EXISTENCE AND BOUNDS OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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CHAPTER 0

INTRODUCTION

The original motivation for the studies in this work is a theorem by W. A. Coppel (see [4, Theorem 3, p. 58]) in which he uses the logarithmic norm of a square matrix A to obtain a bound for the solutions of the linear differential equation $u' = Au$. The logarithmic norm is defined and certain basic properties are derived by S. M. Lozinskii in [11]. If I denotes the identity matrix and $\|\cdot\|$ is a norm on the square matrices such that $\|I\| = 1$, then the logarithmic norm of A--denoted $\mu[A]$--is defined by

$$(0.1) \qquad \mu[A] = \lim_{h \to +0} \frac{\|I+hA\| - 1}{h} .$$

Let E be a Banach space with norm denoted by $|\cdot|$ and let A be a function from E into E. Suppose that there is a number K such that

$$(0.2) \qquad \lim_{h \to +0} \frac{|x-y + h[Ax-Ay]| - |x-y|}{h} \leq K|x-y|$$

for each x and y in E. We extend the notion of logarithmic norm by letting the logarithmic derivative of A--denoted $L[A]$--denote the smallest number K such that the inequality in (0.2) holds for all x and y in E. The notion of logarithmic derivative is used in this work to obtain results on the existence and stability of differential equations in a Banach space.

The basic properties of the logarithmic derivative are derived in Chapter II. Here we also establish a connection between the logarithmic derivative and monotonic and accretive operators defined by T. Kato in [8] and F. E. Browder in [2]. Some existence theorems by ordinary differential equations in a Banach space are given in Chapter V. Theorem 5.1 extends to a general Banach space an existence theorem of F. E. Browder [1, Theorem 3]; Browder's theorem was obtained in a Hilbert space.

In Chapter VI we establish some new results on the generation of semigroups of nonlinear operators (Theorems 6.1 and 6.2) and, in Theorem 6.3, we give sufficient conditions to guarantee the existence of a critical point to an autonomous differential equation which is globally asymptotically stable. This is an improvement of a theorem of L. Markus and H. Yamabe [14, Theorem 1]. In Chapter VII we show how these techniques can be used to extend some of the known results on the stability of differential equations. For example, Theorem 21.1 of N. N. Krasovskii [9, p. 91] is improved (see Example 7.1).

CHAPTER I

PRELIMINARY LEMMAS

In this chapter we prove four lemmas which form the core of the concepts developed in this work. Since the results of this chapter are applicable to several different areas of this work, they are proved in a somewhat general setting; and so some of the notations used here are different from those used in succeeding chapters. Here, K denotes either the field of real or complex numbers, X denotes a vector space over the field K, and $p[\cdot]$ denotes a seminorm on X (i.e. $p[\cdot]$ is a function from X into $[0,\infty)$ such that $p[x+y] \leq p[x] + p[y]$ and $p[ax] = |a|p[x]$ for each x and y in X and a in K).

The space of continuous linear functions from the seminormed space X into the field K is denoted by X^* and if x is in X and f is in X^*, then (x,f) denotes the image of x under f. The vector space X^* is considered as a seminormed space with seminorm $q[\cdot]$ where

$$q[f] = \sup\{|(x,f)| : x\in X,\ p[x] \leq 1\}$$

for each f in X^*. Note that $q[\cdot]$ is a norm on X^* (i.e. $q[f] = 0$ if and only if $(x,f) = 0$ for all x in X).

Definition 1.1. For each x in X define the subset $G(x)$ of X^* by

$$G(x) = \{g \in X^* : q[g] = 1 \text{ and } (x,g) = p[x]\}.$$

Remark 1.1. If x is in X, $p[x] \neq 0$, and $Q = \{ax : a \in K\}$ then Q is a subspace of X; and if $(ax,f) = ap[x]$ for each a in K then f is a continuous linear functional from Q into K such that $\sup\{|(ax,f)| : a \in K, p[ax] = 1\} = 1$. Consequently, by the Hahn-Banach theorem (see e.g. [22, p. 107]) there is a member g of X^* such that $q[g] = 1$ and $(y,g) = (y,f)$ for each y in Q. Since $(x,g) = (x,f) = p[x]$, g is in $G(x)$; and so $G(x)$ is a nonempty subset of X^*. Note that if x is in X and $p[x] = 0$, then $G(x) = \{g \in X^* : q[g] = 1\}$.

Lemma 1.1. If x and y are in X then

(i) $m_+[x,y] = \lim_{h \to +0} (p[x+hy] - p[x])/h$ exists and

$m_+[x,y] \leq (p[x+hy] - p[x])/h$ for each $h>0$.

(ii) $m_-[x,y] = \lim_{h \to -0} (p[x+hy] - p[x])/h$ exists and

$m_-[x,y] \geq (p[x+hy] - p[x])/h$ for each $h<0$.

(iii) $-p[y] \leq m_-[x,y] \leq m_+[x,y] \leq p[y]$.

Proof. For each number $h \neq 0$ let $\phi(h) = (p[x+hy] - p[x])/h$. If k is a positive number less than one, then

$$p[x+khy] = p[k(x+hy) + (1-k)x]$$

$$\leq kp[x+hy] + (1-k)p[x].$$

Thus $p[x+khy] - p[x] \le k(p[x+hy] - p[x])$ and it follows that $\phi(kh) \le \phi(h)$ if $h > 0$ and that $\phi(kh) \ge \phi(h)$ if $h < 0$. In particular, if $0 < h_1 \le h_2$ or $h_1 \le h_2 < 0$, then $\phi(h_1) \le \phi(h_2)$ so that ϕ is non-decreasing on $(-\infty,0)$ and on $(0,\infty)$. Since $|\phi(h)| \le p[y]$, parts (i) and (ii) follow easily. Furthermore, $-p[y] \le m_-[x,y]$ and $m_+[x,y] \le p[y]$. Also, if $h > 0$, then

$$
\begin{aligned}
2p[x] &= p[x+hy+x-hy] \\
&\le p[x+hy] + p[x-hy],
\end{aligned}
$$

so that $p[x+hy] - p[x] \ge -p[x-hy] + p[x]$. Dividing by $h > 0$ and letting $h\to+0$ shows that $m_+[x,y] \ge m_-[x,y]$, and the proof of the lemma is complete.

Example 1.1. Suppose that X is the vector space of complex numbers and $p[x] = |x|$ for each x in X. If z is in X and $h > 0$, then

$$
\begin{aligned}
(|1+hz| - 1)/h &= [(1+hz)(1+h\bar{z}) - 1]/[h(|1+hz| + 1)] \\
&= [2\mathrm{Re}(z) + h|z|^2]/[|1+hz| + 1].
\end{aligned}
$$

Hence $m_-[1,z] = m_+[1,z] = \mathrm{Re}(z)$ and the limits defining $m_-[1,z]$ and $m_+[1,z]$ are uniform for z in a bounded subset of X.

Lemma 1.2. Let m_- and m_+ be as defined in Lemma 1.1 and let x, y and z be in X. Then

(i) $m_+[x,ry] = rm_+[x,y]$ and $m_-[x,ry] = rm_-[x,y]$ for each positive number r.

(ii) $m_+[x,y+z] \leq m_+[x,y] + m_+[x,z]$ and $m_-[x,y+z] \geq m_-[x,y] + m_-[x,z]$.

(iii) $|m_+[x,y]| \leq p[y]$ and $|m_-[x,y]| \leq p[y]$.

(iv) $|m_+[x,y] - m_+[x,z]| \leq p[y-z]$ and $|m_-[x,y] - m_-[x,z]| \leq p[y-z]$.

(v) $m_+[x,y+ax] = m_+[x,y] + \mathrm{Re}(a)p[x]$ and $m_-[x,y+ax] = m_-[x,y] + \mathrm{Re}(a)p[x]$ for each a in K.

Remark 1.2. Note that (i) and (iv) imply that $m_+[x,\cdot]$ and $m_-[x,\cdot]$ are positively homogeneous and continuous functions from X into the real numbers. Part (ii) shows that $m_+[x,\cdot]$ is subadditive. However, if $p[x] \neq 0$, then $m_+[x,-x] = -p[x]$ so that $m_+[x,\cdot]$ is not a seminorm on X.

Proof of Lemma 1.2. If $r > 0$ then $rh \to \pm 0$ as $h \to \pm 0$ so that part (i) follows from the identity

$$(p[x+hry] - p[x])/h = r(p[x+hry] - p[x])/(hr).$$

Since $p[x+h(y+z)] \leq p[x+2hy]/2 + p[x+2hz]/2$, it follows that

$$p[x+h(y+z)] - p[x] \leq (p[x+2hy] - p[x])/2 + (p[x+2hz] - p[x])/2$$

and part (ii) may be seen by dividing each side of the above inequality by h and letting $h \to \pm 0$. Part (iii) is an immediate consequence of part (iii) of Lemma 1.1. From parts (ii) and (iii) of this lemma,

$$m_+[x,y] = m_+[x,z + (y-z)]$$

$$\leq m_+[x,z] + p[y-z],$$

and so $m_+[x,y] - m_+[x,z] \leq p[y-z]$. Interchanging the roles of y and z shows that the first assertion of (iv) is true. The second assertion is proved analogously. It follows easily from Example 1.1 that $m_+[x,ax] = \mathrm{Re}(a)p[x]$ for each a in K. Thus from part (ii) of this lemma,

$$m_+[x,y+ax] \leq m_+[x,y] + \mathrm{Re}(a)p[x]$$

and

$$m_+[x,y] = m_+[x,y+ax-ax]$$

$$\leq m_+[x,y+ax] - \mathrm{Re}(a)p[x]$$

which shows that the first assertion of part (v) is true. The second assertion is proved analogously and this completes the proof of the lemma.

Lemma 1.3. For each x in X let $G(x)$ be as defined in Definition 1.1 and let m_+ and m_- be as in Lemma 1.1. Then if x and y are in X,

(i) $m_+[x,y] = \sup\{\mathrm{Re}(y,g) : g \in G(x)\}$ and

(ii) $m_-[x,y] = \inf\{\mathrm{Re}(y,g) : g \in G(x)\}$.

Remark 1.3. This lemma may be known, but the author has been unable to find it in the literature.

Proof of Lemma 1.3. Let $\Gamma(x,y)$ denote the supremum in (i) and let g be in $G(x)$. If $h > 0$, then

$$\begin{aligned} \mathrm{Re}(y,g) &= (\mathrm{Re}(x+hy,g) - p[x])/h \\ &\le (p[x+hy] - p[x])/h. \end{aligned} \tag{1.1}$$

Here we have used the fact that $p[x] = \mathrm{Re}(x,g)$ and $q[g] = 1$. Letting $h \to +0$ in (1.1) shows that $m_+[x,y] \ge \Gamma(x,y)$. Now, for each $h > 0$, let g_h be a member of $G(x+hy)$. Since $\mathrm{Re}(x+hy,g_h) = p[x+hy]$, we have from (1.1) that

$$\begin{aligned} \mathrm{Re}(y,g) &\le (\mathrm{Re}(x+hy,g_h) - p[x])/h \\ &= \mathrm{Re}(x,g_h)/h + \mathrm{Re}(y,g_h) - p[x]/h. \end{aligned} \tag{1.2}$$

By transposing terms in (1.2) and multiplying by h,

$$p[x] - h[\mathrm{Re}(y,g_h) - \mathrm{Re}(y,g)] \le \mathrm{Re}(x,g_h). \tag{1.3}$$

Since $|(x,g_h)| \le p[x]$ for each $h > 0$, it follows from (1.3) that

$$\lim_{h\to+0} (x,g_h) = p[x]. \tag{1.4}$$

Since the unit ball of X^* is w^*-compact (see e.g. [22, p. 137]) there is an f in X^* such that $q[f] \le 1$ and a sequence of positive numbers $(h_n)_1^\infty$ such that $\lim_{n\to\infty} h_n = 0$ and, if $f_n = g_{h_n}$ for each $n \ge 1$, then $\lim_{n\to\infty}(z,f_n) = (z,f)$ for all z in E. From (1.4), $(x,f) = \lim_{n\to\infty}(x,f_n) = p[x]$ so that $q[f] = 1$ and f is in $G(x)$. Consequently,

$$\begin{aligned}
\Gamma(x,y) &\ge \mathrm{Re}(y,f) \\
&= \lim_{n\to\infty} \mathrm{Re}(y,f_n) \\
&\ge \lim_{n\to\infty} (\mathrm{Re}(x+h_n y,f_n) - p[x])/h_n \\
&= \lim_{n\to\infty} (p[x+h_n y] - p[x])/h_n \\
&= m_+[x,y].
\end{aligned}$$

Here we have used the fact that $\mathrm{Re}(x,f_n) \le p[x]$ and $\mathrm{Re}(x+h_n y,f_n) = p[x+h_n y]$. Thus $\Gamma(x,y) = m_+[x,y]$ and part (i) is proved. Noting that

$$m_-[x,y] = -m_+[x,-y]$$

$$= -\sup\{\mathrm{Re}(-y,g) : g \in G(x)\}$$

$$= \inf\{\mathrm{Re}(y,g) : g \in G(x)\},$$

we see that (ii) is true and the proof of the lemma is complete.

Definition 1.2. Suppose that Y is a normed linear space and $n[\cdot]$ denotes the norm on Y. Then Y is said to be uniformly convex if for each positive number ϵ there is a positive number δ such that if x and y are in Y with $n[x] = n[y] = 1$ and $n[x+y] \geq 2 - \delta$, then $n[x-y] \leq \epsilon$.

Example 1.2. If Y is a complete inner product space the formula

$$n[x+y]^2 + n[x-y]^2 = 2(n[x]^2 + n[y]^2)$$

is valid for all x and y in Y and, as a consequence, Y is uniformly convex.

Lemma 1.4. Suppose that the normed space X^* is uniformly convex and that each of M, β, and ϵ are positive numbers. It follows that there is a positive number $\delta = \delta(M,\beta,\varepsilon)$ such that if x and y are in X with $p[x] \geq \beta$ and $p[y] \leq M$ then

$$|(p[x+hy] - p[x])/h - \mathrm{Re}(y,g)| \leq \varepsilon$$

for the member g of $G(x)$ and all real numbers h such that $0 < |h| \le \delta$.

Remark 1.4. Suppose x is in X, $p[x] \ne 0$, and f and g are in $G(x)$. Then $(x,f+g) = 2p[x]$ so that $q[f+g] = 2$. Hence, if X^* is uniformly convex and x is in X with $p[x] \ne 0$, then the set $G(x)$ consists of exactly one member.

Proof of Lemma 1.4. With the suppositions of Lemma 1.4, let $\varepsilon' > 0$ be such that if f_1 and f_2 are in X^* with $q[f_1] = q[f_2] = 1$ and $q[f_1+f_2] \ge 2 - \varepsilon'$, then $q[f_1-f_2] \le \varepsilon/M$. Choose $\delta = \varepsilon'\beta/(2M)$ and let g be in $G(x)$. If $0 < |h| \le \delta$ and g_h is in $G(x+hy)$ then

$$\begin{aligned}(p[x+hy] - p[x])/h &= (\mathrm{Re}(x+hy,g_h) - p[x])/h \\ &= \mathrm{Re}(x,g_h)/h + \mathrm{Re}(y,g_h) - p[x]/h.\end{aligned}$$

Transposing terms and multiplying by $|h|$ we have

$$p[x] - h\mathrm{Re}(y,g_h) + p[x+hy] - p[x] = \mathrm{Re}(x,g_h)$$

if $h > 0$, and we have

$$-p[x] + h\mathrm{Re}(y,g_h) - p[x+hy] + p[x] = -\mathrm{Re}(x,g_h)$$

if $h < 0$. Since $|\mathrm{Re}(y,g_h)| \le p[y] \le M$ and $|p[x+hy] - p[x]| \le |h|p[y] \le |h|M$, it follows that

$$-2|h|M \le \mathrm{Re}(x,g_h) - p[x] \le 2|h|M.$$

Hence

$$q[g_h+g] \ge |\mathrm{Re}(x,g_h+g)|/p[x]$$

$$= |\mathrm{Re}(x,g_h) + p[x]|/p[x]$$

$$\ge 2 - 2|h|M/p[x].$$

Since $|h| \le \varepsilon'\beta/(2M)$ and $p[x] \ge \beta$, $q[g_h+g] \ge 2-\varepsilon'$ and, by the choice of ε', $q[g_h-g] \le \varepsilon/M$. If $0 < h \le \delta$, then

$$0 \le (p[x+hy] - p[x])/h - \mathrm{Re}(y,g)$$

$$= (\mathrm{Re}(x+hy,g_h) - p[x])/h - \mathrm{Re}(y,g)$$

$$= (\mathrm{Re}(x,g_h) - p[x])/h + \mathrm{Re}(y,g_h-g)$$

$$\le p[y]q[g_h-g]$$

$$\le \varepsilon.$$

Here we have used the fact that $\mathrm{Re}(x,g_h) - p[x] \le 0$. Similarly, if $-\delta \le h < 0$, then

$$0 \le (p[x+hy] - p[x])/h - \mathrm{Re}(y,g)$$

$$\ge -p[y]q[g_h-g]$$

$$\ge -\varepsilon ,$$

and the proof of the lemma is complete.

CHAPTER II

SPACES OF OPERATORS

In this chapter we define four classes of functions which are from a subset $\mathcal{D}$ of a Banach space E into E. One purpose for the construction of these function spaces is to connect the results of this work to previous results in related areas of the study of differential equations. Another is an attempt both to motivate and to provide a unification of the definitions and techniques used in the development of the subsequent theorems. The notations introduced in this chapter are used in each succeeding chapter.

For the remainder of this work K denotes either the field of real or complex numbers and E denotes a Banach space over the field K with the norm on E denoted by $|\cdot|$. The space of continuous linear functionals from E into K is denoted by E^* and if f is in E^* and x is in E, (x,f) denotes the image of x under f. E^* is considered as a Banach space over K with norm $|\cdot|$, where $|f| = \sup\{|(x,f)| : x \in E$ and $|x| = 1\}$ for each f in E^*.

Remark 2.1. It should be noted that $|\cdot|$ denotes the norm on both E and E^* and also the absolute value on K. However, this should not cause any confusion since it will be clear from the context as to how $|\cdot|$ is being used.

Definition 2.1. For each x in E define the subsets $F(x)$ and $G(x)$ of E^* by

(i) $F(x) = \{f \in E^* : (x,f) = |x|^2 = |f|^2\}$ and

(ii) $G(x) = \{g \in E^* : |g| = 1$ and $(x,g) = |x|\}$.

Remark 2.2. Both $F(x)$ and $G(x)$ are nonempty subsets of E^* for each x in E (see Remark 1.1), and if x is a nonzero member of E, then g is in $G(x)$ if and only if $|x|g$ is in $F(x)$.

Notation. Suppose $\mathcal{D}$ is a subset of E and A is a function from $\mathcal{D}$ into E. To keep the number of parentheses to a minimum, for each x in $\mathcal{D}$, Ax denotes the image of x under A. When this notation is ambiguous, parentheses are inserted in the natural places--for example if $x = y + z$ then Ax is denoted $A(y+z)$.

Definition 2.2. If $\mathcal{D}$ is a linear subspace of E, denote by $BL(\mathcal{D},E)$ the class of all bounded linear functions from $\mathcal{D}$ into E. For each member A of $BL(\mathcal{D},E)$ define

$$\|A\| = \sup\{|Ax| : x \in \mathcal{D},\ |x| = 1\}.$$

With addition and scalar multiplication defined in the natural manner $BL(\mathcal{D},E)$ with the norm $\|\cdot\|$ is a Banach space over the field K. We let I denote the identity function from E into E and, for notational convenience, if $\mathcal{D}$ is a subset of E, I also denotes the restriction to $\mathcal{D}$ of the identity function on E. It is immediate that I is in $BL(\mathcal{D},E)$ for each subspace $\mathcal{D}$ of E and that $\|I\| = 1$.

Definition 2.3. For each member A of $BL(\mathcal{D},E)$ define

$$\mu[A] = \lim_{h\to+0} (\|I+hA\| - 1)/h.$$

Remark 2.3. Using the notations of Chapter I we have that if X is the Banach space $BL(\mathcal{D},E)$, $p[\cdot] = \|\cdot\|$, and m_+ is as defined in Lemma 1.1, then $\mu[A] = m_+[I,A]$ for each A in $BL(\mathcal{D},E)$. In particular, $\mu[\cdot]$ satisfies each of the properties of $m_+[I,\cdot]$ in Lemma 1.2.

If $\mathcal{D} = E$ and A and B are in $BL(E,E)$ then $A\cdot B$ denotes the composition of A with B (i.e. $A\cdot B$ is the member C of $BL(E,E)$ defined by $Cx = A(Bx)$ for each x in E). It is immediate that $\|A\cdot B\| \le \|A\| \cdot \|B\|$ so that the Banach space $BL(E,E)$, with multiplication defined by composition, in a Banach algebra over K. A member A of $BL(E,E)$ is said to be invertible if there is a member B of $BL(E,E)$ such that $A\cdot B = B\cdot A = I$. In this case B is denoted A^{-1}. For notational convenience, let $A^0 = I$ and for each positive integer n, define $A^n = A\cdot A^{n-1}$.

Definition 2.4. For each A in $BL(E,E)$ define

$$\exp(A) = \lim_{n\to\infty}(I+n^{-1}A)^n.$$

Remark 2.4. The following properties of $\exp(\cdot)$ are well known and the proofs are routine:

(i) $\exp(A) = \sum_{n=0}^{\infty} A^n/(n!)$.

(ii) $\exp(A)$ is an invertible member of $BL(E,E)$ with

$\exp(A)^{-1} = \exp(-A)$ and $\|\exp(A)\| \le \exp(\|A\|)$.

(iii) $\|\exp(A) - I - A\| \le \|A\|^2 \exp(\|A\|)$.

(iv) If A and B commute then $\exp(A+B) = \exp(A)\cdot\exp(B)$.

Proposition 2.1. If A is in $BL(E,E)$ then

(i) $\mu[A] = \lim_{h\to+0} (\|\exp(hA)\| - 1)/h$.

(ii) $\|\exp(A)\| \le \exp(\mu[A])$

(iii) $1 + h\mu[A] \le \|I+hA\| \le 1 + h\mu[A] + 2h^2\|A\|^2\exp(h\|A\|)$

for each $h > 0$.

Remark 2.5. Lozinskii [11, Lemma 6] shows that (ii) and (iii) are true when E is finite dimensional. The proof of (ii) given here is different but that of (iii) is essentially the same as his.

Proof of Proposition 2.1. Part (i) is immediate from part (iii) of Remark 2.4. Suppose ε is a positive number and choose n_o sufficiently large so that if $n \ge n_o$ then $(\|I+n^{-1}A\| - 1)/n^{-1} \le \mu[A] + \varepsilon$, $\|\exp(A)\| \le \|I+n^{-1}A\|^n + \varepsilon$, and $\{1 + n^{-1}(\mu[A]+\varepsilon)\}^n \le \exp(\mu[A] + \varepsilon) + \varepsilon$. Then

$$\begin{aligned}\|\exp(A)\| &\le \|I + n^{-1}A\|^n + \varepsilon \\ &= \{1 + n^{-1}(\|I+n^{-1}A\| - 1)/n^{-1}\}^n + \varepsilon \\ &\le \{1 + n^{-1}(\mu[A] + \varepsilon)\}^n + \varepsilon\end{aligned}$$

$$\leq \exp(\mu[A] + \varepsilon) + 2\varepsilon.$$

This shows that (ii) is true. From part (ii) of this lemma, part (iii) of Remark 2.4, and since $\mu[hA] = h\mu[A] \leq h\|A\|$ for each $h > 0$ (see parts (i) and (iii) of Lemma 1.2), we have

$$\|I+hA\| \leq \|\exp(hA)\| + \|I+hA - \exp(hA)\|$$

$$\leq \exp(h\mu[A]) + \|hA\|^2\exp(\|hA\|)$$

$$\leq 1 + h\mu[A] + \sum_{n=2}^{\infty} h^2\|A\|^n/(n!) + h^2\|A\|^2\exp(h\|A\|)$$

and the right side of the inequality in (iii) follows. The left side is immediate since $\mu[A] \leq (\|I+hA\| - 1)/h$ for each $h > 0$ (see part (i) of Lemma 1.1).

From parts (i) and (ii) of Proposition 2.1 we have

Corollary 2.1. If A is in $BL(E,E)$ then $\mu[A] \leq 0$ if and only if $\|\exp(hA)\| \leq 1$ for each $h > 0$.

From part (iii) of Proposition 2.1 we have

Corollary 2.2. If A is in $BL(E,E)$ and h is a positive number such that $2h\|A\| \leq 1$, then

$$|(\|I+hA\| - 1)/h - \mu[A]| \leq 4h\|A\|^2.$$

Remark 2.6. Note that Corollary 2.2 implies that the approximations $(\|I+hA\| - 1)/h$ converge to $\mu[A]$ uniformly on bounded subsets of $BL(E,E)$.

Example 2.1. Suppose that A is in $BL(E,E)$ and for each x in E $u_x(t) = \exp(tA)x$ for all t in $[0,\infty)$. Then $u_x(0) = x$, $u_x'(t) = Au_x(t)$, and $|u_x(t)| \le \exp(t\mu[A])|x|$ for all (t,x) in $[0,\infty)\times E$. In particular, $\exp(tA)$ is a nonexpansive semigroup of operators if and only if $\mu[A] \le 0$ (see [13, Theorem 2.1]).

Definition 2.5. For each subset $\mathcal{D}$ of E denote by $Lip(\mathcal{D},E)$ the class of all functions A from $\mathcal{D}$ into E for which there is a number K such that

$$|Ax-Ay| \le K|x-y|$$

for each x and y in $\mathcal{D}$. Denote by $N[A]$ the smallest number K such that this inequality holds.

With addition and scalar multiplication defined in the natural manner $Lip(\mathcal{D},E)$ is a vector space over the field K. $N[\cdot]$ is a seminorm on the vector space $Lip(\mathcal{D},E)$, $N[A] = 0$ if and only if A is constant on $\mathcal{D}$, and the seminormed space $Lip(\mathcal{D},E)$ is complete. Furthermore, if $\mathcal{D}$ is a subspace of E and A is a linear function from $\mathcal{D}$ into E, then A is in $Lip(\mathcal{D},E)$ if and only if A is in $BL(\mathcal{D},E)$ and, in this case, $N[A] = \|A\|$. In particular, $BL(\mathcal{D},E)$ is a closed subspace of $Lip(\mathcal{D},E)$.

Definition 2.6. For each A in $Lip(\mathcal{D},E)$ define

$$M[A] = \lim_{h \to +0} (N[I+hA] - 1)/h.$$

Remark 2.7. If $\mathcal{D}$ is a subspace of E and A is a linear member of $Lip(\mathcal{D},E)$ then $I + hA$ is in $BL(\mathcal{D},E)$ for each $h > 0$ and $N[I+hA] = \|I+hA\|$ so that $M[A] = \mu[A]$.

Remark 2.8. Using the notations of Chapter I we have that if X is the seminormed space $Lip(\mathcal{D},E)$, $p[\cdot] = N[\cdot]$, and m_+ is as defined in Lemma 1.1, then $M[A] = m_+[I,A]$ for each A in $Lip(\mathcal{D},E)$. Consequently, $M[\cdot]$ satisfies each of the properties of $m_+[I,\cdot]$ in Lemma 1.2. For future reference, we list them here: If A and B are in $Lip(\mathcal{D},E)$ then

(i) $M[rA] = rM[A]$ for each positive number r.

(ii) $M[A+B] \leq M[A] + M[B]$.

(iii) $|M[A]| \leq N[A]$.

(iv) $|M[A] - M[B]| \leq N[A-B]$.

(v) $M[A+aI] = M[A] + Re(a)$ for each a in K.

If $\mathcal{D} = E$ and A and B are in $Lip(E,E)$ then $A \cdot B$ denotes the composition of A with B. With addition and multiplication by composition, $Lip(E,E)$ is a near-ring with unity (i.e. $Lip(E,E)$ has each of the properties of a ring with unity except the left distributiveness of multiplication over addition). Also the seminormed near-ring $Lip(E,E)$ is complete and $N[A \cdot B] \leq N[A]N[B]$ for each A and B in $Lip(E,E)$. A member A of $Lip(E,E)$ is said to be invertible if there is a member B of $Lip(E,E)$ such that $A \cdot B = B \cdot A = I$. In this case B is denoted A^{-1}.

Lemma 2.1. If A is in $Lip(E,E)$ and $N[A] < 1$ then $I - A$ is an invertible member of $Lip(E,E)$ with $N[(I-A)^{-1}] \le (1-N[A])^{-1}$.

Proof. This is proved by Neuberger [17, Lemma 1] and we outline it here. Let $B_o = I$ and for each $n \ge 1$ let $B_n = I + A \cdot B_{n-1}$. If x is in E and $n \ge 1$ then $|B_n x - B_{n-1}x| \le N[A]|B_{n-1}x - B_{n-2}x| \le \ldots \le N[A]^{n-1}|Ax|$. If $\beta(x) = |x| + |Ao|N[A]^{-1}$ then $|Ax| \le |Ax-Ao| + |Ao| \le N[A]\beta(x)$ so that $|B_n x - B_{n-1}x| \le N[A]^n \beta(x)$. Thus if $m > n \ge 1$ then

$$|B_m x - B_n x| \le \sum_{i=n+1}^{m} |B_i x - B_{i-1}x|$$

$$\le N[A]^{n+1}\beta(x)(1-N[A])^{-1}.$$

It now follows that $\lim_{n\to\infty} B_n x = (I-A)^{-1}x$ for each x in E and also, $(I-A)^{-1}$ is in $Lip(E,E)$ with $N[(I-A)^{-1}] \le (1-N[A])^{-1}$.

Corollary 2.3. If A is in $Lip(E,E)$ and $M[A] < 0$ (respectively $M[-A] < 0$), then A^{-1} exists and is in $Lip(E,E)$ with $N[A^{-1}] \le -M[A]^{-1}$ (respectively, $N[A^{-1}] \le -M[-A]^{-1}$).

Proof. If $M[A] < 0$ then there is an $h > 0$ such that $(N[I+hA] - 1)/h < 0$ and hence, $N[I+hA] < 1$. By Lemma 2.1, $[I - (I+hA)]^{-1} = [-hA]^{-1}$ exists and is in $Lip(E,E)$ with $N[(-hA)^{-1}] \le (1 - N[I+hA])^{-1}$. Thus A^{-1} exists and since $N[(-hA)^{-1}] = h^{-1}N[A^{-1}]$ we have

$$N[A^{-1}] \leq h(1 - N[I+hA])^{-1}$$

$$= -\{(N[I+hA] - 1)/h\}^{-1}.$$

Since this inequality holds for all sufficiently small $h > 0$, it follows that $N[A^{-1}] \leq -M[A]^{-1}$. The other assertion of the corollary follows in a similar manner.

Corollary 2.4. If K is the complex field, A is in $BL(E,E)$, and λ is in the spectrum of A, then $\mathrm{Re}(\lambda) \leq \mu[A]$.

Proof. It follows from Corollary 2.3 that if λ is in the spectrum of A then $\mu[A-\lambda I] \geq 0$. From part (v) of Remark 2.8, $\mu[A] - \mathrm{Re}(\lambda) \geq 0$ and the corollary follows.

Definition 2.7. If $\mathcal{D}$ is an open subset of E, A is a function from $\mathcal{D}$ into E, and x is in $\mathcal{D}$, then A is said to be Fréchet differentiable at x if there is a U in $BL(E,E)$ such that

$$\lim_{y \to x} \{|Ay-Ax-U(y-x)|/|x-y|\} = 0.$$

U is called the Fréchet derivative of A at x and will be denoted $dA(x)$.

The basic properties of the Fréchet derivative can be found in [5, Chapter VIII]. Here the notion of Fréchet derivative will be used to obtain a further relationship between the functions $M[\cdot]$ and $\mu[\cdot]$. To establish this relationship we need the following:

Lemma 2.2. Suppose x and y are in E and $\mathcal{D}$ is an open subset of E which contains the line segment from x to y. If A is a continuous function from $\mathcal{D}$ into E which is Fréchet differentiable at each point on the open line segment from x to y, then

$$|Ax-Ay| \le |x-y|\sup\{\|dA(x+\beta(y-x))\| : 0 < \beta < 1\}.$$

For a proof of this lemma see [5, p. 155].

Proposition 2.2. Suppose that $\mathcal{D}$ is an open convex subset of E and A is a Fréchet differentiable function from $\mathcal{D}$ into E. Then these are equivalent:

(i) A is in $Lip(\mathcal{D},E)$.

(ii) $\sup\{\|dA(x)\| : x\in\mathcal{D}\}$ is finite.

Furthermore, if (i) is true, then

(iii) $N[A] = \sup\{\|dA(x)\| : x\in\mathcal{D}\}$ and

(iv) $M[A] = \sup\{\mu[dA(x)] : x\in\mathcal{D}\}$.

Proof. Since $\mathcal{D}$ is convex, it is immediate from Lemma 2.2 that (ii) implies (i), and that $N[A] \le \sup\{\|dA(x)\| : x\in\mathcal{D}\}$. Now let ε be a positive number and let x_o be in $\mathcal{D}$. Since $\mathcal{D}$ is open, there is a $\delta > 0$ such that if $|x-x_o| \le \delta$, then x is in $\mathcal{D}$ and if $x \ne x_o$, then

$$|dA(x_o)([x-x_o]/|x-x_o|)| \le |Ax-Ax_o|/|x-x_o| + \varepsilon$$

$$\le N[A] + \varepsilon.$$

Consequently (i) implies (ii) and we also have that $\sup\{\|dA(x)\| : x\in\mathcal{D}\} \le N[A]$. Hence if (i) is true, so is (iii). Let $\Gamma = \sup\{\mu[dA(x)] : x\in\mathcal{D}\}$. If $h > 0$, then $I + hA$ in Fréchet differentiable on $\mathcal{D}$ and $d(I+hA) = I + hdA$. From part (iii) we have that $N[I+hA] \ge \|I+hdA(x)\|$ for each x in $\mathcal{D}$, and it follows that $M[A] \ge \Gamma$. Furthermore, from part (iii), for each $h > 0$ there is an x_h in $\mathcal{D}$ such that $N[I+hA] \le \|I + hdA(x_h)\| + h^2$. If $2hN[A] \le 1$, then $2h\|dA(x_h)\| \le 1$ and, by Corollary 2.2,

$$\begin{aligned} N[I+hA] &\le \|I+hdA(x_h)\| + h^2 \\ &\le 1 + h\mu[dA(x_h)] + 4h^2\|dA(x_h)\|^2 + h^2 \\ &\le 1 + h\Gamma + h^2(4N[A]^2 + 1). \end{aligned}$$

Thus $(N[I+hA] - 1)/h \le \Gamma + h(4N[A]^2 + 1)$ for all sufficiently small $h > 0$ and part (iv) follows.

In the proof of Proposition 2.2 we have shown

Corollary 2.5. If $\mathcal{D}$ is an open convex subset of E, A is a Fréchet differentiable member of $Lip(\mathcal{D},E)$, and h is a positive number such that $2hN[A] \le 1$, then

$$|(N[I+hA] - 1)/h - M[A]| \le h(4N[A]^2 + 1).$$

Example 2.2. Suppose that E is the space of real numbers, A is a continuously differentiable function from E into E, and $\mathcal{D}$ is a bounded open subinterval of E. Then A is in $Lip(\mathcal{D},E)$ (more precisely, the restriction of A to $\mathcal{D}$ is in $Lip(\mathcal{D},E)$), $N[A] = \sup\{|A'(x)| : x\in\mathcal{D}\}$, and $M[A] = \sup\{A'(x) : x\in\mathcal{D}\}$.

Definition 2.8. For each subset $\mathcal{D}$ of E denote by $Ln(\mathcal{D},E)$ the class of all functions A from $\mathcal{D}$ into E for which there is a number K such that

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h \le K|x-y|$$

for each x and y in $\mathcal{D}$. Denote by L[A] the smallest number K such that this inequality holds.

Proposition 2.3. Suppose that $\mathcal{D}$ is a subset of E, K is a number, and A is a function from $\mathcal{D}$ into E. Then these are equivalent:

(i) A is in $Ln(\mathcal{D},E)$ with $L[A] \le K$.

(ii) $\mathrm{Re}(Ax-Ay,g) \le K|x-y|$ for all x and y in $\mathcal{D}$ and all g in $G(x-y)$.

(iii) $\mathrm{Re}(Ax-Ay,f) \le K|x-y|$ for all x and y in $\mathcal{D}$ and all f in $F(x-y)$.

Furthermore, if (i) holds then L[A] is the smallest number K for which the inequalities in (ii) or (iii) hold.

Proof. The fact the (i) and (ii) are equivalent is immediate from Lemma 1.3 and the fact that (ii) and (iii) are equivalent is immediate from the definition of G and F. The last assertion of the proposition is also evident.

Example 2.3. Suppose E is a Hilbert space and let (x,y) denote the inner product of x and y for each x and y in E. Using the natural identification of E^* with E, if x is in E then $F(x)$ is a subset of E. Furthermore, it is immediate that x is in $F(x)$ for each x is E. Since E is uniformly convex (see Example 1.2), we have by Remarks 1.4 and 2.2 that $F(x)$ contains exactly one member, and hence $F(x) = \{x\}$ for each x in E. Consequently, by Proposition 2.3, if $\mathcal{D}$ is a subset of E and A is a function from $\mathcal{D}$ into E then A is in $Ln(\mathcal{D},E)$ if and only if there is a number K such that

$$\mathrm{Re}(Ax-Ay,x-y) \le K|x-y|^2$$

for all x and y in $\mathcal{D}$. Furthermore, L[A] is the smallest number K such that this inequality holds.

Proposition 2.4. If A and B are in $Ln(\mathcal{D},E)$ then

(i) For each $r > 0$, rA is in $Ln(\mathcal{D},E)$ with $L[rA] = rL[A]$.

(ii) $A + B$ is in $Ln(\mathcal{D},E)$ with $L[A+B] \le L[A] + L[B]$.

(iii) For each a in K, $A + aI$ is in $Ln(\mathcal{D},E)$ with $L[A+aI] = L[A] + \mathrm{Re}(a)$.

Proof. With the notations of Chapter I let X be E and let $p[\cdot]$ be $|\cdot|$. Then

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h = m_+[x-y,Ax-Ay]$$

for each x and y in $\mathcal{D}$ so that the assertions of this proposition follow easily from parts (i), (ii), and (v) of Lemma 1.2.

Proposition 2.5. If A is in $Lip(\mathcal{D},E)$ then A is in $Ln(\mathcal{D},E)$ and $L[A] \le M[A]$.

Proof. If x and y are in $\mathcal{D}$ then

$$\begin{aligned}\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h &\le \lim_{h\to+0} (N[I+hA]|x-y| - |x-y|)/h \\ &= M[A]|x-y|\end{aligned}$$

and the assertions of the proposition are immediate.

Definition 2.9. Suppose that $\mathcal{D}$ is a subset of E and A is a function from $\mathcal{D}$ into E. Then

(i) A is said to be accretive on $\mathcal{D}$ if $Re(Ax-Ay,f) \ge 0$ for all x and y in $\mathcal{D}$ and all f in $F(x-y)$.

(ii) A is said to be monotonic on $\mathcal{D}$ if $Re(Ax-Ay,f) \ge 0$ for all x and y in $\mathcal{D}$ and some f in $F(x-y)$.

Remark 2.9. The definition of an accretive operator is given by Browder is [2] and that of a monotonic operator is given by Kato in [8]. It is clear from the definitions that if A is accretive on $\mathcal{D}$ then A is monotonic on $\mathcal{D}$. Furthermore, it is clear from the relationship between F and G (see Remark 2.2) that the following hold:

(i)' A is accretive on $\mathcal{D}$ if and only if $\mathrm{Re}(Ax-Ay,g) \geq 0$ for all x and y in $\mathcal{D}$ and all g in $G(x-y)$.

(ii)' A is monotonic on $\mathcal{D}$ if and only if $\mathrm{Re}(Ax-Ay,g) \geq 0$ for all x and y in $\mathcal{D}$ and some g in $G(x-y)$.

Proposition 2.6. Suppose $\mathcal{D}$ is a subset of E, A is a function from $\mathcal{D}$ into E, and λ is a real number. Then these are equivalent:

(i) A is in $Ln(\mathcal{D},E)$ with $L[A] \leq \lambda$.

(ii) $\lambda I - A$ is accretive on $\mathcal{D}$.

Proof. If (i) is true then $L[A-\lambda I] = L[A] - \lambda \leq 0$ so by Proposition 2.3, $\mathrm{Re}(Ax-\lambda x-Ay+\lambda y,f) \leq 0$ for all x and y in $\mathcal{D}$ and all f in $F(x-y)$. It is now immediate that $\lambda I - A$ is accretive on $\mathcal{D}$, and so (i) implies (ii). Now suppose (ii) is true. If x and y are in $\mathcal{D}$ and f is in $F(x-y)$ then

$$\begin{aligned} 0 &\leq \mathrm{Re}(-Ax+\lambda x+Ay-\lambda y,f) \\ &= -\mathrm{Re}(Ax-Ay,f) + \lambda \mathrm{Re}(x-y,f) \\ &= -\mathrm{Re}(Ax-Ay,f) + \lambda|x-y|^2. \end{aligned}$$

Thus $\mathrm{Re}(Ax-Ay,f) \leq \lambda|x-y|^2$ and (i) is true by Proposition 2.3.

Corollary 2.6. If A is a function from $\mathcal{D}$ into E then $-A$ is accretive on $\mathcal{D}$ if and only if A is in $Ln(\mathcal{D},E)$ and $L[A] \leq 0$.

Remark 2.10. There is a result pertaining to monotonic operators which is analogous to Proposition 2.6. By using part (ii) of Lemma 1.3 and techniques analogous to those used in the proof of Proposition 2.6 one can show that if A is a function from $\mathcal{D}$ into E and λ is a real number, then these are equivalent:

(i) $\lim_{h \to -0} (|x-y + h[Ax-Ay]| - |x-y|)/h \leq \lambda|x-y|$.

(ii) $\lambda I - A$ is monotonic on $\mathcal{D}$.

Since we will be mainly concerned with functions which are in $Ln(\mathcal{D},E)$, we will restrict our attention to accretive operators as opposed to monotone operators. However, note that if $F(x)$ consists of exactly one member for each x in E, then the notions of monotonic operators and accretive operators are the same--for example, if E^* is uniformly convex (see Remark 1.4).

We say that function A from $\mathcal{D}$ into E has a logarithmic derivative on $\mathcal{D}$ if A is in $Ln(\mathcal{D},E)$. The number L[A] is called the logarithmic derivative of A on $\mathcal{D}$. As a consequence of Proposition 2.6 we see that A has a logarithmic derivative on $\mathcal{D}$ if and only if there is a number λ such that $\lambda I - A$ is accretive on $\mathcal{D}$. Furthermore, it follows easily that L[A] is the smallest number λ such that $\lambda I - A$ is accretive on $\mathcal{D}$. Using the notion of accretive operators, several results on the existence of solutions to differential equations have been obtained in Banach spaces whose dual space is uniformly convex (for example, see [3] and [8]). With this in mind we make the following definition:

Definition 2.10. For each subset $\mathcal{D}$ of E denote by $\mathit{ULn}(\mathcal{D},E)$ the class of all functions A from $\mathcal{D}$ into E having the following property: there is a number K such that for each bounded subset $\mathcal{Q}$ of $\mathcal{D}$ for which the image of $\mathcal{Q}$ under A is bounded, and for each pair of positive numbers β and ε, there is a positive number $\delta = \delta(\mathcal{Q},\beta,\varepsilon)$ such that

$$(|x-y + h[Ax-Ay]| - |x-y|)/h \le K|x-y| + \varepsilon$$

whenever $0 < h \le \delta$ and x and y are in $\mathcal{Q}$ with $|x-y| \ge \beta$. Denote by $L'[A]$ the smallest number K for which this inequality holds. If A is a member of $\mathit{ULn}(\mathcal{D},E)$ then A is said to have a uniform logarithmic derivative on $\mathcal{D}$ and $L'[A]$ is called the uniform logarithmic derivative of A on $\mathcal{D}$.

Remark 2.11. Suppose that A is in $\mathit{ULn}(\mathcal{D},E)$ and x and y are in $\mathcal{D}$ with $x \neq y$. By taking $\mathcal{Q} = \{x,y\}$ and $\beta = |x-y|$ in Definition 2.10 we have that

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h \le L'[A]|x-y|.$$

Consequently, A is in $\mathit{Ln}(\mathcal{D},E)$ and $L[A] \le L'[A]$. As in the proof of Proposition 2.5 one can show that if A is in $\mathit{Lip}(\mathcal{D},E)$ then A is in $\mathit{ULn}(\mathcal{D},E)$ and $L'[A] \le M[A]$. Thus we have the following sequence of set inclusions:

$$\mathit{Lip}(\mathcal{D},E) \subset \mathit{ULn}(\mathcal{D},E) \subset \mathit{Ln}(\mathcal{D},E).$$

Proposition 2.7. Suppose $\mathcal{D}$ is a subset of E and A and B are in $\mathcal{ULn}(\mathcal{D},E)$. Then

(i) For each $r > 0$, rA is in $\mathcal{ULn}(\mathcal{D},E)$ with $L'[rA] = rL'[A]$.

(ii) If A and B are bounded on a bounded subset $\mathcal{Q}$ of $\mathcal{D}$ whenever A + B is bounded on $\mathcal{Q}$, then A + B is in $\mathcal{ULn}(\mathcal{D},E)$ with $L'[A+B] \le L'[A] + L'[B]$.

(iii) For each a in K, A + aI is in $\mathcal{ULn}(\mathcal{D},E)$ with $L'[A+aI] = L'[A] + \mathrm{Re}(a)$.

The proof of this proposition is similar to the proof of the analogous parts of Lemma 1.2 and is omitted.

Proposition 2.8. If E^* is uniformly convex and $\mathcal{D}$ is a subset of E, then $\mathcal{Ln}(\mathcal{D},E) = \mathcal{ULn}(\mathcal{D},E)$ and if A is in $\mathcal{ULn}(\mathcal{D},E)$, then $L'[A] = L[A]$.

Proof. We have by Remark 2.11 that $\mathcal{ULn}(\mathcal{D},E) \subset \mathcal{Ln}(\mathcal{D},E)$ and $L[A] \le L'[A]$. Now suppose that A is in $\mathcal{Ln}(\mathcal{D},E)$ and $\mathcal{Q}$ is a bounded subset of $\mathcal{D}$ for which there is a constant Γ such that $|Ax| \le \Gamma$ for each x in $\mathcal{Q}$. Let β and ε be positive numbers and, by Lemma 1.4, choose a positive number δ such that if $0 < h \le \delta$ and x and y are in $\mathcal{Q}$ with $|x-y| \ge \beta$, then

$$(|x-y + h[Ax-Ay]| - |x-y|)/h \le \mathrm{Re}(Ax-Ay,g) + \varepsilon$$

for g in $G(x-y)$. From part (ii) of Proposition 2.3, $\mathrm{Re}(Ax-Ay,g) \le L[A]|x-y|$, and it follows that A is in $\mathcal{ULn}(\mathcal{D},E)$ with $L'[A] \le L[A]$. This completes the proof.

We now give an example to show that $\mathcal{ULn}(\mathcal{D},\mathcal{E})$ is not always equal to $\mathcal{Ln}(\mathcal{D},\mathcal{E})$.

Example 2.4. Let $\mathcal{E}$ denote the space of all continuous functions x from [0,2] into the real numbers such that $x(0) = x(2) = 0$, and, in this example, $|\cdot|_m$ denote the norm on $\mathcal{E}$ defined by $|x|_m = \max\{|x(t)|: t\in[0,2]\}$. Let $\mathcal{D}$ be the set of all x in $\mathcal{E}$ such that x' exists and x' is in $\mathcal{E}$. Define the function A from $\mathcal{D}$ into $\mathcal{E}$ by $Ax = x'$ for each x in $\mathcal{D}$. Let x be a nonzero member of $\mathcal{D}$ and for each $h > 0$ let $t(h)$ be a member of [0,2] such that $|x+hx'|_m = |x(t(h)) + hx'(t(h))|$. Since [0,2] is compact, let $(h_n)_1^\infty$ be a sequence of positive numbers such that $\lim_{n\to\infty} h_n = 0$ and there is a t_o in [0,2] such that $\lim_{n\to\infty} t(h_n) = t_o$. By the choice of $t(h_n)$, it is clear that $|x|_m = |x(t_o)|$ and hence, $x'(t_o) = 0$ and $|x(t(h_n))| \le |x(t_o)|$ for all $n \ge 1$. Thus,

$$\begin{aligned}
\lim_{h\to+0} (|x+hAx|_m - |x|_m)/h &= \lim_{n\to\infty}(|x(t(h_n)) + h_n x'(t(h_n))| - |x(t_o)|)/h_n \\
&\le \lim_{n\to\infty}(|x(t(h_n))|/h_n + |x'(t(h_n))| - |x(t_o)|/h_n) \\
&\le \lim_{n\to\infty}|x'(t(h_n))| \\
&= 0.
\end{aligned}$$

Since A is linear, then A is in $\mathcal{Ln}(\mathcal{D},\mathcal{E})$ with $L[A] \le 0$.

Now assume, for contradiction, that A is in $\mathcal{ULn}(\mathcal{D},\mathcal{E})$. Let $\Gamma = \max\{4|L'[A]|,4\}$ and let $\mathcal{Q} = \{x\in\mathcal{D} : |x|_m \le 2 \text{ and } |x'|_m \le \Gamma\}$.

Since $\mathcal{Q}$ is bounded and $|Ax| \le \Gamma$ for all x in $\mathcal{Q}$, there is a positive number δ less than one such that if x is in $\mathcal{Q}$ with $|x|_m \ge 1$ and $0 < h \le \delta$, then

$$(|x + hAx|_m - |x|_m)/h \le L'[A]|x|_m + 1/2.$$

Define the member x_δ of E as follows: $x_\delta(t) = \Gamma^2 t^2/2$ if t is in $[0,\Gamma^{-1})$; $x_\delta(t) = 1/2 + \Gamma(t-\Gamma^{-1})$ if t is in $[\Gamma^{-1},3\Gamma^{-1}/2)$; $x_\delta(t) = 1 + \delta\sin(\Gamma\delta^{-1}(t-3\Gamma^{-1}/2))$ if t is in $[3\Gamma^{-1}/2,3\Gamma^{-1}/2 + \pi\delta\Gamma^{-1}/2)$; $x_\delta(t) = 1 + \delta$ if t is in $[3\Gamma^{-1}/2 + \pi\delta\Gamma^{-1}/2,1]$; and $x_\delta(t) = x_\delta(2-t)$ if t is in $(1,2]$. Then x_δ is in $\mathcal{Q}$ with $|x_\delta|_m = 1 + \delta$. Thus, by the choice of δ,

$$(|x_\delta + \delta x_\delta'|_m - |x_\delta|_m)/\delta \le L'[A]|x_\delta|_m + 1/2.$$

Furthermore, since $|x_\delta + \delta x_\delta'|_m \ge |x_\delta(3\Gamma^{-1}/2) + \delta x_\delta'(3\Gamma^{-1}/2)| = 1 + \delta\Gamma$ and $|x_\delta|_m = 1 + \delta$, we have

$$\begin{aligned}(|x_\delta+\delta x_\delta'|_m - |x_\delta|_m)/\delta &\ge (1+\delta\Gamma-1-\delta)/\delta \\ &= \Gamma - 1.\end{aligned}$$

Since $\Gamma \ge 4|L'[A]|$ and $|x_\delta|_m \le 2$ we have that $\Gamma - 1 \le L'[A]|x_\delta|_m + 1/2 \le \Gamma/2 + 1/2$. But this implies that $\Gamma/2 \le 3/2$ which is impossible since $\Gamma \ge 4$. This contradiction shows that A is not in $\mathcal{ULn}(\mathcal{D},E)$, and so, in this case, $\mathcal{ULn}(\mathcal{D},E) \ne \mathcal{Ln}(\mathcal{D},E)$.

Remark 2.12. The example above shows that there is a Banach space E, a subset $\mathcal{D}$ of E, and a discontinuous function A from $\mathcal{D}$ into E which is in $Ln(\mathcal{D},E)$ but not in $ULn(\mathcal{D},E)$. The author does not know of an example of a continuous member A of $Ln(\mathcal{D},E)$ which is not in $ULn(\mathcal{D},E)$. However, it will be proved (see Proposition 6.3) that if A is uniformly continuous on bounded subsets of E and A is in $Ln(E,E)$, then A is in $ULn(E,E)$ and $L'[A] = L[A]$.

The spaces $BL(\mathcal{D},E)$ and $Lip(\mathcal{D},E)$ are well-known although the definition of the logarithmic norm $M[\cdot]$ on $Lip(\mathcal{D},E)$ seems to be new. As a consequence of Proposition 2.6, we have that the space $Ln(\mathcal{D},E)$ consists precisely of all functions A from $\mathcal{D}$ into E for which there is a number λ such that $\lambda I - A$ is accretive on $\mathcal{D}$. The notion of accretive operators is well-known, but the limit characterization given here seems to be new. However, in the case that A is linear, Lumer and Phillips [13, Lemma 3.2] give a similar characterization. The limit characterization of the space $ULn(\mathcal{D},E)$ seems to be new and this will be used to prove some existence theorems for differential equations which have previously been proved under the assumption that E is a Hilbert space or that the dual space E^* is uniformly convex.

Remark 2.13. We have that if $\mathcal{D}$ is a subset of E then $Lip(\mathcal{D},E) \subset ULn(\mathcal{D},E) \subset Ln(\mathcal{D},E)$ and that proper containment can occur. We also have that if A is in $Lip(\mathcal{D},E)$ then $M[A] \geq L'[A] \geq L[A]$. The author does not know if $M[A] = L[A]$ in general. However, if $\mathcal{D}$ is a subspace of E and A is in $BL(\mathcal{D},E)$ then Lumer [12, Lemma 12] shows that $\mu[A] = L[A]$. One can then show that if $\mathcal{D}$ is an open convex subset of

E and A is a Fréchet differentiable member of $Lip(D,E)$ then M[A] = L[A].

CHAPTER III

COMPUTATION OF THE LOGARITHMIC NORM

In this chapter we establish some procedures for the computation of the logarithmic norm. These are used both to illustrate some of the applications of the methods developed here and to connect some of these results to those of others.

Let $|\cdot|_o$ be a norm on the vector space $\mathcal{E}$ which is equivalent to the norm $|\cdot|$ on $\mathcal{E}$ and let a_o and b_o be positive numbers such that $a_o|x|_o \le |x| \le b_o|x|_o$ for all x in $\mathcal{E}$. If $\mathcal{D}$ is a subset of $\mathcal{E}$ and A is a function from $\mathcal{D}$ into $\mathcal{E}$, then $|Ax-Ay|_o/|x-y|_o \le b_o a_o^{-1}|Ax-Ay|/|x-y|$ so that E, equipped with the norm $|\cdot|_o$, generates the same classes $BL(\mathcal{D},\mathcal{E})$ and $Lip(\mathcal{D},\mathcal{E})$ as does E equipped with the norm $|\cdot|$. If for each A in $Lip(\mathcal{D},\mathcal{E})$

$$N_o[A] = \sup\{|Ax-Ay|_o/|x-y|_o \ : \ x,y\in D,\ x\neq y\},$$

then N_o is said to be induced by the norm $|\cdot|_o$. If

$$M_o[A] = \lim_{h\to+0} (N_o[I+hA] - 1)/h$$

for each A in $Lip(\mathcal{D},\mathcal{E})$, then M_o is said to be induced by the norm $|\cdot|_o$. Analogous definitions apply to $\|\cdot\|_o$ and $\mu_o[\cdot]$ on the space $BL(\mathcal{D},\mathcal{E})$.

Note that if A is in $Lip(\mathcal{D},E)$ then $a_o b_o^{-1} N[A] \le N_o[A] \le b_o a_o^{-1} N[A]$, so that the seminorms $N[\cdot]$ and $N_o[\cdot]$ are equivalent seminorms on the vector space $Lip(\mathcal{D},E)$.

Example 3.1. Suppose that Q is an invertible member of $BL(E,E)$ and, for each x in E, let $|x|_Q = |Qx|$. It is easy to check that $|\cdot|_Q$ is a norm on E and since $\|Q\|^{-1}|x|_Q \le |x| \le \|Q^{-1}\|\,|x|_Q$ for each x in E, $|\cdot|_Q$ is equivalent to $|\cdot|$. If $\|\cdot\|_Q$ and $\mu_Q[\cdot]$ are induced by the norm $|\cdot|_Q$ and A is in $BL(E,E)$ then

$$\|A\|_Q = \sup\{|Ax|_Q : |x|_Q = 1\}$$

$$= \sup\{|Q\cdot A\cdot Q^{-1}y| : |y| = 1\}$$

$$= \|Q\cdot A\cdot Q^{-1}\|,$$

and hence,

$$\mu_Q[A] = \lim_{h\to+0} (\|I+hQ\cdot A\cdot Q^{-1}\| - 1)/h$$

$$= \mu[Q\cdot A\cdot Q^{-1}].$$

Example 3.2. Suppose the Q and $|\cdot|_Q$ are as in Example 3.1, $\mathcal{D}$ is an open convex subset of E, and A is a Fréchet differentiable member of $Lip(\mathcal{D},E)$. If N_Q and M_Q are induced by the norm $|\cdot|_Q$, then by Proposition 2.2 and Example 3.1

$$N_Q[A] = \sup\{\|Q\cdot dA(x)\cdot Q^{-1}\| \ : \ x\in\mathcal{D}\}$$

and

$$M_Q[A] = \sup\{\mu[Q\cdot dA(x)\cdot Q^{-1}] \ : \ x\in\mathcal{D}\}.$$

Example 3.3. Suppose that n is a positive integer and E is the vector space K^n of column vectors $(\xi_k)_1^n$ where each ξ_k is in K. Associate the vector space $BL(K^n, K^n)$ with the nxn matrices with entries in K. With the following norms on K^n, Lozinskii [11, Lemma 4] derives formulas for computing $\|A\|$ and $\mu[A]$ where $A = (a_{ij})$ is an nxn matrix and a_{ij} is in K.

(i) If $|(\xi_k)_1^n|_1 = \max\{|\xi_k| : 1 \le k \le n\}$ and $\|\cdot\|_1$ and $\mu_1[\cdot]$ are induced by $|\cdot|_1$ then $\|A\|_1 = \max\{\sum_{k=1}^{n} |a_{ik}| : 1 \le i \le n\}$ and $\mu_1[A] = \max\{\mathrm{Re}(a_{ii}) + \sum_{k\neq i} |a_{ik}| : 1 \le i \le n\}$.

(ii) If $|(\xi_k)_1^n|_2 = \sum_{k=1}^{n} |\xi_k|$ and $\|\cdot\|_2$ and $\mu_2[\cdot]$ are induced by $|\cdot|_2$ then $\|A\|_2 = \max\{\sum_{k=1}^{n} |a_{kj}| : 1 \le j \le n\}$ and $\mu_2[A] = \max\{\mathrm{Re}(a_{jj}) + \sum_{k\neq j} |a_{kj}| : 1 \le j \le n\}$.

(iii) If $|(\xi_k)_1^n|_3 = \{\sum_{k=1}^{n} |\xi_k|^2\}^{1/2}$ and $\|\cdot\|_3$ and $\mu_3[\cdot]$ are induced by $|\cdot|_3$ then $\|A\|_3 = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A\cdot A^*\}$ and $\mu_3[A] = \max\{\lambda/2 : \lambda \text{ is an eigenvalue of } A + A^*\}$. (Here A^* is the adjoint of A--i.e. $A^* = (b_{ij})$ where $b_{ij} = \bar{a}_{ji}$).

Example 3.4. As in Example 3.3, let $E = K^n$ and suppose that $\mathcal{D}$ is an open convex subset of K^n and A is a Fréchet differentiable member of $Lip(\mathcal{D},K^n)$. For each integer k in [1,n] let A_k denote the function from $\mathcal{D}$ into K such that $Ax = (A_k x)_1^n$ for each x in $\mathcal{D}$. Since A is Fréchet differentiable on $\mathcal{D}$, for each $x = (\xi_k)_1^n$ and each integer i in [1,n], the partial of A_k with respect to ξ_i at x--denoted $d_i A_k(x)$--exists and dA(x) is associated with the matrix $(d_i A_j(x))$. If $|\cdot|_1$ is the norm on K^n which is defined in part (i) of Example 3.3, and N_1 and M_1 are induced by $|\cdot|_1$, then by Proposition 2.2 and Example 3.3

$$N_1[A] = \sup\{\max\{\sum_{k=1}^{n} |d_i A_k(x)| \ : 1 \le i \le n\} : x \in \mathcal{D}\}$$

and

$$M_1[A] = \sup\{\max\{\mathrm{Re}(d_i A_i(x)) + \sum_{k \ne i} |d_i A_k(x)| \ : 1 \le i \le n\} : x \in \mathcal{D}\}$$

Analogous formulas hold for the norms $|\cdot|_2$ and $|\cdot|_3$ defined in parts (ii) and (iii) of Example 3.3.

Example 3.5. Suppose that K is the field of real numbers, $E = K^2$, and, for notational convenience, let (ξ_1,ξ_2) denote the member $(\xi_k)_1^2$ of K^2. If A is the function from K^2 into K^2 defined by $A(\xi_1,\xi_2) = (-2\xi_1 + \cos(\xi_2), \sin^2(\xi_1) - \xi_2)$ for each (ξ_1,ξ_2) in K^2, then A is Fréchet differentiable on K^2 and $dA(\xi_1,\xi_2)$ is associated with the matrix

$$\begin{bmatrix} -2 & -\sin(\xi_2) \\ \sin(2\xi_1) & -1 \end{bmatrix}.$$

Since dA is bounded on K^2, A is in $Lip(K^2,K^2)$ by Proposition 2.2. Let Q be the member of $BL(K^2,K^2)$ such that $Q(\xi_1,\xi_2) = (\xi_1,2\xi_2/3)$ for each (ξ_1,ξ_2) in K^2. One easily sees that $Q\cdot dA(\xi_1,\xi_2)\cdot Q^{-1}$ is associated with the matrix

$$\begin{bmatrix} -2 & -3\sin(\xi_2)/2 \\ 2\sin(2\xi_1)/3 & -1 \end{bmatrix}.$$

Consequently, if $|(\xi_1,\xi_2)|_1 = \max\{|\xi_1|,|\xi_2|\}$, $|(\xi_1,\xi_2)|_Q = |Q(\xi_1,\xi_2)|_1$, $\mu_1[\cdot]$ is induced by $|\cdot|_1$, and M_Q is induced by $|\cdot|_Q$, then by Example 3.2

$$M_Q[A] = \sup\{\mu_1[Q\cdot dA(\xi_1,\xi_2)\cdot Q^{-1}] : (\xi_1,\xi_2)\epsilon K^2\}.$$

By part (i) of Example 3.3,

$$\mu_1[Q\cdot dA(\xi_1,\xi_2)\cdot Q^{-1}] = \max\{-2 + |3\sin(\xi_2)/2|, -1 + |2\sin(\xi_1)/3|\}$$

$$\leq -1/3.$$

Hence $M_Q[A] \leq -1/3$ and it follows from Corollary 2.3 that A is a bijection, A^{-1} is in $Lip(K^2,K^2)$, and $N_Q[A^{-1}] \leq 3$.

If A is in $BL(E,E)$ the spectrum of A--denoted $\sigma(A)$--is the set of all members λ of K such that $(\lambda I-A)^{-1}$ is not a member of $BL(E,E)$. For the remainder of this chapter we will be interested in the case when E is a Hilbert space. For notational convenience we suppose that H is a Hilbert space over the field K and if x and y are in H, (x,y) denotes the inner product of x with y. If A is in $BL(H,H)$ the adjoint of A--denoted A^*--is defined by the relation $(Ax,y) = (x,A^*y)$ for each x and y in H.

Proposition 3.1. If A is a member of $BL(H,H)$ then

(i) A^* is in $BL(H,H)$ with $\|A^*\| = \|A\|$ and $\mu[A^*] = \mu[A]$.

(ii) $\|A\| = \|A\cdot A^*\|^{1/2} = \sup\{\sqrt{\lambda} : \lambda\epsilon\sigma(A\cdot A^*)\}$.

(iii) $\mu[A] = \sup\{\lambda/2 : \lambda\epsilon\sigma(A+A^*)\}$.

(iv) $\mu[A+A^*] = \mu[A] + \mu[A^*]$.

Proof. A proof of part (ii) and the fact that A^* is in $BL(H,H)$ with $\|A^*\| = \|A\|$ can be found in [20, pp. 250 and 331]. The fact that $\mu[A^*] = \mu[A]$ follows immediately from part (iii) since $A^{**} = A$. If h is a positive number, we have from part (ii) that

$$\|I+hA\|^2 = \|I + h(A+A^*) + h^2A\cdot A^*\|$$

$$= 1 + h \sup\{\lambda : \lambda\epsilon\sigma(A+A^* + hA\cdot A^*)\}.$$

Hence

$$(\|I+hA\| - 1)/h = (\|I+hA\| + 1)^{-1}(\|I+hA\|^2 - 1)/h$$

$$= (\|I+hA\| + 1)^{-1}\sup\{\lambda : \lambda\in\sigma(A+A^* + hA\cdot A^*)\},$$

and part (iii) is established by letting $h\to+0$. Part (iv) is immediate from part (iii).

A member A of $BL(H,H)$ is said to be self-adjoint if $A = A^*$. If A is a self-adjoint member of $BL(H,H)$ and λ is in $\sigma(A)$ then λ is real, (Ax,x) is real for each x in H, and if $\gamma = \inf\{(Ax,x) : |x| = 1\}$ and $\Gamma = \sup\{(Ax,x) : |x| = 1\}$, then $\gamma \le \lambda \le \Gamma$. Furthermore, γ and Γ are in $\sigma(A)$ (see [20, p. 330, Theorem 6.2-B]), and $\|A\| = \max\{|\gamma|,|\Gamma|\}$ (see [20, p. 325, Theorem 6.11-C]). Since $A = A^*$, it follows easily from part (iii) of Proposition 3.1 that $\mu[A] = \Gamma$ and $-\mu[-A] = \gamma$. A member P of $BL(H,H)$ is said to be positive definite self-adjoint if P is self-adjoint and if $\inf\{(Px,x) : |x| = 1\} > 0$ (i.e. if $-\mu[-P] > 0$). If P is a positive definite self-adjoint member of $BL(H,H)$, then there is a unique positive definite self-adjoint member S of $BL(H,H)$ such that $S^2 = P$ (see [19, p. 265]). Furthermore, both P and S are invertible members of $BL(H,H)$, and P^{-1} and S^{-1} are positive definite self-adjoint with $S^{-2} = P^{-1}$. Note also from part (ii) of Proposition 3.1, $\|S\|^2 = \|P\|$ and $\|S^{-1}\|^2 = \|P^{-1}\|$.

Example 3.6. Suppose that P and S are positive definite self-adjoint members of $BL(H,H)$ such that $S^2 = P$. For each x and y in H define $(x,y)_S = (Sx,Sy) = (Px,y)$. This is an inner product on H and if $|\cdot|_S$ is the norm on H induced by this inner product (i.e. $|x|_S = \sqrt{(x,x)_S}$) then

$|x|_S = |Sx|$ for each x in H. Thus, by Example 3.1, if $\|\cdot\|_S$ and $\mu_S[\cdot]$ are induced by $|\cdot|_S$ then $\|A\|_S = \|S\cdot A\cdot S^{-1}\|$ and $\mu_S[A] = \mu[S\cdot A\cdot S^{-1}]$ for each A in BL(H,H).

Proposition 3.2. Suppose that P and S are as in Example 3.6, A is in $BL(H,H)$, and $\Gamma = \sup\{\lambda : \lambda\epsilon\sigma(P\cdot A + A^{*}\cdot P)\}$. Then $\mu_S[A] \leq \Gamma\|P^{-1}\|/2$ if $\Gamma \geq 0$, and $\mu_S[A] \leq \Gamma\|P\|^{-1}/2$ if $\Gamma \leq 0$.

Proof. By Proposition 3.2 and Example 3.6,

$$\mu_S[A] = \mu[S\cdot A\cdot S^{-1}]$$

$$= \sup\{\lambda/2 : \lambda\epsilon\sigma(S\cdot A\cdot S^{-1} + S^{-1}\cdot A^{*}\cdot S)\}.$$

Furthermore, if $z(x) = S^{-1}x/|S^{-1}x|$ for each x in H with $|x| = 1$ and λ is in $\sigma(S\cdot A\cdot S^{-1} + S^{-1}\cdot A^{*}\cdot S)$, then

$$\lambda \leq \sup\{([S\cdot A\cdot S^{-1} + S^{-1}\cdot A^{*}\cdot S]x,x) : |x| = 1\}$$

$$= \sup\{(S^{-1}x,S^{-1}x)([P\cdot A + A^{*}\cdot P]z(x),z(x)) : |x| = 1\}.$$

Since $(S^{-1}x,S^{-1}x) = |S^{-1}x|^2$ and $\|S^{-1}\|^2 = \|P^{-1}\|$, we have $\|P\|^{-1} \leq |S^{-1}x|^2 \leq \|P^{-1}\|$ for all x in H with $|x| = 1$. Since

$$\Gamma = \sup\{[P\cdot A + A^{*}P]z(x),z(x)) : |x| = 1\},$$

it follows that if $\Gamma > 0$ then $\lambda \leq \Gamma\|P^{-1}\|$, and if $\Gamma \leq 0$, then $\lambda \leq \Gamma\|P\|^{-1}$, and the proposition is true.

Example 3.7. Consider the vector space K^2 as defined in Example 3.3, and if $x = (\xi_1,\xi_2)$ and $y = (\eta_1,\eta_2)$ define $(x,y) = \xi_1\eta_1 + \xi_2\eta_2$. Define the 2x2 matrices A, P, and S as follows:

$$A = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix}; \; P = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}; \text{ and } S = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then P and S are positive definite, self-adjoint, and $S^2 = P$. Furthermore, $\mu[A] = 1$, the largest eigenvalue of $P\cdot A + A^*P$ is $4\sqrt{5} - 10$, and $\mu_S[A] = -1/3$.

Example 3.8. Suppose that $\mathcal{D}$ is an open convex subset of the Hilbert space H and A is a Fréchet differentiable member of $Lip(\mathcal{D},H)$. Suppose further that P and S are positive definite self-adjoint members of $BL(H,H)$ such that $S^2 = P$. As in Example 3.6 let $|\cdot|_S$ be the norm on H defined by $|x|_S = |Sx|$ for each x in H. For each x in $\mathcal{D}$ let

$$\Gamma = \sup\{\lambda : \lambda\epsilon\sigma(P\cdot dA(x) + dA(x)^*\cdot P) \text{ and } x\epsilon\mathcal{D}\}.$$

By Proposition 2.2 Γ is finite, and by Propositions 2.2 and 3.2, if M_S is induced by $|\cdot|_S$, then $M_S[A] \leq \Gamma\|P^{-1}\|/2$ if $\Gamma \geq 0$ and $M_S[A] \leq \Gamma\|P\|^{-1}/2$ if $\Gamma \leq 0$.

CHAPTER IV

SOME BASIC DEFINITIONS AND LEMMAS

In this chapter we develop a sequence of definitions and lemmas which are frequently used in establishing existence and stability theorems for differential equations. Most of the lemmas given here are well-known, and those which have long or complicated proofs will be referenced.

Definition 4.1. A sequence $(x_n)_1^\infty$ in E is said to converge weakly to a member x in E if $\lim_{n\to\infty} (x_n,f) = (x,f)$ for each f in E^*. In this case we write $\text{w-}\lim_{n\to\infty} x_n = x$.

Definition 4.2. Suppose that [a,b] is an interval and u is a function from [a,b] into E. Then u is said to be weakly continuous on [a,b] if $\text{w-}\lim_{s\to t} u(s) = u(t)$ for each t in [a,b]. The function u is said to be weakly differentiable on [a,b] if for each t in [a,b] there is a $u'(t)$ such that $\text{w-}\lim_{h\to 0} (u(t+h) - u(t))/h = u'(t)$. If, in addition, the function $t \to u'(t)$ of [a,b] into E is weakly continuous, then u is said to be weakly continuously differentiable on [a,b].

Remark 4.1. Note that if $\lim_{n\to\infty} x_n = x$ then $\text{w-}\lim_{n\to\infty} x_n = x$. Consequently, if u is continuous, differentiable, or continuously differentiable on [a,b] then u is weakly continuous, weakly differentiable, or weakly continuously differentiable on [a,b], respectively.

Some of the theory of Bochner integration will be needed and the reader is referred to [7, pp. 78-88] or [22, pp. 132-136] for a discussion of this theory. A list of the lemmas which will be needed is given below.

Let q be a function from the interval $[a,b]$ into E. Then q is said to be finitely-valued if there is a finite family $\{B_k : 1 \le k \le n\}$ of mutually disjoint measurable subsets of $[a,b]$ and a finite family $\{x_k : 1 \le k \le n\}$ of members of E such that $q(t) = x_k$ for each t in B_k and $q(t) = 0$ for each t not in $\bigcup_k B_k$. The Bochner integral of q over $[a,b]$ is defined as

$$(B) \int_a^b q(s)ds = \sum_{k=1}^{n} m(B_k)x_k$$

where $m(B_k)$ denotes the Lebesque measure of B_k. A function v from $[a,b]$ into E is said to be Bochner integrable on $[a,b]$ if there is a sequence $(q_n)_1^\infty$ of finitely-valued functions on $[a,b]$ such that $\lim_{n\to\infty} q_n(t) = v(t)$ for almost all t in $[a,b]$ and $\lim_{n\to\infty} \int_a^b |v(s)-q_n(s)|ds = 0$. The Bochner integral of v on $[a,b]$ is defined as

$$(B) \int_a^b v(s)ds = \lim_{n\to\infty} (B) \int_a^b q_n(s)ds.$$

Lemma 4.1. If v is a Bochner integrable function on $[a,b]$ then $|v|$ in Lebesque integrable on $[a,b]$ and

$$|(B) \int_a^b v(s)ds| \le \int_a^b |v(s)|ds.$$

(see [7, Theorems 3.7.4 and 3.7.6]).

Lemma 4.2. If $(v_n)_1^\infty$ is a sequence of Bochner integrable functions on $[a,b]$ such that $v(t) = \lim_{n\to\infty} v_n(t)$ for almost all t in $[a,b]$ and there is a Lebesque integrable function p on $[a,b]$ such that $|v_n(t)| \le p(t)$ for each $n \ge 1$ and almost all t in $[a,b]$, then v is Bochner integrable on $[a,b]$ and

$$(B)\int_a^b v(s)ds = \lim_{n\to\infty} (B)\int_a^b v_n(s)ds.$$

(see [7, Theorem 3.7.9]).

Lemma 4.3. If v is a Bochner integrable function on $[a,b]$ and $u(t) = (B)\int_a^t v(s)ds$ for each t in $[a,b]$, then for almost all t in $[a,b]$ $u'(t)$ exists and equals $v(t)$. (see [7, Theorem 3.7.11 and Corollary 2]).

Lemma 4.4. Suppose that u is a Lipschitz continuous function from $[a,b]$ into E which has a weak derivative almost everywhere on $[a,b]$. Then u is differentiable almost everywhere, u' is Bochner integrable on $[a,b]$ and

$$u(t) = u(a) + (B)\int_a^t u'(s)ds$$

for all t in $[a,b]$ (see [7, Theorem 3.8.6]).

Lemma 4.5. Suppose that q is a function from $[a,b]$ into E and $p(t) = |q(t)|$ for each t in $[a,b]$. Then

(i) if $q'_+(t)$ exists then $p'_+(t)$ exists and

$$p'_+(t) = \lim_{h\to+0} (|q(t)+hq'_+(t)| - |q(t)|)/h; \text{ and}$$

(ii) if $q'_-(t)$ exists then $p'_-(t)$ exists and

$$p'_-(t) = \lim_{h\to-0} (|q(t)+hq'_-(t)| - |q(t)|)/h.$$

Proof. The existence of each of these limits follows from Lemma 1.1. If $q'_+(t)$ exists and $h > 0$ is such that $t + h$ is in $[a,b]$ then

$$|[|q(t+h)| - |q(t)|]/h - [|q(t)+hq'_+(t)| - |q(t)|]/h|$$

$$= |[|q(t+h)| - |q(t) + hq'(t)|]/h|$$

$$\le |[q(t+h) - q(t)]/h - q'_+(t)|$$

and part (i) follows by letting $h\to+0$. Part (ii) is proved analogously.

Lemma 4.6. Suppose that u is a continuous function from $[a,b]$ into E which is differentiable almost everywhere on $[a,b]$. Suppose further that $|u|$ is absolutely continuous on $[a,b]$ and there are Lebesque integrable, real valued functions η and γ on $[a,b]$ such that if $p(t) = |u(t)|$ for each t in $[a,b]$ then either

(i) $p'_+(t) \le \eta(t)p(t) + \gamma(t)$ for almost all t in $[a,b]$, or

(ii) $p'_-(t) \le \eta(t)p(t) + \gamma(t)$ for almost all t in $[a,b]$.

It follows that

$$p(t) \le p(a)\exp\left(\int_a^t \eta(s)ds\right) + \int_a^t \gamma(s)\exp\left(\int_s^t \eta(r)dr\right)ds$$

for each t in [a,b].

Proof. If $q(t) = p(t)\exp(-\int_a^t \eta(s)ds)$ then q is absolutely continuous on [a,b] so that $q'(t)$ exists almost everywhere and $q(t) = q(a) + \int_a^t q'(s)ds$ for each t in [a,b]. Suppose that (i) is true. Then for almost all s in [a,b]

$$\begin{aligned} q'(s) &= q'_+(s) \\ &= [p'_+(s) - \eta(s)p(s)]\exp\left(-\int_a^s \eta(r)dr\right) \\ &\le \gamma(s)\exp\left(-\int_a^s \eta(r)dr\right). \end{aligned}$$

Consequently, for each t in [a,b],

$$p(t)\exp\left(-\int_a^t \eta(s)ds\right) \le p(a) + \int_a^t \gamma(s)\exp\left(-\int_a^s \eta(r)dr\right)ds$$

and the assertion of the lemma when (i) holds follows. The proof when (ii) holds is similar.

Remark 4.2. Note that if u is Lipschitz continuous on [a,b] then $|u|$ is absolutely continuous.

Definition 4.3. Suppose that X is a metric space with metric d and A is a function from X into E. The function A is said to be demicontinuous on X if for each x in X and each sequence $(x_n)_1^\infty$ in X such that $\lim_{n\to\infty} d(x,x_n) = 0$, $w\text{-}\lim_{n\to\infty} Ax_n = Ax$.

Definition 4.4. Suppose that X and X' are metric spaces with metrics d and d', respectively, S is a set, and $\{A_\sigma : \sigma\in S\}$ is a family of functions from X into X'. The family $\{A_\sigma : \sigma\in S\}$ is said to be equicontinuous on X if for each $\epsilon > 0$ and each x in X, there is a positive number $\delta = \delta(x,\varepsilon)$ such that if y is in X with $d(y,x) \le \delta$, then $d'(A_\sigma y, A_\sigma x) \le \varepsilon$ for all σ in S. If δ is independent of x in X, the family $\{A_\sigma : \sigma\in S\}$ is said to be uniformly equicontinuous on X.

Definition 4.5. Suppose that $\mathcal{D}$ is a subset of E, S is a set, and $\{A_\sigma : \sigma\in S\}$ is a family of functions from $\mathcal{D}$ into E. The family $\{A_\sigma : \sigma\in S\}$ is said to have an equiuniform logarithmic derivative on $\mathcal{D}$ if there are numbers M and Λ' such that $|A_\sigma x| \le M$ for all σ in S and x in $\mathcal{D}$ and, for each pair of positive numbers β and ε, there is a positive number $\delta = \delta(\beta,\varepsilon)$ such that

$$(|x-y+h[A_\sigma x - A_\sigma y]| - |x-y|)/h \le \Lambda'|x-y| + \varepsilon$$

whenever $0 < h \le \delta$, σ is in S, and x and y are in $\mathcal{D}$ with $|x-y| \ge \beta$.

Remark 4.3. Note that if the family $\{A_\sigma : \sigma\in S\}$ has equiuniform logarithmic derivative on $\mathcal{D}$ and Λ' is as in Definition 4.5, then A_σ is in $ULn(\mathcal{D},E)$ with $L'[A_\sigma] \le \Lambda'$ for all σ in S. Furthermore, if S is finite and, for each σ in S, A_σ is in $ULn(\mathcal{D},E)$ and bounded on $\mathcal{D}$, then the family $\{A_\sigma : \sigma\epsilon\mathcal{S}\}$ has equiuniform logarithmic derivative on $\mathcal{D}$, and Λ' can be taken as $\max\{L'[A_\sigma] : \sigma\epsilon\mathcal{S}\}$.

CHAPTER V

EXISTENCE AND UNIQUENESS THEOREMS FOR DIFFERENTIAL EQUATIONS

Suppose that $[a,b]$ is an interval, $\mathcal{D}$ is an open subset of E, and $\{A(t) : t\epsilon[a,b]\}$ is a family of functions from $\mathcal{D}$ into E. In this chapter we give sufficient conditions to insure that the initial value problem

$$\text{(IVP)} \qquad u'(t) = A(t)u(t), \quad u(a) = z, \; z\epsilon\mathcal{D}$$

has a unique solution on some subinterval $[a,c]$ of $[a,b]$, and also to insure that the solution can be extended to $[a,b]$. We are interested in three notions of solution to (IVP) which are defined as follows:

Definition 5.1. Suppose that $[a,c]$ is a subinterval of $[a,b]$ and u is a Lipschitz continuous function from $[a,c]$ into $\mathcal{D}$ such that $u(a) = z$. Then

(i) u is said to be a solution in the usual sense to (IVP) on $[a,c]$ if u is continuously differentiable and $u'(t) = A(t)u(t)$ for all t in $[a,c]$.

(ii) u is said to be a solution in the weak sense to (IVP) on $[a,c]$ if u is weakly continuously differentiable and $u'(t) = A(t)u(t)$ for all t in $[a,c]$.

(iii) u is said to be a solution in the extended sense to (IVP) on $[a,c]$ if the function $t \to A(t)u(t)$ is Bochner integrable on $[a,c]$ and

$$u(t) = z + (B) \int_a^t A(s)u(s)ds$$

for all t in $[a,c]$.

Theorem 5.1. Suppose that $\mathcal{D}$ is an open subset of E and $\{A(t) : t\epsilon[a,b]\}$ is a family of functions from $\mathcal{D}$ into E which satisfies each of the following conditions:

(i) There is a number M such that $|A(t)x| \leq M$ for all (t,x) in $[a,b]x\mathcal{D}$.

(ii) For each x in $\mathcal{D}$ the function $t \to A(t)x$ is Bochner integrable on $[a,b]$.

(iii) The function $(t,x) \to A(t)x$ is demicontinuous from $[a,b]x\mathcal{D}$ into E.

(iv) The family $\{A(t) : t\epsilon[a,b]\}$ has equiuniform logarithmic derivative on $\mathcal{D}$.

Then for each z in $\mathcal{D}$ there is a positive number $\rho = \rho(z)$ and a unique function u from $[a,a+\rho]$ which is a solution to (IVP) in the extended sense on $[a,a+\rho]$.

The proof of this theorem will be given by a sequence of Lemmas each of which is with the suppositions of Theorem 5.1. Let z be in $\mathcal{D}$ and let $0 < \rho < b-a$ be sufficiently small so that if x is in E and $|x-z| \leq \rho M$, then x is in $\mathcal{D}$. Also, for each positive integer n let

$(t_i^n)_{i=1}^{\lambda(n)}$ be a partition of $[a,a+\rho]$ such that $|t_{i+1}^n - t_i^n| \le n^{-1}$ for each integer i in $[0,\lambda(n)-1]$.

Lemma 5.1. For each $n \ge 1$ there is a function u_n from $[a,a+\rho]$ into $\mathcal{D}$ satisfying each of the following:

(i) $u_n(a) = z$.

(ii) $|u_n(t)-u_n(s)| \le M|t-s|$ for all t and s in $[a,a+\rho]$.

(iii) If $0 \le i \le \lambda(n)-1$, then for almost all t in $[t_i^n, t_{i+1}^n)$, $u_n'(t)$ exists and equals $A(t)u_n(t_i^n)$.

(iv) u_n' is Bochner integrable on $[a,a+\rho]$ and

$$u_n(t) = z + (B)\int_a^t u_n'(s)ds$$

for each t in $[a,a+\rho]$.

Proof. Let $u_n(a) = z$ and for each t in $[a,t_1^n]$ define

$$u_n(t) = z + (B)\int_a^t A(s)zds.$$

Inductively, for each integer i in $[1,\lambda(n)-1]$ and for each t in $[t_i^n, t_{i+1}^n]$ define

$$u_n(t) = u_n(t_i^n) + (B)\int_{t_i^n}^t A(s)u_n(t_i^n)ds.$$

The assertions of the lemma now follow in a routine manner from Lemma 4.3.

Lemma 5.2. The sequence $(u_n)_1^\infty$ constructed in Lemma 5.1 is uniformly Cauchy on $[a,a+\rho]$.

Proof. Since the family $\{A(t) : t\epsilon[a,b]\}$ has equiuniform logarithmic derivative on $\mathcal{D}$, let Λ' be as in Definition 4.5. We can assume without loss that $\Lambda' > 0$. Now let ε be a positive number. For the pair $\beta' = \varepsilon\exp(-\Lambda'\rho)/4$ and $\varepsilon' = \varepsilon\exp(-\Lambda'\rho)/[4(\rho+1)]$ there is, by condition (iv), a positive number $\delta = \delta(\beta',\varepsilon') = \delta(\varepsilon)$ such that if $0 < h \leq \delta$, t is in $[a,a+\rho]$, and x and y are in $\mathcal{D}$ with $|x-y| \geq \beta'$, then

$$(5.1) \qquad (|x-y+h[A(t)x-A(t)y]| - |x-y|)/h \leq \Lambda'|x-y| + \varepsilon'.$$

Now choose a positive integer n_o such that

$$(5.2) \qquad n_o^{-1} \leq \min\{\beta'/(4M),\ \varepsilon\exp(-\Lambda'\rho)/[4(\rho+1)(2\Lambda'M + 4M\delta^{-1})]\}$$

Note that n_o depends only on ε, Λ', M, and ρ. The claim is that whenever $n > m \geq n_o$ and t is in $[a,a+\rho]$, then $|u_n(t)-u_m(t)| \leq \varepsilon$. Assume, for contradiction, that there is a T_1 in $[a,a+\rho]$ and integers n and m such that $n > m \geq n_o$ and

$$(5.3) \qquad |u_n(T_1)-u_m(T_1)| > \varepsilon.$$

Let $p(t) = |u_n(t)-u_m(t)|$ for each t in $[a,a+\rho]$. Then p is continuous, $p(a) = 0$, and $p(T_1) > \varepsilon > 2\beta'$ so there is a number T_o in (a,T_1) such that $p(T_o) = 2\beta'$ and $p(t) \geq 2\beta'$ for all t in $[T_o,T_1]$. Thus, by part

(iii) of Lemma 5.1, if t is in $[T_0,T_1]$ and $u_n'(t)$ and $u_m'(t)$ exist, then there is an integer i in $[0,\lambda(n)-1]$ and an integer j in $[0,\lambda(m)-1]$ such that t is in $[t_i^n,t_{i+1}^n)$, t is in $[t_j^m,t_{j+1}^m)$, $u_n'(t) = A(t)u_n(t_i^n)$, and $u_m'(t) = A(t)u_m(t_j^m)$. By Lemma 4.5 and part (i) of Lemma 1.1,

$$p_+'(t) = \lim_{h\to+0} (|u_n(t)-u_m(t)+h[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]| - |u_n(t)-u_m(t)|)/h$$

$$\le (|u_n(t)-u_m(t)+\delta[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]| - |u_n(t)-u_m(t)|)/\delta.$$

Consequently,

$$p_+'(t) \le (|u_n(t_i^n)-u_m(t_j^m)+\delta[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]| - |u_n(t_i^n)-u_m(t_j^m)|)/\delta$$

$$(5.4) \qquad + 2|u_n(t)-u_n(t_i^n)|/\delta + 2|u_m(t_j^m)-u_m(t)|/\delta.$$

But by part (ii) of Lemma 5.1,

$$(5.5) \qquad 2|u_n(t)-u_n(t_i^n)|/\delta \le 2M|t-t_i^n|\delta^{-1} \le 2Mn_0^{-1}\delta^{-1}$$

and

$$(5.6) \qquad 2|u_m(t)-u_m(t_j^m)|/\delta \le 2M|t-t_j^m|\delta^{-1} \le 2Mn_0^{-1}\delta^{-1}.$$

Furthermore,

$$|u_n(t_i^n)-u_m(t_j^m)| \geq |u_n(t)-u_m(t)| - |u_n(t_i^n)-u_n(t)| - |u_m(t)-u_m(t_j^m)|$$

$$\geq 2\beta' - 2n_o^{-1}M.$$

By (5.2), $2n_o^{-1}M \leq \beta'$ so that $|u_n(t_i^n)-u_m(t_j^m)| \geq \beta'$. Thus, by using (5.1), (5.5), and (5.6), the inequality (5.4) becomes

(5.7) $$p_+'(t) \leq \Lambda'|u_n(t_i^n)-u_m(t_j^m)| + \varepsilon' + 4Mn_o^{-1}\delta^{-1}.$$

But by part (ii) of Lemma 5.1,

$$\Lambda'|u_n(t_i^n)-u_m(t_j^m)| \leq \Lambda'|u_n(t)-u_m(t)| + \Lambda'|u_n(t_i^n)-u_n(t)| + \Lambda'|u_m(t)-u_m(t_j^m)|$$

$$\leq \Lambda'p(t) + 2\Lambda'Mn_o^{-1}$$

and (5.7) becomes

(5.8) $$p_+'(t) \leq \Lambda'p(t) + \varepsilon' + n_o^{-1}(2\Lambda'M + 4M\delta^{-1}).$$

Using (5.2) and the fact that $\varepsilon' = \varepsilon\exp(-\Lambda'\rho)/[4(\rho+1)]$, (5.8) becomes

(5.9) $$p_+'(t) \leq \Lambda'p(t) + \varepsilon\exp(-\Lambda'\rho)/[2(\rho+1)]$$

Since $u_n'(t)$ and $u_m'(t)$ exist for almost all t in $[T_o,T_1]$, the inequality (5.9) holds for almost all t in $[T_o,T_1]$. Since u_n and u_m are Lipschitz continuous on $[T_o,T_1]$, it follows from (5.9) and Lemma 4.6 that

$$p(T_1) \le p(T_o)\exp(\Lambda'(T_1-T_o))$$

$$+ \int_{T_o}^{T_1} \{\varepsilon\exp(-\Lambda'\rho)\exp(\Lambda'(T_1-s))/[2(\rho+1)]\}ds$$

$$\le p(T_o)\exp(\Lambda'\rho) + \varepsilon/2$$

Here we have used the fact that $T_1 - s \le \rho$ for all s is $[T_o,T_1]$. But $p(T_o)\exp(\Lambda'\rho) = 2\beta'\exp(\Lambda'\rho) = \varepsilon/2$ so that $p(T_1) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. This is a contradiction to the assumption (5.3). This contradiction shows that if $n > m > n_o$ then $|u_n(t)-u_m(t)| \le \varepsilon$ and the lemma is proved.

Lemma 5.3. The sequence $(u_n)_1^\infty$ constructed in Lemma 5.1 converges uniformly on $[a,a+\rho]$ to a continuous function u from $[a,a+\rho]$ into $\mathcal{D}$ such that $u(a) = z$ and $|u(t)-u(s)| \le M|t-s|$ for each t and s in $[a,a+\rho]$.

Proof. Since the sequence $(u_n)_1^\infty$ is uniformly Cauchy on $[a,a+\rho]$, it tends uniformly to a continuous function u on $[a,a+\rho]$. Since $u_n(a) = z$ and $|u_n(t)-u_n(s)| \le M|t-s|$ for all $n \ge 1$ and all t and s in $[a,a+\rho]$, it is immediate that $u(a) = z$ and $|u(t)-u(s)| \le M|t-s|$. Furthermore, if t is in $[a,a+\rho]$ then $|u(t)-u(a)| \le M|t-a| \le M\rho$ so that $u(t)$ is in $\mathcal{D}$ and the lemma is proved.

Lemma 5.4. The function $t \to A(t)u(t)$ is Bochner integrable on $[a,a+\rho]$ and for each t in $[a,a+\rho]$

$$u(t) = z + (B) \int_a^t A(s)u(s)ds.$$

Proof. It follows from part (iii) of Lemma 5.1 that for almost all t in $[a,a+\rho]$, $u_n'(t)$ exists for all $n \geq 1$ (one only needs to note that a countable union of sets of measure zero has measure zero). Furthermore, if t is in $[t_i^n, t_{i+1}^n)$ then

$$|u(t)-u_n(t_i^n)| \leq |u(t)-u_n(t)| + |u_n(t)-u_n(t_i^n)|$$

$$\leq |u(t)-u_n(t)| + n^{-1}M.$$

Hence, by the demicontinuity of $A(t)$, if t is in $[a,a+\rho]$ and $u_n'(t)$ exists for all $n \geq 1$ then $u_n'(t) = A(t)u_n(t_i^n)$ for some integer i in $[0,\lambda(n)]$ and it follows that for almost all t in $[a,a+\rho]$,

$$\operatorname{w-lim}_{n\to\infty} u_n'(t) = A(t)u(t).$$

Since the functions u_n' are Bochner integrable on $[a,a+\rho]$, if f is in E^* then the functions $t \to (u_n'(t),f)$ are Lebesque integrable on $[a,a+\rho]$ and

$$\int_a^t (u_n'(s),f)ds = ((B) \int_a^t u_n'(s)ds,f)$$

for each t in $[a,a+\rho]$ and all $n \geq 1$ (see [7, Theorem 3.7.1]). Since $|(u_n'(t),f)| \leq |u_n'(t)||f| \leq M|f|$ for almost all t in $[a,a+\rho]$ and $\lim_{n\to\infty} (u_n'(t),f) = (A(t)u(t),f)$ for almost all t in $[a,a+\rho]$, it follows from the Lebesgue dominated convergence theorem and part (iv) of Lemma 5.1 that

$$
\begin{aligned}
(u(t),f) &= \lim_{n\to\infty} (u_n(t),f) \\
&= \lim_{n\to\infty} \left(z + (B)\int_a^t u_n'(s)ds, f\right) \\
&= (z,f) + \lim_{n\to\infty} \int_a^t (u_n'(s),f)ds \\
&= (z,f) + \int_a^t (A(s)u(s),f)ds.
\end{aligned}
$$

By condition (iii), u is weakly continuously differentiable on $[a,a+\rho]$ and $u'(t) = A(t)u(t)$ for each t in $[a,a+\rho]$. Since u is Lipschitz continuous on $[a,a+\rho]$, the assertions of the lemma are an immediate consequence of Lemma 4.4.

Thus u is a solution to (IVP) in the extended sense on $[a,a+\rho]$. To complete the proof of Theorem 5.1 we need only show that u is unique.

Lemma 5.5. Let u and v be Lipschitz continuous functions from $[a,a+\rho]$ into $\mathcal{D}$ such that $u(a) = z$ and $v(a) = w$. Suppose that for almost all t in $[a,a+\rho]$, $u'(t)$ exists and equals $A(t)u(t)$ and $v'(t)$ exists and equals $A(t)v(t)$. Then

$$|u(t)-v(t)| \le |z-w|\exp(\Lambda'(t-a))$$

for all t in $[a,a+\rho]$. (Here Λ' is as in Lemma 5.2).

Proof. For each t in $[a,a+\rho]$ let $p(t) = |u(t)-v(t)|$. Then by Lemma 4.5, $p_+'(t)$ exists for almost all t in $[a,a+\rho]$ and

$$p_+'(t) = \lim_{h\to+0} (|u(t)-v(t)+h[A(t)u(t)-A(t)v(t)]| - |u(t)-v(t)|)/h$$

$$\le L'[A(t)]|u(t)-v(t)|$$

$$\le \Lambda' p(t).$$

The assertion of the lemma is now an immediate consequence of Lemma 4.6.

Lemma 5.5 shows that the solution u is unique and the proof of Theorem 5.1 is complete. In the proof of Lemma 5.4 we have also shown the following.

Corollary 5.1. The solution u to (IVP) is also a solution in the weak sense on $[a,a+\rho]$.

Corollary 5.2. Instead of condition (iii) of Theorem 5.1 suppose that

(iii)' The function $(t,x) \to A(t)x$ is continuous from $[a,b]\times\mathcal{D}$ into E.

Then the solution u to (IVP) is also a solution in the usual sense on $[a,a+\rho]$.

Proof. Since the function $t \to A(t)u(t)$ is now continuous, we have by Lemma 5.4 that

$$u(t) = z + (B)\int_a^t A(s)u(s)ds$$

$$= z + \int_a^t A(s)u(s)ds$$

and the corollary is immediate.

Example 5.1. Suppose that $\mathcal{D}$ is an open subset of E and $\{A(t) : t\epsilon[a,b]\}$ is a family of members of $Lip(\mathcal{D},E)$ such that the function $(t,x) \to A(t)x$ is continuous from $[a,b]x\mathcal{D}$ into E and there is a number Λ' such that $N[A(t)] \le \Lambda'$ for all t in $[a,b]$. Then if x and y are in $\mathcal{D}$, t is in $[a,b]$, and $h > 0$,

$$(|x-y+h[A(t)x-A(t)y]| - |x-y|)/h \le |A(t)x-A(t)y|$$

$$\le N[A(t)]|x-y|$$

$$\le \Lambda'|x-y|$$

and so the family $\{A(t) : t\epsilon[a,b]\}$ has equiuniform logarithmic derivative on $\mathcal{D}$. Thus each of the conditions of Theorem 5.1 and Corollary 5.2 are fulfilled and so Corollary 5.2 contains the classical Cauchy existence theorem for differential equations.

Example 5.2. Suppose that E^* is uniformly convex, $\mathcal{D}$ is an open subset of E, and $\{A(t) : t\in[a,b]\}$ is a family of functions from $\mathcal{D}$ into E. Suppose further that the function $(t,x) \to A(t)x$ is continuous and bounded on $[a,b]\times\mathcal{D}$ and there is a number Λ' such that $\mathrm{Re}(A(t)x - A(t)y,f) \le \Lambda'|x-y|^2$ for all x and y in $\mathcal{D}$ and f in $F(x-y)$. By Lemma 1.4, for each pair of positive numbers β and ε, there is a positive number δ such that if x and y are in $\mathcal{D}$ with $|x-y| \ge \beta$, t is in $[a,b]$, and $0 < h \le \delta$, then

$$(|x-y+h[A(t)x-A(t)y]| - |x-y|)/h \le \mathrm{Re}(A(t)x-A(t)y,g) + \varepsilon$$

where g is the member of $G(x-y)$. Letting $f = |x-y|g$, f is the member of $F(x-y)$ and

$$\begin{aligned}\mathrm{Re}(A(t)x-A(t)y,g) &= \mathrm{Re}(A(t)x-A(t)y,f)/|x-y| \\ &\le \Lambda'|x-y|.\end{aligned}$$

Substituting this into the previous inequality shows that the family $\{A(t) : t\in[a,b]\}$ has equiuniform logarithmic derivative on $\mathcal{D}$. Thus each of the suppositions of Theorem 5.1 and Corollary 5.2 are fulfilled and so Corollary 5.2 contains the extension of the classical Cauchy existence theorem for a Hilbert space given by Browder [1, Theorem 3].

The next theorem is similar to Theorem 5.1 except that we relax the condition that the family $\{A(t) : t\in[a,b]\}$ have equiuniform logarithmic derivative on $\mathcal{D}$ and place stronger continuity requirements on the family.

Theorem 5.2. Suppose that $\mathcal{D}$ is an open subset of E and $\{A(t) : t\in[a,b]\}$ is a family of functions from $\mathcal{D}$ into E satisfying each of the following conditions:

(i) There is a number M such that $|A(t)x| \le M$ for all (t,x) in $[a,b]\times\mathcal{D}$.

(ii) For each x in $\mathcal{D}$ the function $t \to A(t)x$ is Bochner integrable on $[a,b]$.

(iii) The family $\{A(t) : t\in[a,b]\}$ is uniformly equicontinuous on $\mathcal{D}$.

(iv) There is a positive number Λ such that

$$\lim_{h\to -0} (|x-y+h[A(t)x-A(t)y]| - |x-y|)/h \le \Lambda|x-y|$$

for all x and y in $\mathcal{D}$ and t in $[a,b]$.

Then for each z in $\mathcal{D}$ there is a positive number $\rho = \rho(z)$ and a unique function u from $[a,a+\rho]$ into $\mathcal{D}$ such that u is a solution to (IVP) in the extended sense on $[a,a+\rho]$.

Remark 5.1. It follows from Remark 2.10 that condition (iv) of Theorem 5.2 is fulfilled if and only if $\Lambda I-A(t)$ is monotonic on $\mathcal{D}$ for each t in $[a,b]$.

Remark 5.2. Note that condition (iii) is fulfilled if the function $(t,x) \to A(t)x$ is a uniformly continuous function on $[a,b]\times\mathcal{D}$. However, in [5, p. 287], Dieudonné gives an example which shows that conditions (i), (ii), and (iii) are not sufficient to guarantee a solution to (IVP).

Proof of Theorem 5.2. Let ρ, $(t_i^n)_{i=0}^{\lambda(n)}$, and $(u_n)_1^\infty$ be as in the proof of Theorem 5.1 and suppose that ε is a positive number. By condition (iii) let $\delta > 0$ be sufficiently small so that

$$(5.10) \qquad |A(t)x-A(t)y| \le \varepsilon\exp(-\Lambda\rho)/(2\rho)$$

whenever t is in $[a,b]$ and x and y are in $\mathcal{D}$ with $|x-y| \le \delta$. Now let n_o be a positive integer such that $n_o^{-1}M \le \delta$ and suppose that $n > m \ge n_o$. For each t in $[a,a+\rho]$ let $p(t) = |u_n(t)-u_m(t)|$. Let t be such that $u_n'(t)$ and $u_m'(t)$ exists and let i and j be integers such that t is in $[t_i^n,t_{i+1}^n)$ and t is in $[t_j^m,t_{j+1}^m)$. Since $|u_n(t)-u_n(t_i^n)| \le M|t-t_i^n| \le n^{-1}M \le \delta$ and $|u_m(t)-u_m(t_j^m)| \le \delta$, it follows from (5.10) that

$$|A(t)u_n(t)-A(t)u_n(t_i^n)| + |A(t)u_m(t)-A(t)u_m(t_j^m)| \le \varepsilon\exp(-\Lambda\rho)/\rho.$$

By part (iii) of Lemma 5.1 and Lemma 4.5 $p_-'(t)$ exists; and, using condition (iv),

$$p'_-(t) = \lim_{h\to -0} (|u_n(t)-u_m(t)+h[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]| - |u_n(t)-u_m(t)|)/h$$

$$\le \lim_{h\to -0} (|u_n(t)-u_m(t)+h[A(t)u_n(t)-A(t)u_m(t)]| - |u_n(t)-u_m(t)|)/h$$

$$+ |A(t)u_n(t_i^n)-A(t)u_n(t)| + |A(t)u_m(t)-A(t)u_m(t^m)|$$

$$\le \Lambda p(t) + \varepsilon\exp(-\Lambda\rho)/\rho .$$

Since this inequality holds for almost all t in $[a,a+\rho]$, it follows from Lemma 4.6 that

$$p(t) \le p(a)\exp(\Lambda(t-a)) + \int_a^t \{\varepsilon\exp(-\Lambda\rho)\exp(\Lambda(t-s))/\rho\}ds$$

$$\le p(a)\exp(\Lambda\rho) + \varepsilon$$

for all t in $[a,a+\rho]$. Since $p(a) = 0$ we have $|u_n(t)-u_m(t)| \le \varepsilon$ for all t in $[a,a+\rho]$ and all $n > m \ge n_o$. Hence, the sequence $(u_n)_1^\infty$ is uniformly Cauchy on $[a,a+\rho]$. As in the proof of Lemma 5.3, one can show that the sequence $(u_n)_1^\infty$ tends uniformly to a continuous function u from $[a,a+\rho]$ into $\mathcal{D}$ such that $u(a) = z$ and $|u(t)-u(s)| \le M|t-s|$ for all t and s in $[a,a+\rho]$. Since $A(t)$ is continuous for each t in $[a,b]$, one can show with the techniques used in the proof of Lemma 5.4 that $\lim_{n\to\infty} u_n'(t) = A(t)u(t)$ for almost all t in $[a,a+\rho]$. Since $|u_n'(t)| \le M$ for all $n \le 1$ and almost all t in $[a,a+\rho]$, it follows from Lemma 4.2 and part (iv) of Lemma 5.1 that

$$u(t) = \lim_{n\to\infty} u_n(t)$$

$$= \lim_{n\to\infty} \{z + (B) \int_a^t u_n'(s)ds\}$$

$$= z + (B) \int_a^t A(s)u(s)ds.$$

Thus u is a solution to (IVP) in the extended sense on $[a,a+\rho]$. Now let v be a solution to (IVP) in the extended sense on $[a,a+\rho]$ such that $v(a) = z$ and let $p(t) = |u(t)-v(t)|$ for each t in $[a,a+\rho]$. By Lemma 4.5 and condition (iv) of this theorem,

$$p_-'(t) = \lim_{h\to -0} (|u(t)-v(t)+h[A(t)u(t)-A(t)v(t)]| - |u(t)-v(t)|)/h$$

$$\leq \Lambda p(t)$$

for almost all t in $[a,a+\rho]$, and it follows easily from Lemma 4.6 that $p(t) \leq p(a)\exp(\Lambda(t-a))$ for all t in $[a,a+\rho]$. Since $p(a) = 0$, $u = v$ and the solution u is unique. This completes the proof of Theorem 5.2.

As in Corollaries 5.1 and 5.2 we have

Corollary 5.3. If in addition to the suppositions of Theorem 5.2 we suppose that the function $(t,x) \to A(t)x$ in demicontinuous on $[a,b]\times\mathcal{D}$ then u is a solution to (IVP) in the weak sense on $[a,a+\rho]$.

Corollary 5.4. If in addition to the suppositions of Theorem 5.2 we suppose that the function $(t,x) \to A(t)x$ is continuous on $[a,b]\times\mathcal{D}$ then

u is a solution to (IVP) in the usual sense on $[a,a+\rho]$.

Now let $\{A(t) : t\epsilon[0,\infty)\}$ be a family of functions from E into E. We will give sufficient conditions to insure that the initial value problem

$$\text{(IVP)}' \qquad u'(t) = A(t)u(t), \ u(a) = z$$

has a unique solution on $[a,\infty)$ for each (a,z) in $[0,\infty)\mathrm{x}E$.

Remark 5.3. If b is a number or ∞ and u is a function from $[a,b)$ into E, then we say that u is a solution of (IVP)' in any of the senses of Definition 5.1 on $[a,b)$ if for each c in (a,b), u is a solution of (IVP)' in the corresponding sense on $[a,c]$.

Theorem 5.3. Suppose that $\{A(t) : t\epsilon[0,\infty)\}$ is a family of functions from E into E which satisfy each of the following conditions:

(i) For each x in E the function $t \to A(t)x$ is Bochner integrable on bounded subintervals of $[0,\infty)$.

(ii) The function $(t,x) \to A(t)x$ is demicontinuous on $[0,\infty)\mathrm{x}E$ and maps bounded subsets of $[0,\infty)\mathrm{x}E$ into bounded subsets of E.

(iii) For each t in $[0,\infty)$, $A(t)$ is in $Ln(E,E)$ and there is a continuous function η from $[0,\infty)$ into the real numbers such that $L[A(t)] \le \eta(t)$ for all t in $[0,\infty)$.

(iv) For each (a,z) in $[0,\infty)\mathrm{x}E$ there is a positive number $\rho = \rho(a,z)$ and a Lipschitz continuous function u from

$[a,a+\rho]$ into E such that $u(a) = z$ and u is a solution to (IVP)' in the extended sense on $[a,a+\rho]$.

Then for each (a,z) in $[0,\infty)\times E$ there is a unique function $u(\cdot;a,z)$ from $[a,\infty)$ into E such that $u(a;a,z) = z$ and $u(\cdot;a,z)$ is a solution in the extended sense to (IVP)' on $[a,\infty)$. Furthermore, if w is in E then

$$|u(t;a,z)-u(t;a,w)| \le |z-w|\exp\left(\int_a^t \eta(s)ds\right) \tag{5.11}$$

for all t in $[a,\infty)$.

Remark 5.4. Theorems 5.2 and 5.3 and their corollaries give sufficient conditions for part (iv) of Theorem 5.3 to hold.

Remark 5.5. The inequality (5.11) in Theorem 5.3 shows that the solutions to (IVP)' are uniformly continuous with respect to initial values on bounded subintervals of $[a,\infty)$. Note that if there is a number Γ such that $\int_a^t \eta(s)ds \le \Gamma$ for all t in $[a,\infty)$ then the solutions are uniformly continuous with respect to initial values on $[a,\infty)$. Furthermore, if $\lim_{t\to\infty} \int_a^t \eta(s)ds = -\infty$ then $\lim_{t\to\infty} \{u(t;a,z) - u(t;a,w)\} = 0$ for all z and w in E and the limit is uniform on bounded subsets of E.

Proof of Theorem 5.3. Suppose that (a,z) is in $[0,\infty)\times E$ and u is a solution to (IVP)' which is given by condition (iv) of the theorem. Suppose that u is defined on $[a,T)$ and $T < \infty$. For each t in $[a,T)$ let $p(t) = |u(t)-z|$. Then by Lemma 4.5, for almost all t in $[a,T)$, $p'_+(t)$ exists and

$$p_+'(t) = \lim_{h\to+0} (|u(t)-z+hA(t)u(t)| - |u(t)-z|)/h$$

$$\le \lim_{h\to+0}(|u(t)-z+h[A(t)u(t)-A(t)z]| - |u(t)-z|)/h + |A(t)z|$$

$$\le \eta(t)p(t) + |A(t)z|.$$

By condition (i) and Lemmas 4.1 and 4.6,

$$|u(t)-z| \le \int_a^t |A(s)z|\exp(\int_s^t \eta(r)dr)ds$$

for all t in [0,T) and it follows that u is bounded on [a,T). By condition (ii) let M be a positive number such that $|A(t)u(t)| \le M$ for all t in [a,T). Then if t and s are in [a,T),

$$|u(t)-u(s)| = |(B) \int_s^t A(r)u(r)dr|$$

$$\le M|t-s|.$$

It follows that $u(T) = \lim_{t\to -T} u(t)$ exists. By condition (ii), $\text{w-}\lim_{t\to -T} A(t)u(t) = A(T)u(T)$. Since

$$u(t) = z + (B) \int_a^t A(s)u(s)ds$$

for all t in [a,T), for each f in E^*,

$$(u(t),f) = (z,f) + \int_a^t (A(s)u(s),f)ds \tag{5.12}$$

(see [7, Theorem 3.7.1]). By condition (ii), (5.12) holds for $t = T$. Consequently, u is Lipshitz continuous and weakly differentiable on $[a,T]$ so by Lemma 4.4,

$$u(T) = z + (B) \int_a^T A(s)u(s)ds.$$

This, along with condition (iv), shows that the solution u can be continued past T and it follows that u can be extended to $[a,\infty)$. Now let z and w be in E and let u and v be solutions to (IVP)' in the extended sense on $[a,\infty)$ such that $u(a) = z$ and $v(a) = w$. If $\rho(t) = |u(t)-v(t)|$ for each t in $[a,\infty)$, then by Lemma 4.5, for almost all t in $[a,\infty)$, $p_+'(t)$ exists and

$$p_+'(t) = \lim_{h\to +0} (|u(t)-v(t)+h[A(t)u(t)-A(t)v(t)]| - |u(t)-v(t)|)/h.$$

By condition (iii), $p_+'(t) \le \eta(t)p(t)$ for almost all t in $[a,\infty)$, and the inequality (5.11) follows easily from Lemma 4.6. By taking $w = z$, the uniqueness of $u(\cdot;a,z)$ is immediate from (5.11) and the proof of Theorem 5.3 is complete.

In the proof of Theorem 5.3 we have shown that

Corollary 5.5. The solution $u(\cdot;a,z)$ to (IVP)' is a solution in the weak sense to (IVP)' on $[a,\infty)$ for each (a,z) in $[0,\infty)\times E$.

As in Corollary 5.2 we have

Corollary 5.6. If, in addition to the suppositions of Theorem 5.3, suppose that the function $(t,x) \to A(t)x$ is continuous on $[0,\infty)\times E$, then $u(\cdot;a,z)$ is a solution to (IVP)' in the usual sense on $[a,\infty)$ for each (a,z) in $[0,\infty)\times E$.

Example 5.3. Suppose that E^* is uniformly convex and $\{A(t) : t\in[0,\infty)\}$ is a family of functions from E into E such that the function $(t,x) \to A(t)x$ is continuous and maps bounded subsets of $[0,\infty)\times E$ into bounded subsets of E. Suppose further that there is a continuous function η from $(0,\infty)$ into the real numbers such that $\mathrm{Re}(A(t)x-A(t)y,f) \le \eta(t)|x-y|^2$ for all x and y in E and f in $F(x-y)$. Then, by Example 5.2, condition (iv) of Theorem 5.3 is satisfied and so each of the suppositions of Theorem 5.3 and Corollary 5.6 are satisfied. Thus Corollary 5.6 contains the global existence theorem in the case that E is a Hilbert space given by Browder in [1, Theorem 4].

Now we wish to establish sufficient conditions for the global existence of solutions to an autonomous differential equation on $[0,\infty)$. Let A be a function from E into E and consider the initial value problem

$$\text{(IVP)''} \qquad u'(t) = Au(t),\ u(0) = z$$

where z is in E and t is in $[0,\infty)$.

Theorem 5.4. Suppose that A is a function from E into E which satisfies each of the following conditions:

(i) A is demicontinuous on E.

(ii) A is in $Ln(E,E)$.

(iii) For each z in E there is a positive number $\rho = \rho(z)$ and a Lipschitz continuous function u from $[0,\rho]$ into E such that $u(0) = z$ and u is a solution to (IVP)" in the extended sense on $[0,\rho]$.

Then for each z in E there is a unique function $u(\cdot;z)$ from $[0,\infty)$ into E such that $u(0;z) = z$ and $u(\cdot;z)$ is a solution to (IVP)" in the extended sense on $[0,\infty)$. Furthermore, if w is in E then

$$|u(t;z)-u(t;w)| \leq |z-w|\exp(L[A]t) \tag{5.13}$$

for all t in $[0,\infty)$.

Remark 5.6. Theorems 5.2 and 5.3 and their corollaries give sufficient conditions for part (iv) of Theorem 5.4 to hold.

Proof of Theorem 5.4. Let z be in E and let u be a solution to (IVP)" which is given by condition (iv) of the theorem, and suppose that u is defined on $[0,T)$ where $T < \infty$. Let $0 < h < T$ and for each t in $[0,T-h)$ define $p(t) = |u(t+h)-u(t)|$. By Lemma 4.5 and condition (ii), for almost all t in $[0,T-h)$, $p_+'(t)$ exists and

$$\begin{aligned} p_+'(t) &= \lim_{h\to+0} (|u(t+h)-u(t)+h[Au(t+h)-Au(t)]| - |u(t+h)-u(t)|)/h \\ &\leq L[A]p(t). \end{aligned}$$

By Lemma 4.6,

$$|u(t+h)-u(t)| \leq |u(h)-u(0)|\exp(L[A]t)$$

for all t in $[0,T-h)$. Hence

$$\lim_{t,t+h\to -T} |u(t+h)-u(t)| \leq \lim_{h\to +0} \exp(|L[A]|T)|u(h)-u(0)|$$

$$= 0$$

so that $\lim_{t\to -T} u(t) = u(T)$ exists. The completion of the proof of Theorem 5.4 is now essentially the same as the analogous parts of the proof of Theorem 5.3 (with $\eta(t) = L[A]$ for each t in $[0,\infty)$) and is omitted.

As in Corollary 5.5 we have

Corollary 5.7. The solution $u(\cdot;z)$ to (IVP)" is a solution in the weak sense to (IVP)" on $[0,\infty)$ for each z in E.

As in Corollary 5.6 we have

Corollary 5.8. If, in addition to the suppositions of Theorem 5.4, we suppose that A is continuous on E, then $u(\cdot;z)$ is a solution is the usual sense to (IVP)" on $[0,\infty)$ for each z is E.

Example 5.4. Let K be the real field and let E be the space of all real valued sequences $(\xi_k)_1^\infty$ such that $\lim_{k\to\infty} \xi_k = 0$. In this example let

$|\cdot|_m$ denote the norm on E given by $|(\xi_k)_1^\infty|_m = \max\{|\xi_k| : k \geq 1\}$. For each $k \geq 1$ let A_k be a continuous, nonincreasing function from K into K such that each A_k is uniformly bounded on bounded subsets of K, the family $\{A_k : k \geq 1\}$ is equicontinuous on E, and $A_k 0 = 0$ for each $k \geq 1$. For each $x = (\xi_k)_1^\infty$ in E define $Ax = (\eta_k)_1^\infty$ where $\eta_k = A_k \xi_k$ for each $k \geq 1$. Then A is a continuous function from E into E. Now let R be a positive number and $\mathcal{D}_R = \{x \in E : |x|_m \leq R\}$. Let β and ε be positive numbers and let M be a positive number such that $|Ax| < M$ for all x in $\mathcal{D}_R$. Choose $\delta = \beta/(7M)$ and let $x = (\xi_k)_1^\infty$ and $y = (\eta_k)_1^\infty$ be members of $\mathcal{D}_R$ such that $|x-y|_m \geq \beta$. Since $|Ax-Ay|_m \leq 2M$, if $0 < h \leq \delta$, then

$$|x-y+h[Ax-Ay]| \geq \beta - 2hM \geq 5\beta/7. \tag{5.14}$$

Let q be a positive integer such that

$$|x-y+h[Ax-Ay]|_m = |\xi_q-\eta_q+h[A_q\xi_q-A_q\eta_q]|.$$

Then $|\xi_q-\eta_q| \geq h|A_q\xi_q-A_q\eta_q|$ for if not, $|\xi_q-\eta_q| < h|A_q\xi_q-A_q\eta_q| \leq 2hM \leq 2\beta/7$ which implies that $|x-y+h[Ax-Ay]|_m \leq 2\beta/7 + 2\beta/7 = 4\beta/7$. This is a contradiction to (5.14). Thus,

$$(|x-y+h[Ax-Ay]|_m - |x-y|_m)/h$$

$$= (|\xi_q-\eta_q| - h|A_q\xi_q-A_q\eta_q| - |x-y|_m)/h$$

$$= (|\xi_q - \eta_q| - |x-y|_m)/h - |A_q\xi_q - A_q\eta_q|$$

$$\leq 0.$$

This shows that A is in $\mathcal{ULn}(E,E)$ with $L'[A] \leq 0$. Thus, by Theorem 5.1 and Corollary 5.2, A satisfies each of the conditions of Theorem 5.4 and Corollary 5.8. G. F. Webb [21, Example 3] gives an example of a function A from E into E which satisfies each of the above conditions but is not uniformly continuous on any neighborhood of the origin. Consequently Theorem 5.2 may not apply to this situation.

The theorems presented in this chapter are new and will appear in a paper by the author in the *Journal of the Mathematical Society of Japan* under the title "The Logarithmic Derivative and Equations of Evolution in a Banach Space." It should be noted that, in this paper, the author uses the term logarithmic derivative instead of uniform logarithmic derivative to characterize members of $\mathcal{ULn}(D,E)$.

CHAPTER VI

AUTONOMOUS DIFFERENTIAL EQUATIONS AND SEMIGROUPS OF NONLINEAR OPERATORS

In this chapter the results of Chapter 5 are applied to the autonomous differential equation

$$\text{(ADE)} \qquad u'(t) = Au(t),\ u(0) = z,$$

where A is a function from E into E, z is in E, and t is in $[0,\infty)$. We give some applications of these results to the generation of semigroups of nonlinear operators in $\mathit{Lip}(E,E)$ and also establish sufficient conditions for (ADE) to have a unique critical point in E.

Definition 6.1. A function U from $[0,\infty)$ into $\mathit{Lip}(E,E)$ is called a semigroup of operators in $\mathit{Lip}(E,E)$ if each of the following holds:

(i) $U(0) = I$ and $U(t)\cdot U(s) = U(t+s)$ for all t and s in $[0,\infty)$.

(ii) There is a number σ such that $N[U(t)] \le \exp(\sigma t)$ for all t in $[0,\infty)$.

U is said to be of class $(w\text{-}C_1)$ if in addition to (i) and (ii),

(iii) For each z in E the function $t \to U(t)z$ is weakly continuously differentiable on $[0,\infty)$.

U is said to be of class (C_1) if in addition to (i) and (ii), (iii)' For each z in E the function $t \to U(t)z$ is continuously differentiable on $[0,\infty)$.

Definition 6.2. Let U be a semigroup of operators in $Lip(E,E)$.

(i) If $\mathcal{D}$ is the set of all z in E such that $\text{w-}\lim_{h\to+0} (U(h)z-z)/h$ exists and Az denotes this limit, then A is said to be the weak generator of U.

(ii) If $\mathcal{D}$ is the set of all z in E such that $\lim_{h\to+0} (U(h)z-z)/h$ exists and Az denotes this limit, then A is said to be the strong generator of U.

Remark 6.1. Note that if U is a semigroup of class (C_1), then U is a semigroup of class $(w\text{-}C_1)$. Furthermore, if U is of class (C_1) (respectively, $(w\text{-}C_1)$), then the strong generator (respectively, weak generator) of U is defined on all of E.

Example 6.1. Suppose A is in $BL(E,E)$ and $U(t) = \exp(tA)$ for each t in $[0,\infty)$. Then U is a semigroup of operators in $BL(E,E)$ and A is the strong generator of U. Furthermore, the number σ in part (ii) of Definition 6.1 can be taken as $\mu[A]$ (see part (ii) of Proposition 2.1).

Proposition 6.1. Let U be a semigroup of operators in $Lip(E,E)$ satisfying parts (i) and (ii) of Definition 6.1 and suppose A is the weak generator of U which is defined on a subset $\mathcal{D}$ of E. Then A is in $Ln(\mathcal{D},E)$ and $L[A] \le \sigma$.

Proof. Let x and y be in $\mathcal{D}$ and let g be in $G(x-y)$ (see Definition 2.1). Then

$$\begin{aligned}
\operatorname{Re}(Ax-Ay,g) &= \operatorname{Re}\{\lim_{h\to+0} ([U(h)x-x - U(h)y+y]/h,\ g)\} \\
&= \operatorname{Re}\{\lim_{h\to+0} (U(h)x - U(h)y,g)/h - (x-y,g)/h\} \\
&\le \lim_{h\to+0} (|U(h)x-U(h)y| - |x-y|)/h \\
&\le \lim_{h\to+0} (\exp(\sigma h)|x-y| - |x-y|)/h \\
&= \sigma|x-y|.
\end{aligned}$$

Here, we have used part (ii) of Definition 6.1 and the fact that $(x-y,g) = |x-y|$. The assertion of the proposition now follows from Proposition 2.3.

Definition 6.3. If A is a function from E into E, then

(i) A is called locally bounded on E if for each z in E there is a neighborhood V_z of z such that A is bounded on V_z.

(ii) A is called locally uniformly continuous on E if for each z in E there is a neighborhood V_z of z such that A is uniformly continuous on V_z.

(iii) A is said to have a local uniform logarithmic derivative on E if for each z in E there is a neighborhood V_z of z such that the restriction of A to V_z is in $ULn(V_z,E)$.

Theorem 6.1. Suppose that A is a continuous function from E into E and there is a number σ such that

(i) $\lim_{h\to -0} (|x-y+h[Ax-Ay]| - |x-y|)/h \le \sigma|x-y|$ for all x and y in E.

Suppose further that either of the following is satisfied:

(ii) A is locally uniformly continuous on E.

(ii)' A has a local uniform logarithmic derivative on E.

Then A is in $Ln(E,E)$, $L[A] \le \sigma$, and A is the strong generator of a semigroup of operators U of class (C_1) which satisfies the conditions (i), (ii), and (iii)' of Definition 6.1.

Proof. If (ii) holds then by Corollary 5.4 of Theorem 5.2 (respectively if (ii)' holds then by Corollary 5.2 of Theorem 5.1) for each z in E there is a $\rho(z) > 0$ and a continuous function $u(\cdot;z)$ from $[0,\rho(z)]$ into E such that $u(0;z) = z$ and $u(\cdot;z)$ is a solution to (ADE) in the usual sense on $[0,\rho(z)]$. If x and y are in E, $\rho = \min\{\rho(x),\rho(y)\}$, and $p(t) = |u(t;x)-u(t;y)|$ for each t is $[0,\rho]$, then by Lemma 4.5 $p_-'(t)$ exists for each t in $(0,\rho]$ and, by condition (i) of this theorem,

$$p_-'(t) = \lim_{h\to -0} (|u(t;x) - u(t;y) + h[Au(t;x) - Au(t,y)]| - |u(t;x) - u(t;y)|)/h$$

$$\le \sigma p(t).$$

By Lemma 4.6, $p(t) \le \exp(\sigma(t-s))p(s)$ for all t and s in $[0,\rho]$ with $s \le t$. Consequently, if t is in $[0,\rho)$,

$$p_+'(t) = \lim_{h\to+0} [p(t+h)-p(t)]/h$$

$$\le \lim_{h\to+0} [\exp(\sigma h)p(t)-p(t)]/h$$

$$= p(t)\lim_{h\to+0} [\exp(\sigma h)-1]/h$$

$$= \sigma p(t).$$

Since $u(0;x) = x$, $u(0,y) = y$, $u_+'(0;x) = Ax$, and $u_+'(0;y) = Ay$, we have by Lemma 4.5 that

$$\lim_{h\to+0} (|x-y+h[Ax-Ay]| - |x-y|) = p_+'(0)$$

$$\le \sigma p(0)$$

$$= \sigma|x-y|$$

and so A is in $Ln(E,E)$ with $L[A] \le \sigma$. It is now an immediate consequence of Corollary 5.8 to Theorem 5.4 that for each z in E there is a unique function $u(\cdot;z)$ from $[0,\infty)$ into E which is a solution to (ADE) in the usual since on $[0,\infty)$. By conclusion (5.13) to Theorem 5.4,

$$|u(t;z)-u(t;w)| \le |z-w|\exp(\sigma t)$$

for all t in $[0,\infty)$. Letting $U(t)z = u(t;z)$ for each (t,z) in $[0,\infty)\mathrm{x}E$, it is immediate that U satisfies the conditions (i), (ii), and (iii)' of Definition 6.1 and the proof of Theorem 6.1 is complete.

Proposition 6.2. With the suppositions of Theorem 6.1, condition (ii) implies condition (ii)'.

Proof. Let U be the semigroup generated by A and let z be in E. By condition (ii) let V_z be a neighborhood of z such that A is uniformly continuous on V_z and let $R > 0$ be such that if $|x-z| \le 2R$ then x is in V_z. Suppose further that V_z is chosen so that there is a number M such that $|Ax| \le M$ for all x in V_z. If $T_z = \{x \in E: |x-z| \le R\}$, we will show that the restriction of A to T_z is in $ULn(T_z, E)$. Let $\rho > 0$ be such that $\rho M\exp(|\sigma|\rho) \le R$ and let x be in T_z. If $p(t) = |U(t)x-x|$ for each t in $[0,\rho]$, then, by Lemma 4.5,

$$\begin{aligned} p_+'(t) &= \lim_{h \to +0} (|U(t)x-x + hAU(t)x| - |U(t)x-x|)/h \\ &\le \lim_{h \to +0} (|U(t)x-x + h[AU(t)x-Ax]| - |U(t)x-x|)/h + |Ax| \\ &\le \sigma p(t) + M. \end{aligned}$$

Since $p(0) = 0$ and $t - s \le \rho$ for each s in $[0,t]$, we have by Lemma 4.6 that

$$\begin{aligned} p(t) &\le p(0)\exp(\sigma t) + \int_a^t M\exp(\sigma(t-s))ds \\ &\le tM\exp(|\sigma|\rho). \end{aligned}$$

Thus, for each t in $[0,\rho]$ and each x in T_z,

$$(6.1) \qquad |U(t)x-x| \le tM\exp(|\sigma|\rho).$$

In particular, since $t \le \rho$ and $\rho M\exp(|\sigma|\rho) \le R$, we have $|U(t)x-z| \le |U(t)x-x| + |x-z| \le 2R$ so that $U(t)x$ is in V_z for all t in $[0,\rho]$ and x in T_z. Furthermore, since $U(t)x = x + \int_0^t AU(s)xds$, we have

$$(6.2) \qquad |Ax - (U(t)x-x)/t| = |t^{-1} \int_0^t \{Ax - AU(s)x\}ds|$$

$$\le \sup\{|Ax-AU(s)x| : 0\le s\le t\}$$

for each t in $(0,\rho]$. Now let ε be a positive number. Since A is uniformly continuous on V_z choose $\delta_1 > 0$ so that if w_1 and w_2 are in V_z with $|w_1-w_2| \le \delta_2$, then $|Aw_1-Aw_2| \le \varepsilon/3$. By (6.1) let δ be such that if t is in $[0,\delta]$ then $|U(t)x-x| \le \delta_1$ for all x in T_z and further, choose δ sufficiently small so that $(\exp(\sigma t)-1)/t \le \sigma + \varepsilon/6R$ for each t in $(0,\delta)$. Note that if t is in $(0,\delta)$ then by (6.2) $|Ax-(U(t)x-x)/t| \le \varepsilon/3$ for all x in T_z. Using the above estimates, we have that if $0 < h \le \delta$ and x and y are in T_z then

$$(|x-y+h[Ax-Ay]| - |x-y|)/h \le \{|x-y+h[(U(h)x-x)/h - (U(h)y-y)/h)]|$$

$$- |x-y|\}/h + |Ax - (U(h)x-x)/h| + |Ay - (U(h)y-y)/h|$$

$$\le (|U(h)x - U(h)y| - |x-y|)/h + 2\varepsilon/3$$

$$\le |x-y|(\exp(\sigma h)-1)/h + 2\varepsilon/3$$

$$\le |x-y|(\sigma + \varepsilon/6R) + 2\varepsilon/3$$

$$\le \sigma|x - y| + \varepsilon.$$

Here, we used the fact that $|x-y| \le 2R$. This shows that the restriction of A to T_z is in $ULn(T_z,E)$ and the proof of the proposition is complete.

Using techniques analogous to those in the proof of Proposition 6.2 one can show the following:

Proposition 6.3. If A is a member of $Ln(E,E)$ which is uniformly continuous on bounded subsets of E then A is in $ULn(E,E)$ and $L'[A] = L[A]$.

Remark 6.2. Note that in the proof of Proposition. 3.2 that the number δ was chosen independent of the distance apart x and y were in T_z. If D is a subset of E and A is in $ULn(D,E)$, one can show directly that a necessary and sufficient condition for the number $\delta = \delta(Q,\beta,\varepsilon)$ in Definition 2.10 to be chosen independent of β is that A be uniformly continuous on Q.

Theorem 6.2. Suppose that A is a demicontinuous function from E into E and each of the following is satisfied:

(i) there is a number σ such that

$$\lim_{h\to -0} (|x-y + h[Ax-Ay]| - |x-y|)/h \le \sigma|x-y|$$

for all x and y in E.

(ii) A is locally bounded on E.

(iii) A has a local uniform logarithmic derivative on E.

Then A is in $Ln(E,E)$, $L[A] \le \sigma$, and A is the weak generator of a semigroup U of class (w-C_1) which satisfies the conditions (i), (ii), and (iii) of Definition 6.1.

Proof. Since A is demicontinuous on E we have from conditions (ii) and (iii) and from Theorem 5.1 and Corollary 5.1 that, for each z in E, there is a $\rho(z) > 0$ and a unique continuous function $u(\cdot;z)$ from $[0,\rho(z)]$ into E such that $u(0;z) = z$ and $u(\cdot;z)$ is a solution to (ADE) in both the extended and weak sense on $[0,\rho(z)]$. If x and y are in E, $\rho = \min\{\rho(x),\rho(y)\}$, and $p(t) = |u(t;x) - u(t;y)|$ for each t in $[0,\rho]$, then, by Lemma 4.5, $p'_-(t)$ exists for almost all t in $(0,\rho]$ and, by condition (i) of this theorem,

$$p'_-(t) = \lim_{h\to -0} (|u(t;x)-u(t;y) + h[Au(t;x)-Au(t;y)]| - |u(t;x)-u(t;y)|)/h$$

$$\le \sigma p(t).$$

By Lemma 4.6, $p(t) \le p(0)\exp(\sigma t) = |x-y|\exp(\sigma t)$ for each t in $[0,\rho]$. As in the proof of Proposition 6.1, if g is in $G(x-y)$ then

$$\mathrm{Re}(Ax-Ay,g) = \lim_{h\to +0} \{\mathrm{Re}\,([u(h;x)-x - u(h;y)+y]/h,g)\}$$

$$\le \lim_{h\to +0} (|u(h;x)-u(h;y)| - |x-y|)/h$$

$$\le (\exp(\sigma h)-1)|x-y|/h$$

$$= \sigma|x-y|.$$

Hence A is in $Ln(E,E)$ with $L[A] \le \sigma$. It is now an immediate consequence of Corollary 5.7 to Theorem 5.4 that, for each z in E, there is a unique function $u(\cdot;z)$ from $[0,\infty)$ into E which is a solution to (ADE) in the weak sense on $[0,\infty)$. By conclusion (5.13) to Theorem 5.4,

$$|u(t;z) - u(t;w)| \le |z-w|\exp(\sigma t)$$

for all t in $[0,\infty)$. Let $U(t)z = u(t;z)$ for each (t,z) in $[0,\infty)\times E$; it is immediate that U satisfies the conditions (i), (ii), and (iii) of Definition 6.1 and the proof of Theorem 6.2 is complete.

Proposition 6.4. Suppose that A is a function from E into E and ρ is a nonincreasing function from $[0,\infty)$ into $(0,\infty)$ such that

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h \le -\rho(r)|x-y|$$

whenever x and y are in E with $|x|,|y| \le r$. If x and y are in E with $|x| \ge |y|$ then

$$|Ax-Ay| \ge |x-y|(|x|-|y|)^{-1} \int_{|y|}^{|x|} \rho(r)dr \quad \text{if } |x| > |y|$$

and

$$|Ax-Ay| \geq |x-y|\rho(|y|) \quad \text{if } |x| = |y|.$$

Proof. Let ε be a positive number and let $(s_i)_o^n$ be a subdivision of $[0,1]$ such that

$$(6.3) \qquad \left|\int_o^1 \rho((1-s)|y| + s|x|)ds - \right.$$

$$\left. - \sum_{i=1}^{n} \rho((1-s_i)|y| + s_i|x|)(s_i-s_{i-1})\right| \leq \varepsilon.$$

For each integer i in $[0,n]$ let $z_i = (1-s_i)y + s_i x$ and let $t_i = |y| + s_i(|x|-|y|)$. Note that $|z_i|,|z_{i-1}| \leq t_i$ and $|z_i-z_{i-1}| = (s_i-s_{i-1})|x-y|$. For each integer i in $[1,n]$ there is a $\delta_i > 0$ such that if $0 < h \leq \delta_i$, then

$$(|z_i-z_{i-1} + h[Az_i-Az_{i-1}]| - |z_i-z_{i-1}|)/h$$

$$\leq (-\rho(t_i) + \varepsilon)(s_i-s_{i-1})|x-y|.$$

Consequently, if $\delta = \min\{\delta_i : 1 \leq i \leq n\}$ and $0 < h \leq \delta$, then

$$(6.4) \qquad |z_i-z_{i-1} + h[Az_i-Az_{i-1}]| \leq (1-h\rho(t_i) + h\varepsilon)(s_i-s_{i-1})|x-y|$$

for each integer i in $[1,n]$. Since

$$x-y + h[Ax-Ay] = \sum_{i=1}^{n} \{z_i - z_{i-1} + h[Az_i - Az_{i-1}]\}$$

we have by (6.4), (6.3), and the definition of t_i that if $0 < h \leq \delta$ then

$$|x-y + h[Ax-Ay]| \leq \sum_{i=1}^{n} |z_i - z_{i-1} + h[Az_i - Az_{i-1}]|$$

$$\leq \sum_{i=1}^{n} (1-h\rho(t_i) + h\varepsilon)(s_i - s_{i-1})|x-y|$$

$$\leq |x-y|\{1 - h\int_0^1 \rho(|y| + s(|x|-|y|))ds + 2h\varepsilon\}.$$

Since $|x-y| - h|Ax-Ay| \leq |x-y + h[Ax-Ay]|$, it follows that

$$-h|Ax-Ay| \leq -h|x-y|\int_0^1 \rho(|y| + s(|x|-|y|))ds + 2h\varepsilon|x-y|$$

and hence,

$$|Ax-Ay| \geq |x-y|\int_0^1 \rho(|y| + s(|x|-|y|))ds - 2\varepsilon|x-y|.$$

Since this inequality is true for each $\varepsilon > 0$ the assertions of the lemma follow directly if $|x| = |y|$ and by the change of variable $r = |y| + s(|x|-|y|)$ if $|x| > |y|$.

Corollary 6.1. In addition to the suppositions of Proposition 6.4, suppose that $\int_0^\infty \rho(r)dr = \infty$. If A is bounded on a subset $\mathcal{D}$ of E, then $\mathcal{D}$ is a bounded subset of E.

Proof. If x is in $\mathcal{D}$ and we take y in Proposition 6.4 to be 0 then

$$\int_0^{|x|} \rho(r)dr \le |Ax-Ao|$$

and it is immediate that $\mathcal{D}$ is bounded.

Theorem 6.3. Suppose that A is a function from E into E and ρ is a nonincreasing function from $[0,\infty)$ into $(0,\infty)$ such that each of the following is satisfied:

(i) A is demicontinuous on E.

(ii) $\int_0^{\infty} \rho(r)dr = \infty$.

(iii) For each $r > 0$ and x and y in E with $|x|,|y| \le r$,

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h \le -\rho(r)|x-y|.$$

(iv) For each z in E there is a positive number $T = T(z)$ and a function u from $[0,T]$ into E such that $u(0) = z$ and u is a solution to (ADE) in the extended sense on $[0,T]$.

Then there is a unique member x_c of E such that $Ax_c = 0$, and for each z in E there is a unique function $u(\cdot;z)$ from $[0,\infty)$ into E such that $u(0;z) = z$ and $u(\cdot;z)$ is a solution to (ADE) in the extended sense on $[0,\infty)$. Furthermore,

$$|u(t;z)-x_c| \le |z-x_c|\exp(-\rho(|z-x_c| + |x_c|)t)$$

for each t in $[0,\infty)$.

Proof. Since condition (iii) implies that A is in $Ln(E,E)$ with $L[A] \le 0$ we have, by Theorem 5.4, that for each z in E there is a unique function $u(\cdot;z)$ from $[0,\infty)$ into E such that $u(0;z) = z$ and $u(\cdot;z)$ is a solution to (ADE) in the extended sense on $[0,\infty)$. Furthermore

$$(6.5) \qquad |u(t;z) - u(t;w)| \le |z-w|$$

for all z and w in E. Now let h be in (0,1) and choose $w = u(h;z)$ so that $u(t;w) = u(t+h;z)$. Dividing each side of (6.5) by h we have that

$$(6.6) \qquad \limsup_{h\to+0} |u(t+h;z) - u(t;z)|/h \le \limsup_{h\to+0} |u(h;z) - u(0;z)|/h.$$

Since $u(\cdot;z)$ is Lipschitz continuous on [0,1] there is a number K such that $|u(h;z) - u(0;z)| \le Kh$ for all h in (0,1]. It then follows from (6.6) that for almost all t in $[0,\infty)$, $|Au(t;z)| = |u'(t,z)| \le K$. By condition (ii) and Corollary 6.1, there is a number K' such that $|u(t;z)| \le K'$ for almost all t in $[0,\infty)$ and since $u(\cdot;z)$ is continuous, it is immediate that $|u(t;z)| \le K'$ for all t in $[0,\infty)$. Hence, for each z in E, $u(\cdot;z)$ is bounded on $[0,\infty)$. Let z and w be in E and let r_o be a positive number such that $|u(t;z)|, |u(t;w)| \le r_o$ for all t in $[0,\infty)$. If $p(t) = |u(t;z) - u(t;w)|$ for each t in $[0,\infty)$ then, by condition (iii) and Lemma 4.5, $p_+'(t) \le -\rho(r_o)p(t)$ for almost all t in $[0,\infty)$, and we have by Lemma 4.6 that

$$(6.7) \qquad |u(t;z) - u(t;w)| \le |z-w|\exp(-\rho(r_o)t)$$

for all t in $[0,\infty)$. If h is in $(0,1]$ and w is taken to be $u(h;z)$ then (6.7) becomes

$$(6.8) \qquad |u(t+h;z) - u(t;z)| \le |u(h;z) - z|\exp(-\rho(r_o)t).$$

Since $|u(h;z) - z| \le 2r_o$, (6.8) shows that $u(t;z)$ tends to some limit x_c as t tends to ∞. Dividing both sides of (6.8) by h and letting $h\to+0$, we have that if K is a number such that $|u(h;z) - u(0;z)| \le Kh$ for each h in $(0,1]$, then $|u'(t;z)| \le K\exp(-\rho(r_o)t)$ for almost all t in $[0,\infty)$. Hence, there is a sequence $(t_k)_1^\infty$ in $[0,\infty)$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} u'(t_k;z) = \lim_{k\to\infty} Au(t_k;z) = 0$. Since $\lim_{t\to\infty} u(t;z) = x_c$ and A is demicontinuous on E we have that

$$\begin{aligned} Ax_c &= \text{w-}\lim_{t\to\infty} Au(t;z) \\ &= \text{w-}\lim_{k\to\infty} Au(t_k;z) \\ &= \text{w-}\lim_{k\to\infty} u'(t_k;z) \\ &= 0. \end{aligned}$$

Now, take w to be x_c so that $u(t;w) = x_c$ for all t in $[0,\infty)$. The inequality (6.7) becomes

$$(6.9) \qquad |u(t;z) - x_c| \le |z-x_c|\exp(-\rho(r_o)t)$$

for all t in $[0,\infty)$. In particular, $|u(t;z) - x_c| \le |z-x_c|$ so that $|u(t;z)| \le |u(t;z) - x_c| + |x_c| \le |z-x_c| + |x_c|$ and the number r_o in (6.9) can be taken as $|z-x_c| + |x_c|$. Since (6.9) clearly implies that x_c is the only member of E such that $Ax_c = 0$, the proof of Theorem 6.3 is complete.

Remark 6.3. Theorem 6.3 is related to Theorem 1 of Markus and Yamabe in [14]. Their theorem is done with E being a finite-dimensional, connected, complete Riemannian manifold and with A being continuously differentiable. Instead of condition (ii) of Theorem 6.3 Markus and Yamabe require that

$$(6.10) \qquad \int_0^\infty \exp(-\varepsilon \int_o^s \rho(r)dr)ds < \infty$$

for each $\varepsilon > 0$. Note that (6.10) implies that condition (ii) holds and if $\rho(r) = (1+r)^{-1}$, then ρ satisfies (ii) but not (6.10).

Remark 6.4. Note that condition (ii) of Theorem 6.3 was used only to show that each solution to (ADE) was bounded on $[0,\infty)$. Instead of condition (ii) assume that there exists at least one bounded solution u to (ADE) on $[0,\infty)$. If v is a solution to (ADE) on $[0,\infty)$ and $p(t) = |u(t) - v(t)|$, then $p_+'(t) \le L[A]p(t) \le 0$ for almost all t in $[0,\infty)$ so that $|u(t) - v(t)| \le |u(0) - v(0)|$. Hence each solution to (ADE) is bounded on $[0,\infty)$ and the conclusions of Theorem 6.3 are valid. In particular, if conditions (i), (iii), and (iv) of Theorem 6.3 hold, then either all solutions to (ADE) are unbounded on $[0,\infty)$ or the conclusions of Theorem 6.3 are valid. As a simple illustration, let E be the space of real numbers, let y be in E, and let $Ax = \exp(-x) - y$ for each x in E. Then if $y \leq 0$ all solutions are unbounded on $[0,\infty)$ and if $y > 0$ all solutions are bounded on $[0,\infty)$ and tend to $-\ln(y)$ as t tends to ∞.

Corollary 6.2. For each y in E let $B_y x = Ax - y$ for all x in E and, in addition to the suppositions of Theorem 6.3, suppose that

(v) For each z in E there is a positive number $T = T(z)$ and a function u from $[0,T]$ into E such that $u(0) = z$ and u is a solution in the extended sense to $u'(t) = B_y u(t)$ on $[0,T]$.

Then A is a bijection from E into E and if $\mathcal{D}$ is bounded subset of E, there is an $r_o > 0$ such that $|A^{-1}x - A^{-1}y| \leq \rho(r_o)^{-1}|x-y|$ for all x and y in $\mathcal{D}$.

Proof. It is easy to check that B_y satisfies each of the conditions of A in Theorem 6.3. Consequently, for each y in E there is a unique point x_y in E such that $B_y x_y = 0$. Hence $Ax_y = y$ and it is immediate that A is a bijection. If $\mathcal{D}$ is a bounded subset of E and $\mathcal{D}' = A^{-1}(\mathcal{D})$, then A is bounded on $\mathcal{D}'$ so, by Corollary 6.1, there is an $r_o > 0$ such that $|x| \leq r_o$ for all x in $\mathcal{D}'$. It follows easily from Proposition 6.4 that $|Ax-Ay| \geq \rho(r_o)|x-y|$ for all x and y in $\mathcal{D}'$ and the last assertion of the corollary is now evident.

Corollary 6.3. Suppose that A satisfies the conditions of Theorem 6.1 or Theorem 6.2 and let λ be in K with $\mathrm{Re}(\lambda) > \sigma$. Then $A-\lambda I$ is a bijection from E into E and $(A-\lambda I)^{-1}$ is in $Lip(E,E)$ with $N[(A- I)^{-1}] \leq (\mathrm{Re}(\lambda)-\sigma)^{-I}$.

Proof. Since A is in $Ln(E,E)$ with $L[A] \leq \sigma$, $A-\lambda I$ is in $Ln(E,E)$ with $L[A-\lambda I] = L[A] - \mathrm{Re}(\lambda) \leq \sigma - \mathrm{Re}(\lambda)$. It is now easy to check that $A-\lambda I$ satisfies each of the conditions of Corollary 6.2 with $\rho(r) = (\mathrm{Re}(\lambda) - \sigma)$ for each r in $[0,\infty)$. Thus the assertions of Corollary 6.3 are an immediate consequence of Corollary 6.2.

Example 6.2. Suppose that ρ satisfies the suppositions of Theorem 6.3 and A is a function from E into E which has a Fréchet derivative $dA(x)$ at each point x in E. Suppose further that $\mu[dA(x)] \leq -\rho(|x|)$ for each x in E and that dA maps bounded subsets of E into bounded subsets of $BL(E,E)$. As in the proof of Propositions 2.2 one can show that if x and y are in E with $|x|,|y| \leq r$, then

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]| - |x-y|)/h \leq -\rho(r)|x-y|.$$

Thus each of the suppositions of Theorem 6.3 and Corollary 6.3 hold.

Example 6.3. Let K be the field of real numbers, let $E = K^2$, and let $|\cdot|_1$ be the norm on K^2 defined by $|(\xi_1,\xi_2)|_1 = \max\{|\xi_1|,|\xi_2|\}$ for each (ξ_1,ξ_2) in K^2. Let A and Q be as in Example 3.5--that is

$$A(\xi_1,\xi_2) = (-2\xi_1 + \cos(\xi_2),\ \sin^2(\xi_1) - \xi_2)$$

and

$$Q(\xi_1,\xi_2) = (\xi_1,\ 2\xi_2/3).$$

If $\mu_Q[\cdot]$ is induced by the norm $|\cdot|_Q$ on K^2 where $|(\xi_1,\xi_2)|_Q = |Q(\xi_1,\xi_2)|_1$, then by Example 3.5

$$\mu_Q[dA(\xi_1,\xi_2)] \le -1/3$$

and it follows from Example 6.2 that

$$\lim_{h\to+0} (|x-y + h[Ax-Ay]|_Q - |x-y|_Q)/h \le -|x-y|_Q/3$$

for each x and y in K^2. Now let B be a continuous function from K^2 into K^2 for which there is a nondecreasing function σ from $[0,\infty)$ into $(0,\infty)$ such that $\int_0^\infty (1/3 - 3\sigma(r)/2)dr = +\infty$ and $|Bx-By|_1 \le \sigma(r)|x-y|_1$ whenever x and y are in K^2 with $|x|_1,|y|_1 \le 3r/2$. Since $\|Q\|_1 = 1$ and $\|Q^{-1}\|_1 = 3/2$, we have that if $r > 0$ and x and y are in K^2 with $|x|_Q,|y|_Q \le r$, then $|x|_1,|y|_1 \le 3r/2$ and

$$\lim_{h\to+0} (|x-y + h[Ax+Bx - Ay-By]|_Q - |x-y|_Q)/h$$

$$\le \lim_{h\to+0} (|x-y + h[Ax-Ay]|_Q - |x-y|_Q)/h + |Bx-By|_Q$$

$$\le -|x-y|_Q/3 + |Q\cdot Bx - Q\cdot By|_1$$

$$\leq -|x-y|_Q/3 + \sigma(r)|x-y|_1$$

$$\leq (-1/3 + 3\sigma(r)/2)|x-y|_Q.$$

Consequently, if $\rho(r) = 1/3 - 3\sigma(r)/2$ for each r in $[0,\infty)$, we have that A + B and ρ satisfy each of the conditions of Theorem 6.3 and Corollary 6.2 with $E = K^2$ and the norm $|\cdot|$ on E being the norm $|\cdot|_Q$ defined above.

Remark 6.5. If A is as in Example 6.3, Markus and Yamabe [14, p. 310] show by using the Euclidian norm on K^2 that the differential equation $u'(t) = Au(t)$ has a unique critical point and that each solution tends to this critical point as t tends to ∞.

Remark 6.6. The results established in Theorem 6.3 are new and they will appear in a paper by the author in the *Journal of Mathematical Analysis and Applications* under the title "A Theorem on Critical Points and Global Asymptotic Stability." The results established in Theorems 6.1 and 6.2 also seem to be new but in a remark at the end of section 2 in [21], Webb refers to some recent results of F. Browder and T. Kato which are to appear in the *Proceedings of the Symposium on Nonlinear Function Analysis* (published by the American Mathematical Society) which have considerable overlap with Theorem 6.1. In particular, Kato shows that Theorem 6.1 is true if E^* is uniformly convex and Browder shows that Theorem 6.1 is true if condition (ii) holds.

CHAPTER VII

APPLICATIONS TO THE STABILITY OF DIFFERENTIAL EQUATIONS

Let $\{A(t) : t \in [0,\infty)\}$ be a family of functions from E into E. In this chapter we apply the techniques developed in this work to study the growth of solutions to the differential equation

$$u'(t) = A(t)u(t) \tag{DE}$$

In this chapter we assume that the family $\{A(t) : t \in [0,\infty)\}$ satisfies each of the following conditions:

(7.1)

(1) $A(t)0 = 0$ for each t in $[0,\infty)$.

(2) For each z in E there is a positive number T and a function u from $[0,T)$ into E such that $u(0) = z$ and u is a solution to (DE) in the usual sense on $[0,T)$.

(3) The solution u to (DE) in condition (2) can be continued so long as it remains is a bounded subset of E.

The fundamental theorem used in this Chapter is

Theorem 7.1. Suppose that D is a subset of E and there are continuous functions η and γ from $[0,\infty)$ into the real numbers such that

$$\lim_{h\to+0} (|x+hA(t)x| - |x|)/h \le \eta(t)|x|$$

and

$$-\lim_{h\to+0} (|x-hA(t)x| - |x|)/h \ge \gamma(t)|x|$$

for each (t,x) in $[0,\infty)\times\mathcal{D}$. If u is a solution to (DE) and T is a positive number such that $u(t)$ is in $\mathcal{D}$ for each t in $[0,T)$, then

(i) the function $t \to |u(t)|\exp(-\int_0^t \eta(s)ds)$ is nonincreasing on $[0,T)$,

(ii) the function $t \to |u(t)|\exp(-\int_0^t \gamma(s)ds)$ is nondecreasing on $[0,T)$, and

(iii) $|u(0)|\exp(\int_0^t \gamma(s)ds) \le |u(t)| \le |u(0)|\exp(\int_0^t \eta(s)ds)$ for each t in $[0,T)$.

Remark 7.1. This theorem is closely related to Theorem 2.13.1 of Lakshmikantham and Leela in [10,p 103].

Proof of Theorem 7.1. For each t in $[0,T)$ let $p(t) = |u(t)|$. Then, by Lemma 4.5,

$$p'_+(t) = \lim_{h\to+0} (|u(t) + hA(t)u(t)| - |u(t)|)/h$$

$$\le \eta(t)p(t),$$

and it follows easily that the function $t \to p(t)\exp(-\int_0^t \eta(s)ds)$ is continuous and has a nonpositive right derivative on $[0,T)$. Consequently, part (i) is true. Furthermore, if t is in (0,T) then

$$p'_-(t) = \lim_{h\to -0} (|u(t) + hA(t)u(t)| - |u(t)|)/h$$

$$= -\lim_{h\to +0} (|u(t) - hA(t)u(t)| - |u(t)|)/h$$

$$\geq \gamma(t)p(t)$$

so that the function $t \longrightarrow p(t)\exp(-\int_0^t \gamma(s)ds)$ is continuous and has a nonnegative left derivative on (0,T) and part (ii) is true. Part (iii) is immediate from parts (i) and (ii).

We will now give a sequence of propositions and examples that show how these techniques relate to and sometimes sharpen some of the known results in the stability theory of differential equations.

Proposition 7.1. Suppose that A(t) is in *Lip(E,E)* for each t in $[0,\infty)$ and the function $t \longrightarrow A(t)$ is a continuous function from $[0,\infty)$ into the seminormed space *Lip(E,E)*. If u is a solution to (DE), then u exists on $[0,\infty)$ and each of the following holds:

(i) The function $t \longrightarrow |u(t)|\exp(-\int_0^t M[A(s)]ds)$ is nonincreasing on $[0,\infty)$.

(ii) The function $t \longrightarrow |u(t)|\exp(\int_0^t M[-A(s)]ds)$ is nondecreasing on $[0,\infty)$.

(iii) $|u(0)|\exp(-\int_0^t M[-A(s)]ds) \leq |u(t)| \leq |u(0)|\exp(\int_0^t M[A(s)]ds)$ for each t in $[0,\infty)$.

Remark 7.2. In the case that $A(t)$ is linear for each t in $[0,\infty)$ this is Theorem 3 of Coppel in [4, p. 58]. The author in [15, Theorem 2] shows that there is an analogous result to Theorem 3 of Coppel which bounds solutions of linear Stieltjes integral equations.

Proof of Proposition 7.1. Since the function $t \to A(t)$ is continuous it follows from part (iv) of Remark 2.8 that the functions $t \to M[A(t)]$ and $t \to M[-A(t)]$ are continuous. If x is in E then

$$\begin{aligned} \lim_{h\to+0} (|x+hA(t)x| - |x|)/h &\le \lim_{h\to+0} (N[I+hA(t)]|x| - |x|)/h \\ &= M[A(t)]|x| \end{aligned}$$

and

$$\begin{aligned} -\lim_{h\to+0} (|x-hA(t)x| - |x|)/h &\ge -\lim_{h\to+0} (N[I-hA(t)]|x| - |x|)/h \\ &= -M[-A(t)]|x| \end{aligned}$$

for each t in $[0,\infty)$ so that this proposition is an immediate consequence of Theorem 7.1.

Proposition 7.2. Suppose that r is a positive number, $\mathcal{D}(r) = \{x\epsilon E : |x|<r\}$, and $A(t)$ is Fréchet differentiable in $\mathcal{D}(r)$ for each t in $[0,\infty)$. Let $d_2A(t)(x)$ denote the Fréchet derivative of $A(t)$ at x and suppose for each $T>0$ there is a number $K(T)$ such that $\|d_2A(t)(x)\| \le K(T)$ for each (t,x) in $[0,T]\times\mathcal{D}(r)$. Suppose further that $\alpha(r,\cdot)$ is a continuous function from $[0,\infty)$ into the real numbers such that

(i) $\mu[d_2A(t)(x)] \le \alpha(r,t)$ for each (t,x) in $[0,\infty)\times\mathcal{D}(r)$.

(ii) There is a number $\Gamma(r)$ such that $\int_0^t \alpha(r,s)ds \le \Gamma(r)$ for each t in $[0,\infty)$.

If u is a solution to (DE) such that $|u(0)|\exp(\Gamma(r)) < r$ then u exists on $[0,\infty)$ and

$$|u(t)| \le |u(0)|\exp(\int_0^t \alpha(r,s)ds)$$

for each t in $[0,\infty)$. Furthermore, if these suppositions hold for each $r>0$ and there is a number Γ_o such that $\Gamma(r) \le \Gamma_o$ for each $r > 0$, then each solution to (DE) exists on $[0,\infty)$ and the above bound holds whenever $|u(0)|\exp(\Gamma_o) < r$.

Proof. Using the techniques developed in the proof of Proposition 2.2 it is easy to show that

$$\lim_{h\to+0} (|x+hA(t)x| - |x|)/h \le \alpha(r,t)|x|$$

for all (t,x) in $[0,\infty)\times\mathcal{D}(r)$. By part (iii) of Theorem 7.1 so long as a solution u to (DE) remains in $\mathcal{D}(r)$

$$|u(t)| \le |u(0)|\exp(\int_0^t \alpha(r,s)ds).$$

Thus if $|u(0)|\exp(\Gamma(r))<r$ then $|u(t)|<r$ for all t in $[0,\infty)$ and the assertions of the proposition follow easily.

Example 7.1. Suppose that H is a Hilbert space and $\{A(t) : t\in[0,\infty)\}$ is a family of functions from H into H such that $d_2A(t)(x)$ exists and is bounded on bounded subsets of $[0,\infty)\times H$. Suppose further that P and S are positive definite self-adjoint members of $BL(H,H)$ such that $S^2 = P$, and, for each $r>0$, there is a positive number $\Lambda(r)$ such that if x is in $\mathcal{D}(r)$ and λ is in the spectrum of $P\cdot d_2A(t)(x) + d_2A(t)(x)^*\cdot P$, then $\lambda \leq -\Lambda(r)$ for each t in $[0,\infty)$. By Proposition 3.2, if $\mu_S[\cdot]$ is induced by the norm $|\cdot|_S$ on H (where $|x|_S = |Sx|$), then $\mu_S[d_2A(t)(x)] \leq -\Lambda(r)/(2\|P\|)$ whenever (t,x) is in $[0,\infty)\times\mathcal{D}(r)$. Consequently, by Proposition 7.2 (using the norm $|\cdot|_S$), we can take $\alpha(r,t) = -\Lambda(r)/(2\|P\|)$ so that if u is a solution to (DE) such that $\|S^{-1}\|\,\|S\|\,|u(0)| < r$, then u exists on $[0,\infty)$ and satisfies

$$|u(t)| \leq \|S^{-1}\|\,\|S\|\,|u(0)|\exp(-t\Lambda(r)/(2\|P\|))$$

for each t in $[0,\infty)$. In particular, this shows that Proposition 7.2 contains Theorem 21.1 of Krasovskii [9, p. 91]. Here Krasovskii requires that H is finite dimensional and that $-\Lambda(r) \leq -\Lambda_o < 0$ for each $r>0$.

Proposition 7.3. Suppose that S is a nonempty set and $\{|\cdot|_\sigma : \sigma\in S\}$ is a family of norms on E each of which is equivalent to the norm $|\cdot|$ on E. Also let $\{a_\sigma : \sigma\in S\}$ and $\{b_\sigma : \sigma\in S\}$ be families of positive numbers such that $a_\sigma|x|_\sigma \leq |x| \leq b_\sigma|x|_\sigma$ for each x in E. Furthermore, let $\mathcal{D}$ be a bounded subset of E and suppose that $\{\eta_\sigma : \sigma\in S\}$ and $\{\gamma_\sigma : \sigma\in S\}$ are families of continuous functions from $[0,\infty)$ into the real numbers such that if x is in $\mathcal{D}$, t is in $[0,\infty)$, and σ is in S, then

$$\lim_{h\to+0} (|x + hA(t)x|_\sigma - |x|_\sigma)/h \le \eta_\sigma(t)|x|_\sigma$$

and

$$-\lim_{h\to+0} (|x - hA(t)x|_\sigma - |x|_\sigma)/h \ge \gamma_\sigma(t)|x|_\sigma.$$

If u is a solution to (DE) and T is a positive number such that u(t) is in $\mathcal{D}$ for each t in $[0,T)$, then

(i) $|u(t)| \le |u(0)|\inf\{(b_\sigma/a_\sigma)\exp(\int_0^t \eta_\sigma(s)ds) : \sigma\epsilon S\}$

and

(ii) $|u(t)| \ge |u(0)|\sup\{(b_\sigma/a_\sigma)\exp(\int_0^t \gamma_\sigma(s)ds) : \sigma\epsilon S\}$

for each t in $[0,T)$.

Proof. It is immediate from part (iii) of Theorem 7.1 that if σ is in S and t is in $[0,\infty)$ then

$$|u(t)|_\sigma \le |u(0)|_\sigma \exp(\int_0^t \eta_\sigma(s)ds).$$

Since $|u(t)| \le b_\sigma|u(t)|_\sigma$ and $|u(0)|_\sigma \le a_\sigma^{-1}|u(0)|$ it follows that

$$|u(t)| \le (b_\sigma/a_\sigma)|u(0)|\exp(\int_0^t \eta_\sigma(s)ds)$$

and part (i) is immediate. The proof of part (ii) is analogous.

Example 7.2. Here we give a simple application of Proposition 7.3 and, in the next proposition, we extend this example to a more general situation. Let K be the real field, let $E = K^2$, and let A(t) be the member of $BL(K^2,K^2)$ which is associated with the matrix

$$\begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix}.$$

For each $\epsilon>0$ let $Q_\varepsilon(\xi_1,\xi_2) = (\varepsilon\xi_1,\xi_2)$ for each (ξ_1,ξ_2) in K^2. Then Q_ε is invertible and $Q_\varepsilon \cdot A(t) \cdot Q_\varepsilon^{-1}$ is associated with the matrix

$$\begin{bmatrix} -1 & \varepsilon t \\ 0 & -1 \end{bmatrix}.$$

If $|\cdot|_1$ is the norm on K^2 defined by $|(\xi_1,\xi_2)|_1 = \max\{|\xi_1|,|\xi_2|\}$, $|\cdot|_\varepsilon$ is the norm on K^2 defined by $|(\xi_1,\xi_2)|_\varepsilon = |Q_\varepsilon(\xi_1,\xi_2)|_1$, and μ_ε is induced by $|\cdot|_\varepsilon$, then, by Example 3.1 and part (i) of Example 3.3,

$$\mu_\varepsilon[A(t)] = -1 + \varepsilon t.$$

Since $|x|_\varepsilon \le |x| \le \varepsilon^{-1}|x|_\varepsilon$ for each x in K^2 and ε in (0,1], we have by Proposition 7.3 that if u is a solution to (DE) then

$$|u(t)|_1 \le |u(0)|_1 \inf\{\varepsilon^{-1}\exp(-t+\varepsilon t^2/2) : 0<\varepsilon\le 1\}$$

for each t in $[0,\infty)$. In particular, by taking $\varepsilon = \min\{1, t^{-1}\}$,

$$|u(t)|_1 \le |u(0)|_1 t \exp(-t/2)$$

for each t in $[1,\infty)$.

Proposition 7.4. Suppose that K is the real field, n is a positive integer, $E = K^n$, and $|\cdot|_1$ is the norm on K^n defined by $|(\xi_k)_1^n|_1 = \max\{|\xi_k| : 1 \le k \le n\}$. Let $\{A(t) : t \in [0,\infty)\}$ be a family of differentiable functions from K^n into K^n such that the function $(t,x) \to A(t)x$ of $[0,\infty) x K^n$ into K^n is continuous. Let $A(t)x = (A_k(t)x)_1^n$ for each (t,x) in $[0,\infty) x K^n$ and suppose that $\frac{\partial}{\partial \xi_j} A_k(t)x$ is bounded on bounded subsets of $[0,\infty) x K^n$. Let $J_{ij}(t)x$ denote $\frac{\partial}{\partial \xi_j} A_i(t)x$ for each (t,x) in $[0,\infty) x K^n$ and pair of integers i and j in $[1,n]$, and suppose that each of the following is satisfied:

(i) $A_k(t)0 = 0$ for all t in $[0,\infty)$ and all integer k in $[1,n]$.

(ii) $J_{ij}(t)x = 0$ whenever (t,x) is in $[0,\infty) x K^n$ and $1 \le j < i \le n$.

(iii) For each $r>0$ there is a positive number $\alpha(r)$ such that $J_{ii}(t)x \le -\alpha(r)$ whenever t is in $[0,\infty)$, x is in K^n with $|x|_1 < r$, and $1 \le i \le n$.

(iv) There is a nonnegative number λ such that for each $r>0$ there is a $\Lambda(r)>0$ for which $|J_{ij}(t)x| \le \Lambda(r)(1+t)^\lambda$ whenever t is in $[0,\infty)$, x is in K^n with $|x|_1 < r$, and $1 \le i < j \le n$.

Then each solution u to (DE) exists on $[0,\infty)$ and there are positive numbers Γ and β (which depend on u) such that

$$|u(t)|_1 \le \Gamma\exp(-\beta t)$$

for each t in $[0,\infty)$.

Remark 7.3. This proposition contains Theorem 4 of Markus and Yamabe in [14]. Here they prove this proposition in the case that A does not depend on t.

Proof of Proposition 7.4. The proof will be by induction on n. It is trivial if n = 1 so assume n > 1 and the assertions of the proposition hold for n - 1. Let $u(t) = (u_k(t))_1^n$ be a solution to (DE) and let $v(t) = (0, u_2(t), \ldots, u_n(t))$ for each t for which u(t) is defined. It follows easily from the induction hypothesis that there are positive numbers Γ and β such that $|v(t)|_1 \le \Gamma\exp(-\beta t)$ so long as v(t) exists. If $p(t) = |u_1(t)|$ then

$$p_+'(t) = \lim_{h\to+0} (|u_1(t) + hA_1(t)u(t)| - |u_1(t)|)/h$$

$$\le \{\lim_{h\to+0} (|u_1(t) + h[A_1(t)u(t) - A_1(t)v(t)]| - |u_1(t)|)/h\}$$

$$+ |A_1(t)v(t)|.$$

It follows from condition (iii) that the number in the braces above is nonpositive so that $p_+'(t) \le |A_1(t)v(t)|$ so long as u(t) exists. However, by condition (iv), $|A_1(t)v(t)| \le (n-1)\Lambda(\Gamma)(1+t)^\lambda \exp(-\beta t)$, and hence, so long as u(t) exists,

$$|u_1(t)| \le (n-1)\Lambda(\Gamma)\Gamma\int_0^t (1+s)^\lambda \exp(-\beta s)ds.$$

Thus $u_1(t)$ remains bounded so that $u(t)$ remains in a bounded subset of K^n and consequently, u exists on $[0,\infty)$. Now let $r>0$ be such that $|u(t)|_1<r$ and for each ε in $(0,1]$ let Q_ε be the diagonal matrix $\mathrm{diag}(\varepsilon^{n-1},\varepsilon^{n-2},\ldots,\varepsilon,1)$. Then $Q_\varepsilon^{-1} = \mathrm{diag}(\varepsilon^{1-n},\varepsilon^{2-n},\ldots,\varepsilon^{-1},1)$ and if $J(t)x = (J_{ij}(t)x)_{1\le i,j\le n}$ is the Jacobian matrix of $A(t)$ at x then

$$Q_\varepsilon\cdot J\cdot Q_\varepsilon^{-1} = \begin{bmatrix} J_{11} & \varepsilon J_{12} & \varepsilon^2 J_{13} & \cdots & \varepsilon^{n-1}J_{1n} \\ 0 & J_{22} & \varepsilon J_{23} & \cdots & \varepsilon^{n-2}J_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & J_{nn} \end{bmatrix}$$

where the arguments are suppressed. If $|\cdot|_\varepsilon$ is the norm on K^n defined by $|x|_\varepsilon = |Q_\varepsilon x|_1$, then by Example 3.1 and part (i) of Example 3.3, if $\mu_\varepsilon[\cdot]$ is induced by $|\cdot|_\varepsilon$, we have the estimate

$$\mu_\varepsilon[J(t)x] \le -\alpha(r) + \varepsilon(n-1)\Lambda(r)(1+t)^\lambda$$

for each t in $[0,\infty)$ and x in K^n with $|x|_1<r$. By Proposition 7.2,

$$|u(t)|_\varepsilon \le |u(0)|_\varepsilon \exp(-\alpha(r)t + \varepsilon(n-1)\Lambda(r)(1+t)^{\lambda+1}/(\lambda+1)).$$

Since $|u(t)|_1 \le \varepsilon^{-n+1}|u(t)|_\varepsilon$ and $|u(0)|_\varepsilon \le |u(0)|_1$, it follows that

$$|u(t)|_1 \le |u(0)|_1 \varepsilon^{-n+1} \exp(-\alpha(r)t + \varepsilon(n-1)\Lambda(r)(1+t)^{\lambda+1}/(\lambda+1))$$

for each t in $[0,\infty)$ and ε is $(0,1]$. By taking $\varepsilon = \min\{1, (\lambda+1)\alpha(r)/[2(n-1)\Lambda(r)(1+t)^{\lambda}]\}$ and $\Gamma' = 2(n-1)\Lambda(r)/[\alpha(r)(\lambda+1)]$ we have

$$|u(t)|_1 \le |u(0)|_1 \Gamma'(1+t)^{\lambda} \exp(-\alpha(r)t + \alpha(r)(1+t)/2).$$

Thus if $\Gamma'' = |u(0)|_1 \Gamma' \exp(\alpha(r)/2)$ then

$$|u(t)|_1 \le \Gamma''(1+t)^{\lambda} \exp(-\alpha(r)t/2)$$

for each t in $[0,\infty)$ and it follows that $u(t)$ tends to zero exponentially as t tends to ∞. This completes the proof of Proposition 7.4.

Now let $\{B(t) : t\in[0,\infty)\}$ be a family of functions from E into E and suppose for each z in E there is a positive number $T = T(z)$ and a function u from $[0,T]$ into E such that $u(0) = z$ and u is a solution to the differential equation

$$\text{(PDE)} \qquad u'(t) = A(t)u(t) + B(t)u(t)$$

in the usual sense on $[0,T)$. Suppose further that u can be extended so long as it remains in a bounded subset of E. Also let $\{U(t) : t\in[0,\infty)\}$ be a family of invertible members of $BL(E,E)$ for which there are positive numbers Λ_1 and Λ_2 such that $\|U(t)\| \le \Lambda_1$ and $\|U(t)^{-1}\| \le \Lambda_2$. Suppose further that the function $t \longrightarrow U(t)$ of $[0,\infty)$ into $BL(E,E)$ is continuously differentiable.

Proposition 7.5. Using the notation above let $C(t) = U(t)\cdot A(t)\cdot U(t)^{-1} + U'(t)\cdot U(t)^{-1}$ for each t in $[0,\infty)$. Let r be a positive number, $\mathcal{D}(r) = \{x\epsilon E : |x|<r\}$, and suppose that $\alpha(r,\cdot)$ is a continuous function from $[0,\infty)$ into the real numbers such that

$$\lim_{h\to+0} (|x+hC(t)x| - |x|)/h \le \alpha(r,t)|x|$$

for all (t,x) in $[0,\infty)\mathrm{x}\mathcal{D}(r)$. Now suppose that $\beta(r,\cdot)$ is a continuous function from $[0,\infty)$ into the real numbers such that

$$|B(t)x| \le \beta(r,t)|x|$$

for all (t,x) in $[0,\infty)\mathrm{x}\mathcal{D}(r)$ and that there is a nonnegative number $\Gamma(r)$ such that

$$\int_0^t [\alpha(r,s) + \Lambda_1\Lambda_2\beta(r,s)]ds \le \Gamma(r)$$

for each t in $[0,\infty)$. Then each solution u to (PDE) such that $|U(0)u(0)|\exp(\Gamma(r)) < r$ exists on $[0,\infty)$ and for each t in $[0,\infty)$

$$|u(t)| \le \Lambda_1\Lambda_2|u(0)|\exp(\int_0^t [\alpha(r,s) + \Lambda_1\Lambda_2\beta(r,s)]ds).$$

Proof. Let $v(t) = U(t)u(t)$ so that

$$\begin{aligned} v'(t) &= U(t)\cdot A(t)u(t) + U(t)\cdot B(t)u(t) + U'(t)u(t) \\ &= C(t)v(t) + U(t)\cdot B(t)u(t). \end{aligned}$$

If $p(t) = |v(t)|$ then, so long as $|v(t)| < r$,

$$p'_+(t) = \lim_{h\to+0} (|v(t) + hv'(t)| - |v(t)|)/h$$

$$= \lim_{h\to+0} (|v(t) + h[C(t)v(t) + U(t)\cdot B(t)u(t)]| - |v(t)|)/h$$

$$\leq \lim_{h\to+0} (|v(t) + hC(t)v(t)| - |v(t)|)/h + |U(t)\cdot B(t)u(t)|$$

$$\leq \alpha(r,t)p(t) + \Lambda_1|B(t)\cdot U(t)^{-1}v(t)|$$

$$\leq \alpha(r,t)p(t) + \Lambda_1\Lambda_2\beta(r,t)p(t)$$

We have by Lemma 4.6 that, so long as $|v(t)| < r$,

$$|v(t)| \leq |v(0)|\exp(\int_o^t [\alpha(r,s) + \Lambda_1\Lambda_2\beta(r,s)]ds).$$

Consequently, $|v(t)| \leq |U(0)u(0)|\exp(\Gamma(r))<r$ so that as long as $u(t)$ and $v(t)$ exist, $|v(t)|<r$ and $|u(t)| = |U(t)^{-1}v(t)| \leq \Lambda_2|v(t)| < \Lambda_2 r$. Thus $u(t)$ remains in a bounded subset of E and so $u(t)$ can be extended to $[0,\infty)$. Furthermore,

$$|u(t)| = |U(t)^{-1}v(t)|$$

$$\leq \Lambda_2|v(t)|$$

$$\leq \Lambda_2|U(0)u(0)|\exp(\int_0^t [\alpha(r,s) + \Lambda_1\Lambda_2\beta(r,s)]ds)$$

for each t in $[0,\infty)$ and the assertions of the proposition follow.

Example 7.3. Suppose that H is a Hilbert space, the functions $A(t)$ in Proposition 7.5 (with $E = H$) have a Fréchet derivative on $\mathcal{D}(r)$ and $d_2A(t)(x)$ are bounded on bounded subsets of $[0,\infty)\times\mathcal{D}(r)$. Suppose further that $\{P(t) : t\in[0,\infty)\}$ is a family of positive definite self-adjoint members of $BL(H,H)$ such that the function $t \to P(t)$ of $[0,\infty)$ into $BL(H,H)$ is continuously differentiable and there are positive numbers γ_1 and γ_2 such that each member $\lambda(t)$ of the spectrum of $P(t)$ satisfies $\gamma_1 \leq \lambda(t) \leq \gamma_2$. Also, suppose that there is a positive number $\alpha(r)$ such that if (t,x) is in $[0,\infty)\times\mathcal{D}(r)$ and if $\lambda(t,x)$ is in the spectrum of $d_2A(t)(x) + P(t)^{-1}\cdot d_2A(t)(x)^*\cdot P(t) + P(t)^{-1}P'(t)$, then $\lambda(t,x) \leq -\alpha(r)$. Now let B and β be as in Proposition 7.5 and suppose that there is a number $\Gamma(r)$ such that

$$\int_0^t [-\alpha(r) + 2\sqrt{\gamma_2/\gamma_1}\ \beta(r,s)]ds \leq \Gamma(r)$$

for each t in $[0,\infty)$. Let $S(t)$ denote the positive definite self-adjoint square root of $P(t)$. With the arguments suppressed we have

$$d_2A + P^{-1}\cdot d_2A^*\cdot P + P^{-1}\cdot P'$$

$$= S^{-1}\cdot[S\cdot d_2A\cdot S^{-1} + S^{-1}\cdot d_2A^*\cdot S + S^{-1}\cdot P'\cdot S^{-1}]\cdot S.$$

Since $P' = 2S \cdot S_2'$ we have $S^{-1} \cdot P' \cdot S^{-1} = 2S' \cdot S^{-1}$ so that

$$d_2A + P^{-1}\, d_2A^{*}\, P + P^{-1}\, P'$$

$$= S^{-1}[(S \cdot d_2A \cdot S^{-1} + S' \cdot S^{-1}) + (S \cdot d_2A \cdot S^{-1} + S' \cdot S^{-1})^{*}] \cdot S.$$

Hence, each member λ of the spectrum of $(S \cdot d_2A \cdot S^{-1} + S' \cdot S^{-1}) + (S \cdot d_2A \cdot S^{-1} + S' \cdot S^{-1})^{*}$ satisfies $\lambda \leq -\alpha(r)$. By part (iii) of Proposition 3.1,

$$\mu[S \cdot d_2A \cdot S^{-1} + S' \cdot S^{-1}] \leq -\alpha(r)/2.$$

Thus with $U(t) = S(t)$ and $\alpha(r,t) = -\alpha(r)/2$ we have that $\|U(t)\| \leq \sqrt{\gamma_2}$ and $\|U(t)^{-1}\| \leq \sqrt{1/\gamma_1}$ so that the hypotheses of Proposition 7.5 are fulfilled. In the case that $A(t)$ is linear for each t in $[0,\infty)$, this example is Proposition 1 of Halanay in [6, p. 72]. Note, however, that line 3 of Proposition 1 contains a misprint. It should read $|X(x,t)| \leq \beta(t)|x|$, for $|x| \leq c$.

Remark 7.4. Even though Theorem 7.1 is very similar to Theorem 2.13.1 of Lakshmikantham and Leela in [10, p. 103], the propositions given in this chapter are new. Most of the results of this chapter are contained in a paper by the author which will appear in the *Journal of Differential Equations* under the title "Bounds for Solutions of a Class of Nonlinear Differential Equations."

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