SIAM J. CONTROL AND OPTIMIZATION
Vol. 28, No. 3, pp. 653-670, May 1990

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A SKEW TOEPLITZ APPROACH TO THE H^∞ OPTIMAL CONTROL OF MULTIVARIABLE DISTRIBUTED SYSTEMS*

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Abstract. In this paper the problem of the H^∞ optimization of multivariable distributed systems in the four block setting is studied. This work is based on several previous papers and employs the skew Toeplitz framework developed in [*Operator Theory: Adv. Appl.*, 32 (1988), pp. 21-43], [*Operator Theory: Adv. Appl.*, 32 (1988), pp. 93-112], [*Operator Theory and Integral Equations*, 11 (1988), pp. 726-767], [*J. Functional Anal.*, 74 (1987), pp. 146-159], [SIAM J. Math. Anal., 19 (1988), pp. 1081-1091].

Key words. H^∞-optimal control, distributed system, four block problem, skew Toeplitz operator

AMS(MOS) subject classifications. 93B35, 93C05

1. Introduction. In the past few years, there has been a major research effort devoted to the study of the H^∞ optimization of linear systems. We refer the reader to [13] for an extensive set of references. In this paper we consider the problem of the H^∞-optimization for multivariable distributed systems.

Motivations leading to the H^∞ optimization in systems theory lie in the most natural problems of control engineering such as robust stabilization, sensitivity minimization, and model matching. It can be shown that, in the sense of H^∞ optimality, these problems are equivalent, and can be stated (see [13]) as one *standard problem.* Consider the setup shown in Fig. 1. In this configuration w, u, y, and z are vector-valued signals with w the exogenous input representing the disturbances, measurement noises, etc., u the command signal, z the output to be controlled, and y the measured output. G represents a combination of the plant and the weights in the control system. The standard H^∞ problem is to find a stabilizing controller K such that the H^∞ norm of the transfer function from w to z is minimized. For finite-dimensional systems an expression for a suboptimal controller is given in [2] and [4] using a state-space approach.

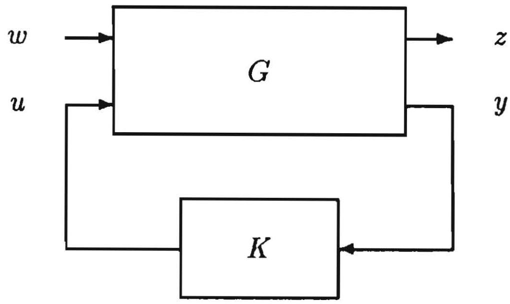

FIG. 1

* Received by the editors November 21, 1988; accepted for publication (in revised form) July 12, 1989. This work was supported by National Science Foundation grants ECS-8704047 and DMS-8811084, and by Air Force Office of Scientific Research grant AFOSR-88-0020.

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Now it is quite well known that an optimal solution of the standard problem can be reduced to finding the singular values of a certain operator (the so-called **four block operator**) that will be defined below. For details we refer the reader to [5]-[7]. Depending on the specific problem considered, the corresponding four block operator can be simplified to a 2-block or a 1-block operator.

This paper is based on several previous papers [6]-[12], [21], and basically employs the skew Toeplitz framework of [3] to study the standard problem. We should note that software for the implementation of the techniques used in this paper has already been written at the Systems Research Center of Honeywell, Minneapolis in collaboration with Blaise Morton, and has been applied to several distributed systems including a flexible beam problem. We plan to write a paper with several such "benchmark" examples with Blaise Morton in the near future.

The present paper is organized as follows. In the next section we set up some notation and give some background on the ideas taken from previous work. In § 3 we derive our main result which is a rank type formula for the singular values of the four block operator. We illustrate a special case of our main result by considering SISO plants in § 4, and by giving an explicit example in § 5. Finally, in § 6 we summarize our results and make some comments.

2. Problem definition and preliminary remarks. We will now state the standard H^∞ problem and define the four block operator. We will also present some preliminary results from earlier work [3], [6], [7]. Throughout the paper all Hardy spaces are defined on the unit disc D in the standard way. For an integer m we denote the canonical unilateral shift (defined by multiplication by z) on $H^2(\mathbf{C}^m)$ by $S: H^2(\mathbf{C}^m) \to H^2(\mathbf{C}^m)$ and the bilateral shift on $L^2(\mathbf{C}^m)$ by $U: L^2(\mathbf{C}^m) \to L^2(\mathbf{C}^m)$. Let W, F, G, J, and M be H^∞ matrices, of sizes $p \times m$, $p \times l$, $q \times m$, $q \times l$, and $p \times p$, respectively, with $p \leqq \max\{m, l\}$, where W, F, G, J have rational entries, and M is a nonconstant inner matrix. These matrices are associated with the weighting matrices and the plant in the usual way of transforming the standard problem to the 4-block framework (i.e., via Youla parametrization and some inner outer factorizations; see, e.g., [13] and [20]). It is important to note that for many problems of interest, in the case of rational weights and distributed stable plants, this reduces to the kind of problem described below. See [15] for all the details. The standard H^∞ problem amounts to finding

$$\mu := \inf\left\{ \left\| \begin{bmatrix} W - MQ & F \\ G & J \end{bmatrix} \right\|_\infty : Q \in H^\infty p \times m \right\},$$

where for a $k \times n$ matrix of the form $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, ($A, B, C, D$ having appropriate sizes with entries in L^∞), we set

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|_\infty = \operatorname{ess\,sup} \left\{ \left\| \begin{bmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{bmatrix} \right\| : |\zeta| = 1 \right\}.$$

(For the norm on the right-hand side the $k \times n$ matrix is taken as a linear operator from $\mathbf{C}^n$ to $\mathbf{C}^k$ for each fixed ζ in ∂D, the unit circle.) Note that if $F = G = J = 0$ then this problem reduces to the classical Nehari problem, which is also known as the *1-block problem.* For $F = J = 0$ we have the *2-block problem.*

To the $p \times p$ inner matrix M, we associate the spaces $H(M) := H^2(\mathbf{C}^p) \ominus MH^2(\mathbf{C}^p)$ and $L(M) := L^2(\mathbf{C}^p) \ominus MH^2(\mathbf{C}^p)$. Let $P_{H(M)}: H^2(\mathbf{C}^p) \to H(M)$, $P_{L(M)}: L^2(\mathbf{C}^p) \to L(M)$, $P_{H^2}: L^2(\mathbf{C}^p) \to H^2(\mathbf{C}^p)$, and $P_{L^2 \ominus H^2}: L^2(\mathbf{C}^p) \to L^2(\mathbf{C}^p) \ominus H^2(\mathbf{C}^p)$ be orthogonal projections.

We now define the 4-block operator (see [5] and [7]):

$$A := \begin{bmatrix} P_{H(M)}W(S) & P_{L(M)}F(U) \\ G(S) & J(U) \end{bmatrix}.$$

Note that $A: H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l) \to L(M) \oplus L^2(\mathbf{C}^q)$.

In the paper, by a slight abuse of notation, ζ will denote a complex variable as well as an element of ∂D. The context will make the meaning clear. Note that $W(S)$ can be seen as the operator defined by multiplication by $W(\zeta)$, and similarly for $G(S)$, $F(U)$, and $J(U)$. Using the commutant lifting theorem [18, pp. 257-259], we can show that μ is equal to $\|A\|$. (See [5] and [7] for the details.) Note that $\|A\|^2$ is the largest element of $\sigma(A^*A)$, the spectrum of A^*A. $\sigma(A^*A)$ consists of the discrete spectrum (i.e., eigenvalues with finite multiplicity), which we denote by $\sigma_d(A^*A)$, and its complement $\sigma_e(A^*A)$, the essential spectrum. The essential spectrum of A^*A consists of those $\lambda \in \mathbf{C}$ for which there exists

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} \in H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l) \quad \text{with} \quad \left\| \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\|_2 = 1 \quad \forall n \geqq 1,$$

and $\left[\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right] \to 0$ weakly as $n \to \infty$, such that

$$(\lambda I - A^*A)\begin{bmatrix} x_n \\ y_n \end{bmatrix} \to 0 \quad \text{as } n \to \infty.$$

The essential norm, denoted by $\|A\|_e$, is defined as

$$\|A\|_e^2 = \max\{\lambda : \lambda \in \sigma_e(A^*A)\}.$$

In the SISO case we have that (see [7, Thm. 3.2])

$$\|A\|_e = \max(\alpha, \beta, \gamma),$$

where

$$\alpha = \max\left\{ \left\| \begin{bmatrix} W(\zeta) & F(\zeta) \\ G(\zeta) & J(\zeta) \end{bmatrix} \right\| : \zeta \in \sigma_e(T) \right\},$$

$$\beta = \max\{\|[G(\zeta) \quad J(\zeta)]\| : \zeta \in \partial D\},$$

$$\gamma = \max\left\{ \left\| \begin{bmatrix} F(\zeta) \\ J(\zeta) \end{bmatrix} \right\| : \zeta \in \partial D \right\}.$$

$\sigma_e(T)$ denotes the essential spectrum of the operator $T := P_{H(M)}S|_{H(M)}$. We let $\mathcal{R}$ be the set of all $\lambda \in \partial D$ that do not lie on any of the open arcs of ∂D on which $M(\zeta)$ is a unitary operator-valued analytic function. Then from [17] and [18], we have that

$$\sigma_e(T) = \mathcal{R}.$$

In the case of infinite-dimensional MIMO systems it may be difficult to find the essential norm of A. Nevertheless, upper and lower bounds can be obtained in terms of α, β, γ. This is discussed in detail in § 3.2.

Note that when $\|A\| > \|A\|_e$, $\|A\|^2$ is an eigenvalue of A^*A. Here we are going to develop a rank type formula for the eigenvalues of A^*A. We will show that this formula is obtained by a certain linear system of equations (called the *singular system* in [7]). These equations are derived from the inversion of two Toeplitz operators and the essential inversion of a skew Toeplitz operator. It is important to note that in the

2-block problem, one of the Toeplitz operator inversions disappears, and in the 1-block case the same is true for both of the Toeplitz operator inversions. The Fredholm conditions on the invertibility of the skew Toeplitz operator (which is essentially invertible) and the coupling between various systems of equations constitute the singular system. See also [3] and [7].

3. Main results.

3.1. Discrete spectrum. Let us begin with the following assumption $W=B/k$, $F=C/k$, $G=D/k$, and $J=E/k$, where B, C, D, E are polynomial matrices and k is a scalar polynomial. We denote by n an upper bound for the degree of the entries of all polynomial matrices appearing throughout the paper.

Now it is easy to see that ρ^2 is an eigenvalue of A^*A if and only if there exists a nonzero

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H^2(\mathbf{C}^m)\oplus L^2(\mathbf{C}^l),$$

such that

$$\begin{aligned}(\rho^2 k(S)^*k(S)I - B(S)^*P_{H(M)}B(S) - D(S)^*D(S))x \\ -(P_{H^2}(B(U)^*P_{L(M)}C(U)+D(U)^*E(U)))y=0,\end{aligned} \tag{1a}$$

and

$$\begin{aligned}-((C(U)^*P_{H(M)}B(S)+E(U)^*D(S)))x \\ +(\rho^2 k(U)^*k(U)I - C(U)^*P_{L(M)}C(U) - E(U)^*E(U))y=0.\end{aligned} \tag{1b}$$

Note that $P_{H(M)}B(S)x = B(\zeta)x - M(\zeta)P_{H^2}M(\zeta)^*B(\zeta)x$. Following the techniques used in [3], we make the factorization

$$M(\zeta)^*B(\zeta)=\Omega_b(\zeta)M_b(\zeta)^*, \tag{f1}$$

where $\Omega_b(\zeta)$ is a polynomial matrix of size $p\times m$ and $M_b(\zeta)$ is an inner matrix of size $m\times m$. We now decompose the space $H^2(\mathbf{C}^m)$ as $H(M_b)\oplus M_bH^2(\mathbf{C}^m)$, and express $x=x_b+M_bx_b'$ where $x_b\in H(M_b)$ and $x_b'\in H^2(\mathbf{C}^m)$. Then we have

$$P_{H^2}M(\zeta)^*B(\zeta)x = P_{H^2}\Omega_b(\zeta)M_b(\zeta)^*(x_b+M_bx_b').$$

Since M_b is inner,

$$P_{H^2}M(\zeta)^*B(\zeta)x=\Omega_b(\zeta)x_b'+P_{H^2}\Omega_b(\zeta)M_b(\zeta)^*x_b.$$

By (f1) we see that the right-hand side of this last equality is equal to

$$M(\zeta)^*B(\zeta)M_b(\zeta)x_b'+P_{H^2}\Omega_b(\zeta)M_b(\zeta)^*x_b.$$

We can write $\Omega_b(\zeta)=\Omega_{b0}+\Omega_{b1}\zeta+\cdots+\Omega_{bn}\zeta^n$. The fact that $x_b\in H(M_b)$ implies

$$M_b(\zeta)^*x_b=\zeta^{-1}u_{-1}+\zeta^{-2}u_{-2}+\cdots$$

for some $u_{-i}\in\mathbf{C}^m$, $i\geqq 1$. Therefore,

$$P_{H^2}\Omega_b(\zeta)M_b(\zeta)^*x_b=\sum_{i=1}^{n}\sum_{j=i}^{n}\Omega_{bj}\zeta^{j-i}u_{-i}=:x_{\omega b}.$$

Combining the above computations we get

$$P_{H(M)}B(S)x=B(S)x_b-Mx_{\omega b}.$$

Similarly, for the computation of

$$P_{L(M)}C(U)y = C(U)y - MP_{H^2}M^*Cy,$$

we use the factorization

$$(2)\qquad M(\zeta)^*C(\zeta) = \Omega_c(\zeta)M_c(\zeta)^*,$$

where $\Omega_c(\zeta)$ is a polynomial matrix of size $p\times l$ and M_c is an inner matrix of size $l\times l$. As before we write $y = y_c + M_c y_c'$ where $y_c \in L(M)$ and $y_c' \in H^2(\mathbf{C}^l)$. Let $\Omega_c(\zeta) = \Omega_{c0} + \Omega_{c1}\zeta + \cdots + \Omega_{cn}\zeta^n$. Then

$$P_{H^2}\Omega_c M_c^* y_c = \sum_{i=1}^{n}\sum_{j=i}^{n}\Omega_{cj}\zeta^{j-i}v_{-i} =: y_{wc}$$

for some $v_{-i}\in\mathbf{C}^l$, $i = 1,\cdots,n$. This leads us to

$$P_{L(M)}C(U)y = C(U)y_c - My_{wc}.$$

Now we see that, with the above factorizations and decompositions, (1a), (1b) are equivalent to

$$\begin{aligned}(2a)\qquad &(\rho^2k(S)^*k(S)I - D(S)^*D(S) - B(S)^*B(S))x_b\\ &-P_{H^2}((B(U)^*C(U) + D(U)^*E(U))y_c + D(U)^*E(U)M_cy_c')\\ &+(\rho^2k(S)^*k(S)I - D(S)^*D(S))M_bx_b' = -B(S)^*Mx_{wb} - P_{H^2}B(U)^*My_{wc},\end{aligned}$$

and

$$\begin{aligned}(2b)\qquad &(\rho^2k(U)^*k(U)I - E(U)^*E(U) - C(U)^*C(U))y_c\\ &-(C(U)^*B(S) + E(U)^*D(S))x_b - E(U)^*D(S)M_bx_b'\\ &+(\rho^2k(U)^*k(U)I - E(U)^*E(U))M_cy_c' = -C(U)^*My_{wc} - C(U)^*Mx_{wb}.\end{aligned}$$

Now we will compute $P_{H^2}(B(U)^*C(U) + D(U)^*E(U))y_c$. First write

$$B(U)^*C(U) + D(U)^*E(U) = Q_{-n}^1U^{*n} + \cdots + Q_0^1 + \cdots + Q_n^1U^n.$$

Then,

$$P_{H^2}Q_{-i}^1U^{*i}y_c = Q_{-i}^1P_{H^2}U^{*i}y_c = Q_{-i}^1S^{*i}(P_{H^2}y_c).$$

Let $y_c = \cdots + y_{c(-1)}\zeta^{-1} + y_{c(0)} + y_{c(1)}\zeta + \cdots$. Then

$$P_{H^2}Q_i^1U^iy_c = Q_i^1S^i(P_{H^2}y_c) + Q_i^1(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).$$

Therefore,

$$\begin{aligned}&P_{H^2}(B(U)^*C(U) + D(U)^*E(U))y_c\\ &\qquad = (B(S)^*C(S) + D(S)^*E(S))(P_{H^2}y_c) + \sum_{i=1}^{n}Q_i^1(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).\end{aligned}$$

Similarly, we have

$$P_{H^2}D(U)^*E(U)M_cy_c' = D(S)^*E(S)M_cy_c'.$$

Hence (2a) is equivalent to

$$\begin{aligned}&(\rho^2k(S)^*k(S)I - D(S)^*D(S) - B(S)^*B(S))x_b + (\rho^2k(S)^*k(S)I - D(S)^*D(S))M_bx_b'\\ (3a)\qquad &-((B(S)^*C(S) + D(S)^*E(S))y_c^+ + D(S)^*E(S)M_cy_c')\\ &= -B(S)^*Mx_{wb} - B(S)^*My_{wc} + \sum_{i=1}^{n}Q_i^1(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}),\end{aligned}$$

where $y_c^+ := P_{H^2}y_c$. Note that we have $y = y_c^- + y_c^+ + M_cy_c'$, where $y_c^- \in L(M_c)\ominus H(M_c)$, $y_c^+\in H(M_c)$, and $y_c'\in H^2(\mathbf{C}^l)$.

We will separate the equation (2b) into two parts by taking the orthogonal projections on $H^2(\mathbf{C}^l)$ and $L^2(\mathbf{C}^l)\ominus H^2(\mathbf{C}^l)$. As in the above discussion, if

$$\rho^2 k(U)^*k(U)I - E(U)^*E(U) - C(U)^*C(U) =: Q^2_{-n}U^{*n} + \cdots + Q^2_0 + \cdots + Q^2_n U^n,$$

then we have

$$P_{H^2}(\rho^2 k(U)^*k(U)I - E(U)^*E(U) - C(U)^*C(U))y_c$$
$$= (\rho^2 k(S)^*k(S)I - E(S)^*E(S) - C(S)^*C(S))y_c^+ + \sum_{i=1}^{n} Q_i^2(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).$$

Hence the projection of (2b) on $H^2(\mathbf{C}^l)$ gives

$$(\rho^2 k(S)^*k(S)I - E(S)^*E(S) - C(S)^*C(S))y_c^+ + (\rho^2 k(S)^*k(S)I - E(S)^*E(S))M_c y_c'$$
$$\text{(3b)} \qquad -(C(S)^*B(S) + E(S)^*D(S))x_b - E(S)^*D(S)M_b x_b'$$
$$= -C(S)^*My_{wc} - C(S)^*Mx_{wb} - \sum_{i=1}^{n} Q_i^2(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).$$

We now study the projection of (2b) on $L^2(\mathbf{C}^l)\ominus H^2(\mathbf{C}^l)$. First note that

$$P_{L^2\ominus H^2}Q^2_{-i}U^{*i}y_c = Q^2_{-i}U^{*i}y_c^- + Q^2_{-i}(\zeta^{-i}y_{c0} + \cdots + \zeta^{-1}y_{c(i-1)}),$$

and

$$P_{L^2\ominus H^2}Q_i^2U^iy_c = Q_i^2U^iy_c^- - Q_i^2(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).$$

Hence

$$P_{L^2\ominus H^2}(\rho^2 k(U)^*k(U)I - E(U)^*E(U) - C(U)^*C(U))y_c$$
$$= (\rho^2 k(U)^*k(U)I - E(U)^*E(U) - C(U)^*C(U))y_c^-$$
$$+ \sum_{i=1}^{n} Q^2_{-i}(\zeta^{-i}y_{c0} + \cdots + \zeta^{-1}y_{c(i-1)})$$
$$- \sum_{i=1}^{n} Q_i^2(\zeta^{i-1}y_{c(-1)} + \cdots + y_{c(-i)}).$$

This takes care of the first term in (2b). For the projections of the other terms we use the following notation:

$$\rho^2 k(U)^*k(U)I - E(U)^*E(U) =: Q^3_{-n}U^{*n} + \cdots + Q^3_0 + \cdots + Q^3_nU^n,$$
$$C(U)^*B(U) + E(U)^*D(U) =: Q^4_{-n}U^{*n} + \cdots + Q^4_0 + \cdots + Q^4_nU^n,$$
$$E(U)^*D(U) =: Q^5_{-n}U^{*n} + \cdots + Q^5_0 + \cdots + Q^5_nU^n,$$
$$M_b(\zeta) =: M_{b0} + M_{b1}\zeta^1 + M_{b2}\zeta^2 + \cdots,$$
$$M_c(\zeta) =: M_{c0} + M_{c1}\zeta^1 + M_{c2}\zeta^2 + \cdots,$$
$$M(\zeta) =: M_0 + M_1\zeta^1 + M_2\zeta^2 + \cdots,$$
$$C(U)^* =: C_0^* + C_1^*U^* + \cdots + C_n^*U^{*n},$$
$$x_b'(\zeta) =: x_{b0}' + x_{b1}'\zeta^1 + \cdots,$$
$$y_c'(\zeta) =: y_{c0}' + y_{c1}'\zeta^1 + \cdots,$$
$$x_b(\zeta) =: x_{b0} + x_{b1}\zeta^1 + \cdots.$$

With this notation, taking the projection of (2b) on $L^2(\mathbf{C}^l)\ominus H^2(\mathbf{C}^l)$, and then multiplying both sides of the resulting equation by ζ^{-n} (this is equivalent to the operation U^{*n}, which is left invertible on $L^2(\mathbf{C}^l)\ominus H^2(\mathbf{C}^l)$) will give us

(4a)
$$X_3(\zeta^{-1})y_c^- := F_3(\zeta^{-1}),$$

where $X_3(\zeta^{-1}) = Q^2_{-n}\zeta^{-2n} + \cdots + Q^2_0\zeta^{-n} + \cdots + Q^2_n$, and

$$\begin{aligned}F_3(\zeta^{-1}) := {} & \sum_{i=1}^{n} Q^2_i \sum_{j=1}^{i} \zeta^{-n+i-j} y_{c(-j)} - \sum_{i=1}^{n} Q^2_{-i} \sum_{j=0}^{i-1} \zeta^{-n+i-j} y_{c(j)} \\ & - \sum_{i=1}^{n} C^*_i \sum_{j=0}^{i-1} \zeta^{-n+j-i} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{c(s+k)} v_{-s} \\ & + \sum_{i=1}^{n} Q^4_{-i} \sum_{j=0}^{i-1} \zeta^{-n-i+j} x_{bj} + \sum_{i=1}^{n} Q^5_{-i} \sum_{j=1}^{i} \zeta^{-n-j} \sum_{k=0}^{i-j} M_{b(i-j-k)} x'_{bk} \\ & - \sum_{i=0}^{n} Q^3_{-i} \sum_{j=1}^{i} \zeta^{-n-j} \sum_{k=0}^{i-j} M_{c(i-j-k)} y'_{ck} \\ & - \sum_{i=1}^{n} C^*_i \sum_{j=0}^{i-1} \zeta^{-n+j-i} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{b(s+k)} u_{-s}.\end{aligned}$$

We now play the same game with (3a) and (3b). Indeed, we multiply both sides of these equations by ζ^n. (This is equivalent to the application of the operator S^n, which is left invertible on $H^2(\mathbf{C}^l)$ and $H^2(\mathbf{C}^m)$.) Set

$$\begin{aligned}\rho^2 k(S)^*k(S)I - D(S)^*D(S) - B(S)^*B(S) &=: Q^6_{-n}S^{*n} + \cdots + Q^6_0 + \cdots + Q^6_n S^n, \\ \rho^2 k(S)^*k(S)I - D(S)^*D(S) &=: Q^7_{-n}S^{*n} + \cdots + Q^7_0 + \cdots + Q^7_n S^n, \\ D(S)^*E(S) &=: Q^8_{-n}S^{*n} + \cdots + Q^8_0 + \cdots + Q^8_n S^n, \\ B(S)^* &=: B^*_0 + \cdots + B^*_n S^{*n}.\end{aligned}$$

For any polynomial of degree $\leqq n$, $P(\zeta)$, we define $\hat{P}(\zeta) := \zeta^n P^*(\zeta^{-1})$. Then it is easy to see that (3a) combined with (3b) is equivalent to

(4b)
$$X_1(\zeta)\begin{bmatrix} x_b \\ y_c^+ \end{bmatrix} + X_2(\zeta)\begin{bmatrix} M_b & 0 \\ 0 & M_c \end{bmatrix}\begin{bmatrix} x'_b \\ y'_c \end{bmatrix} = \begin{bmatrix} F_1(\zeta) \\ F_2(\zeta) \end{bmatrix},$$

where

$$X_1(\zeta) := \begin{bmatrix} (\rho^2\hat{k}kI - \hat{D}D - \hat{B}B) & -(\hat{B}C + \hat{D}E) \\ -(\hat{C}B + \hat{E}D) & (\rho^2\hat{k}kI - \hat{E}E - \hat{C}C) \end{bmatrix},$$

$$X_2(\zeta) := \begin{bmatrix} (\rho^2\hat{k}kI - \hat{D}D) & -\hat{D}E \\ -\hat{E}D & (\rho^2\hat{k}kI - \hat{E}E) \end{bmatrix},$$

$$\begin{aligned}F_1(\zeta) := {} & \sum_{i=1}^{n} Q^6_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} x_{bj} + \sum_{i=1}^{n} Q^7_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{b(j-k)} x'_{bk} \\ & + \sum_{i=1}^{n} Q^1_i \sum_{j=1}^{i} \zeta^{n+i-j} y_{c(-j)} - \sum_{i=1}^{n} Q^1_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} y_{cj} \\ & - \sum_{i=1}^{n} Q^8_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{c(j-k)} y'_{ck} - \hat{B}(\zeta)M(\zeta) \sum_{i=1}^{n} \sum_{j=1}^{i} \Omega_{bi}\zeta^{i-j} u_{-j} \\ & + \sum_{i=1}^{n} B^*_i \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{b(s+k)} u_{-s} - \hat{B}(\zeta)M(\zeta) \sum_{i=1}^{n} \sum_{j=1}^{i} \Omega_{ci}\zeta^{i-j} v_{-j} \\ & + \sum_{i=1}^{n} B^*_i \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{c(s+k)} v_{-s},\end{aligned}$$

and

$$F_2(\zeta)=\sum_{i=1}^{n} Q^2_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} y_{cj}+\sum_{i=1}^{n} Q^3_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{c(j-k)} y'_{ck}$$

$$-\sum_{i=1}^{n} Q^2_i \sum_{j=1}^{i} \zeta^{n+i-j} y_{c(-j)}-\sum_{i=1}^{n} Q^4_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} x_{bj}$$

$$-\sum_{i=1}^{n} Q^5_{-i} \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{b(j-k)} x'_{bk}-\hat{C}(\zeta)M(\zeta)\sum_{i=1}^{n}\sum_{j=1}^{i} \Omega_{bi}\zeta^{i-j}u_{-j}$$

$$+\sum_{i=1}^{n} C_i^* \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{b(s+k)} u_{-s}-\hat{C}(\zeta)M(\zeta)\sum_{i=1}^{n}\sum_{j=1}^{i} \Omega_{ci}\zeta^{i-j}v_{-j}$$

$$+\sum_{i=1}^{n} C_i^* \sum_{j=0}^{i-1} \zeta^{n-i+j} \sum_{k=0}^{j} M_{j-k} \sum_{s=1}^{n-k} \Omega_{c(s+k)} v_{-s}.$$

Let us summarize the above results in the following:

PROPOSITION 1. *ρ^2 is an eigenvalue of A^*A if and only if there exists $x_b \in H(M_b$ $x'_b \in H^2(\mathbf{C}^m)$, $y_c^+ \in H(M_c)$, $y_c^- \in L(M_c) \ominus H(M_c)$, $y'_2 \in H^2(\mathbf{C}^l)$, not all zero, such that* (4 *and* (4b) *hold.*

Defining

$$M_0 := \begin{bmatrix} M_b & 0 \\ 0 & M_c \end{bmatrix},$$

we see that

$$\begin{bmatrix} x_b \\ y_c^+ \end{bmatrix} \in H(M_0) = H^2(\mathbf{C}^N) \ominus M_0 H^2(\mathbf{C}^N), \qquad N = m+l.$$

Now set

$$x_0 := \begin{bmatrix} x_b \\ y_c^+ \end{bmatrix}, \quad x'_0 := \begin{bmatrix} x'_b \\ y'_c \end{bmatrix}, \quad F'_0 := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot p.$$

Then (4b) can be rewritten as

$$X_1(\zeta)x_0 + X_2(\zeta)M_0 x'_0 = F'_0(\zeta). \tag{5}$$

Remark. Equation (5) is exactly the same type of equation that we obtained [12] for the 2-block problem. In the 1-block case, we get a similar equation with $X_2(\zeta$ a scalar. In fact, if we assume that $d_b(\zeta) := \det X_2(\zeta)$ is not identically equal to zer then (5) can be put in the form

$$X_0(\zeta)x_0 + d_b(\zeta)M_0 x'_0 = F_0(\zeta), \tag{6}$$

where $X_0 = X_2^a X_1$, $F_0 = X_2^a F'_0$, and $X_2^a(\zeta)$ is the algebraic adjoint of $X_2(\zeta)$, i.e.,

$$X_2^a(\zeta)X_2(\zeta) = X_2(\zeta)X_2^a(\zeta) = d_b(\zeta)I.$$

For (6), we make the factorization

$$X_0(\zeta)M_1(\zeta) = M_0(\zeta)\Omega_0(\zeta), \tag{f3}$$

where $M_1(\zeta)$ is $N \times N$ inner and $\Omega_0(\zeta)$ is $N \times N$ polynomial. Then, as shown usin skew Toeplitz theory in [3], there exists $X_0^{(-1)}$, an $N \times N$ H^∞-matrix, such that

$$X_0^{(-1)}X_0 = I + M_1 E_0 \quad \text{and} \quad X_0^{(-1)}M_0 = M_1 E_1$$

for some E_0 and E_1, $N\times N$ H^∞-matrices. Multiplying both sides of (6) by $X_0^{(-1)}$ and taking the orthogonal projection, of the resulting equation, on $H(M_1)$ we obtain

$$P_{H(M_1)}x_0 = P_{H(M_1)}X_0^{(-1)}F_0(\zeta).$$

Now we make our first assumption of genericity.

Assumption (a1). The operator $\tau := P_{H(M_1)}|_{H(M_0)}$ is invertible.

With this assumption we obtain

(7a) $$x_0 = \tau^{-1}P_{H(M_1)}X_0^{(-1)}F_0,$$

and

(7b) $$d_b(\zeta)x_0' = P_{H^2}(M_0(\zeta)^*(I - X_0\tau^{-1}P_{H(M_1)}X_0^{(-1)})F_0).$$

Next, applying the algebraic adjoint of $X_3(\zeta^{-1})$, $X_3^a(\zeta^{-1})$ to both sides of (4a) we get

(7c) $$d_d(\zeta^{-1})y_c^- = X_3^a(\zeta^{-1})F_3(\zeta^{-1}),$$

where $d_d(\zeta^{-1}) = \det X_3(\zeta^{-1})$. Equation (7a) gives the conditions for invertibility of a certain skew Toeplitz operator. See also [3]. We see that it is coupled, via F_0, to (7b) and (7c), which give the invertibility conditions of two Toeplitz operators.

We will now show that (7a)-(7c) give finitely many interpolation conditions for ρ^2 to be an eigenvalue of A^*A. From this we will derive the finite matricial rank condition for the determination of the singular values of A. First note that there exists $y_c^- \in L(M_c)\ominus H(M_c)$ satisfying (7c) if and only if there exists $\hat{y}_c^- \in H^2(\mathbf{C}^l)$ satisfying

(7d) $$d_d(\zeta)\hat{y}_c^- = P_{H^2}(\zeta^{-1}X_3^a(\zeta)F_3(\zeta)).$$

Indeed, this follows since $L^2(\mathbf{C}^l)\ominus H^2(\mathbf{C}^l)$ is isomorphic to $SH^2(\mathbf{C}^l) = \zeta H^2(\mathbf{C}^l)$ and the natural isomorphism is given by the reflection operator: $\zeta^{-1}\to\zeta$.

Next it is easy to see that the right-hand sides of (7a), (7b), and (7d) can be put into the form

$$\tau^{-1}P_{H(M_1)}X_0^{(-1)}(\zeta)F_0(\zeta) = K_a(\zeta)\Phi,$$
$$P_{H^2}(M_0(\zeta)^*(I - X_0(\zeta)\tau^{-1}P_{H(M_1)}X_0^{(-1)}(\zeta))F_0(\zeta)) = K_b(\zeta)\Phi,$$
$$P_{H^2}(\zeta^{-1}X_3^a(\zeta)F_3(\zeta)) = K_d(\zeta)\Phi,$$

where $K_a(\zeta)$, $K_b(\zeta)$ are H^∞ matrices of sizes $N\times r$ and $K_d(\zeta)$ is an $l\times r$ polynomial in ζ (these all can be explicitly computed from M_0, M_1, X_0, $X_0^{(-1)}$, X_3, F_0, and F_3),

$$\Phi^T = [x_{b0}^T \cdots x_{b(n-1)}^T x_{b0}'^T \cdots x_{b(n-1)}'^T y_{c0}^T \cdots y_{c(n-1)}^T y_{c0}'^T \cdots y_{c(n-1)}'^T u_{-1}^T \cdots u_{-n}^T v_{-1}^T \cdots v_{-n}^T y_{c(-1)}^T \cdots y_{c(-n)}^T],$$

and $r = 2n(m+l) + n(m+2l)$. With this notation we immediately get the following identities:

(8a) $$K_{ai}\Phi = x_{0i},$$

(8b) $$K_{bi}\Phi = \sum_{j=0}^{i} d_{bj}x'_{0(i-j)},$$

(8d) $$K_{di}\Phi = \sum_{j=0}^{i} d_{dj}\hat{y}^-_{c(i-j)},$$

for all $i=0,\cdots,n$, where

$$K_a(\zeta) =: K_{a0} + K_{a1}\zeta + K_{a2}\zeta^2 + \cdots,$$
$$K_b(\zeta) =: K_{b0} + K_{b1}\zeta + K_{b2}\zeta^2 + \cdots,$$
$$K_d(\zeta) =: K_{d0} + K_{d1}\zeta + K_{d2}\zeta^2 + \cdots,$$
$$x_0(\zeta) =: x_{00} + x_{01}\zeta + x_{02}\zeta^2 + \cdots,$$
$$x_0'(\zeta) =: x_{00}' + x_{01}'\zeta + x_{02}' + \cdots,$$
$$\hat{y}_c^-(\zeta) =: \hat{y}_{c0}^- + \hat{y}_{c1}^-\zeta + \hat{y}_{c2}^-\zeta^2 + \cdots,$$
$$d_b(\zeta) =: d_{b0} + \cdots + d_{b2nN}\zeta^{2nN},$$
$$d_d(\zeta) =: d_{d0} + \cdots + d_{d2nl}\zeta^{2nl}.$$

Rearranging terms in (8a), (8b), (8d) and combining them into one equation we obtain,

(9) $$K\Phi = 0,$$

where K is a constant matrix that can be computed from the K_{ai}, K_{bi}, K_{di}, d_{bi}, and d_{di}, $i=0,\cdots,n$.

We now make our second assumption of genericity.

Assumption (a2). $d_b(\zeta)$ and $d_d(\zeta)$ have distinct roots, all of which are nonzero.

Then, as in [6], [7], and [10], we see that d_b has roots $\alpha_1,\cdots,\alpha_{r_b}$ inside D, $\alpha_{r_b+1},\cdots,\alpha_{(2nN-r_b)}$ on ∂D and $1/\bar{\alpha}_1,\cdots,1/\bar{\alpha}_{r_b}$ outside $\bar{D}$. Similarly, d_d has roots $\beta_1,\cdots,\beta_{r_d}$ inside D, $\beta_{r_d+1},\cdots,\beta_{(2nl-r_d)}$ on ∂D and $1/\bar{\beta}_1,\cdots,1/\bar{\beta}_{r_d}$ outside $\bar{D}$.

We are ready to state our main result.

THEOREM 1. *Assume* (a1) *and* (a2). *Then,* $\rho^2 > \|A\|_e^2$ *is an eigenvalue of* A^*A *if and only if*

$$\operatorname{rank} R < r,$$

where

(9a) $$R := \begin{bmatrix} K \\ K_b(\alpha_1) \\ \vdots \\ K_b(\alpha_{(2nN-r_b)}) \\ K_d(\beta_1) \\ \vdots \\ K_d(\beta_{(2nl-r_d)}) \end{bmatrix}.$$

Proof. By Proposition 1, ρ^2 is an eigenvalue of A^*A if and only if there exists $x_0 \in H(M_0)$, $x_0' \in H^2(\mathbf{C}^N)$ and $\hat{y}_c^- \in H^2(\mathbf{C}^l)$, not zero, such that (7a), (7b), (7d) are satisfied. By an argument similar to the one used in [3], [6], [7], and [11], we see that the existence of such x_0, x_0', $\hat{y}_c^-$ is equivalent to finding a nonzero Φ such that

$$K_b(\alpha_i)\Phi = 0, \qquad i = 1,\cdots,2nN - r_b,$$
$$K_d(\beta_i)\Phi = 0, \qquad i = 1,\cdots,2nl - r_d,$$

and (9) holds. This completes the proof. □

Remark. In the absence of the genericity assumptions, the matrix (9a) takes on a certain degenerate form exactly as in [11]. We see from Theorem 1 that the largest value of ρ that gives a solution for the equation

$$\det R^*R = 0$$

is the norm of the 4-block operator A. From R, we can determine the singular values and singular vectors of the 4-block operator A.

3.2. Essential spectrum. We now give a sufficient condition for ρ to be strictly greater than the essential norm of A; in order to do this we study the essential spectrum of A^*A.

PROPOSITION 2. *Suppose that*

(a3) *the Toeplitz operator* $\tau_2 := P_{H^2} M_1^* M_0|_{H^2}$ *is invertible,*

(a4) $\{z: \det X_1(z) = 0\} \cap \sigma_e(T_0) = \emptyset$,

where $T_0 := P_{H(M_0)} S|_{H(M_0)}$. *Then,* $\rho > \max\{\beta, \gamma\}$ *implies* $\rho^2 \notin \sigma_e(A^*A)$, *where* β *and* γ *are defined as in* § 2.

Proof. Let $\rho > \max\{\beta, \gamma\}$. If ρ^2 were in $\sigma_e(A^*A)$, then there would exist

$$\begin{bmatrix} x^{(n)} \\ y^{(n)} \end{bmatrix} \in H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l) \quad \text{with} \quad \left\| \begin{bmatrix} x^{(n)} \\ y^{(n)} \end{bmatrix} \right\|_2 = 1 \quad \forall n \geqq 1,$$

and $\left[\begin{smallmatrix} x^{(n)} \\ y^{(n)} \end{smallmatrix}\right] \to 0$ weakly as $n \to \infty$, satisfying

$$(4\text{b})_e \qquad X_1(\zeta)\begin{bmatrix} x_b^{(n)} \\ y_c^{+(n)} \end{bmatrix} + X_2(\zeta)\begin{bmatrix} M_b & 0 \\ 0 & M_c \end{bmatrix}\begin{bmatrix} x_b'^{(n)} \\ y_c'^{(n)} \end{bmatrix} \to 0 \quad \text{strongly},$$

and

$$(7\text{d})_e \qquad X_3(\zeta)\hat{y}_c^{-(n)} \to 0 \quad \text{strongly.}$$

(These conditions for $\rho^2 \in \sigma_e(A^*A)$ are sufficient as well.) This follows from Proposition 1 and equations (4b) and (7d). Note that $F_1(\zeta)$, $F_2(\zeta)$, and $F_3(\zeta)$ converge to zero strongly as $x^{(n)}$ and $y^{(n)}$ converge to zero weakly. In the above we have, as before,

$$x^{(n)} = x_b^{(n)} + M_b x_b'^{(n)},$$

$$y^{(n)} = y_c^{-(n)} + y_c^{+(n)} + M_c y_c'^{(n)},$$

with

$$\hat{y}_c^{-(n)}(\zeta) := y_c^{-(n)}(\zeta^{-1}),$$

$$\begin{bmatrix} x_b^{(n)} \\ y_c^{+(n)} \end{bmatrix} =: x_0^{(n)} \in H(M_0),$$

$$\begin{bmatrix} x_b'^{(n)} \\ y_c'^{(n)} \end{bmatrix} =: x_0'^{(n)} \in H^2(\mathbf{C}^m) \oplus H^2(\mathbf{C}^l),$$

and $y_c^{-(n)} \in L^2(\mathbf{C}^l) \ominus H^2(\mathbf{C}^l)$. They all converge to zero weakly as $n \to \infty$.

Note that $(7\text{d})_e$ means that

$$\left(\rho^2 I - [F(S)^* \quad J(S)^*]\begin{bmatrix} F(S) \\ J(S) \end{bmatrix}\right)\hat{y}_c^{-(n)} \to 0 \quad \text{strongly.}$$

Since $\rho > \gamma$, we see that

$$\left(\rho^2 I - [F(S)^* \quad J(S)^*]\begin{bmatrix} F(S) \\ J(S) \end{bmatrix}\right)$$

is invertible, and so $\hat{y}_c^{-(n)}$ converges to zero strongly.

Next from $(4\text{b})_e$ we get that

$$(6)_e \qquad X_0(\zeta)x_0^{(n)} + d_b(\zeta)M_0 x_0'^{(n)} \to 0 \quad \text{strongly.}$$

Taking orthogonal projections on $M_1 H^2(\mathbf{C}^N)$ we see that

$$P_{H^2} M_1^* X_0 x_0^{(n)} + P_{H^2} M_1^* M_0 d_b(\zeta) x_0'^{(n)} \to 0 \quad \text{strongly.}$$

Recall that $P_{H^2} M_1^* X_0 x_0^{(n)} = P_{H^2} \Omega_0 M_0^* x_0^{(n)}$, so it converges to zero strongly as $x_0^{(n)} \in H(M_0)$ converges to zero weakly. Hence using Assumption (a3) we have that

$$(7)_e \qquad d_b(\zeta) x_0'^{(n)} \to 0 \quad \text{strongly.}$$

This implies, by $(6)_e$, that

$$(8)_e \qquad d_b(\zeta) \det X_1(\zeta) x_0^{(n)} \to 0 \quad \text{strongly.}$$

It is easy to see, by definition of β, that for $\rho > \beta$, $d_b(\zeta)$ has no roots on ∂D. Then we can write

$$d_b(\zeta) = (\zeta - \alpha_1)\left(\zeta - \frac{1}{\bar{\alpha}_1}\right) \cdots (\zeta - \alpha_{nN})\left(\zeta - \frac{1}{\bar{\alpha}_{nN}}\right),$$

for some $\alpha_1, \cdots, \alpha_{nN} \in D$. Multiplying $(7)_e$ by

$$\prod_{i=1}^{nN} \frac{\bar{\alpha}_i}{(1-\zeta\bar{\alpha}_i)^2},$$

which is in H^∞ (because all α_i's are in D), we obtain

$$m_1(\zeta) x_0'^{(n)} \to 0 \quad \text{strongly,}$$

where

$$m_1(\zeta) = \prod_{i=1}^{nN} \frac{\zeta - \alpha_i}{(1-\zeta\bar{\alpha}_i)}.$$

This implies that $x_0'^{(n)} \to 0$ strongly, because $m_1(S)^* m_1(S)$ is equal to the identity.

From $(8)_e$, a similar argument gives that

$$(9)_e \qquad \det X_1(\zeta) x_0^{(n)} \to 0 \quad \text{strongly.}$$

Let us assume now that $d_1(\zeta) := \det X_1(\zeta)$ has nonzero distinct roots. So $d_1(\zeta) = 0$ at points $z_1, \cdots, z_{n_1}$ inside D, $1/\bar{z}_1, \cdots, 1/\bar{z}_{n_1}$ outside $\bar{D}$, and $z_{n_1+1}, \bar{z}_{n_1+1}, \cdots, z_{nN}, \bar{z}_{nN}$ on ∂D. Using a similar trick as before, we obtain

$$m_2(\zeta) \prod_{i=n_1+1}^{nN} (\zeta - z_i)(\zeta - \bar{z}_i) x_0^{(n)} \to 0 \quad \text{strongly,}$$

where

$$m_2(\zeta) = \prod_{i=1}^{n_1} \frac{\zeta - z_i}{1 - \bar{z}_i \zeta}.$$

Hence we see that

$$\prod_{i=n_1+1}^{nN} (\zeta - z_i)(\zeta - \bar{z}_i) x_0^{(n)} \to 0 \quad \text{strongly.}$$

Taking the orthogonal projection of this last expression on $H(M_0)$, we get

$$\prod_{i=n_1+1}^{nN} (T_0 - z_i)(T_0 - \bar{z}_i) x_0^{(n)} \to 0 \quad \text{strongly,}$$

for $x_0^{(n)} \to 0$ weakly, and $x_0^{(n)} \in H(M_0)$. By Assumption (a4) none of z_i, $\bar{z}_i$ are in the essential spectrum of T_0, therefore $x_0^{(n)} \to 0$ strongly.

In summary, we have established that $\hat{y}_c^{-(n)} \to 0$ strongly, $x_0'^{(n)} \to 0$ strongly and $x_0^{(n)} \to 0$ strongly. So, $\left[\begin{smallmatrix} x^{(n)} \\ y^{(n)} \end{smallmatrix}\right] \to 0$ strongly, which contradicts that $\|\left[\begin{smallmatrix} x^{(n)} \\ y^{(n)} \end{smallmatrix}\right]\|_2 = 1$ for all $n \geqq 1$. Thus ρ^2 cannot be in $\sigma_e(A^*A)$. □

Remark. Note that a sufficient condition for (a4) to hold is $\rho > \alpha$. So if $\|A\|_e > \alpha$, then the above Proposition 2 gives an upper bound for the essential norm: $\|A\|_e \leqq \max\{\beta, \gamma\}$. Actually, we must prove, if possible, the equality as in the case of SISO plants and MIMO finite-dimensional systems. However this is not easy in our case: All we can show is that if $\rho = \gamma$ then $(7\mathrm{d})_e$ holds for some $\hat{y}_c^{-(n)}$ such that $\|\hat{y}_c^{-(n)}\|_2 = 1$ for all $n \geqq 1$, and $\hat{y}_c^{-(n)} \to 0$ weakly. This implies that $\rho^2 \in \sigma_e(A^*A)$, and $\|A\|_e \geqq \gamma$. But the difficulty is with β: if $\rho = \beta$ then there exists

$$\begin{bmatrix} x^{(n)} \\ y^{(n)} \end{bmatrix} \in H^2(\mathbf{C}^m) \oplus H^2(\mathbf{C}^l) \quad \text{with} \quad \left\| \begin{bmatrix} x^{(n)} \\ y^{(n)} \end{bmatrix} \right\|_2 = 1 \quad \forall n \geqq 1,$$

and $\left[\begin{smallmatrix} x^{(n)} \\ y^{(n)} \end{smallmatrix}\right] \to 0$ weakly as $n \to \infty$, such that

$$(10)_e \qquad X_2(\zeta) \begin{bmatrix} x^{(n)} \\ y^{(n)} \end{bmatrix} \to 0 \quad \text{strongly.}$$

In the SISO case, by multiplying $(10)_e$ by $M_0(\zeta)$ (which then commutes with $X_2(\zeta)$), we get the result that $\|A\|_e \geqq \beta$. Moreover, in the MIMO finite-dimensional case we decompose $\left[\begin{smallmatrix} x^{(n)} \\ y^{(n)} \end{smallmatrix}\right]$ as $x_0^{(n)} + M_0 x_0'^{(n)}$, and as before $x_0^{(n)} \in H(M_0)$. Since in the finite-dimensional case $H(M_0)$ is finite-dimensional, $x_0^{(n)} \to 0$ weakly implies $x_0^{(n)} \to 0$ strongly. Hence we obtain that $(4\mathrm{b})_e$ holds; then $\|A\|_e \geqq \beta$. The infinite-dimensional MIMO case is much more subtle.

We now summarize the above discussion with two corollaries to Proposition 2.

COROLLARY 1. *Assume* (a3) *and* $\alpha \leqq \max\{\beta, \gamma\}$. *Then,*

(i) *If* $\gamma \geqq \beta$ *then* $\|A\|_e = \gamma$.

(ii) *If* $\gamma < \beta$ *then* $\gamma \leqq \|A\|_e \leqq \beta$.

COROLLARY 2. *Consider finite-dimensional* MIMO *case, i.e.,* $M(\zeta)$ *is rational. Then,*

$$\|A\|_e = \max\{\beta, \gamma\}.$$

Proof. Let M^{ad} denote the algebraic adjoint of M. Then

$$\begin{bmatrix} M^{\mathrm{ad}} & 0 \\ 0 & M^{\mathrm{ad}} \end{bmatrix} \begin{bmatrix} W - MQ & F \\ G & J \end{bmatrix} = \begin{bmatrix} M^{\mathrm{ad}} W - mQ & M^{\mathrm{ad}} F \\ M^{\mathrm{ad}} G & M^{\mathrm{ad}} J \end{bmatrix} =: L,$$

where $m := \det M$. Clearly, L has all rational entries. Now let A_L be the 4-block operator associated to L. Then it is easy to see that $\|A_L\|_e = \|A\|_e$. In other words, without loss of generality, we may assume that M is of the form mI where $m \in H^\infty$ is an inner scalar-valued function. But in this case, we have that (a3) is satisfied since we can choose $M_1 = m$ (see also the discussion below in § 4). Hence by Proposition 2, and by the finite dimensionality of $H(m)$, we have the required conclusion. □

Remark. In practice we do not need to compute the essential norm. All we need to know is an upper bound μ_0 for $\|A\|$ with which to start. Then the first zero of $\det R^*R$ (considered as a function of ρ) less than μ_0, will be $\|A\|$. Of course, if $\|A\| = \|A\|_e$, then there is no first eigenvalue. Hence on the computer, if we plot $\det R^*R$ as a function of ρ, the graph of $\det R^*R$ does not cross the ρ axis above $\|A\|_e$, but oscillates near this value, since the eigenvalues accumulate at $\|A\|_e$. In this way we can estimate the essential norm.

4. SISO case. In this section we apply the above theory to SISO plants. The first thing to note in this case is that the factorizations (f1), (f2), (f3) are trivial, because $M(\zeta)$ is scalar, so it commutes with everything:

$$M(\zeta)^*B(\zeta)=B(\zeta)M(\zeta)^*,$$
$$M(\zeta)^*C(\zeta)=C(\zeta)M(\zeta)^*,$$
$$X_0(\zeta)M_0(\zeta)=M_0(\zeta)X_0(\zeta).$$

Here we have $M_0(\zeta)=M(\zeta)\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. Since $M_0(\zeta)=M_1(\zeta)$ Assumption (a1) holds (in fact, τ is the identity). Moreover, we do not really have to compute $X_0^{(-1)}$. Indeed, recall the equation

$$X_1(\zeta)x_0+X_2(\zeta)M_0(\zeta)x_0'=F_0'(\zeta). \tag{5}$$

Taking the projections of (5) on $H(M_0)$ and $M_0H^2(\mathbf{C}^N)$, we obtain

$$X_1(\zeta)x_0=F_0'(\zeta)-M_0(\zeta)P_{H^2}M_0(\zeta)^*F_0'(\zeta)+M_0(\zeta)P_{H^2}M_0(\zeta)^*X_1(\zeta)x_0, \tag{5a}$$

and

$$X_2(\zeta)x_0'=P_{H^2}M_0(\zeta)^*F_0'(\zeta)-P_{H^2}M_0(\zeta)^*X_1(\zeta)x_0. \tag{5b}$$

These equations (5a), (5b), are in the form of the equations (24c), (24d) of [6]. Now we can use similar computations to the ones used in [6] to obtain the final result, namely, a rank type formula as in our main theorem.

In the next section we give an example illustrating the computations for the SISO case.

5. A SISO 2-block example. For simplicity of notation and exposition, the following example is chosen in the 2-block setup and a SISO plant is considered. The 2-block problem for stable SISO distributed plants was first solved in [22]. Motivations for studying the 2-block problem comes from the mixed sensitivity minimization (see, e.g., [14], [19]), which can be stated as follows. Consider the feedback configuration shown in Fig. 2. The mixed sensitivity minimization problem is to find

$$\mu=\inf_{C\text{stabilizing}}\ \sup\left\{\left\|\begin{bmatrix}\tilde e\\ \tilde u\end{bmatrix}\right\|_2 : \|v\|_2\leqq 1\right\}$$
$$=\inf_{C\text{stabilizing}}\left\|\begin{bmatrix}W_1(I+PC)^{-1}W_3\\ W_2C(I+PC)^{-1}W_3\end{bmatrix}\right\|_\infty.$$

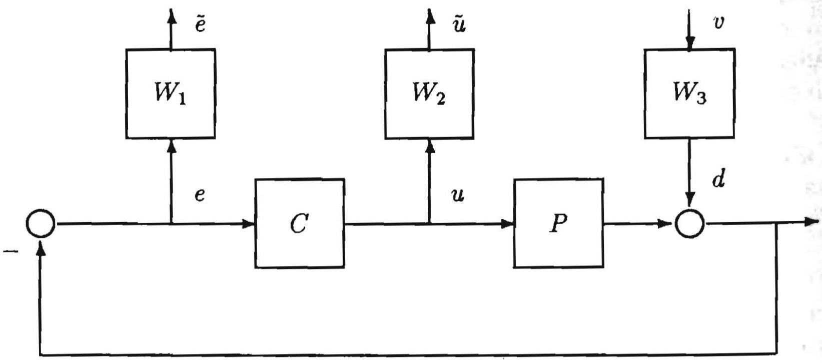

FIG. 2

Invoking the standard Youla parametrization of all stabilizing controllers, we obtain the following expression for μ for P stable:

$$\mu = \inf_{Z\in H^{\infty}} \left\| \begin{bmatrix} W_1(1-P)W_3 \\ W_2 W_3 \end{bmatrix} - \begin{bmatrix} W_1 P W_3 \\ -W_2 W_3 \end{bmatrix} Z \right\|_{\infty}.$$

Let us now choose some specific values for the weights and the plant: $W_1 = 1$, $W_2 = b$, $W_3 = 1/(s+1)$, and $P = e^{-hs}$. Here $0 \leqq b < \infty$ and $0 \leqq h < \infty$ are free parameters. We will find the dependence of μ on b and h. Note that if $b=0$ then the problem reduces to the 1-block case.

Following the factorization techniques used in [15] and [19] we can show that

$$\mu = \inf_{Q\in H^{\infty}} \left\| \begin{bmatrix} \dfrac{1}{\sqrt{1+b^2}} \dfrac{1}{s+1} \\ \dfrac{b}{\sqrt{1+b^2}} \dfrac{1}{s+1} \end{bmatrix} - \begin{bmatrix} e^{-hs} \\ 0 \end{bmatrix} Q \right\|_{\infty}.$$

In terms of our notation

$$W(\zeta) = \frac{1}{\sqrt{1+b^2}} \frac{(1-\zeta)}{2}, \qquad G(\zeta) = \frac{b}{\sqrt{1+b^2}} \frac{(1-\zeta)}{2},$$

and $M(\zeta) = e^{h(\zeta+1)/(\zeta-1)}$. We can compute the lower bound for μ as $\|A\|_e = b/\sqrt{1+b^2}$. Also note that if we set $Q=0$ then we find an upper bound for μ as one. Therefore we seek solutions ρ^2, to the eigenvalue equations (1a), (1b) in the region:

$$\frac{b^2}{1+b^2} \leqq \rho^2 \leqq 1.$$

In this specific example, equation (5) turns out to be

(11) $$X_1(\zeta)x_b + X_2(\zeta)M(\zeta)x'_b = F_1(\zeta),$$

where $x_b \in H(M)$, $x'_b \in H^2$, and where X_1, X_2, and F_1 can be computed to be

(11a) $$X_1(\zeta) = \frac{1}{4}(\zeta^2 + (4\rho^2 - 2)\zeta + 1),$$

(11b) $$X_2(\zeta) = \frac{b^2}{4(1+b^2)}\left(\zeta^2 + \left(\frac{4(1+b^2)}{b^2}\rho^2 - 2\right)\zeta + 1\right),$$

(11c) $$F_1(\zeta) = \frac{1}{4}x_{b0} + \frac{b^2 e^{-h}}{4(1+b^2)} x'_{b0} + \frac{1}{4(1+b^2)}((\zeta-1)M(\zeta) + e^{-h})u_{-1}.$$

If we now take the projection of both sides of (11) on MH^2, we see that

(12) $$X_2(\zeta)x'_b = \frac{b^2}{4(1+b^2)}(x'_{b0} - \zeta u_{-1}).$$

Thus from (11), (11a)-(11c) and (12), we have

(13) $$X_1(\zeta)x_b = \frac{1}{4}x_{b0} + \frac{b^2(e^{-h} - M(\zeta))}{4(1+b^2)} x'_{b0} + \frac{((-1+(1+b^2)\zeta)M(\zeta) + e^{-h})}{4(1+b^2)} u_{-1}.$$

It is easy to check that for $b^2/(1+b^2) \leqq \rho^2 \leqq 1$, $X_2(\zeta)$ has one root, r_2, inside the unit disc D, and that $X_1(\zeta)$ has both roots, r_1, r_1^{-1} on the unit circle ∂D. Therefore we have $x'_{b0} = r_2 u_{-1}$, and so

$$\exp\left(-2h\frac{r_1+1}{r_1-1}\right) = \frac{-1-b^2 r_2+(1+b^2)r_1}{-1-b^2 r_2+(1+b^2)r_1^{-1}}.$$

Hence from (13) we may conclude that

$$hy + \tan^{-1}\frac{y\sqrt{1+b^2}}{\sqrt{1-b^2y^2}} = \pi, \tag{14}$$

where $y := \sqrt{1/\rho^2 - 1}$, $0 \leqq y \leqq 1/b$, and $\tan^{-1} y\sqrt{1+b^2}/\sqrt{1-b^2y^2} \in [0, \pi/2]$. Note that for (14) to have a solution, we need $h \geqq \pi b/2$. Hence, if $h \leqq \pi b/2$ then $\mu = \|A\|_e = b/\sqrt{1+b^2}$; otherwise $\mu = \rho = 1/\sqrt{y^2+1}$ where y is the unique solution of (14) in the range $0 \leqq y \leqq 1/b$. Note that when $b=0$, (14) becomes

$$hy + \tan^{-1} y = \pi, \qquad 0 \leqq y \leqq \infty,$$

which is exactly the same equation obtained previously in [9], [10], [16], and [21] for the 1-block problem. Clearly, as $b \uparrow \infty$, we have that $\mu \uparrow 1$. The physical meaning of this situation is that in this case we infinitely penalize the energy of the command signal u. Indeed, since P is already stable we are allowed to choose $C = 0$, which will make $u = 0$, and hence solve the problem. However, in this situation the tradeoff is that the energy of the worst error signal cannot be less than the energy of the disturbance signal d, so μ will be equal to one. Figure 3 gives an indication on how μ depends on the parameters b and h.

6. Concluding remarks. In this paper we have studied H^∞ optimization of multivariable distributed systems. We took the most general case of the standard H^∞ problem, namely, the so-called 4-block problem. Here, we developed a rank type formula for the computation of the eigenvalues of the operator A^*A. It is important to emphasize once more that the crucial steps of the procedure presented here are: (i) to do the factorizations (f1)–(f3), and (ii) to find $X_0^{(-1)}$. We refer to the paper [3] for the methods of performing these steps. From a computational point of view, the same method may

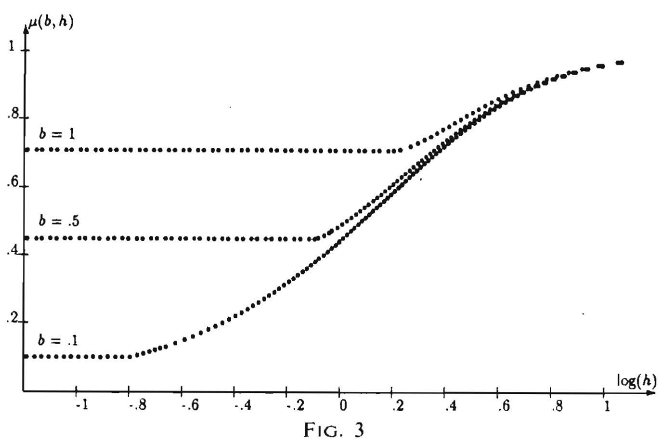

FIG. 3

be used to solve the 4-block problem for MIMO lumped systems, and MIMO stable distributed systems.

At this point we feel that the skew Toeplitz theory gives a satisfactory way of solving the optimal version of the 4-block problem in a very general setting. We should note that these techniques should also lead to the suboptimal solutions as considered in [2] and [4] for finite-dimensional systems using a state-space point of view. Indeed, since the operator A is derived from the commutant lifting theorem, we could in principle get all of the suboptimal solutions via the one-step extension technique of [1], once we know how to do the optimal case. This program has already been carried out for the 1-block case in [8]. Such a suboptimal parametrization would allow us to make contact with the very important work of [2] and [4]. Finally, it would be interesting to explore the possibility of combining state-space and frequency-domain methods in the 4-block problem as was done in [16] and [23] in the 1-block case.

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