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Elastica Solution for the Hygrothermal Buckling of a Beam

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Abstract: An elliptic integral solution for the post-buckling response of a linear-elastic and hygrothermal beam fully restrained against axial expansion is presented. Whereas in the classical solutions the extension of the beam can be neglected, a well-posed formulation of the title problem must include the extension. The solution for the limiting case of a string is presented. The present solution shows that the magnitude of the compressive axial load is a maximum at the onset of buckling and decreases as the potential for free expansion is increased; this is in contrast to the approximate solutions found in the literature.

Keywords: Hygrothermal Buckling, Elastica, Extensible Beam, String.

INTRODUCTION

An analytical solution for the post-buckling response of an elastic and hygrothermal beam subjected to an increase in swelling is presented. There are two interesting features of this problem that differentiate it from the classical solutions of the elastica. First, the extensibility of the beam cannot be ignored. Second, the compressive normal stress field in the beam is not the result of an applied compressive load but is due to an increase in either temperature or moisture content. Thus, the relationship between the axial force and the hygrothermal expansion is not unique but depends on the degree of buckling.

For the present solution, the beam is assumed to exhibit a purely elastic response and dimensional changes in the cross-section are assumed to be negligible. The axial stress is assumed to be proportional to the difference between the stretch and the hygrothermal extension of the beam. The solution is obtained in the form of two coupled integral equations of elliptic form, which can be solved numerically to provide the post-buckling response of the structure.

From a limiting process, the equilibrium state of a string is obtained from the beam solution. The value of this solution for the string is that the two unknown parameters are determined explicitly in terms of elliptic integrals of the first and second kind.

We conclude with a discussion of pertinent results and a comparison with the previous approximate results found in the literature.

BACKGROUND

One would expect that the thermal buckling of a beam has been well studied but, surprisingly, little was found in a search of the literature. The critical buckling temperature of a restrained beam can be obtained from a linear beam buckling theory. The solution given for both the

critical change in the potential for free thermal expansion, $(\alpha\Delta T)_{cr}$, and the compressive load at buckling, P_{cr} , for a simply supported beam of length $2L$ is

$$(\alpha\Delta T)_{cr} = \left(\frac{\pi\rho}{2}\right)^2 \text{ and } P_{cr} = EA\left(\frac{\pi\rho}{2}\right)^2 \quad (1)$$

where $\rho^2 = \frac{I}{AL^2}$, A is the cross sectional area, I is the moment of inertia, and E is Young's modulus.

Boley and Weiner [1], Nowinski [2] and Ziegler and Rammerstorfer [3] each derived the same first order approximation, equation (2), for the post-buckling response by including only the highest order nonlinearity in the expression for axial strain. They [1, 2, 3] obtained

$$\frac{w(0)}{L} = 2\rho\sqrt{\frac{4\alpha\Delta T}{\pi^2\rho^2} - 1} \quad (2)$$

where $w(0)$ is the lateral deflection at the mid-span of the beam.

The derivations yielding equation (2) predict that the axial force given in equation (1) remains constant after initial buckling. Zielger and Rammerstorfer [3] stated that a higher order approximation of the solution yielded a slight decrease in the magnitude of the axial force, but provide neither results nor references to substantiate this claim. Boley and Weiner [1] discuss the general methodology for solving the post-buckling response, but state that the analysis is quite cumbersome, and only give the solution provided by equations (1) and (2).

El Nashie [4] noted that the thermal buckling problem had not been addressed in the literature and presented a variational analysis to show that the post-buckling response is stable. He concluded that the initial post-buckling response is exactly the same as for the inextensible rod subjected to an end thrust and he predicted that the axial load would increase in

the post-buckling response. Unfortunately, El Nashie [4] assumed that the axial strain and angle of rotation are completely independent of each other which, as will be evident from the present analysis, is not true. Thus, his result is not valid.

Recently, Jekot [5], presented a post-buckling solution for a beam made of a nonlinear thermoelastic material. His formulation is the same as mentioned above [1-3] except that he allows for a nonlinear dependence of thermal expansion on temperature; as in the analyses discussed above, his solution is limited to the prediction of a constant axial load after buckling.

As it appears that an analytical solution for the title problem has not been given previously, a solution is presented below. The results of the present analysis determine whether the axial load increases or decreases in the post-buckling regime, and verify the accuracy of equation (2).

In order to obtain a solution of the present problem, one naturally looks to the equations of the elastica. Because the beam cannot be treated as inextensible, a formulation for the elastica including extensibility must be utilized. Both Stoker [6] and Huddleston [7] provide the differential formulation and a numerical solution for the post-buckling response of an elastic extensible beam subjected to an applied axial compressive load. Antman [8] and Stemple [9] each presented a theory for extensional beams, and derived a post-buckling solution for an extensible beam subjected to an applied compressive axial load. In these papers, the solution is determined by two coupled integral equations of elliptic form and a similar approach will be followed here.

PROBLEM FORMULATION

We seek the post-buckling response of an elastic and hygrothermal beam subjected to an increase in temperature and/or moisture content. The solution comprises the out-of-plane

deflection, the axial deflection, and the axial stress distribution as a function of the change in temperature and/or moisture content. Figure 1 illustrates the initial configuration and buckled configuration of the beam. A beam of length $2L$ and cross-sectional area $A = bh$ is pinned at both ends so that axial movement of the beam ends is prohibited. We assume that the beam is made from a linear-elastic and linear-hygrothermal material with a Young's modulus of E , a coefficient of thermal expansion α , and coefficient of hygroexpansion β .

When either the temperature or moisture content is increased, the beam will attempt to expand; however, the pinned ends completely restrain the expansion. Initially, the beam remains undeformed, and an axial compressive force develops. At some critical change in conditions, the axial force is of sufficient magnitude to cause buckling. If the change in temperature or moisture content is increased above this critical value, the undeformed beam is unstable, and a buckling deformation occurs.

In order to determine the post-buckling response of the beam, one must delineate the relationship between both the axial force and the out-of-axis deflection as a function of the change in temperature or moisture content.

As shown in Figure 1, the distance between the pins along the x axis is always $2L$. For the beam to assume a buckled state it must elongate. In classical buckling (due to an applied thrust) the beam is free to move at one end, which allows an inextensible beam to buckle. For most problems, the compressive strains are considered to be negligible and the material is treated as inextensible. In the present problem, ignoring the axial extension violates the geometric constraints, and the classical assumption of inextensibility cannot be invoked.

When the temperature or moisture changes, the magnitude of the compressive force that develops in the beam is proportional to the difference between the actual extension and the potential for free expansion. For a given change in the potential for free expansion, the resulting compressive force is a maximum when the beam has no extension and zero when the

beam is free to expand. Having established that the beam has expanded, it follows that the magnitude of the axial force will be less than the maximum possible force. The solution of the equilibrium equations, coupled with the appropriate kinematic and constitutive relations, will yield both the magnitude of the axial force and the deformation of the beam.

Several assumptions are made in the present formulation. First of all, the cross-sectional area is assumed to remain constant under any loading or changes in temperature or moisture. In addition, it is assumed that there is no transverse shear deformation and, hence, the cross-section remains plane with a normal tangent to the deformed axis of the beam. The displacements are allowed to be large as will be evident from the kinematic relations. Finally, the stress is taken to be proportional to the difference between the strain and the potential strain of free expansion.

The kinematic, equilibrium, and constitutive equations are derived below for the problem described above; details of the formulation are provided so that the impact and limitations of the assumptions will be clear to the reader.

Kinematic Equations

As shown in Figure 1., we choose a reference Cartesian coordinate system with the x axis coinciding with the neutral axis of the undeformed beam. The z axis is taken as the direction of the off-axis deformation. Therefore, all bending of the beam is prescribed to be around the y axis, and the domain of the undeformed beam is taken as

$$\{-L \leq x \leq L, -\frac{b}{2} \leq y \leq \frac{b}{2}, \frac{h}{2} \leq z \leq \frac{h}{2}\}$$

In the deformed state, the domain of the beam can be specified by a $\{s, \eta, \zeta\}$ curvilinear coordinate system where the s axis is the neutral axis of the deformed beam, and the η and ζ axes are in the plane of the cross-section of the beam. The dimensions of the beam's cross-section are assumed to remain constant (i.e. no swelling and Poisson's ratio is taken as zero.) During bending, the cross-sectional area remains plane and perpendicular to the

neutral axis of the beam. The final length of the neutral axis of the beam is unknown but is of magnitude $2(L + \Delta L)$ where ΔL remains to be determined. Therefore, the domain of the deformed beam is

$$\{-(L + \Delta L) \leq s \leq L + \Delta L, -\frac{b}{2} \leq \eta \leq \frac{b}{2}, \frac{h}{2} \leq \zeta \leq \frac{h}{2}\}$$

With the assumptions given above, the neutral axis of the deformed beam can be written in terms of the reference coordinate system as

$$\{s : -L \leq x \leq L, y = 0, z = w(x)\}$$

where $w(x)$ is the vertical deflection of the neutral axis of the deformed beam. The curvature, κ , of the neutral axis can be written as

$$\kappa = \frac{d\theta}{ds} = \frac{\frac{d^2w}{dx^2}}{[1 + (\frac{dw}{dx})^2]^{3/2}} \quad (3)$$

where θ is the angle formed by the x axis and the line tangent to the neutral axis. In the present analysis, the normal strain in the beam is defined as the stretch minus one. Figure 2 shows a line segment of original length dS and deformed length ds .

The strain along the neutral axis is defined as

$$\epsilon_0 = \frac{ds}{dS} - 1 \quad (4)$$

where $-L \leq S \leq L$. Assuming that the cross-sectional area remains plane and perpendicular to the neutral axis, the normal strain at any point can be written in terms of the normal strain and curvature of the neutral axis as

$$\epsilon = \frac{ds}{dS} - 1 - \left(\frac{ds}{dS}\right)\kappa\zeta = \epsilon_0 - (1 + \epsilon_0)\kappa\zeta. \quad (5)$$

For the case of extensible elastica, the total strain is not a linear superposition of ϵ_0 and $\kappa\zeta$. Equation (5) was previously given in References [7-9].

Given the geometry shown in Figure 2, the following additional kinematic relations must hold

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2} = \frac{1}{\cos(\theta)}, \quad \frac{dw}{dx} = \tan(\theta), \quad \text{and} \quad \frac{dS}{dx} = 1 - \frac{du}{dx} \quad (6)$$

A Lagrangian formulation of the above equations can be written by using the S coordinate system instead of the x coordinate system; in this case, the domain is the same, but the length dS is less than the length dx due to the extension of the neutral axis. In the Lagrangian coordinate system, the above kinematic equations are written as:

$$\frac{ds}{dS} = \sqrt{\left(1 + \frac{du}{dS}\right)^2 + \left(\frac{dw}{dS}\right)^2} \quad (7)$$

$$\frac{dw}{dS} = (1 + \epsilon_0) \sin(\theta) \quad (8)$$

and

$$\frac{dx}{dS} = 1 + \frac{du}{dS} = (1 + \epsilon_0) \cos(\theta) \quad (9)$$

Four other geometric conditions that must be satisfied are

$$u(-L) = u(L) = 0 \quad \text{and} \quad w(-L) = w(L) = 0. \quad (10)$$

It is convenient to write some of the above kinematic conditions in integral form. We let the angle of rotation at $x = -L$ equal θ_1 and seek a deformed state that is symmetric about $x = 0$, such that the angle at $x = L$ is $\theta = -\theta_1$. Integrating equation (4), or equivalently the simple identity $dS = \frac{dS}{d\theta} d\theta$, over the domain $-L \leq S \leq L$, yields

$$2L = \int_{-\theta_1}^{\theta_1} \left(-\frac{dS}{d\theta}\right) d\theta \quad (11)$$

Equation (9) can be integrated from $-L$ to some point S to give

$$S + L + u(S) - u(-L) = \int_{\theta}^{\theta_1} (1 + \epsilon_0) \left(-\frac{dS}{d\theta}\right) \cos(\theta) d\theta \quad (12)$$

Evaluating equation (12) at $S = L$ and noting the boundary conditions given in equation (10) yields

$$2L = \int_{-\theta_1}^{\theta_1} (1 + \epsilon_0) \left(-\frac{dS}{d\theta}\right) \cos(\theta) d\theta \quad (13)$$

Integration of equation (8) leads to

$$w(S) - w(-L) = \int_{\theta}^{\theta_1} (1 + \epsilon_0) \left(-\frac{dS}{d\theta}\right) \sin(\theta) d\theta \quad (14)$$

while evaluation of (14) at $s = L$, with the boundary conditions on $w(S)$ given in equation (10), yields

$$0 = \int_{-\theta_1}^{\theta_1} (1 + \epsilon_0) \left(-\frac{dS}{d\theta}\right) \sin(\theta) d\theta \quad (15)$$

Since we have limited our solution to symmetric deflections, equation (15) is identically satisfied, and provides no useful information for the present problem.

The integral equations given above can be evaluated once ϵ_0 and $\frac{dS}{d\theta}$ are expressed as functions of the angle θ ; these relationships are determined from considerations of equilibrium and the constitutive behavior of the beam.

Equations of Static Equilibrium

In the buckled state, the beam is assumed to be in static equilibrium. Equilibrium equations are derived in differential form by satisfying equilibrium on a deformed segment ds . Figure 3 shows the forces acting on the segment ds . The force components are taken as N and V along the x and z axis, respectively. The bending moment, M , acts about the y axis.

The equilibrium equations are

$$\begin{aligned}\frac{dN}{dx} &= 0, \frac{dV}{dx} = 0 \\ \frac{dM}{dx} + N \tan(\theta) + V &= 0\end{aligned}\tag{16}$$

Therefore, N and V are constant. At the pinned ends of the beam, the bending moments must vanish; thus,

$$M(L) = M(-L) = 0\tag{17}$$

Using equation (17) in the global equilibrium balance yields $V = 0$. Therefore, the equilibrium equations (16) reduce to

$$N = \text{constant} \quad \text{and} \quad \frac{dM}{dx} + N \tan(\theta) = 0.\tag{18}$$

Equation (18) can be written in terms of the coordinate s as

$$\frac{dM}{ds} + N \sin(\theta) = 0\tag{19}$$

or in terms of the coordinate S as

$$\frac{dM}{dS} + N(1 + \epsilon_0) \sin(\theta) = 0.\tag{20}$$

Constitutive Equations

The beam is assumed to be made from a linear elastic and hygrothermal material. Therefore, the stress is taken to be proportional to the difference between the actual strain and the potential strain of free expansion. The stress-strain relation is

$$\sigma = E(\epsilon - \alpha \Delta T - \beta \Delta H)\tag{21}$$

where α and β are the coefficients of thermal expansion and hygroexpansion, respectively. The terms ΔT and ΔH are the change in temperature and moisture, respectively. The

potential strain for free hygrothermal expansion is defined as

$$\epsilon_{ht} = \alpha \Delta T + \beta \Delta H \quad (22)$$

Substituting equations (5) and (22) into equation (21) yields

$$\sigma = E[\epsilon_0 - (1 + \epsilon_0)\kappa\zeta - \epsilon_{ht}] \quad (23)$$

The stress given by (23) is normal to the neutral axis of the deformed beam and may be integrated over the cross-sectional area to determine the resultant forces acting on the cross-section. Integrating equation (23) over the cross-sectional area A in the deformed state, with the assumption that the cross-sectional area does not deform, yields

$$N \cos(\theta) + V \sin(\theta) = EA(\epsilon_0 - \epsilon_{ht}) \quad (24)$$

while integrating the product $\sigma\zeta$ over A yields

$$M(x) = -EI(1 + \epsilon_0)\kappa. \quad (25)$$

We note that, for the present problem, $V = 0$. Using equations (3) and (6), the moment expression given in equation (25) can be written as

$$M(x) = -EI \frac{d\theta}{dS}. \quad (26)$$

SOLUTION OF THE BUCKLING PROBLEM

The most convenient method of solution for the system of integral equations delineated in the last section consists of first determining N and ϵ_{ht} (as a function of θ_1) from the integral equations (11) and (13) after which the displacements can be found using equations (12) and (13).

In order to accomplish this task, the strain of the neutral axis and the curvature must be determined as functions of the angle of rotation. The strain, determined directly from equation (24), is

$$\epsilon_0 = \left(\frac{N}{EA}\right) \cos(\theta) + \epsilon_{ht} \quad (27)$$

Substituting equations (26) and (27) into the equilibrium equation (20) yields

$$\frac{d^2\theta}{dS^2} = \left(\frac{N}{EI}\right) \left[\frac{N}{EA} \cos(\theta) + 1 + \epsilon_{ht}\right] \sin(\theta) = 0. \quad (28)$$

We now multiply (28) by $2\frac{d\theta}{dS}$ and integrate once to obtain

$$\left(\frac{d\theta}{dS}\right)^2 = -2\left(\frac{N}{EI}\right) \left[\frac{1}{2}\left(\frac{N}{EA}\right) \cos^2(\theta) + (1 + \epsilon_{ht}) \cos(\theta)\right] + C \quad (29)$$

where C is a constant of integration which is to be evaluated using the boundary conditions (17); these conditions, when combined with equation (26), reduce to

$$\frac{d\theta}{dS}(\pm L) = 0 \quad (30)$$

It then follows that

$$C = 2\left(\frac{N}{EI}\right) \left[\frac{1}{2}\left(\frac{N}{EA}\right) \cos^2(\theta_0) + (1 + \epsilon_{ht}) \cos(\theta_0)\right] \quad (31)$$

For convenience, the following change of variable is now made: we define

$$K = \sin\left(\frac{\theta_1}{2}\right) \text{ and } K \sin(\phi) = \sin\left(\frac{\theta}{2}\right) \quad (32)$$

Then,

$$\cos(\theta) = 1 - 2K^2 \sin^2(\phi), \quad \sin(\theta) = 2K \sin(\phi) \sqrt{1 - K^2 \sin^2(\phi)} \quad (33)$$

and

$$\frac{d\theta}{d\phi} = \frac{2K \cos(\phi)}{\sqrt{1 - K^2 \sin^2(\phi)}} \quad (34)$$

In addition, the following dimensionless parameters are introduced.

$$\rho^2 = \frac{I}{(AL)^2} \text{ and } \lambda^2 = \frac{-NL^2}{EI} \quad (35)$$

Equation (29) can now be re-written and simplified to

$$\frac{d\phi}{dS} = -\left(\frac{\lambda}{L}\right) \sqrt{1 - K \sin^2(\phi)} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2(1 + \sin^2(\phi))]} \quad (36)$$

The negative sign in (36) corresponds to the expectation that the angle of rotation decreases as x increases. For a symmetric buckling mode, $\phi(0) = 0$, and $\phi(-L) = \frac{\pi}{2}$. Equations (27), (33), (34) and (36) are now substituted into the integral equations (11) and (13) and we obtain

$$1 = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\lambda \sqrt{1 - K^2 \sin^2(\phi)} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2(1 + \sin^2(\phi))]} \quad (37)$$

$$1 = \int_0^{\frac{\pi}{2}} \frac{[1 - 2K^2 \sin^2(\phi)][1 + \epsilon_{ht} - \lambda^2 \rho^2 (1 - 2K^2 \sin^2(\phi))]d\phi}{\lambda \sqrt{1 - K^2 \sin^2(\phi)} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2(1 + \sin^2(\phi))]} \quad (38)$$

The expressions on the right-hand sides of the relations (32), (38) are both elliptic integrals; these relations provide two equations which can be solved simultaneously for λ and ϵ_{ht} for a prescribed value of K . The deflections may be determined by introducing equation (36) into equations (12) and (14) so as to yield

$$\frac{w(S)}{L} = \int_{\Phi}^{\frac{\pi}{2}} \frac{2K \sin(\phi)[1 + \epsilon_{ht} - \lambda^2 \rho^2 (1 - 2K^2 \sin^2(\phi))]d\phi}{\lambda \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2(1 + \sin^2(\phi))]} \quad (39)$$

and

$$\frac{u(S)}{L} = \int_{\Phi}^{\frac{\pi}{2}} \frac{[1 - 2K^2 \sin^2(\phi)][1 + \epsilon_{ht} - \lambda^2 \rho^2 (1 - 2K^2 \sin^2(\phi))]d\phi}{\lambda \sqrt{1 - K^2 \sin^2(\phi)} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2(1 + \sin^2(\phi))]} - \left(1 + \frac{S}{L}\right) \quad (40)$$

where Φ is the angle of rotation at position S , which is determined by integrating equation (36) so as to obtain

$$\frac{S}{L} = - \int_0^\Phi \frac{d\phi}{\lambda \sqrt{1 - K^2 \sin^2(\phi)} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 (1 - K^2 (1 + \sin^2(\phi)))}} \quad (41)$$

The integral in equation (39) can be written explicitly in the form

$$\frac{w(S)}{L} = \frac{2K}{\lambda} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 [1 - K^2 (1 + \sin^2(\Phi))] \cos(\Phi)} \quad (42)$$

Thus, the mid-span deflection is

$$\frac{w(0)}{L} = \frac{2K}{\lambda} \sqrt{1 + \epsilon_{ht} - \lambda^2 \rho^2 (1 - K^2)} \quad (43)$$

The complete post-buckling solution has been found and is given by equations (37), (38), (40), (41), and (42). The onset of buckling is determined by the limit as K goes to zero in equations (41) and (42). This limiting process does, indeed, yield equation (1).

THE SOLUTION FOR THE LIMITING CASE OF A STRING

The numerical solution of equations (37) and (38) involves solving simultaneously the two coupled nonlinear equations. Although this can be accomplished on a computer, it would be beneficial to obtain explicit expressions for λ and ϵ_{ht} in terms of K , at least for some special case. Fortunately, such a case exists. If we assume that the beam is very slender, then $\rho \ll 1$. When $\rho = 0$, the beam is infinitely slender, implying that either the length is infinite or the cross sectional area is zero. Thus, the beam acts like a string. Setting $\rho = 0$ in equations (37) and (38) gives

$$\lambda \sqrt{1 + \epsilon_{ht}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - K^2 \sin^2(\phi)}} = F(K, \frac{\pi}{2}) \quad (44)$$

$$\frac{\lambda}{\sqrt{1 + \epsilon_{ht}}} = \int_0^{\frac{\pi}{2}} \frac{[1 - 2K^2 \sin^2(\phi)]d\phi}{\sqrt{1 - K^2 \sin^2(\phi)}} = 2E(K, \frac{\pi}{2}) - F(K, \frac{\pi}{2}) \quad (45)$$

where $E(K, \frac{\pi}{2})$ and $F(K, \frac{\pi}{2})$ are the elliptic integrals of the first and second kind, respectively. Equations (44) and (45) can be re-written as

$$\lambda = \sqrt{F(K, \frac{\pi}{2})[2E(K, \frac{\pi}{2}) - F(K, \frac{\pi}{2})]} \quad (46)$$

$$\epsilon_{ht} = \frac{2[F(K, \frac{\pi}{2}) - E(K, \frac{\pi}{2})]}{2E(K, \frac{\pi}{2}) - F(K, \frac{\pi}{2})} \quad (47)$$

Thus, for a given value of K , the values of λ and ϵ_{ht} can be determined explicitly from equations (46) and (47). The value of λ loses physical meaning in this limit, but this value can be used as an approximation of the axial force in slender beams where $\rho \ll 1$. Finally by substituting equations (46) and (47) into (43) we obtain

$$w(0) = \frac{2K}{2E(K, \frac{\pi}{2}) - F(K, \frac{\pi}{2})} \quad (48)$$

RESULTS

Using equations (37), (38) and (43) for the beam, and (46), (47) and (48) for the string, the respective post-buckling responses can be evaluated. To obtain the results presented below, we used MATHCAD on a PC. The solution was obtained by choosing a value for K and, numerically, finding the values of λ and ϵ_{ht} that satisfy equations (37) and (38) for the beam or equations (46) and (47) for the string. The equations (43) for the beam and (48) for the string were used to find the mid-span deflection.

Figure 4 shows the change in the potential for free expansion versus the mid-span deflection for various values of the slenderness ratio, ρ . As ρ increases, the critical potential for free

expansion increases in proportion to the square of the slenderness ratio. In the post-buckling regime, the curves asymptote to the curve for $\rho = 0$.

Figure 5 shows the compressive load parameter versus the mid-span deflection. From the graph, it is clear that the critical load is indeed a maximum and, after buckling, the magnitude of the load decreases; this finding is in disagreement with the results found by El Nashie [4] and brings into question the validity of his derivation. The decrease in load makes physical sense from the standpoint that buckling relieves a percentage of the load that existed prior to buckling. The present results show that this release of hygrothermal load is larger than the additional load created by the increase in temperature or moisture required for buckling to proceed.

The decrease in load shown in Figure 5 is quite small. For example, at the limit of practical applicability of our solution say, $\frac{w(0)}{2L} = \frac{1}{7}$, and $\rho = 0.1$, the load has dropped by only 1.2%.

In terms of the axial load, a result similar to the current result is obtained by calculating the critical buckling load of a beam that has initially been allowed to freely expand to a length $2L(1 + \epsilon_{ht})$ and is then buckled by a compressive end thrust. For this case, the critical buckling load is given by $\lambda_{cr} = \frac{\pi}{[2(1 + \epsilon_{ht})]}$. This critical buckling load also decreases as ϵ_{ht} increases.

It is of interest to compare the present solution to the approximate solution given in Equation (2). Figure 6 shows a comparison of the solutions for both $\rho = 0.0$ and $\rho = 0.1$. We found that equation (2) underpredicts the mid-span deflection for a given change in ϵ_{ht} but, nonetheless, provides a good prediction of the post-buckling response. In fact, at $\frac{w(0)}{2L} = \frac{1}{7}$ and $r = 0.1$, the elastica solution gives $\epsilon_{ht} = 0.07237$ while equation (2) gives $\epsilon_{ht} = 0.07503$, a 3.7% difference.

CONCLUSIONS

By obtaining the elastica solution for the case of hygrothermal buckling of a simply supported beam, it was shown the previous approximate solutions [1, 2, 3] yield a good prediction of the relationship between the mid-span deflection and the potential strain of hygrothermal expansion, but that the axial load in the post-buckling regime is not constant as predicted by the approximate solution. The present analysis revealed that the magnitude of the axial load decreases in the post-buckling regime. The dimensionless buckling parameter, λ , is fairly insensitive to the slenderness ratio of the beam, and the prediction for this value which was obtained for the limiting string case should provide a good estimate for slender beams.

The present solution pertains to a very idealized case. It is well known that the modulus of a hygrothermal material is dependent on the temperature and moisture content. For most materials, the modulus drops with increases in temperature and moisture content. Inclusion of this effect would produce larger drops in axial load after buckling as compared to the case of constant initial modulus. Accounting for the effect of the lateral swelling would increase both the cross-sectional area and the moment of inertia of the sample and would increase the critical buckling load.

It remains to show that the symmetric buckling mode given in this paper is in fact a stable equilibrium state. This is beyond the scope of the present analysis which was meant to only provide an elastica solution to the problem. The previous paper by El Nashie [4] gives a correct energy formulation, but the perturbation solution leads to the incorrect conclusion that the axial load increases in the post-buckling regime. El Nashie assumed that the axial strain and angle of rotation are independent variables, but, equation (27) shows that, in fact, they are related. El Nashie does show that the energy state which has been determined will be a local minimum.

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Figure Captions:

Fig. 1. Buckling of a simply supported beam subjected to hygrothermal loads.

Fig. 2. Geometry for deformation of the element dS to the element ds .

Fig. 3. Resultant forces acting on an element ds .

Fig. 4. Change in the potential for free expansion as a function of the midspan deflection.

Fig. 5. Axial load as a function of the mid-span deflection.

Fig. 6. Comparison of buckling predictions.

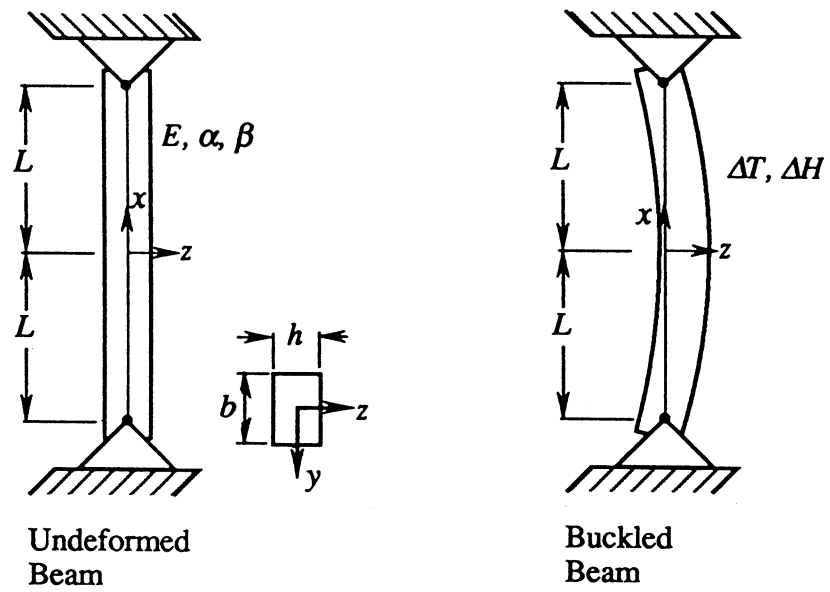


Fig. 1. Buckling of a simply supported beam subjected to hygrothermal loads.

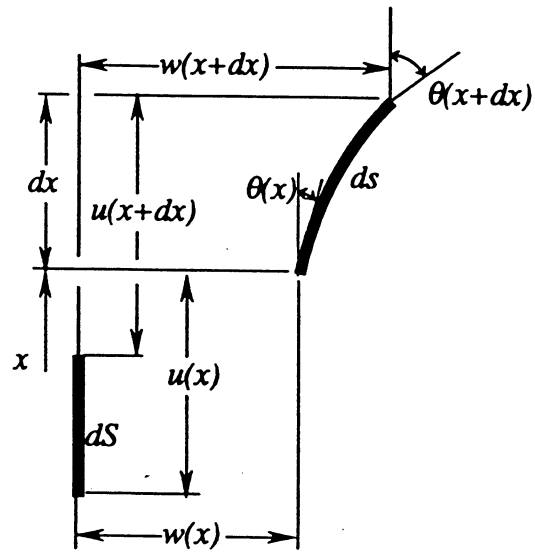


Fig. 2. Geometry for deformation of the element dS to the element ds .

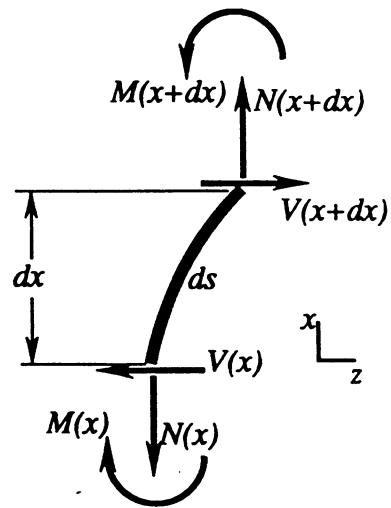


Fig. 3. Resultant forces acting on an element ds .

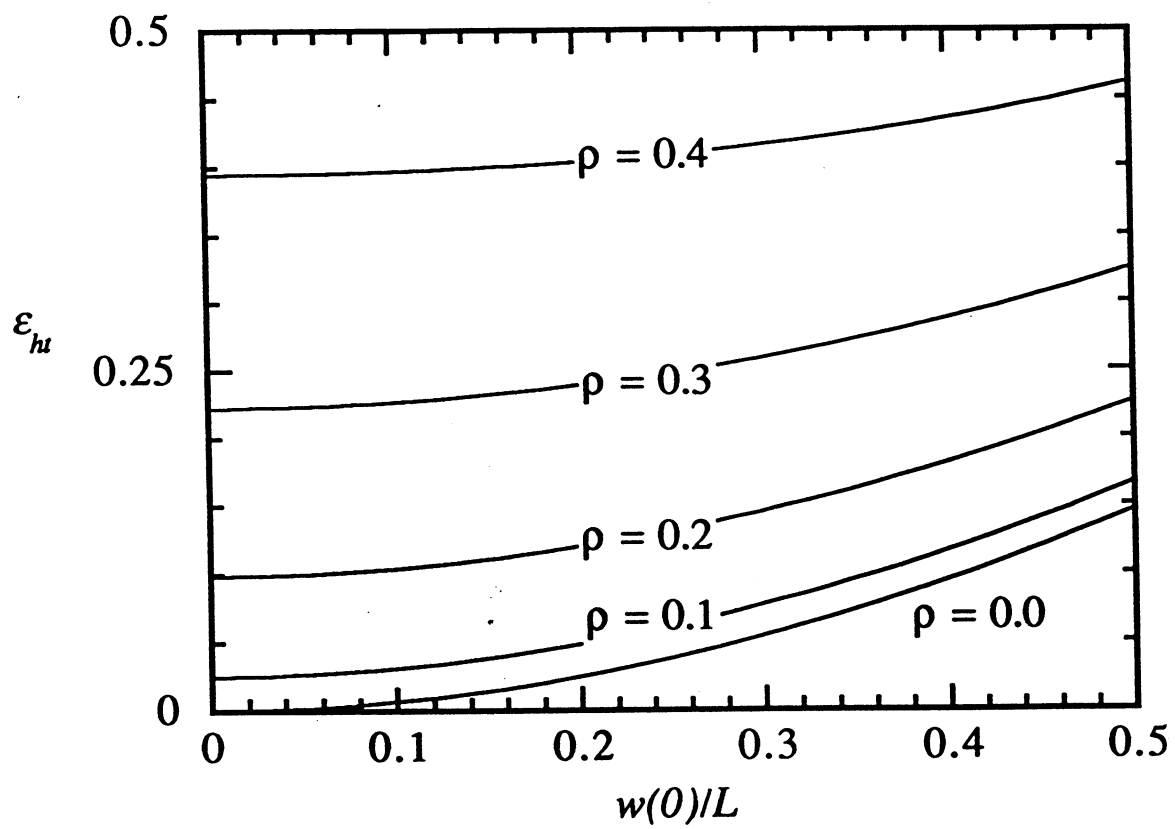


Fig. 4. Change in the potential for free expansion as a function of the midspan deflection.

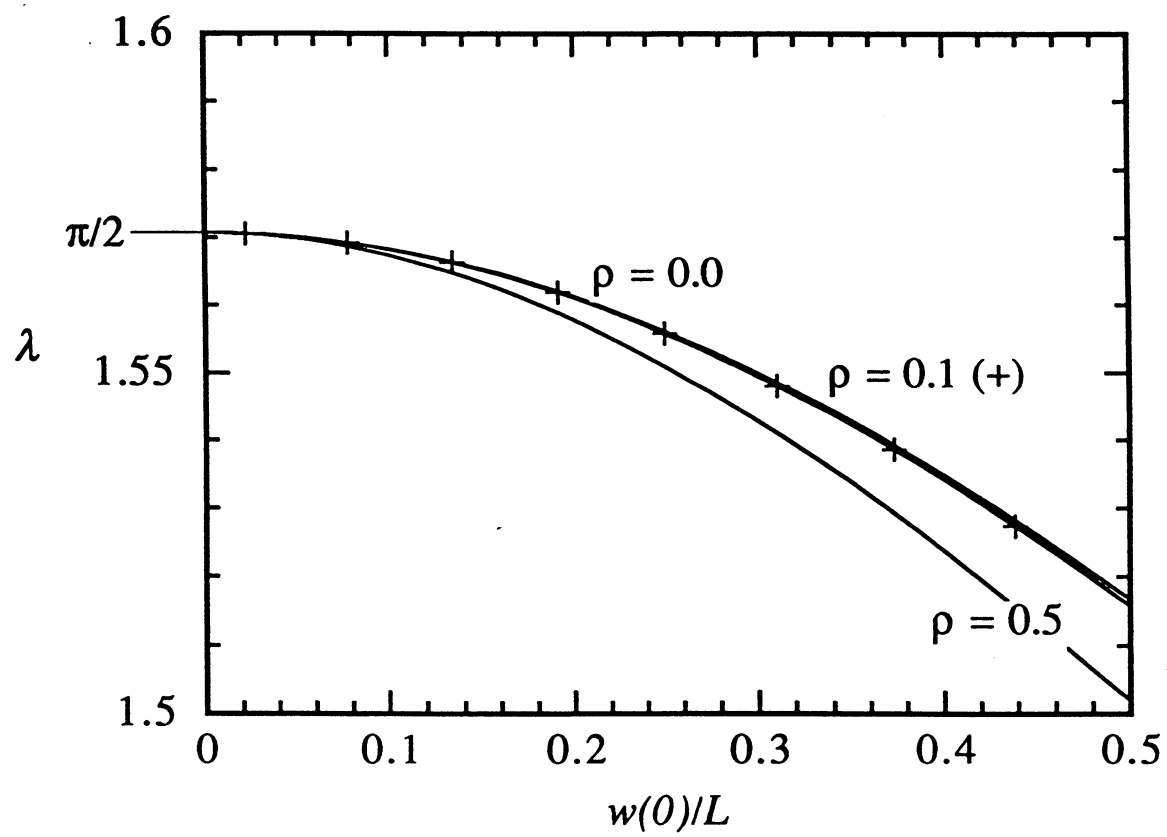


Fig. 5. Axial load as a function of the mid-span deflection.

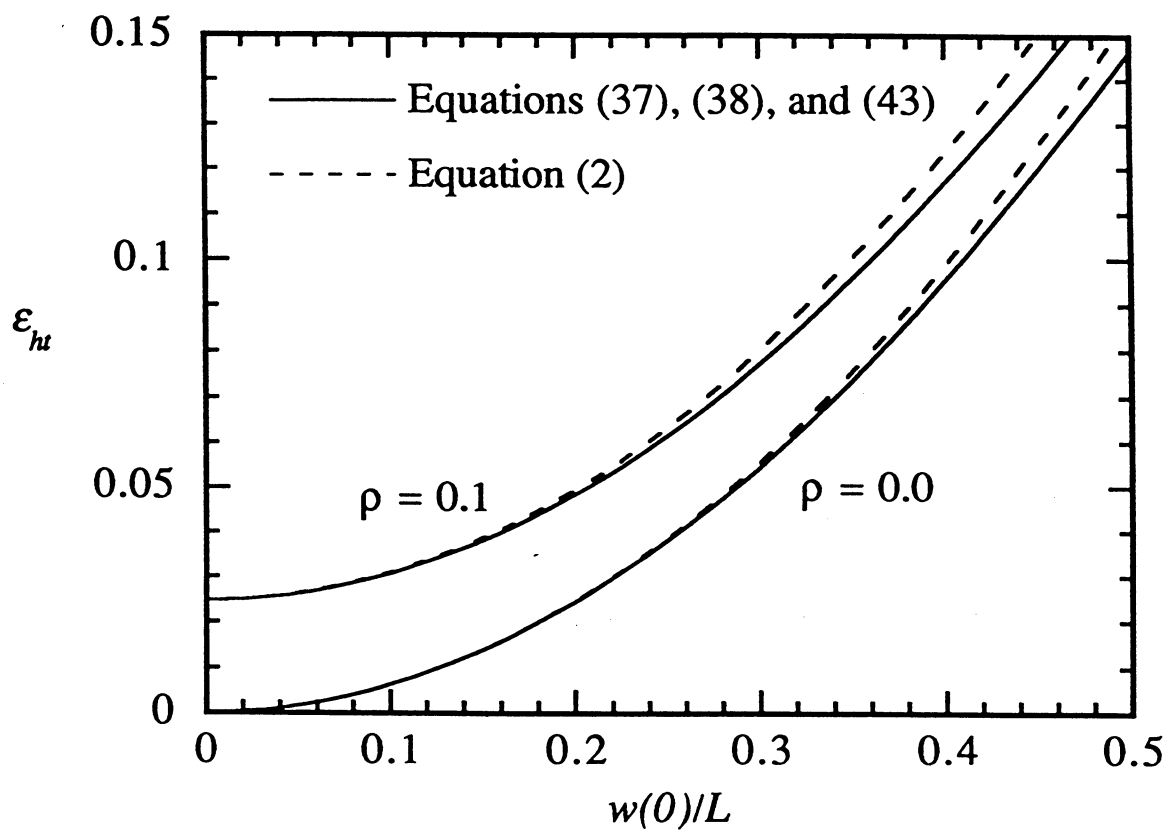


Fig. 6. Comparison of buckling predictions.

NOMENCLATURE

Symbol	Description	Dimension
A	cross-sectional area of beam	$[L^2]$
b	width of beam	$[L]$
E	Young's modulus	$[F/L^2]$
h	depth of beam	$[L]$
H	moisture content	$[\%]$
I	moment of inertia	$[L^4]$
K	beam rotation parameter	$[-]$
L	half length of beam	$[L]$
M	bending moment about y axis	$[FL]$
N	resultant force in x direction	$[F]$
P_{cr}	critical buckling load	$[F]$
S	undeformed neutral axis of beam	$[L]$
s	deformed neutral axis of beam	$[L]$
T	Temperature	$[\theta]$
u	axial deflection	$[L]$
V	resultant force in z direction	$[F]$
w	vertical deflection	$[L]$
(x, y, z)	reference coordinate system	$[L]$
α	coefficient of thermal expansion	$[1/\theta]$
β	coefficient of hygroexpansion	$[\%]$
ϵ	axial normal strain	$[-]$
ϵ_{ht}	potential strain of free expansion	$[-]$
ϵ_0	axial normal strain at neutral axis	$[-]$
ϕ	normalized angle of rotation	$[\text{rad}]$
κ	curvature of neutral axis	$[1/L]$
λ	dimensionless load parameter	$[-]$
ρ	slenderness ratio	$[-]$
θ	angle of rotation	$[\text{rad}]$
θ_1	angle of rotation at ends	$[\text{rad}]$

