SOME RESULTS ON LINEAR DISCREPANCY FOR PARTIALLY ORDERED SETS

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Mitchel Todd Keller

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SOME RESULTS ON LINEAR DISCREPANCY FOR PARTIALLY ORDERED SETS

Approved by:

Dr. William T. Trotter, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Luca Dieci
School of Mathematics
Georgia Institute of Technology

Dr. Prasad V. Tetali
School of Mathematics and
School of Computer Science
Georgia Institute of Technology

Dr. Xingxing Yu
School of Mathematics
Georgia Institute of Technology

Dr. Milena Mihail
School of Computer Science
Georgia Institute of Technology

Date Approved: October 29, 2009

To my parents, Jim and Karol,

who believed a farm kid from North Dakota could do whatever he wanted and helped me achieve whatever I imagined and more.

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SUMMARY

Tanenbaum, Trenk, and Fishburn introduced the concept of linear discrepancy in 2001, proposing it as a way to measure a partially ordered set's distance from being a linear order. In addition to proving a number of results about linear discrepancy, they posed eight challenges and questions for future work. This dissertation completely resolves one of those challenges and makes contributions on two others. This dissertation has three principal components: 3-discrepancy irreducible posets of width 3, degree bounds, and online algorithms for linear discrepancy. The first principal component of this dissertation provides a forbidden subposet characterization of the posets with linear discrepancy equal to 2 by completing the determination of the posets that are 3-irreducible with respect to linear discrepancy. The second principal component concerns degree bounds for linear discrepancy and weak discrepancy, a parameter similar to linear discrepancy. Specifically, if every point of a poset is incomparable to at most Δ other points of the poset, we prove three bounds: the linear discrepancy of an interval order is at most Δ, with equality if and only if it contains an antichain of size $\Delta + 1$; the linear discrepancy of a disconnected poset is at most $\lfloor (3\Delta - 1)/2 \rfloor$; and the weak discrepancy of a poset is at most $\Delta - 1$. The third principal component of this dissertation incorporates another large area of research, that of online algorithms. We show that no online algorithm for linear discrepancy can be better than 3-competitive, even for the class of interval orders. We also give a 2-competitive online algorithm for linear discrepancy on semiorders and show that this algorithm is optimal.

CHAPTER I

INTRODUCTION

In this dissertation, we study a property of partially ordered sets known as linear discrepancy. Given any set of objects and a partial order on it, we can create a linear extension of the partial order, i.e., a total order on the same set that respects the partial order. When two objects are incomparable, we have a choice of where they can be placed in the linear extension relative to each other. Given any two incomparable objects, it is always possible to form a linear extension in which they appear consecutively, possibly at the expense of placing other incomparable objects far apart. On the other hand, for some partial orders, it is possible to form a linear extension in which one incomparable object appears in the lowest position in the linear extension and the other appears in the highest. Linear extensions in which objects between which we are unable to make a comparison are placed far apart are undesirable in many applications since the discrepancy between the objects' positions creates an implicit comparison between two objects that are incomparable. The principal motivation behind the concept of linear discrepancy is to avoid this situation.

With this motivation in hand, we continue this chapter by presenting the formal definitions and background necessary for the subsequent chapters. We also establish some initial results which we will find useful in the dissertation. In Chapter 2, we explore a characterization problem for posets with small linear discrepancy. The focus of Chapter 3 is on bounds for linear discrepancy of particular classes of posets. In Chapter 4, we investigate online algorithms for linear discrepancy.

1.1 Basic definitions and notation

1.1.1 Partially ordered sets

A *partially ordered set* $\mathbf{P} = (X, \leq_P)$ consists of a set X and a reflexive, antisymmetric, and transitive binary relation $\leq_P$ on X. The relation $\leq_P$ is a *partial order*, and we call X the *ground set*. We will usually refer to partially ordered sets as *posets*, although some authors prefer *ordered sets*. In many cases, we will treat the ground set implicitly and write $x \in \mathbf{P}$ for $x \in X$. In this dissertation, we will always assume that the ground set of a poset is finite and will suppress the subscript on the order relation when it is clear from context. If $x, y \in X$, we will write $x \leq_P y$ to indicate that $(x, y) \in \leq_P$ with the possibility that $x = y$. If we insist that $x \neq y$, we write $x <_P y$. We say that x and y are *comparable* in $\mathbf{P}$ if $x <_P y$, $y <_P x$, or $x = y$ and denote this by $x \perp_P y$. We say that x and y are *incomparable* in $\mathbf{P}$ if they are not comparable in $\mathbf{P}$ and write $x \|_P y$. If $x <_P y$ and there is no $z \in X$ such that $x <_P z <_P y$, we say that x is *covered by* y or y *covers* x and write $x \lessdot y$. We say that x is a *maximal element* or simply *maximal* if there is no y such that $x <_P y$. Similarly, x is a *minimal element* or *minimal* if there is no y such that $y <_P x$.

We will frequently find it useful to visualize posets. We do this by identifying the elements of the poset with points in the standard Cartesian plane. In such a visualization, we require that if $x <_P y$, then the vertical coordinate of the point corresponding to x must be smaller than the vertical coordinate of the point corresponding to y. If $x \lessdot_P y$, then we draw a line segment from the point corresponding to x to the point corresponding to y. Such line segments are always drawn so that they do not pass through any point of the plane which corresponds to a point of the poset. (To accommodate this rule, particularly when drawing by hand, we sometimes relax the requirement that line segments be used and allow curves as well.) Such a visualization of a poset is called its *Hasse diagram*, *order diagram*, or often simply *diagram*. In Figure 1.1, we show the Hasse diagram for the poset $\mathbf{P} = (X, \leq_P)$ in

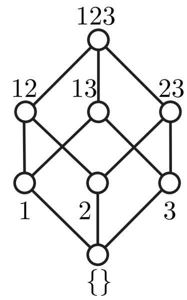

Figure 1.1: The Hasse diagram of the poset of subsets of $\{1, 2, 3\}$

which X is the set of all subsets of $\{1, 2, 3\}$ and the partial order is subset containment, i.e., $A <_P B$ if and only if $A \subseteq B$. (For illustrative purposes, in Figure 1.1, we have written subsets by simply listing the elements, e.g., $\{1, 2\}$ is written as 12.)

If $\mathbf{P} = (X, \leq_P)$ and $\mathbf{Q} = (Y, \leq_Q)$ are posets with $Y \subseteq X$, we say that $\mathbf{Q}$ is a *subposet* of $\mathbf{P}$ if $\leq_Q = \leq_P \cap (Y \times Y)$. If we are given $Y \subseteq X$, we define the subposet of $\mathbf{P}$ induced by Y to be $(Y, (Y \times Y) \cap \leq_P)$. The *dual* of $\mathbf{P}$ is the poset $\mathbf{P}^d = (X, \leq_P^d)$, where $\leq_P^d := \{(y, x) \in X \times X \mid (x, y) \in \leq_P\}$. If $\mathbf{P} = \mathbf{P}^d$, we say that $\mathbf{P}$ is *self-dual.* If $\mathbf{P} = (X, \leq_P)$ and $\mathbf{Q} = (Y, \leq_Q)$ are posets with X and Y disjoint, we define the poset $\mathbf{P} + \mathbf{Q}$, often called the *sum of* $\mathbf{P}$ *and* $\mathbf{Q}$, to be the poset with ground set $X \cup Y$ and partial order $\leq_P \cup \leq_Q$. We say $\mathbf{P} = (X, \leq_P)$ is *connected* if for all $x, y \in X$, there is a sequence $x = x_1, x_2, \ldots, x_n = y$ with $x_i \perp_P x_{i+1}$ for $i = 1, \ldots, n-1$. Otherwise, $\mathbf{P}$ is *disconnected.* If $\mathbf{P} = (X, \leq_P)$ is a poset and $S \subseteq X$, the set

$$D(S) := \{x \in X \mid \text{there is } s \in S \text{ with } x < s\}$$

is called the *down-set* of S. We let $D[S] := D(S) \cup S$. If $S = \{y\}$, we will write $D(y)$ instead of $D(\{y\})$. Similarly, we define the *up-set* of S to be

$$U(S) := \{x \in X \mid \text{there is } s \in S \text{ with } x > s\}$$

and let $U[S] := U(S) \cup S$. We also define $\mathrm{Inc}(x)$ as $\{y \in X \mid x \|_P y\}$, the set of points incomparable to x in $\mathbf{P}$ and let $\Delta(\mathbf{P}) := \max_{x \in X} |\mathrm{Inc}(x)|$. We should point out that this notation is nonstandard, as it is often used in the literature to denote the maximum number of elements with which any point is comparable.

A *chain* in a poset is a set $C \subseteq X$ such that $x \perp_P y$ for all $x, y \in C$. The size of a largest chain in $\mathbf{P}$ is called the *height* of $\mathbf{P}$, denoted $\text{height}(\mathbf{P})$. An *antichain* is a set $A \subseteq X$ such that $x \|_P y$ for all $x, y \in A$. The size of a largest antichain in $\mathbf{P}$ is called the *width* of $\mathbf{P}$, denoted $\text{width}(\mathbf{P})$. If n is a positive integer, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. We also define $\mathbf{n}$ to be the total order $\{0 < 1 < 2 < \cdots < n-1\}$ on n points and refer to it as the *chain on n points*. One of the most celebrated theorems in the theory of partially ordered sets is the following theorem of Dilworth [7].

Theorem 1.1.1 (Dilworth, 1950). *Let $\mathbf{P} = (X, \leq_P)$ be a poset and $w := \text{width}(\mathbf{P})$. Then there exist w disjoint chains $C_1, C_2, \ldots, C_w$ such that $X = C_1 \cup C_2 \cup \cdots \cup C_w$, and X cannot be partitioned into fewer than w chains.*

We also note the following elementary (but useful) result, which is often referred to as the "dual of Dilworth's theorem" because of the way it interchanges the roles of chains and antichains.

Proposition 1.1.2. *Let $\mathbf{P} = (X, \leq_P)$ be a poset and $h := \text{height}(\mathbf{P})$. Then there exist h disjoint antichains $A_1, A_2, \ldots, A_h$ such that $X = A_1 \cup A_2 \cup \cdots \cup A_h$, and X cannot be partitioned into fewer than h antichains.*

A *total order* on a set X is a partial order such that every pair of points in X is comparable. A *linear extension* L of a poset $\mathbf{P} = (X, \leq_P)$ is a total order on X such that if $x <_P y$, then $x <_L y$. If $|X| = n$ and a linear extension L orders the points of X as $x_1, x_2, \ldots, x_n$ and $x = x_i$, we say that i is the *height of x in L* and write $h_L(x) = i$. We write $\mathcal{E}(\mathbf{P})$ for the collection of all linear extensions of $\mathbf{P}$. For convenience, we will often refer to a linear extension of $\mathbf{P}$ as a *labelling* of X ($|X| = n$) using the elements of $[n] := \{1, 2, \ldots, n\}$ such that the ordering created by the labelling is a linear extension of $\mathbf{P}$. Linear extensions have played a central role in the development of the combinatorics of partially ordered sets. In a 1941 paper, Dushnik and Miller [8] showed that every partial order is the intersection of a collection $\mathcal{R} \subseteq \mathcal{E}(\mathbf{P})$ and defined the *dimension* of $\mathbf{P}$, $\dim(\mathbf{P})$, to be the least t such that $\leq_P$ is the intersection of t linear extensions from $\mathcal{E}(\mathbf{P})$. If $\leq_P = \cap_{L\in\mathcal{R}} L$, we call $\mathcal{R}$ a realizer of the partial order $\leq_P$. Dushnik and Miller also showed that finite posets of dimension n exist for every positive integer n by constructing the class of posets now known as the *standard examples* $\mathbf{S}_n$.

It is not difficult to see that $\mathcal{R}$ is a realizer of $\leq_P$ if and only if for each pair $x\|_P y$, there are $L, L' \in \mathcal{R}$ with $x <_L y$ and $y <_{L'} x$. Rabinovitch and Rubin showed in [33] that it is enough to consider only special pairs of incomparable points known as *critical pairs*. We say (x, y) is a critical pair of $\mathbf{P} = (X, \leq_P)$ if $x\|_P y$, $D(x) \subseteq D(y)$, and $U(x) \supseteq U(y)$. In particular, Rabinovitch and Rubin showed that $\mathcal{R}$ is a realizer if and only if for every critical pair (x, y), there exists $L \in \mathcal{R}$ with $y <_L x$. Such a critical pair is said to be *reversed in* L. For well over 30 years, the study of the combinatorics of posets has focused in large part on dimension theory, but in this dissertation we will investigate another property that shares many similarities with dimension, but stands in stark contrast in other ways. Readers interested in learning more about dimension theory should consult Trotter's monograph [40].

1.1.2 Graph theory

Although the primary objects of study in this dissertation are partially ordered sets, there are many instances where posets are closely related to another combinatorial structure: the graph. For our purposes, a *graph* $\mathbf{G} = (V, E)$ consists of a finite set V and a set E of two-element subsets of V. The elements of V are called *vertices* and the elements of E are known as *edges*. Vertices $u, v \in V$ are said to be *adjacent* or *neighbors* if $\{u, v\} \in E$, in which case we will usually simply write $uv \in E$. Note that $uv \in E$ and $vu \in E$ are equivalent. If $e = uv \in E$, we say that u and v are the *endpoints* of e and that e is incident to u and v. The *degree* of a vertex v, denoted $d(v)$ is the number of edges incident to v. The *maximum degree of* $\mathbf{G}$ is

$\Delta(\mathbf{G}) := \max_{v \in V} d(v)$. A graph $\mathbf{H} = (V', E')$ is a *subgraph* of $\mathbf{G} = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$. We say that $\mathbf{H}$ is an *induced subgraph of* $\mathbf{G}$ if $\mathbf{H}$ is a subgraph of $\mathbf{G}$ and every edge in E with both endpoints in V' is also in E'.

Given a poset $\mathbf{P} = (X, \leq_P)$, we can construct a graph $\mathbf{G_P}$ with vertex set $V := X$ and $uv \in E$ if and only if $u \perp_P v$. This graph is called the *comparability graph* of $\mathbf{P}$. The class of graphs that can be comparability graphs was characterized by Gallai [14] in 1967. He identified six infinite families of graphs and 10 other graphs that cannot appear as an induced subgraph in a comparability graph. These are the minimal graphs that cannot be comparability graphs and are also referred to as *forbidden subgraphs* for comparability graphs. Our characterization result of Chapter 2 will have a very similar flavor to Gallai's characterization of comparability graphs. We can also define the *cocomparability graph* of $\mathbf{P} = (X, \leq_P)$ (sometimes called the *incomparability graph*) to be the graph $\mathbf{G}^c_{\mathbf{P}} = (V, E)$ with $V := X$ and $uv \in E$ if and only if $u \|_P v$. It is important to note that although a given poset has only one comparability graph and one cocomparability graph, in general, a (co)comparability graph corresponds to several posets. Any property of a poset that is the same for all posets with the same comparability graph is called a *comparability invariant*. For this dissertation, the graphs we consider will all be cocomparability graphs of posets because we will have great interest in incomparable pairs in posets in our study of linear discrepancy.

1.2 Classes of special posets and graphs

As we shall soon see, one special class of posets (and its corresponding class of cocomparability graphs) has played a central role in the study of linear discrepancy since the first paper [37] by Tanenbaum et al. In this section, we describe this class of posets along with a subclass thereof and their characterizations.

An *interval order* is a poset $\mathbf{P} = (X, \leq_P)$ for which we can associate a closed, bounded interval $I(x) = [l(x), r(x)] \subset \mathbb{R}$ to each element $x \in X$ such that for all $x, y \in X$, $x <_P y$ if and only if $r(x) < l(y)$, i.e., $I(x)$ lies completely to the left of $I(y)$. We call the associated collection of intervals an *interval representation* of $\mathbf{P}$. Note that we do not require that the intervals associated to elements of $\mathbf{P}$ be distinct; however, since we are assuming that X is finite, we may in fact assume when it is convenient that the intervals are all distinct and in fact that no real number appears more than once as the endpoint of an interval. The cocomparability graph of an interval order is called an *interval graph*. Interval orders are one of the most studied special classes of posets, so much so that Fishburn wrote a monograph [11] on interval orders and interval graphs. This monograph contains a nice summary of the origins of interval orders, dating back to a 1914 paper by Wiener [43], and discussing the rise to common study because of interest in the behavior sciences in the middle of the last century.

In [10], Fishburn proved the following characterization of interval orders, which we will find very useful, particularly in Chapter 4.

Theorem 1.2.1. *Let $\mathbf{P} = (X, \leq_P)$ be a poset. Then the following statements are equivalent.*

(1) $\mathbf{P}$ *is an interval order.*

(2) $\mathbf{P}$ *does not contain* $\mathbf{2} + \mathbf{2}$ *as a subposet.*

(3) *For every $x, y \in X$, either $D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$.*

(4) *For every $x, y \in X$, either $U(x) \subseteq U(y)$ or $U(y) \subseteq U(x)$.*

In 1962, five years before Gallai published his characterization of cocomparability graphs, Lekkerkerker and Boland [30] gave a characterization of interval graphs by forbidden subgraphs. Their list consists of four infinite families and two other graphs that are forbidden as induced subgraphs of interval graphs.

A *semiorder* is an interval order having an interval representation in which all intervals have the same (usually taken to be unit) length. Their cocomparability graphs are usually referred to as *unit interval graphs*. Scott and Suppes [36] first characterized semiorders in 1958. The formulation we give below is not their original version, but rather utilizes Fishburn's later characterization of Theorem 1.2.1.

Theorem 1.2.2. *Let* $\mathbf{P} = (X, \leq_P)$ *be an interval order. Then* $\mathbf{P}$ *is a semiorder if and only if* $\mathbf{P}$ *does not contain* $\mathbf{1} + \mathbf{3}$ *as a subposet.*

These characterization theorems will give us powerful tools to use in what follows. We will often be able to deduce facts about the structure of an interval order because if those facts were false, the poset would have to contain a $\mathbf{2} + \mathbf{2}$. We also note that for an interval order, if $D(x) \subsetneq D(y)$, then $l(x) < l(y)$ in any representation. When considering a semiorder, this also tells us about the ordering of the right endpoints, a fact we will repeatedly exploit in Chapter 4.

We close this section with a definition and characterization of an even more restrictive class of posets that will be used as we introduce and discuss the history of linear discrepancy in the next section. A finite poset $\mathbf{P} = (X, \leq_P)$ is a *weak order* if there exists a function $f \colon X \to \mathbb{R}$ such that for all $x, y \in X$ with $x \neq y$, we have

(1) $x <_P y$ if and only if $f(x) < f(y)$ in $\mathbb{R}$, and

(2) $x \|_P y$ if and only if $f(x) = f(y)$.

We can view a weak order as an interval order in which the representation consists entirely of degenerate intervals $[f(x), f(x)]$. The following characterization of weak orders is elementary.

Proposition 1.2.3. *Let* $\mathbf{P} = (X, \leq_P)$ *be a poset. Then the following statements are equivalent.*

(1) $\mathbf{P}$ *is a weak order.*

(2) $\mathbf{P}$ *does not contain* $\mathbf{1} + \mathbf{2}$ *as a subposet.*

(3) X *can be partitioned into into antichains* $A_1, A_2, \ldots A_h$ *so that if* $x \in A_i$ *and* $y \in A_j$ *with* $i < j$*, then* $x <_P y$*.*

1.3 Linear discrepancy and bandwidth

The principal property of posets studied in this dissertation will be linear discrepancy. To place the study of linear discrepancy in its proper historical context, we first introduce a property known as *weak discrepancy*, which Trenk introduced in [38] as *weakness*. (We will uniformly use "weak discrepancy" as this property's name, even when discussing results from the papers where it was referred to as weakness.)

Definition 1.3.1 (Weak discrepancy)**.** Let $\mathbf{P} = (X, \leq_P)$ be a poset and f an integer-valued function on X. We call f a *k-weak labelling* if (1) $f(x) < f(y)$ whenever $x <_P y$ and (2) $|f(x) - f(y)| \leq k$ whenever $x \|_P y$. The smallest k for which there exists a k-weak labelling of $\mathbf{P}$ is called the *weak discrepancy of* $\mathbf{P}$ and denoted $\mathrm{wd}(\mathbf{P})$.

The idea behind weak discrepancy is, on some level, to measure a poset's distance from being a weak order. It is straightforward to see that if $\mathbf{P} = (X, \leq_P)$ is a weak order and $A_1, A_2, \ldots, A_h$ is the antichain partition of X guaranteed by Proposition 1.2.3, then the function $f \colon X \to [h]$ defined by $f(x) = i$ where $x \in A_i$ is a 0-weak labelling, so weak orders have weak discrepancy 0. In fact, they are the only posets with $\mathrm{wd}(\mathbf{P}) = 0$. Trenk gave a polynomial time algorithm for computing weak discrepancy in [38], and then with Gimbel gave another in [16]. Gimbel and Trenk also showed that weak discrepancy is a comparability invariant.

The next work on weak discrepancy came in [37], when Tanenbaum, Trenk, and Fishburn proved several more results about it and also introduced linear discrepancy, which we now define.

Definition 1.3.2 (Linear discrepancy)**.** Let $\mathbf{P} = (X, \le_P)$ be a poset and $L \in \mathcal{E}(\mathbf{P})$. We define the *linear discrepancy of* L, denoted $\mathrm{ld}(\mathbf{P}, L)$ to be $\max_{x \| _P y} |h_L(y) - h_L(x)|$. The *linear discrepancy of* $\mathbf{P}$ is then defined as $\mathrm{ld}(\mathbf{P}) := \min_{L \in \mathcal{E}(P)} \mathrm{ld}(\mathbf{P}, L)$. (If $\mathbf{P}$ is a linear order, we take $\mathrm{ld}(\mathbf{P})$ to be 0.) A linear extension L with $\mathrm{ld}(\mathbf{P}, L) = \mathrm{ld}(\mathbf{P})$ is said to be *optimal*.

What we call $\mathrm{ld}(\mathbf{P}, L)$ here was called the uncertainty of L and denoted $\mathrm{uncert}(L)$ in [37]. We have chosen our notation for its closer alignment to the notation in use for graph bandwidth, which we will soon define. Notice that h_L is a k-weak labelling for $k = \mathrm{ld}(\mathbf{P}, L)$ and is also injective, so we can view linear discrepancy as the restriction of weak discrepancy to require that the labelling function be an injection. Much as weak discrepancy measures the distance of a poset from being a weak order, linear discrepancy is a measure of a poset's distance from being a linear order. In this sense, it is quite similar to dimension. However, there are also many important ways in which linear discrepancy is dissimilar from dimension, and we will explore them as we proceed.

Since it is possible to compute a poset's weak discrepancy in polynomial time, a natural first question for the originators of linear discrepancy to ask was whether the same was true for linear discrepancy. Since it was known that weak discrepancy was a comparability invariant, it also made sense to see if considering the comparability or cocomparability graph of a poset could aid in computing its linear discrepancy. If one pursues that line of research, they quickly discover the similarities between linear discrepancy and a long-studied graph property known as bandwidth.

Definition 1.3.3 (Bandwidth)**.** Let $\mathbf{G} = (V, E)$ be a graph with $|V| = n$ and fix a bijection $\sigma\colon V \to [n]$. We define the *bandwidth of the bijection* σ, denoted $\mathrm{bw}(\mathbf{G}, \sigma)$, to be $\max_{xy \in E} |\sigma(y) - \sigma(x)|$. The *bandwith of* $\mathbf{G}$ is then $\min_\sigma \mathrm{bw}(\mathbf{G}, \sigma)$, where the minimum is taken over all bijections $\sigma\colon V \to [n]$.

The problem of minimizing bandwidth arose in the study of matrices and soon moved into graph theory. In [32], Papadimitriou showed that determining if a graph's bandwidth is at most k is **NP**-complete. Subsequently, Garey, Graham, Johnson, and Knuth showed in [15] that the situation is even worse by proving that it is **NP**-complete to determine if the bandwidth of a tree with maximum degree 3 is at most k. Because of the difficulties of computing the bandwidth of a graph, much work has been done on aproximating it, bounding it, and computing it for special classes of graphs. The survey article by Chinn, Chvátalová, Dwedney, and Gibbs [5] and an update by Lai and Williams [29] provide an overview of this work. The one result of this type that will be of particular interest to us is an algorithm of Kleitman and Vohra, who proved in [27] that for $\mathbf{G}$ an interval graph, the decision problem "Is $\mathrm{bw}(\mathbf{G}) \leq k$?" can be answered in polynomial time.

To see the relationship between linear discrepancy and bandwidth, note that for a poset $\mathbf{P} = (X, \leq_P)$ with $|X| = n$, incomparable pairs correspond to edges in the cocomparability graph $\mathbf{G}^c_{\mathbf{P}}$. Thus, a linear extension L of $\mathbf{P}$ can be interpreted as a bijection from the vertex set of $\mathbf{G}^c_{\mathbf{P}}$ to $[n]$, and so we have $\mathrm{ld}(\mathbf{P}, L) = \mathrm{bw}(\mathbf{G}^c_{\mathbf{P}}, L)$ and $\mathrm{ld}(\mathbf{P}) \geq \mathrm{bw}(\mathbf{G}^c_{\mathbf{P}})$. In [12], Fishburn, Tanenbaum, and Trenk showed that in fact $\mathrm{ld}(\mathbf{P}) = \mathrm{bw}(\mathbf{G}^c_{\mathbf{P}})$. They did this by showing that $\mathbf{P}$ can be transformed to an interval order without changing its linear discrepancy, that the bijection found by the Kleitman-Vohra algorithm is actually a linear extension of the interval order, and that they could modify this linear extension to a linear extension of the original posest without increasing its linear discrepancy. Once they knew that the linear discrepancy of a poset is equal to the bandwidth of its cocomparability graph, the same authors were able to show in [37] that it is **NP**-complete to determine if the linear discrepancy of a poset is at most k since Kloks, Kratsch, and Müller showed in [28] that it is **NP**-complete to determine if the bandwidth of the complement of a bipartite graph is at most k, and therefore this question is **NP**-complete for cocomparability graphs as well.

We conclude this section with a collection of elementary bounds on linear discrepancy and formulas for calculating linear discrepancy. Unless another citation is given, the result can be found in [37].

Lemma 1.3.4. *For every poset* $\mathbf{P}$*,* $\mathrm{ld}(\mathbf{P}) \geq \mathrm{width}(\mathbf{P}) - 1 \geq \dim(\mathbf{P}) - 1$.

The lower bound in the following theorem is trivial. However, the upper bound is not, and is a result of Rautenbach [35].

Theorem 1.3.5. *For every poset* $\mathbf{P}$*,* $\lceil \Delta(\mathbf{P})/2 \rceil \leq \mathrm{ld}(\mathbf{P}) \leq 2\Delta(\mathbf{P}) - 2$.

Theorem 1.3.6. *If* $\mathbf{P} = \mathbf{n_1} + \mathbf{n_2} + \cdots + \mathbf{n_k}$ *with* $k \geq 2$ *and* $n_1 \geq n_2 \geq \cdots \geq n_k$*, then* $\mathrm{ld}(\mathbf{P}) = \lceil n_1/2 \rceil + n_2 + \cdots + n_k - 1$ *and* $\mathrm{wd}(\mathbf{P}) = \lceil (n_1 + n_2)/2 \rceil - 1$.

Theorem 1.3.7. *If* $\mathbf{P}$ *is a semiorder, then* $\mathrm{ld}(\mathbf{P}) = \mathrm{width}(\mathbf{P}) - 1$ *and the linear extension which orders the points according to the left endpoints of their intervals in a unit interval representation is optimal.*

1.4 Useful results on linear discrepancy

This chapter concludes by presenting a number of new results for linear discrepancy that we will find useful in the main portion of the dissertation.

1.4.1 Critical pairs and removing points

One of the unsurprising similarities between linear discrepancy and dimension is that they are both monotonic. That is, if $\mathbf{Q}$ is a subposet of $\mathbf{P}$, then $\mathrm{ld}(\mathbf{Q}) \leq \mathrm{ld}(\mathbf{P})$ and $\dim(\mathbf{Q}) \leq \dim(\mathbf{P})$. This is quite straightforward to see. When studying the dimension of partially ordered sets, one of the first results one encounters after monotonicity is the following theorem of Hiraguchi [17].

Theorem 1.4.1 (Hiraguchi [17])**.** *Let* $\mathbf{P} = (X, \leq_P)$ *be a poset with* $|X| \geq 2$ *and let* $\mathbf{Q}$ *be the subposet of* $\mathbf{P}$ *induced by* $X - \{x\}$*. Then* $\dim(\mathbf{P}) \leq 1 + \dim(\mathbf{Q})$.

It is common to refer to Theorem 1.4.1 by saying that "dimension is continuous," since the removal of a small number of points can only cause a small change in the dimension. In fact, it is a longstanding open question as to whether a stronger form of this result is true. Namely, the question is whether for every poset $\mathbf{P} = (X, \leq_P)$, there exists a pair of points $x, y \in X$ such that the removal of x and y decreases the dimension of $\mathbf{P}$ by at most one. Since linear discrepancy and dimension are both monotonic, we might hope that they are both continuous as well. Unfortunately, this is not the case. As an example, consider the poset $\mathbf{1}+\mathbf{n}$. We know by Theorem 1.3.6 that $\mathrm{ld}(\mathbf{1}+\mathbf{n}) = \lceil n/2 \rceil$, but if we remove the isolated point from $\mathbf{1}+\mathbf{n}$, we are simply left with a chain, which has linear discrepancy 0. Thus, it is not possible to remove an arbitrary point from a poset and decrease its linear discrepancy by a small amount. However, we are able to prove a weaker result, which appears below as Theorem 1.4.4. Before doing this, however, we establish the following pair of lemmas about the role that critical pairs play in the study of linear discrepancy.

Lemma 1.4.2. *Let $\mathbf{P} = (X, \leq_P)$ be a poset and L a linear extension of $\mathbf{P}$. If x and y are incomparable in $\mathbf{P}$ and $h_L(y) - h_L(x) = \mathrm{ld}(\mathbf{P}, L)$, then (x, y) is a critical pair.*

Proof. Suppose $x \|_P y$, $x <_L y$, and $h_L(y) - h_L(x) = \mathrm{ld}(\mathbf{P}, L)$. If $z <_P x$, then $z <_L x$, and thus since $h_L(y) - h_L(x) = \mathrm{ld}(\mathbf{P}, L)$, we must have $z <_P y$. Therefore, $D(x) \subseteq D(y)$. Similarly, if $z >_P y$, then z is above y in L and must therefore also be greater than x in $\mathbf{P}$. Thus, $U(y) \subseteq U(x)$ and (x, y) is a critical pair. $\square$

The preceding result will allow us to focus our search for incomparable pairs witnessing the linear discrepancy of a linear extension to critical pairs, which we will find useful on multiple occasions. Our next lemma shows that critical pairs play a role for linear discrepancy that stands in stark contrast to their role in dimension theory, where any linear extension that does not reverse any critical pair is useless in forming a realizer. For the following, we shall refer to a critical pair (x, y) as *bicritical* if (y, x) is also a critical pair.

Lemma 1.4.3. *Let* $\mathbf{P} = (X, \leq_P)$ *be a poset. There exists a linear extension of* $\mathbf{P}$ *that is optimal with respect to linear discrepancy and reverses no critical pairs that are not bicritical.*

Proof. Consider a linear extension L of $\mathbf{P}$ that reverses at least one non-bicritical critical pair. Among all non-bicritical critical pairs that L reverses, take (x, y) so that $h_L(x) - h_L(y)$ is minimal. We know that L must place $D(y)$ below y, which is below x in L, which must be below $U(x)$ in L. Now notice that any point w satisfiying $y <_L w <_L x$ must be incomparable to both x and y, since (x, y) is a critical pair. Thus, we may form a new linear extension L' from L simply by switching the positions of x and y. Furthermore, since (x, y) is a critical pair, any point less than y in L and incomparable to y must be incomparable to x, and any point greater than x in L and incomparable to x must be incomparable to y. Thus, the distance between a pair of incomparable points in L' is no larger than it is in L, so $\mathrm{ld}(\mathbf{P}, L') = \mathrm{ld}(\mathbf{P}, L)$. Suppose that switching the positions of x and y has introduced a new reversed critical pair (that is not bicritical). Then one point of the critical pair must be x or y, and the other must lie between them in L (and thus in L'). Let this point be z, and without loss of generality, let us assume that (y, z) is a critical pair that is not bicritical. Then $D(y) \subseteq D(z)$ and $U(y) \supseteq U(z)$. But since (x, y) is a critical pair, we have that $D(x) \subseteq D(y) \subseteq D(z)$ and $U(x) \supseteq U(y) \supseteq U(z)$, and thus (x, z) is also a critical pair. Furthermore, since neither (x, y) nor (y, z) is bicritical, (x, z) is not bicritical. Now notice that if (y, z) is reversed in L', we have that (x, z) is reversed in L. Since $y <_L z <_L x$, we have that $h_L(x) - h_L(z) < h_L(x) - h_L(y)$, contradicting our choice of (x, y). Thus, L' reverses fewer non-bicritical critical pairs than L and does not increase its linear discrepancy. Thus, we may take any optimal linear extension of $\mathbf{P}$ and use this process until arriving at an optimal linear extension that does not reverse any non-bicritical critical pairs. $\square$

As previously remarked, it is possible to delete a single point from a poset with linear discrepancy $n/2$ and decrease the poset's linear discrepancy to zero. However, the following weaker result, which one might be tempted to call the "semi-continuity of linear discrepancy," is true.

Theorem 1.4.4. *For any poset there exists a point whose removal reduces the linear discrepancy by at most one.*

Proof. Let $\mathbf{P} = (X, \leq_P)$ be a poset. We first suppose that there are two minimal elements x and x' of $\mathbf{P}$ with the same up-set. Let L be a linear extension of $\mathbf{P} - \{x'\}$ that is optimal with respect to linear discrepancy. Create a new linear extension L' by inserting x' immediately below x in L. It is clear that L' is a linear extension of $\mathbf{P}$. Furthermore, since $\text{Inc}(x) - \{x'\} = \text{Inc}(x') - \{x\}$, the linear discrepancy of L' is at most one more than the linear discrepancy of L. Thus the removal of x decreased $\text{ld}(\mathbf{P})$ by at most one.

Now suppose that no two minimal elements have the same up-set. Then there is a minimal emement z such that there is no critical pair of the form (y, z). (A minimal element z with $|U(z)|$ maximum has this property.) Now consider a linear extension L of $\mathbf{P} - \{z\}$ that is optimal with respect to linear discrepancy and let s be the element of $U(z) \cup \{v \mid (z, v) \text{ is a critical pair}\}$ for which $h_L(s)$ is minimal. Form another linear extension L' by inserting z immediately below s. By construction, L' is a linear extension of $\mathbf{P}$. Since we only wish to show that $\text{ld}(\mathbf{P}, L')$ is at most one more than the linear discrepancy of $\text{ld}(\mathbf{P}, L)$, the only obstructions are of the form $z \| z'$. But by Lemma 1.4.2 and our choice of z, we may restrict our attention to critical pairs (z, z').

If $s \in U(z)$, our choice of s and z' imply that $s <_L z'$, and thus we must have $s \| z'$, as otherwise z and z' are comparable. If $s \notin U(z)$, then (z, s) is a critical pair, so $U(s) \subseteq U(z)$ and in particular $s \| z'$, as otherwise we would have $z' >_P z$. But then $h_{L'}(z') - h_{L'}(z) = h_L(z') - h_L(s) + 1 \leq \text{ld}(\mathbf{P} - \{z\}) + 1$. Hence the linear discrepancy of $\mathbf{P} - \{z\}$ is at least $\mathrm{ld}(\mathbf{P}) - 1$ as desired. $\square$

1.4.2 Ramsey complexity

A common type of question in combinatorial mathematics involves determining what substructures are necessary to force a given parameter to be large. When Dushnik and Miller defined dimension in [8], they showed that posets with arbitrarily large dimension exist, which gave an indication that it might be an interesting property to study, since dimension would not be very interesting if no poset had dimension larger than five, for example. For linear discrepancy, we have already noted in previous sections that $\mathrm{ld}(\mathbf{1} + \mathbf{n}) = \lceil n/2 \rceil$ and that a poset consisting solely of an n-element antichain has linear discrepancy $n - 1$. Of course, these examples merely address the sufficiency of a subposet to force a parameter up. In this section, we will examine the question of necessity. To do so, we will formalize the notion of "forcing" a parameter to be large, following the terminology found in Trotter's monograph [40].

Consider an infinite set $\mathcal{C}$ of discrete structures and a partial order $\prec$ on $\mathcal{C}$. Let $f\colon \mathcal{C} \to \mathbb{R}^+$ be any monotonic parameter on $\mathcal{C}$; that is, if $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, then $\mathbf{A} \prec \mathbf{B}$ implies $f(\mathbf{A}) \leq f(\mathbf{B})$. We say that an infinite chain $\mathbf{A}_1 \prec \mathbf{A}_2 \prec \cdots$ is a *Ramsey trail* if $f(\mathbf{A}_n) \to \infty$ as $n \to \infty$. The *Ramsey complexity* of f is the least t for which there exist t Ramsey trails

$$
\begin{aligned}
T_1 &= \{\mathbf{A}_{11} \prec \mathbf{A}_{12} \prec \mathbf{A}_{13} \prec \cdots\} \\
T_2 &= \{\mathbf{A}_{21} \prec \mathbf{A}_{22} \prec \mathbf{A}_{23} \prec \cdots\} \\
&\vdots \\
T_t &= \{\mathbf{A}_{t1} \prec \mathbf{A}_{t2} \prec \mathbf{A}_{t3} \prec \cdots\}
\end{aligned}
$$

and a function $g\colon \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $f(\mathbf{A}) \geq g(n)$, then there is $s \in [t]$ such that $\mathbf{A}_{sn} \prec \mathbf{A}$.

As our first example of a set of Ramsey trails, we consider the result after which the concept was named—Ramsey's Theorem. To do so, we will use the following terminology and notation. Let $\mathbf{K}_n$ denote the complete graph on n vertices, i.e., the n-vertex graph with all possible edges, and let $\mathbf{I}_n$ denote the independent graph on n vertices, i.e., the n-vertex graph with no edges at all.

Theorem 1.4.5 (Ramsey [34]). *Given positive integers m and n, there exists an integer $R(m,n)$ such that any graph on at least $R(m,n)$ vertices contains either $\mathbf{K}_n$ or $\mathbf{I}_n$ as an induced subgraph.*

A casual explanation of Ramsey's Theorem might be "the only way to force the number of vertices in a graph to be large is to include a big complete graph or large independent graph as an induced subgraph." We will now see how to restate Ramsey's Theorem into the language of Ramsey trails.

Example 1.4.6. Let $\mathcal{C}$ be the collection of all graphs and say $\mathbf{H} \prec \mathbf{G}$ if $\mathbf{H}$ is an induced subgraph of $\mathbf{G}$. For a graph $\mathbf{G} = (V, E)$, define $f(\mathbf{G}) := |V|$. Then it is clear that $T_1 = \{\mathbf{K}_1 \prec \mathbf{K}_2 \prec \mathbf{K}_3 \prec \cdots\}$ and $T_2 = \{\mathbf{I}_1 \prec \mathbf{I}_2 \prec \mathbf{I}_3 \prec \cdots\}$ are Ramsey trails for f. Furthermore, Theorem 1.4.5 establishes that these are the only Ramsey trails needed for f, and so the Ramsey complexity here is 2.

Not all combinatorial parameters have finite Ramsey complexity. For instance, if $\mathcal{C}$ is the class of all posets and $\prec$ is the "is a subposet of" relation, the standard examples $\mathbf{S}_n$ form a Ramsey trail for dimension. Another Ramsey trail for dimension is the class of canonical interval orders $\mathbf{I}_n$, i.e., those determined by all closed intervals with distinct integer endpoints from $[n]$, since Bogart, Rabinovitch, and Trotter showed in [2] that $\dim(\mathbf{I}_n) \to \infty$ as $n \to \infty$. Since $\mathbf{S}_n$ contains a $\mathbf{2}+\mathbf{2}$ for $n \geq 2$, these are very distinct Ramsey trails. In fact, we could go on forever in attempting to assemble a list of Ramsey trails for dimension, as the Ramsey complexity of this parameter is infinite. On the other hand, it is a theorem of Kierstead and Trotter [23] that if $\mathcal{C}$ is the class of interval orders, then the Ramsey complexity of dimension is one, and the single Ramsey trail of canonical interval orders suffices.

With this background, let us now explore the Ramsey complexity of linear discrepancy.

Theorem 1.4.7. *Linear discrepancy has Ramsey complexity* 2 *as evidenced by the Ramsey trails* $T_1 := \{\mathbf{1}+\mathbf{1} \prec \mathbf{1}+\mathbf{2} \prec \mathbf{1}+\mathbf{3} \prec \cdots\}$ *and* $T_2 := \{\mathbf{A}_1 \prec \mathbf{A}_2 \prec \mathbf{A}_3 \prec \cdots\}$, *where* $\mathbf{A}_n$ *is the antichain on* n *points.*

Proof. We have already established that $\mathrm{ld}(\mathbf{1}+\mathbf{n}) = \lceil n/2 \rceil$ and $\mathrm{ld}(\mathbf{A}_n) = n-1$, so T_1 and T_2 are certainly Ramsey trails for linear discrepancy. Now suppose that $\mathbf{P} = (X, \leq_P)$ is a poset that contains neither $\mathbf{1}+\mathbf{t}$ nor $\mathbf{A}_k$. Since $\mathbf{P}$ does not contain $\mathbf{A}_k$, $\mathrm{width}(\mathbf{P}) < k$ and by Dilworth's Theorem $\mathbf{P}$ can be partitioned into $k-1$ chains $C_1, \ldots, C_{k-1}$. Furthermore, since $\mathbf{P}$ does not contain $\mathbf{1}+\mathbf{t}$, any point $x \in C_i$ can be incomparable to at most $t-1$ points from C_j for any $j \neq i$, and thus $\Delta(\mathbf{P}) \leq (k-2)(t-1)$. But by Theorem 1.3.5, we know that $\mathrm{ld}(\mathbf{P}) \leq 2\Delta(\mathbf{P})$, and thus $\mathrm{ld}(\mathbf{P}) \leq 2(k-2)(t-1)$. Therefore, if $\mathrm{ld}(\mathbf{P}) > 2(k-2)(t-1)$, $\mathbf{P}$ must contain $\mathbf{1}+\mathbf{t}$ or $\mathbf{A}_k$, and therefore the Ramsey complexity of linear discrepancy is 2. $\square$

Although Theorem 1.4.7 will not be specifically invoked many times in the following chapters, its general idea will certainly be a recurring theme in our further investigations of linear discrepancy.

CHAPTER II

THE 3-DISCREPANCY-IRREDUCIBLE POSETS OF WIDTH 3

2.1 Introduction

For most monotonic parameters of discrete structures, it makes sense to ask whether it is possible to determine the minimal structures for which the parameter takes on the value k. (Often this is feasible to study only for small values of k.) The most sensible definition of "minimal structures" usually involves the notion that the removal of *any* point results in a structure for which the parameter's value is smaller. Rather than saying such a structure is minimal, we usually say that it is *k-critical* or *k-irreducible* for the property in question. For example, in [42], Trotter and Moore used Gallai's characterization of comparability graphs to determine the posets that are 3-irreducible with respect to dimension, i.e., those posets for which the removal of any point results in a poset of dimension 2. Independently, Kelly provided in [22] the same characterization via a very different approach. The list of posets that are 3-irreducible with respect to dimension contains 17 isolated posets and 10 infinite families. In this chapter, we will provide a similar list of posets that are 3-irreducible with respect to linear discrepancy.

We begin with the following formal definition, which was first used by Chae, Cheong, and Kim in [4].

Definition 2.1.1. We say that a poset $\mathbf{P}$ is *k-discrepancy-irreducible* (or simply *k-irreducible*) if $\mathrm{ld}(\mathbf{P}) = k$ and $\mathrm{ld}(\mathbf{P} - \{x\}) < k$ for all $x \in X$.

The shorter terminology k-irreducible has historically been applied primarily in dimension theory, but since there is little risk of confusion here, we will use k-irreducible to mean k-discrepancy-irreducible unless there is a chance of confusion.

In [37, Theorem 24], Tanenbaum, Trenk, and Fishburn show that the posets with linear discrepancy equal to 1 are precisely the semiorders of width 2. In their Corollary 25, they recast this as a forbidden subposet characterization of the posets with linear discrepancy equal to 1. Specifically, a poset $\mathbf{P}$ has $\mathrm{ld}(\mathbf{P}) = 1$ if and only if it does not contain any of $\mathbf{1} + \mathbf{3}$, $\mathbf{2} + \mathbf{2}$, and $\mathbf{1} + \mathbf{1} + \mathbf{1}$ as an induced subposet. First among the eight open questions with which Tanenbaum et al. concluded [37] was the question of characterizing the posets with linear discrepancy equal to 2. After introducing the idea of k-irreducibility, Chae, Cheong, and Kim recast the result of Tanenbaum et al. as saying that $\mathbf{1} + \mathbf{3}$, $\mathbf{2} + \mathbf{2}$, and $\mathbf{1} + \mathbf{1} + \mathbf{1}$ are the 2-irreducible posets. (It is obvious that the only 1-irreducible poset is the two-element antichain $\mathbf{1} + \mathbf{1}$.) The natural hypothesis at this point seemed to be that there would be a finite list of 3-irreducible posets, and Rautenbach conjectured in [35] that the list of forbidden subposets for linear discrepancy 2 was in fact finite.

Before proceeding, some discussion of the interchangeability of the notion of a k-irreducible poset and of a forbidden subposet of posets with linear discrepancy equal to $k - 1$ is warranted. For $k = 2$, the interchangeability is natural. If linear discrepancy behaved like dimension in that the removal of a point could decrease the linear discrepancy by at most one (and for an irreducible poset exactly one), this would not be a result worth noting. However, we have already noted that linear discrepancy is only semi-continuous in Theorem 1.4.4. When the results of this chapter were published in [19] (with Howard and Young), the result of Theorem 1.4.4 was not yet established, and so a special lemma showing that any poset with linear discrepancy greater than 2 contains a subposet with linear discrepancy equal to 3 was used to show that completing the list of 3-discrepancy-irreducible posets gave a forbidden subposet characterization of posets with linear discrepancy equal to 2. In light of Theorem 1.4.4, we have the following as an immediate corollary.

Corollary 2.1.2. *Let* $\mathbf{P} = (X, \leq_P)$ *be a poset. If* $\mathrm{ld}(\mathbf{P}) \geq k$*, then* $\mathbf{P}$ *contains an induced* k*-discrepancy-irreducible subposet.*

Since posets with linear discrepancy equal to 2 have width at most 3, in order to characterize those with linear discrepancy equal to 2 it suffices by Theorem 1.4.4 to identify the 3-discrepancy-irreducible posets of width 2 and 3. This task was first addressed in [18] by Howard, Chae, Cheong, and Kim. Contrary to Rautenbach's conjecture, they showed that there are infinitely many such posets. In particular, they showed that the 3-irreducible posets of width 2 are $\mathbf{2}+\mathbf{3}$ and an infinite family of posets, each of which has an even number of points. We denote this infinite family by $\mathcal{I}_3^2$ and describe it in the next section. This chapter completes the characterization of linear discrepancy 2 by finding the 3-irreducible posets of width 3. We show that with four exceptions, the 3-discrepancy-irreducible posets of width 3 can all be derived from the list of Howard et al. for width 2 by the removal of comparabilities meeting specific criteria. The four exceptional posets, which are easily verified to be 3-irreducible, are shown in Figure 2.1. The family of 3-discrepancy-irreducible posets of width 3, which we denote by $\mathcal{I}_3^3$, is developed in Sections 2.3 and 2.4. The

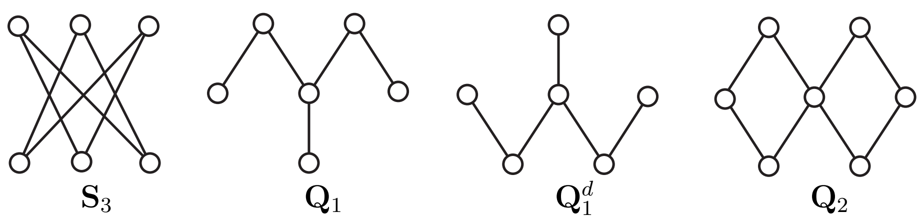

Figure 2.1: The exceptional 3-irreducible posets of width 3

theorem below provides the complete answer to the question of Tanenbaum, Trenk, and Fishburn by characterizing posets of linear discrepancy 2.

Theorem 2.1.3. *A poset has linear discrepancy equal to* 2 *if and only if it contains* $\mathbf{1}+\mathbf{3}$*,* $\mathbf{2}+\mathbf{2}$*, or* $\mathbf{1}+\mathbf{1}+\mathbf{1}$ *and it does not contain any of the following:*

(1) $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$*;*

(2) *any poset obtained from* $\mathbf{2}+\mathbf{3}$ *by the removal of a (possibly empty) subset of cover relations;*

(3) $\mathbf{S}_3$, $\mathbf{Q}_1$, $\mathbf{Q}_1^d$, *or* $\mathbf{Q}_2$; *or*

(4) *any member of the families* $\mathcal{I}_3^2$ *and* $\mathcal{I}_3^3$.

2.2 The infinite family of 3-irreducible posets of width 2

We will denote by $\mathbf{M}_{2n}$ ($n \geq 3$) a special member of $\mathcal{I}_3^2$ on $2n$ points and describe how the other members of the family on $2n$ points are obtained from $\mathbf{M}_{2n}$. Since width$(\mathbf{M}_{2n}) = 2$, we consider it as being made of two chains, which we will call L (left) and R (right), with some comparabilities added between the chains. The construction is dependent on the parity of n. For n even, L has n points and R has n points, while for n odd, L has $n+2$ points and R has $n-2$ points. Let the points of the chain L be $l_1 \lessdot l_2 \lessdot \cdots$ and the points of the chain R be $r_1 \lessdot r_2 \lessdot \cdots$. The cover relations we then add to construct $\mathbf{M}_{2n}$ are

$$l_3 \lessdot r_2 \lessdot l_5 \lessdot r_4 \lessdot l_7 \lessdot r_6 \lessdot \cdots \lessdot l_{n-3} \lessdot r_{n-4} \lessdot l_{n-1} \lessdot r_{n-2} \qquad \text{for } n \text{ even}$$

and

$$l_3 \lessdot r_2 \lessdot l_5 \lessdot r_4 \lessdot l_7 \lessdot r_6 \lessdot \cdots \lessdot l_{n-4} \lessdot r_{n-5} \lessdot l_{n-2} \lessdot r_{n-3} \lessdot l_n \qquad \text{for } n \text{ odd.}$$

The construction of $\mathbf{M}_{2n}$ is completed by adding all relations implied by transitivity after adding the cover relations above. Note that $\mathbf{M}_6$ is simply $\mathbf{1}+\mathbf{5}$ (where L is the 5-point chain). For illustration, Figure 2.2 shows the posets $\mathbf{M}_8$, $\mathbf{M}_{10}$, and $\mathbf{M}_{12}$.

We obtain the remaining $2n$-element members of $\mathcal{I}_3^2$ from $\mathbf{M}_{2n}$ by removing any subset of the cover relations added between L and R while retaining the comparabilities added because of transitivity. For example, Figure 2.3 shows the 3-irreducible poset of width 2 derived from $\mathbf{M}_8$ by removing the only possible cover relation and

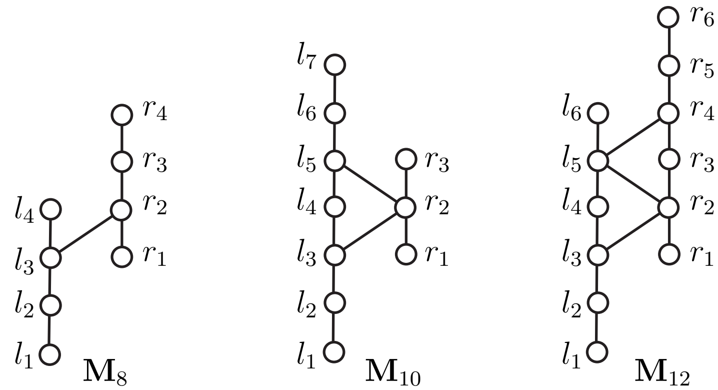

Figure 2.2: Three members of the infinite family $\mathcal{I}_3^2$

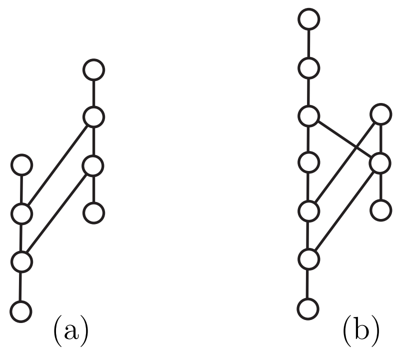

Figure 2.3: Members of the infinite family $\mathcal{I}_3^2$ derived from (a) $\mathbf{M}_8$ and (b) $\mathbf{M}_{10}$

the 3-irreducible poset of width 2 derived from $\mathbf{M}_{10}$ by removing the cover relation $l_3 \lessdot r_2$.

Howard et al. also showed that there are two canonical linear extensions of an element of $\mathcal{I}_3^2$ that witness linear discrepancy 3. Because of the symmetry present in the members of $\mathcal{I}_3^2$, the way these labellings are created is effectively the same, in one case being generated by starting at the "bottom" of the poset with label 1 and in the other starting from the "top" of the poset with label $2n$. To construct the first labelling, which we call $f\colon X \to [2n]$, let $f(l_1) = 1$, $f(l_2) = 2$, and $f(r_1) = 3$. We then proceed to alternately label the two lowest unlabelled from L and R until one chain is exhausted, at which point we complete the labelling of the remaining chain, so $f(l_3) = 4$, $f(l_4) = 5$, $f(r_2) = 6$, $f(r_3) = 7$, and so on. The second labelling g uses the same pattern, labelling the top two elements of the first chain (if n is even,

R is the first chain, and if n is odd, L is the first chain) with $2n$ and $2n-1$, then labelling the top element of the second chain with $2n-2$, and then returning to the first chain to establish the pattern of labelling two consecutive elements from each chain alternately. Figure 2.4 shows the labellings as ordered pairs $(f(x), g(x))$ for the two posets from Figure 2.3.

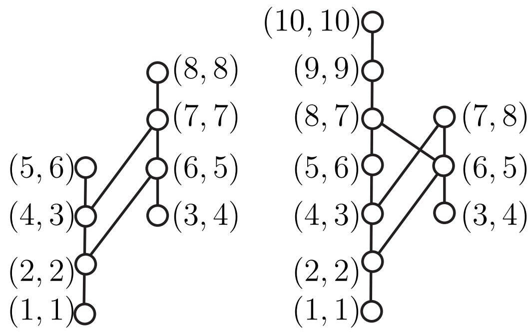

Figure 2.4: Examples of the two labellings of elements of $\mathcal{I}_3^2$

2.3 Removing comparabilities

The removal of comparabilities clearly cannot decrease the linear discrepancy of a poset. However, it is possible to remove comparabilities from a 3-irreducible poset and continue to have a 3-irreducible poset, as was seen in the previous section. In fact, there are more cover relations than just those inserted between the chains L and R to construct $\mathbf{M}_{2n}$ that can be removed while retaining 3-irreducibility. The cost we pay is an increase in width, but only to 3.

Theorem 2.3.1. *Let f and g be the two canonical labellings of $\mathbf{M}_{2n}$ that witness linear discrepancy equal to 3 as defined in Section 2.2. Let C be the set of all cover relations $u \lessdot v$ in $\mathbf{M}_{2n}$ satisfying both $f(v) - f(u) \leq 2$ and $g(v) - g(u) \leq 2$. Then the poset $\mathbf{P}$ formed by removing the comparabilities of any subset D of C is 3-discrepancy-irreducible.*

Proof. Our proof is by induction on $|D|$. If $|D| = 0$, there is nothing to prove. Suppose that for some $k \geq 0$, if $|D| = k$ then deleting the comparabilities of D creates a poset

$\mathbf{P}$ that is 3-irreducible. Now consider $|D| = k+1$. Fix u, v such that $(u, v) \in D$. Let $D' = D - \{(u, v)\}$. By induction, deleting D' gives a 3-irreducible poset $\mathbf{P}'$. We now consider the effect of removing the comparability $u \lessdot v$ from $\mathbf{P}'$, which gives the same poset $\mathbf{P}$ as removing all the comparabilities in D from $\mathbf{M}_{2n}$. The labellings f and g witness $\mathrm{ld}(\mathbf{P}) = 3$, since the only pair of points that are incomparable in $\mathbf{P}$ that were comparable in $\mathbf{P}'$ is $\{u, v\}$, but our constraints on the labels of u and v in the two labellings ensures that this does not increase the linear discrepancy. To see that $\mathbf{P}$ is irreducible, we consider the effect of deleting a point w_0. We begin by considering the poset $\mathbf{Q}$ formed by deleting w_0 from $\mathbf{M}_{2n}$. Since $\mathbf{M}_{2n}$ is an element of $\mathcal{I}_3^2$, it is 3-irreducible, and thus $\mathrm{ld}(\mathbf{Q}) = 2$. We can construct a labelling that witnesses this from the labellings f and g defined above. There are precisely four points for which $f(w_0) = g(w_0)$. Two are located at the top of $\mathbf{M}_{2n}$ and two at the bottom. If w_0 is one of the two points at the top, use f as the labelling, subtracting 1 above the deleted point, if there are any points above it. Similarly, if w_0 is one of the two points at the bottom, use g as the labelling, subtracting 1 from all values higher than that of the deleted point. Since f and g each exhibit linear discrepancy 3 for precisely one pair of incomparable points and that pair is reduced to a difference of 2 under the modified labelling, $\mathbf{M}_{2n} - \{w_0\}$ has linear discrepancy equal to 2. Now suppose that $f(w_0) \neq g(w_0)$. Then there is a point w_1 such that (as sets) $\{w_0, w_1\} = \{l_k, r_{k-2}\}$ for some $3 \leq k \leq n$. Intuitively, w_0 occurs in a position in $\mathbf{M}_{2n}$ where it has the point w_1 opposite it in the other chain. We define a new labelling f' as given below.

$$f'(x) = \begin{cases} f(x) & x = l_i, i < k, \text{ or } x = r_j, j < k-2; \\ g(x) - 1 & x = l_i, i > k \text{ or } x = r_j, j > k-2; \\ \min(f(x), g(x)) & x = w_1. \end{cases}$$

Since f only exhibits linear discrepancy 3 at the top of the poset, where we do not use it in f', and g only exhibits linear discrepancy 3 at the bottom of the poset, where we do not use it in f', we have constructed a labelling that demonstrates that $\mathrm{ld}(\mathbf{M}_{2n} - \{w_0\}) \leq 2$. By induction, this labelling demonstrates that $\mathrm{ld}(\mathbf{P}' - \{w_0\}) \leq 2$. If $u, v \neq w_0$, then removing the cover relation $u \lessdot v$ to form $\mathbf{P}$ is clearly allowed. On the other hand, if $w_0 \in \{u, v\}$, we can still remove the relation $u \lessdot v$, as $f'(w_1)$ agrees with both the labelling used from below and the labelling used from above, adjusting the assignment by subtracting 1 on one side to adjust for the deletion of w_0. Therefore $\mathbf{P}$ is 3-irreducible as desired. $\square$

Theorem 2.3.1 demonstrates that there are an infinite number of 3-irreducible posets of width 3, and we will denote this entire class as $\mathcal{I}_3^3$. In Figure 2.5, the poset on the left shows $\mathbf{M}_{10}$. Cover relations are colored green (solid or dashed) if their removal is allowed by Theorem 2.3.1 and red otherwise. The poset on the right of Figure 2.5 is derived from the one on the left by removing the dashed cover relations.

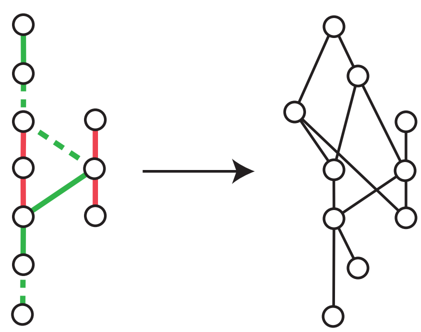

Figure 2.5: Obtaining a 3-irreducible poset of width 3 from $\mathbf{M}_{10}$

The next section will demonstrate that all elements of $\mathcal{I}_3^3$ arise via the approach of Theorem 2.3.1. It is also worthwhile to note that the cover relations described in Theorem 2.3.1 are the *only* cover relations that can be deleted while maintaining 3-irreducibility. It appears that it may be possible to delete a cover relation $u \lessdot v$ for which one of $f(v) - f(u)$ and $g(v) - g(u)$ is equal to 3. However, this is not the case, as the resulting poset is not irreducible, since it contains an induced copy of a smaller 3-irreducible poset.

2.4 Three-irreducible posets of width 3

We are now prepared to provide the complete catalog of 3-discrepancy-irreducible posets of width 3. There are four such posets on five points, all of which can be derived from $\mathbf{2}+\mathbf{3}$ by removing cover relations. Specifically, they are $\mathbf{1}+\mathbf{1}+\mathbf{3}$, $\mathbf{1}+\mathbf{2}+\mathbf{2}$, and the pair of dual posets formed from $\mathbf{2}+\mathbf{3}$ by removing the top comparability or the bottom comparability from the 3-element chain. They are shown, along with $\mathbf{2}+\mathbf{3}$, the 5-point, 3-irreducible poset of width 2, in Figure 2.6, where each poset is enclosed in a box to differentiate these disconnected posets. On six points, there are 15 disconnected, 3-irreducible posets of width 3. They are all elements of $\mathcal{I}_3^3$, as they are derived from $\mathbf{1}+\mathbf{5}$ by deleting a nonempty subset of cover relations. It is easy to verify that these are all 3-irreducible. In Figure 2.1 (in Section 2.1), we give the Hasse diagrams of the remaining three six-point 3-irreducible posets of width 3 (the standard example $\mathbf{S_3}$, $\mathbf{Q}_1$, and $\mathbf{Q}_1^d$) along with the only seven-point 3-irreducible poset of width 3, which we call $\mathbf{Q}_2$. Again, it is easy to verify that these posets are all 3-irreducible.

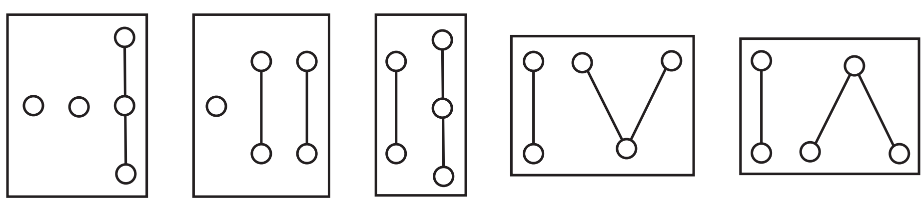

Figure 2.6: The 3-irreducible posets on five points

In the previous section, we saw that we could remove certain cover relations from $\mathbf{M}_{2n}$ to obtain more 3-irreducible posets. We also have the following theorem, which effectively says that the process is reversible and that all members of $\mathcal{I}_3^3$ arise this way.

Theorem 2.4.1. *For $n \geq 4$, every 3-discrepancy-irreducible poset $\mathbf{P}$ of width 3 with $2n$ points can be obtained from $\mathbf{M}_{2n}$ by the removal of comparabilities. Furthermore,* *there are no* 3*-discrepancy-irreducible posets on* $2n + 1$ *points.*

We will prove Theorem 2.4.1 via the following lemmas. The general idea is to make pairs of points comparable, reversing the removal described in the previous section. If this cannot be done and the poset is of width 3, the poset in question is either one of those shown in Figure 2.1 or else is not irreducible.

Remark 2.4.2. Before beginning the proof, we observe that if $|\mathbf{P}| > 6$, $\operatorname{ld}(\mathbf{P}) = 3$, and there exists $x \in \mathbf{P}$ such that $|\operatorname{Inc}(x)| > 4$, then $\mathbf{P}$ is not irreducible, as it contains one of the disconnected 3-irreducible posets.

Our lemmas focus on 3-element antichains with particular properties, first showing what can be done if such antichains exist and then focusing on posets that do not include such antichains.

Definition 2.4.3. A 3-element antichain $A = \{x, y, z\}$ is called a 3-*critical antichain* if $|\operatorname{Inc}(x)| = 4$ and (y, z) is a critical pair.

Lemma 2.4.4. *Let* $\mathbf{P}$ *be a* 3*-discrepancy-irreducible poset of width* 3 *on at least six points. If* $\mathbf{P}$ *contains a* 3*-critical antichain* $A = \{x, y, z\}$*, then the poset obtained by adding the cover relation* $y \lessdot z$ *to* $\mathbf{P}$ *is also* 3*-discrepancy-irreducible.*

Proof. By Remark 2.4.2, we may assume that $|\operatorname{Inc}(x)| = 4$. Let $\operatorname{Inc}(x) = \{y, z, v, w\}$. Since A is an antichain, we know that $y \| z$. The proof proceeds based on the various relations that v and w can have to y and z. Up to duality, there are five cases we must consider as shown in Figure 2.7. Case I is illustrated as having $v \| w$, but this is not used in our argument, so Case I also includes the case where $v \perp w$. We show that (i) the first two configurations cannot exist in a 3-irreducible poset, (ii) the third either allows the addition of $y \lessdot z$ or contains a copy of the excluded configuration of Case II, and (iii) the final two readily allow for the insertion of the relation $y \lessdot z$.

Case I. Suppose that v and w are both greater than y and z. Then the 3-irreducibility of $\mathbf{P}$ implies that the deletion of a point results in a poset of linear

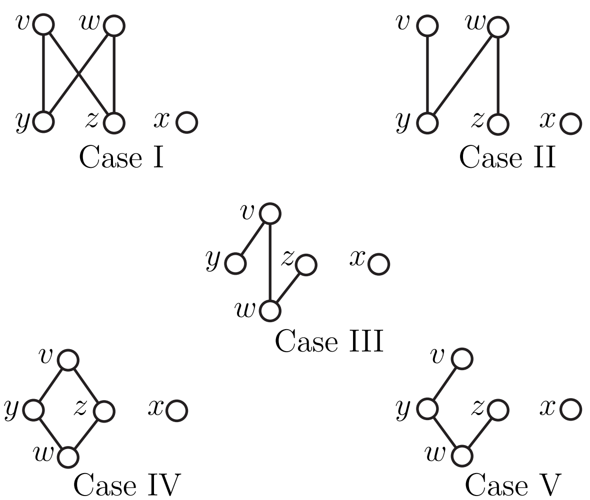

Figure 2.7: The five cases for Lemma 2.4.4

discrepancy equal to 2. We have that v and w are greater than all elements of $D[A] - \{x\}$. This is because, for example, the deletion of w forces x, y, and z to be consecutive (in some order) to maintain linear discrepancy equal to 2, and v must then be greater than all of them in any linear extension witnessing linear discrepancy 2. Thus v cannot be incomparable with anything less than x, y, or z. The argument for w is analogous. Furthermore, we have that y and z are less than all elements of $U[\{v, w, x\}] - \{x\}$. This is because any $u \in U[\{v, w, x\}] - \{x\}$ must be greater than x and therefore greater than z, since if it is incomparable with z, we have a 3-irreducible poset on 5 points induced by $\{u, x, v, w, z\}$. Since (y, z) is a critical pair, we also get $u > y$. Now the 3-irreducibility of $\mathbf{P}$ implies that the posets $\mathbf{U}$ induced by $U[\{x, v, w\}]$ and $\mathbf{D}$ induced by $D[A]$ each have linear discrepancy 2. Without loss of generality, we may assume that the linear extension of $\mathbf{D}$ that witnesses linear discrepancy 2 has x at the top followed immediately by y and z (in some order) and the one for $\mathbf{U}$ has x at the bottom with v and w (in some order) immediately above. (This is because A and $\{x, v, w\}$ are three element antichains, which must be kept consecutive in an extension witnessing linear discrepancy 2, and x is comparable to all points other than y, z, v, w.) Since v and w are both greater than both x and y, we can form a linear extension of $\mathbf{P}$ which witnesses that $\mathrm{ld}(\mathbf{P}) \leq 2$ by starting with an optimal linear extension for $\mathbf{D}$ and following it by an optimal linear extension for $\mathbf{U}$. This is a contradiction.

Case II. Suppose that $y < v$, $y < w$, $z < w$, and $z \| v$. As in the previous case, we have that v and w are greater than all elements of $D[A] - \{x\}$ and that y and z are less than all elements of $U[\{v, w, x\}] - \{x\}$. Again, the 3-irreducibility of $\mathbf{P}$ implies that the posets $\mathbf{U}$ induced by $U[\{x, v, w\}]$ and $\mathbf{D}$ induced by $D[A]$ each have linear discrepancy 2. Since $D(y) \subseteq D(z)$, we may assume without loss of generality that an optimal linear extension of $\mathbf{D}$ ends with $y < z < x$. If there is an optimal linear extension of $\mathbf{U}$ that starts $x < v < w$, then we can use the linear extension of $\mathbf{D}$ followed by the linear extension of $\mathbf{U}$ to find a linear extension L of $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}, L) = 2$. This is because the only pair of points we must check is z and v, which are two apart. This contradiction implies that every linear extension of $\mathbf{U}$ that witnesses linear discrepancy 2 must place $w < v$. Now consider an optimal linear extension L of $\mathbf{P} - \{y\}$, which has linear discrepancy 2. Since restricting L to the points of $\mathbf{U}$ would give an optimal linear extension of $\mathbf{U}$, we must have $w <_L v$. Since $\{v, w, x\}$ is an antichain, we must have that they appear consecutively in L. But since $x \| z$, we cannot have $w <_L v <_L x$, otherwise we would have linear discrepancy 3. We also must have $z <_L x$ to keep x with v and w, but then x and w are between z and v, again creating linear discrepancy 3, a contradiction to the 3-irreducibility of $\mathbf{P}$.

Case III. Suppose that $y < v$, $w < z$, and $w < v$ but $v \| z$ and $y \| w$. (Note here that $w < v$ is forced by the other four relationships, as if this is not true, we have an induced $\mathbf{1} + \mathbf{2} + \mathbf{2}$, contrary to 3-irreducibility.) Form $\mathbf{P}'$ by adding the cover relation $y \lessdot z$ to $\mathbf{P}$. If $\mathrm{ld}(\mathbf{P}') = 3$, we are done, as its irreducibility follows from that of $\mathbf{P}$. Thus, suppose that $\mathrm{ld}(\mathbf{P}') = 2$. If there is an optimal linear extension L of $\mathbf{P}'$ with $w \lessdot_L y \lessdot_L x \lessdot_L z \lessdot_L v$, then we have a contradiction, as the same linear extension witnesses that $\mathrm{ld}(\mathbf{P}) = 2$. Thus, without loss of generality (by considering the dual poset if necessary), we may assume that any optimal linear extension L of $\mathbf{P}'$ orders these five points consecutively as w, y, x, v, z or y, w, x, v, z. There must be a reason that z is forced to be last among these five, so there is another point u that is incomparable to z, and thus $\mathrm{Inc}(z) = \{x, y, v, u\}$. We know that u is greater than x, and the ordering of points in L (witnessing linear discrepancy 2 for $\mathbf{P}'$) also implies that $u > y$. In fact, L forces y to be less than everything in $U(A)$, and thus $U(x) \subseteq U(y)$. Furthermore, it is clear that $D(y) \subseteq D(x)$. Thus, (y, x) is a critical pair in $\mathbf{P}$ and $|\,\mathrm{Inc}(z)| = 4$. Since we have $y < v$, $y < u$, and $x < u$, we are in the excluded configuration of Case II with z playing the role of x, so we are done.

Case IV. Suppose that $w < y < v$ and $w < z < v$. Then adding the relation $y \lessdot z$ to $\mathbf{P}$ clearly forms a poset $\mathbf{P}'$ of linear discrepancy 3. If this were not the case, in order to have linear discrepancy 2, any optimal linear extension L of $\mathbf{P}'$ would have $w \lessdot_L y \lessdot_L x \lessdot_L z \lessdot_L v$, and thus the removal of the cover relation $y \lessdot z$ would not increase the linear discrepancy. The irreducibility of $\mathbf{P}'$ follows trivially from the irreducibility of $\mathbf{P}$.

Case V. Suppose that $w < y < v$ and $w < z$ but $v \| z$. Form $\mathbf{P}'$ by adding $y \lessdot z$ to $\mathbf{P}$. Suppose for a contradiction that $\mathrm{ld}(\mathbf{P}') = 2$. Then if there is an optimal linear extension L of $\mathbf{P}'$ with $w \lessdot_L y \lessdot_L x \lessdot_L z \lessdot_L v$, we have a contradiction, as then L demonstrates that $\mathrm{ld}(\mathbf{P}) = 2$. The only other possible ordering for these five points in an optimal linear extension of $\mathbf{P}'$ is $w \lessdot_L y \lessdot_L x \lessdot_L v \lessdot_L z$. Now there must be a point u incomparable to z in $\mathbf{P}'$ (and therefore in $\mathbf{P}$ as well) that has forced z into this position in L. Furthermore, we note that $x < u$ and $y < u$. Now the 3-irreducibility of $\mathbf{P}$ implies that y, x, z, v, u appear consecutively in that order in any optimal linear extension of $\mathbf{P} - \{w\}$. If there were a point u' other than v and u in $U(A)$, we could delete u' and again force y, x, z, v, u into order consecutively. We could then combine the linear extensions arrived at by deleting w and u' and witness that $\mathbf{P}$ has linear discrepancy 2, a contradiction. But now that $U(A) = \{u, v\}$, we cannot have drawn u up far enough to require z to appear last in the linear extension L previously discussed (i.e., we could reverse v and z in L without increasing the linear discrepancy). This contradiction finally shows that $\mathrm{ld}(\mathbf{P}') = 3$, and its irreducibility follows trivially from that of $\mathbf{P}$. □

Our next step is to resolve the situation where there are no 3-critical antichains because in any antichain $\{x, y, z\}$ with $|\operatorname{Inc}(x)| = 4$, the points y and z do not form a critical pair. In this situation, it turns out that either the poset is not 3-irreducible or else forms one of our exceptional posets. More precisely, we have the following lemma:

Lemma 2.4.5. *Let $\mathbf{P}$ be a 3-discrepancy-irreducible poset of width 3 on at least six points. Suppose that $\mathbf{P}$ does not contain a 3-critical antichain but does contain a 3-element antichain $A = \{x, y, z\}$ such that $|\operatorname{Inc}(x)| = 4$. Then $\mathbf{P}$ is $\mathbf{Q}_1$, $\mathbf{Q}_1^d$, or $\mathbf{Q}_2$.*

Proof. Considering duality, there are essentially two possibilities for how we can ensure that neither (y, z) nor (z, y) is a critical pair. The first is to have points w and u such that $w > y > u$ but $w \| z$ and $u \| z$, and the second is to have points w and u such that $w > y$, $u > z$, $w \| z$, and $u \| y$. Since we require at least one additional point in our poset, in order to ensure 3-irreducibility, we must have in the first case that $w > x$, and in the second case we must have $w > x$ and $u > x$.

Case I. Here we have two subcases, depending on if $x \| u$ or $x > u$. We first consider $x \| u$. Since $|\operatorname{Inc}(x)| = 4$, there is one more point v incomparable to x. We must have $v \perp z$, since otherwise z would have five points incomparable to it. If $v > z$, then we also have $v \gtrdot y$, as otherwise w, y, x and v, z form a 3-irreducible poset on five points. Then these points form $\mathbf{Q}_1$, so there are no more points since $\mathbf{P}$ is 3-irreducible. (Considering the dual case where $w \| x$ and $x > u$ yields $\mathbf{Q}_1^d$.) If $v < z$, we must have $v < y$, otherwise y, u, z, v, and x induce a $\mathbf{2}+\mathbf{2}+\mathbf{1}$. But then we note that in any optimal linear extension, we must have v, u, x, z, y appear consecutively in this fixed ordering, and thus either there are no more points above A or else no more points below A, as if there are additional points on both sides of A we can combine the linear extensions to demonstrate that $\mathrm{ld}(\mathbf{P}) = 2$. If there are no more points below A, consider $\mathbf{P} - \{v\}$, which must have linear discrepancy 2, and then add v back at the bottom and witness $\mathrm{ld}(\mathbf{P}) = 2$. Similarly, if there are no more points above A, consider the result of removing w.

On the other hand, suppose that $x > u$. Again, there is a point v incomparable to x, which must (by duality) be greater than y and z in order to maintain 3-irreducibility. By assumption there is another point t incomparable to x. If $t < z$, then $t < y$ as well, for otherwise $\{y, u, x, t, z\}$ induces a smaller 3-irreducible poset. In this case, we have formed the poset $\mathbf{Q}_2$, so we are done. Thus, suppose that $t > z$. Again, we must have $t > y$ because of irreducibility, but then $\{v, t, z, w, x\}$ induce (regardless of whether t and v are comparable) a 3-irreducible poset on 5 points, showing that we cannot have $t > z$ and concluding this case.

Case II. We first consider where we can insert the two remaining points incomparable to x. Notice that if one of them is above A, then it must be greater than both y and z because of irreducibility, in which case we have formed $\mathbf{S}_3$. Since there is another point required to complete $\mathbf{P}$ ($|\operatorname{Inc}(x)| = 4$ by hypothesis), $\mathbf{P}$ is not 3-irreducible. Thus, the two other points v and t incomparable to x are both less than A, and in fact must both be less than both y and z in order to avoid a $\mathbf{2}+\mathbf{3}$. Although we are not able to force a specific ordering on these seven points in an optimal linear extension, it is clear that v and t must appear in the first two positions (in some order), followed by x, followed by y and z (in some order), finally followed by w and u (u first if y comes before z and w first if z comes before w). This again allows us to say that other than the points already under consideration, there are either no more points above A or no more points below A. In the former case, delete v, find an optimal linear extension of linear discrepancy 2, and reinsert v without increasing the linear discrepancy, a contradiction. In the latter instance, delete w, find an optimal linear extension with linear discrepancy 2, and then reinsert w without increasing the linear discrepancy since it is clear that anything else below A must be comparable to x, y, and z. $\square$

Our final step is to show that with one additional exception, we are able to complete the process described in Lemma 2.4.4 to insert comparabilities, ultimately resulting in the reduction of the width of a 3-irreducible poset of width 3. We do this via the following lemma.

Lemma 2.4.6. *Let* $\mathbf{P}$ *be a poset of width* 3 *on at least* 6 *points. Assume that* $\mathrm{ld}(\mathbf{P}) = 3$*. If for all* 3*-element antichains* A *of* $\mathbf{P}$*,* $|\operatorname{Inc}(x)| \leq 3$ *for all* $x \in A$*, then* $\mathbf{P}$ *is either* $\mathbf{S}_3$ *or is not irreducible with respect to linear discrepancy.*

Proof. As with the previous two lemmas, this proof proceeds by an analysis of cases. By way of contradiction, we assume that $\mathbf{P}$ is 3-irreducible but not isomorphic to $\mathbf{S}_3$. Here, however, we focus on the configuration of the minimal elements M of $U(A)$ and their relationship to $A := \{x, y, z\}$. Since $\mathrm{width}(\mathbf{P}) = 3$, we know that $|M| \leq 3$. We may assume, by considering the dual if necessary, that $|M| > 0$ as well. Additionally, note that it is clear that if all elements of M are comparable to all elements of A, then $\mathbf{P}$ is not irreducible, as there are no incomparabilities between points of $U[M]$ and points of $D[A]$, so one of these sets must induce a smaller 3-irreducible poset.

Case I. Suppose that $M = \{v\}$. Without loss in generality, $v > y$ and $v \| x$. The arguments for $v > z$ and $v \| z$ differ only slightly, so we will just give the argument for the former. Note that anything greater than v must also be greater than x since $|\operatorname{Inc}(x)| = 3$, and anything greater than x is greater than v by the fact that $M = \{v\}$. Thus we have that v and x must be maximal in $\mathbf{P}$, as otherwise we could obtain an optimal linear extension of $U(\{v, x\})$ and an optimal linear extension of $D[\{v, x\}]$ and combine them to obtain a linear extension of $\mathbf{P}$ with linear discrepancy 2. Having established that v and x are maximal in $\mathbf{P}$, consider the effect of deleting v. We may assume that the optimal linear extension L (of linear discrepancy at most 2) does not place x below both y and z, since it is greater than all other elements of the poset, and thus we can place v at the top of L without increasing the linear discrepancy, contradicting that $\mathbf{P}$ is 3-irreducible.

Case II. Suppose that $M = \{v, w\}$. Here there are three configurations: v and w each greater than distinct two-element subsets of A, v greater than all elements of A and w greater than two elements of A, and v greater than all of A and w greater than one element of A. The arguments are all slight variations on the same theme, so we will provide the proof for the last scenario, supposing $w > x$. We first claim that in order to be 3-irreducible, we must have that either $U(\{v, w\})$ or $D(A)$ is empty. If not, deleting an element of $U(\{v, w\})$ or an element of $D(A)$ results in a linear extension L witnessing linear discrepancy 2 and thus must have consecutively x, followed by y and z (in some order), followed by w and then v. Because y and z are each already incomparable to three other points, they are comparable to all other points of the poset, and thus their ordering in L can be freely interchanged. Therefore, we may use the linear extension from deleting an element of $U(\{v, w\})$ to order $D(A)$ and the linear extension from deleting an element of $D(A)$ to order $U(\{v, w\})$ and combine them by placing $x \lessdot y \lessdot z \lessdot w \lessdot v$ in between, achieving linear discrepancy 2.

Having established that either v and w are maximal or $A = \min(\mathbf{P})$, we proceed to argue that this cannot be the case. Suppose that v and w are maximal. If we delete v, we note that $\{y, z, x\}$ and $\{y, z, w\}$ are 3-element antichains, so to witness linear discrepancy 2 we must have w as the last element of an optimal linear extension, which allows for placing v at the top without increasing the linear discrepancy, since v is comparable to all points but w. On the other hand, suppose that $A = \min(\mathbf{P})$. Consider the result of deleting x. Then $\{y, z, w\}$ is still a 3-element antichain, which must be kept consecutive in order to have linear discrepancy 2. Without loss of generality, we may assume that y and z come before w in an optimal linear extension, as they are comparable to all points of the poset with x deleted except w. Since x is less than all points except y and z, we thus may safely add x at the bottom of our linear extension without increasing linear discrepancy, which is our final contradiction.

Case III. Suppose that $M = \{u, v, w\}$. Then we note that, by hypothesis, each element of M is incomparable to at most one element of $\mathbf{P}$. Since we have also assumed that $\mathbf{P}$ is not $\mathbf{S}_3$, we thus have that at least one element of A is comparable to all elements of M and vice versa. Suppose that these elements are w and z. Without loss of generality, take $u \| x$, since we know we are missing at least one comparability. It may also be that $v > y$ or $v \| y$, so we will suppose that $v \| y$, as the following argument only becomes simpler if $v > y$. Since u and x are comparable to all other elements of $\mathbf{P}$, consider the result of deleting x. This poset has linear discrepancy 2, and we may assume that our optimal linear extension L places $u \lessdot v \lessdot w$, since u and v are comparable to all other points. Similarly, deleting u results in an optimal linear extension L' with $z \lessdot y \lessdot x$. Now use L' to order $D[A]$ and L to order $U[M]$ and we clearly have a linear extension that witnesses $\mathrm{ld}(\mathbf{P}) = 2$, a contradiction. □

Having established these three lemmas, we now have Theorem 2.4.1 as a consequence. Lemma 2.4.4 demonstrates that we are, under most circumstances, able to insert particular comparabilities in 3-irreducible posets of width 3. Repeating this process, we are able to reduce the poset to a 3-irreducible poset of width 2 unless we are in the setting of Lemmas 2.4.5 or 2.4.6, but in both of those cases, we must have an irreducible poset of at most 7 points. Thus, we are able to reduce the width of any 3-irreducible poset of width 3 on at least 8 points to a 3-irreducible poset of width 2, implying that there cannot be a 3-irreducible poset of width 3 on an odd number of points greater than seven. □

With Theorem 2.4.1 proved, we combine it with Corollary 2.1.2 and have a complete proof of Theorem 2.1.3, answering the question posed by Tanenbaum, Trenk, and Fishburn.

2.5 Conclusion and future work

Although Theorem 2.1.3 disproves Rautenbach's conjecture by showing that there are infinitely many forbidden subposets required to characterize posets with linear discrepancy equal to 2, it is still a nice result in that the set of posets which generate this collection is nicely describable, similar to Gallai's characterization of comparability graphs and the list of 3-dimension-irreducible posets. Trotter and Ross showed in [39] that for $t \geq 3$, every t-dimensional poset can be embedded in a $(t+1)$-dimensional poset. For this reason, an attempt to catalog the 4-dimension-irreducible posets is unlikely to be successful. For linear discrepancy, however, the situation appears to be better. For instance, the results of Section 2.4 rely on the width and linear discrepancy in a way that suggests they may generalize to posets of higher linear discrepancy. In particular, it would be interesting to find a more general version of Lemma 2.4.4, as the role of critical pairs in linear discrepancy is much larger than previously recognized. Unfortunately, our proofs of these results require fairly intricate case analysis that would quickly become overwhelming with increasing linear discrepancy. Furthermore, computer investigations indicate that the number of "exceptional cases" increases quickly as linear discrepancy increases, which would make the arguments even more complex. Also, it appears that even if similar results can be proved for posets of higher linear discrepancy, the infinite families involved will require more posets to generate them.

CHAPTER III

DEGREE BOUNDS FOR LINEAR AND WEAK DISCREPANCY

3.1 Introduction

In the previous chapter, we mentioned that Tanenbaum et al. gave a list of eight challenges and questions in the conclusion to their initial paper [37] on linear discrepancy. Another of those questions relates to what we call degree bounds because of the role played by the maximum degree of the cocomparability graph of a poset. By Theorem 1.3.6, we know that $\mathrm{ld}(\mathbf{t}+\mathbf{t}) = \lfloor (3t-1)/2 \rfloor$, and since $\Delta(\mathbf{t}+\mathbf{t}) = t$, this is a degree bound. Tanenbaum et al. asked if $\mathrm{ld}(\mathbf{P}) \leq \lfloor (3\Delta(\mathbf{P})-1)/2 \rfloor$ is true for all posets. As we have stated in Theorem 1.3.5, the best known upper bound in terms of $\Delta(\mathbf{P})$ is Rautenbach's [35], which gives an upper bound of $2\Delta(\mathbf{P}) - 2$. This answers the question of Tanenbaum, et al. in the affirmative for posets with $\Delta(\mathbf{P}) = 2$ or 3.

In this chapter, we will answer the question of Tanenbaum et al. in the affirmative for two more classes of posets. First, we show that an even stronger bound is true for interval orders. This result has a flavor similar to Brooks' theorem on the chromatic number of graphs, and we also show that our result is tight even for interval orders of width 2. This also addresses another of the questions posed by Tanenbaum et al., who asked if special results for linear discrepancy could be proved for interval orders. Second, we show that $\mathrm{ld}(\mathbf{P}) \leq \lfloor (3\Delta(\mathbf{P})-1)/2 \rfloor$ for disconnected posets, and this bound is tight by the previous example of $\mathbf{t}+\mathbf{t}$. We conclude the chapter with a degree bound on weak discrepancy.

3.2 *Interval orders*

The principal result of this section is the following theorem.

Theorem 3.2.1. *An interval order* $\mathbf{P}$ *has linear discrepancy at most* $\Delta(\mathbf{P})$*, with equality if and only if it contains an antichain of size* $\Delta(\mathbf{P}) + 1$*.*

After we establish the validity of the result, we will construct a family of examples showing that the result is best possible, even for interval orders of width 2.

Proof. Without loss of generality, we will assume that the interval representations we consideration will have distinct endpoints, i.e., no real number occurs more than once as the endpoint of an interval in the representation. Such an assumption can readily be made since we will only consider finite interval orders. We note that it is implicit in the work of Fomin and Golovach [13] (via a pathwidth argument), that the bandwidth of an interval graph $\mathbf{G}$ is at most $\Delta(\mathbf{G})$. However, there is a very straightforward proof of this fact, which we will express in terms of interval orders. Observe that the ordering of the points of an interval order $\mathbf{P}$ according to right endpoint yields a linear extension L with linear discrepancy at most $\Delta(\mathbf{P})$. This is because if $x \| y$ with $r(x) < r(y)$, then any element placed between x and y in L must be incomparable to y, since $l(y) < r(x) < r(z)$ for all z between x and y in L. Thus, there are at most $\Delta(\mathbf{P}) - 1$ elements placed between them and therefore $h_L(y) - h_L(x) \leq \Delta(\mathbf{P})$. If $\text{width}(\mathbf{P}) = \Delta(\mathbf{P}) + 1$, then trivially $\text{ld}(\mathbf{P}) \geq \Delta(\mathbf{P}) + 1 - 1 = \Delta(\mathbf{P})$, so we must have $\text{ld}(\mathbf{P}) = \Delta(\mathbf{P})$. The remainder of the proof shows that if this is not the case, we can strengthen the upper bound.

Let $\mathbf{P} = (X, \leq_P)$ be an interval order that does not contain an antichain of size $\Delta(\mathbf{P})+1$. By induction, we may assume that P cannot be partitioned into sets D and U such that $d < u$ for all $d \in D$ and $u \in U$, as otherwise $\text{ld}(\mathbf{P}) = \max\{\text{ld}(\mathbf{D}), \text{ld}(\mathbf{U})\}$ where $\mathbf{D}$ and $\mathbf{U}$ are the subposets induced by D and U respectively. Fix an interval representation of $\mathbf{P}$ and let m be the interval with largest left endpoint. We may assume that m also has the largest right endpoint. (Since m must be maximal, we may do this by extending the interval corresponding to m to the right.)

Now form a linear extension L of $\mathbf{P}$ by ordering the intervals by right endpoint. Let x be an arbitrary interval in $X - \{m\}$. Since $\mathbf{P}$ cannot be partitioned as $D \cup U$ with $d < u$ for all $d \in D$ and all $u \in U$, x must overlap an interval having larger right endpoint. Therefore, we must have an element of $\mathrm{Inc}(x)$ that is greater than x in L. Furthermore, the linear extension L has the property that the elements of $\mathrm{Inc}(x)$ less than x in L must precede x immediately as a consecutive block in L. This is because if $y <_P x$, then $r(y) < l(x)$, and thus there cannot be elements of $\mathrm{Inc}(x)$ below any such y in L. Combining these two facts, we see that $h_L(x) - h_L(y) \leq \Delta(\mathbf{P}) - 1$ for any $y \| x$ below x in L, since there are at most $\Delta(\mathbf{P}) - 1$ elements incomparable to x that can appear to its left (for $x \neq m$).

It only remains to address the interval m with largest left endpoint (and also largest right endpoint). We first observe that as above, the elements of $\mathrm{Inc}(m) \cup \{m\}$ are consecutive. Furthermore, m is incomparable only to maximal elements by our choice of m. Since the maximal elements of $\mathbf{P}$ are an antichain and $\mathrm{width}(\mathbf{P}) \leq \Delta(\mathbf{P})$, m is incomparable to at most $\Delta(\mathbf{P}) - 1$ points, and thus $h_L(m) - h_L(z) \leq \Delta(\mathbf{P}) - 1$ for all z incomparable to m. Therefore, $\Delta(\mathbf{P}) - 1 \geq \mathrm{ld}(\mathbf{P}, L) \geq \mathrm{ld}(\mathbf{P})$. $\square$

By the equivalence of the linear discrepancy of a poset with the bandwidth of cocomparability its graph, we may state this result in terms of the bandwidth of interval graphs, yielding the following equivalent result.

Theorem 3.2.2. *The bandwidth of an interval graph $\mathbf{G}$ is at most $\Delta(\mathbf{G})$, with equality if and only if it contains a clique of size $\Delta(\mathbf{G}) + 1$.*

Consider the poset $\mathbf{P} = \mathbf{2} + \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}$ where there are $t-1$ chains of height 1 in the poset. Then $\mathrm{ld}(\mathbf{P}) = t - 1$, $\Delta(\mathbf{P}) = t$, and $\mathrm{width}(\mathbf{P}) = t$, so it is clear that the bound of Theorem 3.2.1 is tight. However, the tightness here is really provided by the fact that $\mathrm{ld}(\mathbf{P}) \geq \mathrm{width}(\mathbf{P}) - 1$ in this case. However, we are able to show that even when the second Ramsey trail for linear discrepancy is the determining factor and the interval order contains no large antichain, the bound is still tight. To do this, we produce for each Δ an infinite family of width-two interval orders that have linear discrepancy $\Delta - 1$. We will find Lemmas 1.4.2 and 1.4.3 regarding critical pairs and linear discrepancy very useful in determining the linear discrepancy of these posets.

We will define a family of interval orders $\mathbf{F}_k^t$ for $k \geq 3$ and $t \geq 1$, and we then show that for $k > t$, we have $\mathrm{ld}(\mathbf{F}_k^t) = \Delta(\mathbf{F}_k^t) - 1$. For each $t \geq 1$ and $k \geq 3$ define the elements of the interval order $\mathbf{F}_k^t$ as follows:

- For $0 \leq i \leq t-1$ and $0 \leq j \leq k-2$, the interval $[2jt + 2i, 2jt + 2i + 1]$ is the element a_i^j.
- For $0 \leq j \leq k$, the interval $\left[2(j-1)t - \frac{1}{2}, 2jt - \frac{1}{2}\right]$ is the element b_j.

Figure 3.1 illustrates the interval representation of a general $\mathbf{F}_k^t$, while Figure 3.2 shows $\mathbf{F}_4^3$. Note that $|\operatorname{Inc}(a_i^j)| = 1$ for all i, j, $|\operatorname{Inc} b_j| = t + 2$ for $1 \leq j \leq k-1$, and $|\operatorname{Inc} b_0| = |\operatorname{Inc} b_k| = 1$, so $\Delta(\mathbf{F}_k^t) = t + 2$. We also observe that $\mathrm{width}(\mathbf{F}_k^t) = 2$.

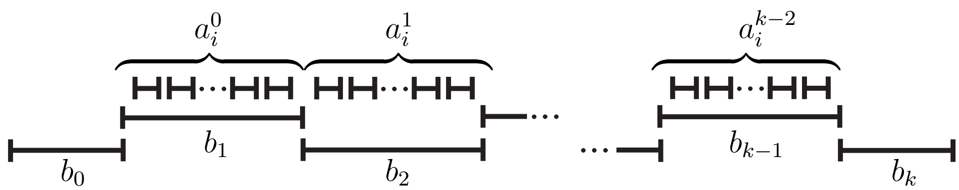

Figure 3.1: The interval order $\mathbf{F}_k^t$

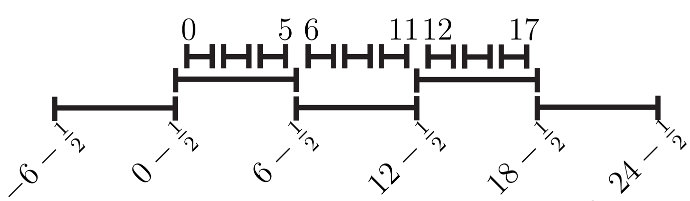

Figure 3.2: The interval order $\mathbf{F}_4^3$

Proposition 3.2.3. *The linear discrepancy of $\mathbf{F}_k^t$ is at least*

$$t + 1 - \lfloor t/k \rfloor = \Delta(\mathbf{F}_k^t) - 1 - \left\lfloor (\Delta(\mathbf{F}_k^t - 2)/k \right\rfloor .$$

Proof. We begin by identifying the critical pairs of $\mathbf{F}_k^t$. First note that $D(a_i^j) \supseteq D(b_i)$ and $U(a_i^j) \supseteq U(b_j)$, and so there are no critical pairs of the form (a_i^j, b_i) or (b_i, a_i^j). On the other hand, $D(b_i) \subseteq D(b_{i+1})$ and $U(b_i) \supseteq U(b_{i+1})$, and thus the only critical pairs in $\mathbf{F}_k^t$ are of the form (b_i, b_{i+1}). Note that the a_i^j form a chain of height $t(k-1)$. By Lemma 1.4.3, we may choose an optimal linear extension L that orders the b_i by index. Further, by Lemma 1.4.2 the distances between these pairs of points completely determine the linear discrepancy. Thus, we wish to distribute the $t(k-1)$ remaining points as equally as possible among the $k+1$ gaps between the elements $\{b_0, b_1, \ldots, b_k\}$. By the pigeonhole principle, this results in one gap containing at least $\lceil t(k-1)/k \rceil = t - \lfloor t/k \rfloor$ elements, implying $\operatorname{ld} \mathbf{F}_k^t \geq t + 1 - \lfloor t/k \rfloor$. $\square$

We note that in particular this implies that for any $k > t$, we have $\Delta(\mathbf{F}_k^t) - 1 = t + 1 \leq \operatorname{ld}(\mathbf{F}_k^t) \leq \Delta - 1$ and so Theorem 3.2.1 is tight even for interval orders of width 2 where the lower bound on linear discrepancy is 1.

3.3 Disconnected posets

In their proof of Theorem 1.3.6, Tanenbaum, Trenk, and Fishburn show that if $\mathbf{P}$ is a sum of chains, then the optimal linear extension divides the longest chain in half as evenly as possible and inserts all the remaining points into the longest chain at that point. This led to the formula for $\operatorname{ld}(\mathbf{t} + \mathbf{t})$ and their proposal that the best upper degree bound on linear discrepancy would be $\lfloor (3\Delta(\mathbf{P}) - 1)/2 \rfloor$. Using Theorem 1.4.4, we are able to show that this bound holds for a large class of posets, namely, the disconnected posets. This is unusual in that there are very few interesting mathematical results known that are specifically proved for disconnected posets, since it is trivial to turn a disconnected poset into a connected poset by adding a maximal (or minimal) element, and doing so has little effect on the order properties of the poset. In fact, our proof below could be extended to some larger class of posets via this method, but we do not see how it might be extended to all posets at this time.

Theorem 3.3.1. *Let* $\mathbf{P} = (X, \leq_P)$ *be a disconnected poset. Then*

$$\operatorname{ld}(\mathbf{P}) \leq \left\lfloor \frac{3\Delta(\mathbf{P}) - 1}{2} \right\rfloor.$$

Proof. We proceed by contradiction. Suppose $\mathbf{P}$ is a minimal counterexample in terms of $|X|$, and hence irreducible with respect to linear discrepancy. Fix $\Delta := \Delta(\mathbf{P})$. Now suppose there is some isolated point $x \in X$. Then $\operatorname{ld}(\mathbf{P}) \leq |X| - 1 = |\operatorname{Inc}(x)| = \Delta$, so $\mathbf{P}$ cannot be a counterexample. Therefore, $\mathbf{P}$ cannot have an isolated point, and thus removing a single point from $\mathbf{P}$ leaves a disconnected poset. In particular, since the removal of a single point does not increase Δ, minimality and Theorem 1.4.4 imply $\operatorname{ld}(\mathbf{P}) = \lfloor (3\Delta - 1)/2 \rfloor + 1$. Furthermore, Theorem 1.4.4 and the irreducibility of $\mathbf{P}$ guarantee the existence of a point whose removal leaves a poset $\mathbf{Q}$ with $\operatorname{ld}(\mathbf{Q}) = \operatorname{ld}(\mathbf{P}) - 1$, and so let $\mathbf{Q}$ be such a subposet. Suppose that $\Delta(\mathbf{Q}) \leq \Delta - 1$. Then by the minimality of $\mathbf{P}$, the desired degree bound holds for $\mathbf{Q}$, and therefore we have

$$\begin{aligned} \left\lfloor \frac{3\Delta - 1}{2} \right\rfloor = \operatorname{ld}(\mathbf{Q}) \leq \left\lfloor \frac{3\Delta(\mathbf{Q}) - 1}{2} \right\rfloor &\leq \left\lfloor \frac{3\Delta - 4}{2} \right\rfloor \\ &= \left\lfloor \frac{3\Delta - 2}{2} \right\rfloor - 1 < \left\lfloor \frac{3\Delta - 1}{2} \right\rfloor. \end{aligned}$$

Hence, it follows that $\Delta(\mathbf{Q}) = \Delta$.

Since we know $\mathbf{Q}$ is disconnected, let (A, B) be a partition of the ground set of $\mathbf{Q}$ witnessing this fact and suppose without loss of generality $|A| \leq |B|$. Let $\mathbf{A}$ and $\mathbf{B}$ be the subposets of $\mathbf{Q}$ induced by A and B, respectively. We observe that $\Delta(\mathbf{A}) \leq \Delta - |B|$ and $\Delta(\mathbf{B}) \leq \Delta - |A|$. Let $L_B \in \mathcal{E}(\mathbf{B})$ be optimal with respect to linear discrepancy. Form a linear extension L of $\mathbf{Q}$ by taking the first $\lceil |B|/2 \rceil$ elements of L_B, followed by all the elements of A (in any order that yields a linear extension), and finally the last $\lfloor |B|/2 \rfloor$ elements of L_B. Then $\operatorname{ld}(\mathbf{Q}) \leq \operatorname{ld}(\mathbf{Q}, L)$, and so in particular,

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq \max\{|A| + \left\lceil \frac{|B|}{2} \right\rceil - 1, |A| + \operatorname{ld}(\mathbf{B})\}.$$

Suppose first that $\lfloor (3\Delta - 1)/2 \rfloor \leq |A| + \mathrm{ld}(\mathbf{B})$. Now $\mathrm{ld}(\mathbf{B}) \leq 2\Delta(\mathbf{B}) - 2$ by Theorem 1.3.5. Therefore, we have $\lfloor (3\Delta - 1)/2 \rfloor \leq |A| + 2\Delta(\mathbf{B}) - 2$. Combining this with the observation that $\Delta(\mathbf{B}) \leq \Delta - |A|$, we obtain the bound

$$|A| \leq 2\Delta - \lfloor (3\Delta - 1)/2 \rfloor - 2.$$

Since $\mathrm{ld}(\mathbf{Q}) = \lfloor (3\Delta - 1)/2 \rfloor$, we must have $|A| + |B| \geq \lfloor (3\Delta - 1)/2 \rfloor + 1$. Therefore, $|B| \geq 2\lfloor (3\Delta - 1)/2 \rfloor + 3 - 2\Delta \geq \Delta + 1$, a contradiction, since each point of A is incomparable to every point of B implying that $|B| \leq \Delta(\mathbf{Q}) = \Delta$.

Now we suppose that $\lfloor (3\Delta - 1)/2 \rfloor \leq |A| + \lceil |B|/2 \rceil - 1$. Since $|A| \leq |B|$ and $|B| \leq \Delta - \Delta(\mathbf{A})$, we then have

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq |A| + \left\lceil \frac{|B|}{2} \right\rceil - 1 \leq \left\lceil \frac{3|B| - 2}{2} \right\rceil \leq \left\lceil \frac{3\Delta - 3\Delta(\mathbf{A}) - 2}{2} \right\rceil.$$

Therefore, we must have $\Delta(\mathbf{A}) = 0$ and $|B| \leq \Delta$. Similarly,

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq \left\lceil \frac{2|A| + |B| - 2}{2} \right\rceil \leq \left\lceil \frac{3\Delta - 2\Delta(\mathbf{B}) - 2}{2} \right\rceil = \left\lceil \frac{3\Delta - 2}{2} \right\rceil - \Delta(\mathbf{B}).$$

Hence $\Delta(\mathbf{B}) = 0$, and $\mathbf{Q}$ is the sum of two chains. Then by Theorem 1.3.6, the $\mathrm{ld}(\mathbf{Q}) = \lceil |B|/2 \rceil + |A| - 1$, and therefore $|A| = |B| = \Delta$. In this situation, we see that we cannot form $\mathbf{P}$ from $\mathbf{Q}$ by the addition of a single point, since $\Delta(\mathbf{P}) = \Delta(\mathbf{Q})$ and $\mathbf{P}$ is also disconnected. Therefore, if $\mathbf{P}$ is a disconnected poset, $\mathrm{ld}(\mathbf{P}) \leq \lfloor (3\Delta(\mathbf{P}) - 1)/2 \rfloor$. $\square$

3.4 Weak discrepancy

Although the focus of this dissertation is linear discrepancy, it makes sense at this time to take a brief diversion into the world of weak discrepancy, as we are able to establish a degree bound for weak discrepancy. First, note that by Theorem 1.3.6, $\mathrm{wd}(\mathbf{t} + \mathbf{t}) = (2t/2) - 1 = t - 1 = \Delta(\mathbf{t} + \mathbf{t}) - 1$. Since $\mathbf{t} + \mathbf{t}$ has motivated the work on the degree bound for linear discrepancy, it is natural to ask if $\mathrm{wd}(\mathbf{P}) \leq \Delta(\mathbf{P}) - 1$ for general posets. The next result answers this in the affirmative.

Theorem 3.4.1. *Let* $\mathbf{P} = (X, \leq_P)$ *be a poset that is not a linear order. Then* $\text{wd}(\mathbf{P}) \leq \Delta(\mathbf{P}) - 1$.

Proof. By the dual of Dilworth's theorem (Proposition 1.1.2), we may partition X into h antichains $A_1 \cup A_2 \cup \cdots \cup A_h$. In fact, we form the partition by letting A_1 be the minimal elements of $\mathbf{P}$, A_2 be the minimal elements of the subposet induced by $X - A_1$, etc. (In general, A_{i+1} is the minimal elements of the subposet induced by $X - (A_1 \cup \cdots \cup A_i)$.) Now define a labelling f of $\mathbf{P}$ by $f(x) = i$ if and only if $x \in A_i$. Consider a pair of points $x, y \in X$ with $x \| y$ and $|f(y) - f(x)| = \text{wd}(\mathbf{P}) = k$. Without loss of generality, say $f(x) = i$ and $f(y) = i + k$. Now by our definition of f, there is $y_{k-1} \in A_{i+k-1}$ such that $y >_P y_{k-1}$. Similarly, there is y_{k-2} in A_{i+k-2} such that $y_{k-1} >_P y_{k-2}$. Continuing this process, we find a chain $y = y_k >_P y_{k-1} >_P \cdots >_P y_0$ with $y_j \in A_{i+j}$ for $j = 0, \ldots, k$. Now notice that we must have $x \| y_j$ for $j = 0, \ldots, k$, and thus $\Delta(\mathbf{P}) \geq k + 1$, and therefore $k = \text{wd}(\mathbf{P}) \leq \Delta(\mathbf{P}) - 1$ as desired. □

We note that for $k \geq t$, the family of interval orders $\mathbf{F}_k^t$ defined earlier in this chapter has $\text{wd}(\mathbf{F}_k^t) = \Delta(\mathbf{F}_k^t)$. Therefore, unlike the case of linear discrepancy, there is not a significant difference between the best possible degree bound for interval orders and that for general posets, and in fact, the bound of Theorem 3.4.1 may be tight for interval orders as well.

3.5 Conclusion and future work

This chapter has addressed three questions concerning degree bounds, but one of the largest interesting questions in linear discrepancy remains open. Can it be shown that $\text{ld}(\mathbf{P}) \leq \lfloor (3\Delta(P) - 1)/2 \rfloor$ for connected posets? The proof of Theorem 3.3.1 and even the motivation behind the question rest on the very large number of incomparable pairs in a disconnected poset. For posets that cannot be built in a simple way from disconnected posets, $\Delta(\mathbf{P})$ may be considerably smaller, which suggests that an affirmative answer to the question will be difficult to arrive at. Recently, Choi and

West [6] have shown that the question can be answered affirmatively for posets of width 2, but there does not seem to be a reasonable path to address the question for posets of width 3. An alternative path to resolving the question could involve seeking results of the form $\mathrm{ld}(\mathbf{P}) \leq (2-\varepsilon)\Delta(\mathbf{P})$ for some $\varepsilon > 0$, and such a result would be very welcome. It should also be noted that for $\Delta(\mathbf{P}) = 4$, Rautenbach's bound gives $\mathrm{ld}(\mathbf{P}) \leq 6$, and it is unknown whether this bound can be lowered to 5. Since Rautenbach's bound for $\Delta(\mathbf{P}) = 2$ and 3 came about as a result of rounding for small values of Δ, it may prove useful for resolving the general question to seek a proof that $\mathrm{ld}(\mathbf{P}) \leq 5$ for posets with $\Delta(\mathbf{P}) = 4$.

CHAPTER IV

ONLINE LINEAR DISCREPANCY

4.1 Introduction

We have already discussed that for fixed k it is **NP**-complete to determine if the linear discrepancy of a general poset is at most k, while this decision problem can be answered in polynomial time for interval orders. In this chapter, we turn to another standard framework for algorithms—online algorithms. Intuitively, an online combinatorial structure (e.g., graph or poset) is one in which the points of the structure come equipped with an arbitrary linear order. An online algorithm is given information about the structure one point at a time, in the associated order, along with complete information about the point's relationship to all points occurring earlier in the order (and only those points). When presented with a new point, the algorithm must make an irrevocable decision about how to treat the point. This is in contrast to an offline algorithm, which is allowed to see the entire structure before making any decisions. The performance of an online algorithm is usually assessed by comparing its output to the output of an optimal offline algorithm. Online algorithms are for coloring graphs have been studied in great detail. A thorough overview can be found in Kierstead's survey article [24].

It is convenient to think of online algorithms for graphs and posets in terms of a game played by two players, whom we will call Assigner and Builder. Builder presents the graph or poset one point at a time and tells Assigner the relationship of the new point to all previously-presented points (the list of vertices to which it is adjacent in the case of a graph and its up-set and down-set in the case of a poset). The i^{th} point Builder presents will be referred to as x_i, and an online poset for which Builder has presented $\{x_1, \ldots, x_i\}$ will be denoted $\mathbf{P}_i$. The up-set and down-set of a point x in $\mathbf{P}_i$ will be denoted $U_i(x)$ and $D_i(x)$, respectively. The information that Builder presents at step i must be consistent with all the information previously presented, so in particular, if $y \| z$ in $\mathbf{P}_{i-1}$, then Builder cannot present x_i with $y < x_i < z$ in $\mathbf{P}_i$, as transitivity would then imply $y < z$ in $\mathbf{P}_i$. (Such matters are not a concern for online graphs, since a graph is defined by adjacency only.) When presented with a new point, Assigner irrevocably determines how to handle that point, e.g., to which chain it should be assigned for online chain partitioning or where it should be positioned in the linear extension of a realizer for online dimension. Assigner's goal is to make assignments so that in the end, she has come as close as possible to the optimal solution, while Builder attempts to force her to be as far from optimal as possible.

Let us consider an example of an online problem for posets. Recall that by Dilworth's Theorem, a poset of width w can be partitioned into w (and no fewer) chains. The online chain partitioning problem is for Assigner to maintain a collection of chains $\mathcal{C} = \{C_j\}$ as builder presents the points. When Builder presents x_i, Assigner irrevocably decides into which of the C_j to insert x_i by using an algorithm $\mathcal{A}$. When Builder has finished presenting the online poset $\mathbf{P}$, Assigner has used some number of chains, which we call the online width of $\mathbf{P}$ for algorithm $\mathcal{A}$ and ordering of the ground set given by $\prec$. We denote this quantity by $\text{width}_{\mathcal{A}}(\mathbf{P}^{\prec})$. The online width of $\mathbf{P}$ is the maximum of $\text{width}_{\mathcal{A}}(\mathbf{P}^{\prec})$ over all orderings $\prec$, which we denote by $\text{width}_{\mathcal{A}}(\mathbf{P})$. We evaluate how successful $\mathcal{A}$ was by comparing this number to $\text{width}(\mathbf{P})$. For a class Π of posets, the maximum of $\text{width}_{\mathcal{A}}(\mathbf{P})$ over all $\mathbf{P} \in \Pi$ is denoted by $\text{width}_{\mathcal{A}}(\Pi)$. The online width of Π, denoted $\text{width}_{\text{ol}}(\Pi)$ is then the minimum of $\text{width}_{\mathcal{A}}(\Pi)$ over all online chain partitioning algorithms $\mathcal{A}$.

Let Π_w denote the collection of all posets of width w. In [25], Kierstead gave an online chain partitioning algorithm $\mathcal{A}$ for which $\text{width}_{\mathcal{A}}(\Pi_w) \leq (5^w - 1)/4$. That is, Assigner needs no more than $(5^w - 1)/4$ chains to find a chain partition of a poset of width w online. Other than for the case $w = 2$, this is still the best known bound. (Felsner showed in [9] that $\text{width}_{\text{ol}}(\Pi_2) = 5$, while Kierstead's algorithm gives a bound of 6.) The best lower bound is that $\text{width}_{\text{ol}}(\Pi_w) \geq \binom{w+1}{2}$. (This result is more or less folklore, often attributed to Szemerédi.) For the class $\mathcal{I}$ of interval orders, much more is known about online chain partitioning, although it is usually discussed in the equivalent context of online coloring of interval graphs. Kierstead and Trotter showed in [26] that there is an online chain partitioning algorithm such that for any interval order $\mathbf{P}$, $\text{width}_{\mathcal{A}}(\mathbf{P}) \leq 3\,\text{width}(\mathbf{P}) - 2$. They also showed that for any online chain partitioning algorithm $\mathcal{A}$ and positive integer w, there is an interval order $\mathbf{P}$ of width w for which $\text{width}_{\mathcal{A}}(\mathbf{P}) \geq 3w - 2$. Despite this result, it has been of particular interest for many years now to analyze the performance of the greedy algorithm First Fit for chain partitioning interval orders online. It is known (see [3, 31]) that First Fit needs no more than $8w - 3$ chains for an interval order of width w. Furthermore, there exists an interval order of width w on which Builder can force First Fit to use at least $4.99w$ chains.

In this chapter, we consider the problem of online algorithms for linear discrepancy. For such problems, we require that Assigner maintains a linear extension L_i of $\mathbf{P}_i$ at each step and further that L_i is formed from L_{i-1} by inserting x_i between two consecutive points of L_{i-1} and leaving the order of $x_1, \ldots, x_{i-1}$ unchanged from their order in L_{i-1}. Our notation here will mirror that used for online chain partitioning above. After Builder has presented the n-point online poset $\mathbf{P}$ revealing the points in the total order given by $\prec$, Assigner has constructed a linear extension L_n using online algorithm $\mathcal{A}$, and we let $\text{ld}_{\mathcal{A}}(\mathbf{P}^{\prec})$ denote $\text{ld}(\mathbf{P}, L_n)$. The online linear discrepancy of $\mathbf{P}$ is the maximum of $\text{ld}_{\mathcal{A}}(\mathbf{P}^{\prec})$ over all orderings $\prec$, which we denote by $\text{ld}_{\mathcal{A}}(\mathbf{P})$. For a class Π of posets, the maximum of $\text{ld}_{\mathcal{A}}(\mathbf{P})$ over all $\mathbf{P} \in \Pi$ is denoted by $\text{ld}_{\mathcal{A}}(\Pi)$. The online linear discrepancy of Π, denoted $\text{ld}_{\text{ol}}(\Pi)$ is then the minimum of $\text{ld}_{\mathcal{A}}(\Pi)$ over all online linear discrepancy algorithms $\mathcal{A}$.

Since the linear discrepancy of a poset is the same as the bandwidth of its cocomparability graph, before delving into online linear discrepancy, it makes sense to discuss what is known about online bandwidth of graphs. In [1], Board addressed the problem of computing graph bandwidth online. He considerd three "protocols" and determined that unless you give Assigner a monumental amount of power, any online bandwidth algorithm outputs a permutation whose bandwidth is far from optimal. In Board's first protocol, Builder begins by informing Assigner that the graph has n vertices and its bandwidth is k. He then presents the graph one point at a time and reveals which of the already-presented points are in its neighborhood. To this point, this is simply the standard framework for problems on online graphs, although giving Assigner the number of vertices is a bit unusual, but not unheard of. Assigner must irrevocably map the new point to an element of $[n]$ not previously mapped to, striving to form a bijection from the vertex set to $[n]$ with bandwidth as small as possible. He showed that regardless of the algorithm employed by Assigner, Builder can always force her to construct a bijection of bandwidth $(k/(k+1))n - 2$ given an n-vertex graph of bandwidth k. He also gave a specific online algorithm that is guaranteed to produce a bijection with bandwidth no more than $((2k-1)n+1)/2k$, which is not much better than the worst case scenario of $n-1$.

On page 181, Board writes "[o]ne difference between our definition and the protocols used in [other problems] is that we allow the algorithm to know the number of vertices in the graph" because "any online algorithm would be severely handicapped if it did not know what the range was required to be." It seems that it did not occur to Board that after seeing the first i points, Assigner should have a bijection from the vertex set to $[i]$ and could extend the bijection when given the next point by inserting the new point between two previous points and then increasing labels as necessary. This would be a very reasonable framework in which to operate and would give Assigner a bit of flexibility but not nearly as much as his two additional protocols. These other protocols operate similarly to the first, except that in the second Builder must give Assigner a new vertex's complete adjacency list (including any unpresented vertices) and in the third Builder must do this as well as allow Assigner to choose which unpresented vertex should be presented next. Board does not present algorithms for either of the latter protocols, only lower bounds on the bandwidth of the bijection Builder can force Assigner to construct. For the second protocol, he gives a lower bound of $n(k-1)/4k - 5/4$ and for the third he proves a lower bound of $(2-\varepsilon)k$ for any $\varepsilon > 0$. It would be interesting to revisit Board's work to see if any significant improvements can be made by using the less restrictive framework described above, but that is not our focus here, since we are able to restrict ourselves to the class of cocomparability graphs and work in terms of linear discrepancy.

4.2 Online Linear Discrepancy

We begin by stating (in slightly different language) a theorem of Kloks, Kratsch, and Müller that will have an immediate implication for linear discrepancy.

Theorem 4.2.1 (Kloks et al. [28, Theorem 4.5])**.** *Let* $\mathbf{P}$ *be a poset and* $\mathbf{G}$ *its cocomparability graph. If* L *is a linear extension of* $\mathbf{P}$ *and* σ *a permutation of the vertex set of* $\mathbf{G}$ *that orders the vertices in the same way as* L*, then* $\operatorname{bw}(\mathbf{G}, \sigma) \leq 3 \operatorname{bw}(\mathbf{G})$*.*

The following proposition is an immediate corollary of Theorem 4.2.1.

Proposition 4.2.2. *Let* $\mathcal{A}$ *be any online linear discrepancy algorithm. Then for any poset* $\mathbf{P}$*,* $\operatorname{ld}_{\mathcal{A}}(\mathbf{P}) \leq 3 \operatorname{ld}(\mathbf{P})$*.*

Proof. An online linear discrepancy algorithm must output a linear extension of the poset, so we are able to apply Theorem 4.2.1, restating the conclusion in terms of linear discrepancy. □

We now show that no online linear discrepancy algorithm can do any better than this.

Theorem 4.2.3. *For every integer $n \geq 2$, there exists an interval order $\mathbf{P}_n$ with $\mathrm{ld}(\mathbf{P}_n) = n$ such that for any online linear discrepancy algorithm $\mathcal{A}$, $\mathrm{ld}_{\mathcal{A}}(\mathbf{P}_n) = 3n-1$.*

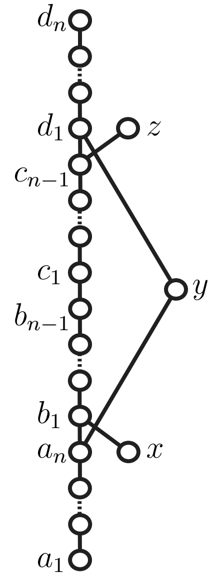

Figure 4.1: The extreme poset $\mathbf{P}_n$

Proof. The poset $\mathbf{P}_n$ is depicted in Figure 4.1. Let L be the linear extension that places z between c_{n-1} and d_1, y between b_{n-1} and c_1, and x between a_n and b_1. Then a quick inspection verifies that $\mathrm{ld}(\mathbf{P}_n, L) = n$ and furthermore L is optimal. It is also easy to see that $\mathbf{P}_n$ contains no $\mathbf{2}+\mathbf{2}$, and thus is an interval order.

The order in which Builder presents the vertices of $\mathbf{P}_n$ will depend in some cases on the algorithm $\mathcal{A}$, and this ordering will emerge through our analysis. We also note that in cases where two points of some $\mathbf{P}_i$ have the same up-sets and down-sets, Builder can determine which points of $\mathbf{P}$ they will represent after seeing how the algorithm handles them. Builder begins by presenting four points $\alpha, \beta, \gamma, \delta$ one at a time, in any order. The relationship between these points is that $\alpha < \gamma, \delta$ and $\beta < \gamma, \delta$ but $\alpha \| \beta$ and $\gamma \| \delta$. Assigner orders these points as $y_1 <_L y_2 <_L y_3 <_L y_4$ with $\{\alpha, \beta\} = \{y_1, y_2\}$ and $\{\gamma, \delta\} = \{y_3, y_4\}$. Builder subsequently decides that $y_1 = x$, $y_2 = a_1$, $y_3 = d_n$, and $y_4 = z$. Builder then presents the points $a_2, \ldots, a_n, d_1, \ldots, d_{n-1}, y$, and Assigner has no option of where to place these points.

Next Builder presents a point that is incomparable to y, above a_n and x, and below d_1 and z. Notice that all of the b_i and c_i fit this description, so if Assigner places this point above y in the linear extension, this point becomes b_1. On the other hand, if Assigner places it below y, the new point becomes c_{n-1}. The remaining c_i and b_i are then presented and there is no choice in their placement. Then the height of y in Assigner's linear extension is either $3n$ or $n+2$ while the height of x is 1 and the height of z is $4n+1$, so the linear discrepancy of the linear extension is $3n-1$ regardless of the height of y in the linear extension. $\square$

It is easy to see that even if Assigner had known in advance that Builder would never construct a $\mathbf{2}+\mathbf{2}$, she still would have been forced to form a linear extension of linear discrepancy $3n-1$. However, if Builder were forced to give Assigner intervals from an interval representation of $\mathbf{P}_n$ (the interval of x_i must be consistent with those presented for x_j for $j < i$) instead of just the up-sets and down-sets at each step, the next theorem shows that Assigner could do better. This is an interesting contrast to the online chain partitioning problem for interval orders, where the work of Kierstead and Trotter shows that being provided with an interval representation does not provide Assigner with any benefit.

Theorem 4.2.4. *Let* $\mathbf{P}$ *be an interval order that Builder presents* $\mathbf{P}$ *to Assigner by giving each point as an interval consistent with the previously-presented intervals. Then there is an online linear discrepancy algorithm* $\mathcal{A}$ *for interval orders presented in this manner such that if* $\mathbf{P}$ *is an interval order,* $\operatorname{ld}_{\mathcal{A}}(\mathbf{P}) \leq 2\operatorname{ld}(\mathbf{P})$. *Furthermore, for each* $n \geq 1$ *there is an interval order* $\mathbf{P}_n$ *such that for any online linear discrepancy algorithm* $\mathcal{A}'$, $\operatorname{ld}_{\mathcal{A}'}(\mathbf{P}_n) = 2\operatorname{ld}(\mathbf{P}_n)$ *when Builder presents* $\mathbf{P}_n$ *via its interval representation.*

Proof. Assigner's algorithm $\mathcal{A}$ simply has to order the intervals by left endpoint, since then as discussed in Chapter 3, $\operatorname{ld}(\mathbf{P}, L) \leq \Delta(\mathbf{P}) \leq 2\operatorname{ld}(\mathbf{P})$. To see that no online linear discrepancy algorithm can do better, we consider the poset $\mathbf{P}_n := \mathbf{1} + \mathbf{2n}$, which has linear discrepancy n. Builder begins by presenting the interval $x_1 = [0, 4n]$ followed by $x_2 = [2n, 2n]$. If Assigner sets $x_1 <_{L_2} x_2$, then Builder presents $[n+1, n+1]$, $[n+2, n+2]$, $\ldots$, $[4n-1, 4n-1]$. Assigner has no choice but to put these above x_2 and construct a linear extension with linear discrepancy $2n$. Similarly, if $x_2 <_{L_2} x_1$, Builder proceeds to present $[1, 1]$, $[2, 2]$, $\ldots$, $[2n-1, 2n-1]$. In both cases, the linear extension L that Assigner produces has $\mathrm{ld}(\mathbf{P}_n, L) = 2n = 2\,\mathrm{ld}(\mathbf{P}_n)$. $\square$

The recurring theme in this section is that Builder has needed to construct posets containing $\mathbf{1} + \mathbf{n}$ for n large in order to force Assigner to construct a linear extension that is as far from optimal as possible. In the next section we show that these long chains really are essential by considering the case of semiorders.

4.3 The Case of Semiorders

We begin our discussion of the online linear discrepancy of semiorders with a proposition that follows directly from Theorem 1.3.7.

Proposition 4.3.1. *Let* $\mathbf{P}$ *be semiorder and assume that Builder presents* $\mathbf{P}$ *to Assigner by giving each point as a unit-length interval consistent with the previously-presented intervals. Then Assigner can always form an optimal linear extension* L *of* $\mathbf{P}$ *online.*

Proof. Since builder is required to present the unit interval representation of $\mathbf{P}$, Assigner's algorithm simply maintains a linear extension by ordering the points by their intervals' left endpoints. By Theorem 1.3.7, this linear extension is optimal. $\square$

Recall that for interval orders, we saw that the performance of an online linear discrepancy algorithm depends on whether or not Builder is required to present an interval representation. It is natural that for semiorders there would also be a difference. In light of Proposition 4.3.1, for the remainder of this section we will focus only on the setting where Builder does *not* present Assigner an interval representation of the semiorder. We first note that for all integers $n \geq 1$, there exists a semiorder $\mathbf{P}_n$ with $\mathrm{ld}(\mathbf{P}_n) = n$ and $\mathrm{ld}_{\mathrm{ol}}(\mathbf{P}_n) \geq 2\,\mathrm{ld}(\mathbf{P}_n) = 2n$. To see this, consider the poset $\mathbf{P}$ consisting of two antichains A and B of size n with $a > b$ for all $a \in A, b \in B$ and a point x incomparable to all elements of $A \cup B$. This poset is clearly a semiorder, so since $\mathrm{width}(\mathbf{P}) = n+1$, it follows that $\mathrm{ld}(\mathbf{P}) = n$. However, if Builder first presents an antichain of size $n+1$ one point at a time and follows this by presenting an antichain of size n in which each point is incomparable to the point Assigner placed lowest in her linear extension and greater than the other n, he has forced a linear extension L of $\mathbf{P}$ having x at the bottom with the points of B above it and the points of A above them. We have $\mathrm{ld}(\mathbf{P}, L) = 2n = 2\,\mathrm{ld}(\mathbf{P})$.

For convenience, let Σ_n denote the class of all semiorders with linear discrepancy n. The primary goal of this section is to prove the following theorem.

Theorem 4.3.2. *For each positive integer n, $\mathrm{ld}_{\mathrm{ol}}(\Sigma_n) = 2n$.*

However, before giving an online linear discrepancy algorithm achieving this, we give examples to show that the simplest algorithms one might propose do not perform this well.

4.3.1 Naïve online linear discrepancy algorithms for semiorders

Perhaps the simplest online linear discrepancy algorithm that Assigner can use works as follows. When presented with a new point x, the algorithm looks at the linear extension L already constructed, identifies the element $d \in D(x)$ appearing highest in L and the element $u \in U(x)$ appearing lowest in L and then places x in position $\lfloor (h_L(d) + h_L(u))/2 \rfloor$ and increments the height in the linear extension of all points above that position. Let us refer to this algorithm as $\mathcal{M}$, since it places x in the middle of its allowable range. Unfortunately, $\mathcal{M}$ is no more effective than an arbitrary algorithm that maintains a linear extension. We see this by considering the semiorder

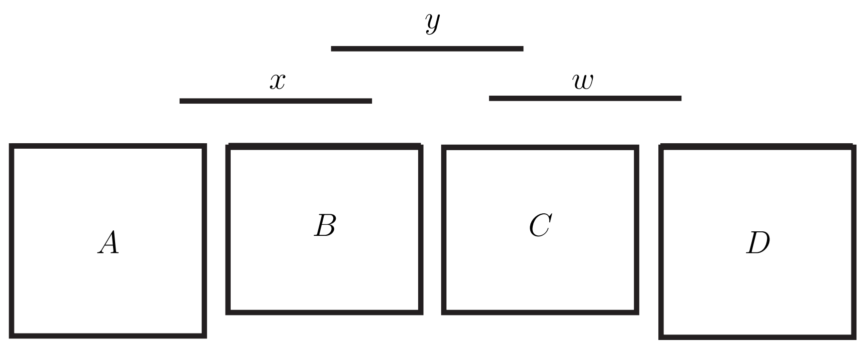

Figure 4.2: A semiorder on which simple algorithms behave badly

suggested in Figure 4.2. To be more precise, the boxes labelled A, B, C, and D are meant to represent antichains of sizes n, $n-1$, $n-1$, and n respectively, with each point in one of the antichains represented by a copy of the same interval. Then this semiorder, which we will call $\mathbf{P}$, has width $n+1$, and therefore $\mathrm{ld}(\mathbf{P}) = n$.

Now suppose that Assigner is using $\mathcal{M}$ to construct her linear extension. Builder begins by presenting, one point at a time, an antichain of size $n+1$ followed by another antichain of size $n+1$ with all points of the second antichain being greater than each point of the first antichain. After seeing the order in which Assigner places the points, Builder decides to henceforth treat the point Assigner placed at the bottom of her linear extension as x and the point she placed at the top of it as w. (The remaining points are the points of A and D.) Builder then presents y, and Assigner has no choice of where to place it, forming a linear extension in which the points are ordered as x, A, y, D, w. Next, Builder presents Assigner with the points of B, which assigner places between B and y, since $\mathcal{M}$ biases toward placing points lower when there are two positions in the "middle." Finally, Builder presents the points of C, which Assigner again places below y. But now Assigner has the linear extension x, A, B, C, y, D, w, which has linear discrepancy $3n-1$, so $\mathrm{ld}_{\mathcal{M}}(\Sigma_n) = 3n-1$.

We now consider a second simple algorithm that fails to perform as well as we desire. When presented with a new point x, the algorithm computes the linear discrepancy of each linear extension that can be formed from the current linear extension by inserting x between two points. It then inserts x into the position that minimizes the linear discrepancy of the new linear extension, breaking ties by placing x as low as possible. Since this algorithm is in some sense greedy in its operation, we denote it by $\mathcal{G}$. Algorithm $\mathcal{G}$ has some aesthetic appeal, but unfortunately, $\mathrm{ld}_{\mathcal{G}}(\Sigma_n) \geq 5n/2-1$. To see this, we again consider the poset $\mathbf{P}$ in Figure 4.2 and assume now that Assigner is using $\mathcal{G}$ to build her linear extension. Builder begins as before, presenting $A \cup D \cup \{x, w\}$ and thereafter treating the lowest point in the linear extension as x and the highest as w. Builder then presents B, and Assigner has no choice of where to place it. Next, Builder presents the point y. Assigner sees her linear extension has x, A, B, D, w and $y \| x, w$; therefore, she places y in the middle of the linear extension by splitting B as evenly as possible. Finally, Builder presents the points of the antichain C, which Assigner has no choice but to place between the highest point of B and the lowest point of D. Now the linear extension that Assigner has constructed using $\mathcal{G}$ has placed $\lceil (n-1)/2 \rceil + (n-1) + n$ points between the incomparable points y and w. Therefore, the linear extension has linear discrepancy $\lceil (n-1)/2 \rceil + 2n$ and $\mathrm{ld}_{\mathcal{G}}(\Sigma_n) \geq 5n/2 - 1$. It would be interesting to know whether this bound is tight or if another family of semiorders could increase it.

4.3.2 An optimal online linear discrepancy algorithm for semiorders

The motivation for our next algorithm is a careful examination of what happens when $\mathcal{G}$ makes the crucial bad decision in constructing the linear extension—the placement of y in the middle of the points of B. According to $\mathcal{G}$'s rule, it has made the right decision. However, even without knowing that the antichain C is yet to arrive, the interval representation of $\mathbf{P}$ in Figure 4.2 and the fact that the optimal linear extension of a semiorder orders the intervals by left endpoint strongly suggest that y should be placed above all of B, despite it producing (for the moment) a linear extension with larger linear discrepancy. Of course, the algorithm doesn't know what the interval representation of $\mathbf{P}$ looks like as Builder presents the points. On the other hand, a smart algorithm can leverage the fact that when y arrives after $A \cup B \cup D \cup \{x, w\}$, $D(b) = D(y)$ for all $b \in B$ while $U(b) \supsetneq U(y)$. The second inclusion implies that the interval representation of $\mathbf{P}$ will have y's interval ending (and hence starting) after the intervals for B, suggesting that y should be placed after the points of B in the linear extension.

With this observation in mind, we are prepared to define algorithm $\mathcal{F}$, which attempts to mimic the left endpoint ordering as closely as possible based on the information available at the time a point is presented. This algorithm will use a function $f_i(x) := |D_i(x)| - |U_i(x)|$, with i denoting the index of the last point of the poset presented. (In contexts where i is not relevant, we will suppress the subscript.) The goal of $\mathcal{F}$ is to place the points so that they are in increasing order according to f_i. However, this is not always possible, so we define it to do the best it can in the situation it is given. Specifically, when presented with a new point x_i, algorithm $\mathcal{F}$ computes $f_i(y)$ for every point y of the poset. It then defines

$$m = \max_{y\colon f_i(y) < f_i(x_i)} h_{L_{i-1}}(y) \qquad \text{and} \qquad m' = \min_{y\colon f_i(y) > f_i(x_i)} h_{L_{i-1}}(y)$$

and forms L_i by placing x_i in position $\lfloor (m + m')/2 \rfloor$ and incrementing the height in the linear extension of the points at least this high in L_{i-1}. Notice that if $m < m'$, then $\mathcal{F}$ has placed x_i in a position that agrees with the left endpoint ordering at step i. However, it is possible that $m > m'$, in which case $\mathcal{F}$ has placed x_i in the middle of the overlap of the region of points with smaller value of f_i and those having larger value of f_i.

Our goal for the remainder of this section is to prove that $\mathrm{ld}_{\mathcal{F}}(\Sigma_n) \leq 2n$, which will prove Theorem 4.3.2. We begin with the following lemma.

Lemma 4.3.3. *Let* $\mathbf{P} = (X, \leq_P)$ *be a semiorder and* $f(x) = |D(x)| - |U(x)|$ *for all* $x \in X$. *If* $|D(x)| < |D(y)|$, *then* $f(x) < f(y)$.

Proof. Since **P** is an interval order, we know by Theorem 1.2.1 that the down-sets are totally ordered by inclusion. Hence, if $|D(x)| < |D(y)|$, then $D(x) \subsetneq D(y)$ and either $x < y$ or there is $z < y$ such that $z \| x$. In the former case, we not only have $|D(x)| < |D(y)|$ but also $|U(x)| > |U(y)|$, so $f(x) < f(y)$. In the latter case, the only way we could not have $f(x) < f(y)$ is if $|U(y)| > |U(x)|$, but then since the up-sets are also totally ordered by inclusion, we would have $U(y) \supsetneq U(x)$ and therefore there would be a point z' with $z' > y$ and $z' \| x$. However then $\{z, y, z', x\}$ would induce a $\mathbf{1} + \mathbf{3}$ in **P**, contradicting that **P** is a semiorder. □

Before continuing, we note that if $f(x) \leq f(y)$, then $l(x) \leq l(y)$ since $D(x)$ can be no larger than $D(y)$.. The remainder of our analysis will focus on the posets **N** and $\mathbf{N}_3$ shown in Figure 4.3 and certain undesirable linear extensions of them when they appear as subposets. The posets in Figure 4.3 are labelled with point names that we will use consistently for convenience.

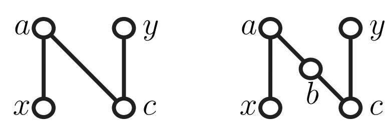

Figure 4.3: The posets **N** (left) and $\mathbf{N}_3$ (right)

Lemma 4.3.4. *Let* **P** *be a semiorder. If* $L \in \mathcal{E}(\mathbf{P})$ *does not order the points of any subposet isomorphic to* **N** *as* $x <_L c <_L a <_L y$*, then* $\mathrm{ld}(\mathbf{P}, L) \leq 2\,\mathrm{ld}(\mathbf{P})$.

Proof. Fix $x \|_P y$. Let $S := \{z \mid x <_L z <_L y\}$. If $S \subseteq \mathrm{Inc}(x)$ or $S \subseteq \mathrm{Inc}(y)$, then it follows that $\mathrm{ld}(\mathbf{P}, L) \leq \Delta(\mathbf{P}) \leq 2\,\mathrm{ld}(\mathbf{P})$. If S contains a point a comparable to x and a point c comparable to y, note that these must be different points since otherwise $y >_P x$. Furthermore, since **P** is a semiorder, we must have $a >_P c$. Thus, if these points exist, they form an **N** in the forbidden order in L, which is a contradiction. □

Lemma 4.3.5. *The online algorithm* $\mathcal{F}$ *never builds a linear extension* L_i *of a* *semiorder* $\mathbf{P}_i$ *with the points of an induced subposet isomorphic to* $\mathbf{N}_3$ *ordered as*

$$x <_{L_i} c <_{L_i} b <_{L_i} a <_{L_i} y$$

and $x_i = b$.

Proof. Assume by way of contradiction that $\mathcal{F}$ has ordered the points of a subposet isomorphic to $\mathbf{N}_3$ in this order. Since there may be more than one choice of points $\{a, c, x, y\}$ that combine with b to form a copy of $\mathbf{N}_3$, we wish to fix a particular one. We do so by choosing x and y to be as far apart as possible, and subject to that, we choose a so that it is as low in L_i as possible, and subject to that, we choose c so that it is as high in L_i as possible.

Now consider the first step j at which all the elements of $\{a, c, x, y\}$ have been presented. We first suppose that $x_j = y$. Then $f_j(y) < f_j(a)$, so there exists a point z such that $z >_{L_j} y$ and $f_j(z) < f_j(y)$. Note that this implies $f_i(z) < f_i(y)$ as well, so we may now use only f_i to complete our analysis. Since $f_i(z) < f_i(y)$ and $z >_{L_i} y$, we have $z \| y$. Similarly, we conclude that z and a are incomparable. Since $f_i(z) < f_i(y)$, $l(z) \leq l(y)$, and since $z \| y$, we have that the interval of z contains the left endpoint of the interval corresponding to y. Furthermore, the same argument allow us to conclude that the interval of x contains $l(y)$. Thus, z is incomparable to x as well. Now since $a > x$, we have $f_i(x) < f_i(z)$, and thus z is also incomparable to b. Then the fact that $\mathbf{P}$ is a semiorder implies that we must have $z > c$. Now note that $\{x, a, b, c, z\}$ forms a copy of $\mathbf{N}_3$ in $\mathbf{P}_i$ with z playing the role of y, and since $z >_{L_i} y$, we contradict our choice of y as being as far from x as possible while forming an $\mathbf{N}_3$.

Next we suppose that $x_j = a$. Then there is a point z such that $z <_{L_j} a$ but $f_j(z) > f_j(a)$ and thus $f_i(z) > f_i(a)$. It immediately follows that z is greater than b, c, and x in $\mathbf{P}_i$. Since $z <_{L_i} y$, we cannot have $z > y$ in $\mathbf{P}_i$, and if $z < y$ in $\mathbf{P}_i$, then we would have $y > x$, so we must have $z \| y$. But then $\{x, b, c, y, z\}$ forms a copy of $\mathbf{N}_3$ in $\mathbf{P}_i$, with z playing the role of a. However, for this copy of $\mathbf{N}_3$, we have z lower than a in L_i, contrary to our choice of a.

The argument for $x_j = x$ is completely dual to that for $x_j = y$, and the argument for $x_j = c$ is dual to that for $x_j = a$, and thus the lemma is proved. □

Theorem 4.3.6. *If* $\mathbf{P}$ *is a semiorder, then* $\mathrm{ld}_{\mathcal{F}}(\mathbf{P}) \leq 2\,\mathrm{ld}(\mathbf{P})$.

Proof. We begin by noting that Lemma 4.3.4 implies the theorem if $\mathcal{F}$ never constructs a linear extension L_i of $\mathbf{P}_i$ with the points of an induced subposet isomorphic to $\mathbf{N}$ ordered as $x <_{L_i} c <_{L_i} a <_{L_i} y$. If $\mathcal{F}$ ever constructed such a linear extension, then Builder could immediately present a point b to form an $\mathbf{N}_3$, and $\mathcal{F}$ would have no choice to put it between c and a in L_i. However, we know by Lemma 4.3.5 that this cannot happen, and thus $\mathcal{F}$ never orders the points of an $\mathbf{N}$ as $x <_{L_i} c <_{L_i} a <_{L_i} y$, which completes our proof. □

Now that Theorem 4.3.6 is proved, Theorem 4.3.2 follows immediately.

4.4 Conclusion and future work

This chapter's results address much of the big picture for online algorithms for linear discrepancy. Interval orders have played a central role in much of the work on linear discrepancy and have reappeared in this chapter. Of particular interest is the distinction between other work on online algorithms for interval orders (or interval graphs), where an online algorithm's performance is independent of whether or not an interval representation is provided, and online algorithms for linear discrepancy, which have a significant performance improvement when provided with an interval representation.

A number of interesting questions remain unanswered by our work in this chapter, however. In the proof of Theorem 4.2.3, we give an interval order with linear discrepancy n for which any online linear discrepancy algorithm constructs a linear extension with linear discrepancy $3n - 1$. Theorem 4.2.2 shows that no linear extension of a poset can have linear discrepancy more than three times the optimal value, and there are examples of interval orders (see for instance Choi and West [6]) for which this bound is tight. However, for these specific interval orders, Builder cannot force Assigner to construct the extreme linear extensions in an online setting. This raises the question of whether there exists an online linear discrepancy algorithm that always constructs a linear extension of linear discrepancy at most $3n - 1$ for a poset of linear discrepancy n or if there is another family of posets for which Builder can force assigner to create a linear extension with linear discrepancy $3n$. The distinction drawn in this chapter between the cases of interval orders and semiorders also suggests the question of determining the online linear discrepancy of the class of interval orders which excludes $\mathbf{1} + \mathbf{t}$ for fixed $t > 3$, as it may be intermediate between that of interval orders and semiorders.

CHAPTER V

CONCLUSION

In this dissertation, we have examined three principal types of questions on linear discrepancy for partially ordered sets. While further work along the lines of the characterization result of Chapter 2 appears at present to be very technical and of limited interest, it is possible that work on other linear discrepancy problems will lead to a deeper understanding that provides insight into that line of inquiry as well. However, Chapters 3 and 4 suggest a number of interesting questions including improving the best known general degree bound for posets and developing a fuller understanding of online algorithms for linear discrepancy. There are also a number of interesting questions regarding online algorithms for other properties of posets.

Moving beyond the work of this dissertation, one of the most intriguing directions for future work would be to explore the relationship between linear discrepancy and dimension through their dependence on critical pairs. There does not seem to be any intuitive reason for the relationship between linear discrepancy and critical pairs, and therefore it is possible that the relationship is simply a fortunate coincidence. However, if there were a natural explanation for this relationship, it would perhaps suggest a proof of the conjecture that if $\mathrm{ld}(\mathbf{P}) = \dim(\mathbf{P}) = n \geq 5$, then $\mathbf{P}$ contains the standard example $\mathbf{S}_n$ as a subposet. (See [37, 41].) Since the class of interval orders contains posets of arbitrarily large dimension but not the standard examples for $n > 1$, it would be interesting to see if this conjecture can be proven for this restricted class of posets.

Recently, Howard and Trenk have introduced in [20] what they call the t-discrepancy of a poset. For t-discrepancy, we seek an order-preserving map from the poset's ground set to the integers so that each integer is the image of at most t points in the poset. Such a function's t-discrepancy is the maximum difference between the labels of incomparable pairs as with linear and weak discrepancy, and the poset's t-discrepancy is the minimum over all such functions. The t-discrepancy is intermediate between linear and weak discrepancy in the sense that a poset's 1-discrepancy is simply its linear discrepancy and weak discrepancy results from removing the restriction of each label being used at most t times altogether. Howard and Trenk have shown that determining if a poset's t-discrepancy is at most k is **NP**-complete and that a polynomial time algorithm exists to compute the t-discrepancy of a semiorder. However, the complexity of determining the t-discrepancy of an interval order remains open. It would be interesting to resolve this question, and many of the questions for linear discrepancy have natural generalizations to t-discrepancy that may be worth exploring as well.

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VITA

Mitchel T. Keller was born in Bismarck, North Dakota, and raised in Halliday, North Dakota, where he graduated at the top of his class of nine students from Halliday High School. In May 2004, He earned a B.S. in Mathematics with a minor in Computer Science from North Dakota State University, where he also completed the University Honors Program. He has been involved with the Mathematics Genealogy Project since 2002, serving as Technical Director, Assistant Director, and Interim Co-Managing Director. At Georgia Tech, Keller served as Executive Vice President of the Graduate Student Government during the 2005–2006 academic year and Graduate Student Body President for 2006–2007. He devoted much of his efforts in student government to ensuring graduate students were provided with quality health care insurance. He also led the organizing team for the Georgia Tech High School Mathematics Competition in 2008 and 2009. His service activities on campus led to him receiving the Helen E. Grenga Friend of the Graduate Student Award, Auxiliary Services IMPACT Scholarship, and John R. Festa Fellowship. In recognition of his teaching and leadership on teaching and learning issues, he won the 2008 Outstanding Teaching Assistant Award, presented by the Center for the Enhancement of Teaching and Learning and BP America, and was one of ten recipients of the 2009 K. Patricia Cross Future Leaders Award from the Association of American Colleges and Universities.