RIEMANNIAN GEOMETRY OF COMPACT METRIC SPACES

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For Jeffrey,

with a glacier's patience.

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SUMMARY

A construction is given for which the Hausdorff measure and dimension of an arbitrary abstract compact metric space (X, d) can be encoded in a spectral triple. By introducing the concept of *resolving sequence* of open covers, conditions are given under which the topology, metric, and Hausdorff measure can be recovered from a spectral triple dependent on such a sequence. The construction holds for arbitrary compact metric spaces, generalizing previous results for fractals, as well as the original setting of manifolds, and also holds when Hausdorff and box dimensions differ—in particular, it does not depend on any self-similarity or regularity conditions on the space. The only restriction on the space is that it have positive s_0-dimensional Hausdorff measure, where s_0 is the Hausdorff dimension of the space, assumed to be finite. Also, X does not need to be embedded in another space, such as $\mathbb{R}^n$.

CHAPTER I

INTRODUCTION

In the present work, a method is given for encoding the data of a compact metric space (X, d) in a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, allowing X to be viewed as a Riemannian manifold, with Hausdorff measure and dimension serving as the volume form and dimension of X. The construction is completely general, extending previous partial results known for fractal spaces, as well as agreeing with the original setting of (commutative) Riemannian geometry. In particular, the construction holds on spaces for which the Hausdorff and box dimensions are not equal. The only condition on the space is that it have positive Hausdorff measure in the dimension of the space, which is necessary to ensure a nontrivial integration theory, and that the dimension is finite and positive.

Main Theorem. *If (X, d) is a compact metric space, there exists a family of spectral triples $(\mathcal{A}, \mathcal{H}, D_\tau)$ from which (X, d) can be recovered, as well as the Hausdorff measure and dimension of X when these are finite and positive.*

The principle innovation of this work is the concept of *resolving sequences* of open covers (Definition 4.1.1). These are fairly intuitive objects and are sufficiently general to encode the space as a topological space (up to homeomorphism—see Theorem 5.1.8). In addition, the choice functions of [51] are adapted to the current, more general setting, giving a family of choices for each resolving sequence. A spectral triple is defined for each choice; the integration theory (trace functional) is independent of the choice, depending only on the resolving sequence (Theorem 5.3.4) so the properties of the resolving sequence govern the integration theory of the triple (Theorem 5.3.4). However, for an arbitrary compact metric space, the original metric can only be recovered by considering a family of choices (Proposition 5.2.2).

Previously, there have been several constructions of spectral triples for fractals [6, 7, 10, 12, 15, 25, 33, 34, 35, 44, 45, 51] as well as conditions for spectral triples to encode the metric data of a compact metric space [6, 14, 15, 49, 59] or the full data of a Riemannian manifold [17, 18, 55]. In fact, analysis of self-similar spaces often relies on a specific resolving sequence determined by the self-similiarity, and a benefit of the more general approach is that it demonstrates the limitations of these individual constructions in cases where they do not recover the complete Hausdorff characteristics (measure and dimension) of the space.

By establishing an operator-theoretic integration theory within the framework of noncommutative geometry, these results provide a conceptual bridge between classical, commutative geometry and geometric topology and fully noncommutative geometry and topology. By encoding metric data in a spectral triple, it is shown that general constructions defined in terms of a Dirac operator in the noncommutative setting correctly generalize the structure of a Riemannian manifold to arbitrary compact metric spaces. In addition, in [51] the Dirac operator of a similar spectral triple determines a Laplace-Beltrami operator on an ultrametric Cantor set, and by a theorem of Fukushima [28], the associated Dirichlet form determines a Markov semigroup on the space giving a notion of Brownian motion on the Cantor set. The goal of the current work is to to provide a framework that would enable such diffusion processes to be defined for arbitrary compact metric spaces.

CHAPTER II

NONCOMMUTATIVE GEOMETRY

Noncommutative geometry provides a framework to study spaces which are "badly-behaved as point sets" [16], and are thus better understood from an algebra that encodes the data of a suitable abstraction of the space. Thus, a space X might be seen as a measure space, a topological space, or a smooth manifold, with the von Neuman algebra $L^{\infty}(X)$, the C*-algebra $C(X)$ of complex-valued continuous functions, or its dense subalgebra $C^{\infty}(X)$ of smooth functions, respectively, providing the algebraic context. This allows the viewpoint of classical Riemannian geometry to be extended both to point-based spaces lacking a smooth structure but possessing a commutative algebra of functions as well as "implied" spaces studied solely from the properties of a noncommutative algebra. The approach of swapping spaces for algebras has its modern origin in the work of Murray and von Neumann on measure spaces and W*-algebras as well as the 1943 paper of Gelfand and Naimark [30], which established the duality between locally compact Hausdorff topological spaces and commutative C*-algebras. The origins of Connes' program to extend this duality to the more refined structures of Riemannian geometry lie in Atiyah's work on K-theory and the subsequent Atiyah-Singer Index Theorem [2].

2.1 C-algebras & Topology: The Gelfand-Naimark Theorems*

Gelfand and Naimark [30] established the properties of C*-algebras and the centrality of these algebras in operator theory.

Definition 2.1.1 (C*-algebra)**.** A Banach algebra is a pair $(\mathcal{A}, \|\cdot\|)$, where $\mathcal{A}$ is an associative algebra over $\mathbb{C}$, $\|\cdot\|$ is a norm on this algebra, and $\mathcal{A}$ is a complete metric space in the norm topology determined by $\|\cdot\|$. A map $* : \mathcal{A} \to S$ is an involution if $*^2 = \mathbf{1}$. A Banach algebra is a *C*-algebra* if it is equipped with an isometric involution $*$ that satisfies the *C^*-condtion*

$$\|a^*a\| = \|a\|^2 \qquad \text{for all } a \in \mathcal{A}$$

There are two primary examples of C*-algebras. The first is the commutative algebra $C_0(X)$ of complex-valued continuous functions on a locally compact Hausdorff space X with the supremum (uniform convergence) norm and involution given by complex-conjugation. If X is compact, then $C(X)$ is a unital algebra (it possesses a multiplicative unit, the constant function 1). A *state* on a C*-algebra $\mathcal{A}$ is a positive linear functional of norm one. For simplicity, assume $\mathcal{A} = C(X)$ for some compact Hausdorff space X (in which case the states represent probability measures on X). The set of all states on $C(X)$ is a closed convex subset of the unit sphere in the dual space of $C(X)$, which is compact by Alaoglu's Theorem. The extreme points of the state space are the *pure states.* A *character* χ on a C*-algebra $\mathcal{A}$ is a multiplicative linear functional on $\mathcal{A}$ (i.e. $\chi(\phi \cdot \psi) = \chi(\phi) \cdot \chi(\psi)$ for all $\phi, \psi \in \mathcal{A}$), and a point $x \in X$ determines a character $\hat{x}$ on $C(X)$ by evaluation: $\hat{x}(f) := f(x)$. Also, x determines a maximal ideal $\mathfrak{m}_x := \{f \in C(X) \mid f(x) = 0\}$ of $C(X)$: those functions vanishing at x. Thus, each point $x \in X$ corresponds to a maximal ideal in $C(X)$ and a pure state on $C(X)$. The first Gelfand-Naimark theorem establishes that all of the pure states on any commutative C*-algebra $\mathcal{A}$ are characters of the form $\hat{x}$, where the points x are elements of the maximal ideal space $\mathrm{spec}(\mathcal{A})$ of $\mathcal{A}$, given the weak-$*$ topology of the state space.

Theorem 2.1.2 (Commutative Gelfand-Naimark Theorem)**.** *Given any commutative C^*-algebra $\mathcal{A}$ there is a locally compact Hausdorff space* $\mathrm{spec}(\mathcal{A})$ *(unique up to homeomorphism), such that $\mathcal{A} \simeq C(\mathrm{spec}(\mathcal{A}))$. $\mathcal{A}$ is unital if and only if* $\mathrm{spec}(\mathcal{A})$ *is compact.*

The second, noncommutative, example of a C*-algebra is $\mathcal{B}(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$ with the operator norm and involution given by taking adjoints. Any subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology is also a C*-algebra.

Theorem 2.1.3 (Noncommutative Gelfand-Naimark Theorem)**.** *Any C^*-algebra is isometrically $*$-isomorphic to a C^*-algebra of bounded operators on a Hilbert Space.*

The content of the Gelfand-Naimark Theorems is that the two examples given above exhaust all possible C*-algebras. The commutative theorem establishes a categorical equivalence between locally compact Hausdorff spaces and commutative C*-algebras. It originally occurred as a lemma in the proof of the noncommutative theorem, which Gelfand and Naimark considered the more important result. The noncommutative theorem ensures that faithful (one-to-one) Hilbert space representations of C*-algebras always exist. The construction was discovered independently by Segal[61], who also established the formal definitions of C*-algebras and their states. For more details, see [19, 63, 1, 60, 21, 52, 39].

Given the relationships between topological spaces, algebras of bounded operators on a Hilbert space, and C*-algebras established by the Gelfand-Naimark Theorems, Gelfand was led to the possibility of a topological basis for the index of an elliptic differential operator [29]. This was resolved by the work of Atiyah and Singer [2], expressing the index of such an operator on a compact manifold in terms of the K-theory of the manifold, establishing the dependence of this index on the topology alone. This led to a theory of abstract elliptic operators, extended by Connes to the concept of a Fredholm module. This laid the groundwork for Connes to show how a Riemannian metric is encoded by a specific elliptic operator (the Dirac operator) on a Riemannian spin manifold.

2.2 Noncommutative Riemannian Geometry

As a motivation for developing an algebraic context for the Riemannian geometry of compact metric spaces, consider the algebras associated with the following two spaces.

2.2.1 The circle $\mathbb{S}^1$

Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and consider the C*-algebra $\mathcal{A} := C(\mathbb{S}^1)$ and the dense subalgebra of smooth functions $C^\infty(X)$. Let $\mathcal{H} := L^2(\mathbb{S}^1, \mu)$, where μ is Lebesque measure, and let $\pi : \mathcal{A} \to \mathcal{H}$ be the (faithful) representation of left multiplication: for any $f \in C(\mathbb{S}^1)$, $\psi(t) \mapsto f(t) \cdot \psi(t)$ for all $\psi \in \mathcal{H}$. Define the unbounded operator $D := -i\frac{d}{dt}$ on the dense subalgabra of smooth functions. On this algebra the identity

$$[D, \pi(f)] = -i\pi\left(\frac{df}{dt}\right)$$

shows that commutation with D represents differentiation. Furthermore,

$$\|[D,\pi(f)]\|_{\mathcal{B}(\mathcal{H})} = \sup_{\|\psi\|_{\mathcal{H}}=1}\left\{\left\|\pi\left(\frac{df}{dt}\right)\psi\right\|_{\mathcal{H}}\right\} = \left\|\frac{df}{dt}\right\|_{\infty} = \mathrm{Lip}(f)$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant for f:

$$\mathrm{Lip}(f) := \sup_{x,y\in X}\left\{\frac{f(x)-f(y)}{x-y}\;\middle|\; x\neq y\right\}$$

As a result, the (Euclidean) metric on $\mathbb{S}^1$ can be recovered via

$$\rho\,(s,t) := \sup_{f\in C^{\infty}(X)}\left\{|f(s)-f(t)| \;\middle|\; \|f'\|_{\infty}\leq 1\right\}$$

since the condition $\|f'\|_{\infty}\leq 1$ is equivalent to selecting functions f such that $\mathrm{Lip}(f)\leq 1$.

2.2.2 Spin Manifolds

Consider a compact Riemannian spin manifold M, with $C^{\infty}\,(M)\subset C(M) =: \mathcal{A}$. Let $\mathcal{H} = L^2\,(M,S)$ be the L^2-sections of the spinor bundle S with inner product

$$\langle\psi,\phi\rangle := \int_M \phi\cdot\psi \;\mathrm{dvol}$$

($\cdot$ represents Clifford multiplication). Let $\mathcal{A}$ act on $\mathcal{H}$ by left multiplication, and let D be the Dirac operator determined by the spin structure (for further details, see[46]). For all $\psi\in\mathcal{H}$,

$$\Big([D,\pi(f)]\,\psi\Big)(x) = (\nabla f)_x\cdot\psi(x)$$

so again, commutation with D represents differentiation. Also, as in the previous example,

$$\rho\,(x,y) := \sup_{f\in C^{\infty}(M)}\left\{|f(x)-f(y)| \;\middle|\; \|[D,\pi(f)]\|_{\mathcal{B}(\mathcal{H})}\leq 1\right\}$$

recovers the original Riemannian metric on M. See [14, 15, 31] for further details.

2.2.3 Spectral Triples

Both of the previous examples feature a faithful representation π of a commutative C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, as well as a differential operator D such that $[D,\pi(f)]$ defines a bounded operator for all f in a dense subalgebra of $\mathcal{A}$. Furthermore, D is an elliptic operator, which implies that it has compact resolvent. From the knowledge of the objects

$(\mathcal{A}, \mathcal{H}, D)$ (together with the representation π) it was possible to recover the Riemannian metric on the original spaces. These properties led Connes to the following axiomatization of this procedure.

Definition 2.2.1 (Spectral Triple [15])**.** A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the data of a C*-algebra $\mathcal{A}$, a faithful $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a densely defined self-adjoint operator D on $\mathcal{H}$ with compact resolvent such that $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ for all a in a dense subalgebra of $\mathcal{A}$.

The two cases above give examples of spectral triples, but the definition makes no assumption on the commutativity of $\mathcal{A}$ or the nature of the dense subalgebra. Further conditions can be placed on $(\mathcal{A}, \mathcal{H}, D)$ to ensure that $\mathcal{A}$ is the algebra of smooth functions on a compact oriented manifold (under further additional conditions it can be shown that $\mathcal{A}$ is the algebra of smooth functions for a compact oriented spin^c manifold) [18]. The investigation of the Riemannian geometry of a compact metric space is thus an intermediate question: $\mathcal{A}$ will be commutative (hence with a dense subalgebra of Lipschitz functions) but no further restrictions will be imposed ab initio. However, recovery of the metric, and the validity of the formula for ρ above, is a subtle question in more general contexts.

2.3 Connes Metric

When $(\mathcal{A}, \mathcal{H}, D)$ is the spectral triple representing a spin manifold as above, the geodesic metric induced by the Riemannian structure on M is fully recoverable by considering the interaction of the Dirac operator and the smooth Lipschitz functions. This construction generalizes to the setting of an arbitrary spectral triple, but may not give a well-defined metric.

Definition 2.3.1 (Connes Metric)**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, and let $\mathcal{S}(\mathcal{A})$ denote the state space of $\mathcal{A}$. For $\phi, \psi \in \mathcal{S}(\mathcal{A})$, the Connes metric is

$$\rho(\phi, \psi) := \sup_{a \in \mathcal{A}} \left\{ |\phi(a) - \psi(a)| \;\middle|\; \|[D, \pi(a)]\|_{\mathcal{B}(\mathcal{H})} \leq 1 \right\}$$

It remains to consider the conditions required for ρ to define a "good" metric on the state space. When $\mathcal{A}$ is commutative, the pure states are identified with $\operatorname{spec}(\mathcal{A})$, which should give a metric space when considered with the restriction of ρ to the pure states. More generally, to be "good", a metric should be a positive-definite symmetric function $\mathcal{S}(\mathcal{A}) \times \mathcal{S}(\mathcal{A}) \to [0, \infty)$, and the metric topology should coincide with the weak-$*$ topology. The conditions for a spectral triple to yield such a metric have been investigated by Pavlović [49] and Rieffel[56, 57, 58, 59], motivating the following definition (proposed by Bellissard [3]).

Definition 2.3.2 (Regular Spectral Triple)**.** A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *regular* if

1. $\mathcal{A}' := \left\{a \in \mathcal{A} \;\middle|\; [D, \pi(a)] = 0\right\} = \mathbb{C}\mathbf{1}$, and
2. $\mathcal{B}_1 := \left\{a \in \mathcal{A} \;\middle|\; \|[D, \pi(a)]\|_{\mathcal{B}(\mathcal{H})} \leq 1\right\}$ is precompact in $\mathcal{A}/\mathcal{A}'$

(If $\mathcal{A}$ is not unital, then $\mathcal{A}'$ must be $\{0\}$.) The set $\mathcal{B}_1$ is called the *Lipschitz ball* of $\mathcal{A}$.

Theorem 2.3.3 (Pavlović, 1998)**.** *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular if and only if ρ is a metric on the state space of $\mathcal{A}$ and the metric topology coincides with the weak-$*$ topology.*

In the commutative setting there are several examples [6, 34, 58, 50, 51] of spectral triples with the property $\|[D, \pi(f)]\|_{\mathcal{B}(\mathcal{H})} = \mathrm{Lip}(f)$ for all $f \in \mathcal{A}$. In the case of a compact metric space X, under this condition the Connes metric coincides with the Kantorovič (Wasserstein) metric, defined on the probability measures on X [42, 43]):

$$\rho(\hat{x}, \hat{y}) := \sup\left\{|\hat{x}(f) - \hat{y}(f)| \;\middle|\; \mathrm{Lip}(f) \leq 1\right\}$$

Rieffel extends this construction by noting that the map $a \mapsto \|[D, \pi(a)]\|_{\mathcal{B}(\mathcal{H})}$ is a seminorm on the algebra $\mathcal{A}$. If L is a seminorm on $\mathcal{A}$, let ρ_L be the metric on the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ derived by replacing $\|[D, \pi(a)]\|_{\mathcal{B}(\mathcal{H})}$ by $L(a)$ in the definition of ρ:

$$\rho_L(\phi, \psi) = \sup_{a \in \mathcal{A}}\left\{|\phi(a) - \psi(a)| \;\middle|\; L(a) \leq 1\right\}$$

A seminorm L is called a *Lip-norm*, and the pair $(\mathcal{A}, L)$ a *compact quantum metric space*, if ρ_L induces the weak-$*$ topology on $\mathcal{S}(\mathcal{A})$ and vanishes precisely on $\mathbb{C}\mathbf{1}$. Rieffel provides necessary and sufficient conditions for a seminorm to be a lipnorm

Theorem 2.3.4 ([58, Theorem 2.1])**.** *If L is a seminorm on $\mathcal{A}$ such that $L(\mathbf{1}) = 0$, then*

1. *$\mathcal{S}(\mathcal{A})$ has finite diameter under ρ_L if and only if the image of $\mathcal{B}_1$ in the quotient $\mathcal{A}/\mathbb{C}\mathbf{1}$ is bounded, and*

2. *ρ_L induces the weak-$*$ topology on $\mathcal{S}(\mathcal{A})$ if and only if the image of $\mathcal{B}_1$ in $\mathcal{A}/\mathbb{C}\mathbf{1}$ is totally bounded.*

In fact, all such lip-norms arise in a manner analagous to the Connes metric: as a commutator with a Dirac operator [58]. Reiffel has subsequently generalized the Gromov-Hausdorff distance introduced by Gromov [32] to a quantum Gromov-Hausdorff distance between compact quantum metric spaces [59].

2.4 Noncommutative Integration and the Dixmier Trace

Given an algebra $\mathcal{A}$ and a theory of noncommutative differential forms implemented by the Dirac operator (see [15]), a trace functional is required in order to compute with them (that is, to provide a theory of cocycles).

As the closure of the space of finite rank operators, the ideal of compact operators provides the "infinitesimals" of noncommutative geometry. For a compact operator T, let $\{\mu_n(T)\}_{n\in\mathbb{N}}$ denote the set of singular values of T (the eigenvalues of the positive operator $|T|$), in decreasing order. A magnitude, or dimension, of such an inifinitesimal is given by the asymptotic behavior of $\mu_n(T)$. Thus, T is of order s if $\mu_n(T) = O(n^s)$. Given a Dirac operator on a Riemannian manifold, the classical length element $\mathrm{d}s$ is replaced by the order-1 infinitesimal $|D|^{-1}$ derived from the Riemannian metric. More generally, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the order of the compact operator $|D|^{-s}$, being the infimum over $s \geq 0$ such that $\mathrm{Tr}\left(|D|^{-s}\right) < \infty$, corresponds to the dimension of the space.

Definition 2.4.1 (Spectral Dimension)**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.

1. $(\mathcal{A}, \mathcal{H}, D)$ is *p-summable* if $|D|^{-p}$ is trace-class ($\mathrm{Tr}\left(|D|^{-p}\right) < \infty$) for some $p > 0$. In general no such p need exist.

2. If $(\mathcal{A}, \mathcal{H}, D)$ is p-summable, then the *zeta-function* of D is the complex function

$$\zeta_D(s) := \mathrm{Tr}\left(|D|^{-s}\right) = \sum_{k=1}^{\infty} \mu_k^s$$

3. The *spectral dimension* of $(\mathcal{A}, \mathcal{H}, D)$ is the (possibly infinite) *abscissa of convergence* s_0 of $\zeta_D(s)$:

$$s_0 := \inf_{s \geq 0} \{\zeta_D(s) < \infty\}$$

Thus, $\zeta_D(s)$ is holomorphic on a half-plane $\{z \in \mathbb{C} \mid \Re(z) > s_0\}$ with a singularity at s_0. In the spin manifold example above, that s_0 is equal to the dimension of the manifold is a result of Weyl.

Interpreting the assignment

$$f \mapsto \int_X f \,\mathrm{d}\mu$$

as a trace on $C(X)$, it becomes clear that in order to develop an integration theory for a spectral triple a trace on a strictly wider class of operators than the trace class operators $\mathcal{L}^1(F)$ is required; if the singular values of $|D|^{-1}$ obey $\mu_n(T) = O(n^{-1})$, then it follows that $\mathrm{Tr}\left(|D|^{-1}\right) = \infty$. The solution to this problem was provided by Dixmier [20]:

Definition 2.4.2 (Dixmier Trace)**.** Given a compact operator T, the limit

$$\mathrm{Tr}_{\mathrm{Dix}}(T) := \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T) \tag{2.4.1}$$

need not exist. For operators such that this limit does exist, the map

$$T \mapsto \mathrm{Tr}_{\mathrm{Dix}}(T)$$

is a trace and T is called *measurable*. The measurable operators form an ideal in $\mathcal{B}(\mathcal{H})$, which contains the ideal $\mathcal{L}^1(\mathcal{H})$ of trace-class operators (since the Dixmier trace vanishes for such operators). Any trace that vanishes on $\mathcal{L}^1(\mathcal{H})$ is called a *singular trace*.

For measurable operators, the Hardy-Littlewood Tauberian Theorem [36, Theorem 98] provides the relationship between the Dixmier trace and the classical (normal) trace (see [15, 31] for further details):

$$\mathrm{Tr}_{\mathrm{Dix}}\left(|D|^{-1}\pi(a)\right) = \operatorname*{Res}_{s=s_0} \mathrm{Tr}\left(|D|^{-s}\pi(a)\right)$$

In general, the limit (2.4.1) need not exist, and a family of singular traces Tr_ω indexed by a limiting process ω can be defined. The question of which operators admit such a trace in general is somewhat involved [8, 9, 34], and it is possible to avoid some of these questions by considering a related construction which does not depend on the order of the pole of ζ_D at s_0. For any $s > s_0$,

$$a \mapsto \frac{1}{\zeta_D(s)} \mathrm{Tr}\left(|D|^{-s}\pi(a)\right)$$

is a state on $\mathcal{A}$—by Alaoglu's Theorem the state space $\mathcal{S}(\mathcal{A})$ is weak-$*$ compact, so there exist limiting states

$$\omega_D(a) = \lim_{s\to s_0} \frac{1}{\zeta_D(s)} \mathrm{Tr}\left(|D|^{-s}\pi(a)\right)$$

Definition 2.4.3. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *spectrally regular* if the limit ω_D is unique.

In the case of the compact spin manifold above, $(\mathcal{A}, \mathcal{H}, D)$ is spectrally regular and ω_D is the (normalized) volume form.

2.5 Noncommutative Fractal Geometry

The immediate predecessor to the current work is the PhD thesis of Pearson ([50]). Motivated by an attempt to understand the properties of metrics on the transversal of the hull of a tiling space, a family of spectral triples was constructed which allowed recovery of an ultrametric on a Cantor set, and, under mild conditions on the space, the upper box dimension. This construction was unique in that it did not rely on an embedding in $\mathbb{R}^n$ to encode the metric data. In addition, in some cases the construction allowed for the definition of a probability measure and a specialized Laplace-Beltrami operator uniquely suited to the context of Cantor sets, where all continuous functions are harmonic. Two important insights of this work are the need to look at "choice functions", which provide an analogue of tangent vectors, and the need to look at a family of spectral triples indexed by these choice functions—as discussed by Buyalo [6] and Rieffel [59], compactness of the resolvent for the Dirac operator prohibits a single spectral triple from encoding the metric for a completely general compact metric space.

There have been several other attempts to use the framework of noncommutative geometry to obtain results in fractal analysis. Connes initiated this approach by showing how to recover some information for a spectral triple for the triadic Cantor set embedded in $\mathbb{R}$, as well as a Julia set [15].

Guido and Isola have constructed a spectral triple for a limit fractal that recovers the metric and the Minkowski dimension and measure (up to a constant), but the construction depends on self-similarity properties of the limit fractal rather than fine topological or algebraic properties [34]. It also requires the fractal to be embedded in $\mathbb{R}$ with the induced metric from $\mathbb{R}$ (extended to $\mathbb{R}^n$ in [35]). Similarly, Falconer and Samuel give an analagous construction for multifractals [25]. In these constructions, the spaces are sufficiently regular to allow a construction with a unique Dixmier trace (independent of the limit functional) implementing the Minkowski measure. Furthermore, the Minkowski dimensions and the Hausdorff dimension coincide, as do the respective measures.

Detailed surveys of the interplay between noncommutative geometry and fractals are found in [44, 45], and careful consideration of which conditions must be sacrificed to recover the metric vs. the dimension and summability properties (particularly for fractals) are found in various papers by Christensen and Ivan and collaborators [11, 13]. Metrics can be approximated arbitrarily well while also recovering the upper box dimension [6, 11]; as will be shown, when these constructions impose a lower bound condition on the open covers in a resolving sequence [11] it is then impossible to recover anything below the box dimensions. Buyalo [6] gives a criterion which is sufficient to ensure that a spectral triple can recover a the metric for an arbitrary compact space with an intrinsic metric.

2.6 Spectral Triples for Compact Metric Spaces

The task remains to build a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for a compact metric space (X, d) such that the order s_0 of $|D|^{-1}$ is equal to the Hausdorff dimension of X, the Hausdorff measure $\mathcal{H}^{s_0}$ is represented by a functional of the form

$$f \mapsto \lim_{s \to s_0} \frac{1}{\zeta_D(s)} \left(|D|^{-s_0} \pi(f)\right)$$

and the metric d is equal to the Connes metric ρ induced by $(\mathcal{A}, \mathcal{H}, D)$. The triple $(\mathcal{A}, \mathcal{H}, D)$ will necessarily have the property that when X is a Riemannian manifold and d is the geodesic metric, these are recovered from $(\mathcal{A}, \mathcal{H}, D)$. (For a detailed discussion of how spectral triples encode the dimension, metric and volume form (as well as the Yang-Mills functional and DeRham cohomology) of a compact Riemannian spin manifold see [15, VI.1].

For results on the necessary conditions on a spectral triple to recover the full data of such a manifold, see [18].)

Attempts to build spectral triples for general (non-fractal) compact metric spaces have been made by Buyalo [6] and Christensen and Ivan [11]. Buyalo's construction is quite general, and conditions are given on the regularity of a space (or q-quasihomogeneity), satisfied by self-similar fractals and Riemannian manifolds, for which the Hausdorff dimension and measure can be recovered. The Minkowski and Hausdorff dimensions coincide for such spaces, and Buyalo shows that in general the upper Minkowki dimension is a lower bound for the spectral dimension in his construction. Christensen and Ivan show that in general it is possible to define different spectral triples for the same space, and both their algorithm and Buyalo's fail to characterize compact metric spaces in general.

It is not the goal of the present work to provide such a characterization. As the work of Connes has showed in the case of smooth manifolds [18], characterizing the conditions on a spectral triple to ensure that it encodes the full data of a manifold is quite involved. Rather, a method is demonstrated to be sufficient for recovering the Hausdorff dimension and measure, as well as the original metric, for (almost) all compact metric spaces, including those for which the Hausdorff and Minkowski dimensions differ. To do this, the construction necessarily does not depend on any regularity or homogeneity properties of the space—no assumptions are made other than the existence of a metric for which the metric topology is compact.

CHAPTER III

BACKGROUND

Let (X, d) be a compact metric space, topologized with the metric topology. For any subset $E \subseteq X$, let $\overline{E}$ denote the closure of E, $E^C = X \backslash E$ its complement in X, and let χ_E denote the characteristic function of E: that function taking the value 1 on E and 0 elsewhere. A *ball* in X is any open set taking the form $B(x, r) := \{x' \in X \mid d(x, x') < r\}$ for some $x \in X$ and some $r > 0$; a ball in $\mathbb{C}$ centered at $z \in \mathbb{C}$ is a *disk* $D(z, r) = \big\{w \in \mathbb{C} \bigm| |z - w| < r\big\}$. The balls in X generate the topology on X.

3.1 Open Covers

Given a subset $E \subseteq X$, a *cover* $\mathcal{U}$ of E is any collection $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ of sets $E_\alpha \subseteq X$ such that $E \subseteq \bigcup_{\alpha \in A} E_\alpha$; in this case, the collection $\mathcal{E}$ *covers* E. Let $|\mathcal{E}|$ denote the cardinality of the cover $\mathcal{E}$. The diameter of a nonempty set $E \subseteq X$ is the nonnegative number

$$\operatorname{diam}(E) := \sup \{d(x, y) \mid x, y \in E\}$$

and the diameter of a cover $\mathcal{E}$ of E is the supremum of the diameters of its members: $\operatorname{diam}(\mathcal{E}) := \sup \{\operatorname{diam}(E) \mid U \in \mathcal{E}\}$. A cover is finite if $|\mathcal{E}| \in \mathbb{N}$, *minimal* if every proper subcollection of $\mathcal{E}$ fails to cover E, and *metric* if it consists of (open) metric balls. Finally, a cover $\mathcal{U}$ is *open* if it consists only of open sets $U \subseteq X$. Henceforth, all covers will be assumed to be open and at most countable; the family of all countable open covers $\mathcal{U}$ of E is denoted $\Gamma(E)$, and for any $\delta > 0$, let

$$\Gamma_\delta(E) := \Big\{\mathcal{U} \in \Gamma(E) \Bigm| \operatorname{diam}(\mathcal{U}) \leq \delta\Big\}$$

Remark. If X is completely disconnected, then a partition of X into clopen sets is also an open cover (cf. [50]). More generally, a partition of X into collections of connected components is an open cover. Such a partition of X is the only possible cover of X into disjoint sets.

Definition 3.1.1 (Refinement)**.** A cover $\mathcal{U}$ is *refined* by a cover $\mathcal{U}'$, written $\mathcal{U} \preccurlyeq \mathcal{U}'$, if for all $U' \in \mathcal{U}'$, there exists some $U \in \mathcal{U}$ such that $U' \subseteq U$. Refinement is a reflexive, transitive relation on $\Gamma(X)$. The refinement is *proper*, written $\mathcal{U} \prec \mathcal{U}'$, if for all $U' \in \mathcal{U}'$ there exists some $U \in \mathcal{U}$ such that $\overline{U'} \subseteq U$.

For $\mathcal{U}, \mathcal{U}' \in \Gamma(X)$ the *join* of $\mathcal{U}$ and $\mathcal{U}'$ is the cover

$$\mathcal{U} \vee \mathcal{U}' := \left\{ U \cap U' \;\middle|\; U \in \mathcal{U}, U' \in \mathcal{U}' \right\}$$

The join of two covers $\mathcal{U}, \mathcal{U}'$ is the least common refinement of $\mathcal{U}$ and $\mathcal{U}'$, in the sense that if $\mathcal{U}''$ refines both $\mathcal{U}$ and $\mathcal{U}'$, then $\mathcal{U}''$ also refines $\mathcal{U} \vee \mathcal{U}'$: given a cover $\mathcal{U}''$ that refines both $\mathcal{U}$ and $\mathcal{U}'$, then any $U'' \in \mathcal{U}''$ is contained in some $U \in \mathcal{U}$ and some $U' \in \mathcal{U}'$, so $U'' \subset U \cap U' \in \mathcal{U} \vee \mathcal{U}'$. Because the join $\mathcal{U} \vee \mathcal{U}'$ refines both $\mathcal{U}$ and $\mathcal{U}'$, $(\Gamma(X), \preccurlyeq)$ is a directed set.

3.2 Hausdorff Dimension and Measure

The following is standard and can be found in books on fractal geometry, such as [24] or more generally, geometric measure theory, such as [26].

The *δ-box number* of a subset $E \subseteq X$ is the real number

$$\mathrm{N}_\delta(E) := \inf \left\{ |\mathcal{U}| \;\middle|\; \mathcal{U} \in \Gamma_\delta(E) \right\}$$

i.e. the minimum number of sets of diameter not greater than δ required to cover E. When E is compact, $\mathrm{N}_\delta(E)$ is finite for all $\delta > 0$.

Definition 3.2.1 (Box Dimension)**.** The *upper box dimension* (also called upper Minkowski dimension) $\overline{\dim}_B(E)$ of E is defined in terms of the exponential growth rate of $\mathrm{N}_\delta(E)$ (with respect to $\frac{1}{\delta}$) as $\delta \downarrow 0$:

$$\overline{\dim}_B(E) := \limsup_{\delta \to 0} \frac{\log \mathrm{N}_\delta(E)}{-\log \delta}$$

The *lower box dimension* (also the lower Minkowski dimension) $\underline{\dim}_B(E)$ of E is defined analogously:

$$\underline{\dim}_B(E) := \liminf_{\delta \to 0} \frac{\log \mathrm{N}_\delta(E)}{-\log \delta}$$

For any $s \geq 0$ and $\delta > 0$, let

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^s \,\middle|\, \mathcal{U} \in \Gamma_\delta(E) \right\}$$

Definition 3.2.2 (Hausdorff Measure)**.** The *s-dimensional Hausdorff measure* of E, is given by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$$

Definition 3.2.3 (Hausdorff Dimension)**.** The *Hausdorff dimension* of E, is

$$\dim_{\mathcal{H}}(E) := \inf \{ s > 0 \mid \mathcal{H}^s(E) = 0 \} = \sup \{ s > 0 \mid \mathcal{H}^s(E) = \infty \}$$

Remark. In some definitions of $\mathcal{H}^s_\delta(E)$ the infimum is taken over all countable covers by **any** sets, not just open sets. While this can result in lower values of $\mathcal{H}^s_\delta(E)$, $\mathcal{H}^s(E)$ is unchanged [26, §2.10]. On the other hand, restricting to metric balls would change $\mathcal{H}^s(E)$, but not the Hausdorff or box dimensions [53]. Though $\Gamma_\delta(E)$ is defined as a collection of *countable* open covers of E, the quantity $\mathcal{H}^s_\delta(E)$ is unchanged if $\Gamma_\delta(E)$ is assumed to consist of *finite* covers of E.

Remark. An important feature of these constructions is the fact that as $\delta \downarrow 0$, $\mathcal{H}^s_\delta(E)$ increases monotonically; since $\mathrm{N}_\delta(E) = \mathcal{H}^0_\delta(E)$, this is true of $\mathrm{N}_\delta(E)$ as well.

In many cases, all of the above-mentioned dimensions coincide. In $\mathbb{R}^n$, they are all equal to n, and n-dimensional Hausdorff measure is equal to Lebesgue measure. In general, the Hausdorff dimension need not be integral, and all of the dimensions are different. To see how the Hausdorff and box dimensions differ, it is useful to pass to a construction of greater generality.

3.3 Caratheodory Structures

In [53], Pesin gives an elegent presentation of Hausdorff and box dimensions in the context of generalized Caratheodory structures [54], which are a generalization of Caratheodory's original method to derive a well-behaved measure from a more arbitrary estimation on the "size" of a set (such as its diameter). A benefit of this approach is that it makes the relationship between Hausdorff dimension and upper and lower box dimensions quite clear.

Definition 3.3.1 (Caratheodory Structure)**.** Let $\mathcal{F}$ be a collection of subsets of a space X, let (ξ, η, ψ) be a triple of nonnegative set functions on $\mathcal{F}$, and for each positive δ, let $\mathcal{F}_\delta := \{E \in \mathcal{F} \mid \psi(E) \leq \delta\}$. Then $\mathcal{F}$ and (ξ, η, ψ) together define a *C-structure* on X if

C1 $\emptyset \in \mathcal{F}$ and $\eta(\emptyset) = \psi(\emptyset) = 0$, and for all nonempty $E \in \mathcal{F}$, $\eta(E), \psi(E) > 0$,

C2 for all $\epsilon > 0$ there exists $\delta > 0$ such that $\eta(E) < \epsilon$ whenever $E \in \mathcal{F}_\delta$, and

C3 $\forall \delta > 0$ there is an at most countable subcollection $\mathcal{G} \subseteq \mathcal{F}_\delta$ covering E.

For any such C-structure on X, a one-parameter family of subadditive set functions $m(\,\cdot\,, \alpha), \alpha \in \mathbb{R}$ can be defined via

$$m(E, \alpha) = \lim_{\delta \to 0} \inf_{\mathcal{G} \subseteq \mathcal{F}_\delta} \sum_{E \in \mathcal{G}} \xi(E) \eta(E)^\alpha$$

(where $\mathcal{G}$ always denotes an at most countable cover of E). The limit is well-defined since the terms are necessarily increasing in δ as $\delta \downarrow 0$. Furthermore, for each $\alpha > 0$, $m(\,\cdot\,, \alpha)$ is an outer measure on X [53, p. 13, Proposition 1.1].

Proposition 3.3.2. *If for some* α_0*,*

1. $m(E, \alpha_0) = \infty$*, then* $m(E, \alpha) = \infty$ *for all* $\alpha < \alpha_0$*, and, if*

2. $m(E, \alpha_0) < \infty$*, then* $m(E, \alpha) = 0$ *for all* $\alpha > \alpha_0$*.*

Proof. For all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\sup\left\{\eta(E) \;\middle|\; E \in \mathcal{F}_{\delta(\epsilon)}\right\} < \epsilon$ by condition C2; therefore, if $\alpha > \alpha_0$,

$$\begin{aligned} m(E, \alpha) &= \lim_{\epsilon \to 0} \inf_{\mathcal{G} \subseteq \mathcal{F}_{\delta(\epsilon)}} \sum_{E \in \mathcal{G}} \xi(E) \eta(E)^\alpha \\ &\leq \lim_{\epsilon \to 0} \inf_{\mathcal{G} \subseteq \mathcal{F}_{\delta(\epsilon)}} \left[\left(\epsilon^{\alpha - \alpha_0}\right) \sum_{E \in \mathcal{G}} \xi(E) \eta(E)^{\alpha_0} \right] = 0 \end{aligned}$$

if $m(E,\alpha_0) < \infty$. Similarly, if $\alpha < \alpha_0$ and $m(E,\alpha_0) = \infty$, then

$$\begin{aligned} m(E,\alpha_0) &= \lim_{\epsilon\to 0} \inf_{\mathcal{G}\subseteq\mathcal{F}_{\delta(\epsilon)}} \sum_{E\in\mathcal{G}} \xi(E)\eta(E)^{\alpha_0} \\ &\leq \lim_{\epsilon\to 0} \inf_{\mathcal{G}\subseteq\mathcal{F}_{\delta(\epsilon)}} \left[\left(\epsilon^{\alpha_0-\alpha}\right) \sum_{E\in\mathcal{G}} \xi(E)\eta(E)^{\alpha} \right] \end{aligned}$$

□

The *Caratheodory dimension* $\dim_C(E)$ of E is the unique value α_0 such that $m(E,\alpha) = \infty$ for all $\alpha < \alpha_0$ and $m(E,\alpha) = 0$ for all $\alpha > \alpha_0$.

Finally, if $\mathcal{F}_\delta^* := \{E \in \mathcal{F} \mid \psi(E) = \delta\}$ condition C3 can be replaced by the more strict condition

C3* $\forall\delta > 0$ there is an at most countable subcollection $\mathcal{G} \subseteq \mathcal{F}_\delta^*$ covering X.

In this case,

$$\inf_{\mathcal{G}\subseteq\mathcal{F}_\delta^*} \sum_{E\in\mathcal{G}} \xi(E)\eta(E)^{\alpha}$$

is no longer monotone as $\delta \downarrow 0$, and the limit may not exist. Thus for each $\alpha > 0$ there are two nonnegative quantities

$$\begin{aligned} \overline{r}(E,\alpha) &:= \limsup_{\delta\to 0} \inf_{\mathcal{G}\subseteq\mathcal{F}_\delta^*} \sum_{E\in\mathcal{G}} \xi(E)\eta(E)^{\alpha} \\ \underline{r}(E,\alpha) &:= \liminf_{\delta\to 0} \inf_{\mathcal{G}\subseteq\mathcal{F}_\delta^*} \sum_{E\in\mathcal{G}} \xi(E)\eta(E)^{\alpha} \end{aligned}$$

and the upper and lower Caratheodory capacities, respectively, are the real numbers

$$\begin{aligned} \overline{\mathrm{Cap}}(E) &:= \inf_{\alpha>0} \left\{ \alpha \,\middle|\, \overline{r}(E,\alpha) = 0 \right\} = \sup_{\alpha>0} \left\{ \alpha \,\middle|\, \overline{r}(E,\alpha) = \infty \right\} \\ \underline{\mathrm{Cap}}(E) &:= \inf_{\alpha>0} \left\{ \alpha \,\middle|\, \underline{r}(E,\alpha) = 0 \right\} = \sup_{\alpha>0} \left\{ \alpha \,\middle|\, \underline{r}(E,\alpha) = \infty \right\} \end{aligned}$$

It follows from the definitions that

$$\dim_C(E) \leq \underline{\mathrm{Cap}}(E) \leq \overline{\mathrm{Cap}}(E)$$

For $\delta > 0$ and $E \subseteq X$, let

$$\Xi_\delta(E) := \inf_{\mathcal{G} \in \mathcal{F}_\delta^*} \sum_{E \in \mathcal{G}} \xi(E)$$

If the C-structure satisfies the additional condition

C4 for all $E_1, E_2 \in \mathcal{F}$, if $\psi(E_1) = \psi(E_2)$, then $\eta(E_1) = \eta(E_2)$.

define $\tilde{\eta}(\delta) = \eta(E)$ for any $E \in \mathcal{F}$ such that $\psi(E) = \delta$. The final result required is the following

Theorem 3.3.3 ([53, p. 18, Theorem 2.2]). *Assuming condition C4 is satisfied, for any $E \subseteq X$*

$$\overline{\mathrm{Cap}}(E) = \limsup_{\delta \to 0} \frac{\log \Xi_\delta(E)}{-\log \tilde{\eta}(\delta)}$$
$$\underline{\mathrm{Cap}}(E) = \liminf_{\delta \to 0} \frac{\log \Xi_\delta(E)}{-\log \tilde{\eta}(\delta)}$$

Remark. Since condition C4 implies that

$$\inf_{\mathcal{G} \subseteq \mathcal{F}_\delta^*} \sum_{E \in \mathcal{G}} \xi(E)\eta(E)^\alpha = \inf_{\mathcal{G} \subseteq \mathcal{F}_\delta^*} \sum_{E \in \mathcal{G}} \xi(E)\tilde{\eta}(\delta)^\alpha$$

the case for the upper capacity is actually a variation of the Hardy-Riesz Formula below (Theorem 3.4.2).

It is now possible to obtain results for the case of Hausdorff dimension and measure. Let (X, d) be a compact metric space, and define a C-structure on X by letting $\mathcal{F}$ be the collection of open sets in X, and $\xi(U) = 1$, $\eta(U) = \psi(U) = \mathrm{diam}(U)$ for all $U \in \mathcal{F}$. Then $m(E, s) = \mathcal{H}^s(E)$, $\dim_{\mathcal{H}}(E) = \dim_C(E)$, and the upper and lower box dimensions are equal to the upper and lower Caratheodory capacities, respectively. Theorem 3.3.3 then gives the result

$$\dim_{\mathcal{H}}(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E)$$

Also, $\Xi_\delta(E)$ is the box number $\mathrm{N}_\delta(E)$ of E and there are two equivalent definitions for the upper and lower box dimensions: for instance, the upper box dimension $\overline{\dim}_B(E)$ of $E \subseteq X$

can be defined in a manner directly analogous to the Hausdorff dimension, but with the covers restricted to sets of a fixed diameter. Since $\Gamma^*_\delta(E) := \{\mathcal{U} \in \Gamma(E) \mid \operatorname{diam}(U) = \delta\ \forall U \in \mathcal{U}\}$, then

$$\overline{\dim}_B(E) = \inf_{s>0}\left\{ s \;\middle|\; \limsup_{\delta\to 0}\left(\inf_{\mathcal{U}\in\Gamma^*_\delta(E)} \sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s\right)\right\}$$

$\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ define outer measures on X. Since for any open subsets $U, U' \subseteq X$

$$\mathcal{H}^s_\delta\left(U \cup U'\right) \geq \mathcal{H}^s_\delta(U) + \mathcal{H}^s_\delta(U') \quad \text{whenever} \quad d_H(E, E') > \delta$$

(where d_H is Hausdorff distance), it follows that $\mathcal{H}^s$ is a finite (hence regular) Borel measure on X (see [24, §1.2] or [26, §2.10] for complete proofs).

Given $\delta > 0$, a cover $\mathcal{U} \in \Gamma_\delta$, and $0 < s' < s$, the following standard inequality will prove useful—it is the trick from the proof of Proposition 3.3.2 specialized to the current context:

$$\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s \leq \delta^{s-s'} \sum_{U\in\mathcal{U}} \operatorname{diam}(U)^{s'} \tag{3.3.1}$$

The definition of dimension (Caratheodory or Hausdorff) does not require that $\mathcal{H}^{s_0}(E)$ be strictly positive when $s_0 = \dim_{\mathcal{H}}(E)$; however, making this assumption will be essential for the construction of spectral triples that recover the Hausdorff measure, and of course to have a nontrivial integration theory on X.

3.4 Dirichlet Series and Tauberian Theory

Noncommutative integration theory depends on the asymptotic properties of compact operators (or more accurately, the asymptotic properties of their sequences of singular values). Spectral dimension is determined by the convergence properties of associated Dirichlet series, requiring Tauberian theorems for proof of convergence.

Definition 3.4.1 (Dirichlet Series and Abscissa of Convergence)**.** For $s \in \mathbb{C}$, a *Dirichlet series* is a series of the form $\sum_{k=1}^\infty a_k e^{-s\lambda_k}$, where $\{a_k\}_{k=1}^\infty$ is any sequence of complex numbers and $\{\lambda_k\}_{k=1}^\infty$ is any sequence of real numbers with the property $\lim_{k\to\infty} \lambda_k = \infty$. The *abscissa of convergence* (a.o.c.) of a Dirichlet series is the (extended) real number

$$s_0 := \inf\left\{ \sigma \in \mathbb{R} \;\middle|\; \Re(s) > \sigma \Longrightarrow \sum_{k=1}^\infty \left|a_k e^{-s\lambda_k}\right| < \infty \right\}$$

The following theorem is an important initial result in the theory of Dirichlet series (see [37, p. 6] for a proof).

Theorem 3.4.2 (Hardy-Riesz Formula)**.** *If $s_0 > 0$ is the a.o.c. of the Dirichlet series $\sum_{k=1}^{\infty} a_k e^{-s\lambda_k}$, then*

$$s_0 = \limsup_{K\to\infty} \frac{1}{\lambda_K} \log \left| \sum_{k=1}^{K} a_k \right|$$

The following lemma is trivial but will be referred to frequently.

Lemma 3.4.3. *Given any two sequences $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty} a_k = a$ and $\lim_{k\to\infty} b_k = b \neq 0$, then*

$$\lim_{K\to\infty} \frac{\sum_{k=1}^{K} a_k}{\sum_{k=1}^{K} b_k} = \frac{a}{b}$$

Given two Dirichlet series $\sum_{k=1}^{\infty} a_k e^{-s\lambda_k}, \sum_{k=1}^{\infty} b_k e^{-s\mu_k}$ with the same abscissa of convergence s_0 (assumed to finite), for any $K \in \mathbb{N}$

$$\lim_{\sigma\to 0} \frac{\sum_{k=1}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k=1}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}} = \lim_{\sigma\to 0} \frac{\sum_{k=K}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}}$$

(i.e. the limit of the ratio is independent of the starting index).

Proof. If $\lim_{k\to\infty} a_k = a$, then the Cesaro means $\frac{1}{K}\sum_{k=1}^{K} a_k$ converge to the same limit and

$$\lim_{K\to\infty} \frac{\sum_{k=1}^{K} a_k}{\sum_{k=1}^{K} b_k} = \lim_{K\to\infty} \frac{\frac{1}{K}\sum_{k=1}^{K} a_k}{\frac{1}{K}\sum_{k=1}^{K} b_k} = \frac{a}{b}$$

proving the first claim.

The second claim is trivial because $\sum_{k<K} a_k e^{-s_0}$ is finite for any K. Thus

$$\begin{aligned}
\lim_{\sigma\to 0} \frac{\sum_{k=1}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k=1}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}} &= \lim_{\sigma\to 0} \frac{\sum_{k<K} a_k e^{-(s_0+\sigma)} + \sum_{k=K}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k<K} b_k e^{-(s_0+\sigma)} + \sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}} \\
&= \lim_{\sigma\to 0} \frac{\frac{\sum_{k<K} a_k e^{-(s_0+\sigma)}}{\sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}} + \frac{\sum_{k=K}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}}}{\frac{\sum_{k<K} b_k e^{-(s_0+\sigma)}}{\sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}} + 1} \\
&= \frac{0 + \lim_{\sigma\to 0} \frac{\sum_{k=K}^{\infty} a_k e^{-(s_0+\sigma)\lambda_k}}{\sum_{k=K}^{\infty} b_k e^{-(s_0+\sigma)\mu_k}}}{0+1}
\end{aligned}$$

$\square$

3.5 Noncommutative Integration Theory

Noncommutative integration theory is the theory of weights, traces and states on von Neumann algebras, which yields the classical measure theory for a commutative algebra. Let $\mathcal{A}^+$ be the positive cone of a von-Neumann algebra $\mathcal{A}$: those elements of the form a^*a for some $a \in \mathcal{A}$. A *weight* is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(\mathcal{A}^+) \subseteq [0, \infty)$, and a *state* is a normalized weight ($\omega(\mathbf{1}) = 1$). For a commutative unital C*-algebra $\mathcal{A}$, a state is a positive linear functional of norm one, and thus corresponds to a probability measure on $\mathrm{spec}\,(\mathcal{A})$. A *trace* is a weight such that $\omega(a^*a) = \omega(aa^*)$ for all $a \in \mathcal{A}$.

For $p \in [1, \infty)$, let

$$\mathcal{L}^p(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) \;\middle|\; \sum_{n=0}^{\infty} \mu_n(T)^p < \infty \right\}$$

be the *p-th Schatten-von Neumann ideal* of compact operators; those operators whose sequence of singular values is p-summable. (That $\mathcal{L}^p(\mathcal{H})$ is a two-sided ideal follows from that fact that $\mu_n(ST), \mu_n(TS) \leq \|S\|_{\mathcal{B}(\mathcal{H})}\, \mu_n(T)$ for all $T \in \mathcal{L}^p(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$.) $\mathcal{L}^1(\mathcal{H})$ is the ideal of *trace-class* operators; those compact operators which have a finite (normal) trace (any normal trace is a scalar multiple of the standard one [22]). A "trace-norm" can be defined on $\mathcal{L}^p(\mathcal{H})$ for each $p \in [1, \infty)$:

$$\||T\||_p := \mathrm{Tr}(T^p) = \sum_{n=0}^{\infty} \mu_n(T)^p$$

Each ideal $\mathcal{L}^p(\mathcal{H})$ is the closure in the corresponding trace-norm of the finite rank operators.

Definition 3.5.1 (Interpolation Ideals)**.** For $p \in [1, \infty)$, let

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) \;\middle|\; \sum_{n=1}^{N} \mu_n(T)^p = O(\log N) \right\}$$

and if $\sigma_N(T) := \sum_{n=1}^{N} \mu_n(T)$, define a norm on this ideal by

$$\|T\|_{p,\infty} := \sup_{N>1} \frac{1}{\log N} \sigma_N(T^p)$$

For a positive operator T in the Dixmier ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, the map

$$T \mapsto \lim_{N\to\infty} \frac{\sigma_N(T)}{\log N}$$

would define a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ that vanishes on the trace-class operators (i.e. $\mathcal{L}^1(\mathcal{H})$), assuming it is both linear and convergent. Linearity follows from convergence, but, in general the sequence $\frac{1}{\log N}\sigma_N(T)$ is merely bounded. As a result, in order to define a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ it is in general necessary to fix a functional ω on $\ell^\infty(\mathbb{N})$ so that

$$\mathrm{Tr}_\omega(T) := \omega\left(\frac{\sigma_N(T)}{\log N}\right)$$

is a well-defined trace, provided ω is positive, scale-invariant, and, when the limit exists, yields $\lim_{N\to\infty}\frac{1}{\log N}\sigma_N(T)$ [15]; by linearity there is a unique extension from such positive operators to a trace on all of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.

In general,

$$\liminf_{N\to\infty}\frac{\sigma_N(T)}{\log N} \leq \omega\left(\frac{\sigma_N(T)}{\log N}\right) \leq \limsup_{N\to\infty}\frac{\sigma_N(T)}{\log N}$$

and the spectral dimension of $(\mathcal{A},\mathcal{H},D)$ is

$$d(\mathcal{A},\mathcal{H},D) := \inf\left\{s\geq 0 \,\middle|\, |D|^{-s}\in\mathcal{L}^{(1,\infty)}(\mathcal{H})\right\} = \sup\left\{s\geq 0 \,\middle|\, |D|^{-s}\notin\mathcal{L}^{(1,\infty)}(\mathcal{H})\right\}$$

It is useful to know when the limit $\lim_{N\to\infty}\frac{\sigma_N(T)}{\log N}$ exists; a useful criterion is a direct consequence of the Littlewood-Hardy Tauberian Theorem. For any positive operator $T\in\mathcal{L}^{(1,\infty)}(\mathcal{H})$ and $s\in\mathbb{C}$ with $\Re(s)>1$ it is always possible to define T^s. Then, on the half-plane $\{z\in\mathbb{C}\mid\Re(z)>1\}$, $\zeta_T(s) := \mathrm{Tr}(T^s) = \sum_{n=1}^\infty \mu_n(T)^s$ is a holomorphic function and $T^s\in\mathcal{L}^1(\mathcal{H})$.

Theorem 3.5.2 (Hardy-Littlewood Tauberian Theorem [36, Theorem 98])**.** *For $T\geq 0$ such that $T\in\mathcal{L}^{(1,\infty)}(\mathcal{H})$, if one of the following limits exists, then the other one does and the two limits are equal:*

$$1)\ \lim_{s\to 1^+}(s-1)\zeta_T(s) = L$$

$$2)\ \lim_{N\to\infty}\frac{\sigma_N(T)}{\log N} = L$$

For operators T such that $\mathrm{Tr}_\omega(T)$ is independent of ω it is customary to refer to Tr_ω as *the* Dixmier trace; the theorem states that this independence occurs if $\zeta_T(s)$ has a simple pole at $s=1$, and in this case $\mathrm{Tr}_{\mathrm{Dix}}(T) := \mathrm{Tr}_\omega(T) = \mathrm{Res}_{s=1}\,\zeta_T(s)$.

Definition 3.5.3 (Measurable operator)**.** An operator T is *measurable* if $\mathrm{Tr}_\omega(T)$ is independent of ω. In this case, $\mathrm{Tr}_{\mathrm{Dix}} := \mathrm{Tr}_\omega$ is called *the* Dixmier trace.

The Cesaro mean of the function $f\colon (1,\infty) \to \mathbb{R}^+$, which is defined piecewise by $f(u) := \frac{1}{\log N}\sigma_N(T)$ for all $u \in (N-1, N]$, is given by

$$M(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda f(u) \, \frac{\mathrm{d}u}{u}$$

Proposition 3.5.4 ([15])**.** *For $T \geq 0$ such that $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, $\mathrm{Tr}_\omega(T)$ is independent of ω if and only if the Cesaro means $M(\lambda)$ of the sequence $\frac{1}{\log N}\sum_{n=1}^N \mu_n(T)$ are convergent for $\lambda \to \infty$. In this case, the family of T such that $\mathrm{Tr}_\omega(T)$ is independent of ω is a closed subspace of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ invariant by conjugation by invertible operators on $\mathcal{H}$.*

As stated in [15], the proposition is a direct consequence of the Hardy-Littlewood Tauberian Theorem. Measurability of an operator T thus depends on the regularity of its zeta-function, and the residue of the zeta-function at the abscissa of convergence is equal to the Dixmier trace. The subject of the next chapter is the construction of resolving sequences and families of spectral triples $(\mathcal{A}, \mathcal{H}, D_\tau)$ related to them. The Dirac operators of these triples are not constructed to be measurable—it is possible that their zeta-functions have poles of higher order, and it is possible to give constructions with arbitrary residue. Nevertheless, unique limiting states are obtainable, and in Chapter 5 it will be shown that resolving sequences exist such that the unique limiting state is equal to the Hausdorff probability measure (normalized Hausdorff measure).

CHAPTER IV

SEQUENCES OF COVERS AND SPECTRAL TRIPLES

A common feature of the various constructions of spectral triples for compact metric spaces, and a natural one in fractal geometry, is some sort of sequence or filtration of successively finer finite approximations of the space. As will be seen, recovering the Hausdorff measure and dimension when they are not equal to the Minkowski measure and dimensions necessitates greater care in this construction.

There have been several constructions of spectral triples for compact metric spaces in which sequences of sets of points with desirable properties are chosen in order for the Dirac operator, for instance, to be summable (more specifically, for its sequence of eigenvalues to have the right asymptotic behaviour) [6]. Here, the concept of a "resolving sequence" of open covers is introduced. This has the dual purpose of retaining the topology on X as well as giving some control on the asymptotics for the spectral triples described later. Furthermore, by dealing with covers first, and point sets second (via choice functions subject to the resolving sequence), it is possible to construct a triple from which the Hausdorff dimension and measure can be recovered when the Hausdorff and box dimensions differ.

By focusing on open covers, as opposed to all covers or metric covers, the construction is both general enough to be flexible and specific enough to encode sufficient data about the space. Even though the metric balls generate the *topology* on X, choosing covers by metric balls, implicit in many previous constructions of spectral triples on fractals, prohibits recovery of the Hausdorff measure. On the other hand, covers by arbitrary sets would be insufficient to encode the topology on X (at least, without the additional knowledge of the metric).

4.1 Resolving Sequences

Definition 4.1.1 (Resolving Sequence)**.** A sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$ of (at most countable open) covers of $E \subseteq X$ is *resolving* if $\lim_{n\to\infty} \operatorname{diam}(\mathcal{U}_n) = 0$. If in addition

$$\operatorname{diam}(\mathcal{U}_n) < \inf_{U \in \mathcal{U}_{n-1}} \{\operatorname{diam}(U)\} \tag{4.1.1}$$

for each $n \in \mathbb{N}$, the resolving sequence is *strict*.

By convention, $\mathcal{U}_0 = \{U\}$ for some open set $U \supseteq E$. In cases when there is no ambiguity about which resolving sequence is being referred to, for each n, the symbols $\Delta_n := \operatorname{diam}(\mathcal{U}_n)$ and $Q_n(s) := \sum_{U \in \mathcal{U}_n} \operatorname{diam}(U)^s$ will be used.

The open sets of a cover can be (not necessarily uniquely) ordered by their diameters. For any resolving sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$, the collection $\coprod_{n=0}^{\infty} \mathcal{U}_n$ of all open sets of the covers in ξ can also be ordered by their diameters. If $\{U_k\}_{k=1}^{\infty}$ is such an ordering, the ordering induces a partition of $\mathbb{N}$.

Definition 4.1.2 (Index Set for a Resolving Sequence)**.** Let $I_n = \{k \in \mathbb{N} \mid U_k \in \mathcal{U}_n\}$ for each $n = 0, \ldots, \infty$, (so that $\mathcal{U}_n = \{U_k\}_{k \in I_n}$). The set I_n is the *index set* for $\mathcal{U}_n$ with respect to the ordering $\{U_k\}_{k=1}^{\infty}$.

If a resolving sequence ξ is strict and consists only of finite covers, then the ordering on the open sets of the individual covers can be extended to an ordering $\{U_k\}_{k=1}^{\infty}$ so that $\operatorname{diam}(U_k) \leq \operatorname{diam}(U_{k-1})$ for all $k \in \mathbb{N}$ in such a way that if $U_k \in \mathcal{U}_n$ and $U_{k'} \in \mathcal{U}_{n'}$, then $k < k' \Rightarrow n \leq n'$. Such an ordering is said to *respect* the resolving sequence ξ.

4.2 Hausdorff Resolving Sequences

A resolving sequence contains a basis for the topology on X (Proposition 5.1.1). The condition that ensures that a resolving sequence encode the topology of X is that $\operatorname{diam}(U_n) \to \infty$ as $n \to \infty$. For the covers of a resolving sequence to encode Hausdorff dimension and measure, it is necessary to choose them in an optimal way. This is done by placing additional conditions on the sequence.

Given a resolving sequence ξ for a subset $E \subseteq X$, its zeta-function is the Dirichlet series

$$\zeta_\xi(s) = \sum_{n=0}^{\infty} \sum_{U \in \mathcal{U}_n} \mathrm{diam}(U)^s = \sum_{n=0}^{\infty} Q_n(s)$$

The following is an immediate corollary of Theorem 3.4.2.

Corollary. *If a resolving sequence ξ is strict and $\{U_k\}_{k=1}^{\infty}$ is an ordering of the open sets that respects ξ (necessitating finiteness of all covers in ξ), then its a.o.c. s_ξ is given by*

$$s_\xi = \limsup_{k \to \infty} \frac{k}{-\log \mathrm{diam}(U_k)} \tag{4.2.1}$$

Remark. Formula 4.2.1 holds for more general resolving sequences of finite covers, but requiring strictness is sufficient for the current situation.

Restricting the families of open covers by placing a lower bound on the diameters in advance restricts the dimension data that can be recovered from a resolving sequence.

Proposition 4.2.1. *Let $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$ be a resolving sequence for $E \subseteq X$ with the property that there exists $\beta > 0$ such that for each cover $\mathcal{U}_n$, $\mathrm{diam}(U) \geq \beta \Delta_n$ for each $U \in \mathcal{U}_n$. Then $\underline{\dim}_B(E) \leq s_\xi$, where s_ξ is the a.o.c. of ζ_ξ.*

Proof. For $\delta > 0$ and any cover $\mathcal{U} \in \Gamma^*_\delta(E)$,

$$\sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s = |\mathcal{U}| \, \mathrm{diam}(\mathcal{U})^s = |\mathcal{U}| \, \delta^s$$

Since

$$\beta \sum_{n=0}^{\infty} |\mathcal{U}_n| \, \Delta_n^s \leq \zeta_\xi(s) \leq \sum_{n=0}^{\infty} |\mathcal{U}_n| \, \Delta_n^s$$

it suffices to assume that $\mathrm{diam}(U) = \Delta_n$ for all $U \in \mathcal{U}_n$ and the problem reduces to considering a resolving sequence of covers $\mathcal{U}_n \in \Gamma^*(E)$ (covers of constant diameter). The a.o.c.of the zeta-function $\sum_{n=0}^{\infty} |\mathcal{U}_n| \, \Delta_n^s$ is

$$s_\xi = \limsup_{N \to \infty} \frac{\log \sum_{n=0}^{N} |\mathcal{U}_n|}{-\log \Delta_N}$$

Clearly for any $s > 0$ such that $\underline{\dim}_B(E) = \infty$, it follows that

$$\liminf_{\delta \to 0} \inf_{\mathcal{U} \in \Gamma^*_\delta(E)} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s = \liminf_{\delta \to 0} \inf_{\mathcal{U} \in \Gamma^*_\delta(E)} |\mathcal{U}| \, \delta^s = \liminf_{\delta \to 0} \delta^s \mathrm{N}_\delta(E)$$

so it is impossible that $\zeta_\xi(s) = \sum_{n=0}^{\infty} |\mathcal{U}_n| \, \Delta_n^s < \infty$, since $|\mathcal{U}_n| \, \Delta_n^s \to \infty$ in the first case ($\underline{\dim}_B(E) = \infty$) and in the second case $\zeta_\xi(s) < \infty$ necessitates that $|\mathcal{U}_n| \, \Delta_n^s \to 0$. ∎

Remark. It follows that in the case where the Hausdorff and box dimensions differ, to recover the Hausdorff dimension from a resolving sequence there must be no lower bound on the diameters of the open sets of a cover.

Theorem 4.2.2. *If ξ is any resolving sequence for $E \subseteq X$ and s_ξ is the a.o.c. of its zeta-function, then $\dim_{\mathcal{H}}(E) \leq s_\xi$. Furthermore, for any compact metric space (X, d), there exists a resolving sequence ξ such that $s_\xi = \dim_{\mathcal{H}}(X)$.*

Proof. Since ξ is resolving, it follows that $\lim_{n\to\infty} \Delta_n = 0$ (where $\Delta_n = \operatorname{diam}(\mathcal{U}_n)$). If $\zeta_\xi(s) = \sum_{n=0}^{\infty} Q_n(s)$ converges for some s, it follows that $\lim_{n\to\infty} Q_n(s) = 0$. Since $\mathcal{H}^s_{\Delta_n}(E) = \inf\left\{\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s \,\middle|\, \mathcal{U} \in \Gamma_{\Delta_n}(E)\right\} < Q_n(s)$, due to the monotonicity of $\mathcal{H}^s_\delta(E)$ in δ, it follows that

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E) = \lim_{n\to\infty} \mathcal{H}^s_{\Delta_n}(E) = 0$$

and thus $\dim_{\mathcal{H}}(E) \leq s$. Conversely, for $s < \dim_{\mathcal{H}}(E)$, since $\lim_{\delta\to 0} \mathcal{H}^s_\delta(E) = \infty$ it follows that for all $M > 0$ there exists $\delta' > 0$ such that $\inf_{\mathcal{U}\in\Gamma_\delta} Q_s(\mathcal{U}) > M$ whenever $\delta < \delta'$. Thus $\sum_{n=0}^{\infty} Q_n(s)$ must diverge and the first statement is proved.

If $\dim_{\mathcal{H}}(E) = \infty$, there is nothing left to show—any resolving sequence would necessarily satisfy $s_\xi = \dim_{\mathcal{H}}(E)$. Suppose that $\dim_{\mathcal{H}}(E) < \infty$.

For $\eta > 0$, let $s > \dim_{\mathcal{H}}(X) + \eta$. Necessarily $\mathcal{H}^s(X) = 0$, so $H^s_\delta(X) = 0$ for all δ. For any $C > 0, p > 1$, and for $n = 1, \ldots, \infty$, choose $\epsilon_n \in \left(0, \frac{1}{n^p}\right]$ such that $\sum_{n=1}^{\infty} \epsilon_n < C$. Then for each ϵ_n there exists δ_n and a cover $\mathcal{U}_n \in \Gamma_{\delta_n}(X)$ such that $Q_n(s) < \epsilon_n$ and $\delta_n < \frac{1}{n}$. As a result, $\xi' := \{\mathcal{U}_n\}_{n=0}^{\infty}$ ($\mathcal{U}_0 = \{X\}$) is a resolving sequence such that $\zeta_{\xi'}(s) < C$. Since convergence at s implies convergence for all real numbers greater than s, for any $\eta > 0$ there is a resolving sequence with abscissa of convergence less than or equal to $\dim_{\mathcal{H}}(X) + \eta$.

The result now follows by a diagonal argument. By the previous construction, for each $m = 1, \ldots, \infty$ there is a resolving sequence $\xi_m = \{\mathcal{U}^m{}_n\}_{n=0}^{\infty}$ with the property that $\zeta_{\xi_m}(s) < \infty$ for all $s > \dim_{\mathcal{H}}(X) + \frac{1}{m}$. Let ξ be the diagonal sequence $\left\{\mathcal{U}^{m+1}{}_m\right\}_{m=0}^{\infty}$. Then for each $s > \dim_{\mathcal{H}}(X)$ there is a unique integer N_s such that

$$\frac{1}{N_s + 1} \leq s - \dim_{\mathcal{H}}(X) < \frac{1}{N_s}$$

and thus

$$\sum_{U \in \mathcal{U}_m^{m+1}} \operatorname{diam}(U) < \epsilon_m^{m+1} \leq \frac{1}{(m+1)^p} \qquad \text{for all } m > N_s$$

As a result,

$$\zeta_\xi(s) = \sum_{m=0}^{N_s} Q_s(\mathcal{U}_m^{m+1}) + \sum_{m=N_s+1}^{\infty} Q_s(\mathcal{U}_m^{m+1}) < \infty$$

since the first sum has a finite number of terms and the second converges. □

Definition 4.2.3 (Hausdorff Resolving Sequence)**.** Let ξ be a resolving sequence with abscissa of convergence s_ξ, and let $s_0 := \dim_{\mathcal{H}}(E)$. In light of Theorem 4.2.2, ξ is called *Hausdorff* if

1. $s_\xi = s_0$, and

2. $\lim_{n\to\infty} Q_n(s_0) = \mathcal{H}^{s_0}(E)$.

Let $\Xi(E)$ denote the collection of all strict Hausdorff resolving sequences of **finite** open covers for a subset $E \subseteq X$.

Proposition 4.2.4. *For any subset $E \subseteq X$, $\Xi(E)$ is nonempty.*

Proof. The proof amounts to an alternative construction of a Hausdorff resolving sequence. Let $s_0 = \dim_{\mathcal{H}}(E)$ and fix two decreasing sequences of real numbers $\{\delta_n\}_{n=0}^{\infty}$, $\{\epsilon_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and the Dirichlet series $\sum_{n=0}^{\infty} \delta_n^s$ has a.o.c. $s_\delta = 0$ (for example, the sequence $\delta_n = e^{-n}$ satisfies this condition). Let $\mathcal{U}_0 = \{U\}$ for some open set $U \supseteq E$ and for each $n \in \mathbb{N}$, choose a finite open cover $\mathcal{U}_n$ of X so that

1. $\Delta_n := \operatorname{diam}(\mathcal{U}_n) < \min\{\delta_n, \inf\{\operatorname{diam}(U) \mid U \in \mathcal{U}_{n-1}\}\}$

2. $Q_n(s_0) < H_{\Delta_n}^{s_0}(E) + \epsilon_n$

Since $\mathcal{U}_{n-1}$ is finite, $\inf\{\operatorname{diam}(U) \mid U \in \mathcal{U}_{n-1}\}$ is necessarily positive, so it is always possible to fulfill condition (1), and (2) is always possible by the definition of $H_{\Delta_n}^{s_0}(E)$. By inequality 3.3.1 and the monotonicity of $\mathcal{H}_\delta^s$ with respect to δ it follows that for $\sigma > 0$,

$$Q_n(s_0+\sigma) \;\leq\; \Delta_n^\sigma Q_n(s_0) \;\leq\; \Delta_n^\sigma \left(\mathcal{H}^{s_0}(X) + \epsilon_n\right)$$

and thus

$$\sum_{n=0}^{\infty} Q_n(s_0+\sigma) \leq \sum_{n=0}^{\infty} \Delta_n^{\sigma} \left(\mathcal{H}^{s_0}(X) + \epsilon_n\right) \leq \left(\mathcal{H}^{s_0}(X) + \sup_{n\in\mathbb{N}} \{\epsilon_n\}\right) \sum_{n=0}^{\infty} \delta_n^{\sigma} < \infty$$

As a result, the sequence $\xi := \{\mathcal{U}_n\}_{n=0}^{\infty}$ is resolving (since $\delta_n \to 0$ as $n \to \infty$ necessarily), and Hausdorff; since its a.o.c. s_ξ is not greater than $s_0 = \dim_{\mathcal{H}}(E)$, by Theorem 4.2.2 it follows that $s_0 = s_\xi$. By construction, the sequence ξ is strict and $\lim_{n\to\infty} Q_n(s_0) = \mathcal{H}^{s_0}(E)$, so $\xi \in \Xi(E)$ and $\Xi(E)$ is nonempty. $\square$

In fact, this construction can be extended to create a system $\{\xi_\alpha\}_{\alpha\in A}$ of decreasing Hausdorff resolving sequences for a finite or countably infinite partition $\{F_\alpha\}_{\alpha\in A}$ of X into Borel sets.

Lemma 4.2.5. *Let $\{F_\alpha\}_{\alpha\in A}, A \subseteq \mathbb{N}$ be an at most countable partition of X into Borel sets such that $\mathcal{H}^{s_0}(F_\alpha) > 0$ for all $\alpha \in A$ (where $s_0 = \dim_{\mathcal{H}}(X)$). Then there exist strict Hausdorff resolving sequences $\xi_\alpha = \{\mathcal{U}^{\alpha}{}_n\}_{n=0}^{\infty}$ for each F_α that together yield a Hausdorff resolving sequence $\widehat{\xi}$ for X given by $\widehat{\mathcal{U}_n} := \coprod_{\alpha\in A} \mathcal{U}_n^{\alpha}$, and for each $\alpha \in A$,*

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} Q_n^{\alpha}(s_0)}{\sum_{n=0}^{N} \widehat{Q_n}(s_0)} = \frac{\mathcal{H}^{s_0}(F_\alpha)}{\mathcal{H}^{s_0}(X)}$$

If A is finite, then ξ can be assumed to be strict as well, but if A is infinite, this cannot occur.

Proof. Suppose that $\{F_\alpha\}_{\alpha\in A}$ is a partition of X into Borel sets of positive measure (i.e. $\mathcal{H}^{s_0}(F_\alpha) > 0$ for all $\alpha \in A$). Fix a decreasing sequence of positive real numbers $\{\delta_n\}_{n=0}^{\infty}$ such that the Dirichlet series $\sum_{0=n}^{\infty} \delta_n$ has a.o.c. $s_\delta = 0$. Fix a seqence of positive real numbers $\{\epsilon_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \epsilon_n = 0$. For each $\alpha \in A$, let $\mathcal{U}_0^{\alpha} = \{U_\alpha\}$, where U_α is an open set in X containing F_α, and for each $n \in \mathbb{N}$ choose a finite open cover $\mathcal{U}_n^{\alpha}$ so that if $\Delta_n^{\alpha} := \sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}_n^{\alpha}\}$,

$$\Delta_n^{\alpha} < \min\left\{\delta_n, \inf_{U\in\mathcal{U}_{n-1}^{\alpha}} \{\operatorname{diam}(U)\}\right\}$$

and

$$Q_n^{\alpha}(s_0) < \mathcal{H}^{s_0}_{\Delta_n^{\alpha}}(F_\alpha) + \epsilon_n \mathcal{H}^{s_0}(F_\alpha) < \mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n) \tag{4.2.2}$$

For $\sigma > 0$,

$$Q_n^\alpha(s_0+\sigma) \ \le\ (\Delta_n^\alpha)^\sigma\, Q_n^\alpha(s_0) \ \le\ (\Delta_n^\alpha)^\sigma\, \mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n)$$

and thus

$$\sum_{n=0}^{\infty} Q_n^\alpha(s_0+\sigma) \le \sum_{n=0}^{\infty} (\Delta_n^\alpha)^\sigma\, \mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n) \le \mathcal{H}^{s_0}(F_\alpha)\left(1+\sup_{n\in\mathbb{N}}\{\epsilon_n\}\right)\sum_{n=0}^{\infty}(\Delta_n^\alpha)^\sigma$$

Since

$$\sum_{n=0}^{\infty}(\Delta_n^\alpha)^\sigma \le \sum_{n=0}^{\infty}\delta_n^\sigma < \infty$$

for each $\alpha \in A$, the resolving sequence $\xi_\alpha := \{\mathcal{U}^\alpha{}_n\}_{n=0}^\infty$ is strict by construction, has a.o.c. $s_\alpha = s_0$, and $\lim_{n\to\infty} Q_n^\alpha(s_0) = \mathcal{H}^{s_0}(F_\alpha)$.

Let $\widehat{\xi} = \left\{\widehat{\mathcal{U}}_n\right\}_{n=0}^\infty$ be the resolving sequence given by $\widehat{\mathcal{U}}_0 = \{X\}$ and $\widehat{\mathcal{U}}_n := \coprod_{\alpha\in A}\mathcal{U}_n^\alpha$ for all $n \in \mathbb{N}$. The covers in $\widehat{\xi}$ are no longer finite if the partition $\{F_\alpha\}_{\alpha\in A}$ is infinite. However,

$$\widehat{Q_n}(s_0) := \sum_{\alpha\in A} Q_n^\alpha(s_0) \le \sum_{\alpha\in A}\mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n) \le \mathcal{H}^{s_0}(X)(1+\epsilon_n)$$

so $\widehat{Q_n}(s_0)$ is finite for $n = 0,\dots,\infty$ and $\lim_{n\to\infty}\widehat{Q_n}(s_0) = \mathcal{H}^{s_0}(X)$.

For $\sigma > 0$

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{Q_n}(s_0+\sigma) &= \sum_{n=0}^{\infty}\sum_{\alpha\in A} Q_n^\alpha(s_0+\sigma)\\
&\le \sum_{n=0}^{\infty}\sum_{\alpha\in A}(\Delta_n^\alpha)^\sigma\,\mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n)\\
&\le \sum_{n=0}^{\infty}\delta_n^\sigma\sum_{\alpha\in A}\mathcal{H}^{s_0}(F_\alpha)(1+\epsilon_n)\\
&\le \mathcal{H}^{s_0}(X)\left(1+\sup_{n\in\mathbb{N}}\{\epsilon_n\}\right)\sum_{n=0}^{\infty}\delta_n^\sigma < \infty
\end{aligned}$$

so $\widehat{\xi}$ is a Hausdorff resolving sequence for X.

Since $\lim_{n\to\infty} Q_n^\alpha(s_0) = \mathcal{H}^{s_0}(F_\alpha) \neq 0$ for all $\alpha \in A$, it follows at once that for any two sets $F_\alpha, F_{\alpha'}$,

$$\lim_{N\to\infty}\frac{\sum_{n=0}^N Q_n^\alpha(s_0)}{\sum_{n=0}^N Q_n^{\alpha'}(s_0)} = \frac{\mathcal{H}^{s_0}(F_\alpha)}{\mathcal{H}^{s_0}(F_{\alpha'})}$$

If the partition $\{F_\alpha\}_{\alpha\in A}$ is not finite, the resolving sequence cannot be strict. If the covers $\widehat{\mathcal{U}}_n$ are infinite, it is possible that $\inf\left\{\operatorname{diam}(U)\;\middle|\; U\in\widehat{\mathcal{U}}_n\right\} = 0$ for some n. In fact, this **must** occur for each $n \in \mathbb{N}$, since $\widehat{Q_n}(s_0) < \infty$. However, if the partition is finite, it is possible to adjust the construction so that $\widehat{\xi}$ is strict by forcing the condition

$$\Delta_n^\alpha < \min\left\{\delta_n, \inf\left\{\operatorname{diam}(U) \;\middle|\; U \in \mathcal{U}_{n-1}^{\alpha'}\right\}\right\} \qquad \text{for all } \alpha' \in A$$

$\square$

Definition 4.2.6. In light of Lemma 4.2.5, if $\mathcal{F}$ is a partition of X into Borel sets of positive measure and $\xi = \{\mathcal{U}_n\}_{n=0}^\infty$ is a Hausdorff resolving sequence for X such that $\mathcal{U}_n = \coprod_{F\in\mathcal{F}} \mathcal{U}_n^F$ for Hausdorff resolving sequences $\xi_F = \left\{{\mathcal{U}^F}_n\right\}_{n=0}^\infty$ for each $F \in \mathcal{F}$, then ξ *respects* the partition $\mathcal{F}$.

Remark. While the construction in Proposition 4.2.4 holds for any subset $E \subseteq X$ (in particular, a subset E with $\dim_{\mathcal{H}}(E) \lneqq \dim_{\mathcal{H}}(X)$), Lemma 4.2.5 places two restrictions on the subsets F_α: that they have the same Hausdorff dimension as the total space X, and furthermore that their Hausdorff measure is positive in this dimension. The second condition necessitates the first.

4.3 Spectral Triples from Resolving Sequences

In the following, the construction for spectral triples on ultrametric Cantor sets in [51] is adapted to the current setting of a general compact metric space (X, d). Each cover $\mathcal{U}_n$ of a resolving sequence is assumed finite and treated as a finite set with elements U. The only restrictions placed on the metric space (X, d) are

1. X is compact (in the metric topology),
2. X contains an infinite set of points, none of which are isolated,
3. X has finite Hausdorff dimension ($s_0 := \dim_{\mathcal{H}}(X) < \infty$), and
4. X has positive s_0-dimensional Hausdorff measure ($\mathcal{H}^{s_0}(X) > 0$)

—henceforth these conditions will be assumed without further comment. Condition 1 and 2 imply that $s_0 > 0$ (since $\mathcal{H}^0$ is counting measure).

4.3.1 Choice Functions and Representations

Definition 4.3.1 (Choice Functions)**.** Given any resolving sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$ for X, let $\Upsilon(\xi)$ denote the space of *choice functions* compatible with ξ: those functions

$$\tau \colon \coprod_{n=0}^{\infty} \mathcal{U}_n \to X \times X$$

such that

1. if $\tau(U) = (\tau_+(U), \tau_-(U))$, then $\tau_\pm(U) \in U$ and,

2. if $U \in \mathcal{U}_n$ for $n \geq 1$, then

$$\operatorname{diam}(U) \geq d\left(\tau_+(U), \tau_-(U)\right) \geq \frac{\operatorname{diam}(U)}{1 + \operatorname{diam}(U)} \qquad (*)$$

   whenever $U \in \mathcal{U}_n$.

Remark. The choice functions of [51], a setting in which all open sets are also closed, have the property $d\left(\tau_+(U), \tau_-(U)\right) = \operatorname{diam}(U)$; in the current setting, where open sets are in general not also closed, this is in general impossible—it is possible that no such choice exists.

Lemma 4.3.2. *For any resolving sequence ξ and any $\tau \in \Upsilon(\xi)$, the set of points*

$$\{\tau_\pm(U) \mid U \in \mathcal{U}_n, n = 0, \ldots, \infty\}$$

is dense in X.

Proof. This follows automatically from the fact that ξ is resolving: for any open set U, there is an $n \in \mathbb{N}$ and $U' \in \mathcal{U}_n$ such that $U' \subseteq U$. It then follows that $\tau_\pm(U') \in U$, so every open set in X contains a point x such that either $\tau_+(U) = x$ or $\tau_-(U) = x$. $\square$

Let $C(X)$ denote the C^*-algebra of complex-valued continuous functions on X, and $C_{\mathrm{Lip}}(X)$ the dense subalgebra of Lipschitz continuous functions. Let $\mathcal{A} := C_{\mathrm{Lip}}(X)$ and, letting $l^2(\xi) := \bigoplus_{n=0}^{\infty} l^2(\mathcal{U}_n)$, let $\mathcal{H} := l^2(\xi) \otimes \mathbb{C}^2$. Given any resolving sequence ξ, any choice $\tau \in \Upsilon(\xi)$, and any (not necessarily continuous) function $f\colon X \to \mathbb{C}$, define a linear transformation $\pi_\tau(f)\colon \mathcal{H} \to \mathcal{H}$ by

$$f \mapsto \left\{ \psi(U) \mapsto \begin{bmatrix} f\left(\tau_+(U)\right) & 0 \\ 0 & f\left(\tau_-(U)\right) \end{bmatrix} \psi(U) \right\}$$

Since $\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_\infty$, it follows that $\pi_\tau(f)$ is a bounded operator on $\mathcal{H}$ whenever f is a bounded function on X.

Proposition 4.3.3. *For $f \in C(X)$, the assignment $f \mapsto \pi_\tau(f)$ is a faithful $*$-representation of $C(X)$ in $\mathcal{B}(\mathcal{H})$.*

Proof. That π_τ is a $*$-representation is straightforward. If $\pi_\tau(f) = \pi_\tau(g)$, then for every $U \in \mathcal{U}_n$, $n = 0, \dots, \infty$, $f\left(\tau_\pm(U)\right) = g\left(\tau_\pm(U)\right)$. Since $\{\mathcal{U}_n\}_{n=0}^\infty$ is a resolving sequence, by Lemma 4.3.2 the continuous functions f and g are equal on a dense set in X, and are thus equal. $\square$

4.3.2 Dirac Operator

If $\sigma_1 := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is the first Pauli matrix, for any $\tau \in \Upsilon(\xi)$, let D_τ be the operator on $\mathcal{H}$ defined by

$$D_\tau \psi(U) = \frac{1}{d\left(\tau_+(U), \tau_-(U)\right)} \sigma_1 \psi(U) \tag{4.3.1}$$

Theorem 4.3.4. *For any choice τ, the operator D_τ is self-adjoint with compact resolvent and $[D_\tau, \pi_\tau(f)]$ is a bounded operator for any Lipschitz continuous function $f \in \mathcal{A}$: i.e. $(\mathcal{A}, \mathcal{H}, D_\tau, \pi_\tau)$ is a spectral triple.*

Proof. The proof follows the proof of [51].

D_τ is self-adjoint:

Because D_τ is defined on any $\psi \in \mathcal{H}$ with finite support, it is densely defined. Let ψ, ψ' be elements of the domain of D_τ. Then

$$\langle D_\tau \psi, \psi' \rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{U \in \mathcal{U}_n} \frac{1}{d\left(\tau_+(U), \tau_-(U)\right)} \langle \sigma_1 \psi(U), \psi'(U) \rangle_{\mathbb{C}^2} = \langle \psi, D_\tau \psi' \rangle_{\mathcal{H}}$$

Since D_τ is defined on a dense subset of $\mathcal{H}$, it is a symmetric operator; if the range of D_τ is $\mathcal{H}$, then it is also self adjoint. Given any $\psi \in \mathcal{H}$, let $\psi'(U) := d\left(\tau_+(U), \tau_-(U)\right) \sigma_1 \psi(U)$.

Since

$$\|\psi'\|^2_{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{U \in \mathcal{U}_n} d\left(\tau_+(U), \tau_-(U)\right)^2 \|\psi(U)\|_{\mathbb{C}^2} \leq \operatorname{diam}(X)^2 \|\psi\|^2_{\mathcal{H}}$$

it follows that $\psi' \in \mathcal{H}$, and since $D_\tau \psi' = \psi$, the range of D_τ is $\mathcal{H}$ and D_τ is self-adjoint.

$[D_\tau, \pi_\tau(f)] \in \mathcal{B}(\mathcal{H})$:

For any $f \in \mathcal{A}$, let $\operatorname{Lip}(f) := \sup\left\{ \frac{|f(x)-f(y)|}{d(x,y)} \;\middle|\; x, y \in X \right\}$ be the Lipschitz constant of f. Then

$$[D_\tau, \pi_\tau(f)]\, \psi(U) =$$

$$\frac{1}{d\left(\tau_+(U), \tau_-(U)\right)} \left[\sigma_1, \begin{bmatrix} f\left(\tau_+(U)\right) & 0 \\ 0 & f\left(\tau_-(U)\right) \end{bmatrix}\right] \psi(U) =$$

$$\frac{f\left(\tau_+(U)\right) - f\left(\tau_-(U)\right)}{d\left(\tau_+(U), \tau_-(U)\right)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(U)$$

so

$$\|[D_\tau, \pi_\tau(f)]\|_{\mathcal{B}(\mathcal{H})} = \sup_{\substack{U \in \mathcal{U}_n \\ n \geq 0}} \left\{ \frac{|f\left(\tau_+(U)\right) - f\left(\tau_-(U)\right)|}{d\left(\tau_+(U), \tau_-(U)\right)} \right\} \leq \operatorname{Lip}(f) \tag{4.3.2}$$

$\left(\mathbf{1} + D_\tau^2\right)^{-1}$ *is compact*:

Since $\left(\mathbf{1} + D_\tau^2\right)^{-1} \psi(U) = \frac{d(\tau_+(U), \tau_-(U))^2}{1 + d(\tau_+(U), \tau_-(U))^2} \psi(U)$, let

$$T_m \psi(U) := \begin{cases} \frac{d(\tau_+(U), \tau_-(U))^2}{1 + d(\tau_+(U), \tau_-(U))^2} \psi(U) & U \in \mathcal{U}_n, n \leq m \\ 0 & \text{otherwise} \end{cases}$$

The covers $\mathcal{U}_n$ are finite, so the operators T_m are finite rank. Since

$$\left[T_m - \left(\mathbf{1} + D_\tau^2\right)^{-1}\right] \psi(U) = \sum_{\substack{U \in \mathcal{U}_n \\ n > m}} \frac{d\left(\tau_+(U), \tau_-(U)\right)^2}{1 + d\left(\tau_+(U), \tau_-(U)\right)^2} \psi(U)$$

and $\sup\left\{ d\left(\tau_+(U), \tau_-(U)\right) \mid U \in \mathcal{U}_n, n > m \right\} \to 0$ as $m \to \infty$, the sequence of finite rank operators $\{T_m\}_{m=0}^{\infty}$ converges to $\left(\mathbf{1} + D_\tau^2\right)^{-1}$ in the topology of $\mathcal{B}(\mathcal{H})$. □

4.4 Trace Functionals

By the Riesz Representation Theorem, the Hausdorff measure, being a positive Radon measure on X, is represented by a positive linear functional on the algebra of continuous functions on X. A state on $C(X)$ can be defined via the evaluation of a trace on $|D_\tau|^{-1}$, where D_τ is the Dirac operator determined by a choice in $\Upsilon(\xi)$. Connes showed (Theorem 3.5.2) that the Dixmier trace of a measurable operator is given by the residue formula

$$\mathrm{Tr}_{\mathrm{Dix}}(T) = \lim_{s \to 1^+} (s-1)\zeta_T(s)$$

A similar approach is followed, but the construction is modified to give a probability measure:

$$\mu(f) = \lim_{\sigma \to 0} \frac{\mathrm{Tr}\left(|D|^{-(s_0+\sigma)}\pi(f)\right)}{\mathrm{Tr}\left(|D|^{-(s_0+\sigma)}\right)}$$

The ratio of traces defines a state for each $\sigma > 0$, so existence of limits is guaranteed by Alaoglu's Theorem. Uniqueness of this limit is much more involved and will be addressed in the next chapter.

Let ξ be a strict Hausdorff resolving sequence of finite covers for X and fix an ordering $\{U_k\}_{k=1}^\infty$ of the open sets of ξ that respects ξ. For a given $\tau \in \Upsilon(\xi)$, for each $k \in N$ let $x_k := \tau_+(U_k), y_k := \tau_-(U_k)$, and $\delta_k := d(x_k, y_k)$. For any (not necessarily continuous) function $f \colon X \to \mathbb{C}$, let $a_k := \frac{1}{2}\left(f(x_k) + f(y_k)\right)$, and let

$$\zeta_\tau(s, f) := \frac{1}{2}\mathrm{Tr}\left(|D_\tau|^{-s}\pi(f)\right) = \sum_{k=1}^{\infty} a_k \delta_k^s$$

be the *spectral zeta-function* corresponding to f. For any subset $E \subseteq X$, let χ_E denote its characteristic function, and let

$$\zeta_\tau(s, E) := \zeta_\tau(s, \chi_E) \qquad \zeta_\tau(s) := \mathrm{Tr}\left(|D_\tau|^{-s}\right) = 2\zeta_\tau(s, X)$$

with corresponding abscissae of convergence s_E and s_τ, respectively, where it follows that $s_\tau = \limsup_{K \to \infty} \frac{K}{-\log \delta_K}$. It follows at once from condition $(*)$ in Definition 4.3.1 that $s_\tau = s_\xi$, and, when ξ is Hausdorff, that $s_\tau = s_\xi = s_0 = \dim_{\mathcal{H}}(X)$. (Note: even though the sequence $\{U_k\}_{k=1}^\infty$ has been ordered by diameter, the sequence $\{\delta_k\}_{k=1}^\infty$ may not be monotonic—the diameters are ordered, but the choices need not satisfy $\delta_k < \delta_{k-1}$ for all $k \in \mathbb{N}$. This does not effect the abscissa of convergence.)

Remark. In general, it is possible that a Dirichlet series converge at its abscissa of convergence, which would pose problems in the definition of the spectral zeta-function. However, this is avoided because the Hausdorff measure of X is assumed to be positive. For any $F \subseteq X$ such that $\mathcal{H}^{s_0}(F) > 0$, if ξ is a strict Hausdorff resolving sequence for F, then necessarily $\lim_{n\to\infty} Q_n(X) = \mathcal{H}^{s_0}(F)$, so that $\sum_{n=0}^{\infty} Q_n(s_0) = \infty$.

Proposition 4.4.1. *For any $\tau \in \Upsilon(\xi)$ and $\sigma > 0$, if*

$$\mu_{\tau,\sigma}(f) := \frac{\mathrm{Tr}\left(|D_\tau|^{-(s_0+\sigma)}\pi(f)\right)}{\mathrm{Tr}\left(|D_\tau|^{-(s_0+\sigma)}\right)} = \frac{\zeta_\tau(s_0+\sigma, f)}{\zeta_\tau(s_0+\sigma)}$$

$\mu_{\tau,\sigma}$ is a state on $C(X)$.

Proof. For any continuous $f \geq 0$, since $a_k \geq 0$ for all $k \in \mathbb{N}$,

$$\frac{\zeta_\tau(s_0+\sigma, f)}{\zeta_\tau(s_0+\sigma)} = \frac{\sum_{k=1}^{\infty} a_k \delta_k^{s_0+\sigma}}{\sum_{k=1}^{\infty} \delta_k^{s_0+\sigma}} > 0$$

for all $\sigma > 0$. Given $\alpha \in \mathbb{C}$ and $g \in C(X)$, if $b_k := \frac{1}{2}\left(g(x_k) + g(y_k)\right)$, then clearly

$$\begin{aligned}
\mu_{\tau,\sigma}(\alpha f + g) &= \frac{\sum_{k=1}^{\infty}(\alpha a_k + b_k)\delta_k^{s_0+\sigma}}{\sum_{k=1}^{\infty} \delta_k^{s_0+\sigma}} \\
&= \alpha \frac{\sum_{k=1}^{\infty} a_k \delta_k^{s_0+\sigma}}{\sum_{k=1}^{\infty} \delta_k^{s_0+\sigma}} + \frac{\sum_{k=1}^{\infty} b_k \delta_k^{s_0+\sigma}}{\sum_{k=1}^{\infty} \delta_k^{s_0+\sigma}} = \alpha\mu_\tau(f) + \mu_\tau(g)
\end{aligned}$$

Finally, if $\|f\|_\infty \leq 1$, then $|a_k| \leq 1$ for all $k \in \mathbb{N}$, so for all $\sigma > 0$,

$$\frac{\lim_{K\to\infty} \sum_{k=1}^{K} |a_k| \delta_k^{s_0+\sigma}}{\lim_{K\to\infty} \sum_{k=1}^{K} \delta_k^{s_0+\sigma}} \leq \frac{\lim_{K\to\infty} \sum_{k=1}^{K} \delta_k^{s_0+\sigma}}{\lim_{K\to\infty} \sum_{k=1}^{K} \delta_k^{s_0+\sigma}} = 1$$

so that $\|\mu_{\tau,\sigma}\| \leq 1$. Since $\|\chi_X\|_\infty = 1$, it follows that $\|\mu_{\tau,\sigma}\| \geq |\mu_{\tau,\sigma}(\chi_X)| = 1$. $\square$

It follows from the proposition and Alaoglu's Theorem that the closure of the set $\{\mu_{\tau,\sigma}(f) \mid \sigma \geq 0\}$ is weak-$*$ compact. Thus, limiting states as $\sigma \to 0$ are guaranteed to exist, and it remains to construct a resolving sequence for which there is a unique limit (and then to show that this limit is the Hausdorff measure). The limit will also be independent of the choice $\tau \in \Upsilon(\xi)$.

CHAPTER V

RESULTS

Assume throughout that (X, d) is a compact metric space without isolated points and that $s_0 := \dim_{\mathcal{H}}(X)$ satisfies $s_0 < \infty$ and $\mathcal{H}^{s_0}(X) > 0$. This implies also that $s_0 > 0$.

At this point, it is possible to show that any resolving sequence encodes the topology on X, and that by taking the suprememum over the Connes metrics on the spectral triples $(\mathcal{A}, \mathcal{H}, D_\tau)$ the metric can be recovered as well. If the resolving sequence is Hausdorff, its a.o.c. is equal to the Hausdorff dimension of X, and it will be shown that this is also equal to the spectral dimension of the triple $(\mathcal{A}, \mathcal{H}, D_\tau)$ for any $\tau \in \Upsilon(\xi)$. The Hausdorff measure can also be recovered, but a specific Hausdorff resolving sequence will be considered to ensure uniqueness of the limit $\lim_{\sigma \to 0} \mu_{\tau,\sigma}$.

5.1 Recovering the Space

From the data of a resolving sequence it is possible to recover the space, not only as a point set but also with the original topology—this follows from the fact that the open sets of the covers of a resolving sequence for a space X comprise a countable basis for the topology on X. Though the topology on X is induced by the metric d, the data of the space as a point set with a topology does not depend on d, so it is no surprise that this information is recovered without reference to any spectral triple—in fact, as will be seen in section 5.1.4, the construction is sufficiently general to hold for any compact Hausdorff space and has an immediate category-theoreic generalization.

Proposition 5.1.1. *If $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a resolving sequence for a subset X, for any open set $U \subset X$ there is a cover $\mathcal{U}_m$ such that $U' \subseteq U$ for some $U' \in \mathcal{U}_m$—i.e. the open sets of the covers in any resolving sequence for X also form a basis for the topology on X.*

Proof. For any $x \in X$ and any open set $U \ni x$, let

$$\epsilon := d\left(x, U^C\right) = \sup\{d(x, y) \mid y \notin U\}$$

Then $B(x, \epsilon') \subseteq U$ for all $\epsilon' < \epsilon$. Since $\Delta_n \to 0$ there exists $n' \in \mathbb{N}$ such that $\Delta_m < \frac{\epsilon'}{2}$ for all $m > n'$. For each such m, $\mathcal{U}_m$ is a cover of X, so there exists $U' \in \mathcal{U}_m$ containing x, and necessarily $x \in U' \subseteq B(x, \epsilon') \subseteq U$. $\square$

Thus, a resolving sequence contains all of the information of the topology on X.

5.1.1 Refining Sequences

For any Hausdorff space, the set of points can be identified with its "neighborhood filter". More precisely, given a point x in an arbitrary Hausdorff space, the intersection of all open sets containing x is the singleton set $\{x\}$. It is possible to make this more precise, using the "sequence of joins" generated by a resolving sequence. In particular, it is possible to compute the Čech cohomology $\hat{\mathrm{H}}(X)$ of X from the resolving sequence via the more restrictive notion of a refining sequence.

Definition 5.1.2 (Refining Sequence)**.** A resolving sequence of covers $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of X is a *refining* sequence for X if $\mathcal{U}_0 = \{X\}$ and $\mathcal{U}_{n-1} \preccurlyeq \mathcal{U}_n$ for every $n \in \mathbb{N}$. The sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is *properly* refining if $\mathcal{U}_{n-1} \prec \mathcal{U}_n$ for every $n \in \mathbb{N}$ (see Definition 3.1.1)

Given any resolving sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$ it is always possible to derive a refining sequence from it by taking its sequence of joins

$$\widehat{\mathcal{U}}_n := \bigvee_{i=0}^{n} \mathcal{U}_i$$

Moreover, Proposition 5.1.1 gives an equivalent condition for a sequence of covers $\{\mathcal{U}_n\}_{n=0}^{\infty}$ with the refining property $\mathcal{U}_{n-1} \preccurlyeq \mathcal{U}_n$ to be a resolving sequence:

Proposition 5.1.3. *If $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a sequence of finite minimal covers of X such that $\mathcal{U}_{n-1} \preccurlyeq \mathcal{U}_n$ for each $n \in \mathbb{N}$, the following are equivalent*

1. *$\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a resolving sequence*

2. *for any open set $U \subset X$ there is a cover $\mathcal{U}_m$ such that $U' \subseteq U$ for some $U' \in \mathcal{U}_m$*

Proof. Since the first condition implies the second by Proposition 5.1.1, the content of the proposition is that for a refining sequence the second condition implies that $\operatorname{diam}(\mathcal{U}_n) \to 0$

as $n \to 0$, giving an equivalent characterization of refining sequences for compact metric spaces.

Any refining sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$, necessarily satisfies $\operatorname{diam}(\mathcal{U}_n) \leq \operatorname{diam}(\mathcal{U}_{n+1})$. (If the sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is strictly refining, the inequality is strict.) Suppose that there exists $\alpha > 0$ such that $\lim_{n\to\infty} \operatorname{diam}(\mathcal{U}_n) = \alpha$. Then for $n = 0, \ldots, \infty$ there is at least one open set $U_n \in \mathcal{U}_n$ such that $\operatorname{diam}(U_n) \geq \alpha$; furthermore, U_n and can be chosen so that $U_n \subset U_{n-1}$ for all $n \in \mathbb{N}$. Since each cover is minimal, in each open set U of any cover $\mathcal{U}_n$ there exists at least one point x_U that is not contained in any other set of $\mathcal{U}_n$. Let x be such a point, chosen so that $x = x_{U_n}$ for all $n = 0, \ldots, \infty$ (if x_U is not contained in any other set $U' \in \mathcal{U}_n$, then it cannot be contained in any subset $U'' \subseteq U'$ for $U'' \in \mathcal{U}_{n'}, n' > n$). Then for any neighborhood U of x there is a cover $\mathcal{U}_m$ such that $x \in U_m \subseteq U$. By construction, $U_{m'} \subseteq U_m \subseteq U$ for all $m' > m$; since U can be chosen to be arbitrarily small, if $\operatorname{diam}(U) < \alpha$, this contradicts the property $\operatorname{diam}(U_n) \geq \alpha$. $\square$

For questions of topology, the refining sequence is a more natural construct. Given any resolving sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$, each $x \in X$ corresponds in a non-unique way to a sequence $\{U_n\}_{n=0}^{\infty}$, where $U_n \in \mathcal{U}_n$ for $n = 0, \ldots, \infty$, via $\{x\} = \bigcap_{n=0}^{\infty} U_n$. As will be seen, this viewpoint is more natural in the context or refining sequences, which yield the additional property $U_n \subseteq U_{n-1}$ for all $n \in \mathbb{N}$ (even though the sequence of neighborhoods is still not unique in general).

For considerations other than topology, the resolving sequence is more appropriate since the covers do not need to form a refining sequence for the quantities $Q_n(s)$ to approximate the Hausdorff measure. The need to choose optimal covers to construct Hausdorff resolving sequences could conflict with choices of covers that refine a previously chosen cover—at any rate, satisfying both a refining property and an optimal Hausdorff property would be both unnecessary and unnecessarily complicated.

5.1.2 The Graph of a Refining Sequence

Following [4], a directed *graph* $\mathcal{G}$ is an ordered pair of sets $(\mathcal{V}, \mathcal{E})$ for which the non-empty countable set $\mathcal{V}$ denotes the set of vertices of $\mathcal{G}$, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of *edges*.

Let $s, t \colon \mathcal{E} \to \mathcal{V}$ be the projections on the first and second components respectively: for $e = (v_1, v_2) \in \mathcal{E}$, the *source* of e is $s(e) = v_1$, and the *target* of e is $t(e) = v_2$. A *path* is a sequence of edges $\{e_i\}_{i=1}^n$ with the property $t(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. Graphs will be assumed to be *simple*: there is no edge e for which $s(e) = t(e)$ and for any $v_1 \neq v_2 \in \mathcal{V}$, there is at most one edge e with $s(e) = v_i$ and $t(e) = v_j$ for $(i, j) = (1, 2)$ or $(2, 1)$. As a result, any path $\{e_i\}_{i=1}^n$ can be represented by a sequence of vertices $\{s(e_1), t(e_1), \ldots, t(e_n)\}$, where $s(e_1)$ is the *beginning* and $t(e_n)$ is the *end* of the path. A vertex x is an *ancestor* of a vertex y, and y is a *descendant* of x, if there is a path beginning at x and ending at y.

Henceforth, assume that $\mathcal{U}_0 = \{X\}$ for any refining sequence $\{\mathcal{U}_n\}_{n=0}^\infty$. Let $\mathcal{G}$ be the directed graph with vertex set $\mathcal{V} = \coprod_{n=0}^\infty \mathcal{U}_n$ and edge set $\mathcal{E}$ consisting of precisely one edge (U, U') whenever $U \supset U'$ for some $U \in \mathcal{U}_n, U' \in \mathcal{U}_{n+1}$. It follows from the definition that $\mathcal{G}$ is a simple graph, so any path is determined by a sequence of vertices $(U_n)_{n=m}^{m'}$ such that $(U_n, U_{n+1}) \in \mathcal{E}$ for $n = m, \ldots, m'-1$. Let $\partial\mathcal{G}$ denote the set of infinite paths and assume that all paths start at X $(s(e_1) = X \in \mathcal{U}_0)$.

Proposition 5.1.4. *The set $\partial\mathcal{G}$ consists of sequences (U_n) such that $\bigcap_{n=0}^\infty U_n$ contains at most one point of X. If $\{\mathcal{U}_n\}_{n=0}^\infty$ is properly refining, then $\bigcap_{n=0}^\infty U_n$ contains precisely one point.*

Proof. The proposition is essentially a rephrasing of the characterization of Hausdorff spaces as those spaces that have well-defined limits (see [5] for details). Since $\{\mathcal{U}_n\}_{n=0}^\infty$ is refining, for no element of $\partial\mathcal{G}$ can $\bigcap_{n=0}^\infty U_n$ contain more than one point, since any two points x, y of X are separated by $\mathcal{U}_m$ for some m, in the sense that there are sets $U, U' \in \mathcal{U}_m$ such that $U \cap U' = \emptyset$ and $x \in U, y \in U'$.

If $\{\mathcal{U}_n\}_{n=0}^\infty$ is properly refining, then for each $n \in \mathbb{N}$ $\overline{U_n} \subseteq U_{n-1}$, so it follows that $\bigcap_{n=1}^\infty \overline{U_n} \subseteq \bigcap_{n=0}^\infty U_n$. Since X is Hausdorff, $\bigcap_{n=1}^\infty \overline{U_n}$ cannot be empty. $\square$

The product $\mathcal{U}^\infty := \prod_{n=0}^\infty \mathcal{U}_n$ of the finite discrete, hence compact, sets $\mathcal{U}_n$ is compact in the product topology. An *elementary cylinder* of $\mathcal{U}^\infty$ is a subset $[U_m]$ consisting of all paths whose m-th term is U_m; and a *cylinder* is a finite intersection of elementary cylinders. For any $n = 0, \ldots, \infty$, the disjoint union $\coprod_{U \in \mathcal{U}_n} [U]$ gives a finite partition of $\mathcal{U}^\infty$ into open, hence clopen, sets and the collection of these partitions over $n = 0, \ldots, \infty$ is a basis for the product topology on $\mathcal{U}^{\infty}$, making it a completely disconnected space.

Lemma 5.1.5. *$\partial\mathcal{G}$ is a closed subspace of the completely disconnected space $\mathcal{U}^{\infty}$, and hence completely disconnected itself. Furthermore, since X has no isolated points, neither does $\partial\mathcal{G}$, giving it the topology of a Cantor set.*

Proof. The path space $\partial\mathcal{G}$ inherits the subspace topology from $\mathcal{U}^{\infty}$. Given a sequence $(U_n) \in \mathcal{U}^{\infty} \backslash \partial\mathcal{G}$, there is an m for which U_m does not contain U_{m+1}. The intersection $[U_m] \bigcap [U_{m+1}]$ is an open neighborhood of (U_n) disjoint from $\partial\mathcal{G}$. Each element of the complement of $\partial\mathcal{G}$ in $\mathcal{U}^{\infty}$ has an open neighborhood disjoint from $\partial\mathcal{G}$, so $\partial\mathcal{G}$ is a closed subspace of the completely disconnected space $\mathcal{U}^{\infty}$.

Since X has no isolated points by assumption, any open set $U \subseteq X$ contains more than one point (in fact, since $\dim_{\mathcal{H}}(X) > 0$, the set X is uncountable, as is any open set of X). Since the elementary cylinders generate the topology on $\partial\mathcal{G}$, it suffices to show that any cylinder contains at least two paths $(U_n), (U'_n)$. Because $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is refining, for any cylinder $[U_n]$, two points $x, x' \in U_n$ each have disjoint, thus nonequal, neighborhoods $U \ni x, U' \ni x'$ contained in U_n, with $U \in \mathcal{U}_m, U' \in \mathcal{U}_{m'}$. The cylinders $[U]$ and $[U']$ are disjoint nonempty subsets of $[U_n]$, so $[U_n]$ contains more than one element. $\square$

Remark. All cylinders of $\partial\mathcal{G}$ are of the form $[U_n]$; for any $U_n \in \mathcal{U}_n$ and $U_m \in \mathcal{U}_m$ with $m \geq n$, either $[U_n] \bigcap [U_m] = \emptyset$ (there is no path beginning at U_n and ending at U_m) or $[U_n] \bigcap [U_m] = [U_m]$ (there is such a path, and thus $U_m \subset U_n$).

Let $\widetilde{X} = \{(U_n) \in \partial\mathcal{G} \mid \bigcap_{n=0}^{\infty} U_n \neq \emptyset\}$. For $(U_n) \in \widetilde{X}$, let $\Phi\colon \widetilde{X} \to X$ be the map $(U_n) \mapsto x$, where $\{x\} = \bigcap_{n=0}^{\infty} U_n$. This map is well-defined by Proposition 5.1.4.

Lemma 5.1.6. *The map Φ is a surjective, continuous, open mapping $\widetilde{X} \to X$.*

Proof. For each $x \in X$, choose any countable collection of open sets $U_k \subset X$ such that $\bigcap_{k\in\mathbb{N}} U_k = \{x\}$ (because X is Hausdorff this can always be done). Because $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is refining, there exists $m_1 \in \mathbb{N}$ and $U_{m_1} \in \mathcal{U}_{m_1}$ such that $U_{m_1} \subseteq U_1$; this allows a choice of the first $(m_1 + 1)$ elements of a path $(U_n) \in \widetilde{X}$. For each $k > 1$, continuing with the neighborhood $U_k \bigcap U_{m_{k-1}}$ gives a neighborhood for which there there is a cover $\mathcal{U}_{m_k}$ and $U_{m_k} \in \mathcal{U}_{m_k}$ with $U_{m_k} \subset U_k \bigcap U_{m_{k-1}}$. The sets U_{m_k}, $k = 1, \ldots, \infty$ therefore allow the selection of an infinite path (U_n) such that $U_{m_k} = U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_{m_{k-1}}$ whenever $m_k = n$, and $\Phi(U_n) = x$. Since x was an arbitrary point of X, Φ is surjective.

For any open set $U \subset X$, let $(U_n) \in \Phi^{-1}(U)$. Because $\Phi(U_n) = x$ for some $x \in U$, there is some $U_m \in (U_n)$ for which $U_m \subset U$. Any path through U_m is contained in $\Phi^{-1}(U)$, so the cylinder $[U_m]$ is a neighborhood of (U_n) contained in $\Phi^{-1}(U)$. Since every point of $\Phi^{-1}(U)$ has a neighborhood contained in $\Phi^{-1}(U)$, it is an open set of $\partial\mathcal{G}$ and Φ is continuous.

Furthermore, the image of any cylinder set $[U_m]$ is the set U_m; for any $x \in U_m$, the proof above that Φ is surjective can be modified so that $(U_n) = \Phi^{-1}\{x\}$ goes through U_m. Because the open sets in the covers of a refining sequence form a basis for the topology on X and the corresponding cylinder sets form a basis on $\partial\mathcal{G}$, Φ is an open map. □

5.1.3 The Path Space

Define a relation $\sim$ on $\widetilde{X}$ by

$$(U_n) \sim (U'_n) \quad \Longleftrightarrow \quad U_n \bigcap U'_n \neq \emptyset, \quad \text{for } n = 0, \ldots, \infty$$

Lemma 5.1.7. *The relation $\sim$ is a closed equivalence relation and $(U_n) \sim (U'_n)$ if and only if $\Phi(U_n) = \Phi(U'_n)$.*

Proof. That $\sim$ is a reflexive and symmetric relation follows from the definition. If $\Phi(U_n) = x$ and $\Phi(U'_n) = x' \neq x$, then there are disjoint neighborhoods $U \ni x$ and $U' \ni x'$. For some m there are open sets $U_m \subset U$ and $U'_m \subset U'$, so U_m and U'_m are disjoint and $(U_n) \nsim (U'_n)$. Conversely, if $\Phi(U_n) = x = \Phi(U'_n)$, then $x \in U_n \bigcap U'_n$ for all n, so $(U_n) \sim (U'_n)$. Since $(U_n) \sim (U'_n)$ if and only if $\Phi(U_n) = \Phi(U'_n)$, it follows that $\sim$ is also transitive, and thus an equivalence relation.

For $(U_n) \nsim (U'_n)$ there is some m for which $U_m \bigcap U'_m = \Phi[U_m] \bigcap \Phi[U'_m]$ is empty. As a result, every path in $[U_m]$ and every path in $[U'_m]$ have the property that $U_m \bigcap U'_m = \emptyset$, so $[U_m] \times [U'_m]$ is an open neighborhood of $((U_n), (U'_n)) \in \widetilde{X} \times \widetilde{X}$ disjoint from the graph of $\sim$, and the relation is closed. □

Corollary. *The equivalence classes of $\sim$ in $\widetilde{X}$ are the sets $\Phi^{-1}\{x\}$ for $x \in X$.*

Let $X_\infty = \widetilde{X}/\sim$, and let $q\colon \widetilde{X} \to X_\infty$ be the quotient map, endowing X_∞ with the quotient topology.

Theorem 5.1.8. *The quotient space X_∞ is homeomorphic to X.*

Proof. Since Φ is a class function, it descends to a map $\widetilde{\Phi}\colon X_\infty \to X$ such that $\Phi = \widetilde{\Phi} \circ q$; since Φ and q are open, continuous, and surjective (Lemma 5.1.6), so must $\widetilde{\Phi}$ be. Since $\widetilde{\Phi}$ is injective by Lemma 5.1.7, it is a homeomorphism between X_∞ and X. □

5.1.4 Čech Cohomology

The following is a (very) brief summary of Čech cohomology (see [38, 23, 47] for details). The key observation is that every compact Hausdorff space is the inverse limit of simplicial complexes. The recovery of X from a refining sequence in the previous section is in fact a reflection of this general property of compact Hausdorff spaces (no reference was made to the metric). A refining sequence is nothing more than a specification of an inverse limit sequence.

Let $\Gamma_0(X)$ denote the collection of finite, minimal open covers of X. To each cover $\mathcal{U} \in \Gamma_0(X)$ is assigned its *nerve* $N(\mathcal{U})$ (a simplicial complex encoding its intersection data) and for $\mathcal{U}_\alpha, \mathcal{U}_\beta \in \Gamma_0(X)$, whenever $\mathcal{U}_\alpha \preccurlyeq \mathcal{U}_\beta$ the inclusion maps on the sets of $\mathcal{U}_\beta$ induce simplicial maps $i_\alpha^\beta : N(\mathcal{U}_\beta) \to N(\mathcal{U}_\alpha)$. As a result, the refinement relation also gives the family of nerves $\{N(\mathcal{U}) \mid \mathcal{U} \in \Gamma_0(X)\}$ the structure of a directed system. The inverse limit of this directed system is a topological space homeomorphic to X [23, Theorem 10.1, p. 284].

Furthermore, to each simplicial complex $N(\mathcal{U})$ is associated its simplicial cohomology group $\mathrm{H}_\Delta\left(N(\mathcal{U})\right)$, and each simplicial map i_α^β induces a map $(i_\alpha^\beta)^*$ on cohomology. The Čech cohomology $\check{\mathrm{H}}(X)$ of X is the direct limit of the simplicial cohomologies of the nerves with respect to these induced maps. Also, to compute $\check{\mathrm{H}}(X)$ it is sufficient to restrict to any cofinal subcollection of $\Gamma_0(X)$.

A refining sequence ξ of finite minimal open covers is cofinal in $\mathcal{U}_0(X)$, so an inverse limit sequence derived from such a refining sequence is sufficient to find $\check{\mathrm{H}}(X)$. Given a refining sequence for X, for each $n \in \mathbb{N}$ there are (not necessarily unique) inclusion maps $i_n \colon \mathcal{U}_n \to \mathcal{U}_{n-1}$ sending each element $U \in \mathcal{U}_n$ onto some $U' \in \mathcal{U}_{n-1}$ containg U. The inclusion maps i_n extend to maps on the nerves $N(\mathcal{U}_n)$, which induce maps $(i_n)^*$ on the simplicial cohomologies $\mathrm{H}_\Delta(N(\mathcal{U}_n))$. Then

$$\check{\mathrm{H}}(X) := \varprojlim \left(H^*_\Delta\left(N(\mathcal{U}_n)\right), (i_n)^*\right)$$

Finally, a refining sequence can always be obtained from a resolving sequence by taking its sequence of joins, which in turn can be reduced to minimal covers, so beginning with a resolving sequence of X it is always possible to recover the Čech cohomology of X.

Since homology groups are not well behaved under inverse limits, computation of homoogy groups for X is more subtle and not addressed here. If the coefficient group is, for example, a field or a compact group, then Čech homology is dual to Čech cohomology and given by the inverse limit of the simplicial homologies with respect to the induced maps $(i_n)_*$. In general, Čech homology satisfies all of the Eileberg-Steenrod axioms except exactness [23]. The related theory of Steenrod homology ([62, 48, 27]), was introduced by Steenrod to address the failure of Čech's theory in the category of compact metric spaces. Steenrod homology agrees with Čech homology in the previously mentioned cases (coefficient groups for which exactness holds) and satisfies the exactness axioms (as well as a pair of addition axioms—see [48]). Because it is suited to "bad" compact metric spaces, Steenrod homology is the appropriate homology theory for operator theoretic considerations on compact metric spaces of interest to noncommutative geometery [40, 41].

5.2 Recovering the Metric

For any resolving sequence ξ and any choice $\tau \in \Upsilon(\xi)$, let

$$\rho_\tau(x, y) := \sup_{f \in \mathcal{A}} \left\{ |f(x) - f(y)| \;\middle|\; \|[D_\tau, \pi_\tau(f)]\|_{\mathcal{B}(\mathcal{H})} \leq 1 \right\}$$

be the Connes metric for the spectral triple $(\mathcal{A}, \mathcal{H}, D_\tau, \pi_\tau)$.

Proposition 5.2.1. *The Connes metric ρ_τ dominates the original metric d on X: for any $x, y \in X$, $\rho_\tau(x, y) \geq d(x, y)$. Also, for each $n = 0, \ldots, \infty$ and each $U \in \mathcal{U}_n$, $d\left(\tau_+(U), \tau_-(U)\right) = \rho_\tau\left(\tau_+(U), \tau_-(U)\right)$.*

Proof. The argument is now standard: for $x \in X$, let $d_x \colon X \to X$ be the map $y \mapsto d(x, y)$. By the triangle inequality, for any $y, y' \in X$, $|d_x(y) - d_x(y')| < d(y, y')$, so $\mathrm{Lip}(d_x) < 1$. Due to inequality (4.3.2) above it follows that $\|[D_\tau, d_x]\|_{\mathcal{B}(\mathcal{H})} \leq 1$, and thus

$$d(x, y) = |d_x(x) - d_x(y)| \leq \rho_\tau(x, y)$$

For any points $x, y \in X$, there exists $f \in C(X)$ such that $f(x) - f(y) = d(x, y)$ and $\mathrm{Lip}(f) = 1$. Therefore for each cover $\mathcal{U}_n$ and each $U \in \mathcal{U}_n$ there is a function $f_{n,U}$ such that $f_{n,U}(\tau_+(U)) - f_{n,U}(\tau_-(U)) = d(\tau_+(U), \tau_-(U))$ and $\|[D_\tau, f_{n,U}]\|_{\mathcal{B}(\mathcal{H})} \leq 1$. □

While in general it is not always possible to recover the metric completely from a single spectral triple (cf. [6, 58]), Pearson [50] introduced an approach that does allow recovery of the metric—considering all choices (hence multiple spectral triples): let

$$\rho_\xi(x, y) := \sup_{f \in \mathcal{A}} \left\{ |f(x) - f(y)| \;\middle|\; \sup_{\tau \in \Upsilon(\xi)} \|[D_\tau, \pi_\tau(f)]\|_{\mathcal{B}(\mathcal{H})} \leq 1 \right\}$$

Proposition 5.2.2. *For any resolving sequence ξ, for all $x, y \in X$,*

$$\rho_\xi(x, y) = d(x, y)$$

Proof. For any $x, y \in X$, $x \neq y$, there exists a choice $\tau \in \Upsilon(\xi)$ such that if $\mathcal{U}_0 = \{U\}$, then $\tau(U) = (x, y)$. It then follows from the previous proposition (Proposition 5.2.1) that $d(x, y) = \rho_\tau(x, y) = \rho_\xi(x, y)$. □

Remark. The lower bound condition $(*)$ in the definition of choice functions places no restriction on the choices for U when $\mathcal{U}_0 = \{U\}$, but its effect is essential for the integration properties of the triple $(\mathcal{A}, \mathcal{H}, D_\tau, \pi_\tau)$ as $n \to \infty$, where the choices become successively more and more restricted. The factor of $\frac{1}{1+\mathrm{diam}(U)}$ could be replaced by any factor that increases to 1 as $\mathrm{diam}(U) \to 0$—it is even possible in some cases to choose a bound that is strictly positive when $n = 0$, though the value of such a bound would depend on the regularity properties of the space (see [6]).

5.3 Recovering Hausdorff Measure

The principle difficulty in showing that the Hausdorff measure can be recovered from the limit $\lim_{\sigma \to 0} \mu_{\tau,\sigma}$ lies in ensuring that the limit as $\sigma \to 0$ is unique and independent of the choice $\tau \in \Upsilon(\xi)$. Uniqueness is forced by ensuring that the resolving sequence respects a countable family of Borel partitions $\mathcal{F}$. This will simultaneously yield the result that the unique limit is in fact the Hausdorff measure.

5.3.1 Independence of Choice

Uniqueness of the limit relies on the independence of the choice $\tau \in \Upsilon(\xi)$ of any limiting state. Thus, any limiting state depends only on the resolving sequence.

Lemma 5.3.1. *Let μ be a limit point of $\{\mu_{\tau,\sigma}(f) \mid \sigma \to 0\}$, so that there exists a sequence $\{\sigma_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \sigma_j = 0$ and $\mu = \lim_{j\to\infty} \mu_{\tau,\sigma_j}$. Then μ is also a state,*

$$\mu(f) = \lim_{j\to\infty} \frac{\sum_{k=1}^{\infty} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=1}^{\infty} \mathrm{diam}(U_k)^{s_0+\sigma_j}} \tag{5.3.1}$$

for all $f \in C(X)$, and μ is independent of the choice $\tau \in \Upsilon(\xi)$.

Proof. That μ_τ is a state is immediate, being the limit of states. For any $\epsilon > 0$, there is an integer K such that $\mathrm{diam}(U_k) < \epsilon$ whenever $k \geq K$, and

$$\frac{\mathrm{diam}(U_k)}{1+\epsilon} < \frac{\mathrm{diam}(U_k)}{1+\mathrm{diam}(U_k)} < \mathrm{diam}(U_k)$$

Therefore,

$$\begin{aligned}
\left(\frac{1}{1+\epsilon}\right)^{s_0} \lim_{j\to\infty} \frac{\sum_{k=K}^{\infty} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=K}^{\infty} \mathrm{diam}(U_k)^{s_0+\sigma_j}}& \\
&\leq \lim_{j\to\infty} \frac{\sum_{k=K}^{\infty} a_k \left(\frac{\mathrm{diam}(U_k)}{1+\mathrm{diam}(U_k)}\right)^{s_0+\sigma_j}}{\sum_{k=K}^{\infty} \mathrm{diam}(U_k)^{s_0+\sigma_j}} \\
&\leq \lim_{j\to\infty} \frac{\sum_{k=K}^{\infty} a_k \delta_k^{s_0+\sigma_j}}{\sum_{k=K}^{\infty} \delta_k^{s_0+\sigma_j}} \\
&\leq \lim_{j\to\infty} \frac{\sum_{k=K}^{\infty} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=K}^{\infty} \left(\frac{\mathrm{diam}(U_k)}{1+\mathrm{diam}(U_k)}\right)^{s_0+\sigma_j}} \\
&\leq (1+\epsilon)^{s_0} \lim_{j\to\infty} \frac{\sum_{k=K}^{\infty} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=K}^{\infty} \mathrm{diam}(U_k)^{s_0+\sigma_j}}
\end{aligned}$$

establishing 5.3.1 (since the Dirichlet series in the numerator and the denominator have the same a.o.c., the limit is independent of the starting index by Lemma 3.4.3).

Independence of the choice for continuous functions follows from the fact the the image of a choice function is a dense set in X. Since the Lipschitz functions are dense in $C(X)$, if

μ is independent of the choice for all Lipschitz functions, it is independent for all continuous functions. For $f \in C_{\mathrm{Lip}}(X)$ and any choice functions $\tau, \tau' \in \Upsilon(\xi)$, where $\tau(U_k) = (x_k, y_k)$, $\tau'(U_k) = (x'_k, y'_k)$ and $a_k = \frac{1}{2}\left(f(x_k) + f(y_k)\right), a'_k = \frac{1}{2}\left(f(x'_k) + f(y'_k)\right)$ for each $k \in \mathbb{N}$, it follows that $a_k - a'_k \leq \mathrm{Lip}(f)\frac{1}{2}\left(d(x_k, x'_k) + d(y_k, y'_k)\right)$. Therefore

$$\lim_{j\to\infty}\left(\mu_{\tau,\sigma_j}(f) - \mu_{\tau',\sigma_j}(f)\right) = \lim_{j\to\infty}\frac{\sum_{k=1}^{\infty}(a_k - a'_k)\,\mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=1}^{\infty}\mathrm{diam}(U_k)^{s_0+\sigma_j}}$$
$$\leq \mathrm{Lip}(f)\lim_{j\to\infty}\frac{\sum_{k=1}^{\infty}\frac{1}{2}\left(d(x_k, x'_k) + d(y_k, y'_k)\right)\mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=1}^{\infty}\mathrm{diam}(U_k)^{s_0+\sigma_j}}$$
$$\leq \mathrm{Lip}(f)\lim_{j\to\infty}\frac{\sum_{k=1}^{\infty}\mathrm{diam}(U_k)^{s_0+\sigma_j+1}}{\sum_{k=1}^{\infty}\mathrm{diam}(U_k)^{s_0+\sigma_j}} = 0$$

The case for general continuous functions now follows by a standard $\frac{\epsilon}{3}$ argument. □

Remark. As previously mentioned, the lower bound condition $(*)$ in the definition of choice functions can be made to be more or less flexible—the important feature to ensure that the proposition holds is that as $k \to \infty$, $\frac{1}{\delta_k}\mathrm{diam}(U_k) \to 1$.

Corollary. *If s_f is strictly less than s_0, then $\mu_\tau(f) = 0$.*

Proof. If $s_f < s_0$, then $\sum_{k=1}^{\infty} a_k \delta_k^{s_0}$ is finite; since $\sum_{k=1}^{\infty}\delta_k^{s_0+\sigma} \to \infty$ as $\sigma \to 0$ (because $s_\tau = s_\xi = s_0$), the result follows. □

5.3.2 Uniqueness of the Limit

For the limiting state μ to be unique (independent of the sequence $\{\sigma_j\}_{j=1}^{\infty}$), it remains to construct a resolving sequence ξ so that

$$\mu(f) = \int_X f\,\mathrm{d}\mathcal{H}^{s_0} \tag{5.3.2}$$

for all Borel functions f on X. The construction depends on a diagonal argument; first it is shown that if f is a step function, then (5.3.2) holds for a specific resolving sequence related to the function. More precisely when f is a linear combination of characteristic functions for the sets of a finite partition of X into Borel sets of positive measure, then a Hausdorff resolving sequence satisfies the desired condition. Then a sequence of partitions will be constructed so that a diagonal resolving sequence related to the partitions will give the desired result for any Borel partition, and thus (5.3.2) will hold for all measurable functions on X.

Lemma 5.3.2. *Given a partition $\mathcal{F}$ of X into Borel sets of positive measure, if ξ is any strict Hausdorff resolving sequence that respects $\mathcal{F}$, then for any $\tau \in \Upsilon(\xi)$ and any simple function ϕ of the form $\phi = \sum_{F\in\mathcal{F}} \varphi_F \chi_F$ with $\varphi_F \in \mathbb{C}$ for each $F \in \mathcal{F}$,*

$$\mu(\phi) := \lim_{\sigma\to 0} \frac{\mathrm{Tr}\left(|D_\tau|^{-(s_0+\sigma)}\pi(\phi)\right)}{\mathrm{Tr}\left(|D_\tau|^{-(s_0+\sigma)}\right)} = \int_X \phi \, \mathrm{d}\mathcal{H}^{s_0}$$

Proof. Since μ is linear, $\mu(\phi) = \sum_{F\in\mathcal{F}} \varphi_F \mu(\chi_F)$, so it suffices to consider the case where ϕ is the characteristic function of a Borel set F with $\mathcal{H}^{s_0}(F) > 0$, and the partition $\mathcal{F}$ is $\left\{F, F^C\right\}$. In this case, (5.3.1) in Lemma 5.3.1 gives

$$\begin{aligned}\mu(\chi_F) &= \lim_{j\to\infty} \frac{\zeta_\tau(s_0+\sigma_j, \chi_F)}{\zeta_\tau(s_0+\sigma_j)} \\ &= \lim_{j\to\infty} \frac{\sum_{k=K}^\infty a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}}{\sum_{k=K}^\infty \mathrm{diam}(U_k)^{s_0+\sigma_j}}\end{aligned}$$

for any $K \in \mathbb{N}$ (by Lemma 3.4.3).

Since $\xi = \{\mathcal{U}_n\}_{n=0}^\infty$ is a resolving sequence that respects the partition $\mathcal{F} = \left\{F, F^C\right\}$, there are resolving sequences $\xi_F = \left\{\mathcal{U}^F{}_n\right\}_{n=0}^\infty$ for F and $\xi_C = \left\{\mathcal{U}^C{}_n\right\}_{n=0}^\infty$ for F^C such that $\mathcal{U}_n = \mathcal{U}_n^F \coprod \mathcal{U}_n^C$ for $n = 0, \ldots, \infty$. Since ξ is strict,

$$\begin{aligned}\mu(\chi_F) &= \lim_{j\to\infty} \frac{\sum_{n=N}^\infty \left(\sum_{k\in I_n} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}\right)}{\sum_{n=N}^\infty Q_n(s_0+\sigma_j)} \\ &= \lim_{j\to\infty} \frac{\sum_{n=N}^\infty \left(\sum_{k\in I_n^F} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j} + \sum_{k\in I_n^C} a_k \mathrm{diam}(U_k)^{s_0+\sigma_j}\right)}{\sum_{n=N}^\infty Q_n(s_0+\sigma_j)}\end{aligned}$$

for any $N \in \mathbb{N}$, where $\left\{I_n^F\right\}_{n=0}^\infty$ and $\left\{I_n^C\right\}_{n=0}^\infty$ are the index sets for the resolving sequences ξ_F and ξ_C, respectively (see Definition 4.1.2). The expression for $\mu(\chi_F)$ depends on the choice via the a_k; to recover the Hausdorff measure of F, it is sufficient that $a_k = 1$ for $k \in I_n^F$ and $a_k = 0$ for $k \in I_n^C$. In general, this need not occur because the choice selects points in an element of a cover of F, and these points need not be contained in F. By

Lemma 5.3.1, it is possible to fix a choice that only selects points contained in F, showing that the points chosen by any $\tau \in \Upsilon(\xi_F)$ that lie outside of F are negligible with respect to the asymptotics of the zeta-function.

Given the lower bound condition $(*)$ on the choices, it is possible that no choice exists that selects two points in F for every open set of ever cover of ξ_F. This requires the additional assumption on ξ_F that each open set of each cover $\mathcal{U}_n^F$ be chosen so that

$$\sup_{x \in F^C} d(x, F) < \frac{\operatorname{diam}(U)^2}{1 + \operatorname{diam}(U)}$$

—because this leaves the diameters of the open sets U either unchanged or smaller, this does not have an effect on the other properties of the resolving sequence ξ_F (i.e. it can only reduce the quantities $Q_n(s_0)$, bringing them closer to $\mathcal{H}_\delta^{s_0}(F)$). With this assumption, it is possible to fix a choice that only selects points in F, and thus, letting $Q_n^F(s) := \sum_{k \in I_n^F} \operatorname{diam}(U_k)^s$,

$$\mu(\chi_F) = \lim_{j \to \infty} \frac{\sum_{n=N}^{\infty} Q_n^F(s_0 + \sigma_j)}{\sum_{n=N}^{\infty} Q_n(s_0 + \sigma_j)} = \frac{\mathcal{H}^{s_0}(F)}{\mathcal{H}^{s_0}(X)}$$

In addition, $\chi_F + \chi_{F^C} = \chi_X$, so $\mu(\chi_F) + \mu(\chi_{F^C}) = 1$, and $\mu(\chi_{F^C}) = \frac{\mathcal{H}^{s_0}(F^C)}{\mathcal{H}^{s_0}(X)}$ as well. $\square$

Thus far, a limiting state $\mu = \lim_{j \to \infty} \mu_{\tau, \sigma_j}$ is independent of the limiting process only on characteristic functions of the sets of a partition respected by the resolving sequence ξ. Thus it remains to construct a resolving sequence that respects an (almost) arbitrary Borel partition. In fact, it is sufficient that $\mu(\chi_B)$ give the Hausdorff measure of any ball B in a countable family of balls that generate the topology on X.

Given a Borel partition $\mathcal{F}$ of X, a resolving sequence $\xi = \{\mathcal{U}_n\}_{n=0}^{\infty}$ respects the partition if there are resolving sequences $\xi_F = \left\{\mathcal{U}^F{}_n\right\}_{n=0}^{\infty}$ for each $F \in \mathcal{F}$ (Definition 4.2.6). It follows that if $F = F_1 \coprod F_2$ for Borel sets $F_1, F_2 \in \mathcal{F}$, then $\lim_{n \to \infty} Q_n(s_0, F) = \mathcal{H}^{s_0}(F)$, where $Q_n(s_0, F) = Q_n^{F_1}(s_0) + Q_n^{F_2}(s_0)$. Henceforth, this notation will be used whenever F is a union of sets F_i in a partition $\mathcal{F}$: F occurs as a superscript on Q_n^F to indicate that the set F is an element of a partition, while $Q_n(s_0, F)$ indicates that F is a union of sets of a partition (respected by ξ).

Since X is a compact metric space, it is separable, so it is possible to extract a countable dense subset $\hat{S} \subset X$. Let $\mathcal{B} := \left\{ B(x, q) \;\middle|\; x \in \hat{S}, q \in \mathbb{Q} \right\}$.

Proposition 5.3.3. *The cover $\mathcal{B}$ is a countable basis for the topology on X.*

Proof. The proof is standard. That $\mathcal{B}$ is countable is automatic, as the balls in $\mathcal{B}$ are indexed by (a subset of) the countable product $\left\{(x,q) \;\middle|\; x \in \hat{S}, q \in \mathbb{Q}\right\}$. Because the topology on X is generated by metric balls, for any open set $U \subseteq X$ and $x \in U$, there is a ball $B(x',r)$ such that $x \in B(x',r) \subseteq U$. Since $\hat{S}$ is dense there is a point $x_0 \in \hat{S}$ contained in the open set $B(x',r)$ and for any rational number q in the interval $(d(x_0,x),r)$, the ball $B(x_0,q)$ contains x and is contained in $B(x,r) \subseteq U$. □

Theorem 5.3.4. *There exists a Hausdorff resolving sequence ξ for X such that the limit $\mu = \lim_{\sigma\to 0} \mu_{\tau,\sigma}$ exists and is independent of $\tau \in \Upsilon(\xi)$, and*

$$\mu(f) = \int_X f \, \frac{\mathrm{d}\mathcal{H}^{s_0}}{\mathcal{H}^{s_0}(X)}$$

for all Borel functions f.

Proof. Choose a sequence $\{r_k\}_{k=1}^{\infty}$ of rational numbers strictly decreasing to 0. Let S_1 be a finite subset of $\hat{S}$ such that $\mathcal{B}_1 := \{B(x,r_1) \mid x \in S_1\}$ is a minimal open cover of X. Fix an ordering $x_1, \ldots, x_{|S_1|}$ of the elements of S_1. For each $k \in \mathbb{N}$, proceed inductively as follows:

1. Whenever $\{B(x,r_{k+1}) \mid x \in S_k\}$ covers X, remove r_{k+1} from the sequence and reindex.
2. Let S_{k+1} be a finite subset of $\hat{S}$ such that $\mathcal{B}_{k+1} := \{B(x,r_{k+1}) \mid x \in S_{k+1}\}$ is a minimal cover of X, with $S_{k+1} \supset S_k$.
3. Fix an ordering $x_{|S_k|+1}, \ldots, x_{|S_{k+1}|}$ of the elements of $S_{k+1} \backslash S_k$.

Let $S = \bigcup_{k=1}^{\infty} S_k$. Since $\lim_{k\to\infty} r_k = 0$, the sequence of minimal covers $\{\mathcal{B}_n\}_{n=0}^{\infty}$ is necessarily a resolving sequence for X (where $\mathcal{B}_0 = \{X\}$).

Let $\mathcal{F}_k$ be the finest partition generated by $\mathcal{B}_k$: for $i = 1, \ldots, |S_k|$, if

$$\mathcal{F}_k^i = \left\{ \bigcap_{B\in\alpha} B \;\middle|\; \alpha \subseteq \mathcal{B}_k, |\alpha| = i \right\}$$

consists of i-fold intersections of elements of $\mathcal{B}_k$, then

$$\mathcal{F}_k := \coprod_{i=1}^{|S_k|} \left\{ F \backslash \left(\bigcup_{j>i} \bigcup_{F'\in\mathcal{F}_k^j} F' \right) \;\middle|\; F \in \mathcal{F}_k^i \right\}$$

is a partition of X into minimal intersections. Let $\mathcal{P}_k := \bigvee_{i=1}^{k} \mathcal{F}_i = \mathcal{F}_k \vee \mathcal{F}_{k-1}$.

For each $k \in \mathbb{N}$, let ξ_k be the strict Hausdorff resolving sequence that respects the partition $\mathcal{P}_k$ given by $\mathcal{U}_n = \coprod_{P \in \mathcal{P}_k} \mathcal{U}_n^F$, where $\xi_F = \left\{\mathcal{U}^F{}_n\right\}_{n=0}^{\infty}$ is a strict Hausdorff resolving sequence for F.

For $k \in \mathbb{N}$, let $\xi_k = \left\{\mathcal{U}^k{}_n\right\}_{n=0}^{\infty}$ be a resolving sequence of X that respects the partition $\mathcal{P}_k$. This determines a bisequence $\left\{\mathcal{U}_n^k\right\}_{n=0,k=1}^{\infty}$ of covers of X. Let ξ be the diagonal resolving sequence given by $\mathcal{U}_n := \mathcal{U}_n^n$ for $n = 0, \ldots, \infty$. Each cover $\mathcal{U}_n$ can therefore be decomposed as a union of covers $\mathcal{U}_n^F$ of each set $F \in \mathcal{P}_n$ (since $\mathcal{U}_n \in \xi_n$ and ξ_n respects $\mathcal{P}_n$):

$$\mathcal{U}_n = \mathcal{U}_n^n = \coprod_{F \in \mathcal{P}_n} \mathcal{U}_n^F$$

Since $\mathcal{P}_{k+1}$ refines $\mathcal{P}_k$ for each $k \in \mathbb{N}$, given $F \in \mathcal{P}_k$ for some fixed k, for all $n \geq k$ there is a subset $\mathcal{P}_n^F \subseteq \mathcal{P}_n$ such that

$$\coprod_{F' \in \mathcal{P}_n^F} F' = F$$

Let $Q_n(s, F) = \sum_{F' \in \mathcal{P}_n^F} Q_n^{F'}(s)$. Since $Q_n^F(s_0) < \mathcal{H}^{s_0}(F)\,(1 + \epsilon_n)$ by construction (inequality (4.2.2)),

$$\lim_{n \to \infty} Q_n^F(s_0) = \mathcal{H}^{s_0}(F)$$

In fact this holds for any F that is a disjoint union of elements of the various $\mathcal{P}_k$. This includes $B(x, r_k)$ for any $x \in S$ and $k \in \mathbb{N}$. Since the $\{\mathcal{B}_n\}_{n=0}^{\infty}$ is a resolving sequence, the balls $B(x, r_k)$ form a basis for the topology on X, and thus any Borel set F is measured by μ and $\mu(F) = \mathcal{H}^{s_0}(F)$. $\square$

Remark. As opposed to the analogous construction in [51], the existence of μ does not depend on the properties of a specific zeta-function derived from X (and hence does not depend on X), but rather the resolving sequence is chosen so that the construction always yields the desired property. Essentially, rather than selecting spaces for which a canonical resolving sequence gives the desired result, resolving sequences are now determined in a non-canonical (and non-unique) way so that no limitations on the space X are made.

CHAPTER VI

CONCLUDING REMARKS

In [51], a family of spectral triples $(\mathcal{A}, \mathcal{H}, D_\tau)$ are constructed based on choice functions for an ultrametric Cantor set. The construction relies on the Michon graph of the Cantor set, which is equivalent to a resolving sequence of covers by clopen sets. As opposed to $(*)$, the choice functions of this construction satisfy the more strict condition

$$d\left(\tau_+(U), \tau_-(U)\right) = \mathrm{diam}(U)$$

This condition is possible in the setting of Cantor sets because all of the open sets are also closed—asymptotically, this condition is equivalent to the condition $(*)$. However, the upper box dimension is a lower bound for the spectral dimension, so the resolving sequence determined by the Michon graph is not Hausdorff. The metric is recovered precisely if and only if the metric on the Cantor set is an ultrametric.

Also in [51], by defining a measure ν over the space of choice functions $\Upsilon(\xi)$, a closable sesquilinear form (in fact, a one-parameter family of such forms) on $\mathcal{H}$ is constructed, determining an analogue of the Laplace-Beltrami operator on a compact manifold:

$$Q_s\left(f, g\right) := \int_{\tau(\xi)} \mathrm{Tr}\left(|D|^{-s}\left[D_\tau, \pi(f)\right]^*\left[D_\tau, \pi(g)\right]\right)\, \mathrm{d}\nu(\tau)$$

This operator generates a semigroup of operators on $\mathcal{H}$, which determines a diffusion process on the Cantor set C. On Cantor sets, the topology is generated by clopen sets and all continuous functions are in effect harmonic. It is expected that in spaces with connected components of positive measure there will be non-harmonic continuous functions and $Q_D\left(\cdot, \cdot\right)$ will take a slightly different form.

For any Hilbert space $\mathcal{H}$, if T is an operator on $\mathcal{H}$, then

$$Q_T\left(\psi, \phi\right) := \langle \psi, T\phi \rangle_{\mathcal{H}}$$

is a bilinear form on $\mathcal{H}$, and for every bilinear form there is a corresponding operator T

satisfying this condition. Moreover, if the bilinear form is symmetric nonpositive-definite, then T is a self-adjoint nonpositive operator.

Let (X, μ) be a locally compact separable Radon space such that μ is positive and $\operatorname{supp}(\mu) = X$. Let $Q_D(\cdot, \cdot)$ be a symmetric nonnegative-definite bilinear form on $L^2(X, \mu)$. For any measurable function $f : X \to \mathbb{R}$, let

$$\hat{f} = \begin{cases} 0 & f(x) \leq 0 \\ f(x) & f(x) \in [0, 1] \\ 1 & f(x) \geq 1 \end{cases}$$

Definition 6.0.5 (Dirichlet Form)**.** If the symmetric form $Q_D(\cdot, \cdot)$ on $L^2(X, \mu)$ satisfies the *Markovian* condition

$$Q_D\left(\hat{f}, \hat{f}\right) \leq Q_D(f, f) \qquad \forall \text{ measurable } f : X \to \mathbb{R}$$

then $Q_D(\cdot, \cdot)$ is a *Dirichlet form* on $L^2(X, \mu)$.

Definition 6.0.6 (Generator)**.** An *infinitesimal generator* of a (one-parameter) semigroup $\{\Phi_t \mid t \geq 0\}$ of operators on a Hilbert space $\mathcal{H}$ is an operator A such that the limit:

$$A := \lim_{t \to 0} \frac{1}{t} \left(\Phi_t(f) - f\right)$$

exists for all $f \in \mathcal{H}$. A semigroup $\left\{e^{t\Delta} \mid t \geq 0\right\}$ is *strongly continuous* if the assignment

$$t \mapsto \Phi_t$$

is continuous in the strong operator topology on $\mathcal{B}(\mathcal{H})$

If Δ_Q is the operator corresponding to a Dirichlet form $Q_D(\cdot, \cdot)$, it generates a strongly continuous symmetric Markov contraction semigroup [28].

Theorem 6.0.7 (Fukushima, 1971)**.** *A contraction semigroup on a Hilbert space $\mathcal{H}$ is a Markov semigroup if and only if its generator is defined by a Dirichlet form.*

As a result, defining a diffusion process on the Radon space (X, μ) amounts to defining a Dirichlet form $Q_D(\cdot, \cdot)$ on $L^2(X, \mu)$. Motivated by [51], it should be possible to construct an analagous Dirichlet form for the resolving seqeunce of Theorem 5.3.4. Bellissard [3] has outlined a general procedure to define a Laplace-Beltrami operator for spectral triples that are both regular and spectrally regular, such as the spectral triples $(\mathcal{A}, \mathcal{H}, D_\tau, \pi_\tau)$ from the previous chapter.

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