DETERMINING THE MINIMAL COVERING SET OF PARAMETER SPACES FOR PHENOMENOLOGICAL GRAVITATIONAL WAVEFORMS

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Dustin Burns

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Submitted by:

Dustin Burns
School of Sciences
Georgia Institute of Technology

Approved by:

Dr. Deirdre Shoemaker, Advisor
College of Science
Georgia Institute of Technology
Deirdre Shoemaker

Dr. Pablo Laguna, Second Reader
College of Science
Georgia Institute of Technology
PLaguna

Dr. Brian Kennedy
College of Science
Georgia Institute of Technology
T.A.B. Kennedy

Date Approved: 4/29/11

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SUMMARY

Gravitational wave observatories are now trying to detect gravitational waves, ripples in space time predicted by Einstein's theory of General Relativity, from sources such as merging binary star and black hole systems. Numerical relativists create template banks of gravitational waves from merging black hole binaries in an effort to confirm a gravitational wave detection by solving Einstein's field equations. These waveforms are then compared to the raw data collected by gravitational wave detectors. Since it is computationally expensive to produce the full numerical relativity waveforms, theorists have created approximation techniques called phenomenological waveforms, in which analytical functions approximate the numerical solutions over a finite space of parameters. It is computationally expensive to match the waveform template banks to the data from the observatories. In an effort to minimize the number of waveforms in the template banks, I determine the minimal covering set of the parameter space for non-spinning binary black hole phenomenological waveforms. This is accomplished by marching through a very fine mesh of the parameter space, ensuring that the match between adjacent waveforms is above a given threshold. I determine this minimal covering set for the non-spinning case and discuss how to generalize the program to the full spinning case.

CHAPTER 1

INTRODUCTION

Background

Einstein's theory of general relativity predicts that an accelerated massive body will emit gravitational waves, oscillations in space-time, which can be detected on Earth. For example, a binary black hole (BBH) system will emit gravitational waves until the black holes eventually merge. Although the predictions of general relativity have been repeatedly confirmed experimentally, there has yet to be a confirmed gravitational wave detection(1). However, there have been indirect confirmations of gravitational wave radiation.

Currently, there is an international effort to detect gravitational waves at land based detectors such as LIGO(2), Virgo(3), TAMA(4), and GEO600(5) . Essentially, these detectors are giant laser interferometers. A light ray is split into two beams. The beams are sent down two very long perpendicular arms of equal length L. Through a series of reflections, the light rays reach their origin. If the time it takes each light beam to reach its origin is the same, the lengths of the arms remained constant during the travel time of the light rays. If the time varies, the lengths of the arms changed by an amount δL due to a gravitational wave passing through the earth in the neighborhood of the detector(6). Although this is an extreme simplification of the complex engineering used to build the detectors, the general idea of how the detector works is shown.

Once a detection is confirmed, these detectors will effectively become one of the most powerful telescopes on the planet, able to observe objects further away in space than ever before(7). Scientists in this field are also excited about what these detectors will teach us about the universe. We will undoubtedly observe things that we cannot explain, motivating theorists to expand or reform current theories to account for the new observations. Since the data from the detectors is dominated by noise, the waveforms for the gravitational waves must be produced. The following expression gives a simple relation between the theoretically produced waveforms, denoted Ψ_4, and the strain on an arm of length L of a land based detector, denoted by h(t):

$$\frac{\delta L}{L} = h(t) = \iint \Psi_4 dt^2$$

One of the main motivations of Numerical Relativity (NR) is the production and analysis of gravitational waveforms. Numerical relativists solve Einstein's equations for a binary black hole merger to produce waveform template banks, large collections of waveforms with a similar set of parameters, to match to the detectors' data. Since the data from the detectors is very noisy, scientists must match these template banks of gravitational waveforms to the raw data. Without this information, it is nearly impossible to identify the waveforms for even the most catastrophic astronomical events in the data. The merging of BBH systems provides one of the strongest sources of gravitational waves. Since strong sources of gravitational waves are easier to detect, the waveforms describing these systems are sought (1).

Unfortunately, it is computationally expensive to solve Einstein's equations for the full numerical relativity waveforms, and although many groups do it on supercomputer clusters, there is a strong desire to be able to produce valid waveforms using analytical approximation techniques in order to save time and money producing the waveform template banks, the groups of waveforms describing all of the physically possible configurations of the system for a certain set of parameters. It is relatively inexpensive computationally and monetarily to produce a waveform from an analytical function. Several approximation techniques exist, including one class of approximations known as phenomenological waveforms (p-waveforms). This approximation scheme is desirable because of the simplicity of its implementation and the simplicity of creating the waveforms. Unlike other approximation techniques such as post-Newtonian (PN) approximations, p-waveforms have little theoretical derivation or justification. Their creation is based on fitting a function to the NR waveforms. Although the idea is simple, it is still a very difficult process to choose the correct functions. The main computational cost in generating the p-waveforms comes from determining the fitting factors, the coefficients and exponents of the candidate functions which produce the highest match to the numerical relativity waveforms they are approximating.

The most efficient way of covering of the parameter space with as few p-waveforms as possible in not known. Since a matched filtering technique (to be discussed further in the methods section) must be run between the raw data and each waveform in the template bank, it is useful to know the minimum number of waveforms which "cover" the parameter space of the template bank being used. This would allow for the minimum amount of computational expense.

I will now develop an analogy using basic calculus that will aid in understanding the problem of covering the parameter space of p-waveforms studied in this paper. Let A be the set of real valued continuous functions f of one variable with a single parameter a given by

$$f_a(x) = x^a \ \ for\ x \in [0,1]$$

Where a is a natural number. For a and n natural numbers, define a metric by

$$< f_a(x), f_{a+n}(x) > = (2a+1)\int_0^1 f_a(x) f_{a+n}(x)\, dx$$

A metric is just a function which takes in two functions, outputs a real number, and obeys certain properties irrelevant to this discussion. This metric has been normalized, meaning

$$< f_a(x), f_{a+n}(x) > \in (0,1)$$

Now, we wish to find the subset P of natural numbers which "cover" the parameter space of A in the following since: for functions parameterized by any two adjacent elements a and a+1 of P,

$$< f_a(x), f_{a+1}(x) > \geq 1 - \varepsilon$$

For some ε ϵ (0,1). Note that P could have just one or infinitely many elements depending on ε. Also, there may be multiple sets which obey the properties of P, so we will seek the smallest or most sparse set obeying the properties of P. To illustrate, let ε = 0.4. We want to find P such that for adjacent elements of P

$$< f_a(x), f_{a+1}(x) > \geq 0.6$$

We have

$$< f_a(x), f_{a+n}(x) > = (2a+1)\int_0^1 x^a x^{a+n}\, dx$$

$$= (2a+1)\int_0^1 x^{2a+n}\, dx = (2a+1)\frac{1}{2a+n+1}[x^{2a+n+1}]|_0^1$$

$$= \frac{(2a+1)}{2a+n+1}$$

In particular,

$$< f_1(x), f_{1+1}(x) > = \frac{(2*1+1)}{2*1+1+1} = 0.75$$

$$< f_1(x), f_{1+2}(x) > = \frac{(2*1+1)}{2*1+2+1} = 0.6$$

So the first two elements in P are 1 and 3. Continuing,

$$< f_3(x), f_{3+1}(x) > = \frac{(2*3+1)}{2*3+1+1} = 0.875$$

$$< f_3(x), f_{3+2}(x) > = \frac{(2*3+1)}{2*3+2+1} = 0.778$$

$$< f_3(x), f_{3+3}(x) > = \frac{(2*3+1)}{2*3+3+1} = 0.7$$

$$< f_3(x), f_{3+4}(x) > = \frac{(2*3+1)}{2*3+4+1} = 0.636$$

$$< f_3(x), f_{3+5}(x) > = \frac{(2*3+1)}{2*3+5+1} = 0.583$$

So, the third element of P is 7. This process can be continued ad infinitum or until some predetermined maximum integer is reached.

Now, the analogy is as follows: the p-waveforms are analogous to the set of functions $f_a(x)$, just the p-waveform functions are much more complicated. The metric $\langle f,g \rangle$ is analogous to what is called the match of two waveforms, and is just a more complicated metric which is maximized over internal parameters of the p-waveforms. Also, the match functional is not explicit in the parameters of the system; it is completely numerical. In the simplest case of non-spinning unequal mass black holes, there is only one parameter, the symmetric mass ratio, so the analogy is even stronger, but in more complex cases there are multiple parameters, so there is a higher dimensional parameter space. The idea is the same however. The main difference between P in the example above and the subset Q of all possible values of a parameter of a p-waveform is that Q is a subset of the real numbers instead of integers, so some small minimum step size between adjacent elements in Q must be specified. We will still want to find the sparsest set Q such that the match between functions parameterized by adjacent elements is greater than or equal to some predefined number between zero and one. Since we cannot maximize over the infinite set of internal parameters, the set Q will not be minimal as desired, but will still give a sparse covering very close to the optimal subset.

The purpose of this study is as follows: given the finite range of parameters T over which the p-waveforms are known to be valid, I wish to find the smallest subset S of T such that the match between adjacent elements in S is greater than or equal to $1-\varepsilon$ for some $\varepsilon \in (0,1)$. This partition, called the minimal covering set, will be determined for the case of unequal mass, non-spinning BBH waveforms within the range of parameters previously determined to parameterize valid p-waveforms. Next, a prescription based on the simpler case will be given to determine the minimal covering set of the two dimensional parameter space in the spinning, non-precessing case.

The Phenomenological Waveform

To begin this discussion, we must first understand BBH systems. Although black holes are relatively simple objects, a system of two black holes is drastically more complicated than a single black hole. According to the no hair theorem, a single black hole can be completely specified by its mass, charge, and angular momentum (8). When considering a system of two black holes however, we must consider the mass ratio between them, the spin ratio between them, the orientation of their angular momentum axes, and whether their angular momentum axes are fixed with respect to each other or whether one precesses about the other(9). The only remaining parameter is the system's electrical charge. In this study, we will consider both black holes to be electrically neutral, as would most likely be the case in free space.

The different cases of complexity for binary black holes are as follows: The parameter space for the simplest case, for non-spinning black holes of unequal mass, consists of a single parameter, the symmetric mass ratio

$$\eta = \frac{M_1 M_2}{(M_1 + M_2)^2}$$

Where M_1 and M_2 are the masses of each black hole.

For the next simplest case, spinning but non-precessing unequal mass black holes, the parameter space consists of the symmetric mass ratio η and the symmetric spin ratio

$$\chi = \frac{1}{2}\left(1 + \frac{M_1 - M_2}{M_1 + M_2}\right)\frac{S_1}{M_1^2} + \frac{1}{2}\left(1 - \frac{M_1 - M_2}{M_1 + M_2}\right)\frac{S_2}{M_2^2}$$

Where S_1 and S_2 are the spin magnitudes of each black hole.

The most general case of unequal mass, precessing spin binary systems will not be analyzed here, but will be commented on further in the Conclusions section.

Using techniques elaborated on by Ajith et al., p-waveforms for the simplest merging BBH system, that is, when neither black hole is spinning but their masses aren't necessarily equal, can be produced(9). Analytical formulas are also known for the next simplest case, when the two black holes are spinning, have unequal mass, but the spin axes are non-precessing(10). Finally, a function for the full precessing spin, unequal mass p-waveform is known for a small subspace of the possible spin orientations(11).

In each of these cases, the p-waveforms were compared to hybrid waveforms (a combination of the full numerical relativity waveforms and Post Newtonion waveforms) over a finite range of parameters. By their construction, the p-waveforms agree with NR waveforms over their domain of definition, but the full range of mass ratios over which the unequal mass, non-spinning p-waveforms were found to be valid has been found. The minimal partition of the mass ratio parameter space has not been determined. Furthermore, neither the range of parameters over which the spinning but non-precessing waveforms are valid, nor the minimal partition of the mass ratio and spin ratio parameter space has been determined. A simple way to corroborate the data in Figure 1 is to simply match p-waveforms parametrized by adjacent data points and confirm their match is above the desired threshold.

CHAPTER 2

METHOD

The method used to define the parameter space over which the p-waveforms are valid can be divided into three main sections. First, the programs defining the p-waveforms for the different levels of complexity must be written in Matlab. Second, a program must be written to convert the units of the numerical relativity and candidate p-waveforms to natural units and move them from the time domain to the frequency domain. This code will also take the Fourier transform of the strain of the waveforms, and match the waveforms to one another. This match is detailed below. Third, a program will be written to maximize the match over the parameters of the p-waveforms.

One additional program, which employs the programs used to accomplish steps one through three, is required to find the minimal covering set of the parameter spaces.

The Waveform Code

First, programs must be written to produce the p-waveforms. The first, which produces the unequal mass non-spinning waveforms, will implement equation 4.12 in Ref. 9

$$\boldsymbol{u(f) = A_{eff}(f)e^{i\Psi_{eff}}}$$

where,

$$\boldsymbol{A_{eff}(f) = C\left(f/f_{merg}\right)^{-7/6} \quad if\ f < f_{merg}}$$

$$A_{eff}(f) = C\left({}^{f}\!/_{f_{merg}}\right)^{-2/3} \quad if\ f_{merg} \leq f < f_{ring}$$

$$A_{eff}(f) = \omega\mathcal{L}(f, f_{ring}, \sigma) \quad if\ f_{ring} \leq f < f_{cut}$$

$$\Psi_{eff}(f) = 2\pi f t_0 + \varphi_0 + \sum_{k=0}^{7} \psi_k f^{(k-5)/3}$$

Where f_{merg}, f_{ring}, and f_{cut} are the frequencies when the BBH system begins its merger phase and ring down phase – the phase after the black holes have merged, and when the simulation is to stop, C, σ, t_0, φ_0, and the ψ_k are numerical parameters and $\mathcal{L}$ is a Lorentzian function, a predefined mathematical function similar to the Gaussian.

The p-waveforms for the unequal mass spinning but non- precessing case are produced similarly by implementing equation 1 in Ref. 10.

Matching the Waveforms over the Parameter Space

The maximization of matches over different parameters is discussed in this section. The match between two waveforms is defined similar to formula (3.10) in Ref. 9.

$$M[f, g] = \max_{U} \frac{(f|g)}{\sqrt{(f|f)(g|g)}}$$

Where U is the internal parameter space of the p-waveforms (for the non-spinning case, U consists of t_0, φ_0, and M_{total}) and,

$$(\boldsymbol{g}|\boldsymbol{h}) = \ \mathbf{2}\int_{\mathbf{0}}^{\infty}\frac{\widetilde{\boldsymbol{g}}^{*}(\boldsymbol{f})\widetilde{\boldsymbol{h}}(\boldsymbol{f}) + \ \widetilde{\boldsymbol{g}}^{*}(\boldsymbol{f})\widetilde{\boldsymbol{h}}(\boldsymbol{f})}{\boldsymbol{S}_{\boldsymbol{n}}(\boldsymbol{f})}\boldsymbol{df}$$

Where, a tilde denotes a Hermitian conjugate and S is the noise spectral density. Since these simulations are ran in finite time, the entire space U, which is very large, cannot be maximized over, so an appropriate subspace which takes advantage of certain periodicities in the matches is used. The matched filtering algorithm, implemented in Matlab, will return a number between zero and one. The closer the match is to one, the better the match between the waveforms is, and therefore, the better the p-waveform approximates the numerical relativity waveform. When the minimal covering set of a parameter space is being determined, p-waveforms whose parameters are adjacent in the parameter space will be matched.

It is necessary to convert the NR and p-waveforms to proper units and to frequency space. In order to implement the match formula, these conversions must be made because the noise spectral density is in frequency space. Both waveforms must be put in natural units, which are set to the black hole system’s mass and distance from the detector.

Finally, graphs can be produced showing the values in the minimal covering set S of the parameter space attainable versus the match between p-waveforms parameterized by adjacent elements in S. In this way, we can display both the minimal covering subset of the parameter space, and how well the adjacent waveforms match.

CHAPTER 3

RESULTS AND DISCUSSION

For the simplest case of non-spinning unequal mass BBH systems, the programs described in the methods section can be ran to produce Figure 1, which shows a subset of the symmetric mass ratio interval [0.08, 0.25] which covers the parameter space (this interval corresponds to a range of mass ratios from 1/10 to 1/2). As discussed in the introduction section, this set is possibly not minimal because the match between adjacent waveforms can only be maximized over a finite range of the intrinsic p-waveform parameters.

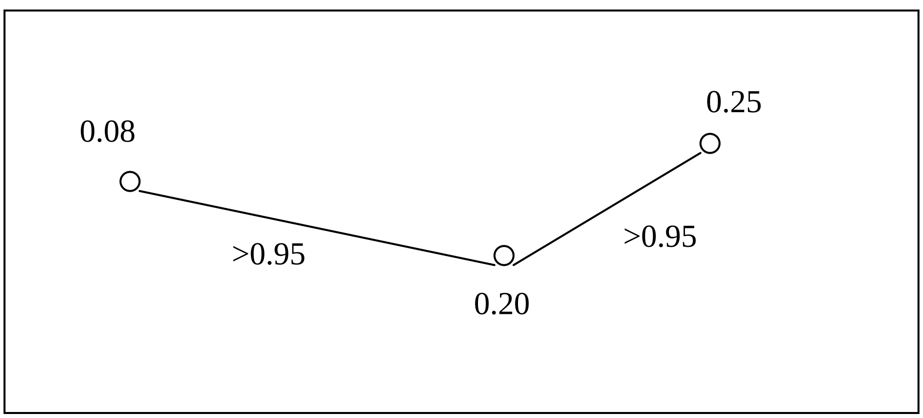

Figure 1. The subset which covers the parameter space of non-spinning unequal mass BBH systems. Nodes represent elements of the minimal covering set and edges represent the match between connected nodes.

The graph is presented in this slightly unusual way, where the data points from the minimal covering set (in this case specific values of the symmetric mass ratio) represent nodes and the edges represent the match between adjacent nodes, to suggest how the graph would be presented for a higher dimensional parameter space. For example, in two dimensions, the graph could be a triangulation of the plane with nodes for elements of the minimal covering set and edges for the matches between them.

The implementation I used of the matching algorithm given in the methods section turns out to be particularly sensitive to several inputs, but after much experimentation, the desired range of intrinsic parameters to match over can be found. It is expected that the matches vary smoothly, so the inputs such as mass ratio step size are adjusted until this occurs. After the matches are observed to vary smoothly, the appropriate range of intrinsic parameters to match over (t_0 and total mass in the case of Figure 1), can be found.

CHAPTER 4

CONCLUSION

In an effort to help data analysts confirm the first gravitational wave detection, numerical relativists have been producing waveform template banks so the data analysts have a basis for comparison. It is computationally expensive to produce NR waveforms, so approximations to them are sought. Since it takes time to compare the template banks to the raw data, the template banks should be made as sparse as possible. We have seen that this notion of sparsity can be quantified by the minimal covering set which we found for the non-spinning BBH p-waveforms. Although this analysis is interesting and relevant, there are several prospects for future work which hold more promise.

One prospect for future work is to find the minimal covering set for spinning non-precessing p-waveforms. The process for carrying this out is nearly identical to the one described in the methods section of this paper: the function describing the waveforms is known, so they can be produced, their parameter space partitioned, and matches can be ran between them, just as before. The main difference is that the parameter space includes the symmetric mass ratio and the symmetric spin ratio, whereas before we only had the one dimensional parameter space consisting of the symmetric mass ratio. This effectively squares the run-time of the program used to find the minimal covering set.

In an effort to increase efficiency in finding the minimal covering set, an interesting mathematical problem arises, worthy of analysis on its own. That is, are there different ways to partition the parameter space so that fewer matches have to be run to find the minimal covering set? For example, if the two dimensional parameter space of spinning non-precessing p-waveforms is initially partitioned into a triangulation rather than a mesh, would this increase efficiency? Although the development of p-waveforms for the most general spinning precessing BBH case is still in its infancy, we can discuss the problem of finding their minimal covering set, since in the end we are going to have a formula approximating them just as before. The problem of maximizing efficiency for a parameter space of dimension n greater than two seems astronomically more complex. We could of course just partition the space into a mesh grid and proceed as in the one dimensional case, but it may increase efficiency to partition the space with a regular polyhedron other than the cube.

Another prospect that would be very easy to adapt the programs written to find the minimal covering set to finding the range of the parameter space over which the p-waveforms accurately describe NR waveforms. Essentially, this amounts to matching a p-waveform with an NR waveform with corresponding parameters instead of matching two p-waveforms. We know, of course, that the p-waveforms are valid over the range of parameters of the NR waveforms from which they were constructed. However, since it is difficult to determine the p-waveform fitting factors, it is useful to know if this given range can be extended. This task has been accomplished in Ref. (12) for unequal mass, non-spinning waveforms, but has not yet been done for the spinning non-precessing case.

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