_{r_{bc}}^{r_{tc}} \sigma_{\theta\theta}^c b dr \;, \tag{2.12a}$$

and moment stress resultants:

$$M_{ss}^t = \int_{-f_t/2}^{f_t/2} \sigma_{\theta\theta}^t z_t b dz_t \;; \quad M_{ss}^b = \int_{-f_b/2}^{f_b/2} \sigma_{\theta\theta}^b z_b b dz_b \;,$$

$$M_{ss}^c = \int_{r_{bc}}^{r_{tc}} \sigma_{\theta\theta}^c r b dr \ , \tag{2.12b}$$

and shear stress resultants:

$$V_{rs}^c = \int_{r_{bc}}^{r_{tc}} \tau_{r\theta}^c b dr \ , \tag{2.12c}$$

lastly, associated high-order stress resultants, these are non physical quantities:

$$Q_{ss1}^c = \int_{r_{bc}}^{r_{tc}} \frac{1}{r} \sigma_{\theta\theta}^c b dr; \quad Q_{ss2}^c = \int_{r_{bc}}^{r_{tc}} \sigma_{\theta\theta}^c \ln r b dr \ , \tag{2.12d}$$

$$Q_{sr1}^c = \int_{r_{bc}}^{r_{tc}} \frac{1}{r} \tau_{r\theta}^c b dr; \quad Q_{sr2}^c = \int_{r_{bc}}^{r_{tc}} \tau_{r\theta}^c \ln r b dr \ , \tag{2.12e}$$

$$Q_{rr1}^c = \int_{r_{bc}}^{r_{tc}} \frac{1}{r} \sigma_{rr}^c b dr; \quad N_{rr1}^c = \int_{r_{bc}}^{r_{tc}} \sigma_{rr}^c b dr \ . \tag{2.12f}$$

Using the definition of stress resultants, substituting into the first variation of beam (2.9b), then performing integration by parts, governing equations and associated boundary conditions can also be written in stress resultants instead of displacement functions, see Appendix C. Note that the stress resultants and the corresponding governing equations are EHSAPT logarithmic version while EHSAPT polynomial version is not shown here.

2.1.5 Solution Procedure

In the following, we outline the solution procedure for a simply supported curved panel subjected to a distributed load on the top face sheet $q_t(\theta)$, which can be expressed as a Fourier series:

$$q_t(\theta) = \sum_{n=1}^{\infty} q_n \sin(\frac{n\pi\theta}{\alpha}) \ . \tag{2.13a}$$

The solution that satisfies the simply supported boundary conditions is:

$$w_0^{t,b,c}(\theta) = \sum_{n=1}^{\infty} W_{0n}^{t,b,c} \sin(\frac{n\pi\theta}{\alpha}) \; ; \quad u_0^{t,b,c}(\theta) = \sum_{n=1}^{\infty} U_{0n}^{t,b,c} \cos(\frac{n\pi\theta}{\alpha}) \; , \tag{2.13b}$$

$$u_1^c(\theta) = \sum_{n=1}^{\infty} U_{1n}^c \cos(\frac{n\pi\theta}{\alpha}) \; . \tag{2.13c}$$

Substituting the foregoing equations (2.13) into the governing differential equations (2.10) results for each n in a system of linear algebraic equations, $[K_n]\{X_n\} = \{F_n\}$, where $[K_n]$ is a 7×7 stiffness matrix, $\{F_n\}$ is a 1×7 force matrix, and $\{X_n\}$ is a 1×7 unknown displacement matrix, namely: $\{W_{0,n}^t, W_{0,n}^b, W_{0,n}^c, U_{0,n}^t, U_{0,n}^b, U_{0,n}^c, U_{1,n}^c\}$. Each individual n linear algebraic equations system can be easily solved obtaining $\{X_n\}$, then the analytical solution is obtained from the series (2.13b,2.13c). In equations (2.13), $\sum\limits_{n=1}^{\infty}$ is replaced by $\sum\limits_{n=1}^{N}$ where N is the total number of terms included in the Fourier series equation (2.13a) and corresponding solutions equations (2.13b,2.13c); the no of terms, N, is determined from a study of the series convergence.

2.2 Elasticity Solution for Curved Sandwich Beams/Panels

The linear elasticity problem formulation and solution for a generally asymmetric sandwich curved beam/panel consisting of orthotropic core and face sheets, which is subjected to a top face distributed transverse loading is presented. The displacement approach is used and the panel is assumed to be simply supported at the ends. Closed form solutions for the displacements and stresses are derived.

$$q(\theta) = q_0 \; \sin k\theta \; ; \quad k = \frac{n\pi}{\alpha} \; . \tag{2.14}$$

We denote by $c_{ij}^{t,b,c}$ the stiffness constants in each layer, where the superscript t is for the top face, b is for the bottom face, and c is for the core. In the following, we outline the derivation within each layer and thus, for convenience, we drop the superscript.

Both face sheets and the core are assumed to be orthotropic and the constitutive relations are:

$$\begin{bmatrix} \sigma_{\theta\theta} \\ \sigma_{rr} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{55} \end{bmatrix} \begin{bmatrix} \epsilon_{\theta\theta} \\ \epsilon_{rr} \\ \gamma_{r\theta} \end{bmatrix} , \tag{2.15}$$

Similar to previous section, stiffness coefficients are derived from the inverse of compliance matrix (2.8b), they are the same as equation (2.8c) but valid for all t, b, c.

The three components of strain are expressed in terms of w and u, the two components of the displacement field (radial and circumferential, respectively):

$$\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r} \; ; \quad \epsilon_{rr} = \frac{\partial w}{\partial r} \; ; \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial r} - \frac{u}{r} \; . \tag{2.16}$$

The two equilibrium equations, to be satisfied, are as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \; , \tag{2.17a}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \; . \tag{2.17b}$$

We are seeking displacements in the form:

$$w = f(r) \; \sin k\theta \; , \tag{2.18a}$$

$$u = g(r) \; \cos k\theta \; . \tag{2.18b}$$

By using (2.15), (2.16), and (2.18) and substituting into (2.17), the following two ordinary differential equations in r are obtained:

$$c_{33} f'' + c_{33} \frac{f'}{r} - (c_{11} + c_{55} k^2) \frac{f}{r^2} - (c_{13} + c_{55}) k \frac{g'}{r} + (c_{11} + c_{55}) k \frac{g}{r^2} = 0 \; , \tag{2.19a}$$

$$c_{55} g'' + c_{55} \frac{g'}{r} - (c_{55} + c_{11} k^2) \frac{g}{r^2} + (c_{13} + c_{55}) k \frac{f'}{r} + (c_{11} + c_{55}) k \frac{f}{r^2} = 0 \; . \tag{2.19b}$$

These are, indeed, two set of Cauchy-Euler equation where the assumed trial solution are:

$$f(r) = A r^{\lambda} \; ; \quad g(r) = B r^{\lambda} \; , \tag{2.20}$$

the system (2.19) results in the following system of homogeneous algebraic equations

$$A \left(c_{33} \lambda^2 - c_{55} k^2 - c_{11} \right) + Bk \left[c_{11} + c_{55} - \lambda (c_{13} + c_{55}) \right] = 0 \; , \tag{2.21a}$$

$$Ak \left[c_{11} + c_{55} + \lambda (c_{13} + c_{55}) \right] + B \left[c_{55} (\lambda^2 - 1) - c_{11} k^2 \right] = 0 \; . \tag{2.21b}$$

Thus, λ is determined by setting the determinant of the coefficients of (2.21) to zero. This results in the fourth order polynomial characteristic equation:

$$c_{33} c_{55} \lambda^4 + \left[k^2 (c_{13}^2 + 2 c_{13} c_{55} - c_{33} c_{11}) - c_{55} (c_{33} + c_{11}) \right] \lambda^2 + c_{11} c_{55} (k^2 - 1)^2 = 0 \; . \tag{2.22}$$

Setting,

$$\mu = \lambda^2 \; , \tag{2.23a}$$

results in a quadratic equation, which can be solved in closed form, i.e.,

$$a\mu^2 + b\mu + c = 0 \ , \tag{2.23b}$$

where

$$a = c_{33}c_{55} \ ; \quad b = k^2(c_{13}^2 + 2c_{13}c_{55} - c_{33}c_{11}) - c_{55}(c_{33} + c_{11}) \ , \tag{2.23c}$$

$$c = c_{11}c_{55}(k^2 - 1)^2 \ . \tag{2.23d}$$

In general, the two roots $\mu_{1,2}$ in (2.23b) are distinct; then, $\lambda_{1,2,3,4}$ are distinct as well. And the displacement field in each layer j, $j = t, c, b$ is in the form:

for $b^2 - 4ac \neq 0$

$$w^{(j)} = \sin k\theta \sum_{i=1,2,3,4} A_{ij} r^{\lambda_{ij}} \ , \tag{2.24a}$$

$$u^{(j)} = \cos k\theta \sum_{i=1,2,3,4} B_{ij} r^{\lambda_{ij}} \ . \tag{2.24b}$$

In this, most common case, these 8 unknowns are the A_{ij}, B_{ij}, $i = 1, 2, 3, 4$. There are three layers, the top face, the core, and the bottom face, $j = t, c, b$ for a total of 24 unknowns. These unknowns are not all independent, though. Of the 24 constants, only 12 are independent. By substituting into the equilibrium equations, the following relations exist:

$$B_{ij} = -A_{ij} f_{ij} \ ; \quad i = 1, 2, 3, 4 \ ; \quad j = t, c, b \tag{2.25a}$$

where

$$f_{ij} = \frac{\left[c_{33}^{(j)}\lambda_{ij}^2 - c_{55}^{(j)}k^2 - c_{11}^{(j)}\right]}{k\left[c_{11}^{(j)} + c_{55}^{(j)} - \lambda_{ij}(c_{13}^{(j)} + c_{55}^{(j)})\right]} \ . \tag{2.25b}$$

Notice that the λ_i's can be real or be in complex conjugate pairs, thus the numerical calculations have to carried out in complex numbers.

A special case, the two roots in (2.23b) are repeated when the discriminant of the quadratic equation is zero, i.e., when $b^2 - 4ac = 0$. For an isotropic material layer, such a case is not possible. For an orthotropic material, however, such repeated roots could exist for certain combinations of $c_{33}, c_{13}, c_{11}, c_{55}$, and k. In this case, μ_1=μ_2=$-b/(2a)$; hence, $\lambda_1 = \lambda_3$, and $\lambda_2 = \lambda_4$, and the corresponding displacement field is:

for $b^2 - 4ac = 0$

$$w^{(j)} = \sin k\theta \left(\sum_{i=1,2} A_{ij} r^{\lambda_{ij}} + \sum_{i=3,4} A_{ij} r^{\lambda_{ij}} \ln r \right) , \tag{2.26a}$$

$$u^{(j)} = \cos k\theta \left(\sum_{i=1,2} B_{ij} r^{\lambda_{ij}} + \sum_{i=3,4} B_{ij} r^{\lambda_{ij}} \ln r \right) , \tag{2.26b}$$

where the B_{ij} are in terms of the A_{ij} from (2.25) where the λ_{1j}, λ_{2j}, $\lambda_{3j} = \lambda_{1j}$ and $\lambda_{4j} = \lambda_{2j}$ are substituted for each layer j.

Therefore in both cases there are 8 unknowns for each layer j. We outline in the following the solution for the case of distinct roots, i.e. displacement field given by (2.24). In this, most common, case, these 4 unknowns are the A_{ij}, $i = 1, 2, 3, 4$. Notice that the B_{ij} are not independent but are related to the A_{ij} through relations (2.25). There are three layers, the top face, the core, and the bottom face, $j = t, c, b$ for a total of 12 unknowns.

Then the strains are from (2.16)

$$\epsilon_{rr} = \sin k\theta \sum_{i=1,2,3,4} A_{ij} \lambda_{ij} r^{\lambda_{ij}-1} \ ; \quad \epsilon_{\theta\theta} = \sin k\theta \sum_{i=1,2,3,4} A_{ij} \left(k f_{ij} + 1 \right) r^{\lambda_{ij}-1} , \tag{2.27a}$$

$$\gamma_{r\theta} = \cos k\theta \sum_{i=1,2,3,4} A_{ij} \left[k + f_{ij}(1 - \lambda_{ij}) \right] r^{\lambda_{ij}-1} , \tag{2.27b}$$

and the corresponding stresses at each layer $j = t, c, b$, are from (2.15):

$$\sigma_{rr} = \sin k\theta \sum_{i=1,2,3,4} A_{ij} \left[c_{33}^{(j)} \lambda_{ij} + c_{13}^{(j)} \left(k f_{ij} + 1 \right) \right] r^{\lambda_{ij}-1} \, , \tag{2.28}$$

$$\sigma_{\theta\theta} = \sin k\theta \sum_{i=1,2,3,4} A_{ij} \left[c_{13}^{(j)} \lambda_{ij} + c_{11}^{(j)} \left(k f_{ij} + 1 \right) \right] r^{\lambda_{ij}-1} \, , \tag{2.29}$$

$$\tau_{r\theta} = \cos k\theta \sum_{i=1,2,3,4} A_{ij} c_{55}^{(j)} \left[k + f_{ij}(1 - \lambda_{ij}) \right] r^{\lambda_{ij}-1} \, . \tag{2.30}$$

The 12 constants A_{ij} that are in the stress expressions (2.28)-(2.30) can be found from the displacement and traction continuity face/core interfaces conditions, the traction-free lower bounding surface, $r = R_1$, and the traction loading at the upper bounding surface, $r = R_2$.

There are two traction conditions at the bottom face-sheet/core interface, $r = R_1 + f_b$.

$$\sigma_{rr}^{(c)} = \sigma_{rr}^{(b)} \; ; \quad \tau_{r\theta}^{(c)} = \tau_{r\theta}^{(b)} \, . \tag{2.31a}$$

There are two displacement continuity conditions at the bottom face-sheet/core interface:

$$w^{(c)} = w^{(b)} \; ; \quad u^{(c)} = u^{(b)} \, . \tag{2.31b}$$

Similarly, there are two traction conditions at the top face-sheet/core interface, $r = R_2 - f_t$:

$$\sigma_{rr}^{(c)} = \sigma_{rr}^{(t)} \; ; \quad \tau_{r\theta}^{(c)} = \tau_{r\theta}^{(t)} \, . \tag{2.31c}$$

There are two displacement continuity conditions at the top face-sheet/core interface:

$$w^{(t)} = w^{(c)} \; ; \quad u^{(t)} = u^{(c)} \; . \tag{2.31d}$$

There are two traction-free conditions at the bottom bounding surface, $r = R_1$:

$$\sigma_{rr}^{(b)} = 0 \; ; \quad \tau_{r\theta}^{(b)} = 0 \; . \tag{2.31e}$$

Finally, there are two traction conditions at the top bounding surface, $r = R_2$, that represent the loading of the top face:

$$\sigma_{rr}^{(t)} = q(\theta) \; ; \quad \tau_{r\theta}^{(t)} = 0 \; . \tag{2.31f}$$

Substituting displacement solution form (2.24), corresponding strains (2.27), and stresses (2.28) into twelve boundary conditions (2.31a)-(2.31f); consequently, $\sin k\theta$ and $\cos k\theta$ are cancelled out in these twelve equations. Results are twelve linear algebraic equations which can be solved for the twelve unknown constants: A_{ij}, $i = 1, 2, 3, 4$, $j = t, b, c$.

2.3 First Order Shear Deformation Theory for Curved Sandwich Beams/Panels

Due to the curvature, the first order shear deformation (FOSD) theory for curved sandwich panels is not a direct extension of the corresponding one for flat panels and thus, it is formulated accordingly, and its unique features, such as the reference curve, are discussed. Three versions of the FOSD theory are formulated: the one based on direct variational formulation based on the assumed through-thickness displacement field (termed “basic”), one based on the definition of an equivalent shear modulus for the section (termed “G_{eq}”) and one based on derivation of a shear correction factor, which is considered in conjunction with the equivalent shear modulus.

2.3.1 "Basic Version" Formulation

In the following, the First Order Shear Deformation Theory (FOSD) will be applied to the sandwich panel configuration. It should be noted the curvature introduces complications, to be discussed in the following, and therefore, this is not a straightforward application of the corresponding theory for flat plates. As will be seen in the following, several unique features exist when it comes to curved panels, among them the fact that the shear strain is not constant and that the reference line is a function of the curvature. In the Euler Bernoulli theory, the panel cross-section remains perpendicular to the normal line and its rotation is given as $[u_0(\theta) - w_0'(\theta)]/R$, where R is the reference curvature, resulting in zero shear strain. The FOSD theory, also known as the Timoshenko theory, allows for an independent plane rotation, denoted as $\psi(\theta)$, consequently resulting in a non-zero shear strain and corresponding non-zero shear stress.

The displacement field is given in terms of the radial distance z defined from a radius R, to be determined,

$$w(z,\theta) = w_0(\theta)\ ; \quad u(z,\theta) = u_0(\theta) + z\psi(\theta)\ . \tag{2.32}$$

Since $r = R + z$, the strains from (2.16) are:

$$\epsilon_{rr}(z,\theta) = 0\ , \tag{2.33a}$$

$$\epsilon_{\theta\theta}(z,\theta) = \frac{1}{R+z}\left[w_0(\theta) + u_0'(\theta) + z\psi'(\theta)\right]\ , \tag{2.33b}$$

$$\gamma_{r\theta}(z,\theta) = \psi(\theta) + \frac{1}{R+z}\left[w_0'(\theta) - u_0(\theta) - z\psi(\theta)\right]\ . \tag{2.33c}$$

Notice that unlike flat plates, in which the shear strain in the FOSD is constant through the thickness, the curvature introduces a shear strain that has a variation with z, see eqn

(2.33c). The curvature also introduces the z coordinate in the denominator of all strains, unlike the flat plates case. Thus, the theory needs to be applied with new area integral definitions.

The corresponding stresses are:

$$\sigma_{\theta\theta}(z,\theta) = c_{11}^{j}\epsilon_{\theta\theta}\ ; \quad \tau_{r\theta}(z,\theta) = c_{55}^{j}\gamma_{r\theta}; \quad j = t, b, c \tag{2.33d}$$

where, $c_{11}^{j} = E_1^{j}$ and $c_{55}^{j} = G^{j}$

Governing equations and associated boundary conditions are derived from the Principle of Minimum Total Potential energy:

$$\delta(U + V) = 0\ . \tag{2.34a}$$

where U is the strain energy of the sandwich panel and V is the external potential due to the applied loads.

The first variation of the strain energy of the sandwich beam involves stresses multiplied by variations of strains and since the radial strain ϵ_{rr} is zero, the radial stress σ_{rr} term will not appear. In the following, we assume unit width. The variation of the internal potential is:

$$\delta U = \int_0^{\alpha} \int_A \left(\sigma_{\theta\theta}\delta\epsilon_{\theta\theta} + \tau_{r\theta}\delta\gamma_{r\theta}\right)(R + z)dAd\theta\ , \tag{2.34b}$$

and the external potential is:

$$\delta V = -\int_0^{\alpha} q_t(\theta)\delta w_0(\theta)R_2 d\theta - (H\delta u_0)\ |_0^{\alpha} - (N\delta w_0)\ |_0^{\alpha} - (M\delta\psi)\ |_0^{\alpha}\ , \tag{2.34c}$$

where $N_{0,\alpha}$ are the radial, $H_{0,\alpha}$ are the tangential (along θ) forces at the ends θ_e = 0, α, respectively, and $M_{0,\alpha}$ are the moments at theses ends.

Next we discuss the reference curve. From the stress eqn (2.33d), the moment at any section is:

$$M = \int_A \sigma_{\theta\theta} z dA = (u_0' + w_0 + z\psi') \int_A \frac{c_{11} z}{R+z} dA \; . \tag{2.35}$$

Since in the FOSDT the moment is the derivative of the ψ, the above eqn implies that the reference curve should be defined not by the static moment, as in flat plates, but by the the "extensional static moment modified by the radius" [35], i.e., by:

$$\int_A \frac{c_{11}^i z}{R+z} dA = 0 \; . \tag{2.36}$$

Accordingly, the equation for the moment involves not the moment of inertia, but the "extensional moment of inertia modified by the radius", I_{E1}, defined as:

$$I_{E1} = \int_A \frac{c_{11}^i z^2}{R+z} dA \; . \tag{2.37}$$

In this case, by using (2.36) and (2.37), the moment is

$$M = \left(\int_A \frac{c_{11} z^2}{R+z} dA \right) \psi' = I_{E1} \psi' \; , \tag{2.38}$$

in accordance with the FOSD postulates.

Let's determine now the reference radius, R, from (2.36). Let's denote by e the distance from the mid-plane of the core, thus $z = z_c - e$, in which case $R = R_c + e$. Thus, R is defined at a radial distance e from the mid-curve of the core, given by

$$e \int_A \frac{c_{11}^i}{R_c + z_c} dz_c = \int_A \frac{c_{11}^i z_c}{R_c + z_c} dz_c \; , \tag{2.39a}$$

i.e., e is found from:

$$e \left[c_{11}^t \ln \frac{R_c + c + f_t}{R_c + c} + c_{11}^c \ln \frac{R_c + c}{R_c - c} + c_{11}^b \ln \frac{R_c - c}{R_c - c - f_b} \right]$$

$$= c_{11}^t \left(f_t - R_c \ln \frac{R_c + c + f_t}{R_c + c} \right) + c_{11}^c \left(2c - R_c \ln \frac{R_c + c}{R_c - c} \right) + c_{11}^b \left(f_b - R_c \ln \frac{R_c - c}{R_c - c - f_b} \right) . \tag{2.39b}$$

The connection can be made for the traditional reference plane definition for flat plates as follows: The right-hand-side of (2.39b) can be written as

$$\begin{aligned} c_{11}^t \left[f_t - R_c \ln \left(1 + \frac{f_t}{R_c + c} \right) \right] + c_{11}^c \left[2c - R_c \ln \left(1 + \frac{2c}{R_c - c} \right) \right] \\ + c_{11}^b \left[f_b - R_c \ln \left(1 + \frac{f_b}{R_c - c - f_b} \right) \right] . \end{aligned} \tag{2.40a}$$

Applying the Taylor series expansion

$$\ln(1 + x) \simeq x - \frac{x^2}{2} , \tag{2.40b}$$

to the first term in (2.40a) leads to:

$$\begin{aligned} & c_{11}^t \left[f_t - R_c \left(\frac{f_t}{R_c + c} - \frac{f_t^2}{2(R_c + c)^2} \right) \right] \\ & \quad = c_{11}^t \left\{ f_t - [(R_c + c) - c] \left(\frac{f_t}{R_c + c} - \frac{f_t^2}{2(R_c + c)^2} \right) \right\} \\ & \quad = \left(c + \frac{f_t}{2} \right) \frac{c_{11}^t f_t}{R_c + c} - \frac{c_{11}^t f_t^2 c}{2(R_c + c)^2} . \end{aligned} \tag{2.40c}$$

Similarly, the second term in (2.40a) leads to:

$$\begin{aligned} c_{11}^c \left[2c - R_c \left(\frac{2c}{R_c - c} - \frac{(2c)^2}{2(R_c - c)^2} \right) \right] &= c_{11}^c \left\{ 2c - [(R_c - c) + c] \left(\frac{2c}{R_c - c} - \frac{(2c)^2}{2(R_c - c)^2} \right) \right\} \\ &= c_{11}^c \frac{2c^3}{(R_c - c)^2} . \end{aligned} \tag{2.40d}$$

and the third term leads to:

$$c_{11}^b\left[f_b - R_c\left(\frac{f_b}{R_c-c-f_b} - \frac{f_b^2}{2(R_c-c-f_b)^2}\right)\right]$$
$$= c_{11}^b\left\{f_b - [(R_c-c-f_b)+(c+f_b)]\left(\frac{f_b}{R_c-c-f_b} - \frac{f_b^2}{2(R_c-c-f_b)^2}\right)\right\}$$
$$= -\left(c+\frac{f_b}{2}\right)\frac{c_{11}^b f_b}{R_c-c-f_b} + \frac{c_{11}^b f_b^2(c+f_b)}{2(R_c-c-f_b)^2}\ . \quad (2.40e)$$

Ignoring the higher order terms, the right-hand-side of eqn (2.39b) becomes

$$c_{11}^t\left(c+\frac{f_t}{2}\right)\frac{f_t}{R_c+c} - c_{11}^b\left(c+\frac{f_b}{2}\right)\frac{f_b}{R_c-c-f_b}\ . \quad (2.40f)$$

Applying the Taylor series expansion to the left-hand-side of (2.39b) and ignoring higher order terms would lead to:

$$e\left[c_{11}^t\frac{f_t}{R_c+c} + c_{11}^c\frac{(2c)}{R_c-c} + c_{11}^b\frac{f_b}{R_c-c-f_b}\right]\ . \quad (2.40g)$$

Thus, from (2.40g) and (2.40f), if we further assume that f_t, c, f_b are all $<< R_c$, the definition for e would be the same as in the flat plate case [17]. Moreover, under these assumptions, for symmetric construction, $e = 0$.

For the sandwich construction in figure 2.1, the “extensional moment of inertia modified by the radius”, I_{E1}, defined in eqn (2.37), is found to be:

$$I_{E1} = c_{11}^t\left[f_t\left(R-e+c+\frac{f_t}{2}\right) - 2Rf_t + R^2\ln\frac{R-e+c+f_t}{R-e+c}\right]$$
$$- c_{11}^c\left[2c(R+e) - R^2\ln\frac{R-e+c}{R-e-c}\right]$$
$$+ c_{11}^b\left[f_b\left(R-e-c-\frac{f_b}{2}\right) - 2Rf_b + R^2\ln\frac{R-e-c}{R-e-c-f_b}\right]\ . \quad (2.41a)$$

We also define

the "extensional area modified by the radius", A_{E1}, defined as:

$$A_{E1} = \int_A \frac{c_{11}^i}{R+z} dA = c_{11}^t \ln \frac{R-e+c+f_t}{R-e+c} + c_{11}^c \ln \frac{R-e+c}{R-e-c} + c_{11}^b \ln \frac{R-e-c}{R-e-c-f_b} \ , \tag{2.41b}$$

In performing the integration (2.34b), we also need the following area quantities associated with shear:

the "shear area", A_{G0},

$$A_{G0} = \int_A c_{55}^i dA = c_{55}^t f_t + c_{55}^c (2c) + c_{55}^b f_b \ , \tag{2.42a}$$

the "shear static moment", S_{G0},

$$S_{G0} = \int_A c_{55}^i z dA = c_{55}^t f_t \left(c - e + \frac{f_t}{2} \right) - c_{55}^c 2ce - c_{55}^b f_b \left(c + e + \frac{f_b}{2} \right) \ . \tag{2.42b}$$

Furthermore, due to the curvature, additional quantities are needed as follows:

the "shear area modified by the radius", A_{G1}

$$A_{G1} = \int_A \frac{c_{55}^i}{R+z} dA = c_{55}^t \ln \frac{R_2}{R-e+c} + c_{55}^c \ln \frac{R-e+c}{R-e-c} + c_{55}^b \ln \frac{R-e-c}{R_1} \ , \tag{2.42c}$$

the "shear static moment modified by the radius", S_{G1},

$$\begin{aligned} S_{G1} = \int_A \frac{c_{55}^i z}{R+z} dA = c_{55}^t \left(f_t - R \ln \frac{R-e+c+f_t}{R-e+c} \right) + c_{55}^c \left(2c - R \ln \frac{R-e+c}{R-e-c} \right) \\ + c_{55}^b \left(f_b - R \ln \frac{R-e-c}{R-e-c-f_b} \right) \ , \end{aligned} \tag{2.42d}$$

and

the "shear moment of inertia modified by the radius", I_{G1}

$$I_{G1} = \int_A \frac{c_{55}^i z^2}{R+z} dA = c_{55}^t \left[f_t \left(R - e + c + \frac{f_t}{2} \right) - 2Rf_t + R^2 \ln \frac{R-e+c+f_t}{R-e+c} \right]$$

$$- c_{55}^{c}\left[2c(R+e) - R^2 \ln\frac{R-e+c}{R-e-c}\right]$$
$$+ c_{55}^{b}\left[f_b\left(R-e-c-\frac{f_b}{2}\right) - 2Rf_b + R^2 \ln\frac{R-e-c}{R-e-c-f_b}\right] \ , \quad (2.42e)$$

Performing the integrations in (2.34b) results in the following three governing equations for the FOSDT (for $0 \le \theta \le \alpha$):

δw_0 :

$$-[A_{G1}(w_{0,\theta} - u_0) + (A_{G0} - S_{G1})\psi]_{,\theta} + A_{E1}(w_0 + u_{0,\theta}) = R_2 q_t \ , \quad (2.43a)$$

δu_0 :

$$-[A_{E1}(w_0 + u_{0,\theta})]_{,\theta} - A_{G1}(w_{0,\theta} - u_0) + (S_{G1} - A_{G0})\psi = 0 \ , \quad (2.43b)$$

and

$\delta\psi$:

$$-(I_{E1}\psi_{,\theta})_{,\theta} + (A_{G0}R + I_{G1} - S_{G0})\psi + (A_{G0} - S_{G1})(w_{0,\theta} - u_0) = 0 \ , \quad (2.43c)$$

Associated boundary conditions (three at each end, θ_e = 0, α) for the FOSDT are;

Either $\delta w_0 = 0$ *or*,

$$A_{G1}\,(w_{0,\theta} - u_0) + (A_{G0} - S_{G1})\psi = N_e \ , \quad (2.44a)$$

Either $\delta u_0 = 0$ *or*,

$$A_{E1}\,(u_{0,\theta} + w_0) = H_e \ , \quad (2.44b)$$

and

Either $\delta\psi = 0$ *or*,

$$I_{E1}\psi_{,\theta} = M_e \ , \tag{2.44c}$$

For a curved beam/panel is simply-supported, as shown in figure 2.1, and is loaded by a sinusoidal distributed load of the form:

$$q(\theta) = q_0 \ \sin k\theta \ ; \quad k = \frac{n\pi}{\alpha} \ , \tag{2.45}$$

analytical solutions are sought in the form:

$$w_0(\theta) = W_0 \sin k\theta \ ; \quad u_0(\theta) = U_0 \cos k\theta \ ; \quad \psi(\theta) = \Psi \cos k\theta \ ; \quad k = \frac{n\pi}{\alpha} \ . \tag{2.46}$$

The displacement field (2.46) satisfies the simply-supported boundary conditions, (w_0 being zero at the ends and no moment, i.e., $\psi_{,\theta}$=0 at the ends).

Assuming constant properties (i.e., independent of θ) and substituting into the governing equations (2.43) leads to:

$$(A_{G1}k^2 + A_{E1})W_0 - (A_{G1} + A_{E1})kU_0 + (A_{G0} - S_{G1})k\Psi = R_2 q_0 \ , \tag{2.47a}$$

$$-(A_{E1} + A_{G1})kW_0 + (A_{E1}k^2 + A_{G1})U_0 + (S_{G1} - A_{G0})\Psi = 0 \ , \tag{2.47b}$$

$$(A_{G0} - S_{G1})kW_0 - (A_{G0} - S_{G1})U_0 + (I_{E1}k^2 + A_{G0}R + I_{G1} - S_{G0})\Psi = 0 \ . \tag{2.47c}$$

This system of three algebraic equations can be solved for W_0, U_0, and Ψ.

2.3.2 First Order Shear Deformation Theory with an Equivalent Shear Modulus

It is well known that, when it comes to the extensional properties, i.e. the c_{11}, the different layers in a multi-layered section can be considered as "springs in series" but when it comes to the shear properties, i.e. the c_{55}, the layers should be considered as "springs in parallel" [52]. In this case, an equivalent shear modulus, $\bar{c}_{55}$ for the entire section is defined as

$$\frac{h}{\bar{c}_{55}} = \frac{f_t}{c_{55}^t} + \frac{2c}{c_{55}^c} + \frac{f_b}{c_{55}^b} \ . \tag{2.48a}$$

Instead of (2.33d), where the shear stress is defined in the faces and core with the different shear modulus of the corresponding layer, now the shear stress over the entire section will be based on a single shear modulus, the $\bar{c}_{55}$:

$$\tau_{r\theta}(z,\theta) = \bar{c}_{55}\gamma_{r\theta} \ . \tag{2.48b}$$

The variational principle will lead again to the system of governing differential equations (2.43) and associated boundary conditions (2.44) but in this case, the shear area properties will be defined as: the "shear area", A_{G0},

$$A_{G0} = \bar{c}_{55}\int_A dA = \bar{c}_{55}h \ , \tag{2.49a}$$

the "shear static moment", S_{G0},

$$S_{G0} = \bar{c}_{55}\int_A zdA = \bar{c}_{55}\int_{-e-c-f_b}^{-e+c+f_t} zdz = \bar{c}_{55}\left(f_t - f_b - 2e\right)\left(c + \frac{f_t + f_b}{2}\right) \ , \tag{2.49b}$$

the "shear area modified by the radius", A_{G1}

$$A_{G1} = \bar{c}_{55}\int_A \frac{dA}{R+z} = \bar{c}_{55}\ln\frac{R-e+c+f_t}{R-e-c-f_b} \ , \tag{2.49c}$$

the "shear static moment modified by the radius", S_{G1},

$$S_{G1} = \bar{c}_{55} \int_A \frac{z}{R+z} dA = \bar{c}_{55} \left(h - R \ln \frac{R - e + c + f_t}{R - e - c - f_b} \right) , \tag{2.49d}$$

and

the "shear moment of inertia modified by the radius", I_{G1}

$$I_{G1} = \bar{c}_{55} \int_A \frac{z^2}{R+z} dA = \bar{c}_{55} \left[h \left(R - e + \frac{f_t - f_b}{2} \right) - 2Rh + R^2 \ln \frac{R - e + c + f_t}{R - e - c - f_b} \right] , \tag{2.49e}$$

The extensional area properties can stay the same as before, i.e., the reference curve will again be defined from (2.39b), the "extensional area modified by the radius", A_{E1}, will be as in (2.41b) and the "extensional moment of inertia modified by the radius", I_{E1}, will be as in (2.41a).

In this version, the differential equations of the FOSD (2.43)-(2.47) can be applied with the area properties defined as above (in terms of $\bar{c}_{55}$).

Shear stiffness in thin face sheets of sandwich beam are generally neglected, while thick sandwich core is solely responsible for the shear deformations [53]. The equivalent shear modulus, G_{eq} ,(2.48a), also reflects shear stiffness face sheets exclusion; because $f_{t,b} << c$ and $c_{55}^{t,b} >> c_{55}^{c}$, $f_{t,b}/c_{55}^{t,b}$ are very small and can be neglected i.e.

$$\bar{c}_{55} \approx c_{55}^{c} \tag{2.50}$$

Adopting the assumption that the shear stiffness contribution solely depends on sandwich core, this could implies the G_{eq} is redundant. Its merits; however, are general and extended beyond the scope of sandwich configuration.

Considering a generic 3 layered isotropic curved beam of equivalent thickness of 10 mm (total thickness $h = 30$mm). The beam has an angular span of $\alpha = 0.3178$ rad; and it

Table 2.1: Comparisons of radial displacement ,w, at $\theta = \alpha/2$ with elasticity in percentage difference

	Classical	"Basic version"	G_{eq}	βG_{eq}
$r = R_o$	16.41%	13.03 %	6.89 %	0.23 %
$r = (R_o + R_i)/2$	16.72%	13.32 %	7.20 %	0.10 %
$r = R_i$	16.37%	12.96 %	6.82 %	0.32 %

is bounded by outermost and innermost radii, $R_o = 818$ mm and $R_i = R_o - h$, respectively .The outermost and innermost layer are made of, $E^{o,i} = 61.9$ GPa, $\nu^{o,i} = 0.3$; while middle layer, denotes by m, is made of an artificial isotropic whose $E^m = 1/10E^{o,i}, \nu^m = \nu^{o,i}$. The curved beam has simply supported on both ends and subjected to a half sine distributed load as depicts in figure 2.4.

Table 2.1 shows comparisons of mid-span (maximum, $\theta = \alpha/2$) radial displacement ,w, at different through thickness locations: top/bottom surface, and mid-thickness. The results are presented in terms of absolute percentage error comparing against elasticity benchmark presented in 2.2. The "basic version" FOSD shows a little improvement from classical theory, while, the equivalent shear modulus FOSD shows a significant improvement. Introducing shear correction factor β, see next section, the deflection is extremely accurate, below 1% difference from elasticity. Figure 2.3 presents through thickness normalized shear stress distribution at $\theta = 0$. The "basics version" shear distribution shows misleading results as in middle layer; it predicts very little stress when elasticity indicates maximum. However, the "equivalent shear modulus", G_{eq} unifies the cross sectional properties and predicts acceptable shear stress.

2.3.3 Shear Correction Factor with G_{eq}

The shear correction factor is a concept that attempts to correct the accuracy of the FOSD theory. It is based on employing equilibrium to calculate the shear stresses and the corresponding shear resultant and them comparing with the one from the FOSD theory.

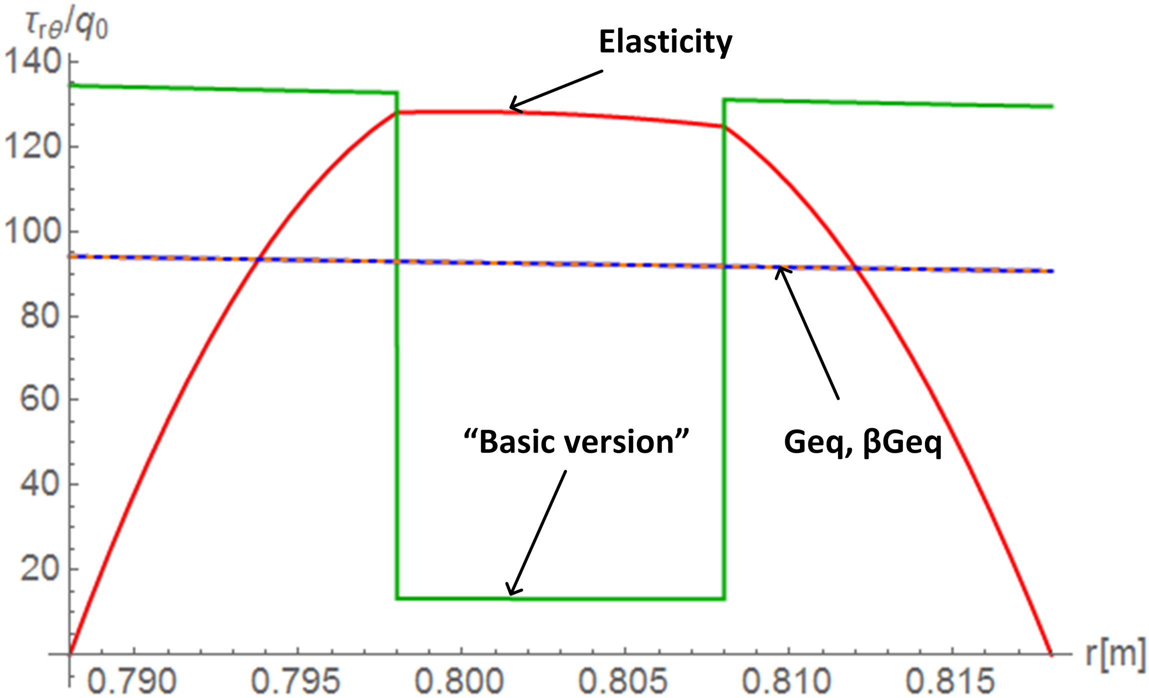

Figure 2.3: Through thickness shear stress distributions ,$\tau_{r\theta}$, at $\theta = 0$

The two equilibrium equations in polar coordinates are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \,, \tag{2.51a}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \,. \tag{2.51b}$$

The second equilibrium equation (2.51b), together with the assumed from the FOSD theory normal stress $\sigma_{\theta\theta}$ from (2.33b) and (2.33d) and substituting $z = r - R$ gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) = -\frac{c_{11}}{r^2}\left[w_{0,\theta} + u_{0,\theta\theta} + (r-R)\psi_{,\theta\theta}\right] \,, \tag{2.52a}$$

from which we obtain the shear stress in the face sheets and core in terms of a function

$C_k(\theta)$, $k = t, c, b$:

$$\tau_{r\theta}^{(k)} = -\frac{c_{11}^k}{r}\left(w_{0,\theta} + u_{0,\theta\theta} - R\psi_{,\theta\theta}\right) - \frac{c_{11}^k}{2}\psi_{,\theta\theta} + \frac{C_k(\theta)}{r^2} \ ; \quad k = t, b, c \tag{2.52b}$$

Imposing the outer boundary condition $\tau_{r\theta}^{(t)} = 0$ at $r = R_2$ gives:

$$C_t(\theta) = c_{11}^t R_2 \left[w_{0,\theta} + u_{0,\theta\theta} + \left(\frac{R_2}{2} - R\right)\psi_{,\theta\theta}\right] \ . \tag{2.52c}$$

Next, imposing the inner boundary condition $\tau_{r\theta}^{(b)} = 0$ at $r = R_1$, gives:

$$C_b(\theta) = c_{11}^b R_1 \left[w_{0,\theta} + u_{0,\theta\theta} + \left(\frac{R_1}{2} - R\right)\psi_{,\theta\theta}\right] \ . \tag{2.52d}$$

The traction conditions at the bottom face/core interface, $\tau_{r\theta}^{(b)} = \tau_{r\theta}^{(c)}$ at $r = R_1 + f_b$ gives:

$$C_c(\theta) = C_b(\theta) + \left(c_{11}^c - c_{11}^b\right)(R_1 + f_b)\left[w_{0,\theta} + u_{0,\theta\theta} + \left(\frac{R_1 + f_b}{2} - R\right)\psi_{,\theta\theta}\right] \ . \tag{2.52e}$$

whereas the traction condition at the top face/core interface, $\tau_{r\theta}^{(t)} = \tau_{r\theta}^{(c)}$ at $r = R_2 - f_t$, gives the relation:

$$\begin{aligned} -\left[c_{11}^t f_t\left(R_2 - \frac{f_t}{2} - R\right) + c_{11}^b f_b\left(R_1 + \frac{f_b}{2} - R\right) + c_{11}^c 2c\left(R_1 + f_b + c - R\right)\right]\psi_{,\theta\theta} \\ = \left[c_{11}^t f_t + c_{11}^b f_b + c_{11}^c(2c)\right]\left(w_{0,\theta} + u_{0,\theta\theta}\right) \ . \end{aligned} \tag{2.52f}$$

Thus, if we define

$$D = \frac{c_{11}^t f_t\left(R_2 - \frac{f_t}{2} - R\right) + c_{11}^b f_b\left(R_1 + \frac{f_b}{2} - R\right) + c_{11}^c 2c\left(R_1 + f_b + c - R\right)}{c_{11}^t f_t + c_{11}^b f_b + c_{11}^c(2c)} \ , \tag{2.52g}$$

substituting (2.52c) into (2.52b) gives the shear stresses as:

$$\tau_{r\theta}^{t} = c_{11}^{t}\left[-\frac{1}{2} + \frac{B}{r} + \frac{R_2}{r^2}\left(\frac{R_2}{2} - B\right)\right]\psi_{,\theta\theta}\ , \tag{2.53a}$$

$$\tau_{r\theta}^{b} = c_{11}^{b}\left[-\frac{1}{2} + \frac{B}{r} + \frac{R_1}{r^2}\left(\frac{R_1}{2} - B\right)\right]\psi_{,\theta\theta}\ , \tag{2.53b}$$

$$\tau_{r\theta}^{c} = c_{11}^{c}\left[-\frac{1}{2} + \frac{B}{r} + \frac{c_{11}^{c}(R_1+f_b) - c_{11}^{b}f_b}{c_{11}^{c}r^2}\left(R_1 + \frac{f_b}{2} - B\right) - \frac{R_1(R_1+f_b)}{2r^2}\right]\psi_{,\theta\theta}\ . \tag{2.53c}$$

where

$$B = D + R\ . \tag{2.53d}$$

Integrating over the cross section the shear force is:

$$V = \int_{R_1}^{R_1+f_b}\tau_{r\theta}^{b}dr + \int_{R_1+f_b}^{R_1+f_b+2c}\tau_{r\theta}^{c}dr + \int_{R_1+f_b+2c}^{R_2}\tau_{r\theta}^{t}dr = E\psi_{,\theta\theta}\ , \tag{2.54a}$$

where

$$\begin{aligned} E = {} & c_{11}^{b}\left[-\frac{f_b}{2} + B\ln\left(1+\frac{f_b}{R_1}\right) + \left(\frac{R_1}{2} - B\right)\frac{f_b}{R_1+f_b}\right] \\ & + c_{11}^{c}\left\{-c + B\ln\left(1+\frac{2c}{R_1+f_b}\right) + \left[\left(c_{11}^{c} - c_{11}^{b}\frac{f_b}{R_1+f_b}\right)\left(R_1+\frac{f_b}{2} - B\right) - c_{11}^{c}\frac{R_1}{2}\right]\frac{2c}{R_1+f_b+2c}\right\} \\ & + c_{11}^{t}\left[-\frac{f_t}{2} + B\ln\left(1+\frac{f_t}{R_1+f_b+2c}\right) + \left(\frac{R_2}{2} - B\right)\frac{f_t}{R_1+f_b+2c}\right]\ . \end{aligned} \tag{2.54b}$$

Then we can write the shear stresses in each layer as:

$$\tau_{r\theta}^{k} = \frac{V}{E}c_{11}^{k}\left(-\frac{1}{2} + \frac{B}{r} + \frac{d_k}{r^2}\right)\ ; \quad k = b, c, t \tag{2.55a}$$

where

$$d_t = R_2\left(\frac{R_2}{2} - B\right) \ ; \quad d_b = R_1\left(\frac{R_1}{2} - B\right) \ , \tag{2.55b}$$

$$d_c = \frac{\left[c_{11}^c(R_1+f_b) - c_{11}^b f_b\right]}{c_{11}^c}\left(R_1 + \frac{f_b}{2} - B\right) - \frac{R_1(R_1+f_b)}{2} \ . \tag{2.55c}$$

The corresponding strain energy is

$$U_{EQ} = \int_{R_1}^{R_1+f_b} \frac{\tau_{r\theta}^{b2}}{2c_{55}^b} dr + \int_{R_1+f_b}^{R_1+f_b+2c} \frac{\tau_{r\theta}^{c2}}{2c_{55}^c} dr + \int_{R_2-f_t}^{R_2} \frac{\tau_{r\theta}^{t2}}{2c_{55}^t} dr \ . \tag{2.55d}$$

Let us define the radius at the inner and outer boundaries of each layer by r_{ki} and r_{ko} and the radius at the mid-surface of each layer by r_{km}; also the thickness of each layer by h_k where k = b, c, t. In particular, for the bottom face,

$$r_{bi} = R_1 \ ; \quad r_{bo} = R_1 + f_b \ ; \quad r_{bm} = R_1 + \frac{f_b}{2} \ ; \quad h_b = f_b \ . \tag{2.56a}$$

For the core

$$r_{ci} = R_1 + f_b \ ; \quad r_{co} = R_1 + f_b + 2c \ ; \quad r_{cm} = R_1 + f_b + c \ ; \quad h_c = 2c \ , \tag{2.56b}$$

and for the top face

$$r_{ti} = R_2 - f_t \ ; \quad r_{to} = R_2 \ ; \quad r_{tm} = R_2 - \frac{f_t}{2} \ ; \quad h_t = f_t \ . \tag{2.56c}$$

Then

$$U_{EQ} = \frac{V^2}{2E^2} \sum_{k=b,c,t} \frac{c_{11}^{k2}}{c_{55}^k} \left[\frac{h_k}{4} - B \ln\left(1 + \frac{h_k}{r_{ki}}\right) + \frac{(B^2 - d_k)h_k}{r_{ki} r_{ko}} \right.$$
$$\left. + 2Bd_k \frac{h_k r_{km}}{r_{ki}^2 r_{ko}^2} + d_k^2 \frac{(r_{ko}^3 - r_{ki}^3)}{3 r_{ki}^3 r_{ko}^3} \right] \ . \tag{2.57}$$

Now the FOSD shear stress from (2.33c) and (2.48b) is

$$\tau_{r\theta} = \frac{\bar{c}_{55}}{r}\left(w_{0,\theta} - u_0 + R\psi\right) \ . \tag{2.58a}$$

Integrating over the cross-section gives

$$V = \int \tau_{r\theta} dr = \bar{c}_{55}\left(w_{0,\theta} - u_0 + R\psi\right) \ln\frac{R_2}{R_1} \ . \tag{2.58b}$$

Thus the shear force from the FOSDT can be expressed as

$$\tau_{r\theta} = \frac{V}{r\ln(R_2/R_1)} \ , \tag{2.58c}$$

and the corresponding energy is

$$U_{FOSD} = \int_{R_1}^{R_2} \frac{\tau_{r\theta}^2}{2\bar{c}_{55}} dr = \frac{V^2}{2\bar{c}_{55}[\ln(R_2/R_1)]^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \ , \tag{2.59}$$

resulting in a shear correction factor β = U_{FOSD}/U_{EQ}:

$$\frac{1}{\beta} = \frac{\bar{c}_{55}[\ln(R_2/R_1)]^2 R_1 R_2}{E^2(R_2 - R_1)} \sum_{k=b,c,t} \frac{c_{11}^{k2}}{c_{55}^k}\left[\frac{h_k}{4} - B\ln\left(1 + \frac{h_k}{r_{ki}}\right) + \frac{(B^2 - d_k)h_k}{r_{ki}r_{ko}}\right.$$
$$\left. + 2Bd_k\frac{h_k r_{km}}{r_{ki}^2 r_{ko}^2} + d_k^2\frac{(r_{ko}^3 - r_{ki}^3)}{3r_{ki}^3 r_{ko}^3}\right] \ . \tag{2.60}$$

Thus, the differential equations of the FOSD (2.43)-(2.47) can be applied with the area properties defined as in the G_{eq} version (previous section) and with $\beta\bar{c}_{55}$ in place of $\bar{c}_{55}$.

A connection of curved beam shear correction factor ,β, and Timoshenko beam theory (flat) which, normally, the shear correction factor is $\kappa = 5/6$ for a rectangular cross section [53].

Considering a single material curved beam whose thickness is h and bounded by outer radius R_2 and inner radius R_1 , similar to 2.1. Using (2.52) and two traction free surface conditions: $\tau_{r\theta} = 0$ at $r = R_1$ and $r = R_2$, gives

$$\tau_{r\theta} = -\frac{c_{11}(r - R_2)(r - R_1)\psi_{\theta\theta}}{2r^2} \,. \tag{2.61a}$$

Integrating over the cross section to get shear force V, then the shear stress can be written in term of shear force as

$$\tau_{r\theta} = \frac{(r - R_1)(r - R_2)V}{r^2(2R_2 - 2R_1 + (R_1 + R_2)\ln \frac{R_1}{R_2})} \,. \tag{2.61b}$$

Corresponding strain energy is

$$U_{EQ} = \int_{R_1}^{R_2} \frac{\tau_{r\theta}^2}{2c_{55}} dr \;=\; \frac{V^2}{2c_{55}\left(2(R_2 - R_1) + (R_1 + R_2)\ln \frac{R_1}{R_2}\right)^2} A, \tag{2.61c}$$

where,

$$\begin{aligned} A = (R_2 - R_1) - 2(R_1 + R_2)\ln \frac{R_2}{R_1} + \left((R_1 + R_2)^2 + 2R_1R_2\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ + R_1R_2(R_1 + R_2)\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right) + \frac{R_1^2R_2^2}{3}\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) \,. \end{aligned} \tag{2.61d}$$

Now the FOSD shear stress and corresponding strain energy are as previously shown in (2.58), and (2.59).

The outer radius $R_2 = R_1 + h$, substituting the relation into (2.61c), then the shear correction factor is

$$\beta = U_{FOSD}/U_{EQ} = \frac{\left(2h + (h + 2R_1)\ln \frac{R_1}{R_1 + h}\right)^2}{AR_1(1 + \frac{R_1}{h})\ln^2 \frac{R_1 + h}{R_1}} \tag{2.62a}$$

where,

$$A = \frac{h^3}{3R_1(R_1 + h)} + 4h - 2(h + 2R_1)\ln \frac{R_1 + h}{R_1} \,. \tag{2.62b}$$

Let us define,

$$\alpha = \frac{h}{R_1} \ , \tag{2.62c}$$

and divide both numerator and denominator of (2.62a) by R_1^2 becomes:

$$\beta = \frac{\left(2\alpha - (\alpha+2)\ln(1+\alpha)\right)^2}{\left(\frac{\alpha^2}{3} + 4(1+\alpha) - 2(3+\alpha+\frac{2}{\alpha})\ln(1+\alpha)\right)\ln^2(1+\alpha)} \ . \tag{2.62d}$$

To make a connection with flat beam, taking limit the inner radius R_1 to infinity i.e. $\alpha << 1$. Using the Taylor series expansion

$$\ln(1+\alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \frac{\alpha^5}{5} + O(\alpha^6) \ , \tag{2.62e}$$

and substituting into (2.62d), and keeping all terms up to $O(\alpha^6)$, resulting in:

$$\beta = \frac{\alpha^6/36 + O(\alpha^7)}{\alpha^6/30 + O(\alpha^7)} = 5/6 \ , \tag{2.62f}$$

i.e., the shear correction factor of the Timoshenko beam has been recovered.

2.3.4 Classical Theory Formulation for Curved Sandwich Panels

In the classical theory, the displacement field is in the form:

$$w(z,\theta) = w_0(\theta) \ ; \qquad u(z,\theta) = u_0(\theta) + \frac{z}{R}\left[u_0(\theta) - w_0'(\theta)\right] \ . \tag{2.63}$$

where z is the distance from the reference radius R, as defined in the previous section. Accordingly, the strains, since $r = R + z$, are from (2.33):

$$\epsilon_{\theta\theta}(z,\theta) = \frac{1}{R+z}\left\{w_0(\theta) + u_0'(\theta) + \frac{z}{R}\left[u_0'(\theta) - w_0''(\theta)\right]\right\} \ . \tag{2.64a}$$

$$\epsilon_{rr}(z,\theta) = 0\ ; \qquad \gamma_{r\theta}(z,\theta) = 0\ . \tag{2.64b}$$

i.e., there is only one non-zero strain, the $\epsilon_{\theta\theta}$.

Governing equations and associated boundary conditions are again derived from the Principle of Minimum Total Potential energy, eqn (2.34a), where now

$$\delta U = \int_0^\alpha \int_A \sigma_{\theta\theta}\delta\epsilon_{\theta\theta}(R+z)dAd\theta\ , \tag{2.65}$$

and δV is given by (2.34c).

In this theory we need two additional area quantities: the “extensional area”, A_{E0},

$$A_{E0} = \int_A c_{11}^i dA = c_{11}^t f_t + c_{11}^c(2c) + c_{11}^b f_b\ , \tag{2.66a}$$

and the “extensional static moment”, S_{E0},

$$S_{E0} = \int_A c_{11}^i z dA = c_{11}^t f_t\left(\frac{f_t}{2}+c-e\right) - 2c_{11}^c ce - c_{11}^b f_b\left(\frac{f_b}{2}+c+e\right)\ , \tag{2.66b}$$

Performing the integration by parts, results in the following three governing equations for the classical theory (for $0 \leq \theta \leq \alpha$):

δw_0 :

$$\left(\frac{I_{E1}}{R^2}w_{0,\theta\theta} - \frac{S_{E0}}{R^2}u_{0,\theta}\right)_{,\theta\theta} + \frac{A_{E0}}{R}u_{0,\theta} + A_{E1}w_0 = bq_t\ . \tag{2.67a}$$

δu_0 :

$$\left[\left(\frac{A_{E0}}{R}+\frac{S_{E0}}{R^2}\right)u_{0,\theta} + \frac{A_{E0}}{R}w_0 - \frac{S_{E0}}{R^2}w_{0,\theta\theta}\right]_{,\theta} = 0\ . \tag{2.67b}$$

Associated boundary conditions (three at each end, θ_e = 0, α) for the Classical theory are;

Either $\delta w_0 = 0$ *or*,

$$-\left(\frac{I_{E1}}{R^2}w_{0,\theta\theta} - \frac{S_{E0}}{R^2}u_{0,\theta}\right)_{,\theta} = N_e \ , \tag{2.68a}$$

Either $\delta w_{0,\theta} = 0$ *or*,

$$\frac{I_{E1}}{R^2}w_{0,\theta\theta} - \frac{S_{E0}}{R^2}u_{0,\theta} = M_e \ , \tag{2.68b}$$

Either $\delta u_0 = 0$ *or*,

$$\left(\frac{A_{E0}}{R} + \frac{S_{E0}}{R^2}\right)u_{0,\theta} + \frac{A_{E0}}{R}w_0 - \frac{S_{E0}}{R^2}w_{0,\theta\theta} = H_e \ , \tag{2.68c}$$

Again, assuming constant properties and using the form (2.32) for the displacements w_0 and u_0 leads to:

$$\left(\frac{I_{E1}}{R^2}k^4 + A_{E1}\right)W_0 - k\left(\frac{S_{E0}}{R^2}k^2 + \frac{A_{E0}}{R}\right)U_0 = R_2 q_0 \ , \tag{2.69a}$$

$$\left(\frac{S_{E0}}{R^2}k^2 + \frac{A_{E0}}{R}\right)W_0 - k\left(\frac{S_{E0}}{R^2} + \frac{A_{E0}}{R}\right)U_0 = 0 \ . \tag{2.69b}$$

from which U_0 and W_0 can be directly determined.

2.4 Results

Simply supported curved sandwich panels subjected to a half sine distributed load (figure 2.4),

$$q_t(\theta) = q_0 \sin\frac{\pi\theta}{\alpha} \ , \tag{2.70}$$

are studied.

The solutions from three different theories and their variants in previous chapter are presented. They are: two versions of the Extended High Order Sandwich Panel Theory(EHSAPT) with logarithmic and polynomial core displacement functions, Theory of

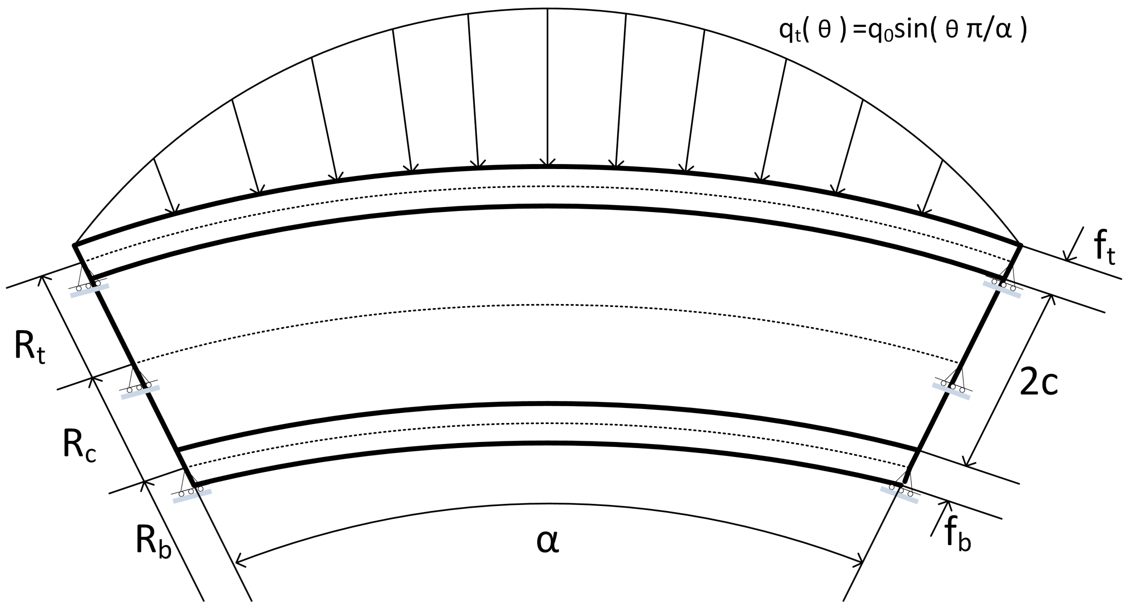

Figure 2.4: Simply supported curved sandwich panels subjected to a half sine distributed load

Elasticity(ELST), the First Order Shear Deformation Theory (FOSDT) basic variant, equivalent shear modulus variant, and shear correction factor variant, and Classical Theory (CLSS). In addition, solutions from high order sandwich panel theory HSAPT, which assumes a compressible core [21],[37] are also presented. The Elasticity solution serves as the benchmark to assess the accuracy of all theories, whereas the First Order Shear Deformation Theory is widely used in the sandwich structures community due to its simplicity. Solutions correspond to these different theories are then compared for various sandwich geometries and core materials in order to validate and assess the relative merits of the two versions of the EHSAPTs and other accuracy of all structural theories.

Various geometries of the sandwich curved beam/panel in figure 2.1 are analyzed. In particular, we consider a symmetric construction with thin faces of thickness $f_{t,b} = 1$ mm, made out of isotropic aluminium (2024-T3) with modulus $E_{t,b} = 69.13$ GPa and a thick core of thickness $2c$ = 25 mm. In addition, the (out-of-plane) width is $b = 30$ mm.

Cases 1 and 2 consist of a curved beam of angular span $\alpha = 3\pi/4$ and a radius of the top face mid-line R_t = 813 mm. Case 1 has a core made out of the relatively stiffer Balsawood

(Gurit Balsaflex) with $E_c = 5199$ MPa, $G_c = 206$ MPa and and ν_c = 0.30 whereas case 2 has a core made out of the relatively flexible foam Divinycell H35, with $E_c = 40$ MPa, $G_c = 12$ MPa and ν_c = 0.30.

Cases 3 of a curved beam of, again, angular span is $\alpha = 3\pi/4$ but with a (relatively small) radius of the top face mid-line R_t = 40.65 mm and a core made out of the soft foam Divinycell H35 (properties given in the previous paragraph).

Case 4 consists of a curved beam of a relatively small angular span, $\alpha = \pi/8$ rad, and a radius of the top face mid-line R_t = 81.3 mm and a core made out of the soft foam Divinycell H35 (with $E_c = 40$ MPa, $G_c = 12$ MPa and ν_c = 0.30).

In the following we shall normalize the stresses with $|q_0|$, and the transverse displacement with the quantities that scale the maximum transverse displacement of a flat plate of length $R_t\alpha$ and bending rigidity $EI = 2E_t f_t c^2$ under distributed load q_0, i.e., the

$$w_{norm} = \frac{q_0 R_t^4 \alpha^4}{2E_t f_t c^2} \; ; \quad \tilde{w}(r,\theta) = \frac{w(r,\theta)}{w_{norm}} \; , \tag{2.71}$$

and the circumferential (hoop) displacement with the quantities that scale the shortening of a flat plate of length $R_t\alpha$ and bending rigidity $EI = 2E_t f_t c^2$ under distributed load q_0, i.e., the

$$u_{norm} = \frac{q_0^2 R_t^7 \alpha^7}{[2E_t f_t c^2]^2} \; ; \quad \tilde{u}(r,\theta) = \frac{u(r,\theta)}{u_{norm}} \; . \tag{2.72}$$

Radial through-thickness and angular span-wise coordinates are presented in the dimensionless quantities:

$$\tilde{r} = \frac{r - R_1}{R_2 - R_1} \; ; \quad \tilde{\theta} = \frac{\theta}{\alpha} \; , \tag{2.73}$$

i.e., the sandwich top panel surface is at $\tilde{r} = 1$ and bottom surface is at $\tilde{r} = 0$.

Figures 2.5 and 2.6 show the radial (transverse) normal stress, σ_{rr}, through the thickness and at mid-span ($\theta = \alpha/2$), from the elasticity, logarithmic EHSAPT, polynomial

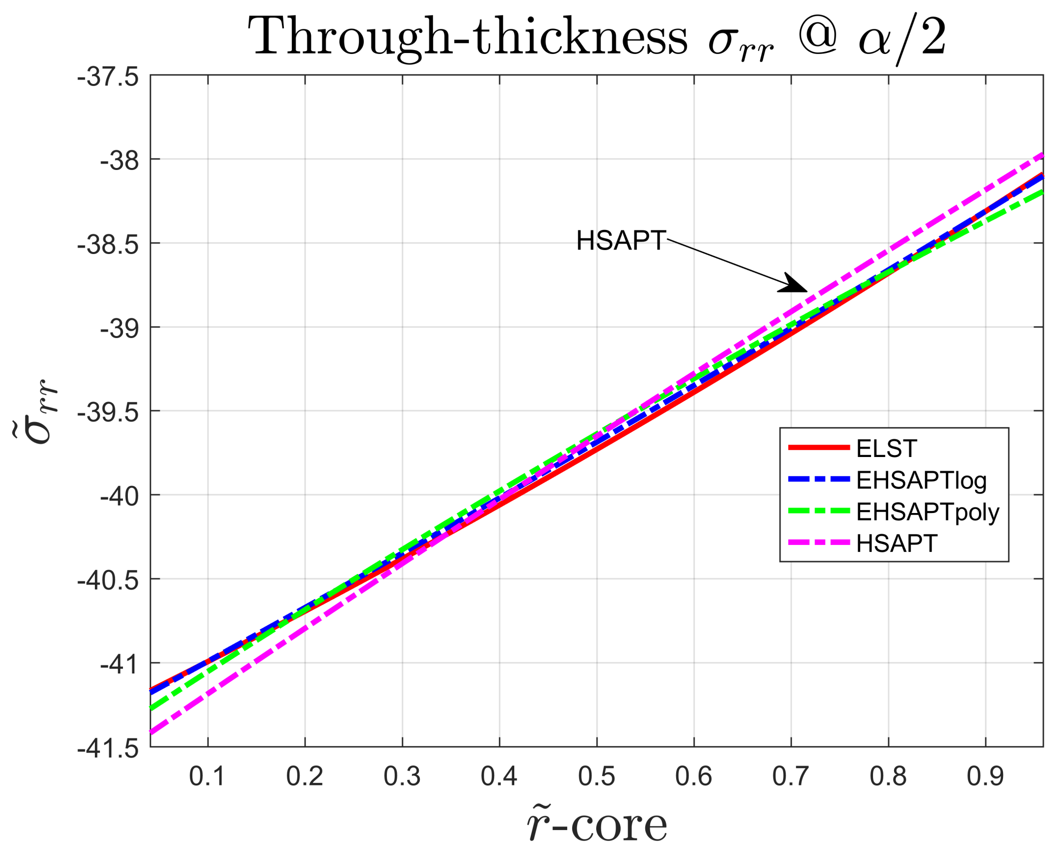

Figure 2.5: The radial normal stress distribution, σ_{rr} for case 1

EHSAPT, and the HSAPT, for case 1 and 2. Notice that neither the FOSD nor the classical theory can provide σ_{rr}, because they are incompressible theories. However, the two EHSAPT variants and HSAPT, which are compressible core theory, can provide the radial normal stress. Figure 2.5, stiff core case 1, show clearly that the logarithmic EHSAPT (2.3) has a superior accuracy over the polynomial EHSAPT (2.7) and HSAPT. When the core is very flexible, figure 2.6, the difference between the two EHSPATs and HSAPT is negligible (case 2). Accurate determination of the radial stresses is needed because significant compressive stresses can develop within the core, for example, under impact loading, which can result in core crushing failure modes [16]-[20]. It should also be noted that the core σ_{rr} is significant in value and, in both cases, the logarithmic EHSAPT can capture the stress very as predicted by elasticity .

Figures 2.7 and 2.8 show the shear stress distribution, $\tau_{r\theta}$, through the thickness and

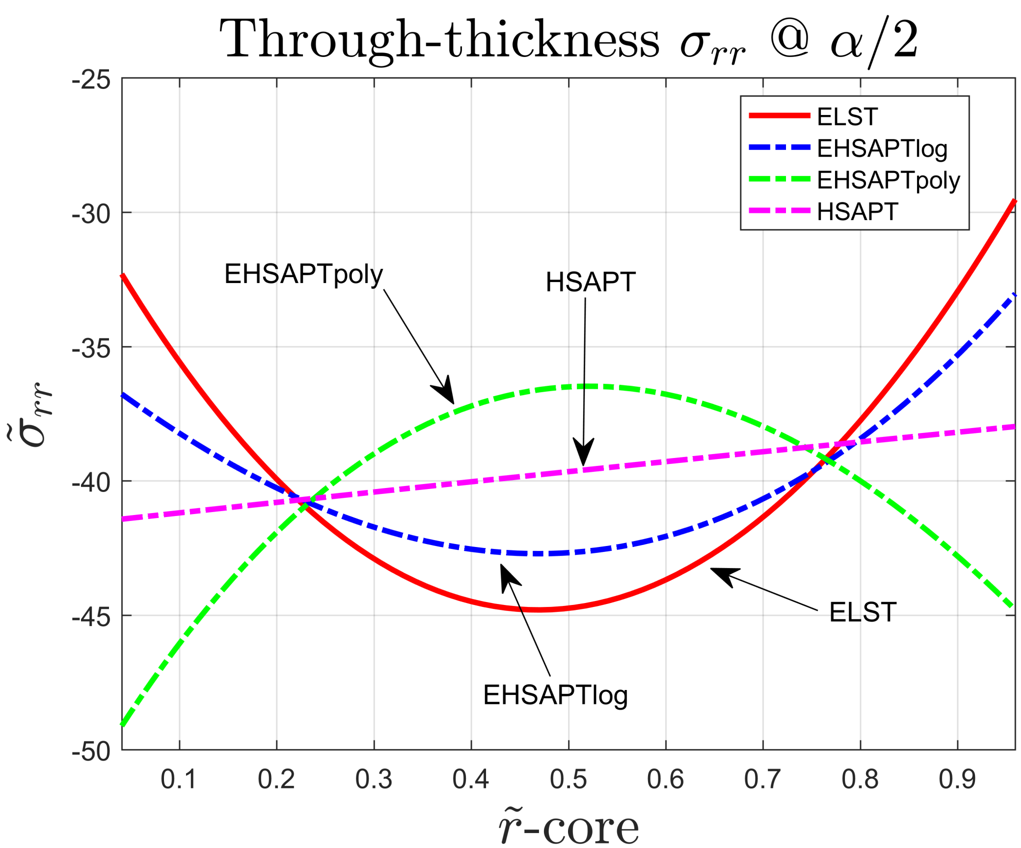

Figure 2.6: The radial normal stress distribution, σ_{rr} for case 2

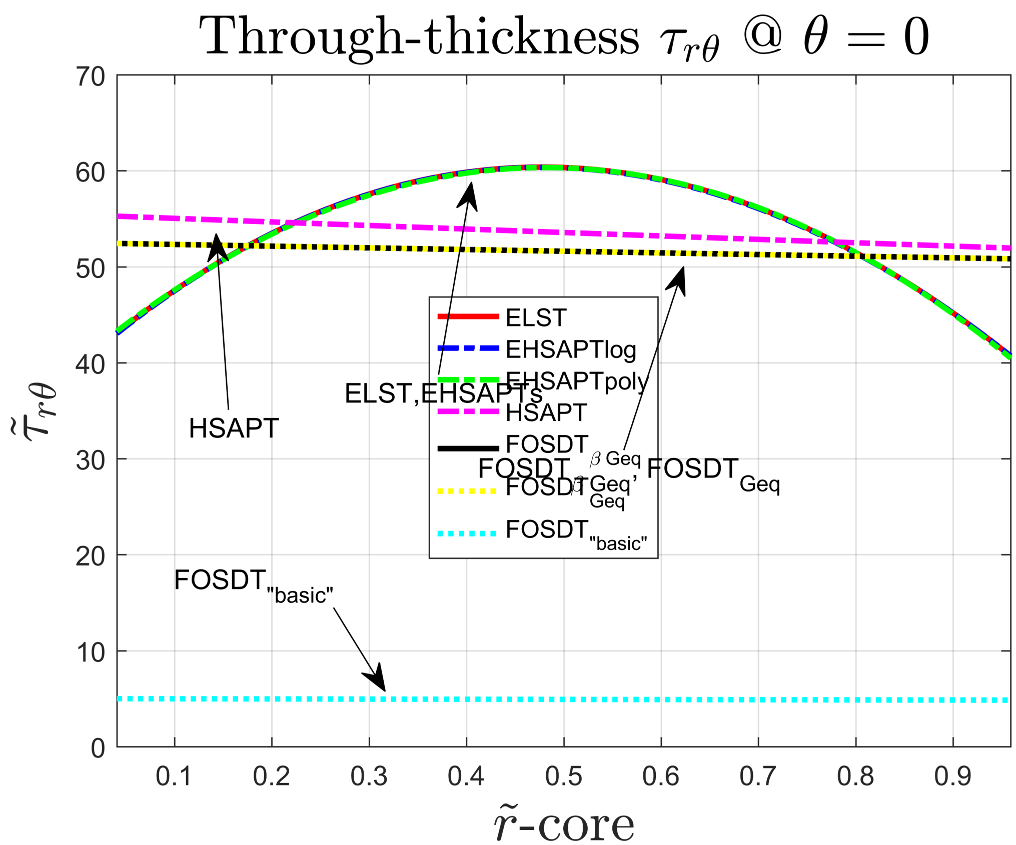

Figure 2.7: The shear stress distribution, $\tau_{r\theta}$ for case 1

at $\theta = 0$, from elasticity, the FOSD theory with an equivalent shear modulus, and the high order HSAPT for cases 1 (stiffer balsa wood core) and 2 (softer Divinycell H35), respectively. First thing can be noticed the distribution is that highly nonlinear for the stiffer core case but rather flat for the softer core; moreover, neither the FOSD theory nor the HSPAT can capture the nonlinear profile in the stiffer core case. Secondly, Both EHSAPT variants accurately capture the shear distribution for both cases; while, the prediction by either the equivalent shear modulus FOSD theory or the HSPAT is, in general, of the same order of magnitude as elasticity and the “basic” FOSD theory largely underestimate shear stress. Also, notice that the classical theory would predict no shear.

The hoop normal stress, $\sigma_{\theta\theta}$ is of most interest in the face sheets where it is of significant magnitude. Figure 2.9 show the distribution of the $\sigma_{\theta\theta}$ at mid-span ($\theta = \alpha/2$) in the bottom and top faces, respectively for case 3 (soft core). Both the FOSD theory and the

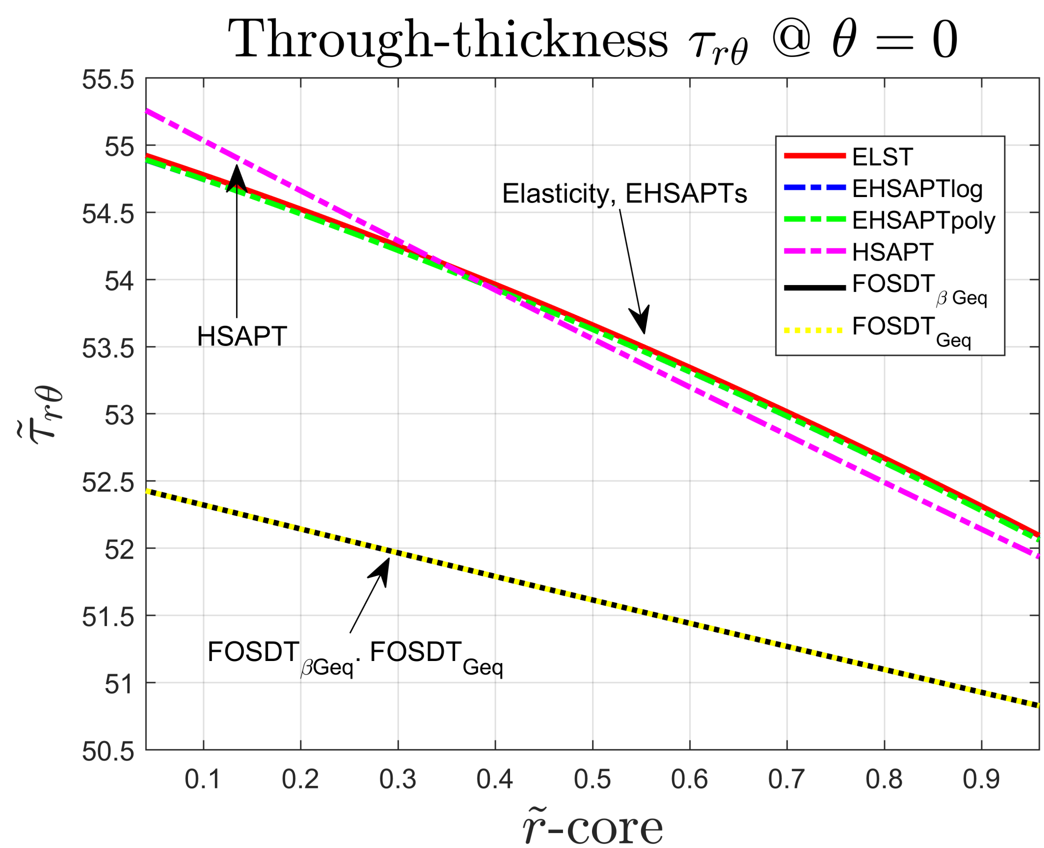

Figure 2.8: The shear stress distribution, $\tau_{r\theta}$ for case 2

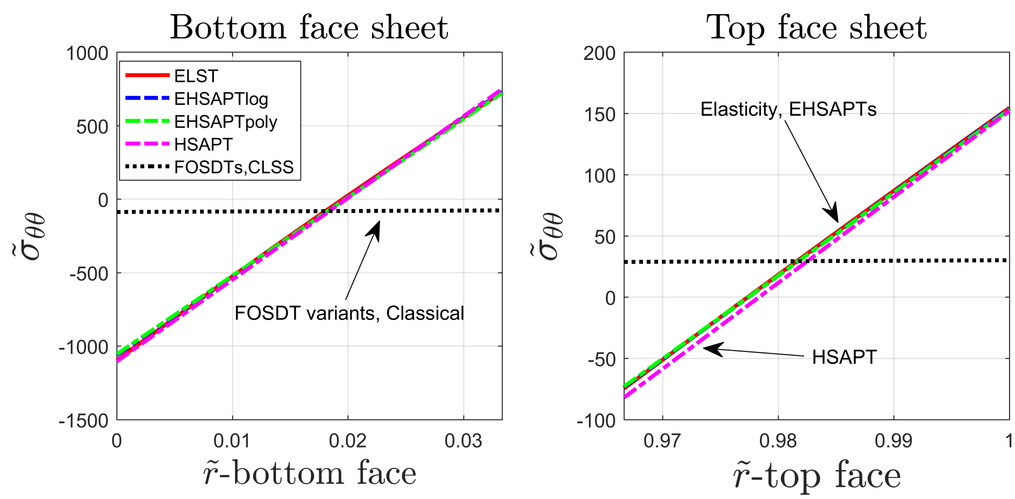

Figure 2.9: The circumferential (hoop) stress distribution for bottom face (left) and top face (right), σ_{ss} for case 3

Classical theory are rather flat unlike the elasticity which shows a line with a rather significant gradient; thus, the FOSD theory and the Classical theory cannot capture the extreme values of the $\sigma_{\theta\theta}$ and this can have significant implications for the prediction of failure. The two EHSAPT are; however, very close to the elasticity, and, shows a gradient similar to elasticity.

Figures 2.10 and 2.11 present the transverse displacement distribution (spanwise and through the thickness, respectively) for case 4. From Figures 2.10, it can be seen that both the EHSAPT-logarithmic and the EHSAPT-polynomial are in very close agreement with the elasticity while HSAPT slightly overestimate. The equivalent shear modulus FOSD theory underestimate the displacement; however, with shear correction factor, the theory shows an improvement in accuracy. The basic FOSD and classical theory seems to be very inaccurate and they are not in the same order of magnitude. From figure 2.11, it can be seen that the through-thickness profile of the transverse displacement is nonlinear and the EHSAPT is capable of accurately capturing this nonlinear profile, unlike the FOSD theory or the Classical theory.

Finally, figure 2.12 and 2.13 present the circumferencial (hoop) displacement distribution (spanwise and through the thickness, respectively) for case 4. Again both versions

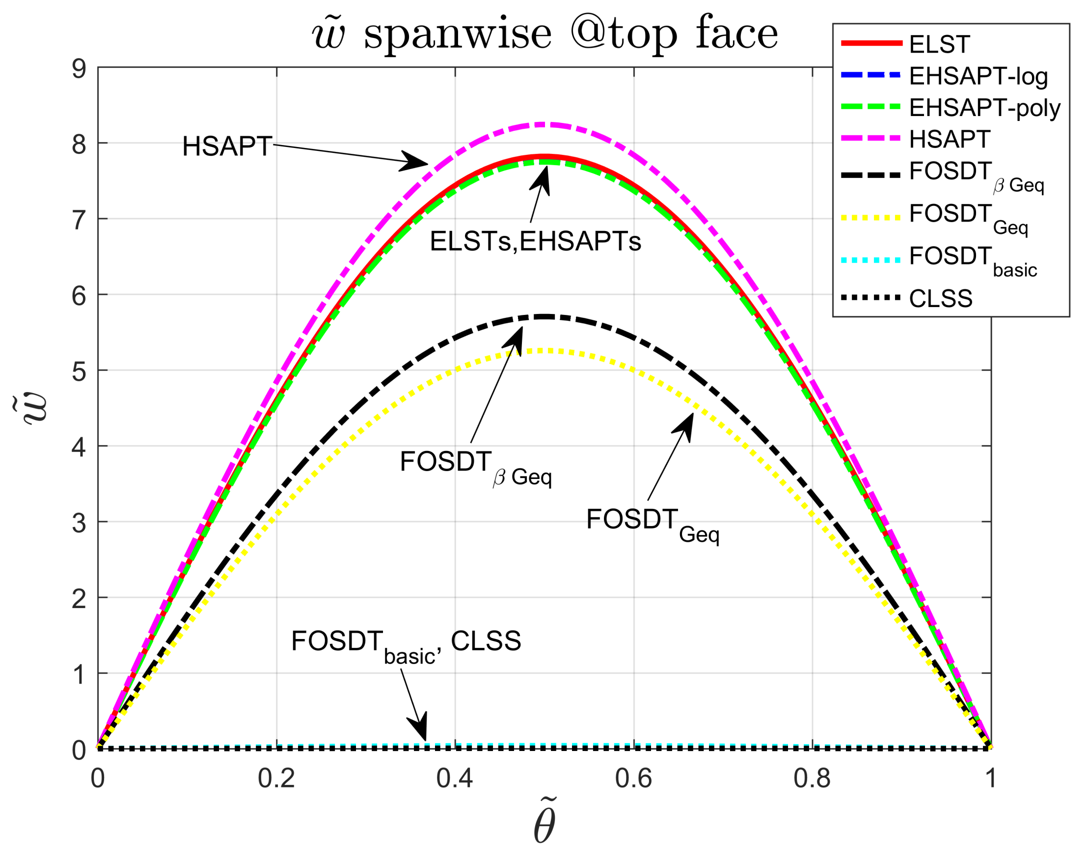

Figure 2.10: The top face transverse displacement distribution through the span of the curved panel, w for case 4

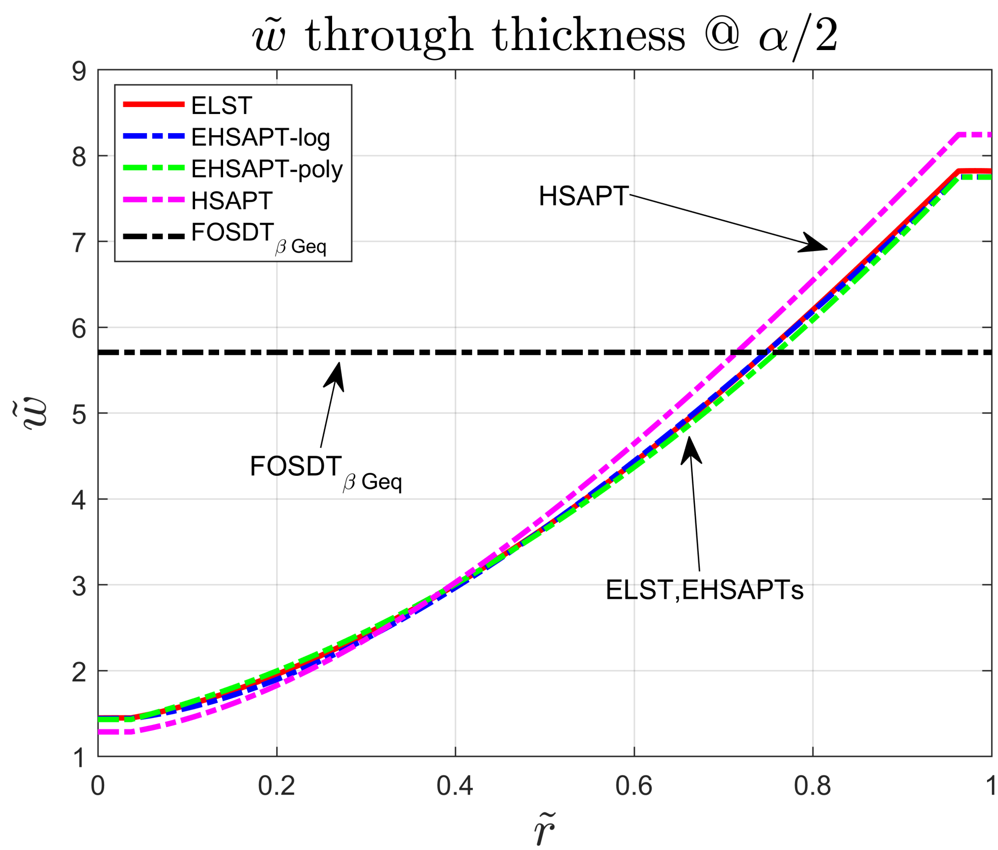

Figure 2.11: The transverse displacement distribution through the thickness of the curved panel, w, at $\theta = \alpha/2$, for case 4

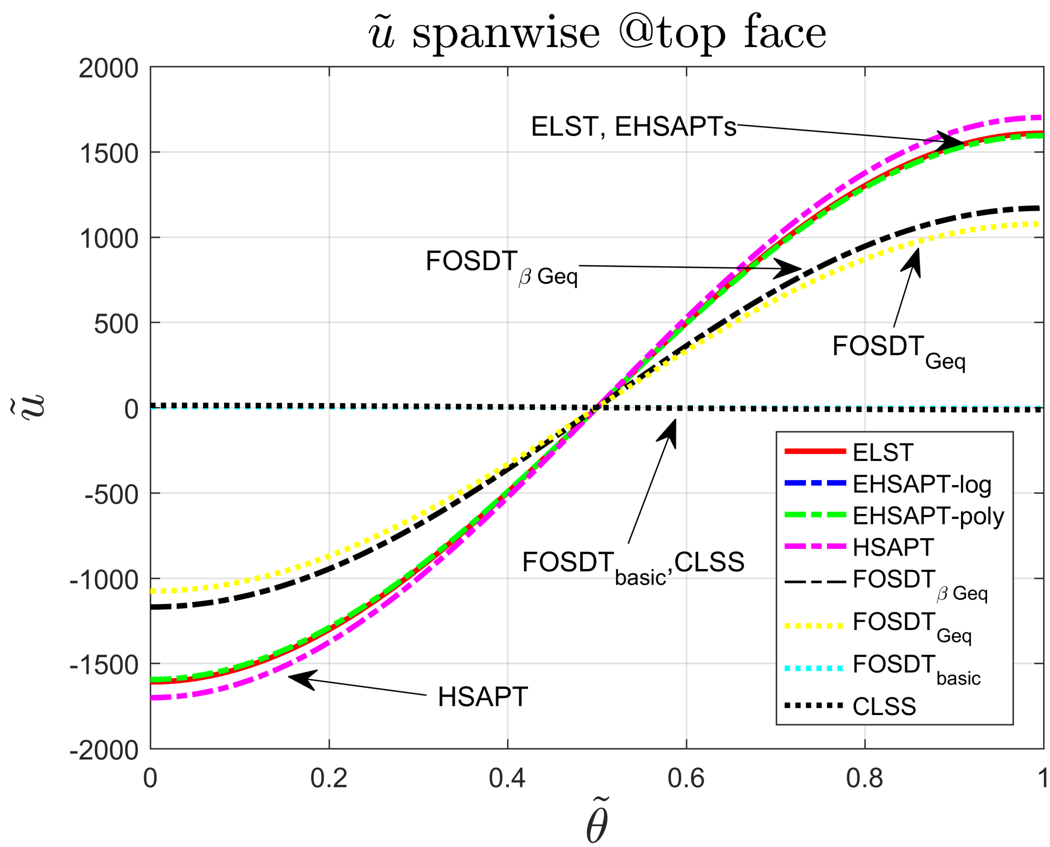

Figure 2.12: The top face circumferential displacement distribution through the span of the curved panel, u, for case 4

of EHSAPT precisely predict circumferential displacement and its through thickness non-linear profile. And similar other conclusions can be drawn except for through thickness displacement of HSAPT in figure 2.13, which excessively estimate the displacement profile while the equivalent shear modulus FOSD with shear correction factor seems to better predict the average values.

The results shown above, demonstrate clearly excellent performance of EHSAPTs, in particular, logarithmic-EHSAPT; and, the limitations of the FOSD theory, which can be significant, even with shear correction factors included. In this regard, advanced higher order compressible theories, i.e. EHSAPT can make up for the shortcomings of the FOSD theory.

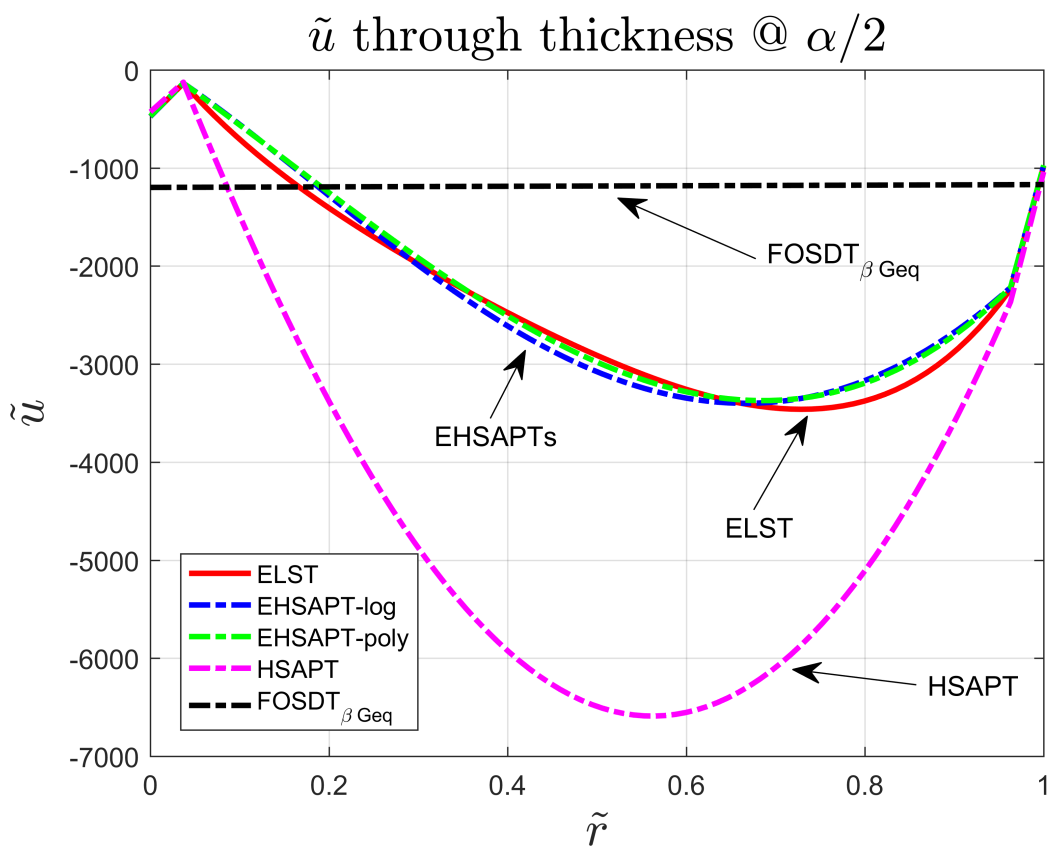

Figure 2.13: The transverse displacement distribution through the thickness of the curved panel, u, at $\theta = 0$, for case 4

2.5 Conclusion

In summary, simply support curved sandwich panels subjected to a distributed load acting on the top face are studied and closed form analytical solution and its procedure are outlined for following theories. Firstly, two variants of the Extended High order Sandwich Panel Theory (EHSAPT) are formulated. One is based on a logarithmic displacement field for the core, while the other is based on polynomial functions of the thickness-wise coordinate. The system of governing differential equations and the associated boundary conditions are derived via the Principle of Minimum Total Potential Energy. Secondly, the displacement approach for Theory of Elasticity was used to derive a closed-form elasticity solution for a simply-supported curved sandwich beam/panel with orthotropic faces and orthotropic core. Certain cases of the orthotropic material constants result in different mathematical functions describing the displacement and stresses, which can include logarithmic functions or combinations or powers of the radial coordinate and logarithmic functions. Thirdly, three variants of the first order shear deformation (FOSD) theory for curved sandwich panels are formulated. The three variants are: the "basic", in which the theory is derived by direct application of the variational principles on the assumed displacement field (similar to the derivation of the Timoshenko beam theory); the "G_{eq}", in which an equivalent shear modulus for the section is defined and the "shear correction factor", in which a shear correction factor is derived and used on the "G_{eq}" version. Notice that due to the curvature, the FOSD theory for curved sandwich panels is not a direct extension of the corresponding one for flat panels and has several unique features, such as a reference curve which needs to be defined accordingly. The classical theory for curved panels (which does not include transverse shear) is also derived.

Results from various geometries and core materials are compared to Theory of Elasticity which regard as an accuracy benchmark. Elasticity results give radial the normal stress, σ_{rr}, which cannot be predicted by the incompressible core theories, the shear stress, $\tau_{r\theta}$, which has a highly nonlinear profile and is not constant for moderately stiff cores, as well as the hoop normal stress, $\sigma_{\theta\theta}$ in the faces; and circumferential and transverse displacement which may have a nonlinear distribution through the thickness. Both EHSAPTs capture all these three stress components: σ_{rr}, $\tau_{r\theta}$, $\sigma_{\theta\theta}$ in sandwich core and show exceptional accuracy (unlike other theories) but the logarithmic EHSAPT has an edge over the polynomial EHSAPT especially with regard to the radial normal stresses σ_{rr} and for configurations involving stiffer cores. HSAPT ([21],[37]) also show certain level of accuracy but not as good as EHSAPT. On the other hand, FOSD can not predict the, rather significant, radial normal stress, σ_{rr} (since they are all incompressible theories) and the FOSD theory absolutely needs to be used with an equivalent shear modulus, G_{eq}, as the "basic" version offers little, if any, improvement over the classical theory. Further improved accuracy can be achieved by embedding a shear correction factor.

CHAPTER 3

FREE VIBRATION ANALYSIS OF CURVED SANDWICH COMPOSITES BEAM/WIDE PANEL

In this chapter, we consider the same curved sandwich configuration as previously seen in chapter 2, figure 2.1. Free vibration analysis of the Extended High order Sandwich Theory(EHSAPT), the First Order Shear Theory (FOSD), and Classical Theory are formulated using Hamilton Principle; and dynamic Elasticity solution for simply supported boundary conditions are derived using the method of Frobenius series.

3.1 Extended High order Sandwich Panel Theory dynamic formulation

In the following, free vibration dynamic formulations will be presented with logarithmic (2.3) and polynomial core displacements (2.7) under the same assumptions as presented in section 2.1 static EHSAPT formulation. Kinematic descriptions of top/bottom face sheets and core, strain displacement relationships and constitutive laws are also derived with the same procedure (2.1)-(2.8) with addition independent variable t i.e. the dynamic seven dependent variables are a function of (θ, t): two for the top face sheet; $w_0^t(\theta, t)$, $u_0^t(\theta, t)$, two for the bottom face sheet; $w_0^b(\theta, t)$, $u_0^b(\theta, t)$, and three for the core; $w_0^c(\theta, t)$, $u_0^c(\theta, t)$, and $u_1^c(\theta, t)$.

3.1.1 Governing differential equations and associated boundary conditions

Governing equations and associated boundary conditions are derived from Hamilton's Principle, the principle states a dynamic equilibrium.

$$\delta \int_t (K - U + V)\, dt = 0 \,. \tag{3.1a}$$

where K is kinetic energy, U is strain energy of the sandwich panel, and V is the external potential due to applied loads which is none, $V = 0$, because the problem is free vibration analysis.

The first variation of the kinetic energy is:

$$\delta K = \int_0^\alpha \Bigg[\int_{-f_b/2}^{f_b/2} \rho_b \left(\dot{w}^b \delta \dot{w}^b + \dot{u}^b \delta \dot{u}^b \right) R_b dz_b \\ + \int_{r_{bc}}^{r_{tc}} \rho_c \left(\dot{w}^c \delta \dot{w}^c + \dot{u}^c \delta \dot{u}^c \right) r dr + \int_{-f_t/2}^{f_t/2} \rho_t \left(\dot{w}^t \delta \dot{w}^t + \dot{u}^t \delta \dot{u}^t \right) R_t dz_t \Bigg] b d\theta \ , \quad (3.1b)$$

where $\dot{}$ and $\ddot{}$ denote d/dt and d^2/dt^2 respectively.

The first variation of the strain energy was written in (2.9b).

Substituting (3.1b) and (2.9b) into (3.1a), and applying integration by part on δK respect to dt removing time derivative of displacement's variation, then the first variation of the kinetic energy becomes:

$$\delta K = -\int_0^\alpha \Bigg[\int_{-f_b/2}^{f_b/2} \rho_b \left(\ddot{w}^b \delta w^b + \ddot{u}^b \delta u^b \right) R_b dz_b \\ + \int_{r_{bc}}^{r_{tc}} \rho_c \left(\ddot{w}^c \delta w^c + \ddot{u}^c \delta u^c \right) r dr + \int_{-f_t/2}^{f_t/2} \rho_t \left(\ddot{w}^t \delta w^t + \ddot{u}^t \delta u^t \right) R_t dz_t \Bigg] b d\theta \ , \quad (3.1c)$$

Hamilton's principle states that the integral (3.1a) must vanishes for all arbitrary choice of displacement's variation, they are: $\delta w_0^{t,b,c}$, $\delta u_0^{t,b,c}$ and δu_1^c. Then, the statement yields:

$$(\delta K + \delta U) = 0 \quad (3.1d)$$

Subsequently, the same procedure of deriving governing equations and associated boundary conditions that was carried out in Chapter 2 static EHSAPT is followed here. Indeed, Left-Hand-Side terms of (2.10)-(2.11) which contributed from δU remain identical. And all contribution from δK will be presented in Right-Hand-Side terms. Corresponding seven partial differential equations are as follows:

Top Face Sheet

δw_0^t :

$$
\begin{aligned}
&\left[A_3^c+\frac{c_{11}^t bf_t}{R_t}+(A_3^a-A_7^b+A_9^c)\frac{\partial^2}{\partial\theta^2}+(A_9^a+\frac{c_{11}^t bf_t^3}{12R_t^3})\frac{\partial^4}{\partial\theta^4}\right]w_0^t+\left[A_1^c+(A_1^a-A_5^b+A_8^c)\frac{\partial^2}{\partial\phi^2}+A_8^a\frac{\partial^4}{\partial\theta^4}\right]w_0^b \\
&\quad+\left[A_2^c+(A_2^a-A_6^b)\frac{\partial^2}{\partial\theta^2}\right]w_0^c+\left[(A_6^c-A_3^b+\frac{c_{11}^t bd_t}{R_t})\frac{\partial}{\partial\theta}+(A_6^a+\frac{c_{11}^t bd_t^3}{12R_t^3})\frac{\partial^3}{\partial\theta^3}\right]u_0^t \\
&+\left[(A_4^c-A_1^b)\frac{\partial}{\partial\theta}+A_4^a\frac{\partial^3}{\partial\theta^3}\right]u_0^b+\left[(A_5^c-A_2^b)\frac{\partial}{\partial\theta}+A_5^a\frac{\partial^3}{\partial\theta^3}\right]u_0^c+\left[(A_7^c-A_4^b)\frac{\partial}{\partial\theta}+A_7^a\frac{\partial^3}{\partial\theta^3}\right]u_1^c \\
&\quad=\left((A_7^m-\rho_t bf_tR_t)\frac{\partial^2}{\partial t^2}-\left(A_6^m-\frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^4}{\partial\theta^2\partial t^2}\right)w_0^t \\
&\quad+\left(A_8^m\frac{\partial^2}{\partial t^2}-A_5^m\frac{\partial^4}{\partial\theta^2\partial t^2}\right)w_0^b+A_9^m\frac{\partial^2}{\partial t^2}w_0^c-\left(A_1^m+\frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^3}{\partial\theta\partial t^2}u_0^t \\
&\quad-A_2^m\frac{\partial^3}{\partial\theta\partial t^2}u_0^b-A_3^m\frac{\partial^3}{\partial\theta\partial t^2}u_0^c-A_4^m\frac{\partial^3}{\partial\theta\partial t^2}u_1^c\,, \qquad (3.2a)
\end{aligned}
$$

δu_0^t :

$$
\begin{aligned}
&\left[(D_7^c-D_3^b-\frac{c_{11}^t bd_t}{R_t})\frac{\partial}{\partial\theta}+(\frac{c_{11}^t bd_t^3}{12R_t^3}-D_9^b)\frac{\partial^3}{\partial\theta^3}\right]w_0^t+\left[(D_5^c-D_1^b)\frac{\partial}{\partial\theta}-D_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b \\
&\quad+(D_6^c-D_2^b)\frac{\partial}{\partial\theta}w_0^c+\left[D_3^c-(D_6^b+\frac{c_{11}^t bd_t}{R_t}+\frac{c_{11}^t bd_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]u_0^t \\
&\quad+(D_1^c-D_4^b\frac{\partial^2}{\partial\theta^2})u_0^b+(D_2^c-D_5^b\frac{\partial^2}{\partial\theta^2})u_0^c+(D_4^c-D_7^b\frac{\partial^2}{\partial\theta^2})u_1^c \\
&=\left(D_6^m+\frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^3}{\partial\theta\partial t^2}w_0^t+D_5^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^b+\left(D_1^m-\rho_t bf_tR_t-\frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^2}{\partial t^2}u_0^t \\
&\quad+D_2^m\frac{\partial^2}{\partial t^2}u_0^b+D_3^m\frac{\partial^2}{\partial t^2}u_0^c+D_4^m\frac{\partial^2}{\partial t^2}u_1^c\,. \qquad (3.2b)
\end{aligned}
$$

Bottom Face Sheet

δw_0^b :

$$\left[B_3^c+(B_3^a-B_7^b+B_9^c)\frac{\partial^2}{\partial\theta^2}+B_9^a\frac{\partial^4}{\partial\theta^4}\right]w_0^t+\left[B_1^c+\frac{c_{11}^b bd_b}{R_b}+(B_1^a-B_5^b+B_8^c)\frac{\partial^2}{\partial\theta^2}+(B_8^a+\frac{c_{11}^b bd_b^3}{12R_b^3})\frac{\partial^4}{\partial\theta^4}\right]w_0^b$$
$$+\left[B_2^c+(B_2^a-B_6^b)\frac{\partial^2}{\partial\theta^2}\right]w_0^c+\left[(B_6^c-B_3^b)\frac{\partial}{\partial\theta}+B_6^a\frac{\partial^3}{\partial\theta^3}\right]u_0^t$$
$$+\left[(B_4^c-B_1^b+\frac{c_{11}^b bd_b}{R_b})\frac{\partial}{\partial\theta}+(B_4^a-\frac{c_{11}^b bd_b^3}{12R_b^3})\frac{\partial^3}{\partial\theta^3}\right]u_0^b+\left[(-B_2^b+B_5^c)\frac{\partial}{\partial\theta}+B_5^a\frac{\partial^3}{\partial\theta^3}\right]u_0^c$$
$$+\left[(-B_4^b+B_7^c)\frac{\partial}{\partial\theta}+B_7^a\frac{\partial^3}{\partial\theta^3}\right]u_1^c=\left(B_7^m\frac{\partial^2}{\partial t^2}-B_6^m\frac{\partial^4}{\partial\theta^2\partial t^2}\right)w_0^t$$
$$+\left((B_8^m-\rho_b bf_bR_b)\frac{\partial^2}{\partial t^2}-\left(B_5^m-\frac{\rho_b bf_b^3}{12R_b}\right)\frac{\partial^4}{\partial\theta^2\partial t^2}\right)w_0^b+B_9^m\frac{\partial^2}{\partial t^2}w_0^c$$
$$-B_1^m\frac{\partial^3}{\partial\theta\partial t^2}u_0^t-\left(B_2^m+\frac{\rho_b bf_b^3}{12R_b}\right)\frac{\partial^3}{\partial\theta\partial t^2}u_0^b-B_3^m\frac{\partial^3}{\partial\theta\partial t^2}u_0^c-B_4^m\frac{\partial^3}{\partial\theta\partial t^2}u_1^c\ , \quad (3.2c)$$

δu_0^b :

$$\left[(E_7^c-E_3^b)\frac{\partial}{\partial\theta}-E_9^b\frac{\partial^3}{\partial\theta^3}\right]w_0^t+\left[(E_5^c-E_1^b-\frac{c_{11}^b bd_b}{R_b})\frac{\partial}{\partial\theta}+(\frac{c_{11}^b bd_b^3}{12R_b^3}-E_8^b)\frac{\partial^3}{\partial\theta^3}\right]w_0^b$$
$$+(E_6^c-E_2^b)\frac{\partial}{\partial\theta}w_0^c+(E_3^c-E_6^b\frac{\partial^2}{\partial\theta^2})u_0^t+\left[E_1^c-(E_4^b+\frac{c_{11}^b bd_b}{R_b}+\frac{c_{11}^b bd_b^3}{12R_b^3})\frac{\partial^2}{\partial\theta^2}\right]u_0^b$$
$$+(E_2^c-E_5^b\frac{\partial^2}{\partial\theta^2})u_0^c+(E_4^c-E_7^b\frac{\partial^2}{\partial\theta^2})u_1^c$$
$$=E_6^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^t+\left(E_5^m+\frac{\rho_b bf_b^3}{12R_b}\right)\frac{\partial^3}{\partial\theta\partial t^2}w_0^b+E_1^m\frac{\partial^2}{\partial t^2}u_0^t$$
$$+\left(E_2^m-\rho_b bf_bR_b-\frac{\rho_b bf_b^3}{12R_b}\right)\frac{\partial^2}{\partial t^2}u_0^b+E_3^m\frac{\partial^2}{\partial t^2}u_0^c+E_4^m\frac{\partial^2}{\partial t^2}u_1^c\ . \quad (3.2d)$$

Core

δw_0^c :

$$\left[C_3^c+(C_9^c-C_7^b)\frac{\partial^2}{\partial\theta^2}\right]w_0^t+\left[C_1^c+(C_8^c-C_5^b)\frac{\partial^2}{\partial\theta^2}\right]w_0^b+\left(C_2^c-C_6^b\frac{\partial^2}{\partial\theta^2}\right)w_0^c$$
$$+(C_6^c-C_3^b)\frac{\partial}{\partial\theta}u_0^t+(C_4^c-C_1^b)\frac{\partial}{\partial\theta}u_0^b+(C_5^c-C_2^b)\frac{\partial}{\partial\theta}u_0^c+(C_7^c-C_4^b)\frac{\partial}{\partial\theta}u_1^c$$
$$=C_7^m\frac{\partial^2}{\partial t^2}w_0^t+C_8^m\frac{\partial^2}{\partial t^2}w_0^b+C_9^m\frac{\partial^2}{\partial t^2}w_0^c\ , \quad (3.2e)$$

δu_0^c :

$$\left[(F_7^c - F_3^b)\frac{\partial}{\partial\theta} - F_9^b\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(F_5^c - F_1^b)\frac{\partial}{\partial\theta} - F_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (F_6^c - F_2^b)\frac{\partial}{\partial\theta}w_0^c$$
$$+ (F_3^c - F_6^b\frac{\partial^2}{\partial\theta^2})u_0^t + (F_1^c - F_4^b\frac{\partial^2}{\partial\theta^2})u_0^b + (F_2^c - F_5^b\frac{\partial^2}{\partial\theta^2})u_0^c + (F_4^c - F_7^b\frac{\partial^2}{\partial\theta^2})u_1^c$$
$$= F_6^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^t + F_5^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^b + F_1^m\frac{\partial^2}{\partial t^2}u_0^t + F_2^m\frac{\partial^2}{\partial t^2}u_0^b + F_3^m\frac{\partial^2}{\partial t^2}u_0^c + F_4^m\frac{\partial^2}{\partial t^2}u_1^c\ , \tag{3.2f}$$

δu_1^c :

$$\left[(G_7^c - G_3^b)\frac{\partial}{\partial\theta} - G_9^b\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(G_5^c - G_1^b)\frac{\partial}{\partial\theta} - G_8^b\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (G_6^c - G_2^b)\frac{\partial}{\partial\theta}w_0^c$$
$$+ (G_3^c - G_6^b\frac{\partial^2}{\partial\theta^2})u_0^t + (G_1^c - G_4^b\frac{\partial^2}{\partial\theta^2})u_0^b + (G_2^c - G_5^b\frac{\partial^2}{\partial\theta^2})u_0^c + (G_4^c - G_7^b\frac{\partial^2}{\partial\theta^2})u_1^c$$
$$= G_6^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^t + G_5^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^b + G_1^m\frac{\partial^2}{\partial t^2}u_0^t + G_2^m\frac{\partial^2}{\partial t^2}u_0^b + G_3^m\frac{\partial^2}{\partial t^2}u_0^c + G_4^m\frac{\partial^2}{\partial t^2}u_1^c\ . \tag{3.2g}$$

The corresponding boundary conditions are at $\theta = 0$ and $\theta = \alpha$, read as follows (at each end there are nine boundary conditions, three for each of the two face sheets and three for the core):

Top Face Sheet

Either $\delta w_0^t = 0$ *or,*

$$\left[(A_7^b - A_3^a)\frac{\partial}{\partial\theta} - (A_9^a + \frac{c_{11}^t bf_t^3}{12R_t^3})\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(A_5^b - A_1^a)\frac{\partial}{\partial\theta} - A_8^a\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (A_6^b - A_2^a)\frac{\partial}{\partial\theta}w_0^c$$
$$+\left(A_3^b - A_6^a + \frac{c_{11}^t bf_t^3}{12R_t^3}\right)\frac{\partial^2}{\partial\theta^2}u_0^t + \left(A_1^b - A_4^a\frac{\partial^2}{\partial\theta^2}\right)u_0^b + \left(A_2^b - A_5^a\frac{\partial^2}{\partial\theta^2}\right)u_0^c + \left(A_4^b - A_7^a\frac{\partial^2}{\partial\theta^2}\right)u_1^c$$
$$= \left(A_6^m - \frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^3}{\partial\theta\partial t^2}w_0^t + A_5^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^b + \left(A_1^m + \frac{\rho_t bf_t^3}{12R_t}\right)\frac{\partial^2}{\partial t^2}u_0^t$$
$$+ A_2^m\frac{\partial^2}{\partial t^2}u_0^b + A_3^m\frac{\partial^2}{\partial t^2}u_0^c + A_4^m\frac{\partial^2}{\partial t^2}u_1^c\ , \tag{3.3a}$$

Either $\delta w_0^{t\,\prime} = 0$ *or,*

$$\left[A_3^a + (A_9^a + \frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^t + \left(A_1^a + A_8^a\frac{\partial^2}{\partial\theta^2}\right)w_0^b + A_2^a w_0^c$$
$$+ \left[A_6^a\frac{\partial}{\partial\theta} - \frac{c_{11}^t b f_t^3}{12R_t^3}\frac{\partial^3}{\partial\theta^3}\right]u_0^t + A_4^a\frac{\partial}{\partial\theta}u_0^b + A_5^a\frac{\partial}{\partial\theta}u_0^c + A_7^a\frac{\partial}{\partial\theta}u_1^c = 0\ , \quad (3.3b)$$

Either $\delta u_0^t = 0$ *or,*

$$\left[D_3^b + \frac{c_{11}^t b f_t}{R_t} + (D_9^b - \frac{c_{11}^t b f_t^3}{12R_t^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^t + \left(D_1^b + D_8^b\frac{\partial^2}{\partial\theta^2}\right)w_0^b + D_2^b w_0^c$$
$$+ \left[D_6^b + \frac{c_{11}^t b f_t^3}{12R_t^3} + \frac{c_{11}^t b f_t}{R_t}\right]\frac{\partial}{\partial\theta}u_0^t + D_4^b\frac{\partial}{\partial\theta}u_0^b + D_5^b\frac{\partial}{\partial\theta}u_0^c + D_7^b\frac{\partial}{\partial\theta}u_1^c = 0\ , \quad (3.3c)$$

<u>*Bottom Face Sheet*</u>

Either $\delta w_0^b = 0$ *or,*

$$\left[(B_7^b - B_3^a)\frac{\partial}{\partial\theta} - B_9^a\frac{\partial^3}{\partial\theta^3}\right]w_0^t + \left[(B_5^b - B_1^a)\frac{\partial}{\partial\theta} - (\frac{c_{11}^b b f_b^3}{12R_b^3} + B_8^a)\frac{\partial^3}{\partial\theta^3}\right]w_0^b + (B_6^b - B_2^a)\frac{\partial}{\partial\theta}w_0^c$$
$$+ \left(B_3^b - B_6^a\frac{\partial^2}{\partial\theta^2}\right)u_0^t + \left[B_1^b + (\frac{c_{11}^b b f_b^3}{12R_b^3} - B_4^a)\frac{\partial^2}{\partial\theta^2}\right]u_0^b + \left(B_2^b - B_5^a\frac{\partial^2}{\partial\theta^2}\right)u_0^c + \left(B_4^b - B_7^a\frac{\partial^2}{\partial\theta^2}\right)u_1^c$$
$$= B_6^m\frac{\partial^3}{\partial\theta\partial t^2}w_0^t + \left(B_5^m + \frac{\rho_t b f_t^3}{12R_t}\right)\frac{\partial^3}{\partial\theta\partial t^2}w_0^b + B_1^m\frac{\partial^2}{\partial t^2}u_0^t$$
$$+ \left(B_2^m + \frac{\rho_t b f_t^3}{12R_t}\right)\frac{\partial^2}{\partial t^2}u_0^b + B_3^m\frac{\partial^2}{\partial t^2}u_0^c + B_4^m\frac{\partial^2}{\partial t^2}u_1^c\ , \quad (3.3d)$$

Either $\delta w_0^{b\,\prime} = 0$ *or,*

$$\left(B_3^a + B_9^a\frac{\partial^2}{\partial\theta^2}\right)w_0^t + \left[B_1^a + (B_8^a + \frac{c_{11}^b b f_b^3}{12R_b^3})\frac{\partial^2}{\partial\theta^2}\right]w_0^b + B_2^a w_0^c + B_4^a\frac{\partial}{\partial\theta}u_0^t$$
$$+ \left(B_4^a\frac{\partial}{\partial\theta} - \frac{c_{11}^b b f_b^3}{12R_b^3}\frac{\partial^3}{\partial\theta^3}\right)u_0^b + B_5^a\frac{\partial}{\partial\theta}u_0^c + B_7^a\frac{\partial}{\partial\theta}u_1^c$$
$$= 0\ , \quad (3.3e)$$

Either $\delta u_0^b = 0$ *or,*

$$\left(E_3^b + E_9^b \frac{\partial^2}{\partial\theta^2}\right) w_0^t + \left(E_1^b + \frac{c_{11}^b b f_b}{R_b} + (E_8^b - \frac{c_{11}^b b f_b^3}{12 R_b^3}) \frac{\partial^2}{\partial\theta^2}\right) w_0^b + E_2^b w_0^c + E_5^b \frac{\partial}{\partial\theta} u_0^t$$
$$\left(E_4^b + \frac{c_{11}^b b f_b^3}{12 R_b^3} + \frac{c_{11}^b b f_b}{R_b}\right) \frac{\partial}{\partial\theta} u_0^b + E_5^b \frac{\partial}{\partial\theta} u_0^c + E_7^b \frac{\partial}{\partial\theta} u_1^c = 0\,, \quad (3.3f)$$

<u>*Core*</u>

Either $\delta w_0^c = 0$ *or,*

$$C_7^b \frac{\partial}{\partial\theta} w_0^t \ + \ C_5^b \frac{\partial}{\partial\theta} w_0^b \ + \ C_6^b \frac{\partial}{\partial\theta} w_0^c \ + \ C_3^b u_t^c \ + \ C_1^b u_0^b \ + \ C_2^b u_0^c \ + \ C_4^b u_1^c \ = \ 0\,, \quad (3.3g)$$

Either $\delta u_0^c = 0$ *or,*

$$\left(F_3^b + F_9^b \frac{\partial^2}{\partial\theta^2}\right) w_0^t + \left(F_1^b + F_8^b \frac{\partial^2}{\partial\theta^2}\right) w_0^b + F_2^b w_0^c + F_6^b \frac{\partial}{\partial\theta} u_0^t + F_4^b \frac{\partial}{\partial\theta} u_0^b + F_5^b \frac{\partial}{\partial\theta} u_0^c + F_7^b \frac{\partial}{\partial\theta} u_1^c = 0\,, \quad (3.3h)$$

Either $\delta u_1^c = 0$ *or,*

$$\left(G_3^b + G_9^b \frac{\partial^2}{\partial\theta^2}\right) w_0^t + \left(G_1^b + G_8^b \frac{\partial^2}{\partial\theta^2}\right) w_0^b + G_2^b w_0^c + G_6^b \frac{\partial}{\partial\theta} u_0^t + G_4^b \frac{\partial}{\partial\theta} u_0^b + G_5^b \frac{\partial}{\partial\theta} u_0^c + G_7^b \frac{\partial}{\partial\theta} u_1^c = 0\,, \quad (3.3i)$$

where $A_i^{a,b,c,m}$, $B_i^{a,b,c,m}$, $C_i^{b,c,m}$, $D_i^{b,c,m}$, $E_i^{b,c,m}$, $F_i^{b,c,m}$, $G_i^{b,c,m}$ are constants which include both geometric and material properties and are defined in Appendix A and Appendix B for logarithmic and polynomial variant, respectively.

3.1.2 Solution Procedure

In the following, free vibration solution procedure for a simply supported curved panel is outlined.

$$w_0^{t,b,c}(\theta) = W_0^{t,b,c} e^{\mathrm{i}\omega t} \sin(\frac{n\pi\theta}{\alpha}) \; ; \quad u_0^{t,b,c}(\theta) = U_0^{t,b,c} e^{\mathrm{i}\omega t} \cos(\frac{n\pi\theta}{\alpha}) \; , \tag{3.4a}$$

$$u_1^c(\theta) = U_1^c e^{\mathrm{i}\omega t} \cos(\frac{n\pi\theta}{\alpha}) \; . \tag{3.4b}$$

where n is wave number and ω is frequency.

Substituting the solution form (3.4) into the dynamic governing differential equations (3.2) results in a system of eigenvalue problems can be written in following form:

$$(-\omega^2[M_n] + [K_n])\{X_n\} = 0; \tag{3.5a}$$

where elements of 7×7 $[K_n]$ represents Left-Hand-Side terms from (3.2), 7×7 $[M_n]$ represents Right-Hand-Side terms and $\{X_n\}$ is a 1×7 unknown displacement matrix, namely: $\{W_{0,n}^t, W_{0,n}^b, W_{0,n}^c, U_{0,n}^t, U_{0,n}^b, U_{0,n}^c, U_{1,n}^c\}$.

For nontrivial solutions, the determinant of (3.5a) is equate to zero:

$$\left| - \omega^2[M_n] + [K_n] \right| = 0; \tag{3.5b}$$

Given wave number i.e. n is specified by any positive integer, results are seven natural frequencies, ω_i, corresponding to seven modes of radial vibrations, $\{X_n\}_i$.

3.2 Free vibration using Elasticity for Curved Sandwich composite Panel

The linear dynamic elasticity problem formulation and solution for a generally asymmetric sandwich curved beam/panel consisting of orthotropic core and face sheets. The displacement approach is used and the panel is assumed to be simply supported at the ends. Closed form solutions for the displacements and stresses are derived using method of Frobenius series.

In each individual sandwich's layer, dynamic equilibrium equations in polar coordinates are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 w}{\partial t^2} \ , \tag{3.6a}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2} \ . \tag{3.6b}$$

Cylindrical orthotropic constitutive law and strain-displacement relations are as previously stated in (2.15), and (2.16)

Considering a curved sandwich panel with simply supported on both ends, solutions are:

$$w(r, \theta, t) = W(r)e^{\mathrm{i}\omega t}\sin(k\theta) \ , \tag{3.7a}$$

$$u(r, \theta, t) = U(r)e^{\mathrm{i}\omega t}\cos(k\theta) \ , \tag{3.7b}$$

where $k = n\pi/\alpha$, n is wave number ,($n = 1, 2, 3, \dots$); and ω is natural frequency which later will be determined.

Using (2.15),(2.16),and (3.7) substituting into (3.6), following two linear second-order ordinary differential equations in r are obtained:

$$r^2 c_{33} W'' + r c_{33} W' - (c_{11} + c_{55} k^2) W + r^2 \rho \omega^2 W - r(c_{13} + c_{55}) k U' + (c_{11} + c_{55}) k U = 0 \ , \tag{3.8a}$$

$$r^2 c_{55} U'' + r c_{55} U' - (c_{55} + c_{11} k^2) + r^2 \rho \omega^2 U + r(c_{13} + c_{55}) k W' + (c_{11} + c_{55}) k W = 0 \ . \tag{3.8b}$$

Multiplying r^2 to (3.8a) and (3.8b), consequently, the equations can be written in matrix form as:

$$r^2[M]\{X''\} + r[C]\{X'\} + ([K] + r^2\rho\omega^2[I])\{X\} = 0\ , \tag{3.8c}$$

where ,

$$\{X\} = \begin{Bmatrix} W(r) \\ U(r) \end{Bmatrix}\ , \quad [M] = \begin{bmatrix} c_{33} & 0 \\ 0 & c_{55} \end{bmatrix}\ , \quad [C] = \begin{bmatrix} c_{33} & -(c_{12}+c_{55})k \\ (c_{13}+c_{55})k & c_{55} \end{bmatrix}\ , \tag{3.8d}$$

$$[K] = \begin{bmatrix} -(c_{11}+c_{55}k^2) & (c_{11}+c_{55})k \\ (c_{11}+c_{55})k & -(c_{55}+c_{11}k^2) \end{bmatrix}\ , \quad [I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ . \tag{3.8e}$$

Unlike having a system of Cauchy Euler equations (2.19) as in static elasticity where solution form is simple single term (2.20), the presence of dynamic quantities, time-derivative terms, in (3.6) induces complexities to the solution procedure. The dynamic elasticity displacement formulations (3.8a) and (3.8b) are presented in matrix form (3.8c). The method of Frobenius series are then applied to solve the equations. The solution has the following form:

$$\{X\} = \sum_{n=0}^{\infty}\{X_n\}r^{s+n}\ , \tag{3.9a}$$

The first and second derivative respect to r of (3.9) are:

$$\{X\}' = \sum_{n=0}^{\infty}(s+n)\{X_n\}r^{s+n-1}\ , \tag{3.9b}$$

$$\{X\}'' = \sum_{n=0}^{\infty}(s+n)(s+n-1)\{X_n\}r^{s+n-2}\ , \tag{3.9c}$$

Substituting solution forms (3.9a)- (3.9c) into the displacement formulation (3.8c), results are:

$$
\begin{aligned}
&(s(s-1)[M]+s[C]+[K])r^s\{X_0\} \\
&\qquad\qquad + \big((s+1)s[M]+(s+1)[C]+[K]\big)r^{s+1}\{X_1\} \\
&+\sum_{n=2}^{\infty}\Bigg(\big((s+n)(s+n-1)[M]+(s+n)[C]+[K]\big)\{X_n\}+\rho\omega^2[I]\{X_{n-2}\}\Bigg)r^{s+n}=0\ ,
\end{aligned}
\tag{3.10}
$$

Then using method of comparing coefficients:

r^s $(n=0)$;

$$
(s(s-1)[M]+s[C]+[K])\{X_0\}=0\ , \tag{3.11a}
$$

r^{s+1} $(n=1)$,

$$
((s+1)(s)[M]+(s+1)[C]+[K])\{X_1\}=0\ , \tag{3.11b}
$$

$r^{s+2}(n=2)$,

$$
((s+2)(s+1)[M]+(s+2)[C]+[K])\{X_2\}+\rho\omega^2[I]\{X_0\}=0\ , \tag{3.11c}
$$

$$
\vdots
$$

r^{s+n},

$$
((s+n)(s+n-1)[M]+(s+n)[C]+[K])\{X_n\}+\rho\omega^2[I]\{X_{n-2}\}=0\ . \tag{3.11d}
$$

3.2.1 Indicial equation

Considering the very first coefficient term r^s (3.11a), for nontrivial solution i.e. $\{X_0\} \neq 0$, the determinant is equate to zero:

$$
\Big| s(s-1)[M]+s[C]+[K] \Big| = 0\ , \tag{3.12a}
$$

Then, the result is the "indicial equation":

$$c_{33}c_{55}s^4 + (k^2(c_{13}^2 + 2c_{13}c_{55} - c_{11}c_{33}) - c_{55}(c_{11} + c_{33}))s^2 + c_{11}c_{55}(k^2 - 1)^2 = 0 \ , \quad (3.12b)$$

It is also note that the indicial equation, indeed, is the characteristic equation in static elasticity formulation (2.22).

Applying similar approach, the 4th order indicial equation (3.12b) can be reduce to quadratic equation by let $s^2 = \mu$; and solving (3.12b) 4 indicial roots. As a result, $s_{1st,2nd,3rd,4th}$ are obtained. Substituting 4 roots into (3.11a) and solving for $\{X_0\}$:

$$\{X_0\}^i = \left\{ \frac{-\left(c_{11} + c_{55}\right)k + \left(c_{13} + c_{55}\right)ks^i}{-c_{11} - c_{55}k^2 + c_{33}s^i + c_{33}(-1 + s^i)s^i}, 1 \right\}^T , \quad (3.12c)$$

where $i = 1st, 2nd, 3rd, 4th$

Next, considering the second term $r^s + 1$ (3.11b) with $s_{1st,2nd,3rd,4th}$ substitution, the determinant is non-zero; hence, (3.11b) is only true when $\{X_1\} = 0$

3.2.2 Recurrence relations

Let us consider even number $n = 2, 4, \ldots$ coefficient matrices of r^{s+n} ; (3.11c) is coefficient matrices corresponding to $n = 2$, r^{s+2}. Solving the equation gives:

$$\{X_2\} = -\rho\omega^2 \left[(s+2)(s+1)[M] + (s+2)[C] + [K]\right]^{-1} \{X_0\} \ . \quad (3.13a)$$

Similarly, the next even n terms (3.11d) gives $\{X_{n=2,4,\ldots}\}$. This repetitive process is recurrence relations:

$$\{X_n\} = -\rho\omega^2 [R_n]^{-1} \{X_{n-2}\} \ ; \quad \text{for } n = 2, 4, 6, \ldots \quad (3.13b)$$

where,

$$[R_n] = \big((s+n)(s+n-1)[M] + (s+n)[C] + [K]\big) \ . \quad (3.13c)$$

And the inverse of $[R_n]$ is:

$$[R_n]^{-1} = \frac{1}{\left|[R_n]\right|}\begin{bmatrix} -c_{11}k^2 + c_{55}(n+s-1)(n+s+1) & -(c_{11}+c_{55})k + (c_{13}+c_{55})(n+s)k \\ -k\left(c_{11} + c_{13}\left(n+s\right) + c_{55}\left(1+n+s\right)\right) & -\left(c_{11} + c_{55}k^2 - c_{33}\left(n+s\right)^2\right) \end{bmatrix}, \tag{3.13d}$$

where,

$$\begin{aligned}\left|[R_n]\right| = {} & (n+s)^2\left(\left(c_{13}\left(c_{13}+2c_{55}\right)k^2 + c_{33}c_{55}\left(n+s-1\right)\left(n+s+1\right)\right)\right. \\ & \left. + c_{11}\left(-c_{33}k^2(n+s)^2 + c_{55}\left(k^2-1-n-s\right)\left(k^2-1+n+s\right)\right)\right) \ .\end{aligned} \tag{3.13e}$$

Alternatively, (3.13b) implies that any $\{X_{n=even}\}$ can be determined from,

$$\{X_n\} = [Q_n]\{X_0\} \ ; \quad \text{for } n = 2, 4, 6, \ldots \tag{3.14a}$$

where,

$$[Q_n] = (-1)^{n/2}\rho^{n/2}\omega^n[R_n]^{-1}[R_{n-2}]^{-1}[R_{n-4}]^{-1}\ldots[R_4]^{-1}[R_2]^{-1} \ . \tag{3.14b}$$

In similar manner, $\{X_{3,5,7,\ldots}\}$, these odd number terms are solved by coefficient matrices of $r^{s+3,5,7,\ldots}$:

$$\{X_n\} = [Q_n]\{X_1\} \ ; \quad \text{for } n = 3, 5, 7, \ldots \tag{3.14c}$$

As previously determine, however, $\{X_1\} = 0$; therefore,

$$\{X_n\} = 0 \ ; \quad \text{for } n = 1, 3, 5, 7, \ldots \tag{3.14d}$$

3.2.3 General series solution

At this point, s and $\{X_n\}$ are determined. For orthotropic material in general, $s_{1st,2nd,3rd,4th}$ are four distinct roots or 2 pairs of complex conjugates. Considering following cases:

First case, 4 distinct roots and a following condition is satisfied,

$$s_i - s_j \neq 2m \; ; \quad i \neq j \; , \tag{3.15a}$$

where $i, j = 1st, 2nd, 3rd, 4th$; and m is any positive integer $m = 1, 2, 3, \ldots$. A general series solution then has the form,

$$\{W(r), U(r)\}^T = \sum_{j=1}^{4} \sum_{n=0,2,4,\ldots}^{\infty} A_j \{X_n\}^j r^{s_j+n} \; , \tag{3.15b}$$

where A_j are constants need to be determined.

Second case, the indicial roots are two pairs of complex conjugate,

$$s_{1st,3rd} = a_{1st} \pm \mathrm{i} b_{1st} \; ; \quad s_{2nd,4th} = a_{2nd} \pm \mathrm{i} b_{2nd} \; . \tag{3.16a}$$

Considering the first pair of complex conjugates in (3.16a), a partial general solution is:

$$\{W(r), U(r)\}^T = A_{1st} \sum_{n=0,2,4,\ldots}^{\infty} \{X_n\}^{1st} r^{(a_{1st}+\mathrm{i}b_{1st}+n)} + A_{3rd} \sum_{n=0,2,4,\ldots}^{\infty} \{X_n\}^{2nd} r^{a_{1st}-\mathrm{i}b_{1st}+n}, \tag{3.16b}$$

corresponding X_n from (3.13b) are also complex conjugate,

$$\{X\}_n^{1st,3rd} = \{Y\}_n^{1st} \pm \mathrm{i}\{Z\}_n^{1st} \; , \quad \text{for } n = 0, 2, 4, 6, \ldots . \tag{3.16c}$$

And note that Euler's formula is:

$$r^{\mathrm{i}(\pm b_j)} = e^{\mathrm{i}(\pm b_j \ln r)} = \cos(\pm b_j \ln r) + \mathrm{i} \sin(\pm b_j \ln r). \tag{3.16d}$$

Applying Euler's formula (3.16d) to the complex conjugate pair of (3.16b) and (3.16c); and applying the same approach for the second pair in (3.16a), the general solution (3.16b) is then rewritten as,

$$\{X\} = \sum_{j=1}^{2} \sum_{n=0,2,4,\ldots}^{\infty} \Big[B_j(\{Y_n\}^j \cos(b_j \ln r) - \{Z_n\}^j \sin(b_j \ln r)) \\ + C_j(\{Z_n\}^j \cos(b_j \ln r) + \{Y_n\}^j \sin(b_j \ln r)) \Big]. \quad (3.16e)$$

where B_j and C_j are four constants need to be determined.

Third case: in addition, an isotropic sandwich layer result in a special case. 4 distinct roots are obtained but the pair is differed by 2 (m=1) i.e. (3.15a) is not satisfied. Here, we derive solution for any even,

$$\begin{aligned} s_{1st} - s_{2nd} &= 2m\ , \quad s_{1st} > s_{2nd} \\ s_{3rd} - s_{4th} &= 2m\ , \quad s_{3rd} > s_{4th}\ , \quad m = 0, 1, 2, 3, \ldots \end{aligned} \quad (3.17)$$

The condition in (3.17) implies that terms $n^{th} + 2m$ of the series (3.9a) correspond to s_{2nd} are linearly dependent with n^{th} term of the series correspond to s_{1st}. While $\{X\}^{1st}$ and $\{X\}^{3rd}$ are identical to those presented in (3.15b), $\{X\}^{2nd}$ and $\{X\}^{4th}$ are modified. Consequently, the series solution correspond to s_{2nd} is:

$$\{Y\}^{2nd} = c_k \left(\sum_{n=0}^{\infty} \{X_n\}^{1st} r^{s_{1st}+n} \right) \ln r + \sum_{n=0}^{\infty} \{Y_n\}^{2nd} r^{s_{2nd}+n}\ , \quad (3.18)$$

where c_k is an unknown constant which might be 0 unless $m = 0$. Substituting (3.18) into (3.8c) and realizing that $s_{1st} = s_{2nd} + 2m$, results are:

$$\sum_{j=0}^{\infty} \Big(c_k(s_{1st} + j)[M] + [C] \Big) \{X_j\}^{1st} r^{s_{1st}+j}$$

$$+\sum_{n=0}^{\infty}\Big((s_{2nd}+n)(s_{2nd}+n-1)[M]+(s_{2nd}+n)[C]+[K]\Big)\{Y_n\}^{2nd}r^{s_{2nd}+n}$$

$$+\sum_{n=0}^{\infty}\rho\omega^2[I]\{Y_n\}^{2nd}r^{s_{2nd}+n+2}=0 \quad (3.19)$$

Then using method of comparing coefficient to (3.19). For $0 \geq n > 2m$, there is no contribution from $\{X\}^{1st}$ solution; hence, similar solution process from (3.11)-(3.13) is applied and results are as follow:

For $0 \geq n > 2m$,

$$\{Y_0\}^{2nd}=\left\{\frac{-\left(c_{11}+c_{55}\right)k+\left(c_{13}+c_{55}\right)ks_{2nd}}{-c_{11}-c_{55}k^2+c_{33}s_{2nd}+c_{33}(-1+s_{2nd})s_{2nd}},1\right\}^T, \quad (3.20a)$$

$$\{Y_n\}^{2nd}=[R_n]^{2nd}\{Y_{n-2}\}^{2nd}\,; \quad \text{for } n=2,4,\ldots,n-2m \quad (3.20b)$$

where, $[R_n]^{2nd}$ is as defined in (3.13c)

For $n = 2m$ and $j = 0$,

$$c_k\Big(s_{1st}[M]+[C]\Big)\{X_0\}^{1st}+\Big((s_{2nd}+2m)(s_{2nd}+2m-1)[M]+(s_{2nd}+2m)[C]+[K]\Big)\{Y_n\}^{2nd}$$

$$+[\Omega]\{Y_{2m-2}\}^{2nd}=0\,. \quad (3.21a)$$

Indeed, the determinant matrices coefficient of $\{Y_{2m}\}^{2nd}$ is identical to equation (3.11a) where $s = s_{1st}$ which has zero determinant. If we choose,

$$\{Y_{2m}\}^{2nd}=\left\{X_0^{1st}(1),d_k\right\}^T \;, \quad (3.21b)$$

where $X_0^{1st}(1)$ is the first element of (3.12c) and d_k is another unknown constant. Then, both c_k and d_k are determined by solving (3.21a).

For $n > 2m$ and $j > 0$,

The recurrence relations are revised as follow:

$$\{Y_n\}^{2nd} = -([R_n]^{2nd})^{-1}(c_k[P_{n-2}]^{1st} + \rho\omega^2[I])\{Y_{n-2}\}^{2nd} , \tag{3.21c}$$

where,

$$[P_n]^{1st} = (s_{1st} + n - 2m)[M] + [C] . \tag{3.21d}$$

And the series solution correspond to s_{4th} is:

$$\{Y\}^{4th} = c_k \left(\sum_{n=0}^{\infty} \{X_n\}^{3rd} r^{s_{3rd}+n} \right) \ln r + \sum_{n=0}^{\infty} \{Y_n\}^{4th} r^{s_{4th}+n} , \tag{3.22}$$

where c_k and $\{Y_n\}^{4th}$ are determined by similar procedure as in (3.20)-(3.21).

Therefore, a general series solution has the form,

$$\{W(r), U(r)\}^T = A_{1st}\{X\}^{1st} + A_{2nd}\{Y\}^{2nd} + A_{3rd}\{X\}^{3rd} + A_{4th}\{Y\}^{4th} . \tag{3.23}$$

3.2.4 Boundary Conditions

With simply supported displacement solution form (3.7), stresses are explicitly written as:

$$\sigma_{rr} = \left[c_{13} \left(-k\frac{U(r)}{r} + \frac{W(r)}{r} \right) + c_{33} W'(r) \right] \sin(k\theta) e^{\mathrm{i}\omega t} , \tag{3.24a}$$

$$\tau_{r\theta} = \left[c_{55} \left(-\frac{U(r)}{r} + k\frac{W(r)}{r} + U'(r) \right) \right] \cos(k\theta) e^{\mathrm{i}\omega t} . \tag{3.24b}$$

We shall introduce superscript notations t, c, b corresponding to each sandwich layer, namely: top face, core, and bottom face respectively. Hence, we have: $c_{ij}^{t,b,c}$, $s_{1st,2nd,3rd,4th}^{t,b,c}$ and $\{W(r), U(r)\}^{t,b,c}$.

In this curved sandwich panel configuration, there are 12 unknowns constants: $A_{1st}^{t,b,c}$, $A_{2nd}^{t,b,c}$, $A_{3rd}^{t,b,c}$, $A_{4th}^{t,b,c}$, and additional 1 unknown frequency ω; hence, there are total 13 unknowns. There are 12 homogeneous facial boundary conditions; four are traction free surface of top and bottom face sheet, another four are upper/lower inter-facial continuity of displacements, and remaining four are inter-facial traction conditions.

Two traction free at top face, $r = R_2$, are:

$$\sigma^t_{rr} = 0 \; ; \quad \tau^t_{rs} = 0 \; . \tag{3.25a}$$

Two traction free at bottom face, $r = R_1$, are:

$$\sigma^b_{rr} = 0 \; ; \quad \tau^b_{rs} = 0 \; . \tag{3.25b}$$

Four upper face/core displacement and stress continuities, $r = r_{tc}$, are:

$$w^t = w^c \; ; \quad u^t = u^c \; , \tag{3.25c}$$

$$\sigma^t_{rr} = \sigma^c_{rr} \; ; \quad \tau^t_{rs} = \tau^c_{rs} \; . \tag{3.25d}$$

Four lower face/core displacement and stress continuities, $r = r_{bc}$, are:

$$w^b = w^c \; ; \quad u^b = u^c \; , \tag{3.25e}$$

$$\sigma^b_{rr} = \sigma^c_{rr} \; ; \quad \tau^b_{rs} = \tau^c_{rs} \; . \tag{3.25f}$$

In (3.25a)-(3.25f), $e^{\mathrm{i}\omega t}$, $\sin(k\theta)$, and $\cos(k\theta)$ are canceled out resulting in a system of 12 equations and is then written in matrix form,

$$\Big[\Xi(\omega)\Big]\Big\{\underline{A}\Big\} = 0 \; , \tag{3.26}$$

where, $\{\underline{A}\} = \{A^t_{1st}, A^t_{2nd}, A^t_{3rd}, A^t_{4th}, A^c_{1st}, A^c_{2nd}, A^c_{3rd}, A^c_{4th}, A^b_{1st}, A^b_{2nd}, A^b_{3rd}, A^b_{4th}\}^T$.

For nontrivial solution,

$$\Big|[\Xi(\omega)]\Big| = 0 \,, \tag{3.27}$$

The determinant of (3.27) gives a polynomial equation of ω which is solved by numerical method i.e. Root-finding. The order of the polynomial equations depend on number of terms included in a partial summation of (3.9a). To determine higher frequencies, higher number of terms is needed for solution convergence and computational resource could become an issue.

The behavior of series convergence is studied by comparing n^{th} and $n^{th}-2$ terms in (3.9a),

$$\{X_n\}r^n = -\rho\omega^2[R_n]^{-1}\{X_{n-2}\}r^{n-2}r^2 \,. \tag{3.28a}$$

Right hand side terms $\{X_{n-2}\}r^{n-2}$ is indeed the $n^{th}-2$, and the coefficient matrix is $-\rho\omega^2[R_n]^{-1}r^2$. Applying Taylor series's expansion,

$$-\rho\omega^2[R_n]^{-1}r^2 = -\frac{\rho\omega^2 r^2}{n^2}\begin{bmatrix} \frac{1}{c_{33}} - \frac{2s}{c_{33}n} + O(1/n^2) & \frac{k(c_{13}+c_{55})}{c_{33}c_{55}n} + O(1/n^2) \\ \frac{-k(c_{13}+c_{55})}{c_{33}c_{55}n} + O(1/n^2) & \frac{1}{c_{55}} - \frac{2s}{c_{55}n} + O(1/n^2) \end{bmatrix} \tag{3.28b}$$

Taking limit $\lim_{n\to\infty}$ (3.28b) ,the result is a zero; hence, the series is converged. The quantity ω^2/n^2 indicates that as ω increase, n has to proportionally increase. Numerical convergence analysis is discussed later in results section.

3.3 First Order Shear Deformation Theory

In Chapter 2, three variants of static FOSD theory were presented: basic, equivalent shear modulus, shear correction factor. The latter two are modification of shear modulus and corresponding correction factor which improve accuracy. In the following, free vibration dynamic formulations of First Order Shear Deformation Theory will be presented. Kinematic description, strain displacement relationships and constitutive laws follow (2.32)-(2.33) with addition independent variable t i.e. the dynamic three dependent variables are a function of $(\theta, t) : w_0(\theta, t)$, $u_0(\theta, t)$, and $\Psi_0(\theta, t)$

Governing equations and associated boundary conditions are derived from Hamilton's Principle, the principle states a dynamic equilibrium:

$$\delta \int_t (K - U + V) = 0 \, , \tag{3.29a}$$

where K is kinetic energy, U is strain energy of the sandwich panel, and V is the external potential due to applied loads which is none, $V = 0$, because the problem is free vibration analysis.

The first variation of the kinetic energy is:

$$\delta K = \int_0^\alpha \int_A \rho \left(\dot{w}\delta\dot{w} + \dot{u}\delta\dot{u}\right) dA d\theta, \tag{3.29b}$$

where $\dot{}$ and $\ddot{}$, denote d/dt and d^2/dt^2 respectively. Substituting (3.29b) into (3.29a) and performing integration by part respect to time dt, the first variation of kinetic energy becomes:

$$\delta K = -\int_0^\alpha \int_A \rho \left(\ddot{w}\delta w + \ddot{u}\delta u\right) dA d\theta, \tag{3.29c}$$

The first variation of strain energy, δU, is as stated in (2.34b). Let us define following quantities:

$$M_0 = \int_A \rho(R{+}z)dz = \frac{1}{2}\rho_t f_t(2c{-}2e{+}f_t{+}2R){+}2c(R{-}e)\rho_c{-}\frac{1}{2}\rho_b f_b(2c{+}2e{+}f_b{-}2R) \, , \tag{3.30a}$$

$$M_1 = \int_A \rho(R{+}z)zdz = \frac{1}{6}\rho_t f_t \left(6c^2 + 6c(-2e + f_t + R) + 6e^2 - 6e(f_t + R) + f_t(2f_t + 3R)\right) + \frac{2}{3}c\rho_c \left(c^2 + 3e(e - R)\right) + \frac{1}{6}f_b \left(-3R(2(c + e) + f_b) + 6f_b(c + e) + 6(c + e)^2 + 2f_b^2\right) \, , \tag{3.30b}$$

$$M_2 = \int_A \rho(R{+}z)z^2 dz = \frac{1}{12}\rho_t f_t \left(4R\left(3f_t(c-e)+3(c-e)^2+f_t^2\right)+3(2c-2e+f_t)\left(2f_t(c-e)+2\right.\right.$$

$$+\frac{2}{3}\rho_c c\left(c^2(R-3e)+3e^2(R-e)\right)$$

$$+\frac{1}{12}\rho_b f_b\left(4R\left(3f_b(c+e)+3(c+e)^2+f_b^2\right)-3(2(c+e)+f_b)\left(2f_b(c+e)+2(c+e)^2+f_b^2\right)\right) \ . \tag{3.30c}$$

Performing integration by parts in (3.29a) respect to θ; results are dynamic governing equations and associated boundary condition (for $0 \leq \theta \leq \alpha$):

δw_0 :

$$-[A_{G1}(w_{0,\theta}-u_0)+(A_{G0}-S_{G1})\psi]_{,\theta}+A_{E1}(w_0+u_{0,\theta})+M_0 w_{0,tt}=0 \ , \tag{3.31a}$$

δu_0 :

$$-[A_{E1}(w_0+u_{0,\theta})]_{,\theta}-A_{G1}(w_{0,\theta}-u_0)+(S_{G1}-A_{G0})\psi+M_0 u_{0,tt}+M_1\psi_{0,tt}=0 \ , \tag{3.31b}$$

and

$\delta\psi$:

$$-(I_{E1}\psi_{,\theta})_{,\theta}+(A_{G0}R+I_{G1}-S_{G0})\psi+(A_{G0}-S_{G1})(w_{0,\theta}-u_0)+M_1 u_{0,tt}+M_2\psi_{0,tt}=0 \ , \tag{3.31c}$$

Associated boundary conditions (three at each end, θ_e = 0, α) for the dynamic FOSD are:

Either $\delta w_0 = 0$ *or*,

$$A_{G1}\left(w_{0,\theta}-u_0\right)+(A_{G0}-S_{G1})\psi=0 \ , \tag{3.32a}$$

Either $\delta u_0 = 0$ *or*,

$$A_{E1}\left(u_{0,\theta} + w_0\right) = 0\ , \tag{3.32b}$$

and *Either* $\delta\psi = 0$ *or*,

$$I_{E1}\psi_{,\theta} = 0\ , \tag{3.32c}$$

Other sectional properties, A_{G1}, A_{E1}, S_{G1} etc. are previously defined in section 2.3. Note that the two variants, ”equivalent shear modulus” and shear correction factor can be applied via modification of shear modulus $G \to G_{eq}$ (2.48) and $G \to \beta G_{eq}$ (2.60).

For a curved beam/panel is simply-supported, as shown in figure 2.1, analytical solutions are sought in the form:

$$w_0(\theta, t) = W_0 e^{\mathrm{i}\omega t} \sin k\theta\ ; \quad u_0(\theta, t) = U_0 e^{\mathrm{i}\omega t} \cos k\theta\ ,$$

$$\psi(\theta, t) = \Psi e^{\mathrm{i}\omega t} \cos k\theta\ ; \quad k = \frac{n\pi}{\alpha}\ , \tag{3.33}$$

where n is a wave number which is any positive integers. The displacement field (3.33) satisfies the simply-supported boundary conditions, (w_0 being zero at the ends and no moment, i.e., $\psi_{,\theta}$=0 at the ends) and substituting into the governing equations (3.31) leads to:

$$\begin{bmatrix} A_{G1}k^2 + A_{E1} - M_0\omega^2 & -(A_{G1} + A_{E1})k & \dots \\ -(A_{E1} + A_{G1})k & A_{E1}k^2 + A_{G1} - M_0\omega^2 & \dots \\ (A_{G0} - S_{G1})k & -A_{G0} + S_{G1} - M_1\omega^2 & \dots \end{bmatrix}$$

$$\begin{matrix} (A_{G0} - S_{G1})k \\ S_{G1} - A_{G0} - M_1\omega^2 \\ (I_{E1}k^2 + A_{G0}R + I_{G1} - S_{G0}) - M_2\omega^2 \end{matrix}\Bigg] \begin{Bmatrix} W_0 \\ U_0 \\ \Psi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.34}$$

For nontrivial solution, the determinant of (3.34) is a zero; result is 6^{th} order polynomial equation which reduces to cubic equation by setting $\mu = \omega^2$,

$$a\mu^3 + b\mu^2 + c\mu + d = 0 \; , \tag{3.35a}$$

where,

$$\begin{aligned}
a =& M_0 \left(M_1^2 - M_0 M_2 \right) \; , && (3.35\text{b}) \\
b =& A_{E1} k^2 M_0 M_2 + A_{E1} M_0 M_2 - A_{E1} M_1^2 + A_{G0} M_0^2 R + 2 A_{G0} M_0 M_1 \\
& + A_{G1} k^2 M_0 M_2 - A_{G1} k^2 M_1^2 + A_{G1} M_0 M_2 + I_{E1} k^2 M_0^2 + I_{G1} M_0^2 \\
& - M_0^2 S_{G0} - 2 M_0 M_1 S_{G1} \; , && (3.35\text{c}) \\
c =& A_{G0} \left(-A_{E1} \left(k^2 + 1 \right) M_0 R + 2 A_{E1} \left(k^2 - 1 \right) M_1 - \left(k^2 + 1 \right) M_0 (A_{G1} R + 2 S_{G1}) \right) \\
& - A_{E1} \Bigg(k^2 \left(A_{G1} \left(k^2 - 2 \right) M_2 - M_0 S_{G0} + 2 M_1 S_{G1} \right) + A_{G1} M_2 \\
& + I_{E1} \left(k^2 + 1 \right) k^2 M_0 + I_{G1} \left(k^2 + 1 \right) M_0 - M_0 S_{G0} - 2 M_1 S_{G1} \Bigg) \\
& + A_{G0}^2 \left(k^2 + 1 \right) M_0 - \left(k^2 + 1 \right) M_0 \left(A_{G1} \left(I_{E1} k^2 + I_{G1} - S_{G0} \right) - S_{G1}^2 \right) \; , && (3.35\text{d}) \\
d =& A_{E1} (k^2 - 1)^2 (-A_{G0}^2 + A_{G1} (I_{G1} + I_{E1} k^2 - S_{G0}) - S_{G1}^2 \\
& + A_{G0} (A_{G1} R + 2 S_{G1})) \; . && (3.35\text{e})
\end{aligned}$$

Given a positive integer wave number, for instance $n = 1$, the cubic equation (3.35a) is then solved and results in three roots $\mu_{1,2,3}$. Next, three frequencies are obtained by calculating positive square roots of:

$$\omega_i = +\sqrt{\mu_i} \; ; \quad \text{where } i = 1, 2, 3 \; . \tag{3.36}$$

Choosing any two out of three algebraic equations in (3.34) and letting one out of three unknowns W_0, U_0, and Ψ_0 becomes a constant. In this case, the first two equations are chosen, and $W_0 = 1$. Corresponding three mode shapes are:

$$\begin{Bmatrix} W_0 \\ U_0 \\ \Psi_0 \end{Bmatrix}_i = \begin{Bmatrix} 1 \\ \frac{A_{E1}\left(k^2-1\right)(A_{G0}-S_{G1})-A_{E1}\mu_i M_1+\mu_i\left(M_0(A_{G0}+\mu_i M_1-S_{G1})-A_{G1}k^2M_1\right)}{A_{E1}k(k^2-1)(A_{G0}-S_{G1})-k\mu_i(M_1(A_{E1}+A_{G1})+A_{G0}M_0-M_0S_{G1})} \\ \frac{\left(k^2+1\right)\mu_i M_0(A_{E1}+A_{G1})-A_{E1}A_{G1}\left(k^2-1\right)^2-\mu_i^2 M_0^2}{A_{E1}k(k^2-1)(A_{G0}-S_{G1})-k\mu_i(M_1(A_{E1}+A_{G1})+A_{G0}M_0-M_0S_{G1})} \end{Bmatrix} . \quad (3.37)$$

3.4 Classical Theory Formulation

In the following, free vibration dynamic formulations of Classical Theory will be presented. Kinematic description, strain displacement relationships and constitutive laws are following (2.63)-(2.64) with addition independent variable t i.e. the dynamic two dependent variables are a function of (θ, t): $w_0(\theta, t)$, and $u_0(\theta, t)$.

Governing equations and associated boundary conditions are derived from Hamilton's Principle (3.29). The first variation of strain energy δU is previously stated in (2.65), the first variation of external potential energy is $\delta V = 0$ for free vibration analysis. The first variation of kinetic energy is:

$$\delta K = -\int_0^\alpha \int_A \rho \left(\ddot{w}\delta w + \ddot{u}\delta u\right) dA d\theta, \quad (3.38)$$

Substituting displacement field (2.63) into (3.38) and performing integration (3.29a) with classical displacement field respect to θ; results are dynamic governing equations and associated boundary condition (for $0 \le \theta \le \alpha$):

δw_0 :

$$\left(\frac{I_{E1}}{R^2} w_{0,\theta\theta} - \frac{S_{E0}}{R^2} u_{0,\theta}\right)_{,\theta\theta} + \frac{A_{E0}}{R} u_{0,\theta} + A_{E1} w_0 - \left(\frac{M_2}{R^2} w_{0,\theta tt}\right)_{,\theta} + M_0 w_{0,tt} + \left[\left(\frac{M_1}{R} + \frac{M_2}{R^2}\right) u_{0,tt}\right]_{,\theta} = 0 \,. \quad (3.39a)$$

δu_0 :

$$-\left[\left(\frac{A_{E0}}{R}+\frac{S_{E0}}{R^2}\right)u_{0,\theta}+\frac{A_{E0}}{R}w_0-\frac{S_{E0}}{R^2}w_{0,\theta\theta}\right]_{,\theta} + \left(M_0+\frac{2M_1}{R}+\frac{M_2}{R^2}\right)u_{0,tt}-\left(\frac{M_1}{R}+\frac{M_2}{R}\right)w_{0,\theta tt}=0\ . \quad (3.39b)$$

Associated boundary conditions (three at each end, θ_e = 0, α) for the Classical theory are;

Either $\delta w_0 = 0$ *or*,

$$-\left(\frac{I_{E1}}{R^2}w_{0,\theta\theta}-\frac{S_{E0}}{R^2}u_{0,\theta}\right)_{,\theta}-\left(\frac{M_1}{R}+\frac{M_2}{R^2}\right)u_{0,tt}+\frac{M_2}{R^2}w_{0,\theta tt}=0\ , \quad (3.40a)$$

Either $\delta w_{0,\theta} = 0$ *or*,

$$\frac{I_{E1}}{R^2}w_{0,\theta\theta}-\frac{S_{E0}}{R^2}u_{0,\theta}=0\ , \quad (3.40b)$$

Either $\delta u_0 = 0$ *or*,

$$\left(\frac{A_{E0}}{R}+\frac{S_{E0}}{R^2}\right)u_{0,\theta}+\frac{A_{E0}}{R}w_0-\frac{S_{E0}}{R^2}w_{0,\theta\theta}=0\ , \quad (3.40c)$$

Again, assuming constant properties and using the form (3.33) for the displacements w_0 and u_0 leads to:

$$\begin{bmatrix} \left(\frac{I_{E1}}{R^2}k^4+A_{E1}\right)-\omega^2\left(M_0+k^2\frac{M_2}{R^2}\right) & -k\left(\frac{S_{E0}}{R^2}k^2+\frac{A_{E0}}{R}\right)+k\omega^2\left(\frac{M_1}{R}+\frac{M_2}{R^2}\right) \\ -k\left(\frac{S_{E0}}{R^2}k^2+\frac{A_{E0}}{R}\right)+\omega^2 k\left(\frac{M_1}{R}+\frac{M_2}{R^2}\right) & k^2\left(\frac{S_{E0}}{R^2}+\frac{A_{E0}}{R}\right)-\omega^2\left(M_0+\frac{2M_1}{R}+\frac{M_2}{R^2}\right) \end{bmatrix} \begin{Bmatrix} W_0 \\ U_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.41)$$

For nontrivial solution, the determinant of (3.41) is a zero; result is 4^{th} order polynomial equation which reduces to quadratic equation by setting $\mu = \omega^2$,

$$a\mu^2 + b\mu + c = 0 \ , \tag{3.42a}$$

where,

$$a = \frac{k^2}{R^2}\left(M_0M_2 - M_1^2\right) + M_0(R(M_0R + 2M_1) + M_2) \ , \tag{3.42b}$$

$$\begin{aligned} b =& \frac{1}{R^4}\Big(- I_{E1}k^4(R(M_0R + 2M_1) + M_2) \\ & - R\left(A_{E0}k^2\left(\left(k^2 - 2\right)M_2 + M_0R^2 - 2M_1R\right) + A_{E1}R\left(M_0R^2 + 2M_1R + M_2\right)\right) \\ & + k^2S_{E0}\left(k^2(2M_1R + M_2) - M_0R^2\right)\Big) \ , \end{aligned} \tag{3.42c}$$

$$\begin{aligned} c =& \frac{k^2}{R^4} - A_{E0}^2R^2 + A_{E0}R\left(A_{E1}R^2 + I_{E1}k^4 - 2k^2S_{E0}\right) \\ & + S_{E0}\left(A_{E1}R^2 + k^4(I_{E1} - S_{E0})\right) \ . \end{aligned} \tag{3.42d}$$

Given a wave number i.e n is specified, the quadratic equation (3.35a) is then solved result in two roots $\mu_{1,2}$. Next, two frequencies are obtained by calculating positive square roots of:

$$\omega_i = +\sqrt{\mu_i} \ ; \quad \text{where } i = 1, 2 \ . \tag{3.43}$$

Choosing one of the two algebraic equations in (3.41) and letting one of the two unknowns, W_0 and U_0, becomes a constant. In this case, the first equation is selected, and $W_0 = 1$. Corresponding two mode shapes are:

$$\begin{Bmatrix} W_0 \\ U_0 \end{Bmatrix}_i = \begin{Bmatrix} 1 \\ \frac{A_{E1}R^2 + I_{E1}k^4 - k^2\mu_iM_2 - \mu_iM_0R^2}{k(A_{E0}R + k^2S_{E0} - \mu_iM_1R - \mu_iM_2)} \end{Bmatrix} \ . \tag{3.44}$$

3.5 Results

A free vibration numerical case is studied. Solutions from two versions of the Extended High Order Sandwich Panel Theory(EHSAPT) with logarithmic and polynomial core displacement functions, Theory of elasticity(ELST), the equivalent shear modulus First Order Shear Deformation Theory (FOSD) with shear correction factor. In addition, solutions from the High Order Sandwich Panel Theory (HSAPT), which assumes a compressible core [39], is presented. The dynamic elasticity solution serves as a benchmark to assess the accuracy of all theories.

A simply supported sandwich beam/panel configuration in figure 2.1 is studied. The sandwich consists of two thin top and bottom face sheet of equivalent thickness $f_{t,b} = 1$ mm; separates by a thick sandwich core $2c = 25$ mm. Both face sheets and core have cylindrical orthotropic material. The two face sheets are made out of uni-direction high modulus carbon fiber epoxy resin whose properties are : elastic modulus $E_1^{t,b} = 175$ GPa, $E_3^{t,b} = 8$ GPa, poisson's ratio $v_{13}^{t,b} = 0.3$, shear modulus $G^{t,b} = 5$ GPa, and density $\rho_{t,b} = 1,600$ kg/m^3. The sandwich core are made out of Divinycell foam core H160 whose properties are : $E_1^c = E_3^c = 170$ MPa, $v_{13}^c = 0.3$, $G^c = 66$ MPa, and $\rho_c = 170$ kg/m^3. The curved beam has angular span $\alpha = 0.317815$ rad, a radius of the top face mid-line R_t = 813 mm, and a width (out-of-plane) 30 mm.

Considering the first wave number case i.e. $n = 1$, displacements are in the form:

$$
\begin{aligned}
w(r,\theta,t) &= W(r) sin(\frac{\theta\pi}{\alpha}) e^{\mathrm{i}\omega t} \ , \\
u(r,\theta,t) &= U(r) sin(\frac{\theta\pi}{\alpha}) e^{\mathrm{i}\omega t} \ .
\end{aligned}
\tag{3.45}
$$

The two EHSAPTs results in 7 frequencies. The HSAPT [39] provides 4 frequencies, the equivalent shear modulus FOSDT with shear correction factor results 3 frequencies, the Classical theory gives 2 frequencies. And the elasticity while results in infinite frequencies, 6 of them are presented. Frequencies comparisons are shown in table 3.1-3.2.Again,

Table 3.1: Natural frequencies (Hz) 1^{st} to 4^{th} mode for wave number $n = 1$

	1^{st}	2^{nd}	3^{rd}	4^{th}
Elasticity	866.01	10,043.28	12,735.03	17,390.04
EHSAPT-log	866.13	10,075.61	12,845.43	17,430.49
EHSAPT-poly	866.43	10,084.59	12,846.10	17,451.52
HSAPT	864.73	12,160.49	13,538.85	18,870.97
FOSDT-$G_{eq,\beta}$	866.37	13,620.81	19,010.10	N/A
Classical	2120.98	13,623.05	N/A	N/A

Table 3.2: Natural frequencies (Hz) 5^{th} to 7^{th} mode for wave number $n = 1$

	5^{th}	6^{th}	7^{th}
Elasticity	22,425.59	28,329.23	29,865.35
EHSAPT-log	23,412.76	29,142.82	31,608.84
EHSAPT-poly	23,441.83	29,145.50	31,613.69
HSAPT	N/A	N/A	N/A
FOSDT-$G_{eq,\beta}$	N/A	N/A	N/A
Classical	N/A	N/A	N/A

as state in static analysis, both EHSAPTs are shown to be the most accurate with logarithmic variant always slightly closer to elasticity. HSAPT also show good predictions but not as much as EHSAPT, it underestimates the first frequency while overestimates the rest. In this particular numerical case, the equivalent shear modulus FOSD theory with shear correction factor surprisingly accurate finding first frequency (better than EHSAPT-poly but worse than EHSAPT-log). While Classical theory significantly inaccurate in first frequency prediction, almost 3 times off, the second frequency is acceptable.

Figure 3.1 shows through thickness radial(transverse) displacements, w, at $\alpha/2$ for the first mode. In order to present the displacement distributions within the same plot frame, their magnitudes are needed to be in the same order because mode shapes, hence displacements, can have arbitrary magnitude. Each theory is then normalized by each artificial constant such that $w_{\alpha/2} = 1$ at $r = R_t$ and these particular artificial constants are carried along to normalize circumferential displacements in figure 3.2. Due to the normalization, the plot shall be interpret as only distribution profile for each individual theory; and it

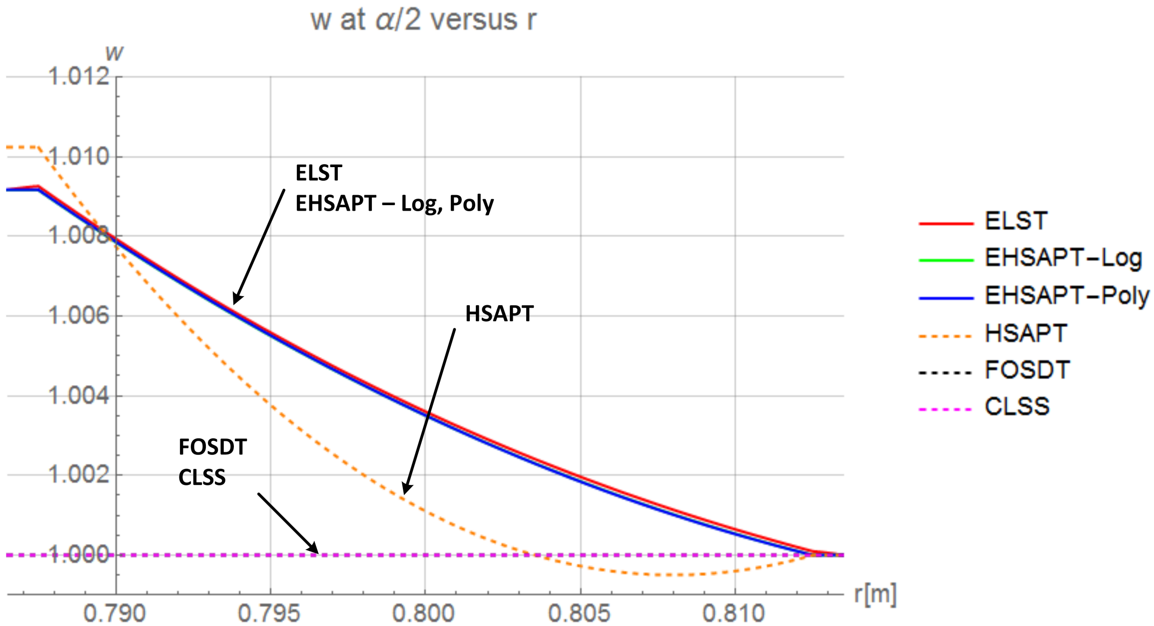

Figure 3.1: Through-thickness distribution of radial displacement, w, at $\alpha/2$ 1st mode and wave number $n = 1$

cannot regard as accuracy comparison among theories. To some extent, however, figure 3.2 presents through thickness circumferential displacement, u, at $\alpha = 0$; it is observed that the distributions (both w and u) from EHSAPTs capture the elasticity profile perfectly unlike others.

The same normalization procedure is followed to present stresses corresponding to first mode. New artificial constants are applied such that circumferential stress $\sigma_{\theta\theta} = 1$ at $r = R_t$ and these new set of constants are applied to shear,$\tau_{r\theta}$, and radial normal stress, σ_{rr}. Figure 3.3 shows through thickness distribution of $\sigma_{\theta\theta}$ at $\alpha/2$ within bottom face sheet (left-figure) and top face sheet (right-figure). Transverse normal stress, σ_{rr}, at $\alpha/2$ core distributions are shown in figure 3.4; notice an absence of FOSDT and Classical as they are incompressible theory. Lastly, shear stress, $\tau_{r\theta}$, core distributions at $\alpha = 0$ are presented in figure 3.5; note that classical theory excludes shear deformation.

Figure 3.6 and 3.7 present curved beam deformations(right-most column in the figure) and through thickness displacements plot from EHSAPT logarithmic (red-solid line) and polynomial (blue-dashed line) version correspond to all seven modes for the given wave

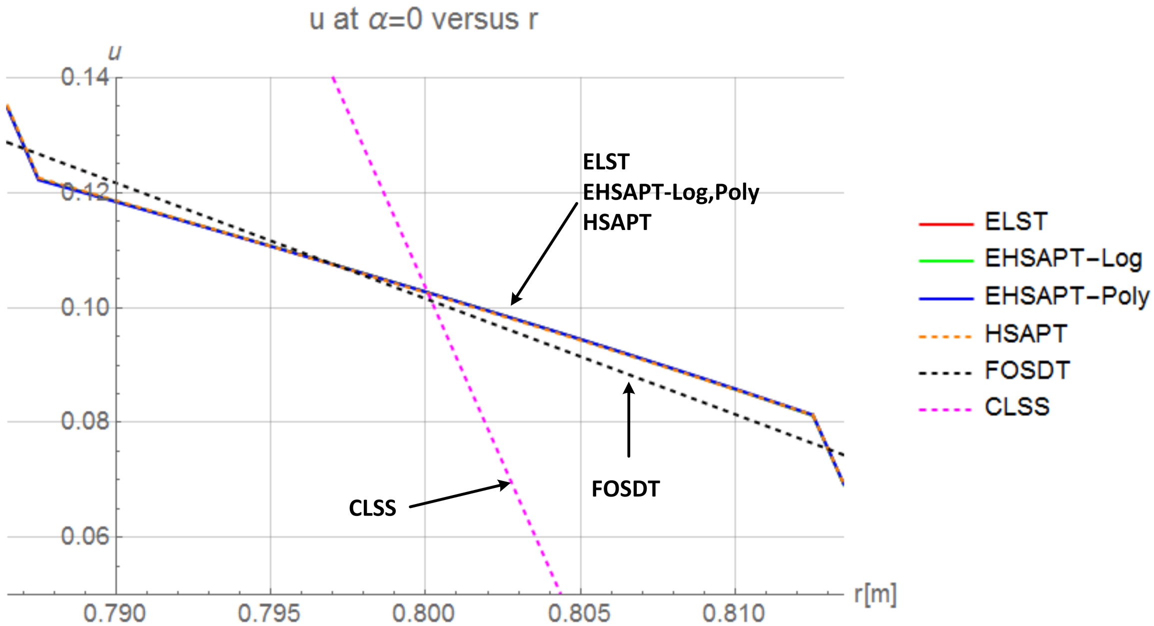

Figure 3.2: Through-thickness distribution of circumferential displacement, u at $\alpha = 0$ 1st mode and wave number $n = 1$

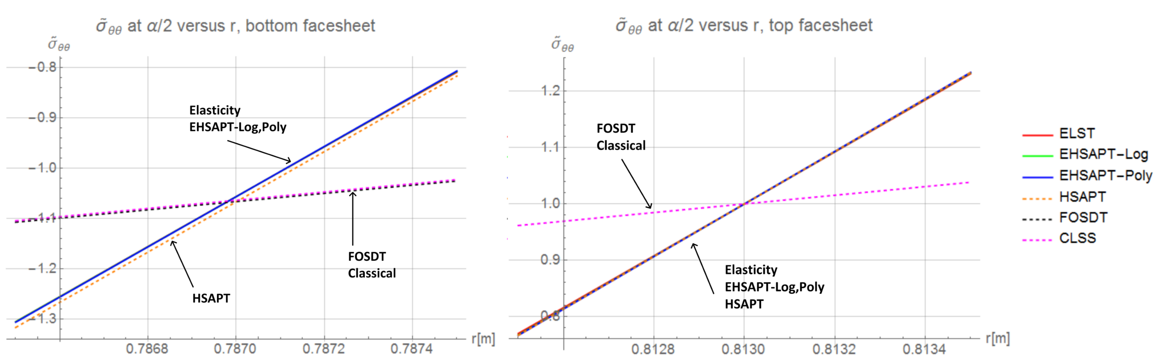

Figure 3.3: Face sheet through-thickness distribution of circumferential stress, $\sigma_{\theta\theta}$, at $\alpha/2$, 1st mode and wave number $n = 1$

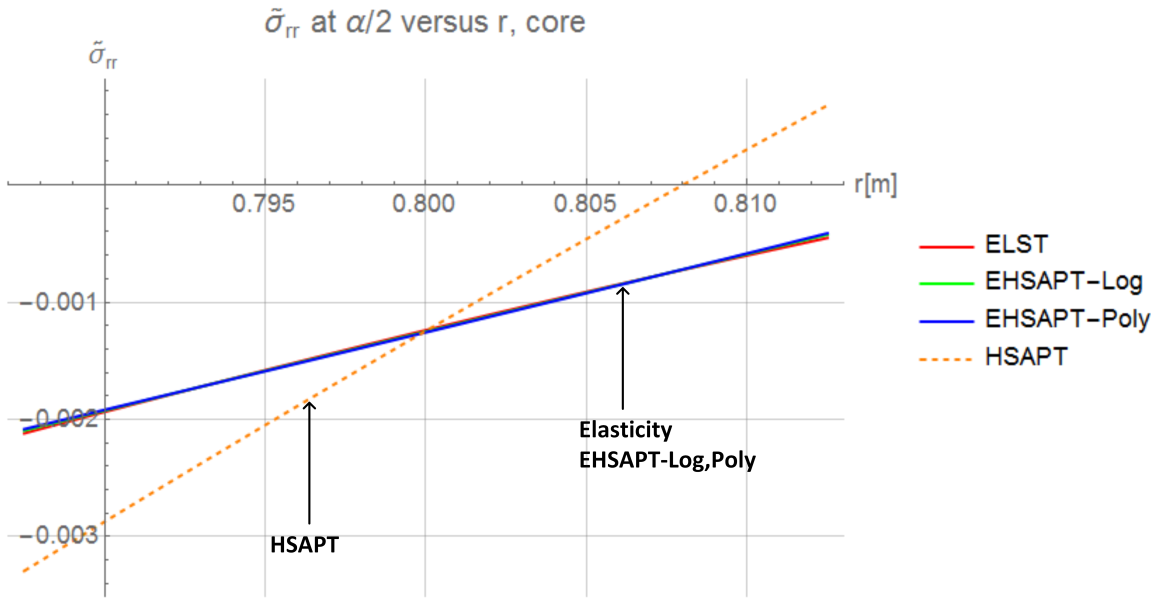

Figure 3.4: Core through-thickness distribution of radial normal stress, σ_{rr}, at $\alpha/2$ 1st mode and wave number $n = 1$

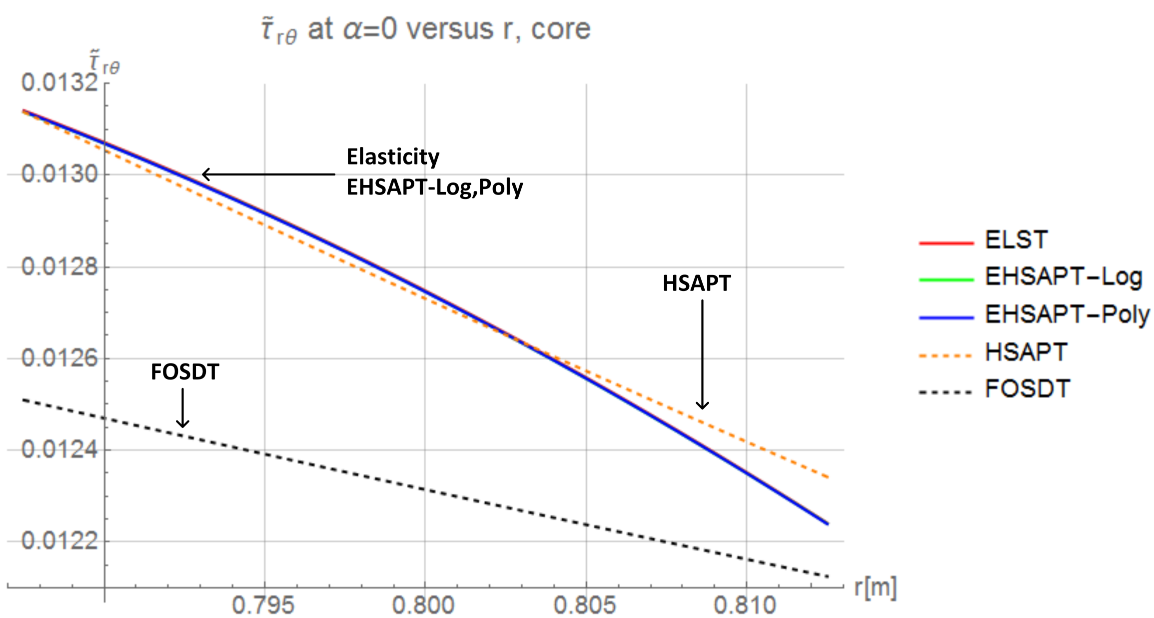

Figure 3.5: Core through-thickness distribution of shear stress, τ_{rs}, at $\alpha = 0$ 1st mode and wave number $n = 1$

number, $n = 1$. The beam deformation plots are not in true scale, they are exaggerated to enhanced visualization; and displacement plot are normalized as previously done. It can be seen that free vibration modes can be categorized into radial dominance, and circumferential dominance. 1st, 3rd and 6th mode are radial dominance because w component is significantly larger than u. And 2nd, 4th, 5th, and 7th mode are circumferential dominance due to the same inverse reasoning. It is also noticed that 5th, 6th, and 7th mode are core dominant vibration as large mid-core w shown in left-most column in the figure.

Solution procedures for the other structural theories are straight forward; however, elasticity's solution procedure is rather complicated and poses numerical issues. It shall be further discussed here.

With the numerical case, top/bottom face sheets, the indicial roots (3.12b) are $s_{1,2}^{t,b} = \pm 7.8473$ and $s_{3,4}^{t,b} = \pm 57.6421$; sandwich core, the roots are $s_{1,2}^{c} = \pm 9.2164$ and $s_{3,4}^{c} = \pm 10.4936$. These indicial roots are distinct and satisfying (3.15a). Therefore, a general series solution takes the form of (3.15b). The infinite Frobenius series (3.9a) is replaced by a partial summation to produce numerical results,

$$\{\tilde{X}\}_N^j = \sum_{n=0,2,4,\ldots}^{N} \{X_n\}^j r^{s_j+n} \text{ , where } j = 1st, 2nd, 3rd, 4th \; . \tag{3.46}$$

As frequency ω increase, two major numerical issues are raised. Firstly, higher number of terms are needed to include in the partial summation (3.46) for convergence; in this report, $N = 500$ is used to produce results. Figure 3.8 shows a semi Log-Scale plot of a vector magnitude of absolute error $\%_{error}$ versus number of term N using face sheet material and evaluating at $r = R_t$,

$$\%_{error}^i = \left\| \frac{\{\tilde{X}\}_N^j - \{\tilde{X}\}_{N=1000}^j}{\{\tilde{X}\}_{N=1000}^j} \times 100\% \right\| \; . \tag{3.47}$$

Minimum limit on y-axis, $\%_{error}$, is at 10^{-20} and if this is set as tolerance criteria. Figure

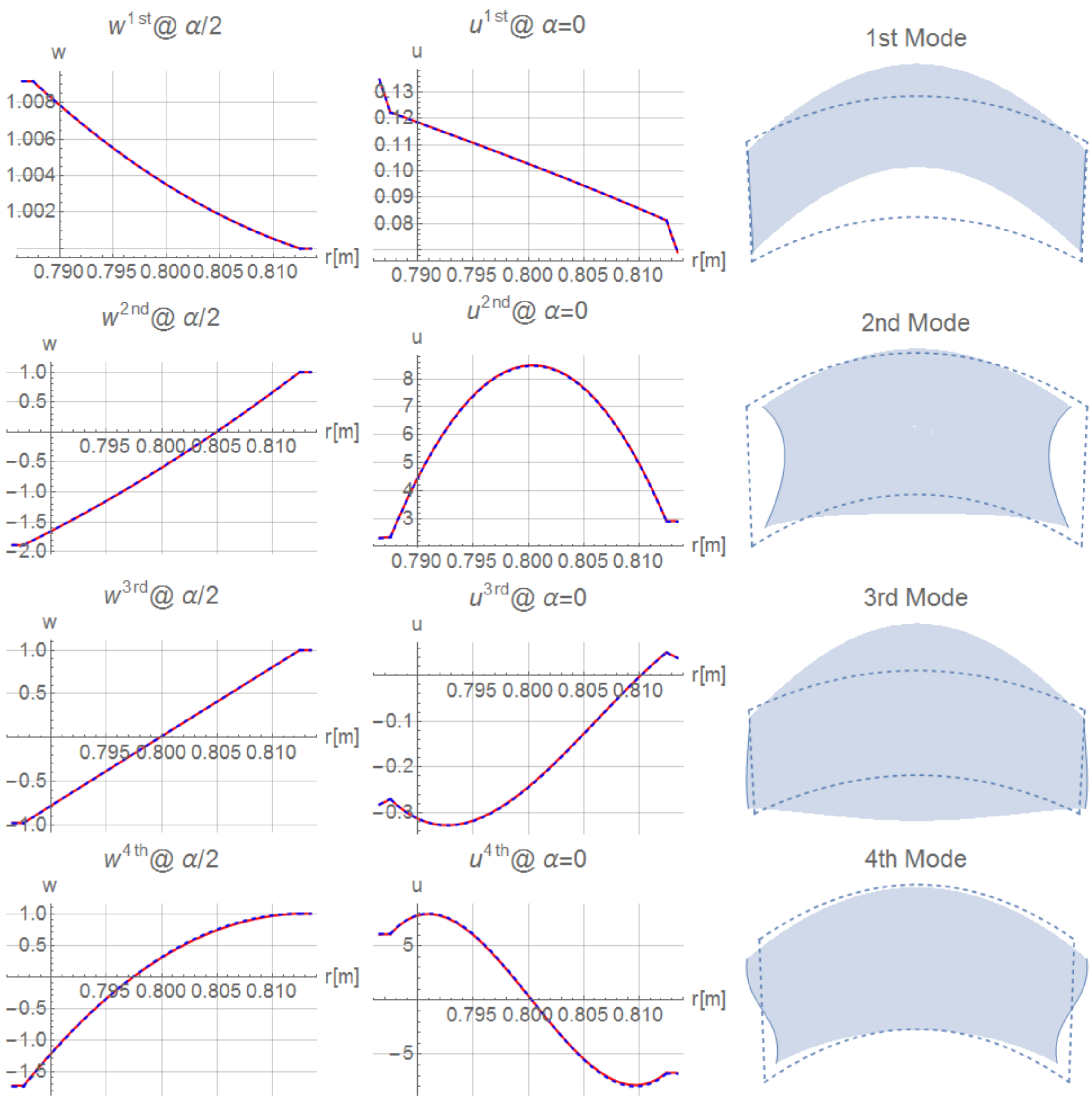

Figure 3.6: EHSAPT-log(red-solid line), poly(blue-dashed line) displacements(w-left, u-middle) of 1st to 4th mode for wave number $n = 1$. And mode deformation(right); deformed (blue shaded area), and undeformed (light blue dashed line) configuration

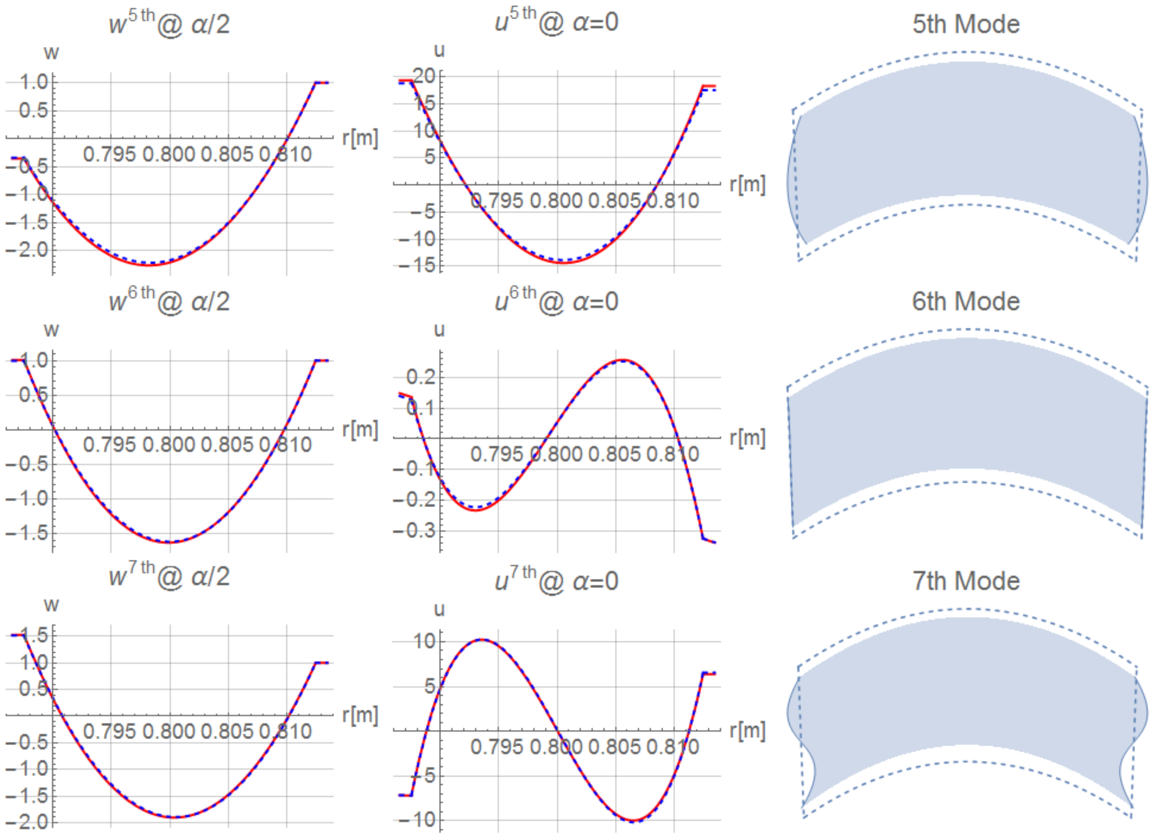

Figure 3.7: EHSAPT-log(red-solid line), poly(blue-dashed line) displacements(w-left, u-middle)of 5th to 7th mode for wave number $n = 1$. And mode deformation(right); deformed (blue shaded area), and undeformed (light blue dashed line) configuration

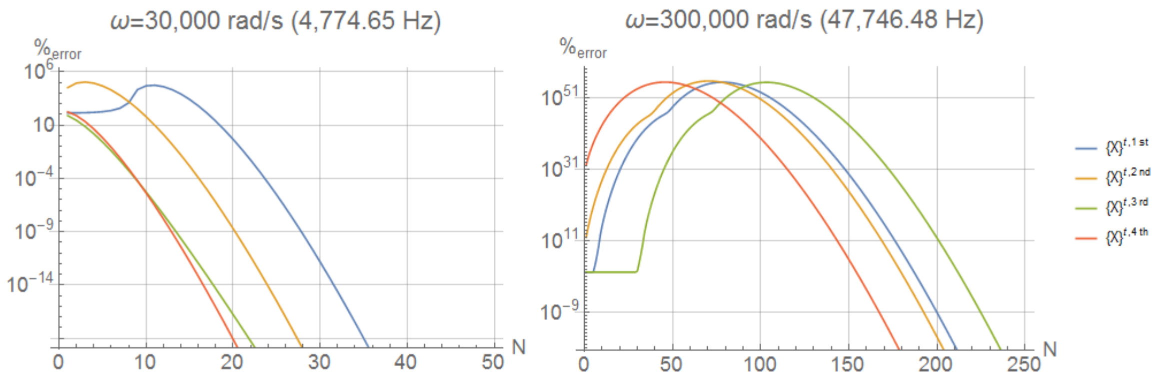

Figure 3.8: Semi-log plot free vibration elasticity solution convergence, face sheet at $r = R_t$

3.8-left shows that lower frequency $\omega = 30,000$ rad/s, $N = 40$ would be sufficiently enough; while, figure-3.8 right indicates that for higher frequency $\omega = 300,000$ rad/s, $N = 250$ are needed. In addition, figure 3.8-right shows that, initially, $\%_{error}$ is increasing as N increase; this strange behavior is explained by the term ω^2/n^2 in the recurrence relation (3.28b) as n is not substantial to overcome contribution from ω.

The second issue is truncation error. The numerical value becomes very large, specifically, when the determinant (3.27) is being evaluated. Increase digits precision would reduce the error; 100 digits are kept to produce the elasticity results.

As shown in static result, the free vibration results underline again the remarkable performance of EHSAPTs, in particular logarithmic-EHSAPT, among other theories. In addition, considering elasticity solution complexity, specifically the Frobenius series solution; EHSAPTs are rather simple and straight forward.

3.6 Conclusion

To conclude, free vibration analysis of a curved sandwich beam with simply supported condition on both ends is studied. Dynamic governing equations and associated boundary conditions are derived using Hamilton's Principle for following theories: the new Extended High order Sandwich Panel Theory (EHSAPT), the First Order Shear Deformation theory (FOSD), and Classical theory; and closed form analytical solution are obtained. Then, free vibration elasticity closed form analytical solutions for simply supported condition are derived using the method of Frobenius series; common solution characteristics of orthotropic material are addressed. Subsequently, the elasticity serves as a benchmark to assess accuracy of all theories. In addition, results from the High order Sandwich Panel Theory [39] are produced to compare.

A numerical case of a curved sandwich beam vibrates in the first wave number is presented. Corresponding natural frequencies and mode shapes are determined and compared. Through thickness displacements and stresses distribution are presented for the first mode for all theories. It has shown that EHSAPTs frequencies are the most accurate, especially the logarithmic variants slightly outperform as previously shown in static analysis. HSAPT is under performed when comparing with EHSAPTs. The equivalent shear modulus FOSD with shear correction factor also shows accurate the first natural frequency. Elasticity theoretically provides infinite number of natural frequencies but associated numerical issues limit the performance. Meanwhile EHSAPTs provide very accurate and up to seven natural frequencies for each wave number more than the other theories. Deformations of seven EHSAPTs mode shapes are shown capturing unique sandwich beam vibration type: radial/circumferential dominance, and face sheets/core dominance. Again in free vibration analysis, logarithmic EHSAPT is proved to have superior performance.

Appendices

APPENDIX A

LOGARITHMIC EHSAPT CONSTANTS

$$A_i^a = k_0 \frac{f_t r_{tc}}{4R_t}(H_{1i}^d - H_{1i}^a r_{bc} \ln r_{bc}) \ , \qquad (I.1a)$$

$$A_i^b = k_0 \frac{c_{55}^c r_{tc}}{4R_b R_t^2}[f_t H_{3i}^b - (f_t - 2R_t)H_{3i}^c + 2H_{3i}^a(f_t - R_t)r_{bc} \ln r_{bc}] \ , \qquad (I.1b)$$

$$A_i^c = k_0 \frac{r_{tc}}{2}(H_{2i}^b + H_{1i}^d + H_{2i}^a - H_{1i}^a) r_{bc} \ln r_{bc} \ , \qquad (I.1c)$$

$$B_i^a = k_0 \frac{f_b r_{bc}}{4R_b}(H_{1i}^d - H_{1i}^a r_{tc} \ln r_{tc}) \ , \qquad (I.2a)$$

$$B_i^b = k_0 \frac{c_{55}^c r_{bc}}{4R_b^2 R_t}[f_b H_{3i}^b - (f_b + 2R_b)H_{3i}^c + 2H_{3i}^a(f_b + R_b)r_{tc} \ln r_{tc}] \ , \qquad (I.2b)$$

$$B_i^c = -k_0 \frac{r_{bc}}{2}[H_{2i}^b + H_{1i}^d + (H_{2i}^a - H_{1i}^a) r_{tc} \ln r_{tc}] \ , \qquad (I.2c)$$

$$C_i^b = k_0 \frac{c_{55}^c}{2R_b R_t}[H_{3i}^c(r_{bc} - r_{tc}) + H_{3i}^a r_{bc} r_{tc} \ln \frac{r_{bc}}{r_{tc}} + H_{3i}^b(-r_{bc} \ln r_{bc} + r_{tc} \ln r_{tc})] \ , \quad (I.3a)$$

$$C_i^c = k_0 \frac{1}{2}\left[(H_{2i}^b + H_{1i}^d)(r_{bc} - r_{tc}) - H_{1i}^b r_{bc} \ln r_{bc} + (H_{1i}^a - H_{2i}^a) r_{bc} r_{tc} \ln \frac{r_{bc}}{r_{tc}} + H_{1i}^b r_{tc} \ln r_{tc}\right] , \quad (I.3b)$$

$$D_i^b = k_0 \frac{(f_t - 2R_t) r_{tc}}{4R_t}(-H_{1i}^d + H_{1i}^a r_{bc} \ln r_{bc}) , \quad (I.4a)$$

$$D_i^c = -k_0 \frac{c_{55}^c (f_t - 2R_t) r_{tc}}{4R_b R_t^2}(H_{3i}^b - H_{3i}^c + 2H_{3i}^a r_{bc} \ln r_{bc}) , \quad (I.4b)$$

$$E_i^b = k_0 \frac{(f_b + 2R_b) r_{bc}}{4R_b}(-H_{1i}^d + H_{1i}^a r_{tc} \ln r_{tc}) , \quad (I.5a)$$

$$E_i^c = -k_0 \frac{c_{55}^c (f_b + 2R_b) r_{bc}}{4R_b^2 R_t}(H_{3i}^b - H_{3i}^c + 2H_{3i}^a r_{tc} \ln r_{tc}) , \quad (I.5b)$$

$$F_i^b = k_0 \frac{1}{2}\left(H_{1i}^d (r_{bc} - r_{tc}) + H_{1i}^a r_{bc} r_{tc} \ln \frac{r_{bc}}{r_{tc}} + H_{1i}^b(-r_{bc} \ln r_{bc} + r_{tc} \ln r_{tc})\right) , \quad (I.6a)$$

$$F_i^c = k_0 \frac{c_{55}^c}{2R_b R_t}\left((H_{3i}^b + H_{3i}^c)(r_{bc} - r_{tc}) + H_{3i}^b r_{bc} \ln r_{bc} - H_{3i}^b r_{tc} \ln r_{tc} + 2H_{3i}^a r_{bc} r_{tc} \ln \frac{r_{tc}}{r_{bc}}\right) , \quad (I.6b)$$

$$G_i^b = k_0 \frac{1}{2}\left(H_{1i}^d (r_{bc} - r_{tc})(r_{bc} + r_{tc}) + r_{bc}(-H_{1i}^c + H_{1i}^a r_{tc}^2) \ln r_{bc} + (H_{1i}^c - H_{1i}^a r_{bc}^2) r_{tc} \ln r_{tc}\right) , \quad (I.7a)$$

$$G_i^c = k_0 \frac{c_{55}^c}{2R_b R_t}((H_{3i}^b - H_{3i}^c)(r_{bc} - r_{tc})(r_{bc} + r_{tc}) + 2H_{3i}^a r_{bc} r_{tc}(-r_{tc}\ln r_{bc} + r_{bc}\ln r_{tc})) \ , \tag{I.7b}$$

where

$$k_0 = \frac{b}{(r_{bc}\ln(r_{bc}) - r_{tc}\ln(r_{tc}))^2} \ , \tag{I.8}$$

$$H_{11}^a = \frac{1}{r_{bc} r_{tc}} \Big[2r_{bc}\left(c_{11}^c + c_{13}^c\right)\left(r_{bc} - r_{tc}\right) - 2r_{bc} r_{tc} c_{11}^c \ln\left(r_{bc}\right) + \ln\left(r_{tc}\right)\left(r_{bc}^2\left(c_{11}^c + c_{13}^c\right) + r_{tc}^2\left(c_{11}^c - c_{13}^c\right)\right) \Big] \ , \tag{I.9a}$$

$$H_{12}^a = \frac{\left(c_{11}^c + c_{13}^c\right)\left(r_{bc} - r_{tc}\right)}{r_{bc} r_{tc}} \left[\left(r_{bc} + r_{tc}\right) \ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc} \right] \ , \tag{I.9b}$$

$$H_{13}^a = \frac{1}{r_{bc} r_{tc}} \Big[-2r_{tc}\left(c_{11}^c + c_{13}^c\right)\left(r_{bc} - r_{tc}\right) - 2r_{bc} r_{tc} c_{11}^c \ln\left(r_{tc}\right) + \ln\left(r_{bc}\right)\left(r_{bc}^2\left(c_{11}^c - c_{13}^c\right) + r_{tc}^2\left(c_{11}^c + c_{13}^c\right)\right) \Big] \ , \tag{I.9c}$$

$$H_{14}^a = \frac{c_{11}^c\left(f_b + 2R_b\right)}{2R_b r_{bc} r_{tc}} \left[2r_{bc}\left(-r_{tc}\ln\left(r_{bc}\right) + r_{bc} - r_{tc}\right) + \left(r_{bc}^2 + r_{tc}^2\right)\ln\left(r_{tc}\right) \right] \ , \tag{I.9d}$$

$$H_{15}^a = \frac{c_{11}^c (r_{bc} - r_{tc})}{r_{bc} r_{tc}} \left[\left(r_{bc} + r_{tc}\right) \ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc} \right] \ , \tag{I.9e}$$

$$H_{16}^a = \frac{c_{11}^c (f_t - 2R_t)}{2R_t r_{bc} r_{tc}} \left[-2r_{tc}\left(r_{bc}\left(-\ln\left(r_{tc}\right)\right) - r_{bc} + r_{tc}\right) - \left(r_{bc}^2 + r_{tc}^2\right)\ln\left(r_{bc}\right) \right] , \tag{I.9f}$$

$$H_{17}^{a}=\frac{c_{11}^{c}}{r_{bc}r_{tc}}\left[2r_{bc}r_{tc}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left(r_{bc}\ln\left(r_{bc}\right)-r_{tc}\ln\left(r_{tc}\right)\right)\right.$$

$$\left.-\left(r_{bc}-r_{tc}\right)\left(r_{bc}+r_{tc}\right)\left(r_{bc}\left(\ln\left(r_{tc}\right)+2\right)-r_{tc}\left(\ln\left(r_{bc}\right)+2\right)\right)\right], \qquad (I.9g)$$

$$H_{18}^{a}=\frac{c_{11}^{c}f_{b}}{2R_{b}r_{bc}r_{tc}}\left[2r_{bc}\left(r_{tc}\ln\left(r_{bc}\right)-r_{bc}+r_{tc}\right)-\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{tc}\right)\right], \qquad (I.9h)$$

$$H_{19}^{a}=\frac{c_{11}^{c}f_{t}}{2R_{t}r_{bc}r_{tc}}\left[2r_{tc}\left(r_{bc}\left(-\ln\left(r_{tc}\right)\right)-r_{bc}+r_{tc}\right)+\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{bc}\right)\right], \qquad (I.9i)$$

$$H_{11}^{b}=c_{11}^{c}\left(r_{bc}\left(\ln\left(r_{bc}\right)^{2}-\ln\left(r_{tc}\right)\left(\ln\left(r_{tc}\right)+2\right)\right)+2r_{tc}\ln\left(r_{tc}\right)\right)$$

$$+2c_{13}^{c}\left(r_{bc}\ln\left(r_{bc}\right)-r_{tc}\ln\left(r_{tc}\right)\right), \qquad (I.10a)$$

$$H_{12}^{b}=c_{11}^{c}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right)-2r_{bc}+2r_{tc}\right], \qquad (I.10b)$$

$$H_{13}^{b}=c_{11}^{c}\left(r_{tc}\left(\ln\left(r_{tc}\right)^{2}-\ln\left(r_{bc}\right)\left(\ln\left(r_{bc}\right)+2\right)\right)+2r_{bc}\ln\left(r_{bc}\right)\right)$$

$$+2c_{13}^{c}\left(r_{tc}\ln\left(r_{tc}\right)-r_{bc}\ln\left(r_{bc}\right)\right), \qquad (I.10c)$$

$$H_{14}^{b}=\frac{c_{11}^{c}(f_{b}+2R_{b})}{2R_{b}}\left[r_{bc}\left(\ln\left(r_{bc}\right)^{2}-\ln\left(r_{tc}\right)\left(\ln\left(r_{tc}\right)+2\right)\right)+2r_{tc}\ln\left(r_{tc}\right)\right], \qquad (I.10d)$$

$$H_{15}^{b}=c_{11}^{c}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right)-2r_{bc}+2r_{tc}\right], \qquad (I.10e)$$

$$H_{16}^{b}=\frac{c_{11}^{c}(f_t-2R_t)}{2R_t}\left[r_{tc}\left(\ln\left(r_{bc}\right)\left(\ln\left(r_{bc}\right)+2\right)-\ln\left(r_{tc}\right)^{2}\right)-2r_{bc}\ln\left(r_{bc}\right)\right], \quad (I.10f)$$

$$H_{17}^{b}=c_{11}^{c}\left(r_{bc}-r_{tc}\right)\left(-\ln\left(r_{bc}r_{tc}\right)\right)\left[\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right)-2r_{bc}+2r_{tc}\right], \quad (I.10g)$$

$$H_{18}^{b}=\frac{c_{11}^{c}f_b}{2R_b}\left[\ln\left(r_{tc}\right)\left(r_{bc}\left(\ln\left(r_{tc}\right)+2\right)-2r_{tc}\right)-r_{bc}\ln\left(r_{bc}\right)^{2}\right], \quad (I.10h)$$

$$H_{19}^{b}=\frac{c_{11}^{c}f_t}{2R_t}\left[r_{tc}\ln\left(r_{tc}\right)^{2}-\ln\left(r_{bc}\right)\left(r_{tc}\left(\ln\left(r_{bc}\right)+2\right)-2r_{bc}\right)\right], \quad (I.10i)$$

$$H_{11}^{c}=\frac{1}{2R_b}\left[-(c_{11}^{c}-c_{13}^{c})(r_{bc}-r_{tc})+c_{11}^{c}r_{bc}\ln(r_{bc})+r_{tc}\ln(r_{tc})(-c_{11}^{c}+(c_{11}^{c}-c_{13}^{c})\ln\left(\frac{r_{tc}}{r_{bc}}\right))\right], \quad (I.11a)$$

$$H_{12}^{c}=2(c_{11}^{c}-c_{13}^{c})\left[(r_{bc}-r_{tc})^{2}-r_{bc}r_{tc}\ln\left(\frac{r_{bc}}{r_{tc}}\right)^{2}\right], \quad (I.11b)$$

$$H_{13}^{c}=\frac{1}{2R_t}\left[(c_{11}^{c}-c_{13}^{c})(r_{bc}-r_{tc})+c_{11}^{c}r_{tc}\ln(r_{tc})-r_{bc}\ln(r_{bc})(c_{11}^{c}+(c_{11}^{c}-c_{13}^{c})\ln\left(\frac{r_{tc}}{r_{bc}}\right))\right], \quad (I.11c)$$

$$H_{14}^{c}=\frac{c_{11}^{c}r_{bc}(f_b+2R_b)}{R_b}\left[-r_{bc}+r_{tc}+r_{bc}\ln(r_{bc})+r_{tc}\ln(r_{tc})(-1-\ln(r_{bc})+\ln(r_{tc}))\right], \quad (I.11d)$$

$$H_{15}^{c} = 2c_{11}^{c}\left[(r_{bc}-r_{tc})^2 - r_{bc}r_{tc}\ln\left(\frac{r_{bc}}{r_{tc}}\right)^2\right], \qquad (I.11e)$$

$$H_{16}^{c} = \frac{c_{11}^{c}r_{tc}(f_t-2R_t)}{R_t}\left[-r_{bc}+r_{tc}+r_{bc}\ln(r_{bc})-r_{tc}\ln(r_{tc})(-1-\ln(r_{bc})+\ln(r_{tc}))\right], \qquad (I.11f)$$

$$H_{17}^{c} = c_{11}^{c}\left[2r_{bc}r_{tc}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left(r_{bc}\ln\left(r_{tc}\right)-r_{tc}\ln\left(r_{bc}\right)\right)\right.$$
$$\left.-\left(r_{bc}-r_{tc}\right)\left(r_{bc}+r_{tc}\right)\left(r_{bc}\left(\ln\left(r_{bc}\right)-2\right)-r_{tc}\left(\ln\left(r_{tc}\right)-2\right)\right)\right], \qquad (I.11g)$$

$$H_{18}^{c} = -\frac{c_{11}^{c}f_b r_{bc}}{R_b}\left[-r_{bc}+r_{tc}+r_{bc}\ln(r_{bc})+r_{tc}\ln(r_{tc})(-1-\ln(r_{bc})+\ln(r_{tc}))\right], \quad (I.11h)$$

$$H_{19}^{c} = \frac{c_{11}^{c}f_t r_{tc}}{R_t}\left[r_{bc}-r_{tc}+r_{tc}\ln(r_{tc})+r_{bc}\ln(r_{bc})(-1+\ln(r_{bc})-\ln(r_{tc}))\right], \quad (I.11i)$$

$$H_{11}^{d} = -\frac{2}{3}r_{bc}c_{11}^{c}\ln\left(r_{tc}\right)^{3} + r_{bc}\left(c_{13}^{c}-2c_{11}^{c}\right)\ln\left(r_{tc}\right)^{2}$$
$$+2\left(c_{13}^{c}-c_{11}^{c}\right)\ln\left(r_{tc}\right)\left(-r_{tc}\ln\left(r_{bc}\right)+r_{bc}-r_{tc}\right)+\frac{1}{3}r_{bc}\ln\left(r_{bc}\right)^{2}\left(2c_{11}^{c}\ln\left(r_{bc}\right)+3c_{13}^{c}\right), \qquad (I.12a)$$

$$H_{12}^{d} = \frac{1}{3}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[-6\left(c_{11}^{c}-c_{13}^{c}\right)\left(r_{bc}-r_{tc}\right)+c_{11}^{c}\left(r_{bc}+2r_{tc}\right)\ln\left(r_{bc}\right)^{2}\right.$$
$$+c_{11}^{c}\ln\left(r_{bc}\right)\left(\left(r_{bc}-r_{tc}\right)\ln\left(r_{tc}\right)+6r_{tc}\right)-c_{11}^{c}\ln\left(r_{tc}\right)\left(\left(2r_{bc}+r_{tc}\right)\ln\left(r_{tc}\right)+6r_{bc}\right)$$

$$+3c_{13}^{c}\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{tc}}{r_{bc}}\right)\Bigg], \tag{I.12b}$$

$$H_{13}^{d}=-\frac{2}{3}r_{tc}c_{11}^{c}\ln\left(r_{bc}\right)^{3}+r_{tc}\left(c_{13}^{c}-2c_{11}^{c}\right)\ln\left(r_{bc}\right)^{2}$$

$$+2\left(c_{11}^{c}-c_{13}^{c}\right)\ln\left(r_{bc}\right)\left(r_{bc}\ln\left(r_{tc}\right)+r_{bc}-r_{tc}\right)+\frac{1}{3}r_{tc}\ln\left(r_{tc}\right)^{2}\left(2c_{11}^{c}\ln\left(r_{tc}\right)+3c_{13}^{c}\right)\ , \tag{I.12c}$$

$$H_{14}^{d}=\frac{c_{11}^{c}(f_b+2R_b)}{3R_b}\Big[3r_{tc}\ln\left(r_{bc}\right)\ln\left(r_{tc}\right)-\ln\left(r_{tc}\right)\big(3\left(r_{bc}-r_{tc}\right)$$

$$+r_{bc}\ln\left(r_{tc}\right)\left(\ln\left(r_{tc}\right)+3\right)\big)+r_{bc}\ln\left(r_{bc}\right)^{3}\Big], \tag{I.12d}$$

$$H_{15}^{d}=\frac{1}{3}c_{11}^{c}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\Big[\left(r_{bc}+2r_{tc}\right)\ln\left(r_{bc}\right)^{2}+\ln\left(r_{bc}\right)\left(\left(r_{bc}-r_{tc}\right)\ln\left(r_{tc}\right)+6r_{tc}\right)$$

$$-\ln\left(r_{tc}\right)\left(\left(2r_{bc}+r_{tc}\right)\ln\left(r_{tc}\right)+6r_{bc}\right)-6r_{bc}+6r_{tc}\Big], \tag{I.12e}$$

$$H_{16}^{d}=\frac{c_{11}^{c}(f_t-2R_t)}{3R_t}\Big[r_{tc}\ln\left(r_{bc}\right)^{3}+3r_{tc}\ln\left(r_{bc}\right)^{2}-3\ln\left(r_{bc}\right)\left(r_{bc}\ln\left(r_{tc}\right)+r_{bc}-r_{tc}\right)-r_{tc}\ln\left(r_{tc}\right)^{3}\Big], \tag{I.12f}$$

$$H_{17}^{d}=\frac{2}{3}c_{11}^{c}\Big[\left(r_{tc}^{2}-r_{bc}^{2}\right)\ln\left(r_{bc}\right)^{3}+3\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{bc}\right)^{2}-3\ln\left(r_{bc}\right)\left(4r_{bc}r_{tc}\ln\left(r_{tc}\right)+r_{bc}^{2}-r_{tc}^{2}\right)$$

$$+\ln\left(r_{tc}\right)\left(r_{bc}^{2}\left(\ln\left(r_{tc}\right)^{2}+3\ln\left(r_{tc}\right)+3\right)-r_{tc}^{2}\left(\ln\left(r_{tc}\right)^{2}-3\ln\left(r_{tc}\right)+3\right)\right)\Big], \tag{I.12g}$$

$$H_{18}^{d} = \frac{c_{11}^{c} f_b}{3R_b}\left[r_{bc}\left(\ln\left(r_{tc}\right)^3 - \ln\left(r_{bc}\right)^3\right) + 3\ln\left(r_{tc}\right)\left(r_{bc}\ln\left(r_{tc}\right) - r_{tc}\ln\left(r_{bc}\right) + r_{bc} - r_{tc}\right)\right], \tag{I.12h}$$

$$H_{19}^{d} = \frac{c_{11}^{c} f_t}{3R_t}\left[3r_{bc}\ln\left(r_{bc}\right)\ln\left(r_{tc}\right) - \ln\left(r_{bc}\right)\left(r_{tc}\ln\left(r_{bc}\right)\left(\ln\left(r_{bc}\right) + 3\right) - 3r_{bc} + 3r_{tc}\right) + r_{tc}\ln\left(r_{tc}\right)^3\right], \tag{I.12i}$$

$$\begin{aligned} H_{21}^{a} = \frac{1}{r_{bc}r_{tc}}\Big[&2r_{bc}\left(c_{13}^{c} + c_{33}^{c}\right)\left(r_{bc} - r_{tc}\right) - 2r_{bc}r_{tc}c_{13}^{c}\ln\left(r_{bc}\right) \\ &+ \ln\left(r_{tc}\right)\left(r_{bc}^2\left(c_{13}^{c} + c_{33}^{c}\right) + r_{tc}^2\left(c_{13}^{c} - c_{33}^{c}\right)\right)\Big], \end{aligned} \tag{I.13a}$$

$$H_{22}^{a} = \frac{(c_{13}^{c} + c_{33}^{c})(r_{bc} - r_{tc})}{r_{bc}r_{tc}}\left[\left(r_{bc} + r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc}\right], \tag{I.13b}$$

$$\begin{aligned} H_{23}^{a} = \frac{1}{r_{bc}r_{tc}}\Big[&-2r_{tc}\left(c_{13}^{c} + c_{33}^{c}\right)\left(r_{bc} - r_{tc}\right) - 2r_{bc}r_{tc}c_{13}^{c}\ln\left(r_{tc}\right) \\ &+ \ln\left(r_{bc}\right)\left(r_{bc}^2\left(c_{13}^{c} - c_{33}^{c}\right) + r_{tc}^2\left(c_{13}^{c} + c_{33}^{c}\right)\right)\Big], \end{aligned} \tag{I.13c}$$

$$H_{24}^{a} = \frac{c_{13}^{c}(f_b + 2R_b)}{2R_b r_{bc} r_{tc}}\left[2r_{bc}\left(-r_{tc}\ln\left(r_{bc}\right) + r_{bc} - r_{tc}\right) + \left(r_{bc}^2 + r_{tc}^2\right)\ln\left(r_{tc}\right)\right], \tag{I.13d}$$

$$H_{25}^{a} = \frac{c_{13}^{c}(r_{bc} - r_{tc})}{r_{bc}r_{tc}}\left[\left(r_{bc} + r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc}\right], \tag{I.13e}$$

$$H_{26}^{a}=-\frac{c_{13}^{c}(f_t-2R_t)}{2R_t r_{bc} r_{tc}}\left[2r_{tc}\left(r_{bc}\left(-\ln\left(r_{tc}\right)\right)-r_{bc}+r_{tc}\right)+\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{bc}\right)\right], \quad (I.13f)$$

$$H_{27}^{a}=\frac{c_{13}^{c}}{r_{bc}r_{tc}}\left[-2\left(r_{bc}+r_{tc}\right)\left(r_{bc}-r_{tc}\right)^{2}+r_{tc}\ln\left(r_{bc}\right)\left(2r_{bc}^{2}\ln\left(\frac{r_{bc}}{r_{tc}}\right)+r_{bc}^{2}-r_{tc}^{2}\right)\right.$$
$$\left.-r_{bc}\ln\left(r_{tc}\right)\left(2r_{tc}^{2}\ln\left(\frac{r_{bc}}{r_{tc}}\right)+r_{bc}^{2}-r_{tc}^{2}\right)\right], \quad (I.13g)$$

$$H_{28}^{a}=\frac{c_{13}^{c}f_b}{2R_b r_{bc} r_{tc}}\left[2r_{bc}\left(r_{tc}\ln\left(r_{bc}\right)-r_{bc}+r_{tc}\right)-\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{tc}\right)\right], \quad (I.13h)$$

$$H_{29}^{a}=\frac{c_{13}^{c}f_t}{2R_t r_{bc} r_{tc}}\left[2r_{tc}\left(r_{bc}\left(-\ln\left(r_{tc}\right)\right)-r_{bc}+r_{tc}\right)+\left(r_{bc}^{2}+r_{tc}^{2}\right)\ln\left(r_{bc}\right)\right], \quad (I.13i)$$

$$H_{21}^{b}=-\ln\left(r_{tc}\right)\left(r_{bc}c_{13}^{c}\left(\ln\left(r_{tc}\right)+2\right)+2r_{tc}\left(c_{33}^{c}-c_{13}^{c}\right)\right)+r_{bc}c_{13}^{c}\ln\left(r_{bc}\right)^{2}+2r_{bc}c_{33}^{c}\ln\left(r_{bc}\right), \quad (I.14a)$$

$$H_{22}^{b}=c_{13}^{c}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right)-2r_{bc}+2r_{tc}\right], \quad (I.14b)$$

$$H_{23}^{b}=-\ln\left(r_{bc}\right)\left(r_{tc}c_{13}^{c}\left(\ln\left(r_{bc}\right)+2\right)-2r_{bc}\left(c_{13}^{c}-c_{33}^{c}\right)\right)+r_{tc}c_{13}^{c}\ln\left(r_{tc}\right)^{2}+2r_{tc}c_{33}^{c}\ln\left(r_{tc}\right), \quad (I.14c)$$

$$H_{24}^{b} = \frac{c_{13}^{c}(f_b + 2R_b)}{2R_b}\left[r_{bc}\ln\left(r_{bc}\right)^{2} - \ln\left(r_{tc}\right)\left(r_{bc}\ln\left(r_{tc}\right) + 2r_{bc} - 2r_{tc}\right)\right], \qquad (I.14d)$$

$$H_{25}^{b} = c_{13}^{c}\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[\left(r_{bc} + r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc}\right], \qquad (I.14e)$$

$$H_{26}^{b} = \frac{c_{13}^{c}(f_t - 2R_t)}{2R_t}\left[r_{tc}\ln\left(r_{bc}\right)^{2} + 2\left(r_{tc} - r_{bc}\right)\ln\left(r_{bc}\right) - r_{tc}\ln\left(r_{tc}\right)^{2}\right], \qquad (I.14f)$$

$$H_{27}^{b} = c_{13}^{c}\left(r_{bc} - r_{tc}\right)\left(-\ln\left(r_{bc}r_{tc}\right)\right)\left[\left(r_{bc} + r_{tc}\right)\ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc}\right], \qquad (I.14g)$$

$$H_{29}^{b} = \frac{c_{13}^{c}f_b}{2R_b}\left[\ln\left(r_{tc}\right)\left(r_{bc}\ln\left(r_{tc}\right) + 2r_{bc} - 2r_{tc}\right) - r_{bc}\ln\left(r_{bc}\right)^{2}\right], \qquad (I.14h)$$

$$H_{29}^{b} = \frac{c_{13}^{c}f_t}{2R_t}\left[r_{tc}\ln\left(r_{tc}\right)^{2} - \ln\left(r_{bc}\right)\left(r_{tc}\ln\left(r_{bc}\right) - 2r_{bc} + 2r_{tc}\right)\right], \qquad (I.14i)$$

$$H_{31}^{a} = \frac{1}{r_{bc}}R_t\left(f_b + 2R_b\right)\left(r_{bc}\ln\left(r_{bc}\right) - r_{tc}\ln\left(r_{tc}\right)\right)\ ; \quad H_{32}^{a} = 0\,, \qquad (I.15a)$$

$$H_{33}^{a} = \frac{1}{r_{tc}}R_b\left(f_t - 2R_t\right)\left(r_{bc}\ln\left(r_{bc}\right) - r_{tc}\ln\left(r_{tc}\right)\right)\ ; \quad H_{34}^{a} = 0\,, \qquad (I.15b)$$

$$H_{35}^{a} = \frac{R_t}{r_{bc}r_{tc}}\left[-r_{bc}r_{tc}\left(f_b + 2R_b\right)\ln\left(r_{bc}\right) + \ln\left(r_{tc}\right)\left(R_b r_{bc}^{2} + r_{tc}^{2}\left(f_b + R_b\right)\right) + 2R_b r_{bc}\left(r_{bc} - r_{tc}\right)\right], \qquad (I.15c)$$

$$H_{36}^{a}=\frac{R_bR_t(r_{bc}-r_{tc})}{r_{bc}r_{tc}}\left[(r_{bc}+r_{tc})\ln\left(\frac{r_{bc}}{r_{tc}}\right)-2r_{bc}+2r_{tc}\right], \qquad (I.15d)$$

$$H_{37}^{a}=\frac{R_b}{r_{bc}r_{tc}}\left[r_{bc}r_{tc}\left(f_t-2R_t\right)\ln\left(r_{tc}\right)+\ln\left(r_{bc}\right)\left(R_t\left(r_{bc}^2+r_{tc}^2\right)-r_{bc}^2f_t\right)+2R_tr_{tc}\left(r_{tc}-r_{bc}\right)\right], \qquad (I.15e)$$

$$H_{31}^{b}=r_{bc}R_t\left(f_b+2R_b\right)\left[\frac{1}{2}\ln\left(r_{tc}\right)\left(-\frac{4r_{tc}}{r_{bc}}+\ln\left(r_{tc}\right)+2\right)-\frac{1}{2}\ln\left(r_{bc}\right)^2+\ln\left(r_{bc}\right)\right], \qquad (I.16a)$$

$$H_{32}^{b}=R_bR_t\ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[2\left(r_{bc}-r_{tc}\right)+\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{tc}}{r_{bc}}\right)\right], \qquad (I.16b)$$

$$H_{33}^{b}=\frac{1}{2}R_b\left(f_t-2R_t\right)\left[-r_{tc}\ln\left(r_{bc}\right)^2+r_{tc}\left(\ln\left(r_{tc}\right)^2-2\ln\left(r_{bc}r_{tc}\right)\right)+4r_{bc}\ln\left(r_{bc}\right)\right], \qquad (I.16c)$$

$$H_{34}^{b}=R_bR_t\left(r_{bc}-r_{tc}\right)\left(-\ln\left(r_{bc}r_{tc}\right)\right)\left[2\left(r_{bc}-r_{tc}\right)+\left(r_{bc}+r_{tc}\right)\ln\left(\frac{r_{tc}}{r_{bc}}\right)\right], \qquad (I.16d)$$

$$H_{35}^{b}=\frac{1}{2}R_t\Big[-\ln\left(r_{tc}\right)\left(r_{bc}\left(f_b+2R_b\right)\ln\left(r_{tc}\right)+2r_{bc}\left(f_b+2R_b\right)-4r_{tc}\left(f_b+R_b\right)\right)$$

$$+r_{bc}\left(f_b+2R_b\right)\ln\left(r_{bc}\right)^2-2f_br_{bc}\ln\left(r_{bc}\right)\Big], \qquad (I.16e)$$

$$H^b_{36} = R_b R_t \ln\left(\frac{r_{bc}}{r_{tc}}\right)\left[(r_{bc}+r_{tc})\ln\left(\frac{r_{bc}}{r_{tc}}\right) - 2r_{bc} + 2r_{tc}\right], \qquad (I.16f)$$

$$H^b_{37} = \frac{1}{2}R_b\Big[\ln(r_{bc})\left(2f_t(r_{tc}-2r_{bc}) + r_{tc}\ln(r_{bc})(f_t-2R_t) + 4R_t(r_{bc}-r_{tc})\right)$$

$$+r_{tc}\left(-(f_t-2R_t)\right)\ln(r_{tc})^2 + 2f_t r_{tc}\ln(r_{tc})\Big], \qquad (I.16g)$$

$$H^c_{31} = \frac{1}{6}R_t(f_b+2R_b)\Big[2r_{bc}\ln(r_{tc})^3 + 9r_{bc}\ln(r_{tc})^2 + 12\ln(r_{tc})\left(-r_{tc}\ln(r_{bc}) + r_{bc} - r_{tc}\right)$$

$$+r_{bc}\left(3-2\ln(r_{bc})\right)\ln(r_{bc})^2\Big], \qquad (I.17a)$$

$$H^c_{32} = \frac{1}{3}R_b R_t \ln\left(\frac{r_{bc}}{r_{tc}}\right)\Big[12(r_{bc}-r_{tc}) + (2r_{bc}+r_{tc})\ln(r_{tc})^2 + \ln(r_{tc})\big(3(3r_{bc}+r_{tc})$$

$$+(r_{tc}-r_{bc})\ln(r_{bc})\big) - \ln(r_{bc})\left(3(r_{bc}+3r_{tc}) + (r_{bc}+2r_{tc})\ln(r_{bc})\right)\Big], \qquad (I.17b)$$

$$H^c_{33} = \frac{1}{6}R_b(f_t-2R_t)\Big[-2r_{tc}\ln(r_{bc})^3 - 9r_{tc}\ln(r_{bc})^2 + 12\ln(r_{bc})\left(r_{bc}\ln(r_{tc}) + r_{bc} - r_{tc}\right)$$

$$+r_{tc}\ln(r_{tc})^2\left(2\ln(r_{tc})-3\right)\Big], \qquad (I.17c)$$

$$H^c_{34} = -\frac{1}{3}R_b R_t\Big[2\left(r_{tc}^2 - r_{bc}^2\right)\ln(r_{bc})^3 + 3\left(r_{bc}^2 + 3r_{tc}^2\right)\ln(r_{bc})^2$$

$$+12r_{tc}\ln(r_{bc})\left(-2r_{bc}\ln(r_{tc}) - r_{bc} + r_{tc}\right) + \ln(r_{tc})\big(12r_{bc}(r_{bc}-r_{tc}) + 2(r_{bc}-r_{tc})(r_{bc}+r_{tc})\ln(r_{tc})^2$$

$$+3\left(3r_{bc}^2+r_{tc}^2\right)\ln\left(r_{tc}\right)\Big)\Big],\qquad (I.17d)$$

$$H_{35}^c=\frac{1}{6}R_t\Big[-2r_{bc}\left(f_b+2R_b\right)\ln\left(r_{tc}\right)^3-3r_{bc}\left(3f_b+4R_b\right)\ln\left(r_{tc}\right)^2$$

$$-12\left(f_b+R_b\right)\ln\left(r_{tc}\right)\left(-r_{tc}\ln\left(r_{bc}\right)+r_{bc}-r_{tc}\right)+r_{bc}\ln\left(r_{bc}\right)^2\left(2\left(f_b+2R_b\right)\ln\left(r_{bc}\right)-3f_b\right)\Big],\qquad (I.17e)$$

$$H_{36}^c=\frac{1}{3}R_bR_t\ln\left(\frac{r_{bc}}{r_{tc}}\right)\Big[\left(r_{bc}+2r_{tc}\right)\ln\left(r_{bc}\right)^2+\ln\left(r_{bc}\right)\left(\left(r_{bc}-r_{tc}\right)\ln\left(r_{tc}\right)+6r_{tc}\right)$$

$$-\ln\left(r_{tc}\right)\left(\left(2r_{bc}+r_{tc}\right)\ln\left(r_{tc}\right)+6r_{bc}\right)-6r_{bc}+6r_{tc}\Big],\qquad (I.17f)$$

$$H_{37}^c=\frac{1}{6}R_b\Big[2r_{tc}\ln\left(r_{bc}\right)^3\left(f_t-2R_t\right)+3r_{tc}\ln\left(r_{bc}\right)^2\left(3f_t-4R_t\right)$$

$$-12\ln\left(r_{bc}\right)\left(f_t-R_t\right)\left(r_{bc}\ln\left(r_{tc}\right)+r_{bc}-r_{tc}\right)+r_{tc}\ln\left(r_{tc}\right)^2\left(3f_t-2\left(f_t-2R_t\right)\ln\left(r_{tc}\right)\right)\Big],\qquad (I.17g)$$

Dynamic - Logarithmic EHSAPT

$$k_1=\rho_c*k_0\ ,\qquad (I.18)$$

$$A_1^m=k_1\frac{f_tr_{\mathrm{tc}}^2}{16R_t^2}(f_t-2R_t)\Big(2r_{\mathrm{bc}}^2\log^2\left(r_{\mathrm{bc}}\right)\left(2\log\left(r_{\mathrm{tc}}\right)+3\right)$$
$$-2r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(3r_{\mathrm{bc}}-4r_{\mathrm{tc}}+4r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\right)-r_{\mathrm{bc}}^2-4r_{\mathrm{bc}}^2\log^3\left(r_{\mathrm{bc}}\right)$$
$$+r_{\mathrm{tc}}^2+2r_{\mathrm{tc}}^2\left(\log\left(r_{\mathrm{tc}}\right)-1\right)\log\left(r_{\mathrm{tc}}\right)\Big),\qquad \textbf{(I.19a)}$$

$$A_2^m = k_1 \frac{r_{\mathrm{bc}} f_t r_{\mathrm{tc}}}{16 R_b R_t} (f_b + 2R_b) \Bigg(2r_{\mathrm{bc}} \log^2(r_{\mathrm{bc}}) (r_{\mathrm{bc}} - 2r_{\mathrm{tc}} \log(r_{\mathrm{tc}}))$$

$$- 2r_{\mathrm{bc}} \log(r_{\mathrm{bc}}) \left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} - 2r_{\mathrm{tc}} \log^2(r_{\mathrm{tc}})\right) + 2r_{\mathrm{tc}} \log(r_{\mathrm{tc}}) (-2r_{\mathrm{bc}} + r_{\mathrm{tc}} - r_{\mathrm{tc}} \log(r_{\mathrm{tc}}))$$

$$- r_{\mathrm{bc}}^2 + r_{\mathrm{tc}}^2 \Bigg) , \quad \text{(I.19b)}$$

$$A_3^m = -k_1 \frac{f_t r_{\mathrm{tc}}}{8R_t} \Bigg(4r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \log^3(r_{\mathrm{bc}}) - 2r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \log^2(r_{\mathrm{bc}}) (4 \log(r_{\mathrm{tc}}) + 1)$$

$$+ r_{\mathrm{bc}} \log(r_{\mathrm{bc}}) (2r_{\mathrm{tc}} \log(r_{\mathrm{tc}}) (2r_{\mathrm{bc}} \log(r_{\mathrm{tc}}) + r_{\mathrm{bc}} + r_{\mathrm{tc}}) - (3r_{\mathrm{bc}} - 7r_{\mathrm{tc}}) (r_{\mathrm{bc}} - r_{\mathrm{tc}}))$$

$$- (r_{\mathrm{bc}} - r_{\mathrm{tc}} + r_{\mathrm{tc}} \log(r_{\mathrm{tc}})) \left(2r_{\mathrm{bc}} r_{\mathrm{tc}} \log(r_{\mathrm{tc}}) + r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right) \Bigg) , \quad \text{(I.19c)}$$

$$A_4^m = k_1 \frac{f_t r_{\mathrm{tc}}}{72 R_t} \Bigg(9 \left(r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)^2 - 36 r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^2 \log^3(r_{\mathrm{bc}})$$

$$- 12 r_{\mathrm{bc}}^2 \log^2(r_{\mathrm{bc}}) \left(-3r_{\mathrm{tc}} (r_{\mathrm{bc}} + r_{\mathrm{tc}}) \log(r_{\mathrm{tc}}) + r_{\mathrm{bc}}^2 - 3r_{\mathrm{tc}}^2\right)$$

$$+2r_{\mathrm{bc}} \log(r_{\mathrm{bc}}) \left(-18 r_{\mathrm{bc}}^2 r_{\mathrm{tc}} - 27 r_{\mathrm{bc}} r_{\mathrm{tc}}^2 - 18 r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \log^2(r_{\mathrm{tc}}) - 3r_{\mathrm{tc}} \left(r_{\mathrm{bc}}^2 + 7r_{\mathrm{tc}}^2\right) \log(r_{\mathrm{tc}}) + 11 r_{\mathrm{bc}}^3 + 34 r_{\mathrm{tc}}^3\right)$$

$$+ 6 \left(3 r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^2 + r_{\mathrm{tc}}^4\right) \log^2(r_{\mathrm{tc}}) - 2r_{\mathrm{tc}} \left(9 r_{\mathrm{bc}}^2 r_{\mathrm{tc}} - 16 r_{\mathrm{bc}}^3 + 7 r_{\mathrm{tc}}^3\right) \log(r_{\mathrm{tc}}) \Bigg) , \quad \text{(I.19d)}$$

$$A_5^m = -k_1 \frac{f_b r_{\mathrm{bc}} f_t r_{\mathrm{tc}}}{16 R_b R_t} \Bigg(2r_{\mathrm{bc}} \log^2(r_{\mathrm{bc}}) (r_{\mathrm{bc}} - 2r_{\mathrm{tc}} \log(r_{\mathrm{tc}})) - 2r_{\mathrm{bc}} \log(r_{\mathrm{bc}}) \left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} - 2r_{\mathrm{tc}} \log^2(r_{\mathrm{tc}})\right)$$

$$+ 2r_{\mathrm{tc}} \log(r_{\mathrm{tc}}) (-2r_{\mathrm{bc}} + r_{\mathrm{tc}} - r_{\mathrm{tc}} \log(r_{\mathrm{tc}})) - r_{\mathrm{bc}}^2 + r_{\mathrm{tc}}^2 \Bigg) , \quad \text{(I.19e)}$$

$$A_6^m = -k_1 \frac{f_t^2 r_{\text{tc}}^2}{16R_t^2}\Bigg(2r_{\text{bc}}^2 \log^2(r_{\text{bc}})(2\log(r_{\text{tc}})+3) - 2r_{\text{bc}}\log(r_{\text{bc}})(3r_{\text{bc}} - 4r_{\text{tc}} + 4r_{\text{tc}}\log(r_{\text{tc}}))$$
$$- r_{\text{bc}}^2 - 4r_{\text{bc}}^2 \log^3(r_{\text{bc}}) + r_{\text{tc}}^2 + 2r_{\text{tc}}^2(\log(r_{\text{tc}})-1)\log(r_{\text{tc}})\Bigg), \quad \text{(I.19f)}$$

$$A_7^m = -k_1 \frac{r_{\text{tc}}^2}{4}\Bigg(2r_{\text{bc}}^2 \log^2(r_{\text{bc}})(2\log(r_{\text{tc}})+3) - 2r_{\text{bc}}\log(r_{\text{bc}})(3r_{\text{bc}} - 4r_{\text{tc}} + 4r_{\text{tc}}\log(r_{\text{tc}}))$$
$$- r_{\text{bc}}^2 - 4r_{\text{bc}}^2 \log^3(r_{\text{bc}}) + r_{\text{tc}}^2 + 2r_{\text{tc}}^2(\log(r_{\text{tc}})-1)\log(r_{\text{tc}})\Bigg), \quad \text{(I.19g)}$$

$$A_8^m = -k_1 \frac{1}{4} r_{\text{bc}} r_{\text{tc}}\Bigg((r_{\text{bc}} - r_{\text{tc}})(r_{\text{bc}} + r_{\text{tc}}) - 2r_{\text{bc}}\log^2(r_{\text{bc}})(r_{\text{bc}} - 2r_{\text{tc}}\log(r_{\text{tc}}))$$
$$+ 2r_{\text{bc}}\log(r_{\text{bc}})\left(r_{\text{bc}} - 2r_{\text{tc}} - 2r_{\text{tc}}\log^2(r_{\text{tc}})\right) + 2r_{\text{tc}}\log(r_{\text{tc}})(2r_{\text{bc}} - r_{\text{tc}} + r_{\text{tc}}\log(r_{\text{tc}}))\Bigg),$$
$$\text{(I.19h)}$$

$$A_9^m = -k_1 \frac{r_{\text{tc}}}{4}\Bigg(4r_{\text{bc}}^2 r_{\text{tc}} \log^3(r_{\text{bc}}) - 2r_{\text{bc}}^2 r_{\text{tc}} \log^2(r_{\text{bc}})(4\log(r_{\text{tc}})+1)$$
$$+ r_{\text{bc}}\log(r_{\text{bc}})(2r_{\text{tc}}\log(r_{\text{tc}})(2r_{\text{bc}}\log(r_{\text{tc}}) + r_{\text{bc}} + r_{\text{tc}}) - (3r_{\text{bc}} - 7r_{\text{tc}})(r_{\text{bc}} - r_{\text{tc}}))$$
$$- (r_{\text{bc}} - r_{\text{tc}} + r_{\text{tc}}\log(r_{\text{tc}}))\left(2r_{\text{bc}} r_{\text{tc}}\log(r_{\text{tc}}) + r_{\text{bc}}^2 - r_{\text{tc}}^2\right)\Bigg), \quad \text{(I.19i)}$$

$$B_1^m = k_1 \frac{f_b r_{\text{bc}} r_{\text{tc}}}{16R_b R_t}(f_t - 2R_t)\Bigg(2r_{\text{bc}}\log^2(r_{\text{bc}})(r_{\text{bc}} - 2r_{\text{tc}}\log(r_{\text{tc}}))$$
$$-2r_{\text{bc}}\log(r_{\text{bc}})\left(r_{\text{bc}} - 2r_{\text{tc}} - 2r_{\text{tc}}\log^2(r_{\text{tc}})\right) + 2r_{\text{tc}}\log(r_{\text{tc}})(-2r_{\text{bc}} + r_{\text{tc}} - r_{\text{tc}}\log(r_{\text{tc}})) - r_{\text{bc}}^2 + r_{\text{tc}}^2\Bigg),$$
$$\text{(I.20a)}$$

$$B_2^m = k_1 \frac{f_b r_{\text{bc}}^2}{16R_b^2}(f_b + 2R_b)\Bigg(- 2r_{\text{tc}}^2 \left(2\log\left(r_{\text{bc}}\right) + 3\right)\log^2\left(r_{\text{tc}}\right)$$

$$+2r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(-4r_{\text{bc}} + 4r_{\text{bc}}\log\left(r_{\text{bc}}\right) + 3r_{\text{tc}}\right) - r_{\text{bc}}^2 - 2r_{\text{bc}}^2\left(\log\left(r_{\text{bc}}\right) - 1\right)\log\left(r_{\text{bc}}\right) + r_{\text{tc}}^2 + 4r_{\text{tc}}^2\log^3\left(r_{\text{tc}}\right)\Bigg), \tag{I.20b}$$

$$B_3^m = -k_1 \frac{f_b r_{\text{bc}}}{8R_b}\Bigg(- \left(r_{\text{bc}} - r_{\text{tc}}\right)^2\left(r_{\text{bc}} + r_{\text{tc}}\right) - 2r_{\text{bc}}r_{\text{tc}}\log^2\left(r_{\text{bc}}\right)\left(r_{\text{bc}} - 2r_{\text{tc}}\log\left(r_{\text{tc}}\right)\right)$$

$$+ r_{\text{bc}}\log\left(r_{\text{bc}}\right)\left(\left(r_{\text{bc}} - r_{\text{tc}}\right)\left(r_{\text{bc}} + 3r_{\text{tc}}\right) + 2r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(r_{\text{bc}} + r_{\text{tc}} - 4r_{\text{tc}}\log\left(r_{\text{tc}}\right)\right)\right)$$

$$+ r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(2r_{\text{bc}}r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(2\log\left(r_{\text{tc}}\right) - 1\right) - \left(7r_{\text{bc}} - 3r_{\text{tc}}\right)\left(r_{\text{bc}} - r_{\text{tc}}\right)\right)\Bigg), \tag{I.20c}$$

$$B_4^m = -k_1 \frac{f_b r_{\text{bc}}}{72R_b}\Bigg(- 9\left(r_{\text{bc}}^2 - r_{\text{tc}}^2\right)^2 + 36r_{\text{bc}}^2 r_{\text{tc}}^2\log^3\left(r_{\text{tc}}\right) + 12r_{\text{tc}}^2\left(r_{\text{tc}}^2 - 3r_{\text{bc}}^2\right)\log^2\left(r_{\text{tc}}\right)$$

$$- 6r_{\text{bc}}\log^2\left(r_{\text{bc}}\right)\left(3r_{\text{bc}}r_{\text{tc}}^2 + r_{\text{bc}}^3 - 6r_{\text{tc}}^3\log\left(r_{\text{tc}}\right)\right)$$

$$+2r_{\text{bc}}\log\left(r_{\text{bc}}\right)\left(9r_{\text{bc}}r_{\text{tc}}^2 - 18r_{\text{tc}}^2\left(r_{\text{bc}} + r_{\text{tc}}\right)\log^2\left(r_{\text{tc}}\right) + 3\left(7r_{\text{bc}}^2 r_{\text{tc}} + r_{\text{tc}}^3\right)\log\left(r_{\text{tc}}\right) + 7r_{\text{bc}}^3 - 16r_{\text{tc}}^3\right)$$

$$+ \left(36r_{\text{bc}}r_{\text{tc}}^3 + 54r_{\text{bc}}^2 r_{\text{tc}}^2 - 68r_{\text{bc}}^3 r_{\text{tc}} - 22r_{\text{tc}}^4\right)\log\left(r_{\text{tc}}\right)\Bigg), \tag{I.20d}$$

$$B_5^m = -k_1 \frac{f_b^2 r_{\text{bc}}^2}{16R_b^2}\Bigg(-2r_{\text{tc}}^2\left(2\log\left(r_{\text{bc}}\right) + 3\right)\log^2\left(r_{\text{tc}}\right) + 2r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(-4r_{\text{bc}} + 4r_{\text{bc}}\log\left(r_{\text{bc}}\right) + 3r_{\text{tc}}\right)$$

$$- r_{\text{bc}}^2 - 2r_{\text{bc}}^2\left(\log\left(r_{\text{bc}}\right) - 1\right)\log\left(r_{\text{bc}}\right) + r_{\text{tc}}^2 + 4r_{\text{tc}}^2\log^3\left(r_{\text{tc}}\right)\Bigg), \tag{I.20e}$$

$$B_6^m = -k_1 \frac{f_b r_{\text{bc}} f_t r_{\text{tc}}}{16 R_b R_t} \Bigg(2 r_{\text{bc}} \log^2 (r_{\text{bc}}) (r_{\text{bc}} - 2 r_{\text{tc}} \log (r_{\text{tc}})) - 2 r_{\text{bc}} \log (r_{\text{bc}}) \left(r_{\text{bc}} - 2 r_{\text{tc}} - 2 r_{\text{tc}} \log^2 (r_{\text{tc}}) \right) \\ + 2 r_{\text{tc}} \log (r_{\text{tc}}) (-2 r_{\text{bc}} + r_{\text{tc}} - r_{\text{tc}} \log (r_{\text{tc}})) - r_{\text{bc}}^2 + r_{\text{tc}}^2 \Bigg) \,, \tag{I.20f}$$

$$B_7^m = -k_1 \frac{1}{4} r_{\text{bc}} r_{\text{tc}} \Bigg((r_{\text{bc}} - r_{\text{tc}}) (r_{\text{bc}} + r_{\text{tc}}) - 2 r_{\text{bc}} \log^2 (r_{\text{bc}}) (r_{\text{bc}} - 2 r_{\text{tc}} \log (r_{\text{tc}})) \\ + 2 r_{\text{bc}} \log (r_{\text{bc}}) \left(r_{\text{bc}} - 2 r_{\text{tc}} - 2 r_{\text{tc}} \log^2 (r_{\text{tc}}) \right) + 2 r_{\text{tc}} \log (r_{\text{tc}}) (2 r_{\text{bc}} - r_{\text{tc}} + r_{\text{tc}} \log (r_{\text{tc}})) \Bigg) \,, \tag{I.20g}$$

$$B_8^m = -k_1 \frac{r_{\text{bc}}^2}{4} \Bigg(-2 r_{\text{tc}}^2 (2 \log (r_{\text{bc}}) + 3) \log^2 (r_{\text{tc}}) + 2 r_{\text{tc}} \log (r_{\text{tc}}) (-4 r_{\text{bc}} + 4 r_{\text{bc}} \log (r_{\text{bc}}) + 3 r_{\text{tc}}) \\ - r_{\text{bc}}^2 - 2 r_{\text{bc}}^2 (\log (r_{\text{bc}}) - 1) \log (r_{\text{bc}}) + r_{\text{tc}}^2 + 4 r_{\text{tc}}^2 \log^3 (r_{\text{tc}}) \Bigg) \,, \tag{I.20h}$$

$$B_9^m = -k_1 \frac{r_{\text{bc}}}{4} \Bigg((r_{\text{bc}} - r_{\text{tc}})^2 (r_{\text{bc}} + r_{\text{tc}}) + 2 r_{\text{bc}} r_{\text{tc}} \log^2 (r_{\text{bc}}) (r_{\text{bc}} - 2 r_{\text{tc}} \log (r_{\text{tc}})) \\ - r_{\text{bc}} \log (r_{\text{bc}}) ((r_{\text{bc}} - r_{\text{tc}}) (r_{\text{bc}} + 3 r_{\text{tc}}) + 2 r_{\text{tc}} \log (r_{\text{tc}}) (r_{\text{bc}} + r_{\text{tc}} - 4 r_{\text{tc}} \log (r_{\text{tc}}))) \\ + r_{\text{tc}} \log (r_{\text{tc}}) ((r_{\text{bc}} - r_{\text{tc}}) (7 r_{\text{bc}} - 3 r_{\text{tc}}) + 2 r_{\text{bc}} r_{\text{tc}} \log (r_{\text{tc}}) (1 - 2 \log (r_{\text{tc}}))) \Bigg) \,, \tag{I.20i}$$

$$C_7^m = -k_1 \frac{r_{\text{tc}}}{4} \Bigg(4 r_{\text{bc}}^2 r_{\text{tc}} \log^3 (r_{\text{bc}}) - 2 r_{\text{bc}}^2 r_{\text{tc}} \log^2 (r_{\text{bc}}) (4 \log (r_{\text{tc}}) + 1) \\ + r_{\text{bc}} \log (r_{\text{bc}}) (2 r_{\text{tc}} \log (r_{\text{tc}}) (2 r_{\text{bc}} \log (r_{\text{tc}}) + r_{\text{bc}} + r_{\text{tc}}) - (3 r_{\text{bc}} - 7 r_{\text{tc}}) (r_{\text{bc}} - r_{\text{tc}})) \\ - (r_{\text{bc}} - r_{\text{tc}} + r_{\text{tc}} \log (r_{\text{tc}})) \left(2 r_{\text{bc}} r_{\text{tc}} \log (r_{\text{tc}}) + r_{\text{bc}}^2 - r_{\text{tc}}^2 \right) \Bigg) \,, \tag{I.21a}$$

$$C_8^m = -k_1 \frac{r_{\text{bc}}}{4}\Bigg(\left(r_{\text{bc}} - r_{\text{tc}}\right)^2 \left(r_{\text{bc}} + r_{\text{tc}}\right) + 2 r_{\text{bc}} r_{\text{tc}} \log^2\left(r_{\text{bc}}\right)\left(r_{\text{bc}} - 2 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\right)$$

$$- r_{\text{bc}} \log\left(r_{\text{bc}}\right)\left(\left(r_{\text{bc}} - r_{\text{tc}}\right)\left(r_{\text{bc}} + 3 r_{\text{tc}}\right) + 2 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\left(r_{\text{bc}} + r_{\text{tc}} - 4 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\right)\right)$$

$$+ r_{\text{tc}} \log\left(r_{\text{tc}}\right)\left(\left(r_{\text{bc}} - r_{\text{tc}}\right)\left(7 r_{\text{bc}} - 3 r_{\text{tc}}\right) + 2 r_{\text{bc}} r_{\text{tc}} \log\left(r_{\text{tc}}\right)\left(1 - 2\log\left(r_{\text{tc}}\right)\right)\right)\Bigg) , \quad \text{(I.21b)}$$

$$C_9^m = k_1 \frac{1}{4}\left(\left(r_{\text{bc}} - r_{\text{tc}}\right)^3 \left(r_{\text{bc}} + r_{\text{tc}}\right) - 2 r_{\text{bc}} r_{\text{tc}} \log\left(\frac{r_{\text{bc}}}{r_{\text{tc}}}\right)\left(3\left(r_{\text{bc}} - r_{\text{tc}}\right)^2 - 2 r_{\text{bc}} r_{\text{tc}} \log^2\left(\frac{r_{\text{tc}}}{r_{\text{bc}}}\right)\right)\right) , \quad \text{(I.21c)}$$

$$D_1^m = -k_1 \frac{r_{\text{tc}}^2}{16 R_t^2}\left(f_t - 2R_t\right)^2 \Bigg(2 r_{\text{bc}}^2 \log^2\left(r_{\text{bc}}\right)\left(2\log\left(r_{\text{tc}}\right) + 3\right)$$

$$-2 r_{\text{bc}} \log\left(r_{\text{bc}}\right)\left(3 r_{\text{bc}} - 4 r_{\text{tc}} + 4 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\right) - r_{\text{bc}}^2 - 4 r_{\text{bc}}^2 \log^3\left(r_{\text{bc}}\right) + r_{\text{tc}}^2 + 2 r_{\text{tc}}^2 \left(\log\left(r_{\text{tc}}\right) - 1\right)\log\left(r_{\text{tc}}\right)\Bigg) , \quad \text{(I.22a)}$$

$$D_2^m = -k_1 \frac{r_{\text{bc}} r_{\text{tc}}}{16 R_b R_t}\left(f_b + 2R_b\right)\left(f_t - 2R_t\right)\Bigg(2 r_{\text{bc}} \log^2\left(r_{\text{bc}}\right)\left(r_{\text{bc}} - 2 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\right)$$

$$-2 r_{\text{bc}} \log\left(r_{\text{bc}}\right)\left(r_{\text{bc}} - 2 r_{\text{tc}} - 2 r_{\text{tc}} \log^2\left(r_{\text{tc}}\right)\right) + 2 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\left(-2 r_{\text{bc}} + r_{\text{tc}} - r_{\text{tc}} \log\left(r_{\text{tc}}\right)\right) - r_{\text{bc}}^2 + r_{\text{tc}}^2\Bigg) , \quad \text{(I.22b)}$$

$$D_3^m = -k_1 \frac{r_{\text{tc}}}{8 R_t}\left(f_t - 2R_t\right)\Bigg(- 4 r_{\text{bc}}^2 r_{\text{tc}} \log^3\left(r_{\text{bc}}\right) + 2 r_{\text{bc}}^2 r_{\text{tc}} \log^2\left(r_{\text{bc}}\right)\left(4\log\left(r_{\text{tc}}\right) + 1\right)$$

$$+ r_{\text{bc}} \log\left(r_{\text{bc}}\right)\left(\left(3 r_{\text{bc}} - 7 r_{\text{tc}}\right)\left(r_{\text{bc}} - r_{\text{tc}}\right) - 2 r_{\text{tc}} \log\left(r_{\text{tc}}\right)\left(2 r_{\text{bc}} \log\left(r_{\text{tc}}\right) + r_{\text{bc}} + r_{\text{tc}}\right)\right)$$

$$+\left(r_{\rm bc}-r_{\rm tc}+r_{\rm tc}\log\left(r_{\rm tc}\right)\right)\left(2r_{\rm bc}r_{\rm tc}\log\left(r_{\rm tc}\right)+r_{\rm bc}^2-r_{\rm tc}^2\right)\Bigg), \quad \text{(I.22c)}$$

$$D_4^m=-k_1\frac{r_{\rm tc}}{72R_t}(f_t-2R_t)\Bigg(9\left(r_{\rm bc}^2-r_{\rm tc}^2\right)^2+6r_{\rm tc}\log^2\left(r_{\rm tc}\right)\left(3r_{\rm bc}^2r_{\rm tc}-6r_{\rm bc}^3\log\left(r_{\rm bc}\right)+r_{\rm tc}^3\right)$$
$$-2r_{\rm tc}\log\left(r_{\rm tc}\right)\left(9r_{\rm bc}^2r_{\rm tc}+3r_{\rm bc}\log\left(r_{\rm bc}\right)\left(-6r_{\rm bc}\left(r_{\rm bc}+r_{\rm tc}\right)\log\left(r_{\rm bc}\right)+r_{\rm bc}^2+7r_{\rm tc}^2\right)-16r_{\rm bc}^3+7r_{\rm tc}^3\right)$$
$$-2r_{\rm bc}\log\left(r_{\rm bc}\right)\left(6r_{\rm bc}\log\left(r_{\rm bc}\right)\left(3r_{\rm tc}^2\log\left(r_{\rm bc}\right)+r_{\rm bc}^2-3r_{\rm tc}^2\right)-\left(r_{\rm bc}-r_{\rm tc}\right)\left(-7r_{\rm bc}r_{\rm tc}+11r_{\rm bc}^2-34r_{\rm tc}^2\right)\right)\Bigg), \quad \text{(I.22d)}$$

$$D_5^m=-k_1\frac{f_br_{\rm bc}r_{\rm tc}}{16R_bR_t}(f_t-2R_t)\Bigg(\left(r_{\rm bc}-r_{\rm tc}\right)\left(r_{\rm bc}+r_{\rm tc}\right)-2r_{\rm bc}\log^2\left(r_{\rm bc}\right)\left(r_{\rm bc}-2r_{\rm tc}\log\left(r_{\rm tc}\right)\right)$$
$$+2r_{\rm bc}\log\left(r_{\rm bc}\right)\left(r_{\rm bc}-2r_{\rm tc}-2r_{\rm tc}\log^2\left(r_{\rm tc}\right)\right)+2r_{\rm tc}\log\left(r_{\rm tc}\right)\left(2r_{\rm bc}-r_{\rm tc}+r_{\rm tc}\log\left(r_{\rm tc}\right)\right)\Bigg), \quad \text{(I.22e)}$$

$$D_6^m=k_1\frac{f_tr_{\rm tc}^2}{16R_t^2}(f_t-2R_t)\Bigg(2r_{\rm bc}^2\log^2\left(r_{\rm bc}\right)\left(2\log\left(r_{\rm tc}\right)+3\right)-2r_{\rm bc}\log\left(r_{\rm bc}\right)\left(3r_{\rm bc}-4r_{\rm tc}+4r_{\rm tc}\log\left(r_{\rm tc}\right)\right)$$
$$-r_{\rm bc}^2-4r_{\rm bc}^2\log^3\left(r_{\rm bc}\right)+r_{\rm tc}^2+2r_{\rm tc}^2\left(\log\left(r_{\rm tc}\right)-1\right)\log\left(r_{\rm tc}\right)\Bigg), \quad \text{(I.22f)}$$

$$E_1^m=-k_1\frac{r_{\rm bc}r_{\rm tc}}{16R_bR_t}\left(f_b+2R_b\right)\left(f_t-2R_t\right)\Bigg(2r_{\rm bc}\log^2\left(r_{\rm bc}\right)\left(r_{\rm bc}-2r_{\rm tc}\log\left(r_{\rm tc}\right)\right)$$
$$-2r_{\rm bc}\log\left(r_{\rm bc}\right)\left(r_{\rm bc}-2r_{\rm tc}-2r_{\rm tc}\log^2\left(r_{\rm tc}\right)\right)+2r_{\rm tc}\log\left(r_{\rm tc}\right)\left(-2r_{\rm bc}+r_{\rm tc}-r_{\rm tc}\log\left(r_{\rm tc}\right)\right)-r_{\rm bc}^2+r_{\rm tc}^2\Bigg), \quad \text{(I.23a)}$$

$$E_2^m = -k_1 \frac{r_{\mathrm{bc}}^2}{16R_b^2} \left(f_b + 2R_b\right)^2 \Bigg(- 2r_{\mathrm{tc}}^2 \left(2\log\left(r_{\mathrm{bc}}\right) + 3\right) \log^2\left(r_{\mathrm{tc}}\right)$$

$$+2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(-4r_{\mathrm{bc}} + 4r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) + 3r_{\mathrm{tc}}\right) - r_{\mathrm{bc}}^2 - 2r_{\mathrm{bc}}^2 \left(\log\left(r_{\mathrm{bc}}\right) - 1\right) \log\left(r_{\mathrm{bc}}\right) + r_{\mathrm{tc}}^2 + 4r_{\mathrm{tc}}^2 \log^3\left(r_{\mathrm{tc}}\right) \Bigg) , \tag{I.23b}$$

$$E_3^m = k_1 \frac{r_{\mathrm{bc}}}{8R_b}(f_b+2R_b)\Bigg(-\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^2 \left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - 2r_{\mathrm{bc}} r_{\mathrm{tc}} \log^2\left(r_{\mathrm{bc}}\right) \left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right)$$

$$+ r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) \left(\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + 3r_{\mathrm{tc}}\right) + 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(r_{\mathrm{bc}} + r_{\mathrm{tc}} - 4r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right)\right)$$

$$+ r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(2r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(2\log\left(r_{\mathrm{tc}}\right) - 1\right) - \left(7r_{\mathrm{bc}} - 3r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\right) \Bigg) , \tag{I.23c}$$

$$E_4^m = k_1 \frac{r_{\mathrm{bc}}}{72R_b}(f_b+2R_b)\Bigg(-9\left(r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)^2 + 36r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^2 \log^3\left(r_{\mathrm{tc}}\right) + 12r_{\mathrm{tc}}^2 \left(r_{\mathrm{tc}}^2 - 3r_{\mathrm{bc}}^2\right) \log^2\left(r_{\mathrm{tc}}\right)$$

$$- 6r_{\mathrm{bc}} \log^2\left(r_{\mathrm{bc}}\right) \left(3r_{\mathrm{bc}} r_{\mathrm{tc}}^2 + r_{\mathrm{bc}}^3 - 6r_{\mathrm{tc}}^3 \log\left(r_{\mathrm{tc}}\right)\right)$$

$$+2r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) \left(9r_{\mathrm{bc}} r_{\mathrm{tc}}^2 - 18r_{\mathrm{tc}}^2 \left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) \log^2\left(r_{\mathrm{tc}}\right) + 3\left(7r_{\mathrm{bc}}^2 r_{\mathrm{tc}} + r_{\mathrm{tc}}^3\right) \log\left(r_{\mathrm{tc}}\right) + 7r_{\mathrm{bc}}^3 - 16r_{\mathrm{tc}}^3\right)$$

$$+ \left(36r_{\mathrm{bc}} r_{\mathrm{tc}}^3 + 54r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^2 - 68r_{\mathrm{bc}}^3 r_{\mathrm{tc}} - 22r_{\mathrm{tc}}^4\right) \log\left(r_{\mathrm{tc}}\right) \Bigg) , \tag{I.23d}$$

$$E_5^m = k_1 \frac{f_b r_{\mathrm{bc}}^2}{16R_b^2}(f_b+2R_b)\Bigg(-2r_{\mathrm{tc}}^2 \left(2\log\left(r_{\mathrm{bc}}\right) + 3\right) \log^2\left(r_{\mathrm{tc}}\right) + 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(-4r_{\mathrm{bc}} + 4r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) + 3r_{\mathrm{tc}}\right)$$

$$- r_{\mathrm{bc}}^2 - 2r_{\mathrm{bc}}^2 \left(\log\left(r_{\mathrm{bc}}\right) - 1\right) \log\left(r_{\mathrm{bc}}\right) + r_{\mathrm{tc}}^2 + 4r_{\mathrm{tc}}^2 \log^3\left(r_{\mathrm{tc}}\right) \Bigg) , \tag{I.23e}$$

$$E_6^m = -k_1 \frac{r_{\mathrm{bc}} f_t r_{\mathrm{tc}}}{16R_b R_t}(f_b+2R_b)\Bigg(\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - 2r_{\mathrm{bc}} \log^2\left(r_{\mathrm{bc}}\right) \left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right)$$

$$+ 2r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) \left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} - 2r_{\mathrm{tc}} \log^2\left(r_{\mathrm{tc}}\right)\right) + 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(2r_{\mathrm{bc}} - r_{\mathrm{tc}} + r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right) \Bigg) , \tag{I.23f}$$

$$\begin{aligned} F_1^m = -k_1 \frac{r_{\mathrm{tc}}}{8R_t}(f_t - 2R_t)\Bigg(&- 4r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \log^3\left(r_{\mathrm{bc}}\right) + 2r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \log^2\left(r_{\mathrm{bc}}\right) \left(4\log\left(r_{\mathrm{tc}}\right) + 1\right) \\ &+ r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right) \left(\left(3r_{\mathrm{bc}} - 7r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right) - 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\left(2r_{\mathrm{bc}} \log\left(r_{\mathrm{tc}}\right) + r_{\mathrm{bc}} + r_{\mathrm{tc}}\right)\right) \\ &+ \left(r_{\mathrm{bc}} - r_{\mathrm{tc}} + r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right)\left(2r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) + r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right) \Bigg) , \end{aligned} \tag{I.24a}$$

$$\begin{aligned} F_2^m = k_1 \frac{r_{\mathrm{bc}}}{8R_b}(f_b + 2R_b)\Bigg(&-\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^2 \left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - 2r_{\mathrm{bc}} r_{\mathrm{tc}} \log^2\left(r_{\mathrm{bc}}\right)\left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right) \\ &+ r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right)\left(\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + 3r_{\mathrm{tc}}\right) + 2r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + r_{\mathrm{tc}} - 4r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\right)\right) \\ &+ r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\left(2r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right)\left(2\log\left(r_{\mathrm{tc}}\right) - 1\right) - \left(7r_{\mathrm{bc}} - 3r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\right) \Bigg) , \end{aligned} \tag{I.24b}$$

$$F_3^m = k_1 \frac{1}{4}\left(\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^3 \left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - 2r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(\frac{r_{\mathrm{bc}}}{r_{\mathrm{tc}}}\right)\left(3\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^2 - 2r_{\mathrm{bc}} r_{\mathrm{tc}} \log^2\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right)\right)\right) , \tag{I.24c}$$

$$\begin{aligned} F_4^m = k_1 \frac{1}{36}\Bigg\{ &9\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^3 \left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right)^2 - 36 r_{\mathrm{bc}}^3 r_{\mathrm{tc}}^2 \log^3\left(r_{\mathrm{tc}}\right) \\ &- 12 r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^3 \log^2\left(r_{\mathrm{bc}}\right)\left(3\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right) - 2\right) + 12 r_{\mathrm{bc}} r_{\mathrm{tc}}^2 \left(r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)\log^2\left(r_{\mathrm{tc}}\right) \\ -r_{\mathrm{bc}} \log\left(r_{\mathrm{bc}}\right)\Bigg[&22 r_{\mathrm{bc}}^3 r_{\mathrm{tc}} - 86 r_{\mathrm{bc}} r_{\mathrm{tc}}^3 - 36 r_{\mathrm{bc}}^2 r_{\mathrm{tc}}^2 \log^2\left(r_{\mathrm{tc}}\right) + 18 r_{\mathrm{bc}}^3 r_{\mathrm{tc}} \log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right) - 18 r_{\mathrm{bc}} r_{\mathrm{tc}}^3 \log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right) \\ +6 r_{\mathrm{bc}} r_{\mathrm{tc}} \left(r_{\mathrm{bc}}^2 + 3r_{\mathrm{tc}}^2\right)\log\left(\frac{r_{\mathrm{bc}}}{r_{\mathrm{tc}}}\right) &- 12 r_{\mathrm{tc}}^2 \log\left(r_{\mathrm{tc}}\right)\left(-2 r_{\mathrm{bc}} r_{\mathrm{tc}} + 3 r_{\mathrm{bc}}\left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right)\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right) - r_{\mathrm{bc}}^2 + r_{\mathrm{tc}}^2\right) \end{aligned}$$

$$+ 5r_{\mathrm{bc}}^4 + 59r_{\mathrm{tc}}^4\Big] + r_{\mathrm{tc}}\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^2\left(32r_{\mathrm{bc}}r_{\mathrm{tc}} + 59r_{\mathrm{bc}}^2 + 5r_{\mathrm{tc}}^2\right)\log\left(r_{\mathrm{tc}}\right)\Bigg\}\,, \quad \text{(I.24d)}$$

$$F_5^m = -k_1\frac{f_b r_{\mathrm{bc}}}{8R_b}\Bigg(-\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)^2\left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - 2r_{\mathrm{bc}}r_{\mathrm{tc}}\log^2\left(r_{\mathrm{bc}}\right)\left(r_{\mathrm{bc}} - 2r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\right)$$
$$+ r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + 3r_{\mathrm{tc}}\right) + 2r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} + r_{\mathrm{tc}} - 4r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\right)\right)$$
$$+ r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(2r_{\mathrm{bc}}r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(2\log\left(r_{\mathrm{tc}}\right) - 1\right) - \left(7r_{\mathrm{bc}} - 3r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\right)\Bigg)\,, \quad \text{(I.24e)}$$

$$F_6^m = -k_1\frac{f_t r_{\mathrm{tc}}}{8R_t}\Bigg(4r_{\mathrm{bc}}^2 r_{\mathrm{tc}}\log^3\left(r_{\mathrm{bc}}\right) - 2r_{\mathrm{bc}}^2 r_{\mathrm{tc}}\log^2\left(r_{\mathrm{bc}}\right)\left(4\log\left(r_{\mathrm{tc}}\right) + 1\right)$$
$$+ r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(2r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(2r_{\mathrm{bc}}\log\left(r_{\mathrm{tc}}\right) + r_{\mathrm{bc}} + r_{\mathrm{tc}}\right) - \left(3r_{\mathrm{bc}} - 7r_{\mathrm{tc}}\right)\left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\right)$$
$$- \left(r_{\mathrm{bc}} - r_{\mathrm{tc}} + r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\right)\left(2r_{\mathrm{bc}}r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right) + r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)\Bigg)\,, \quad \text{(I.24f)}$$

$$G_1^m = -k_1\frac{r_{\mathrm{tc}}}{72R_t}(f_t - 2R_t)\Bigg(9\left(r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)^2 + 6r_{\mathrm{tc}}\log^2\left(r_{\mathrm{tc}}\right)\left(3r_{\mathrm{bc}}^2 r_{\mathrm{tc}} - 6r_{\mathrm{bc}}^3\log\left(r_{\mathrm{bc}}\right) + r_{\mathrm{tc}}^3\right)$$
$$-2r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(9r_{\mathrm{bc}}^2 r_{\mathrm{tc}} + 3r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(-6r_{\mathrm{bc}}\left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right)\log\left(r_{\mathrm{bc}}\right) + r_{\mathrm{bc}}^2 + 7r_{\mathrm{tc}}^2\right) - 16r_{\mathrm{bc}}^3 + 7r_{\mathrm{tc}}^3\right)$$
$$-2r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(6r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(3r_{\mathrm{tc}}^2\log\left(r_{\mathrm{bc}}\right) + r_{\mathrm{bc}}^2 - 3r_{\mathrm{tc}}^2\right) - \left(r_{\mathrm{bc}} - r_{\mathrm{tc}}\right)\left(-7r_{\mathrm{bc}}r_{\mathrm{tc}} + 11r_{\mathrm{bc}}^2 - 34r_{\mathrm{tc}}^2\right)\right)\Bigg)\,, \quad \text{(I.25a)}$$

$$G_2^m = -k_1\frac{r_{\mathrm{bc}}}{72R_b}(f_b + 2R_b)\Bigg(9\left(r_{\mathrm{bc}}^2 - r_{\mathrm{tc}}^2\right)^2 + 6r_{\mathrm{bc}}\log^2\left(r_{\mathrm{bc}}\right)\left(3r_{\mathrm{bc}}r_{\mathrm{tc}}^2 + r_{\mathrm{bc}}^3 - 6r_{\mathrm{tc}}^3\log\left(r_{\mathrm{tc}}\right)\right)$$
$$-2r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(9r_{\mathrm{bc}}r_{\mathrm{tc}}^2 + 3r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(-6r_{\mathrm{tc}}\left(r_{\mathrm{bc}} + r_{\mathrm{tc}}\right)\log\left(r_{\mathrm{tc}}\right) + 7r_{\mathrm{bc}}^2 + r_{\mathrm{tc}}^2\right) + 7r_{\mathrm{bc}}^3 - 16r_{\mathrm{tc}}^3\right)$$

$$-2r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(6r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(3r_{\mathrm{bc}}^2\log\left(r_{\mathrm{tc}}\right)-3r_{\mathrm{bc}}^2+r_{\mathrm{tc}}^2\right)-\left(r_{\mathrm{bc}}-r_{\mathrm{tc}}\right)\left(7r_{\mathrm{bc}}r_{\mathrm{tc}}+34r_{\mathrm{bc}}^2-11r_{\mathrm{tc}}^2\right)\right)\Bigg),$$

(I.25b)

$$G_3^m=k_1\frac{1}{36}\Bigg[9\left(r_{\mathrm{bc}}-r_{\mathrm{tc}}\right)^3\left(r_{\mathrm{bc}}+r_{\mathrm{tc}}\right)^2-36r_{\mathrm{bc}}^3r_{\mathrm{tc}}^2\log^3\left(r_{\mathrm{tc}}\right)-12r_{\mathrm{bc}}^2r_{\mathrm{tc}}^3\log^2\left(r_{\mathrm{bc}}\right)\left(3\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right)-2\right)$$

$$+12r_{\mathrm{bc}}r_{\mathrm{tc}}^2\left(r_{\mathrm{bc}}^2-r_{\mathrm{tc}}^2\right)\log^2\left(r_{\mathrm{tc}}\right)$$

$$-r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\Big[22r_{\mathrm{bc}}^3r_{\mathrm{tc}}-86r_{\mathrm{bc}}r_{\mathrm{tc}}^3-36r_{\mathrm{bc}}^2r_{\mathrm{tc}}^2\log^2\left(r_{\mathrm{tc}}\right)+18r_{\mathrm{bc}}^3r_{\mathrm{tc}}\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right)$$

$$-18r_{\mathrm{bc}}r_{\mathrm{tc}}^3\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right)+6r_{\mathrm{bc}}r_{\mathrm{tc}}\left(r_{\mathrm{bc}}^2+3r_{\mathrm{tc}}^2\right)\log\left(\frac{r_{\mathrm{bc}}}{r_{\mathrm{tc}}}\right)$$

$$-12r_{\mathrm{tc}}^2\log\left(r_{\mathrm{tc}}\right)\left(-2r_{\mathrm{bc}}r_{\mathrm{tc}}+3r_{\mathrm{bc}}\left(r_{\mathrm{bc}}+r_{\mathrm{tc}}\right)\log\left(\frac{r_{\mathrm{tc}}}{r_{\mathrm{bc}}}\right)-r_{\mathrm{bc}}^2+r_{\mathrm{tc}}^2\right)+5r_{\mathrm{bc}}^4+59r_{\mathrm{tc}}^4\Big]$$

$$+r_{\mathrm{tc}}\left(r_{\mathrm{bc}}-r_{\mathrm{tc}}\right)^2\left(32r_{\mathrm{bc}}r_{\mathrm{tc}}+59r_{\mathrm{bc}}^2+5r_{\mathrm{tc}}^2\right)\log\left(r_{\mathrm{tc}}\right)\Bigg],\quad\text{(I.25c)}$$

$$G_4^m=k_1\frac{1}{36}\Bigg[9\left(r_{\mathrm{bc}}^2-r_{\mathrm{tc}}^2\right)^3-36r_{\mathrm{bc}}^4r_{\mathrm{tc}}^2\log^3\left(r_{\mathrm{tc}}\right)+36r_{\mathrm{bc}}^2r_{\mathrm{tc}}^4\log^3\left(r_{\mathrm{bc}}\right)$$

$$+3r_{\mathrm{bc}}^2\log^2\left(r_{\mathrm{bc}}\right)\left(8r_{\mathrm{bc}}^2r_{\mathrm{tc}}^2-12r_{\mathrm{tc}}^3\left(2r_{\mathrm{bc}}+r_{\mathrm{tc}}\right)\log\left(r_{\mathrm{tc}}\right)+r_{\mathrm{bc}}^4-9r_{\mathrm{tc}}^4\right)$$

$$-2r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\Big[\left(10r_{\mathrm{bc}}^2r_{\mathrm{tc}}+37r_{\mathrm{bc}}r_{\mathrm{tc}}^2+5r_{\mathrm{bc}}^3+32r_{\mathrm{tc}}^3\right)\left(r_{\mathrm{bc}}-r_{\mathrm{tc}}\right)^2$$

$$-18r_{\mathrm{bc}}^2r_{\mathrm{tc}}^2\left(r_{\mathrm{bc}}+2r_{\mathrm{tc}}\right)\log^2\left(r_{\mathrm{tc}}\right)+15r_{\mathrm{tc}}\left(r_{\mathrm{bc}}^4-r_{\mathrm{tc}}^4\right)\log\left(r_{\mathrm{tc}}\right)\Big]$$

$$-3r_{\mathrm{tc}}^2\left(8r_{\mathrm{bc}}^2r_{\mathrm{tc}}^2-9r_{\mathrm{bc}}^4+r_{\mathrm{tc}}^4\right)\log^2\left(r_{\mathrm{tc}}\right)+2r_{\mathrm{tc}}\left(r_{\mathrm{bc}}-r_{\mathrm{tc}}\right)^2\left(37r_{\mathrm{bc}}^2r_{\mathrm{tc}}+10r_{\mathrm{bc}}r_{\mathrm{tc}}^2+32r_{\mathrm{bc}}^3+5r_{\mathrm{tc}}^3\right)\log\left(r_{\mathrm{tc}}\right)\Bigg],$$

(I.25d)

$$G_5^m=-k_1\frac{f_br_{\mathrm{bc}}}{72R_b}\Bigg(-9\left(r_{\mathrm{bc}}^2-r_{\mathrm{tc}}^2\right)^2-6r_{\mathrm{bc}}\log^2\left(r_{\mathrm{bc}}\right)\left(3r_{\mathrm{bc}}r_{\mathrm{tc}}^2+r_{\mathrm{bc}}^3-6r_{\mathrm{tc}}^3\log\left(r_{\mathrm{tc}}\right)\right)$$

$$+2r_{\mathrm{bc}}\log\left(r_{\mathrm{bc}}\right)\left(9r_{\mathrm{bc}}r_{\mathrm{tc}}^2+3r_{\mathrm{tc}}\log\left(r_{\mathrm{tc}}\right)\left(-6r_{\mathrm{tc}}\left(r_{\mathrm{bc}}+r_{\mathrm{tc}}\right)\log\left(r_{\mathrm{tc}}\right)+7r_{\mathrm{bc}}^2+r_{\mathrm{tc}}^2\right)+7r_{\mathrm{bc}}^3-16r_{\mathrm{tc}}^3\right)$$

$$+2r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(6r_{\text{tc}}\log\left(r_{\text{tc}}\right)\left(3r_{\text{bc}}^2\log\left(r_{\text{tc}}\right)-3r_{\text{bc}}^2+r_{\text{tc}}^2\right)-\left(r_{\text{bc}}-r_{\text{tc}}\right)\left(7r_{\text{bc}}r_{\text{tc}}+34r_{\text{bc}}^2-11r_{\text{tc}}^2\right)\right)\Bigg), \tag{I.25e}$$

$$\begin{aligned}G_6^m = -k_1\frac{f_t r_{\text{tc}}}{72R_t}\Bigg(&-9\left(r_{\text{bc}}^2-r_{\text{tc}}^2\right)^2+36r_{\text{bc}}^2r_{\text{tc}}^2\log^3\left(r_{\text{bc}}\right)\\ &+12r_{\text{bc}}^2\log^2\left(r_{\text{bc}}\right)\left(-3r_{\text{tc}}\left(r_{\text{bc}}+r_{\text{tc}}\right)\log\left(r_{\text{tc}}\right)+r_{\text{bc}}^2-3r_{\text{tc}}^2\right)\\ &+2r_{\text{bc}}\log\left(r_{\text{bc}}\right)\left(18r_{\text{bc}}^2r_{\text{tc}}+27r_{\text{bc}}r_{\text{tc}}^2+18r_{\text{bc}}^2r_{\text{tc}}\log^2\left(r_{\text{tc}}\right)+3r_{\text{tc}}\left(r_{\text{bc}}^2+7r_{\text{tc}}^2\right)\log\left(r_{\text{tc}}\right)-11r_{\text{bc}}^3-34r_{\text{tc}}^3\right)\\ &-6\left(3r_{\text{bc}}^2r_{\text{tc}}^2+r_{\text{tc}}^4\right)\log^2\left(r_{\text{tc}}\right)+2r_{\text{tc}}\left(9r_{\text{bc}}^2r_{\text{tc}}-16r_{\text{bc}}^3+7r_{\text{tc}}^3\right)\log\left(r_{\text{tc}}\right)\Bigg),\end{aligned} \tag{I.25f}$$

APPENDIX B

POLYNOMIAL EHSAPT CONSTANTS

$$A_i^a = \frac{bf_t}{8c^5 R_t}\left((c-R_c)\left(R_c\left(R_c H_{1i}^b - H_{1i}^a\right) + H_{1i}^c\right) + H_{1i}^d\right) \ , \qquad (II.1a)$$

$$A_i^b = \frac{bc_{55}^c}{16c^6 R_b R_t^2}\Bigg(f_t\left(-R_c^2\left(H_{3i}^a + cH_{3i}^b\right) + R_c\left(cH_{3i}^a + H_{3i}^c\right) + R_c^3 H_{3i}^b + cH_{3i}^c + 2H_{3i}^d\right)$$

$$+2cR_t\left((c-R_c)\left(H_{3i}^a - R_c H_{3i}^b\right) + H_{3i}^c\right)\Bigg) \ , \qquad (II.1b)$$

$$A_i^c = \frac{b}{8c^4}\left((c-R_c)\left(-R_c H_{1i}^a + R_c^2 H_{1i}^b + H_{1i}^c\right) + H_{1i}^d\right) \ , \qquad (II.1c)$$

$$B_i^a = \frac{bf_b}{8c^5 R_b}\left(H_{1i}^d - (R_c + c)\left(R_c\left(R_c H_{1i}^b - H_{1i}^a\right) + H_{1i}^c\right)\right) \ , \qquad (II.2a)$$

$$B_i^b = \frac{bc_{55}^c}{16c^6 R_b^2 R_t}\Bigg(-2c^2 R_b\left(H_{3i}^a - R_c H_{3i}^b\right) + f_b\left(R_c\left(-R_c H_{3i}^a + R_c^2 H_{3i}^b + H_{3i}^c\right) + 2H_{3i}^d\right)$$

$$+c\left(2R_b\left(-R_c H_{3i}^a + R_c^2 H_{3i}^b + H_{3i}^c\right) - f_b\left(R_c\left(H_{3i}^a - R_c H_{3i}^b\right) + H_{3i}^c\right)\right)\Bigg) \ , \qquad (II.2b)$$

$$B_i^c = \frac{b}{8c^4}\left(c^2\left(2H_{2i}^b - cH_{2i}^a\right) - 2\left(R_c + c\right)H_{1i}^a + 2R_c\left(R_c + c\right)H_{1i}^b + 2H_{1i}^c\right) \ , \qquad (II.2c)$$

$$C_i^b = \frac{bc_{55}^c}{4c^5 R_b R_t}\left(R_c H_{3i}^a + \left(c^2 - R_c^2\right)H_{3i}^b - H_{3i}^c\right) \ , \qquad (II.3a)$$

$$C_i^c = \frac{b}{2c^4}\left(R_c H_{1i}^a + \left(c^2 - R_c^2\right) H_{1i}^b - c^2 H_{2i}^b - H_{1i}^c\right) \;, \tag{II.3b}$$

$$D_i^b = -\frac{b\left(f_t - 2R_t\right)}{8c^5 R_t}\left(\left(c - R_c\right)\left(-R_c H_{1i}^a + R_c^2 H_{1i}^b + H_{1i}^c\right) + H_{1i}^d\right) \;, \tag{II.4a}$$

$$D_i^c = -\frac{b c_{55}^c\left(f_t - 2R_t\right)}{16c^6 R_b R_t^2}\left(\left(c - R_c\right) R_c H_{3i}^a - \left(c - R_c\right) R_c^2 H_{3i}^b + \left(R_c + c\right) H_{3i}^c + 2H_{3i}^d\right) \;, \tag{II.4b}$$

$$E_i^b = \frac{b\left(f_b + 2R_b\right)}{8c^5 R_b}\left(\left(R_c + c\right)\left(-R_c H_{1i}^a + R_c^2 H_{1i}^b + H_{1i}^c\right) - H_{1i}^d\right) \;, \tag{II.5b}$$

$$E_i^c = \frac{b c_{55}^c\left(f_b + 2R_b\right)}{16c^6 R_b^2 R_t}\left(R_c\left(R_c + c\right) H_{3i}^a - R_c^2\left(R_c + c\right) H_{3i}^b + \left(c - R_c\right) H_{3i}^c - 2H_{3i}^d\right) \;, \tag{II.5c}$$

$$F_i^b = \frac{b}{2c^4}\left(R_c H_{1i}^a + \left(c^2 - R_c^2\right) H_{1i}^b - H_{1i}^c\right) \;, \tag{II.6a}$$

$$F_i^c = -\frac{b c_{55}^c}{4c^5 R_b R_t}\left(R_c H_{3i}^a + \left(c^2 - R_c^2\right) H_{3i}^b + H_{3i}^c\right) \;, \tag{II.6b}$$

$$G_i^b = \frac{b}{2c^4}\left(\left(c^2 - R_c^2\right) H_{1i}^a + \left(R_c^3 - c^2 R_c\right) H_{1i}^b + R_c H_{1i}^c - H_{1i}^d\right) \;, \tag{II.7a}$$

$$G_i^c = \frac{b c_{55}^c}{4c^5 R_b R_t}\left(R_c\left(R_c H_{3i}^a + \left(c^2 - R_c^2\right) H_{3i}^b - H_{3i}^c\right) - 2H_{3i}^d\right) \;, \tag{II.7b}$$

where

$$H_{11}^{a}=\frac{2}{3}c^{2}\left(c\left(c_{11}^{c}+2c_{13}^{c}\right)-3R_{c}c_{13}^{c}\right)\ ;\quad H_{12}^{a}=\frac{8}{3}c^{3}\left(c_{11}^{c}-c_{13}^{c}\right)\ , \tag{II.8a}$$

$$H_{13}^{a}=\frac{2}{3}c^{2}\left(3R_{c}c_{13}^{c}+c\left(c_{11}^{c}+2c_{13}^{c}\right)\right)\ ;\quad H_{14}^{a}=\frac{c^{3}c_{11}^{c}\left(f_{b}+2R_{b}\right)}{3R_{b}}\ , \tag{II.8b}$$

$$H_{15}^{a}=\frac{8}{3}c^{3}c_{11}^{c}\ ,;\quad H_{16}^{a}=-\frac{c^{3}c_{11}^{c}\left(f_{t}-2R_{t}\right)}{3R_{t}}\ , \tag{II.8c}$$

$$H_{17}^{a}=0\ ;\quad H_{18}^{a}=-\frac{c^{3}f_{b}c_{11}^{c}}{3R_{b}}\ ;\quad H_{19}^{a}=\frac{c^{3}f_{t}c_{11}^{c}}{3R_{t}}\ , \tag{II.8d}$$

$$H_{11}^{b}=-2c^{2}c_{13}^{c}-\left(R_{c}+c\right)c_{11}^{c}\left(R_{c}\ln\left(\frac{R_{c}+c}{R_{c}-c}\right)+2c\right)\ , \tag{II.9a}$$

$$H_{12}^{b}=4c_{11}^{c}\left(\frac{1}{2}\left(c^{2}-R_{c}^{2}\right)\ln\left(\frac{R_{c}-c}{R_{c}+c}\right)+cR_{c}\right)\ , \tag{II.9b}$$

$$H_{13}^{b}=2c^{2}c_{13}^{c}+\left(c-R_{c}\right)c_{11}^{c}\left(R_{c}\ln\left(\frac{R_{c}+c}{R_{c}-c}\right)+2c\right)\ , \tag{II.9c}$$

$$H_{14}^{b}=\frac{c_{11}^{c}\left(f_{b}+2R_{b}\right)}{3cR_{b}}\left(\frac{3}{2}R_{c}^{2}\left(R_{c}+c\right)\ln\left(\frac{R_{c}-c}{R_{c}+c}\right)-c\left(c^{2}+3cR_{c}+3R_{c}^{2}\right)\right)\ , \tag{II.9d}$$

$$H_{15}^{b}=4c_{11}^{c}\left(\frac{1}{2}\left(c^{2}-R_{c}^{2}\right)\ln\left(\frac{R_{c}-c}{R_{c}+c}\right)+cR_{c}\right)\ , \tag{II.9e}$$

$$H^b_{16} = \frac{c^c_{11}\left(f_t - 2R_t\right)}{3cR_t}\left(f_t - 2R_t\right)\left(\frac{3}{2}\left(c - R_c\right)R_c^2 \ln\left(\frac{R_c + c}{R_c - c}\right)\right.$$
$$\left. -c\left(c^2 - 3cR_c + 3R_c^2\right)\right), \tag{II.9f}$$

$$H^b_{17} = \frac{4}{3}c^c_{11}\left(2c^3 + \frac{3}{2}R_c\left(c^2 - R_c^2\right)\ln\left(\frac{R_c + c}{R_c - c}\right) - 3cR_c^2\right), \tag{II.9g}$$

$$H^b_{18} = \frac{f_b c^c_{11}}{3cR_b}\left(c\left(c^2 + 3cR_c + 3R_c^2\right) + \frac{3}{2}\left(R_c + c\right)R_c^2 \ln\left(\frac{R_c + c}{R_c - c}\right)\right), \tag{II.9h}$$

$$H^b_{19} = \frac{f_t c^c_{11}}{3cR_t}\left(c\left(c^2 - 3cR_c + 3R_c^2\right) + \frac{3}{2}\left(c - R_c\right)R_c^2 \ln\left(\frac{R_c - c}{R_c + c}\right)\right), \tag{II.9i}$$

$$H^c_{11} = -\frac{2}{3}c^3\left(\left(c - 2R_c\right)c^c_{13} + cc^c_{11}\right); \quad H^c_{12} = -\frac{8}{3}c^3 R_c c^c_{13}, \tag{II.10a}$$

$$H^c_{13} = \frac{2}{3}c^3\left(\left(2R_c + c\right)c^c_{13} + cc^c_{11}\right); \quad H^c_{14} = -\frac{c^4 c^c_{11}\left(f_b + 2R_b\right)}{5R_b}, \tag{II.10b}$$

$$H^c_{15} = 0; \quad H^c_{16} = -\frac{c^4 c^c_{11}\left(f_t - 2R_t\right)}{5R_t}; \quad H^c_{17} = \frac{8}{15}c^5 c^c_{11}, \tag{II.10c}$$

$$H^c_{18} = \frac{c^4 f_b c^c_{11}}{5R_b}; \quad H^c_{19} = \frac{c^4 f_t c^c_{11}}{5R_t}, \tag{II.10d}$$

$$H^d_{11} = \frac{2}{15}c^4\left(\left(6c - 5R_c\right)c^c_{13} + 3cc^c_{11}\right), \tag{II.11a}$$

$$H^d_{12} = \frac{8}{15} c^5 \left(c^c_{11} - 3c^c_{13} \right) \ , \qquad (II.11b)$$

$$H^d_{13} = \frac{2}{15} c^4 \left((5R_c + 6c)\, c^c_{13} + 3cc^c_{11} \right) \ , \qquad (II.11c)$$

$$H^d_{14} = \frac{c^5 c^c_{11} \left(f_b + 2R_b \right)}{5R_b} \ ; \quad H^d_{15} = \frac{8}{15} c^5 c^c_{11} \ , \qquad (II.11d)$$

$$H^d_{16} = -\frac{c^5 c^c_{11} \left(f_t - 2R_t \right)}{5R_t} \ ; \quad H^d_{17} = 0 \ , \qquad (II.11e)$$

$$H^d_{18} = -\frac{c^5 f_b c^c_{11}}{5R_b} \ ; \quad H^d_{19} = \frac{c^5 f_t c^c_{11}}{5R_t} \ , \qquad (II.11f)$$

$$H^a_{21} = \frac{4}{3} \left((2c - 3R_c)\, c^c_{33} + cc^c_{13} \right) \ ; \quad H^a_{22} = \frac{16}{3} c \left(c^c_{13} - c^c_{33} \right) \ , \qquad (II.12a)$$

$$H^a_{23} = \frac{4}{3} \left((3R_c + 2c)\, c^c_{33} + cc^c_{13} \right) \ ; \quad H^a_{24} = \frac{2cc^c_{13} \left(f_b + 2R_b \right)}{3R_b} \ , \qquad (II.12b)$$

$$H^a_{25} = \frac{16}{3} cc^c_{13} \ ; \quad H^a_{26} = -\frac{2cc^c_{13} \left(f_t - 2R_t \right)}{3R_t} \ , \qquad (II.12c)$$

$$H^a_{27} = 0 \ ; \quad H^a_{28} = -\frac{2cf_b c^c_{13}}{3R_b} \ ; \quad H^a_{29} = \frac{2cf_t c^c_{13}}{3R_t} \ , \qquad (II.12d)$$

$$H^b_{21} = -\frac{4}{3} c \left((c - 2R_c)\, c^c_{33} + cc^c_{13} \right) \ ; \quad H^b_{22} = -\frac{16}{3} cR_c c^c_{33} \ , \qquad (II.13a)$$

$$H^b_{23} = \frac{4}{3} c \left((2R_c + c)\, c^c_{33} + cc^c_{13} \right) \ ; \quad H^b_{24} = -\frac{2c^2 c^c_{13} \left(f_b + 2R_b \right)}{5R_b} \ , \qquad (II.13b)$$

$$H^b_{25} = 0\,; \quad H^b_{26} = -\frac{2c^2 c^c_{13}\left(f_t - 2R_t\right)}{5R_t}\ ; \quad H^b_{27} = \frac{16}{15}c^3 c^c_{13}\,, \tag{II.13c}$$

$$H^b_{28} = \frac{2c^2 f_b c^c_{13}}{5R_b}\ ; \quad H^b_{29} = \frac{2c^2 f_t c^c_{13}}{5R_t}\ , \tag{II.13d}$$

$$H^a_{31} = \frac{2}{3}c^3\left(c - 3R_c\right)R_t\left(f_b + 2R_b\right)\ ; \quad H^a_{32} = -\frac{32}{3}c^4 R_b R_t\,, \tag{II.14a}$$

$$H^a_{33} = -\frac{2}{3}c^3 R_b\left(3R_c + c\right)\left(f_t - 2R_t\right)\ ; \quad H^a_{34} = 0\,, \tag{II.14b}$$

$$H^a_{35} = -\frac{2}{3}c^3 R_t\left(f_b\left(c - 3R_c\right) - 2cR_b\right)\ ; \quad H^a_{36} = \frac{16}{3}c^4 R_b R_t\,, \tag{II.14c}$$

$$H^a_{37} = \frac{2}{3}c^3 R_b\left(\left(3R_c + c\right)f_t + 2cR_t\right)\ , \tag{II.14g}$$

$$H^b_{31} = R_t\left(f_b + 2R_b\right)\left(-\frac{4c^3}{3} + 2c^2 R_c + 2cR_c^2 + R_c^2\left(R_c + c\right)\ln\left(\frac{R_c + c}{R_c - c}\right)\right)\ , \tag{II.15a}$$

$$H^b_{32} = -4cR_b R_t\left(2cR_c + \left(c - R_c\right)\left(R_c + c\right)\ln\left(\frac{R_c - c}{R_c + c}\right)\right)\ , \tag{II.15b}$$

$$H^b_{33} = -\frac{2}{3}R_b\left(c\left(2c^2 + 3cR_c - 3R_c^2\right) + \frac{3}{2}\left(R_c - c\right)R_c^2\ln\left(\frac{R_c - c}{R_c + c}\right)\right)\left(f_t - 2R_t\right)\ , \tag{II.15c}$$

$$H^b_{34} = -\frac{8}{3} c R_b R_t \left(2c^3 + \frac{3}{2} R_c \left(c^2 - R_c^2\right) \ln\left(\frac{R_c + c}{R_c - c}\right) - 3cR_c^2 \right) \ , \qquad (II.15d)$$

$$H^b_{35} = \frac{1}{3} R_t \Bigg(4c^3 \left(f_b - 3R_b\right) - 6c^2 R_c \left(f_b + 2R_b\right) - 6c f_b R_c^2$$
$$+ 3R_c \left(R_c + c\right) \ln\left(\frac{R_c - c}{R_c + c}\right) \left(f_b R_c + 2cR_b\right) \Bigg) \ , \qquad (II.15e)$$

$$H^b_{36} = 8cR_b R_t \left(\frac{1}{2} \left(c^2 - R_c^2\right) \ln\left(\frac{R_c - c}{R_c + c}\right) + cR_c \right) \ , \qquad (II.15f)$$

$$H^b_{37} = \frac{2}{3} R_b \Bigg(c \left(\left(2c^2 + 3cR_c - 3R_c^2\right) f_t + 6c\left(c - R_c\right) R_t \right)$$
$$+ \frac{3}{2} \left(c - R_c\right) R_c \ln\left(\frac{R_c + c}{R_c - c}\right) \left(R_c f_t + 2cR_t\right) \Bigg) \ , \qquad (II.15g)$$

$$H^c_{31} = -\frac{4}{15} c^4 \left(3c - 5R_c\right) R_t \left(f_b + 2R_b\right) \ ; \quad H^c_{32} = -\frac{16}{3} c^4 R_b R_c R_t \ , \qquad (II.16a)$$

$$H^c_{33} = -\frac{4}{15} c^4 R_b \left(5R_c + 3c\right) \left(f_t - 2R_t\right) \ ; \quad H^c_{34} = -\frac{16}{5} c^6 R_b R_t \ , \qquad (II.16b)$$

$$H^c_{35} = \frac{4}{15} c^4 R_t \left(f_b \left(3c - 5R_c\right) - 5cR_b\right) \ ; \quad H^c_{36} = 0 \ , \qquad (II.16c)$$

$$H^c_{37} = \frac{4}{15} c^4 R_b \left(\left(5R_c + 3c\right) f_t + 5cR_t \right) \ , \qquad (II.16g)$$

$$H^d_{31} = \frac{2}{5}c^5\left(c - 3R_c\right)R_t\left(f_b + 2R_b\right)\ ;\quad H^d_{32} = -\frac{64}{15}c^6R_bR_t\ , \tag{II.17a}$$

$$H^d_{33} = -\frac{2}{5}c^5R_b\left(3R_c + c\right)\left(f_t - 2R_t\right)\ ;\quad H^d_{34} = -\frac{32}{15}c^6R_bR_cR_t\ , \tag{II.17b}$$

$$H^d_{35} = -\frac{2}{5}c^5R_t\left(f_b\left(c - 3R_c\right) - 2cR_b\right)\ ;\quad H^d_{36} = \frac{16}{15}c^6R_bR_t\ , \tag{II.17c}$$

$$H^d_{37} = \frac{2}{5}c^5R_b\left(\left(3R_c + c\right)f_t + 2cR_t\right)\ , \tag{II.17d}$$

Dynamic - Polynomial EHSAPT

$$A^m_1 = \frac{bc\rho_c\left(6R_c + 5c\right)f_t\left(f_t - 2R_t\right)}{140R_t^2}\ ;\quad A^m_2 = b\rho_c\left(-\frac{cf_bR_cf_t}{140R_bR_t} - \frac{cR_cf_t}{70R_t}\right)\ , \tag{II.18a}$$

$$A^m_3 = b\rho_c\left(-\frac{c^2f_t}{35R_t} - \frac{cR_cf_t}{15R_t}\right)\ ;\quad A^m_4 = -\frac{bc^2\rho_c\left(R_c + c\right)f_t}{35R_t}\ , \tag{II.18b}$$

$$A^m_5 = \frac{bcf_b\rho_cR_cf_t}{140R_bR_t}\ ;\quad A^m_6 = b\rho_c\left(-\frac{c^2f_t^2}{28R_t^2} - \frac{3cR_cf_t^2}{70R_t^2}\right)\ , \tag{II.18c}$$

$$A^m_7 = b\rho_c\left(-\frac{c^2}{5} - \frac{4cR_c}{15}\right)\ ;\quad A^m_8 = \frac{1}{15}bc\rho_cR_c\ , \tag{II.18d}$$

$$A^m_9 = \frac{1}{15}(-2)bc\rho_c\left(R_c + c\right)\ , \tag{II.18e}$$

$$B_1^m = b\rho_c \left(\frac{cf_b R_c}{70 R_b} - \frac{cf_b R_c f_t}{140 R_b R_t} \right) \; ; \quad B_2^m = -\frac{bcf_b \rho_c \left(5c - 6R_c\right)\left(f_b + 2R_b\right)}{140 R_b^2} \; , \tag{II.19a}$$

$$B_3^m = b\rho_c \left(\frac{cf_b R_c}{15 R_b} - \frac{c^2 f_b}{35 R_b} \right) \; ; \quad B_4^m = b\rho_c \left(\frac{c^3 f_b}{35 R_b} - \frac{c^2 f_b R_c}{35 R_b} \right) \; , \tag{II.19b}$$

$$B_5^m = b\rho_c \left(\frac{c^2 f_b^2}{28 R_b^2} - \frac{3c f_b^2 R_c}{70 R_b^2} \right) \; ; \quad B_6^m = \frac{bcf_b \rho_c R_c f_t}{140 R_b R_t} \; , \tag{II.19c}$$

$$B_7^m = \frac{1}{15} bc\rho_c R_c \; ; \quad B_8^m = b\rho_c \left(\frac{c^2}{5} - \frac{4cR_c}{15} \right) \; , \tag{II.19d}$$

$$B_9^m = b\rho_c \left(\frac{2c^2}{15} - \frac{2cR_c}{15} \right) \; , \tag{II.19e}$$

$$C_7^m = \frac{1}{15}(-2) bc\rho_c \left(R_c + c\right) \; ; \quad C_8^m = b\rho_c \left(\frac{2c^2}{15} - \frac{2cR_c}{15} \right) \; , \tag{II.20a}$$

$$C_9^m = \frac{1}{15}(-16) bc\rho_c R_c \; , \tag{II.20b}$$

$$D_1^m = -\frac{bc\rho_c \left(6R_c + 5c\right)\left(f_t - 2R_t\right)^2}{140 R_t^2} \; ; \quad D_2^m = \frac{bc\rho_c R_c \left(f_b + 2R_b\right)\left(f_t - 2R_t\right)}{140 R_b R_t} \; , \tag{II.21a}$$

$$D_3^m = \frac{bc\rho_c \left(7R_c + 3c\right)\left(f_t - 2R_t\right)}{105 R_t} \; ; \quad D_4^m = \frac{bc^2 \rho_c \left(R_c + c\right)\left(f_t - 2R_t\right)}{35 R_t} \; , \tag{II.21b}$$

$$D_5^m = b\rho_c \left(\frac{cf_b R_c}{70R_b} - \frac{cf_b R_c f_t}{140R_b R_t} \right) \; ; \quad D_6^m = \frac{bc\rho_c \left(6R_c + 5c\right) f_t \left(f_t - 2R_t\right)}{140R_t^2} \; , \quad \text{(II.21c)}$$

$$E_1^m = \frac{bc\rho_c R_c \left(f_b + 2R_b\right) \left(f_t - 2R_t\right)}{140R_b R_t} \; ; \quad E_2^m = -\frac{bc\rho_c \left(6R_c - 5c\right) \left(f_b + 2R_b\right)^2}{140R_b^2} \; , \quad \text{(II.22a)}$$

$$E_3^m = -\frac{bc\rho_c \left(7R_c - 3c\right) \left(f_b + 2R_b\right)}{105R_b} \; ; \quad E_4^m = -\frac{bc^2\rho_c \left(c - R_c\right) \left(f_b + 2R_b\right)}{35R_b} \; , \quad \text{(II.22b)}$$

$$E_5^m = -\frac{bcf_b\rho_c \left(5c - 6R_c\right) \left(f_b + 2R_b\right)}{140R_b^2} \; ; \quad E_6^m = b\rho_c \left(-\frac{cf_b R_c f_t}{140R_b R_t} - \frac{cR_c f_t}{70R_t} \right) \; , \quad \text{(II.22c)}$$

$$F_1^m = \frac{bc\rho_c \left(7R_c + 3c\right) \left(f_t - 2R_t\right)}{105R_t} \; ; \quad F_2^m = -\frac{bc\rho_c \left(7R_c - 3c\right) \left(f_b + 2R_b\right)}{105R_b} \; , \quad \text{(II.23a)}$$

$$F_3^m = \frac{1}{15}(-16)bc\rho_c R_c \; ; \quad F_4^m = \frac{1}{105}(-16)bc^3\rho_c \; , \quad \text{(II.23b)}$$

$$F_5^m = b\rho_c \left(\frac{cf_b R_c}{15R_b} - \frac{c^2 f_b}{35R_b} \right) \; ; \quad F_6^m = b\rho_c \left(-\frac{c^2 f_t}{35R_t} - \frac{cR_c f_t}{15R_t} \right) \; , \quad \text{(II.23c)}$$

$$G_1^m = \frac{bc^2\rho_c \left(R_c + c\right) \left(f_t - 2R_t\right)}{35R_t} \; ; \quad G_2^m = -\frac{bc^2\rho_c \left(c - R_c\right) \left(f_b + 2R_b\right)}{35R_b} \; , \quad \text{(II.24a)}$$

$$G_3^m = \frac{1}{105}(-16)bc^3\rho_c\,; \quad G_4^m = \frac{1}{105}(-16)bc^3\rho_c R_c\,, \tag{II.24b}$$

$$G_5^m = b\rho_c\left(\frac{c^3 f_b}{35R_b} - \frac{c^2 f_b R_c}{35R_b}\right)\,; \quad G_6^m = -\frac{bc^2\rho_c\,(R_c + c)\,f_t}{35R_t}\,, \tag{II.24c}$$

APPENDIX C

LOGARITHMIC EHSAPT WITH STRESS RESULTANTS

Let us define,

$$k_2 = \frac{1}{r_{tc}\ln r_{tc} - r_{bc}\ln r_{bc}} \ , \tag{III.1}$$

logarithmic EHSAPT governing equations for $0 < \theta < \alpha$ are:

Top Face Sheet

δw_0^t:

$$\begin{aligned}
&-\frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) \left(Q_{\mathrm{rs1}}^c\right)'(\theta)}{R_t} - \frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) \left(Q_{\mathrm{ss2}}^c\right)''(\theta)}{2R_t} \\
&\quad + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) \left(Q_{\mathrm{rs1}}^c\right)'(\theta) + \frac{k_2 f_t r_{\mathrm{tc}} \left(Q_{\mathrm{rs2}}^c\right)'(\theta)}{2R_t} + \frac{k_2 f_t r_{\mathrm{tc}} \left(Q_{\mathrm{ss2}}^c\right)''(\theta)}{2R_t} \\
&\quad - \frac{k_2 f_t r_{\mathrm{tc}} \left(V_{\mathrm{rs}}^c\right)'(\theta)}{2R_t} - k_2 r_{\mathrm{tc}} \left(Q_{\mathrm{rs2}}^c\right)'(\theta) - \frac{\left(M_{\mathrm{ss}}^t\right)''(\theta)}{R_t} + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) Q_{\mathrm{rr1}}^c(\theta) \\
&\quad - k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) Q_{\mathrm{ss2}}^c(\theta) + k_2 r_{\mathrm{tc}} Q_{\mathrm{ss2}}^c(\theta) + k_2 r_{\mathrm{tc}} N_{\mathrm{ss}}^c(\theta) + N_{\mathrm{ss}}^t(\theta) \\
&\quad = -m_t'(\theta) - R_t q_t(\theta) \ ,
\end{aligned} \tag{III.2a}$$

δu_0^t:

$$\begin{aligned}
&-\frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) \left(Q_{\mathrm{ss2}}^c\right)'(\theta)}{2R_t} + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) \left(Q_{\mathrm{ss2}}^c\right)'(\theta) + \frac{k_2 f_t r_{\mathrm{tc}} \left(Q_{\mathrm{ss2}}^c\right)'(\theta)}{2R_t} \\
&\quad - k_2 r_{\mathrm{tc}} \left(Q_{\mathrm{ss2}}^c\right)'(\theta) - \frac{\left(M_{\mathrm{ss}}^t\right)'(\theta)}{R_t} - \left(N_{\mathrm{ss}}^t\right)'(\theta) - \frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) Q_{\mathrm{rs1}}^c(\theta)}{R_t} \\
&\quad + 2k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln(r_{\mathrm{bc}}) Q_{\mathrm{rs1}}^c(\theta) + \frac{k_2 f_t r_{\mathrm{tc}} Q_{\mathrm{rs2}}^c(\theta)}{2R_t} - \frac{k_2 f_t r_{\mathrm{tc}} V_{\mathrm{rs}}^c(\theta)}{2R_t} - k_2 r_{\mathrm{tc}} Q_{\mathrm{rs2}}^c(\theta) + k_2 r_{\mathrm{tc}} V_{\mathrm{rs}}^c(\theta) \\
&\quad = -m_t(\theta) - R_t n_t(\theta) \ ,
\end{aligned} \tag{III.2b}$$

Bottom Face Sheet

δw_0^b:

$$
\begin{aligned}
&-\frac{k_2 f_b r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{rs1}}^c\right)'(\theta)}{R_b} + \frac{k_2 f_b r_{\text{bc}} \left(Q_{\text{rs2}}^c\right)'(\theta)}{2R_b} + \frac{k_2 f_b r_{\text{bc}} \left(Q_{\text{ss2}}^c\right)''(\theta)}{2R_b} \\
&\qquad - \frac{k_2 f_b r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{ss2}}^c\right)''(\theta)}{2R_b} - \frac{k_2 f_b r_{\text{bc}} \left(V_{\text{rs}}^c\right)'(\theta)}{2R_b} - \frac{\left(M_{\text{ss}}^b\right)''(\theta)}{R_b} \\
&\qquad - k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{rs1}}^c\right)'(\theta) + k_2 r_{\text{bc}} \left(Q_{\text{rs2}}^c\right)'(\theta) + N_{\text{ss}}^b(\theta) \\
&+ k_2 r_{\text{bc}} r_{\text{tc}} \left(-\ln(r_{\text{tc}})\right) Q_{\text{rr1}}^c(\theta) - k_2 r_{\text{bc}} Q_{\text{ss2}}^c(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) Q_{\text{ss2}}^c(\theta) - k_2 r_{\text{bc}} N_{\text{ss}}^c(\theta) \\
&= R_b q_b(\theta) + m_b'(\theta)\ , \quad \text{(III.2c)}
\end{aligned}
$$

δu_0^b:

$$
\begin{aligned}
&\frac{k_2 f_b r_{\text{bc}} \left(Q_{\text{ss2}}^c\right)'(\theta)}{2R_b} - \frac{k_2 f_b r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{ss2}}^c\right)'(\theta)}{2R_b} - \frac{\left(M_{\text{ss}}^b\right)'(\theta)}{R_b} \\
&- \left(N_{\text{ss}}^b\right)'(\theta) + k_2 r_{\text{bc}} \left(Q_{\text{ss2}}^c\right)'(\theta) - k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{ss2}}^c\right)'(\theta) - \frac{k_2 f_b r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) Q_{\text{rs1}}^c(\theta)}{R_b} \\
&+ \frac{k_2 f_b r_{\text{bc}} Q_{\text{rs2}}^c(\theta)}{2R_b} - \frac{k_2 f_b r_{\text{bc}} V_{\text{rs}}^c(\theta)}{2R_b} - 2k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) Q_{\text{rs1}}^c(\theta) + k_2 r_{\text{bc}} Q_{\text{rs2}}^c(\theta) - k_2 r_{\text{bc}} V_{\text{rs}}^c(\theta) \\
&= R_b n_b(\theta) + m_b(\theta)\ , \quad \text{(III.2d)}
\end{aligned}
$$

Core

δw_0^c:

$$
\begin{aligned}
&- k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{bc}}) \left(Q_{\text{rs1}}^c\right)'(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) \left(Q_{\text{rs1}}^c\right)'(\theta) - k_2 r_{\text{bc}} \left(Q_{\text{rs2}}^c\right)'(\theta) \\
&+ k_2 r_{\text{tc}} \left(Q_{\text{rs2}}^c\right)'(\theta) - \left(V_{\text{rs}}^c\right)'(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \left(-\ln(r_{\text{bc}})\right) Q_{\text{rr1}}^c(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) Q_{\text{rr1}}^c(\theta) \\
&+ k_2 r_{\text{bc}} Q_{\text{ss2}}^c(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{bc}}) Q_{\text{ss2}}^c(\theta) - k_2 r_{\text{bc}} r_{\text{tc}} \ln(r_{\text{tc}}) Q_{\text{ss2}}^c(\theta) + k_2 r_{\text{bc}} N_{\text{ss}}^c(\theta) \\
&- k_2 r_{\text{tc}} Q_{\text{ss2}}^c(\theta) - k_2 r_{\text{tc}} N_{\text{ss}}^c(\theta) + N_{\text{ss}}^c(\theta) \\
&= 0\ , \quad \text{(III.2e)}
\end{aligned}
$$

δu_0^c:

$$-k_2 r_{\mathrm{bc}} \left(Q_{\mathrm{ss2}}^c\right)'(\theta) - k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln\left(r_{\mathrm{bc}}\right) \left(Q_{\mathrm{ss2}}^c\right)'(\theta) + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln\left(r_{\mathrm{tc}}\right) \left(Q_{\mathrm{ss2}}^c\right)'(\theta)$$
$$+ k_2 r_{\mathrm{tc}} \left(Q_{\mathrm{ss2}}^c\right)'(\theta) - \left(N_{\mathrm{ss}}^c\right)'(\theta) - 2k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln\left(r_{\mathrm{bc}}\right) Q_{\mathrm{rs1}}^c(\theta) + 2k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \ln\left(r_{\mathrm{tc}}\right) Q_{\mathrm{rs1}}^c(\theta)$$
$$- k_2 r_{\mathrm{bc}} Q_{\mathrm{rs2}}^c(\theta) + k_2 r_{\mathrm{bc}} V_{\mathrm{rs}}^c(\theta) + k_2 r_{\mathrm{tc}} Q_{\mathrm{rs2}}^c(\theta) - k_2 r_{\mathrm{tc}} V_{\mathrm{rs}}^c(\theta) - V_{\mathrm{rs}}^c(\theta)$$
$$= 0\,, \quad \text{(III.2f)}$$

δu_1^c:

$$-k_2 r_{\mathrm{bc}}^2 \left(Q_{\mathrm{ss2}}^c\right)'(\theta) + k_2 r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \ln\left(r_{\mathrm{tc}}\right) \left(Q_{\mathrm{ss2}}^c\right)'(\theta) - k_2 r_{\mathrm{bc}} r_{\mathrm{tc}}^2 \ln\left(r_{\mathrm{bc}}\right) \left(Q_{\mathrm{ss2}}^c\right)'(\theta)$$
$$+ k_2 r_{\mathrm{tc}}^2 \left(Q_{\mathrm{ss2}}^c\right)'(\theta) - \left(M_{\mathrm{ss}}^c\right)'(\theta) + 2k_2 r_{\mathrm{bc}}^2 r_{\mathrm{tc}} \ln\left(r_{\mathrm{tc}}\right) Q_{\mathrm{rs1}}^c(\theta)$$
$$- 2k_2 r_{\mathrm{bc}} r_{\mathrm{tc}}^2 \ln\left(r_{\mathrm{bc}}\right) Q_{\mathrm{rs1}}^c(\theta) - k_2 r_{\mathrm{bc}}^2 Q_{\mathrm{rs2}}^c(\theta) + k_2 r_{\mathrm{bc}}^2 V_{\mathrm{rs}}^c(\theta) + k_2 r_{\mathrm{tc}}^2 Q_{\mathrm{rs2}}^c(\theta) - k_2 r_{\mathrm{tc}}^2 V_{\mathrm{rs}}^c(\theta)$$
$$= 0\,. \quad \text{(III.2g)}$$

Boundary conditions are at θ = 0 and $\theta = \alpha$, read as follows (at each end there are nine boundary conditions, three for each of the two face sheets and three for the core):

Top Face Sheet

Either $\delta w_0^t = 0$ *or*,

$$\frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) \left(Q_{\mathrm{ss2}}^c\right)'(\theta)}{2R_t} - \frac{k_2 f_t r_{\mathrm{tc}} \left(Q_{\mathrm{ss2}}^c\right)'(\theta)}{2R_t} + \frac{\left(M_{\mathrm{ss}}^t\right)'(\theta)}{R_t} + \frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) Q_{\mathrm{rs1}}^c(\theta)}{R_t}$$
$$+ k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \left(-\log\left(r_{\mathrm{bc}}\right)\right) Q_{\mathrm{rs1}}^c(\theta) - \frac{k_2 f_t r_{\mathrm{tc}} Q_{\mathrm{rs2}}^c(\theta)}{2R_t} + \frac{k_2 f_t r_{\mathrm{tc}} V_{\mathrm{rs}}^c(\theta)}{2R_t} + k_2 r_{\mathrm{tc}} Q_{\mathrm{rs2}}^c(\theta)$$
$$= m_t(\theta) - P_t\,, \quad \text{(III.3a)}$$

Either $\delta w_0^{t\,\prime} = 0$ *or* ,

$$-\frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) Q_{\mathrm{ss2}}^c(\theta)}{2R_t} + \frac{k_2 f_t r_{\mathrm{tc}} Q_{\mathrm{ss2}}^c(\theta)}{2R_t} - \frac{M_{\mathrm{ss}}^t(\theta)}{R_t} = -\frac{M_t}{R_t}\,, \quad \text{(III.3b)}$$

Either $\delta u_0^t = 0$ *or,*

$$\frac{k_2 r_{\mathrm{bc}} f_t r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) Q^c_{\mathrm{ss2}}(\theta)}{2R_t} - k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) Q^c_{\mathrm{ss2}}(\theta) - \frac{k_2 f_t r_{\mathrm{tc}} Q^c_{\mathrm{ss2}}(\theta)}{2R_t} + k_2 r_{\mathrm{tc}} Q^c_{\mathrm{ss2}}(\theta) + \frac{M^t_{\mathrm{ss}}(\theta)}{R_t} + N^t_{\mathrm{ss}}(\theta) = \frac{M_t}{R_t} - N_t \ , \quad \text{(III.3c)}$$

<u>*Bottom Face Sheet*</u>

Either $\delta w_0^b = 0$ *or,*

$$-\frac{k_2 f_b r_{\mathrm{bc}} \left(Q^c_{\mathrm{ss2}}\right)'(\theta)}{2R_b} + \frac{k_2 f_b r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) \left(Q^c_{\mathrm{ss2}}\right)'(\theta)}{2R_b} + \frac{\left(M^b_{\mathrm{ss}}\right)'(\theta)}{R_b} + \frac{k_2 f_b r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{rs1}}(\theta)}{R_b} - \frac{k_2 f_b r_{\mathrm{bc}} Q^c_{\mathrm{rs2}}(\theta)}{2R_b} + \frac{k_2 f_b r_{\mathrm{bc}} V^c_{\mathrm{rs}}(\theta)}{2R_b} + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{rs1}}(\theta) - k_2 r_{\mathrm{bc}} Q^c_{\mathrm{rs2}}(\theta) = -m_b(\theta) - P_b \ , \quad \text{(III.3d)}$$

Either $\delta w_0^{b\,\prime} = 0$ *or,*

$$\frac{k_2 f_b r_{\mathrm{bc}} Q^c_{\mathrm{ss2}}(\theta)}{2R_b} - \frac{k_2 f_b r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{ss2}}(\theta)}{2R_b} - \frac{M^b_{\mathrm{ss}}(\theta)}{R_b} = -\frac{M_b}{R_b} \ , \quad \text{(III.3e)}$$

Either $\delta u_0^b = 0$ *or,*

$$-\frac{k_2 f_b r_{\mathrm{bc}} Q^c_{\mathrm{ss2}}(\theta)}{2R_b} + \frac{k_2 f_b r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{ss2}}(\theta)}{2R_b} + \frac{M^b_{\mathrm{ss}}(\theta)}{R_b} + N^b_{\mathrm{ss}}(\theta) - k_2 r_{\mathrm{bc}} Q^c_{\mathrm{ss2}}(\theta) + k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{ss2}}(\theta) = \frac{M_b}{R_b} - N_b \ , \quad \text{(III.3f)}$$

<u>*Core*</u>

Either $\delta w_0^c = 0$ *or,*

$$k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{bc}}\right) Q^c_{\mathrm{rs1}}(\theta) - k_2 r_{\mathrm{bc}} r_{\mathrm{tc}} \log\left(r_{\mathrm{tc}}\right) Q^c_{\mathrm{rs1}}(\theta) + k_2 r_{\mathrm{bc}} Q^c_{\mathrm{rs2}}(\theta) - k_2 r_{\mathrm{tc}} Q^c_{\mathrm{rs2}}(\theta) + V^c_{\mathrm{rs}}(\theta) = 0 \ , \quad \text{(III.3g)}$$

Either $\delta u_0^c = 0$ *or,*

$$k_2 r_{\text{bc}} Q_{\text{ss2}}^c(\theta) + k_2 r_{\text{bc}} r_{\text{tc}} \log\left(r_{\text{bc}}\right) Q_{\text{ss2}}^c(\theta) - k_2 r_{\text{bc}} r_{\text{tc}} \log\left(r_{\text{tc}}\right) Q_{\text{ss2}}^c(\theta)$$
$$- k_2 r_{\text{tc}} Q_{\text{ss2}}^c(\theta) + N_{\text{ss}}^c(\theta) = 0 \,, \quad \text{(III.3h)}$$

Either $\delta u_1^c = 0$ *or,*

$$k_2 r_{\text{bc}}^2 Q_{\text{ss2}}^c(\theta) - k_2 r_{\text{bc}}^2 r_{\text{tc}} \log\left(r_{\text{tc}}\right) Q_{\text{ss2}}^c(\theta) + k_2 r_{\text{bc}} r_{\text{tc}}^2 \log\left(r_{\text{bc}}\right) Q_{\text{ss2}}^c(\theta)$$
$$- k_2 r_{\text{tc}}^2 Q_{\text{ss2}}^c(\theta) + M_{\text{ss}}^c(\theta) = 0 \,, \quad \text{(III.3i)}$$

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