"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

A MODEL FOR DETERMINING THE EFFECT OF IN-PROCESS STORAGE ON THE OUTPUT OF A SERIES OF MACHINES

A THESIS

Presented to

the Faculty of the Graduate Division

By

Paul Andler Weber, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Industrial Engineering

Georgia Institute of Technology

June, 1962

3 b
12 R

A MODEL FOR DETERMINING THE EFFECT OF IN-PROCESS STORAGE ON THE OUTPUT OF A SERIES OF MACHINES

Approved:

Harrison M. Wadsworth, Chairman

John H. MacKay

James M. Apple

Date Approved by Chairman June 1, 1962

ACKNOWLEDGEMENTS

The subject of this research grew out of discussions with Dr. Harrison M. Wadsworth. Dr. Wadsworth also provided valuable suggestions and guidance, and his interest and encouragement as thesis advisor are gratefully acknowledged.

Dr. John H. MacKay and Professor James M. Apple served on the reading committee, and both contributed valuable guidance and advice.

Special credit is due Mrs. Jeanne M. Crawford for her excellent typing and help in the preparation of the final manuscript.

The author would also like to take this opportunity to express his appreciation and gratitude to his wife Margaret, for her patience, understanding, and encouragement during some trying periods in the development of this thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

A model is developed to describe the mean output of a group of machines or work stations which are arranged in series and subject to breakdowns or interruptions, and the effects on this output of varying the amount and location of storage space between the work stations.

The model assumes no runouts in the initial supply, exponentially distributed service and repair times, Poisson distributed breakdown arrival times, and immediate removal of material from the system on completion of service at the last station. The system is considered as a steady state queueing process, and material moves through the series of stations in order. No unit may leave the system until it has completed service at all stations.

General expressions are derived for determining the mean output of a two machine series with any amount of in-process storage capacity. In this case it is found that P_n, the steady state probability that there are n units of material in the storage space between the two machines, is equal to P_o, the steady state probability that there are no units in this space, multiplied by a ratio which has been raised to the n^{th} power. This ratio is the ratio of the product of the mean production rate of the first machine when it is producing and the proportion of the time it is not broken down (unity minus the ratio of its mean breakdown rate to its mean repair rate) to the same product for the second machine. This ratio must be less than unity because of the

assumption of a steady state. This relation holds for any value of n and for any capacity of the storage space.

No general expressions were derived when there are three or more machines, but a procedure is developed for writing the specific expressions for any specified number of machines and capacity and arrangement of storage space. Specific expressions are derived for the case of three machines in series and all possible combinations of five or fewer units of storage space.

The rapidly increasing complexity of the expressions as the size of the system increases suggests that economical application of the results may be limited to fairly simple systems.

It is suggested that the procedure developed may be of use in the development of future decision rules for optimum in-process storage capacities and arrangements.

CHAPTER I

INTRODUCTION

Background.--With the advent of production line and assembly line manufacturing methods in the past century, problems have arisen in arranging systems of facilities and furnishing equipment to accommodate a complex manufacturing process which is subject to interdependence among the elements of the system and to varying conditions imposed upon it, some by design and some by chance. Many graphical and analytical techniques have been devised to assist in designing and evaluating such systems; perhaps the most familiar are the Gantt chart and its modifications, the use of scale models and templates in layout work, and the conventional methods of machine shop and production estimating. These methods have proved highly successful in industry, as is witnessed by their continued widespread use more than half a century after their introduction. They do, nowever, have some fundamental limitations. One of the most important of these is that, while the methods describe the situations under normal conditions and can accommodate most of those changes that are intentionally imposed, they do not consider random or chance fluctuations within the system. In actuality, of course, such systems are dynamic in nature and involve continuous small changes. The traditional manner of handling these small changes is to allow a straight percentage or "safety factor" to provide for them.

More recently, particularly since World War II, efforts have been made to evaluate the effects of these chance fluctuations. This work has largely fallen into two categories. The first is that of system simulation; this approach is often used where the problems are too complex or cumbersome to be handled in a mathematical model at a reasonable cost in time or money and when full-scale manipulations of the system are not feasible. The simulation may be physical, numerical, or by some other means such as by an analogue computer.

The second approach, and that used in this study, is that of the mathematical analysis of congestion. The early work in this area was done by Erlang (1) in the 1920's on the problems related to the switching of calls in a large telephone exchange. More recently this approach has been applied to many other problems, and the general area of knowledge has come to be known as "queuing" theory.*

The Specific Problem.--When a group of machines or work stations is arranged in series as in a normal production line, the entire line becomes interdependent in the sense that a malfunction, breakdown, or other disruption at any station can disrupt the entire system. This can happen in two ways: the stations following the stopped station can run out of work while those before it can be "blocked," that is, they have no place in which to dispose of their finished material and cannot undertake more work until the path is cleared. In either case the consequence is lost production time.

*Other approaches (which were not used in this study) have also shown considerable success. Among these the most widely known are various modifications of the "critical path" technique.

Part of the cost of this lost production time is unavoidable. At the present state of technology, machinery, even with the best of maintenance, is subject to breakdowns. Even if machinery could be built that would never break down, no one has yet proposed a workable way to eliminate all the disruptions and irregularities inherent in any activity where the human element is present, although much progress has been made, particularly in the area of automated production facilities. It may be, however, that some of the effects of these occurrences can be minimized. The effects are, again, the blocking of stations before the station which is stopped and the run out of work of those following it. It is the specific objective of this study to develop a mathematical model which, in certain cases and within specified limitations, will describe the output of a series of machines subject to breakdowns and interruptions, and will show how varying the amount and location of in-process storage space will affect this output. The model may be useful in further studies for developing decision rules for economically optimum arrangements of facilities.

In the remainder of this thesis, in order to provide more concise terminology, the word "machine" will be used to describe "work station" regardless of the physical arrangement of the station, and the word "breakdown" will be used to indicate "stoppage or interruption." This should present no difficulty if the assumptions regarding the nature of the distributions of "breakdown," repair times, and operation service times are carefully noted.

<u>General Considerations</u>.--This study is not an engineering study to determine the output of actual manufacturing plants, but is instead intended as an analytical attempt to develop a mathematical model which will approximate certain characteristics of such plants. As such, it is subject to many limitations, and many real features of such plants must be neglected in order to provide expressions which may be solved with any reasonable amount of effort. Certain of these limitations are concerned with the distributions assumed for service times, arrival of breakdowns, and repair times. These are explained in Chapter III. Others will be mentioned in the following paragraphs.

A characteristic of machines in series is that each machine must have an average output rate less than or equal to that of the machine immediately following it. If this were not so and the system were run for a long time, there would eventually be a large quantity of material which had been finished by one machine waiting for service by the next. If material were not regularly removed from the system, this amount would continually increase, and, in theory at least, would eventually reach infinity. The normal procedure (2) is to combine the operations so that the time needed at each station is as close as possible to the longest operation time so that faster operations will not be blocked by slower ones. In most cases it is impossible to "balance" the line exactly in this manner and the stations capable of faster operation must reduce their average output to that of the slowest station. This utilization of less than full capacity leads to real and significant costs ("balance loss"), but these costs are not considered in this thesis.

Another limitation of the model is that it deals only with the steady state behavior of the machine system. In effect this requires that the system has been running for a considerable period of time and has settled into steady behavior. This precludes consideration of lead times in setting up machines in order or of any other feature of the process which is dependent on the length of time since the process was started.

In practice there are, of course, many causes of interrupted production, including such things as actual breakdown of machinery, tool changes, temporary absence of the operator, and many others. The model here assumes that all of these causes of "breakdowns" produce a net result which can be characterized by a single Poisson distribution, and that the times required for restoration of service can be described by a single exponential distribution.

In effect, the model describes an idealized process where all the assumptions mentioned are satisfied, in which no machine produces faster than the machine following it, and which has been operating continuously for a long enough period of time to have settled into a steady behavior independent of conditions which existed when the process was started.

CHAPTER II

LITERATURE SURVEY

A search of the literature indicates that considerable work has been done in areas related to the topic of this research. One of the areas most thoroughly studied is that of the problem of machine interference. This problem has been studied by Jones (3), Palm (4), Benson and Cox (5), Naor (6), and others. The problem is similar in that groups of machines are subject to random breakdown or stoppage, but the machines work independently or in parallel rather than in series. Loss beyond that time required to repair a machine arises because there may be more machines stopped at one time than there are repairmen available to service them. The stoppage of one machine can affect the production of another only by reducing its probability of immediate service if it should break down. Solutions are given in terms of overall production from the groups of machines and of average number of repairmen occupied or per cent utilization of repairmen's time for different assignments of repairmen to machines.

A problem more closely related to that considered in this thesis is that considered by Jackson (7). He assumes a system of k service stations in series and allows there to be r_i different identical servers (machines) at the i^{th} station, each of which can service incoming units. Service times are exponentially distributed with mean μ_i at the i^{th} station, and queues of any length are allowed before each station. Units arrive at the first station at random with mean arrival rate λ and proceed through the system where they must be served by one of the servers at each station in order. The solution is limited to the steady state where $\frac{\lambda}{r_i \mu_i} < 1$ for all i. He studies the behavior of the queues before each station and derives the steady-state solution

$$P(n_1, n_2, \ldots, n_k) = P(0) \prod_{j=1}^{k} b(n_j)$$

where

$$b(n_j) = \begin{cases} \dfrac{1}{n_j!} \left(\dfrac{\lambda}{\mu_j}\right)^{n_j}, & \text{for } n_j < r_j \\[2ex] \dfrac{1}{r_j} \left(\dfrac{\lambda}{\mu_j}\right)^{r_j} \left(\dfrac{\lambda}{\mu_j r_j}\right)^{n_j - r_j}, & \text{for } n_j \geq r_j , \end{cases}$$

n_j is the number of units at the j^{th} station, and $P(0) = P(0, 0, \ldots, 0)$ is found from the normalizing equation

$$\sum P(n_1, n_2, \ldots, n_k) = 1$$

in the following manner.

The $b(n_j)$ are all positive and $\sum_{n_j=0}^{\infty} b(n_j)$ is a convergent series $(j = 1, 2, \ldots, k)$; hence

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \left[\prod_{j=1}^{k} b(n_j) \right] = \prod_{j=1}^{k} \left[\sum_{n_j=0}^{\infty} b(n_j) \right],$$

whence writing

$$\sum_{n_j=0}^{\infty} b(n_j) = A_j, \quad j = 1, 2, \ldots, k,$$

it follows that

$$P(0) = \prod_{j=1}^{k} A_j^{-1} .$$

The main differences between Jackson's problem and that considered in this thesis are that Jackson is concerned with the behavior of the queues in the system, the in-process inventory and not the output, which in his steady-state solution is bound to have the same average rate as the input. He is not concerned with the effects of breakdowns and does not consider them, instead dealing with the system in normal operation.

He also makes some assumptions different from those which will be made here. He allows unbounded queues in front of every station, thus eliminating blocking, and more than one service channel at each station. He also assumes Poisson input to the system, while here it will be assumed that no runouts are allowed in the initial supply. This is often the case in industrial situations.

Hunt (8) considers three situations of interest. He treats service stations in series and allows blocking, but does not consider breakdowns. As measures of effectiveness he uses the average number of units in the system (again the in-process inventory) and maximum possible utilization in the steady state, which is defined as the ratio of mean arrival rate to mean service rate. Poisson arrivals to the system and exponential service times are assumed. Since in the steady state of such a system the average output rate will be the same as the mean input rate and since the denominator of the utilization ratio is known, the output can be readily determined.

Hunt's first case is the same as that treated by Jackson, where unbounded queues are allowed before each station. As before, no blocking can occur, and in the steady state which exists if the mean input rate is less than or equal to the mean service rate the output equals the input. In Hunt's terminology this is expressed by saying the maximum possible utilization is unity.

The second case is that in which an unbounded queue is allowed before the first station, but no queues are allowed before any others. Hunt finds expressions for the maximum possible utilization for two and three stations in series with unequal service rates and actual values of the maximum possible utilization for two, three, and four stations when all service rates are equal. The expressions for unequal service rates are:

For two stations,

$$\rho_{max} = \frac{\mu_2 (\mu_1 + \mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2} .$$

For three stations,

$$\rho_{max} = \frac{N}{D} ,$$

where

$$N = \mu_2\mu_3(\mu_2 + \mu_3)(\mu_1^4 + 2\mu_1^3\mu_2 + 3\mu_1^3\mu_3 + \mu_1^2\mu_2^2 + 4\mu_1^2\mu_2\mu_3$$

$$+ 3\mu_1^2\mu_3^2 + \mu_1\mu_2^2\mu_3 + 4\mu_1\mu_2\mu_3^2 + \mu_1\mu_3^3 + \mu_2^2\mu_3^2 + \mu_3^3\mu_2), \quad \text{and}$$

$$D = \mu_1^5(\mu_2^2 + \mu_2\mu_3 + \mu_3^2) + \mu_1^4(2\mu_2^3 + 5\mu_2^2\mu_3 + 5\mu_2\mu_3^2 + 3\mu_3^3) +$$

$$+ \mu_1^3(\mu_2^4 + 5\mu_2^3\mu_3 + 8\mu_2^2\mu_3^2 + 7\mu_2\mu_3^3 + 3\mu_3^4)$$

$$+ \mu_1^2(\mu_2^4\mu_3 + 5\mu_2^3\mu_3^2 + 8\mu_2^2\mu_3^3 + 5\mu_2\mu_3^4 + \mu_3^5)$$

$$+ \mu_1(\mu_2^4\mu_3^2 + 5\mu_2^3\mu_3^3 + 5\mu_2^2\mu_3^4 + \mu_2\mu_3^5)$$

$$+ (\mu_2^4\mu_3^3 + 2\mu_2^3\mu_3^4 + \mu_2^2\mu_3^5) ,$$

where μ_i is the mean service rate at the i^{th} station. For the special cases where all service rates are equal, the values of ρ_{max} are:

For two stations,

$$\rho_{max} = \frac{2}{3} .$$

For three stations,

$$\rho_{max} = \frac{22}{39} .$$

For four stations,

$$\rho_{max} \doteq 0.5115 .$$

Hunt's third case allows an unbounded queue before the first station and finite queues before each of the remaining stations. This system does not guarantee the absence of runouts in the initial supply, and it allows unlimited storage space before the first machine. Except for the assumption of Poisson input and the lack of consideration of breakdowns, this is the problem considered in this thesis. For two stations, Hunt finds the maximum possible utilization

$$\rho_{max} = \frac{\mu_2(\mu_1^{q+1} - \mu_2^{q+1})}{\mu_1^{q+2} - \mu_2^{q+2}}$$

where $q - 1$ is the length of the queue allowed before the second station. For three stations only the case where queues of length one are allowed to form before the second and third stations and where all service rates are equal is considered. The maximum utilization is given as approximately 0.6705.

It might be noted that Hunt has not derived a general expression for N stations in series except in the first case, where no blocking can occur. Instead, he has proceeded in a step-by-step approach, adding one machine at a time, and it may be that this is the only possible approach. It is also worthwhile to note that he states

> In the general N-stage problem, blocking occurs more frequently in the first stage than in any succeeding stage, and the maximum possible utilization for the first stage is the maximum possible utilization for the entire system. In the remainder of this work we shall refer to this quantity as ρ_{max}, the maximum possible utilization, but it should be remembered that ρ_{max} really refers only to the first stage.

While this statement is not supported by analysis it seems intuitively correct, and can logically be extended to cases where breakdowns are considered and where an infinite supply to the first stage is assumed. This is so because blocking can occur at all stations except the last and blocking at any station will affect all preceding stations. Since a steady state is assumed and no material is allowed to leave the system until it has passed through all stations, the average output of any machine will be the same as that of any other. It is therefore clear that the overall output of the system will be the same as the average output of the first station during the time it is neither blocked or broken down.

White and Christie (8) consider the case where there is a single service station and more than one priority class of units arriving for service, each class having its own Poisson arrival rate and exponential service rate. When a unit of higher priority class than the unit in service arrives it preempts the service facility and the unit in service is returned to the head of the queue for items of its class, where it must wait until service is completed on the preempting unit, or even longer if any other units of higher priority arrive in the meantime. A unit may therefore be repeatedly displaced by units of higher priority classes. The authors point out that a regular service facility servicing only one type of customer, but subject to breakdown, can be considered as a system of two priority classes with the breakdowns treated as a higher priority class with preemptive privileges. They make the assumption that the arrival process of breakdowns is cut off for the duration of repair periods so that "no latent breakdowns can build up at the facility when it is under repair." The system can then be treated as one of two priority classes with a maximum of one higher priority unit present at a time. They derive steady state equations for the queue length and average time in the system. In this steady state the output would again have the same average rate as the input. The case of several such stations in series with only finite queues allowed is not considered.

A valuable feature of this work is the author's discussion of the effects of preemption on service time distributions Since some units are preempted they have to enter service repeatedly, and there is a possibility that the exponential service rates for each class in isolation might not characterize the situation when several classes are considered together. The critical assumption is, of course, that the probability of a unit in service completing service in the next interval Δt is a constant regardless of the time the item has been in service and the queue length. One might intuitively expect units requiring long service times to be overrepresented since they have a greater probability of being preempted, and the mean of the distribution could be expected to depend on queue length, since a long queue could imply that the unit at the head of the queue is likely to be one requiring long service which has been displaced repeatedly (and needs as much time to complete service as it did on its first entry).

The authors consider the extreme possibilities. At one extreme all units are alike and the exponential service distribution results from the unpredictability of the server, as in the case of waiting for a particle from a constant radioactive source to strike or for an indifferent clerk to stamp a form. In this case, obviously all units have identical service time characteristics whether entering for the first time or after preemption. The other extreme is that the server operates at a constant rate on units whose intrinsic service requirements are exponentially distributed. This extreme has two alternatives, depending on whether service is started at the beginning of a unit each time it reenters or whether it is resumed at the point where it was interrupted. The authors show that in the latter case the service time characteristics are unchanged, but in the former case the mean of the service time distribution does in fact depend on queue length.

To meet the requirements of the critical assumption, it is necessary that all the entries from each queue length have exponentially distributed remaining service times independent of queue length. They show, however, that the service time distribution averaged over all queue lengths is exponential, and they use the inverse of this distribution, the service rate of the lower class in isolation minus the arrival rate of the higher class, as the effective service rate of the lower class when the two are considered together. For the purposes of this study this should not be a critical point, since in most industrial situations one would expect service to be resumed at the point where it was interrupted.

Bedworth (10) has attacked a problem similar to the one considered here, although his approach is one of simulation rather than analysis. He has designed and built a simulator to simulate a system of four machines in series with three interconnecting conveyors, all subject to breakdown, and an infinite supply to the first machine. Distributions of breakdowns and of service times are taken as desired and this information is fed into the simulator on punched paper tape. Counters keep track of the number of units in each queue and continuous recordings of the output and the queue states can be made on an oscillograph. A test program is provided to assure proper operation of the simulator before actual programs are run. The simulator effectively employs a Monte Carlo technique, giving a continuous recording of a queuing problem having been fed punched tapes prepared with the probability distributions desired by computers. The information sought is of the same type as that sought here. The differences are that the approach is one of simulation, that no limits are set on the size of the queues, that no more than four machines in series can be considered, and that Bedworth's simulator can use any distributions desired, even empirical distributions, while the analytical solution attempted here will be limited to Poisson arrivals of failures and exponential repair times and service times.

CHAPTER III

THE MODEL

Description of the Process.--A group of machines is arranged in series and raw material is fed to the first machine. No runouts are allowed in this initial supply, so the first machine will always have material when it is otherwise able to undertake work. Once a unit enters the system it must proceed through the entire series in order. Finite storage space is provided between each two machines. When a unit finishes service in a machine it proceeds immediately to the next machine where it commences service at once if the machine is in operating condition and there are no units ahead of it; otherwise it must wait in a queue in the storage space provided. When a queue reaches the capacity of its storage space the machine feeding into it is shut off so that no machine may complete a unit when there is no room in which to dispose of it. The capacity of each storage space includes the space represented by the unit which may be in service in the machine following that storage space. The last machine in the series can always dispose of its production, so it is never blocked. The service times in each machine are exponentially distributed.

Each machine is subject to random breadkdown, and the arrival times of breakdowns are Poisson distributed for each machine and thus are independent of the number of units waiting, number of other machines broken down, or any other considerations. A machine may break down while working or while idle but operable, but the Poisson arrival process of breakdowns is cut off while the machine is broken down, so that no latent breakdowns build up while the machine is under repair. The repair times for each machine are exponentially distributed.

In order for a steady state to exist it is also necessary that the average production rate of each machine multiplied by the mean proportion of time that it is not broken down be less than or equal to the average production rate of the following machine, multiplied by the mean proportion of the time it is operable. Otherwise there would be no time-independent solution, but instead the queue between the two machines would continuously increase if unbounded. If it were bounded it would tend to remain at its capacity whenever the first machine was not broken down, and the first machine would complete a unit each time the second machine did. This assumption is also made. This is similar to the requirement, mentioned in the introduction, that all machines must reduce their output to that of the slowest machine. Here, though, we allow faster operation as long as the effects of breakdown reduce the overall output of the faster machines to that of the slowest one. The maximum production possible from the system is the lowest value of production rate times the proportion of time operable when all machines are considered.

Method of Attack.--Since each machine is either broken down or operable, there are 2^n possible arrangements of broken down and operable machines when n machines are arranged in series; the probability that any particular combination exists at any given time is independent of the number of units waiting before any of the machines and is in fact dependent only on the breakdown and repair rates. The states of the individual machines may be further classified to indicate whether they are blocked or run out of material.

The overall production rate of the system may be determined by analyzing the behavior of the queues between the machines to determine the proportion of time that runout and blocking will occur. The probabilities of increases or decreases in queue lengths from any given lengths, given the conditions of all the machines, may be calculated, and since the probabilities of the machine conditions are known and independent of queue states, the total probability of any transition in queue state may be expressed as a sum of conditional probabilities. The derivatives with respect to time of the queue state probabilities may then be set equal to zero for the steady state, and the resulting expressions solved for all queue state probabilities in terms of any one of them. The final absolute probabilities are then determined by use of the normalizing equation which requires the sum of all probabilities to be unity. This general procedure will be followed and explained in detail in the work which follows.

Notation.--The following notation will be used:

μ_i = the mean service rate of the i^{th} machine

Λ_i = the mean breakdown rate of the i^{th} machine

M_i = the mean repair rate (reciprocal of mean repair time) of the i^{th} machine

$R_i = \frac{\Lambda_i}{M_i}$ = the mean proportion of time the i^{th} machine is broken down

n_i = the number of units waiting in the queue between the i^{th} and $i + 1^{st}$ machines

N_i = the maximum capacity of the queue between the i^{th} and $i + 1^{st}$ machines

$E_{n_1, n_2, \ldots, n_{z-1}}$ = the queue state where there are n_1 units waiting in the first queue, n_2 units waiting in the second queue, ..., and n_{z-1} units waiting in the $z - 1^{st}$ queue; $n_i = 1,2, \ldots, N_i$ and $i = 1,2, \ldots, z - 1$ when there are z machines in the series.

In addition, the condition of all machines is shown by the machine condition symbol $C(m_1\ m_2\ m_3\ \ldots\ m_z)$; $m_i \varepsilon$ 1, 0, b, x, m_i describes the condition of the i^{th} machine, and

1 = normal operation

0 = run out but not broken down

b = blocked but not broken down

x = broken down .

The following characteristics of the machine conditions may be seen:

(a) m_1 = 0 is impossible, since the first machine is never run out.

(b) m_i = 0 $(i > 1)$ only if the queue is in state

$$E_{n_1, \ldots, n_{i-1}, n_i, \ldots, n_{z-1}} \text{ with } n_{i-1} = 0.$$

(c) $m_z = b$ is impossible since the last machine is never blocked.

(d) $m_i = b$ $(i < z)$ only if the queue is in state $E_{n_1, \ldots, n_i, \ldots, n_{z-1}}$ with $n_i = N_i$.

(e) $m_i = b$ when $m_{i+1} = 0$ is impossible because a machine cannot be blocked and the next machine run out simultaneously.

(f) $m_i = 1$ $(1 < i < z)$ only if the queue is in state $E_{n_1, \ldots, n_{i-1}, n_i, \ldots, n_{z-1}}$ with $n_{i-1} \neq 0$ and $n_i \neq N_i$.

(g) $m_1 = 1$ only if the queue is in state $E_{n_1, \ldots, n_{z-1}}$ with $n_1 \neq N_1$.

(h) $m_z = 1$ only if the queue is in state $E_{n_1, \ldots, n_{z-1}}$ with $n_{z-1} \neq 0$.

(i) $m_i = x$ is possible for all values of i and in any queue state.

Figure 1 is a schematic diagram of the model.

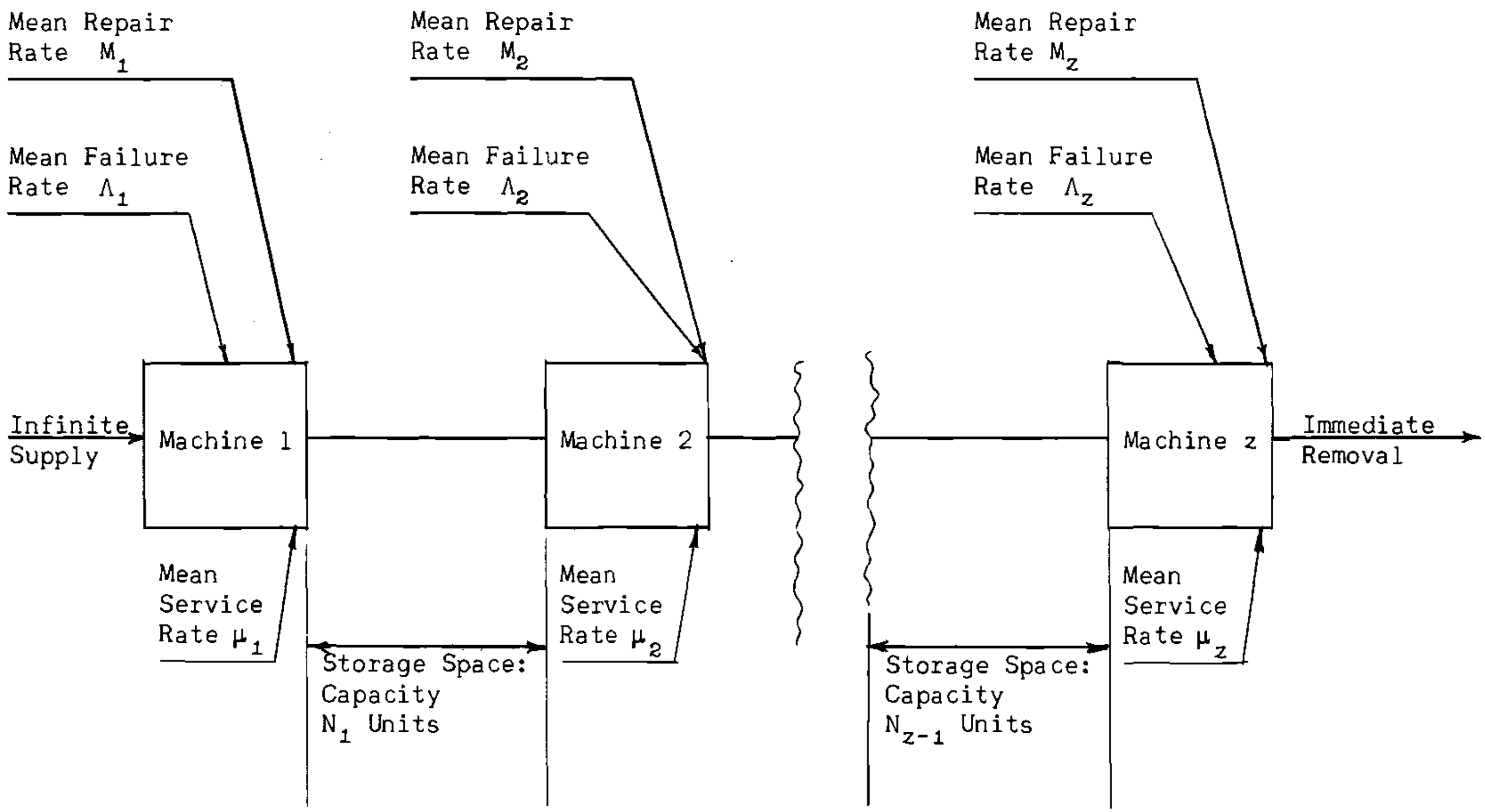

Figure 1. Schematic Diagram of the Model

CHAPTER IV

THE TWO MACHINE CASE

Transitions.--It will be assumed for the present that the capacity of the queue between the two machines is greater than two units. This restriction will later be relaxed. It may immediately be seen that only machine conditions $C(11)$, $C(10)$, $C(1x)$, $C(x1)$, $C(b1)$, $C(bx)$, $C(x0)$, and $C(xx)$ can exist, since the first machine never runs out, the second is never blocked, and $C(b0)$ is impossible. Furthermore, given any queue state E_{n_1}, only four of these eight machine conditions can exist; one corresponds to the first machine being broken down, another to the second, a third to both, and the fourth to neither broken down. Since the probability of a machine being broken down is independent of the queue state we can list the probabilities of the various machine conditions given the queue state. These are tabulated in Table 1.

Table 1. Possible Machine Conditions Given Queue States with Their Probabilities

Queue State E_o	$E_{n_1}(1 \le n_1 \le N_1-1)$	E_{N_1}	Probability of Machine Condition
$C(10)$	$C(11)$	$C(b1)$	$1 - R_1 - R_2 + R_1R_2$
$C(1x)$	$C(1x)$	$C(bx)$	$R_2 - R_1R_2$
$C(x0)$	$C(x1)$	$C(x1)$	$R_1 - R_1R_2$
$C(xx)$	$C(xx)$	$C(xx)$	R_1R_2

Given the queue state and machine condition, only certain transitions in machine condition are possible, and the probabilities of these transitions can be calculated. It is first noted that during a time interval Δt, the probability that the i^{th} machine will complete a unit is $\mu_i \Delta t$ if it is working on a unit at the start of the interval, the probability that it will break down is $\Lambda_i \Delta t$ if it is not broken down, and the probability that it will complete repairs if it is broken down is $M_i \Delta t$. In calculating the transition probabilities, it is assumed that the probability of two or more breakdowns, completions of repairs, or completions of service during time Δt is negligible, and these probabilities are ignored. This assumption is implicit in the assumption of Poisson and exponential distributions, and is here justified by a quotation from Saaty (11), referring to a Poisson process with parameter λ:

> . . . we note that, during a time interval t, the probability of no arrivals is $e^{-\lambda t}$ and that of a single arrival is $\lambda t e^{-\lambda t}$; hence the probability of more than one arrival is
>
> $$1 - (e^{-\lambda t} + \lambda t e^{-\lambda t}) = 1 - \{[1 - \lambda t + \frac{(\lambda t)^2}{2!} - \cdots] + \lambda t[1 - \lambda t + \frac{\lambda t^2}{2!} - \cdots]\}$$
>
> $$= \frac{(\lambda t)^2}{2!} + \cdots = O(t^2) ,$$
>
> a function which behaves at t^2.
>
> Thus if t is small, terms with t^2 are negligible compared with terms without t or with the first power of t. Hence for small t the probability of more than one arrival is negligible.
>
> . . ., let us assume these properties, i.e., that the probability of a single arrival during a small interval Δt is $\lambda \Delta t$ and that of more than a single arrival during Δt is negligible; then we can derive the Poisson distribution, which of course has these properties.

To illustrate the calculations, consider the case where the queue is in state E_{n_1} with n_1 greater than 1 and less than N_1-1. It is first noted that of the eight possible machine conditions, it is impossible to reach C(10), C(b1), C(bx), or C(x0) with only one completion of service, and the probability of two or more completions in time Δt is considered negligible. Considering the remaining four machine conditions and starting in C(11), in order to remain in C(11) neither machine may break down and the probability of this is $(1 - \Lambda_1\Delta t)(1 - \Lambda_2\Delta t) \cong 1 - \Lambda_1\Delta t - \Lambda_2\Delta t$. To accomplish a transition to C(1x), the first machine may not break down but the second must, and this probability is $(1 - \Lambda_1\Delta t)(\Lambda_2\Delta t) \cong \Lambda_2\Delta t$. The probability of a transition to C(x1) is $(\Lambda_1\Delta t)(1 - \Lambda_2\Delta t) \cong \Lambda_2\Delta t$, and to C(xx) it is $(\Lambda \Delta t)(\Lambda \Delta t) \cong 0$. The sum of these probabilities is unity, which provides a check. Additional sample calculations for starting queue state E_{n_1} with $1 < n_1 < N_1-1$ and starting machine conditions C(1x), C(x1), and C(xx) and for starting queue state E_1 with starting machine conditions C(11) and C(1x) are given in Appendix A. By similar calculations the machine condition transition probabilities in Tables 2 through 6 were derived.

Table 2. Transition Probabilities from Starting Queue State E_0

Starting Machine Condition	Final Machine Condition C(11)	C(10)	C(1x)	C(x1)	C(x0)	C(xx)
C(10)	$\mu_1\Delta t$	$1-\Lambda_1\Delta t-\Lambda_2\Delta t$ $-\mu_1\Delta t$	$\Lambda_2\Delta t$	0	$\Lambda_1\Delta t$	0
C(1x)	0	$M_2\Delta t$	$1-\Lambda_1\Delta t-M_2\Delta t$	0	0	$\Lambda_1\Delta t$
C(x0)	0	$M_1\Delta t$	0	0	$1-M_1\Delta t-\Lambda_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	0	$M_1\Delta t$	0	$M_2\Delta t$	$1-M_1\Delta t-M_2\Delta t$

Table 3. Transition Probabilities from Starting Queue State E_1

Starting Machine Condition	Final Machine Condition C(11)	C(10)	C(1x)	C(x1)	C(x0)	C(xx)
C(11)	$1 - \Lambda_1 \Delta t - \Lambda_2 \Delta t - \mu_2 \Delta t$	$\mu_2 \Delta t$	$\Lambda_2 \Delta t$	$\Lambda_1 \Delta t$	0	0
C(1x)	$M_2 \Delta t$	0	$1 - \Lambda_1 \Delta t - M_2 \Delta t$	0	0	$\Lambda_1 \Delta t$
C(x1)	$M_1 \Delta t$	0	0	$1 - M_1 \Delta t - \Lambda_2 \Delta t - \mu_2 \Delta t$	$\mu_2 \Delta t$	$\Lambda_2 \Delta t$
C(xx)	0	0	$M_1 \Delta t$	$M_2 \Delta t$	0	$1 - M_1 \Delta t - M_2 \Delta t$

Table 4. Transition Probabilities from Starting Queue State E_{n_1} $(1 < n_1 < N_1 - 1)$

Starting Machine Condition	Final Machine Condition C(11)	C(1x)	C(x1)	C(xx)
C(11)	$1 - \Lambda_1 \Delta t - \Lambda_2 \Delta t$	$\Lambda_2 \Delta t$	$\Lambda_1 \Delta t$	0
C(1x)	$M_2 \Delta t$	$1 - \Lambda_1 \Delta t - M_2 \Delta t$	0	$\Lambda_1 \Delta t$
C(x1)	$M_1 \Delta t$	0	$1 - M_1 \Delta t - \Lambda_2 \Delta t$	$\Lambda_2 \Delta t$
C(xx)	0	$M_1 \Delta t$	$M_2 \Delta t$	$1 - M_1 \Delta t - M_2 \Delta t$

Table 5. Transition Probabilities from Starting Queue State $E_{N_1} - 1$

Starting Machine Condition	Final Machine Condition C(11)	C(1x)	C(b1)	C(bx)	C(x1)	C(xx)
C(11)	$1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_1\Delta t$	$\Lambda_2\Delta t$	$\mu_1\Delta t$	0	$\Lambda_1\Delta t$	0
C(1x)	$M_2\Delta t$	$1 - \Lambda_1\Delta t - M_2\Delta t - \mu_1\Delta t$	0	$\mu_1\Delta t$	0	$\Lambda_1\Delta t$
C(x1)	$M_1\Delta t$	0	0	0	$1 - M_1\Delta t - \Lambda_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	$M_1\Delta t$	0	0	$M_2\Delta t$	$1 - M_1\Delta t - M_2\Delta t$

Table 6. Transition Probabilities from Starting Queue State E_{N_1}

Starting Machine Condition	Final Machine Condition C(11)	C(1x)	C(b1)	C(bx)	C(x1)	C(xx)
C(b1)	$\mu_2\Delta t$	0	$1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t$	$\Lambda_2\Delta t$	$\Lambda_1\Delta t$	0
C(bx)	0	0	$M_2\Delta t$	$1 - \Lambda_1\Delta t - M_2\Delta t$	0	$\Lambda_1\Delta t$
C(x1)	0	0	$M_1\Delta t$	0	$1 - M_1\Delta t - \Lambda_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	0	0	$M_1\Delta t$	$M_2\Delta t$	$1 - M_1\Delta t - M_2\Delta t$

The conditional probabilities for all possible transitions, given the initial queue state and machine condition, have now been derived. The machine conditions must now be eliminated so that the transition probabilities between queue states may be determined. In transitions to machine conditions where the first machine is blocked or the second is run out, the resultant queue state is fixed. In some other transitions the resultant queue state is also fixed, as for example in the transition from E_1 and $C(x1)$ to $C(x1)$. Here the queue state must remain E_1 since the first machine did not complete repairs and could not have completed a unit and the second machine has not run out and so must still be working on the unit that was in the queue before the transition. In still other transitions there are two or more possible resultant queue states, but their probabilities can be calculated. It may be noted that changes of more than one unit in queue state involve terms of order $(\Delta t)^2$ and higher, so their probabilities are negligible.

To illustrate the method of calculating queue state transition probabilities, consider the case where the initial queue state is E_1 and the initial machine condition is $C(11)$. The probability of this machine condition, given E_1, is, from Table 1, $1 - R_1 - R_2 + R_1R_2$. From Table 3, the possible machine conditions after transition are determined, along with their probabilities. The transitions to $C(11)$ and $C(x1)$ can result in queue states of E_1 or E_2, the transition to $C(1x)$ can result in E_0, E_1, or E_2, and the transition to $C(10)$ must result in E_0. Given that a transition to $C(11)$ or $C(x1)$

has occurred, the probability that state E_1 resulted is $(1 - \mu_1 \Delta t)$, the probability that the first machine did not complete a unit, while the probability that state E_2 resulted is $\mu_1 \Delta t$. Given that a transition to $C(1x)$ occurred, the probability of state E_1 resulting is $(\mu_1 \Delta t)(\mu_2 \Delta t) + (1 - \mu_1 \Delta t)(1 - \mu_2 \Delta t) \cong 1 - \mu_1 \Delta t - \mu_2 \Delta t$, the probability of E_0 is $(\mu_2 \Delta t)(1 - \mu_1 \Delta t) \cong \mu_2 \Delta t$, and the probability of E_2 is $(\mu_1 \Delta t)(1 - \mu_2 \Delta t) \cong \mu_1 \Delta t$. Therefore, the probability of remaining in state E_1 given that the initial conditions were E_1 and $C(11)$ is the sum of the probabilities of the transitions to each machine condition, each multiplied by the respective probability that state E_1 resulted given that the transition to that machine condition occurred. In this example, the probability of a transition to state E_0 is $\mu_2 \Delta t$, the probability of remaining in E_1 is

$$(1 - \mu_1 \Delta t)(1 - \Lambda_1 \Delta t - \Lambda_2 \Delta t - \mu_2 Pt) + (1 - \mu_1 \Delta t - \mu_2 \Delta t)(\Lambda_2 \Delta t)$$
$$+ (1 - \mu_1 \Delta t)(\Lambda_1 \Delta t)$$
$$\cong 1 - \mu_1 \Delta t - \mu_2 \Delta t ,$$

and the probability of a transition to E_2 is

$$(\mu_1 \Delta t)(1 - \Lambda_1 \Delta t \; \Lambda_2 \Delta t - \mu_2 \Delta t) + (\mu_1 \Delta t)(\Lambda_2 \Delta t) + (\mu_1 \Delta t)(\Lambda_1 \Delta t) \cong \mu_1 \Delta t .$$

This process is then extended over the remaining possible machine conditions in state E_1, and the probabilities of the transitions from E_1 to other states are derived using the principles of conditional probability.

By definition the conditional probability that an event, A, will occur, under the assumption that a second event, B, has occurred, denoted $P(A|B)$, is equal to the probability that A and B occur together, denoted $P(A,B)$, divided by the unconditional probability of B, provided that the probability of B is not zero. If it is zero, $P(A|B)$ is undefined. This definition can therefore be written

$$P(A,B) = P(A|B)\ P(B), \quad \text{so} \quad \sum_i P(A,B_i) = \sum_i P(A|B_i)\ P(B_i) \ .$$

The unconditional probability of A, $P(A)$, can be written

$$P(A) = \sum_i P(A,B_i)$$

if the events B_i are exhaustive and mutually exclusive, that is, if the sum of their unconditional probabilities is unity and the occurrence of any one of them precludes the occurrence of any of the remaining events at the same time. If these conditions are met, then

$$P(A) = \sum_i P(A|B_i)\ P(B_i) \ .$$

Since there are four possible machine conditions in each queue state and these machine conditions are mutually exclusive and exhaustive, the absolute probability of a transition from any queue state to any other can be expressed as the sum of the conditional probabilities for all four machine conditions that the transition in question will occur under the assumption that a particular machine condition is in effect at the start of the period Δt, each multiplied by the unconditional probability that that particular machine condition is in fact in effect. For example, the transition probability from E_1 to E_2, denoted p_{12}, is expressed

$$p_{12} = \sum_{i=C(11),C(1x),C(x1),C(xx)} (p_{12} \mid \text{initial machine condition } i)\, P(\text{initial machine condition } i).$$

p_{oo} and p_{o1} are calculated below.

$$\begin{aligned} p_{oo} = {} & (1 - R_1 - R_2 + R_1R_2)[(1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_1\Delta t) + \Lambda_2\Delta t(1 - \mu_1\Delta t) + \Lambda_1\Delta t] \\ & + (R_2 - R_1R_2)[M_2\Delta t + (1 - \Lambda_1\Delta t - M_2\Delta t)(1 - \mu_1\Delta t) + (\Lambda_1\Delta t)(1 - \mu_1\Delta t)] \\ & + (R_1 - R_1R_2)[M_1\Delta t + 1 - M_1\Delta t - \Lambda_2\Delta t + \Lambda_2\Delta t] \\ & + (R_1R_2)[(M_1\Delta t)(1 - \mu_1\Delta t) + M_2\Delta t + 1 - M_1\Delta t - M_2\Delta t] \end{aligned}$$

$$p_{oo} = 1 - \mu_1\Delta t + R_1\mu_1\Delta t = 1 - \mu_1\Delta t(1 - R_1)$$

$$\begin{aligned} p_{o1} = {} & (1 - R_1 - R_2 + R_1R_2)[\mu_1\Delta t + (\Lambda_2\Delta t)(\mu_1\Delta t)] + (R_2 - R_1R_2)(\mu_1\Delta t) \\ & + (R_1 - R_1R_2)(0) + (R_1R_2)(\mu_1\Delta t)(M_1\Delta t) \end{aligned}$$

$$p_{o1} = \mu_1\Delta t - R_1\mu_1\Delta t = \mu_1\Delta t(1 - R_1) .$$

As a check on the calculations, it may be seen that $p_{oo} + p_{o1} = 1$. Similarly, $p_{1o} = \mu_2\Delta t(1 - R_2)$, $p_{11} = 1 - \mu_2\Delta t(1 - R_2) - \mu_1\Delta t(1-R_1)$, $p_{12} = \mu_1\Delta t(1 - R_1)$, and $p_{1o} + p_{11} + p_{12} = 1$.

Upon extending this procedure to all other initial queue states and machine conditions, the transition probabilities are determined to be

$$p_{oo} = 1 - \mu_1 \Delta t(1 - R_1)$$

$$p_{o1} = \mu_1 \Delta t(1 - R_1)$$

$$\left.\begin{aligned} p_{n_1,n_1} &= 1 - \mu_1 \Delta t(1 - R_1) - \mu_2 \Delta t(1 - R_2) \\ p_{n_1,n_1-1} &= \mu_2 \Delta t(1 - R_2) \\ p_{n_1,n_1+1} &= \mu_1 \Delta t(1 - R_1) \end{aligned}\right\} 1 \leq n_1 \leq N_1 - 1$$

$$p_{N_1,N_1} = 1 - \mu_2 \Delta t(1 - R_2)$$

$$p_{N_1,N_1-1} = \mu_2 \Delta t(1 - R_2) .$$

The transition probabilities between all queue states have now been derived. They were, however, derived under the assumption that N_1 is greater than two units. This restriction will now be relaxed, and it will be shown that this relaxation does not affect the probabilities. First consider the case where $N_1 = 2$. Here transitions starting in E_o and E_2 $(E_2 = E_{N_1})$ are not affected and their probabilities are the same as those where N_1 is greater than 2. Transitions starting in E_1 are affected, however, since E_1 is now also E_{N_1-1}, and all the machine conditions which could previously be reached from either E_1 (Table 3) or E_{N_1-1} (Table 5) can now be reached from E_1. Using the same procedure as before, the machine condition probabilities starting from E_1 are derived and shown in Table 7.

Table 7. Transition Probabilities from Starting Queue State E_1 when $N_1 = 2$

Starting Machine Condition	Final Machine Condition C(11)	C(10)	C(1x)	C(b1)	C(bx)	C(x1)	C(x0)	C(xx)
C(11)	$1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t - \mu_2\Delta t$	$\mu_2\Delta t$	$\Lambda_2\Delta t$	$\mu_1\Delta t$	0	$\Lambda_1\Delta t$	0	0
C(1x)	$M_2\Delta t$	0	$1 - \Lambda_1\Delta t - M_2\Delta t - \mu_1\Delta t$	0	$\mu_1\Delta t$	0	0	$\Lambda_1\Delta t$
C(x1)	$M_1\Delta t$	0	0	0	0	$1 - M_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t$	$\mu_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	0	$M_1\Delta t$	0	0	$M_2\Delta t$	0	$1 - M_1\Delta t - M_2\Delta t$

By a comparison of Table 7 and Table 3 it can be seen that with N_1 greater than two units, only machine conditions C(11), C(10), C(1x), and C(x1) could be reached from E_1 and C(11), but with $N_1 = 2$, C(b1) can also be reached. The probability of remaining in C(11) is reduced by $\mu_1 \Delta t$, which is also the probability of a transition to C(b1). The probabilities of transitions to all other machine conditions remain unchanged. The resultant queue state probabilities for transitions to C(10) and C(x1) are unchanged, but a transition to C(11) now fixes the queue state at E_1, and E_o and E_1 are now the only possible resultant queue states for transitions to C(1x), since E_2 would lead to C(bx). The probability of E_o resulting from a transition to C(11) is now $\mu_2 \Delta t$, and of E_1, $(1 - \mu_2 \Delta t)$. The sum of the probabilities of all transitions to E_o is now $\mu_2 \Delta t + (\mu_2 \Delta t)(\Lambda_2 \Delta t) = \mu_2 \Delta t$, as it was with N_1 greater than 2, and similarly the probability of remaining in E_1 remains $1 - \mu_1 \Delta t - \mu_2 \Delta t$ and that of a transition to E_2 remains $\mu_1 \Delta t$. By extending this process to all the other possible initial machine conditions in E_1, it is found that similar changes occur, but that the total probabilities of remaining in E_1, making a transition to E_o, or making a transition to E_2 remain the same.

When $N_1 = 1$ there are only two possible queue states, E_o and E_1, and machine condition C(11) does not exist. Possible initial machine conditions in E_o are C(10), C(1x), C(x0), and C(xx). In E_1 they are C(b1), C(bx), C(x1), and C(xx). Again using the same procedure, machine condition transition probabilities starting from E_o and E_1 were derived and tabulated in Table 8 and Table 9.

Table 8. Transition Probabilities from Starting Queue State E_o when $N_1 = 1$

Starting Machine Condition	Final Machine Condition C(10)	C(1x)	C(b1)	C(bx)	C(x1)	C(x0)	C(xx)
C(10)	$1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_1\Delta t$	$\Lambda_2\Delta t$	$\mu_1\Delta t$	0	0	$\Lambda_1\Delta t$	0
C(1x)	$M_2\Delta t$	$1 - \Lambda_1\Delta t - M_2\Delta t - \mu_1\Delta t$	0	$\mu_1\Delta t$	0	0	$\Lambda_1\Delta t$
C(x0)	$M_1\Delta t$	0	0	0	0	$1 - M_1\Delta t - \Lambda_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	$M_1\Delta t$	0	0	0	$M_2\Delta t$	$1 - M_1\Delta t - M_2\Delta t$

Table 9. Transition Probabilities from Starting Queue State E_1 when $N_1 = 1$

Starting Machine Condition	Final Machine Condition C(10)	C(1x)	C(b1)	C(bx)	C(x1)	C(x0)	C(xx)
C(b1)	$\mu_2\Delta t$	0	$1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t$	$\Lambda_2\Delta t$	$\Lambda_1\Delta t$	0	0
C(bx)	0	0	$M_2\Delta t$	$1 - \Lambda_1\Delta t - M_2\Delta t$	0	0	$\Lambda_1\Delta t$
C(x1)	0	0	$M_1\Delta t$	0	$1 - M_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t$	$\mu_2\Delta t$	$\Lambda_2\Delta t$
C(xx)	0	0	0	$M_1\Delta t$	$M_2\Delta t$	0	$1 - M_1\Delta t - M_2\Delta t$

Upon examining the case where the initial conditions are E_o and $C(10)$, one may see that the resultant queue state will be E_o when transitions are made to machine states $C(10)$, $C(1x)$, and $C(x0)$, and E_1 when transitions to $C(b1)$ are made. The probability of remaining in E_o given initial conditions of E_o and $C(10)$ is therefore $1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_1\Delta t + \Lambda_2\Delta t + \Lambda_1\Delta t = 1 - \mu_1\Delta t$, and when multiplied by $1 - R_1 - R_2 + R_1R_2$, the probability of $C(10)$ given E_o, it becomes $(1 - \mu_1\Delta t)(1 - R_1 - R_2 + R_1R_2)$. The probability of a transition to E_1 is $(\mu_1\Delta t)(1 - R_1 - R_2 + R_1R_2)$. With initial machine condition $C(x0)$ or $C(xx)$ all possible final machine conditions result in queue state E_o, so the probability of remaining in E_o in this way is simply the sum of their original probabilities, $R_1 - R_1R_2 + R_1R_2 = R_1$. With initial machine condition $C(1x)$, transitions to $C(10)$ and $C(1x)$ lead to resultant queue state E_o, those to $C(bx)$ result in E_1, and those to $C(xx)$ to E_o with probability $(1 - \mu_1\Delta t)$ and to E_1 with probability $\mu_1\Delta t$. Upon multiplying these by $R_2 - R_1R_2$, the probability of $C(1x)$ given E_o, and combining all terms, p_{oo} is seen to be $\mu_1\Delta t(1 - R_1)$, the same as it was when N_1 was 2 or greater. Similar calculations show that p_{o1} remains the same and p_{1o} and p_{11} are the same as p_{N_1,N_1-1} and p_{N_1,N_1} when N_1 is 2 or greater. The queue state transition probabilities which were derived under the assumption that N_1 was greater than 2 are thus valid for all values of N_1.

<u>The Equations and Their Solution</u>.--Letting $P_n(t)$ denote the probability that the queue is in state E_n at time t, we write the equations for the probabilities of each queue state:

$$P_o(t + \Delta t) = P_o(t)\ p_{oo} + P_1(t)\ p_{1o}$$

$$P_n(t + \Delta t) = P_n(t)\ p_{nn} + P_{n-1}(t)\ p_{n-1,n} + P_{n+1}(t)\ p_{n+1,n}$$

$$(1 \leq n \leq N_1 - 1)$$

$$P_{N_1}(t + \Delta t) = P_{N_1}(t)\ p_{N_1,N_1} + P_{N_1-1}(t)\ p_{N_1-1,N_1}.$$

These equations are solved recursively for all state probabilities in terms of P_o in the following manner.

$$P_o(t + \Delta t) = P_o(t)[1 - (\mu_1 \Delta t)(1 - R_1)] + P_1(t)(\mu_2 \Delta t)(1 - R_2)$$

Upon rearranging terms and taking the limit as Δt approaches zero, the result is

$$\lim_{\Delta t \to o} \frac{P_o(t + \Delta t) - P_o(t)}{\Delta t} = - P_o(t)\ \mu_1(1 - R_1) + P_1(t)\ \mu_2(1 - R_2)$$

$$= \frac{d\ P_o(t)}{dt}.$$

The derivative with respect to time is set equal to zero, eliminating dependence on time, and the time-independent or steady state probabilities, denoted P_n, are then determined.

$$- P_o\ \mu_1(1 - R_1) + P_1\ \mu_2(1 - R_2) = 0$$

$$P_1 = \frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\ P_o.$$

Further recursive solutions will indicate that $P_n = \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^n P_o$

for all values of n. However, starting with the result that

$P_1 = \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^1 P_o$, an inductive proof is given to establish the

general case for all values of n including N_1.

For any value of n greater than one, the derivative with respect to time of $n - 1$, when set equal to zero, is

$$P'_{n-1} = P_{n-2}\, p_{n-2,n-1} + P_{n-1}[p_{n-1,n-1} - 1] + P_n\, p_{n,n-1} = 0.$$

Then

$$\left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n-2} P_o \mu_1(1 - R_1) + \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n-2} P_o[-\mu_1(1 - R_1) - \mu_2(1 - R_2)]$$

$$+ P_n\, \mu_2(1 - R_2) = 0$$

and

$$P_n = -\left[\frac{\mu_1(1-R_1)}{\mu_2(1-R_2)}\right]^{n-2} P_o\, \frac{\mu_1(1-R_1)}{\mu_2(1-R_2)} + \left[\frac{\mu_1(1-R_1)}{\mu_2(1-R_2)}\right]^{n-1} P_o\, \frac{\mu_1(1-R_1)+\mu_2(1-R_2)}{\mu_2(1-R_2)}$$

$$P_n = -\left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n-1} P_o + \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n} P_o + \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n-1} P_o$$

$$P_n = \left[\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)}\right]^{n} P_o\,.$$

The probabilities for all values of P_n in terms of P_o for any value of N_1 may now be calculated, and the absolute values of all state probabilities may be found by use of the normalizing equation,

$$\sum_{n=o}^{N_1} P_n = 1 .$$

It may be noted that Λ_i and M_i do not appear in the expression for queue state probabilities, but only their ratio, R_i . It is not necessary to know the exact values of Λ_i and M_i as long as R_i is known, but it should be pointed out that the model assumes that they are Poisson and exponentially distributed, respectively. When the queue state probabilities are known the output of the system may be computed by the procedure used in the example in the next section.

An Example.--Two machines are to be arranged in series. The first has a capacity of 150 units per hour when operating, breaks down on the average once every two and a half hours, and requires an average of 15 minutes repair time. The second has a capacity of 200 units per hour, averages a breakdown every hour and a quarter, and requires an average of 15 minutes repair time. It is desired to determine the optimum storage capacity between the two machines. It has been established that the cost of providing an additional unit of capacity will be justified if that unit increases the output of the system by at least three units per hour.

It is seen that $\Lambda_1 = 0.4$ breakdowns per hour and $M_1 = 4$ repairs per hour, so $R_1 = 0.10$. Similarly, $R_2 = 0.20$. $\mu_1(1-R_1) = 135$

units per hour and $\mu_2(1 - R_2) = 160$ units per hour, so the maximum capacity of the system is 135 units per hour. $\frac{\mu_1(1 - R_1)}{\mu_2(1 - R_2)} = 0.844$, and the powers of 0.844 are listed below:

$$(0.844)^2 = 0.712 \qquad (0.844)^5 = 0.428$$
$$(0.844)^3 = 0.601 \qquad (0.844)^6 = 0.361$$
$$(0.844)^4 = 0.507 \qquad (0.844)^7 = 0.304$$

The queue state probabilities are now computed when N_1 takes on values of 1 through 7.

$N_1 = 1$	$N_1 = 2$	$N_1 = 3$
$1.844\ P_o = 1.0$	1.844 0.712 $2.566\ P_o = 1.0$	2.556 0.601 $3.157\ P_o = 1.0$
$P_o = 0.542$	$P_o = 0.391$	$P_o = 0.317$
$P_1 = 0.457$	$P_1 = 0.330$	$P_1 = 0.264$
	$P_2 = 0.278$	$P_2 = 0.226$
		$P_3 = 0.191$

$N_1 = 4$	$N_1 = 5$	$N_1 = 6$
3.157 0.507 $3.664\ P_o = 1.0$	3.664 0.428 $4.092\ P_o = 1.0$	4.092 0.361 $4.453\ P_o = 1.0$
$P_o = 0.273$	$P_o = 0.244$	$P_o = 0.225$
$P_1 = 0.230$	$P_1 = 0.206$	$P_1 = 0.190$
$P_2 = 0.194$	$P_2 = 0.174$	$P_2 = 0.160$
$P_3 = 0.164$	$P_3 = 0.147$	$P_3 = 0.135$
$P_4 = 0.138$	$P_4 = 0.124$	$P_4 = 0.114$
	$P_5 = 0.104$	$P_5 = 0.096$
		$P_6 = 0.081$

$\underline{N_1 = 7}$

$$
\begin{aligned}
&4.453\\
&\underline{0.064}\\
&4.758\ P_o = 1.0\\
&P_o = 0.210\\
&P_1 = 0.177\\
&P_2 = 0.150\\
&P_3 = 0.126\\
&P_4 = 0.106\\
&P_5 = 0.090\\
&P_6 = 0.076\\
&P_7 = 0.064
\end{aligned}
$$

It may already be seen that each additional unit of capacity decreases the probability of machine 1 being blocked (P_{N_1}) and of machine 2 running out (P_o), but that each additional unit reduces these probabilities less than the unit before it. To determine the optimum point the actual output of the system must be calculated.

Since in the steady state the production that goes through either machine must also go through the other, the output of either machine can be calculated to determine the output of the system. In this example, however, the production of both machines will be calculated to illustrate the method and also to provide a check.

The output of the first machine is μ_1 times the proportion of time it is not broken down or blocked. The probability that it is broken down is R_1, and the probability that it is blocked is P_{N_1}. These conditions are not mutually exclusive; the machine can be blocked and broken down at the same time. Therefore, the total

Page missing from thesis

The calculations for machine 2 confirm those for machine 1. In this example the optimum plan is to provide six units of storage capacity (including the space for the unit in machine 2) between the two machines, since the seventh unit fails to increase the production by the desired three units per hour.

CHAPTER V

THE THREE MACHINE CASE

Transitions.--With three machines there are eight general classes of machine conditions, each class corresponding to a different arrangement of broken down and operable machines. These classes, together with their absolute probabilities and the possible machine conditions in each, are listed in Table 10.

Table 10. Machine Condition Classes, Their Probabilities, and Their Possible Machine Conditions: Three Machine Case

Class	Probability	Possible Machine Conditions
1.	$R_1R_3 - R_1R_2R_3$	C(x1x), C(x0x), C(xbx)
2.	$R_1R_2 - R_1R_2R_3$	C(xx1), C(xx0)
3.	$R_1R_2R_3$	C(xxx)
4.	$R_1 - R_1R_2 - R_1R_3 + R_1R_2R_3$	C(x11), C(x10), C(x01), C(x00), C(xb1)
5.	$R_3 - R_1R_3 - R_2R_3 + R_1R_2R_3$	C(11x), C(10x), C(1bx), C(b1x), C(bbx)
6.	$R_2 - R_1R_2 - R_2R_3 + R_1R_2R_3$	C(1x1), C(1x0), C(bx1), C(bx0)
7.	$R_2R_3 - R_1R_2R_3$	C(1xx), C(bxx)
8.	$1 - R_1 - R_2 - R_3 + R_1R_2$	C(111), C(110), C(101), C(100),
	$+ R_1R_3 + R_2R_3 - R_1R_2R_3$	C(1b1), C(b11), C(b10), C(bb1)

The queue states here are denoted by E_{n_1,n_2}, and with one exception there is one and only one machine condition from each class which may exist with any given queue state. The exception is in queue state E_{o,N_2}. Here conditions $C(xOx)$ and $C(xbx)$, $C(xO1)$ and $C(xb1)$, $C(1Ox)$ and $C(1bx)$, and $C(1O1)$ and $C(1b1)$ exist simultaneously in pairs since in E_{o,N_2} the second machine is both blocked and run out. This presents no real problem, but it does necessitate the introduction of some additional terminology for this particular state. The new initial machine states $C(x|{}^{o}_{b}|x)$, $C(x|{}^{o}_{b}|1)$, $C(1|{}^{o}_{b}|x)$, and $C(1|{}^{o}_{b}|1)$, possible only in E_{o,N_2} are here defined. The same procedure that was used in the two machine case may then be used to calculate the machine condition transition probabilities. Consideration must be given here to the fact that from these "ambiguous" conditions, transitions may be made to the same or other ambiguous conditions or to an "unambiguous" condition which may have been represented in the initial ambiguous condition. For example, transitions possible from $C(x|{}^{o}_{b}|1)$ include the ones to $C(x|{}^{o}_{b}|x)$, to $C(x|{}^{o}_{b}|1)$, and also to $C(xO1)$ with machine 2 no longer blocked. If all such possibilities are recognized the original procedure may be used in a straightforward manner to determine all machine condition transition probabilities. In the same manner as before, the machine conditions are then eliminated by calculating the conditional probabilities of resultant queue states given initial queue states and machine conditions and taking the queue state transition probabilities as sums of conditional probabilities.

The derivation proceeds by first assuming N_1 and N_2 both greater than two units and determining the queue state transition probabilities, then relaxing the restrictions on N_1 and N_2 and showing that the probabilities are not affected. Sample calculations are given in Appendix B for machine condition transition probabilities starting in E_{n_1,n_2} with $1 < n_1 < N_1 - 1$ and $1 < n_2 < N_2 - 1$. Appendix C gives sample calculations for resultant queue state probabilities with initial conditions E_{n_1,n_2} $(1 < n_1 < N_1 - 1)$, $(1 - n_2 < N_2 - 1)$ and $C(xx1)$. The queue state transition probabilities are found to be as follows:

$$p_{n_1,n_2;n_1+1,n_2} = \mu_1 \Delta t(1 - R_1) \quad \text{for } n_1 < N_1 \text{ and zero otherwise,}$$

$$p_{n_1,n_2;n_1-1,n_2+1} = \mu_2 \Delta t(1 - R_2) \quad \text{for } 1 \leq n_1 \leq N_1, \; n_2 < N_2, \text{ and zero otherwise,}$$

$$p_{n_1,n_2;n_1,n_2-1} = \mu_3 \Delta t(1 - R_3) \quad \text{for } n_2 > 0 \text{ and zero otherwise,}$$

$$p_{n_1,n_2;n_1,n_2} = 1 - (p_{n_1,n_2;n_1+1,n_2} + p_{n_1,n_2;n_1-1,n_2+} + p_{n_1,n_2;n_1,n_2-1}) \, .$$

The relations between transition probabilities may be more readily seen with the aid of a diagram. Figure 2 shows the possible transitions between queue states when $N_1 = 3$ and $N_2 = 3$. For more concise terminology, let $a = \mu_1(1 - R_1)$, $b = \mu_2(1 - R_2)$, and $c = \mu_3(1 - R_3)$. The probabilities of the various transitions are shown on the arrows indicating the transition. The probability of remaining in any state is unity minus the sum of the probabilities of the transitions which may be made from it. For example, the probability of remaining in E_{23} is $1 - a\Delta t - c\Delta t$, the probability of remaining in E_{30} is $1 - b\Delta t$, and the probability of remaining in E_{22} is $1 - a\Delta t - b\Delta t - c\Delta t$.

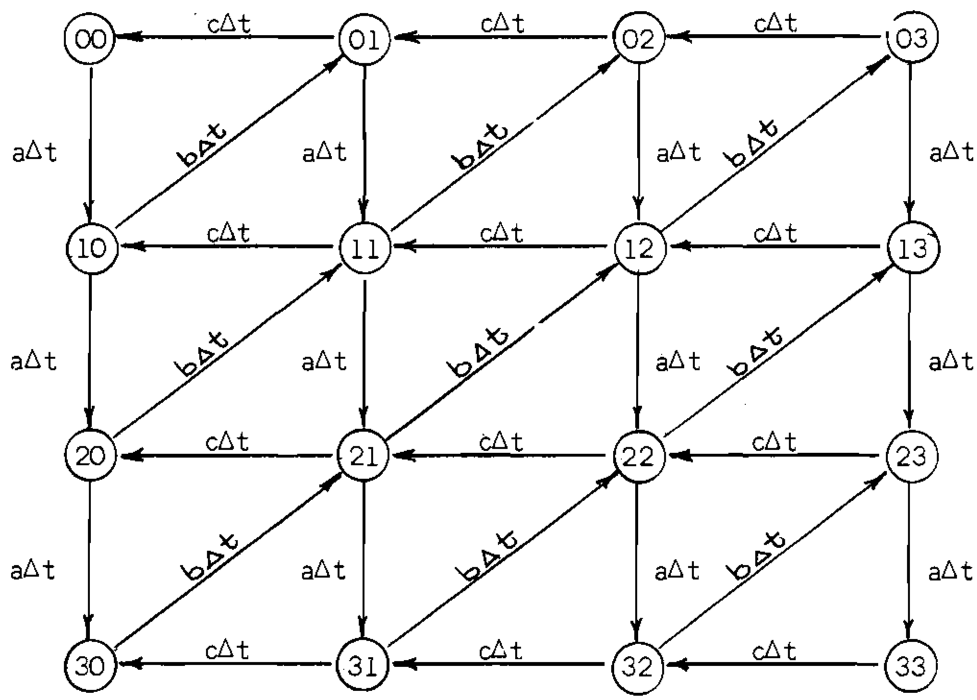

Figure 2. Possible Queue State Transitions when $N_1 = 3$ and $N_2 = 3$.

Although Figure 1 shows the case where N_1 and N_2 are both three, it also describes the general N_1, N_2 case. It may be seen that the transition from n_1, n_2 to n_1+1, n_2 may be made from any state where $n_1 \neq N_1$, the transition from n_1, n_2 to n_1-1, n_2+1 may be made from any state where $n_1 \neq 0$ and $n_2 \neq N_2$, and the transition from n_1, n_2 to n_1,$n_2 - 1$ may be made from any state where $n_2 \neq 0$.

The Equations and Their Solution.--Unfortunately, the three machine case is considerably more complex than the two machine case. In the latter it was possible to derive a general expression for P_n in terms of P_o regardless of the value of N_1. In the three machine case, though, the addition of a unit of capacity in either queue changes the relationships between the previously existing probabilities, so no general expression can be written. It is possible, however, to derive a particular expression for any specific arrangement of capacities, and the method will be indicated. Particular expressions for certain selected arrangements will be derived.

The method of solution is to express the steady state probabilities in terms of the probabilities of other queue states, then by a series of eliminations and substitutions to express all probabilities in terms of one of them, and finally to apply the normalizing equation. Once the basic relationships between the states are understood the initial expressions may be written by inspection. This may be seen by noting that the probability of being in any state at time $t + \Delta t$ is the probability of being in that state at time t times the probability of remaining in that state during Δt, plus the sum of the probabilities of being, at time t, in each of the states from which the state in question can be reached in a single transition, each multiplied by the probability of such a transition. Since the probability of remaining in any state is always of the form $1 - Z$, where Z is the sum of one, two, or three of the terms $a\Delta t$, $b\Delta t$, and $c\Delta t$, the derivative with respect to time of the probability of any queue state can always be expressed as $-Z/\Delta t$ plus the sum of probabilities of states from which the state in question may be reached, each multiplied by the probability of transition to that state and divided by Δt. Upon setting the derivative equal to zero, $Z/\Delta t$ times the probability of the state in question is seen to equal the sum just mentioned. To illustrate, consider the state E_{23} in Figure 1. Here, $Z = (a\Delta t + c\Delta t)$, and $Z/\Delta t = a + c$; E_{23} can be reached from E_{13} with probability $a\Delta t$ and from E_{32} with probability $b\Delta t$. By inspection this indicates that

$$(a + c)P_{23} = aP_{13} + bP_{32} .$$

Similarly, for state E_{11}, $Z = (a\Delta t + b\Delta t + c\Delta t)$, and

$$(a + b + c)P_{11} = aP_{01} + bP_{20} + cP_{12} .$$

To arrive at these values analytically, the following steps would be necessary:

$$P_{23}(t + \Delta t) = P_{23}(t)\ p_{23;23} + P_{32}(t)\ p_{32;23} + P_{13}(t)\ p_{13;23}$$

$$P_{23}(t + \Delta t) = P_{23}(t)(1 - a\Delta t - c\Delta t)+P_{13}(t)\ a\Delta t + P_{32}(t)\ b\Delta t$$

$$\lim_{\Delta t \to 0} \frac{P_{23}(t+\Delta t) - P_{23}(t)}{\Delta t} = \frac{d}{dt} P_{23} = 0 = -(a+c)\ P_{23} + aP_{13} + bP_{32}$$

$$(a + c)\ P_{23} = aP_{13} + bP_{32}$$

$$P_{11}(t+\Delta t) = P_{11}(t)\ p_{11;11} + P_{01}(t)\ p_{01;11} + P_{20}(t)\ p_{20;11} + P_{12}(t)\ p_{12;11}$$

$$P_{11}(t+\Delta t) = P_{11}(t)(1 - a\Delta t - b\Delta t - c\Delta t) + P_{01}(t)\ a\Delta t + P_{20}\ b\Delta t + P_{12}\ c\Delta t$$

$$\lim_{\Delta t \to 0} \frac{P_{11}(t+\Delta t) - P_{11}(t)}{\Delta t} = \frac{d}{dt} P_{11} = 0 = -\ (a+b+c)P_{11} + aP_{01} + bP_{20} + cP_{12}$$

$$(a + b + c)\ P_{11} = aP_{01} + bP_{20} + cP_{12}\ .$$

Using this method, expressions for all state probabilities will be derived for the cases $N_1 = 1$ and $N_2 = 1$, $N_1 = 1$ and $N_2 = 2$, $N_1 = 2$ and $N_2 = 1$, $N_1 = 2$ and $N_2 = 2$, $N_1 = 1$ and $N_2 = 3$, $N_1 = 3$ and $N_2 = 1$, $N_1 = 1$ and $N_2 = 4$, $N_1 = 4$ and $N_2 = 1$, $N_1 = 2$ and $N_2 = 3$, and $N_1 = 3$ and $N_2 = 2$. The expressions for any other specific case may be drived in the same manner. Although the desired end is to express all queue state probabilities in terms of one of them, the expressions in this form become extremely complicated. It is more practical, both in deriving the expressions and in the numerical computation, to express each queue state probability in terms of other previously derived probabilities, which in turn have been expressed in terms of the one desired. This allows step-by-step computation of the probabilities with simpler equations, fewer substitutions, smaller numbers, and fewer opportunities for error. The expressions will accordingly be given in such a form.

Figure 3 represents the case $N_1 = 1$, $N_2 = 1$. By inspection and substitution,

$$aP_{00} = cP_{01}, \qquad P_{01} = \frac{a}{c} P_{00} ,$$

$$(a + c)\, P_{01} = bP_{10} , \qquad P_{10} = \frac{a + c}{b} P_{01} ,$$

$$cP_{11} = aP_{01} , \qquad P_{11} = \frac{a}{c} P_{01} .$$

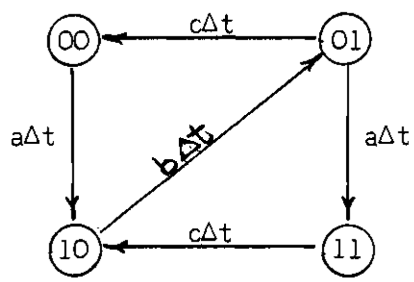

Figure 3. Transition Probabilities when $N_1 = 1$ and $N_2 = 1$

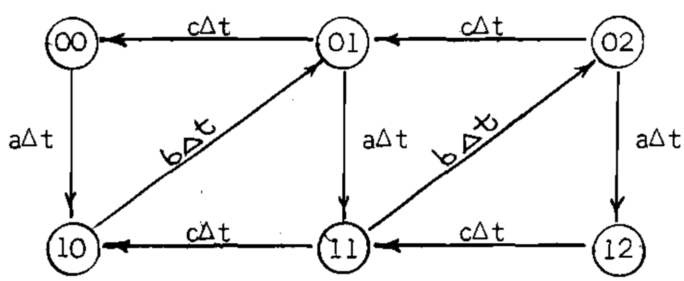

Figure 4. Transition Probabilities when $N_1 = 1$, $N_2 = 2$

Figure 4 represents the case $N_1 = 1$, $N_2 = 2$. Here it may be seen immediately that

$$P_{01} = \frac{a}{c} P_{00} .$$

It is then necessary to eliminate P_{11} from the equations

$$(a + c) P_{02} - bP_{11} = 0$$

$$- cP_{11} + bP_{10} - aP_{00} = 0$$

by multiplying the first by c and the second by $-b$ and adding them. This yields P_{02} in terms of P_{10} and P_{00}. Upon substituting this value of P_{02} in the equation

$$(a + c) P_{01} - bP_{10} - cP_{02} = 0$$

P_{10} is found in terms of P_{01} and P_{00}:

$$P_{10} = \frac{(a + c)^2}{b(a + b + c)} P_{01} + \frac{a}{a + b + c} P_{00} .$$

Then, from the equation

$$bP_{10} = aP_{00} + cP_{11} ,$$

P_{11} is seen to be:

$$P_{11} = \frac{b}{c} P_{10} - \frac{a}{c} P_{00} .$$

From $(a + c) P_{02} = bP_{11}$, P_{02} is found:

$$P_{02} = \frac{b}{a + c} P_{11} .$$

By inspection, $cP_{12} = aP_{02}$, and

$$P_{12} = \frac{a}{c} P_{02} .$$

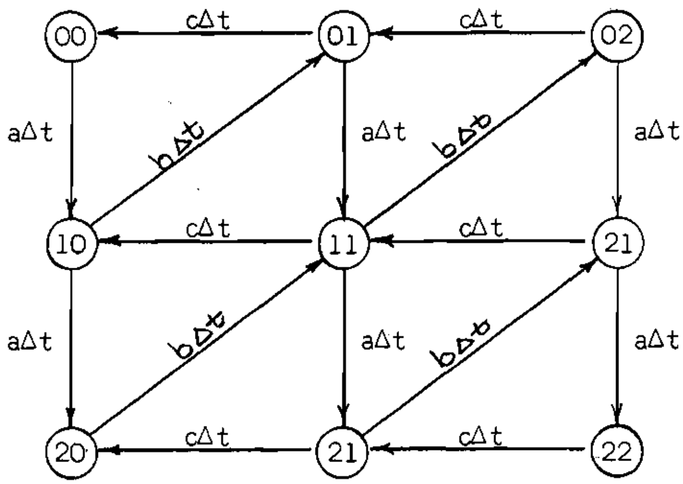

Figure 5. Transition Probabilities when $N_1 = 2$, $N_2 = 2$

Figure 5 represents the case $N_1 = 2$, $N_2 = 2$. Here it is immediately seen that

$$P_{01} = \frac{a}{c} P_{00} .$$

By suing the two equations

$$(a + c)\, P_{01} - bP_{10} - cP_{02} = 0$$

and

$$(a + b)\, P_{10} - aP_{00} - cP_{11} = 0 ,$$

multiplying the first by $(a + b)$ and the second by b, adding them, and substituting $\frac{a + c}{b} - P_{02}$ for P_{11} in the result, the expression for P_{02} is obtained:

$$P_{02} = \frac{(a + b)(a + c)}{c\,(2a + b + c)} P_{01} - \frac{ab}{c(2a + b + c)} P_{00} .$$

The expressions for P_{11} and P_{10} are obtained immediately as:

$$P_{11} = \frac{a + c}{b} P_{02} ,$$

and

$$P_{10} = \frac{a}{a + b} P_{00} + \frac{c}{a + b} P_{11} .$$

It is then necessary to eliminate P_{21} from

$$(a + c) P_{12} - a P_{02} - b P_{21} = 0$$

and

$$(b + c) P_{21} - a P_{11} - c P_{22} = 0$$

and to substitute $\frac{c}{a} P_{22}$ for P_{12} in the result to obtain

$$P_{22} = \frac{a^2(b + c)}{c(a + c)(b + c) - abc} P_{02} + \frac{a^2 b}{c(a + c)(b + c) - abc} P_{11} .$$

The remaining expressions then follow immediately as:

$$P_{12} = \frac{c}{a} P_{22}$$

$$P_{21} = \frac{a}{b + c} P_{11} + \frac{c}{b + c} P_{22}$$

$$P_{20} = \frac{a}{b} P_{10} + \frac{c}{b} P_{21} .$$

The expressions for the queue state probabilities in the remaining cases solved are derived by a similar procedure. They are:

For $N_1 = 1$ and $N_2 = 3$:

$$P_{13} = \frac{a}{c} P_{03}$$

$$P_{12} = \frac{a + c}{b} P_{03}$$

$$P_{02} = \frac{b + c}{a} P_{12} - \frac{c}{a} P_{13}$$

$$P_{11} = \frac{a + c}{b} P_{02} - \frac{c}{b} P_{03}$$

$$P_{01} = \frac{b + c}{a} P_{11} - \frac{c}{a} P_{12}$$

$$P_{00} = \frac{c}{a} P_{01}$$

$$P_{10} = \frac{a}{b} P_{00} + \frac{c}{b} P_{11}$$

For $N_1 = 3$ and $N_2 = 1$:

$$P_{01} = \frac{a}{c} P_{00}$$

$$P_{10} = \frac{a + c}{b} P_{01}$$

$$P_{11} = \frac{a + b}{c} P_{10} - \frac{a}{c} P_{00}$$

$$P_{20} = \frac{a + c}{b} P_{11} - \frac{a}{b} P_{01}$$

$$P_{21} = \frac{a + b}{c} P_{20} - \frac{a}{c} P_{10}$$

$$P_{31} = \frac{a}{c} P_{21}$$

$$P_{30} = \frac{a}{b} P_{20} + \frac{c}{b} P_{31}$$

For $N_1 = 1$ and $N_2 = 4$:

$$P_{14} = \frac{a}{c} P_{04}$$

$$P_{13} = \frac{a + c}{b} P_{04}$$

$$P_{03} = \frac{b + c}{a} P_{13} - \frac{c}{a} P_{14}$$

$$P_{12} = \frac{a+c}{b} P_{03} - \frac{c}{b} P_{04}$$

$$P_{02} = \frac{b+c}{a} P_{12} - \frac{c}{a} P_{13}$$

$$P_{11} = \frac{a+c}{b} P_{02} - \frac{c}{b} P_{03}$$

$$P_{01} = \frac{b+c}{a} P_{11} - \frac{c}{a} P_{12}$$

$$P_{10} = \frac{a+c}{b} P_{01} - \frac{c}{b} P_{02}$$

$$P_{00} = \frac{c}{a} P_{01}$$

For $N_1 = 4$ and $N_2 = 1$:

$$P_{01} = \frac{a}{c} P_{00}$$

$$P_{10} = \frac{a+c}{b} P_{01}$$

$$P_{11} = \frac{a+b}{c} P_{10} - \frac{a}{c} P_{00}$$

$$P_{20} = \frac{a+c}{b} P_{11} - \frac{a}{b} P_{01}$$

$$P_{21} = \frac{a+b}{c} P_{20} - \frac{a}{c} P_{10}$$

$$P_{30} = \frac{a+c}{b} P_{21} - \frac{a}{b} P_{11}$$

$$P_{31} = \frac{a+b}{c} P_{30} - \frac{a}{c} P_{20}$$

$$P_{41} = \frac{a}{c} P_{31}$$

$$P_{40} = \frac{a}{b} P_{30} + \frac{c}{b} P_{41}$$

For $N_1 = 2$ and $N_2 = 3$:

$$P_{12} = \frac{a + c}{b} P_{03}$$

$$P_{13} = \frac{a(b + c)}{(a + c)(b + c) - ab} P_{03} + \frac{ab}{(a + c)(b + c) - ab} P_{12}$$

$$P_{22} = \frac{a + c}{b} P_{13} - \frac{a}{b} P_{03}$$

$$P_{23} = \frac{a}{c} P_{13}$$

$$P_{21} = \frac{(a + c)(a + b + c)}{b(a + b + 2c)} P_{12} + \frac{c}{(a + b + 2c)} P_{22} - \frac{c(a + c)}{b(a + b + 2c)} P_{13} - \frac{ac}{b(a + b + 2c)} P_{03}$$

$$P_{11} = \frac{b + c}{a} P_{21} - \frac{c}{a} P_{22}$$

$$P_{02} = \frac{b}{a + c} P_{11} + \frac{c}{a + c} P_{03}$$

$$P_{01} = \frac{b(a + b + c)}{ab + a(a + c)} P_{11} - \frac{bc}{ab + a(a + c)} (P_{21} + P_{12}) + \frac{c}{a + b + c} P_{02}$$

$$P_{10} = \frac{a + c}{b} P_{01} - \frac{c}{b} P_{02}$$

$$P_{00} = \frac{c}{a} P_{01}$$

$$P_{20} = \frac{a}{b} P_{10} + \frac{c}{b} P_{21}$$

For $N_1 = 3$ and $N_2 = 2$:

$$P_{01} = \frac{a}{c} P_{00}$$

$$P_{02} = \frac{(a + b)(a + c)}{c(2a + b + c)} P_{01} - \frac{ab}{c(2a + b + c)} P_{00}$$

$$P_{11} = \frac{a + c}{b} P_{02}$$

$$P_{10} = \frac{a}{a + b} P_{00} + \frac{c}{a + b} P_{11}$$

$$P_{12} = \frac{(a + b)(a + b + c)}{c(2a + b + c)} P_{11} + \frac{a}{2a + b + c} P_{02} - \frac{ab}{c(2a + b + c)} P_{10}$$

$$- \frac{a(a + b)}{c(2a + b + c)} P_{01}$$

$$P_{21} = \frac{a + c}{b} P_{12} - \frac{a}{b} P_{02}$$

$$P_{20} = \frac{a}{a + b} P_{10} + \frac{c}{a + b} P_{21}$$

$$P_{22} = \frac{a(b + c)}{(a + c)(b + c) - ab} P_{12} + \frac{ab}{(a + c)(b + c) - ab} P_{21}$$

$$P_{32} = \frac{a}{c} P_{22}$$

$$P_{31} = \frac{a}{b + c} P_{21} + \frac{c}{b + c} P_{32}$$

$$P_{30} = \frac{a}{b} P_{20} + \frac{c}{b} P_{31} .$$

An Example.--A third machine is to be added in series following the two machines of the example of Chapter IV. The third machine has a capacity of 200 units per hour when operating, breaks down on the average once an hour, and requires an average of six minutes to repair. This gives Λ_3 and 1.0, M_3 as 10.0, and R_3 as 0.10. $c = \mu_3(1 - R_3) = 180$ units per hour, and from Chapter IV, $a = \mu_1(1 - R_1) = 135$ units per hour and $b = \mu_2(1 - R_2) = 160$ units per hour. The average production rate of the system for all possible combinations of five or less total units of storage capacity will be calculated.

Here again the lowest value of $\mu_i(1 - R_i)$ is 135 units per hour, and this is the maximum possible output of the system. As before, the output of any machine is the same as the output of the system, but the outputs of all three machines will be separately calculated as a check and to illustrate the method. The average output of the first machine is again the probability that it is neither blocked nor broken down, multiplied by its production rate when operating. This may be expressed as

$\mu_1 \left[1 - R_1 - (1 - R_1) \sum_{n_2=o}^{N_2} P_{N_1,n_2} \right]$. The average output of the second machine is the probability that it is neither broken down, blocked, nor run out, multiplied by its production rate when operating, or

$\mu_2 \left[1 - R_2 - (1 - R_2) \left(\sum_{n_1=1}^{N_1} P_{n_1,N_2} + \sum_{n_2=o}^{N_2} P_{o,n_2} \right) \right]$. The output of the third machine is, similarly, $\mu_3[1-R_3-(1-R_3) \sum_{n_1=o}^{N1} P_{n_1,o}]$. The calculations are shown below, the results in Table 11. For $N_1 = 1$ and $N_2 = 1$:

$3.790\ P_{00} = 1.0$

$$\begin{aligned}
P_{00} &= 1.000\ P_{00} = 0.264 \\
P_{01} &= 0.750\ P_{00} = 0.198 \\
P_{10} &= 1.477\ P_{00} = 0.390 \\
P_{11} &= \underline{0.563}\ P_{00} = 0.149 \\
&\quad\ 3.790
\end{aligned}$$

Output, machine 1 $= 150\ [1 - 0.10 - 0.90(0.390 + 0.149)] = 62.3$ units per hr.

Output, machine 2 $= 200\ [1 - 0.20 - 0.80(0.264 + 0.198 + 0.390) = 62.3$

Output, machine 3 $= 200\ [1 - 0.10 - 0.90(0.264 + 0.390)] = 62.3$

For $N_1 = 1$ and $N_2 = 2$:

$3.7177\ P_{00} = 1.0$

P_{00}	=	1.000	P_{00}	=	0.2690
P_{01}	=	0.7500	P_{00}	=	0.2018
P_{10}	=	1.2634	P_{00}	=	0.3398
P_{02}	=	0.1895	P_{00}	=	0.0510
P_{11}	=	0.3730	P_{00}	=	0.1003
P_{12}	=	0.1421	P_{00}	=	0.0382
		3.7177			

Output, machine 1 = 150 [1 - 0.10 - 0.90(0.3398 + 0.1003 + 0.0382)] = 70.43

Output, machine 2 = 200 [1 - 0.20 - 0.80(0.2690 + 0.2018 + 0.0510 + 0.0382)] = 70.43

Output, machine 3 = 200 [1 - 0.10 - 0.90(0.2690 + 0.3398)] = 70.43

For $N_1 = 2$ and $N_2 = 1$:

$8.808\ P_{00} = 1.0$

P_{00}	=	1.000	P_{00}	=	0.114
P_{01}	=	0.750	P_{00}	=	0.086
P_{10}	=	1.477	P_{00}	=	0.168
P_{11}	=	1.670	P_{00}	=	0.190
P_{21}	=	1.253	P_{00}	=	0.143
P_{20}	=	2.658	P_{00}	=	0.303
		8.808			

Output, machine 1 = 74.85

Output, machine 2 = 74.80

Output, machine 3 = 74.80

For $N_1 = 2$ and $N_2 = 2$:

$6.7663\ P_{00} = 1.0$

P_{00}	=	1.0000	P_{00}	=	0.1478
P_{01}	=	0.7500	P_{00}	=	0.1109
P_{02}	=	0.4380	P_{00}	=	0.0647
P_{11}	=	0.8623	P_{00}	=	0.1274
P_{10}	=	0.9837	P_{00}	=	0.1454
P_{22}	=	0.3397	P_{00}	=	0.0502
P_{12}	=	0.4529	P_{00}	=	0.0669
P_{21}	=	0.5222	P_{00}	=	0.0772
P_{20}	=	1.4175	P_{00}	=	0.2095
		6.7663			

Output, machine 1 = 89.52

Output, machine 2 = 89.52

Output, machine 3 = 89.52

For $N_1 = 1$ and $N_2 = 3$:

72.4231 $P_{03} = 1.0$

P_{03}	=	1.0000	P_{03}	=	0.0140	Output, machine 1 = 72.45
P_{13}	=	0.7500	P_{03}	=	0.0105	Output, machine 2 = 72.46
P_{12}	=	1.9688	P_{03}	=	0.0276	Output, machine 3 = 72.40
P_{02}	=	3.9584	P_{03}	=	0.0554	
P_{11}	=	6.7213	P_{03}	=	0.0941	
P_{01}	=	14.3026	P_{03}	=	0.2002	
P_{00}	=	19.0701	P_{03}	=	0.2670	
P_{10}	=	23.6519	P_{03}	=	0.3311	
		74.4231				

For $N_1 = 3$ and $N_2 = 1$:

18.2056 $P_{00} = 1.0$

P_{00}	=	1.0000	P_{00}	=	0.0549	Output, machine 1 = 80.09
P_{01}	=	0.7500	P_{00}	=	0.0412	Output, machine 2 = 80.08
P_{10}	=	1.4766	P_{00}	=	0.0811	Output, machine 3 = 80.10
P_{11}	=	1.6700	P_{00}	=	0.0917	
P_{20}	=	2.6551	P_{00}	=	0.1458	
P_{21}	=	3.2439	P_{00}	=	0.1781	
P_{31}	=	2.4330	P_{00}	=	0.1336	
P_{30}	=	4.9771	P_{00}	=	0.2732	
		18.2056				

For $N_1 = 1$ and $N_2 = 4$:

251.7588 $P_{04} = 1.0$

P_{04}	=	1.0000	P_{04}	=	0.00397	
P_{14}	=	0.7500	P_{04}	=	0.00298	
P_{13}	=	1.9688	P_{04}	=	0.00782	Output, machine 1 = 73.034
P_{03}	=	3.9585	P_{04}	=	0.01572	
P_{12}	=	6.6683	P_{04}	=	0.02647	Output, machine 2 = 73.046
P_{02}	=	14.1691	P_{04}	=	0.05625	
P_{11}	=	23.4421	P_{04}	=	0.09307	Output, machine 3 = 73.058
P_{01}	=	50.1483	P_{04}	=	0.19909	
P_{10}	=	82.7893	P_{04}	=	0.32867	
P_{00}	=	66.8644	P_{04}	=	0.26545	
		251.7588				

For $N_1 = 4$, $N_2 = 1$:

$35.9639 \; P_{00} = 1.0$

P_{00}	= 1.0000	P_{00}	= 0.02780
P_{01}	= 0.7500	P_{00}	= 0.02085
P_{10}	= 1.4766	P_{00}	= 0.04105
P_{11}	= 1.6700	P_{00}	= 0.04643
P_{20}	= 2.6550	P_{00}	= 0.07381
P_{21}	= 3.2438	P_{00}	= 0.09018
P_{30}	= 4.9771	P_{00}	= 0.13836
P_{31}	= 6.1656	P_{00}	= 0.17140
P_{41}	= 4.6242	P_{00}	= 0.12855
P_{40}	= 9.4016	P_{00}	= 0.26136
	35.9639		

Output, machine 1 = 82.362

Output, machine 2 = 82.368

Output, machine 3 = 82.372

For $N_1 = 2$, $N_2 = 3$:

$37.54225 \; P_{03} = 1.0$

P_{03}	= 1.00000	P_{03}	= 0.02663
P_{12}	= 1.96875	P_{03}	= 0.05243
P_{13}	= 1.03420	P_{03}	= 0.02754
P_{22}	= 1.19233	P_{03}	= 0.03175
P_{23}	= 0.77565	P_{03}	= 0.02066
P_{21}	= 2.34708	P_{03}	= 0.06250
P_{11}	= 4.32139	P_{03}	= 0.11508
P_{02}	= 2.76641	P_{03}	= 0.07367
P_{01}	= 4.23163	P_{03}	= 0.11269
P_{10}	= 5.21881	P_{03}	= 0.13898
P_{00}	= 5.64217	P_{03}	= 0.15025
P_{20}	= 7.04383	P_{03}	= 0.18758
	37.54225		

Output, machine 1 = 94.164

Output, machine 2 = 94.170

Output, machine 3 = 94.174

For $N_1 = 3$, $N_2 = 2$:

$10.49751 \; P_{00} = 1.0$

P_{00}	= 1.00000	P_{00}	= 0.09526
P_{01}	= 0.75000	P_{00}	= 0.07145
P_{02}	= 0.43801	P_{00}	= 0.04172
P_{11}	= 0.86233	P_{00}	= 0.08215
P_{10}	= 0.98372	P_{00}	= 0.09371
P_{12}	= 0.73189	P_{00}	= 0.06972
P_{21}	= 1.07133	P_{00}	= 0.10205
P_{20}	= 1.10386	P_{00}	= 0.10515
P_{22}	= 0.66355	P_{00}	= 0.06321
P_{32}	= 0.49766	P_{00}	= 0.04741
P_{31}	= 0.68884	P_{00}	= 0.06542
P_{30}	= 1.70632	P_{00}	= 0.16254
	10.49751		

Output, machine 1 = 97.799

Output, machine 2 = 97.796

Output, machine 3 = 97.802

Table 11 shows the performance of the system for the various arrangements of storage capacities, and Table 12 indicates the best arrangement for any given value of total storage capacity up to five units, and the gain by adding an additional unit.

Table 11. Output of Three Machine System for Various Values of N_1 and N_2

N_1	N_2	Output	Total Units Capacity
1	1	62.3	2
1	2	70.4	3
2	1	74.8	3
2	2	89.5	4
1	3	72.4	4
3	1	80.1	4
1	4	73.0	5
4	1	82.4	5
2	3	94.2	5
3	2	97.8	5

Table 12. Best Values of Output for Individual Values of Total Storage Capacity

Total Capacity	N_1	N_2	Output	Gain
2	1	1	62.3	-
3	2	1	74.8	12.5
4	2	2	89.5	14.7
5	3	2	97.8	8.3

Here the addition of a unit of capacity does not necessarily increase the output, unless the unit is placed in the correct location; a change from a total of three units capacity, two in the first queue and one in the second, to four units, one in the first queue and three in the second, actually reduces the average output of the system. In evaluating the effects of a proposed addition of capacity, all possible combinations must be evaluated to determine the optimum position for it to be placed.

CHAPTER VI

THE GENERAL CASE

It has been shown that in the two machine case it is possible to derive a general expression for P_n in terms of P_0 for all values of n, but that this is not possible in the three machine case. It may be deduced that it would also be impossible when four or more machines are involved, again because the addition of a unit of capacity in any queue will change the relationships between the other queue state probabilities. Expressions similar to those derived in Chapters IV and V for queue state transition probabilities could be derived by the same procedure for a series of any number of machines, and queue state probabilities for any given values of queue capacities could be calculated using the algebraic methods of Chapter V. This procedure is extremely laborious, however. A pattern has occurred in the two and three machine cases which may considerably reduce the labor of obtaining the transition probabilities. This will eliminate much, though by no means all, of the work of solving for the queue state probabilities for any specific arrangement. The findings of the first two cases will be summarized to illustrate the nature of these cases and to extend certain features to the general case. Other features which may be reasonably expected to be generalized will be indicated. It may be helpful to the reader to refer to Fig. 1 at this point.

Aside from "transitions" involving no change, there are three types of transitions which may occur in the first two cases. These are:

Type I: An increase of one unit in the first queue with no changes in any other queue. The vertical arrows in Fig. 1 represent Type I transitions.

Type II: A decrease of one unit in any queue except the last, a simultaneous increase of one unit in the next queue, and no changes in any other queue. The diagonal arrows in Fig. 1 represent Type II transitions.

Type III: A decrease of one unit in the last queue, with no changes in any other queue. The horizontal arrows in Fig. 1 represent Type III transitions.

Only Type I and Type III transitions occur in the two machine case, since the first queue is also the last queue.

The general case will also be limited to these three types of transitions, since any other type of transition would involve either the completion of more than one unit by a single machine or completions of units by more than one machine during time Δt. The probabilities of these occurrences are of order $(\Delta t)^2$ or higher and are taken as negligible.

Type I transitions can occur from any queue state where the first machine is not blocked; they can occur from any state where $n_1 \neq N_1$.

Type II transitions can occur from any queue state except those where the queue which is to increase is not already at its capacity, or where the queue which is to decrease is already at zero.

Type III transitions can occur from any queue state where the last machine is not run out.

It is apparent from the physical nature of such systems that these limitations on queue states from which the various types of transitions can originate must apply to the general case as well as to the two particular cases solved.

Given that the initial queue states are ones from which the transitions in question can originate, the probability of a Type I transition is $\mu_1\Delta t(1 - R_1)$, the probability of a Type II transition involving the queues before and after the ith machine is $\mu_1\Delta t(1 - R_i)$, and the probability of a Type III transition is $\mu_z\Delta t(1 - R_z)$ when there are z machines in the series.

There is apparently no way to prove that these probabilities extend to the general case. However, they might reasonably be expected to be valid in the general case because any transition must be caused by a completion of service. Since the probabilities derived in the two cases solved are simply the probability that a machine will complete service in the next interval Δt, given that it is operating, multiplied by the probability that it is in fact operating, it seems logical that these probabilities would also hold in the general case.

If the reader is not willing to assume that these probabilities extend to the general case on the basis of this justification, the transition probabilities for any specific case may be derived by an extension of the procedure of Chapters IV and V and the relations between queue states then obtained by the inspection procedure introduced in Chapter V. If he is willing to make the assumption, the relations may be written immediately using the inspection procedure. The general rule may be expressed:

> The steady state probability of any queue state, multiplied by unity minus the probability of remaining in that state and then divided by Δt, is equal to the sum of the steady state probabilities of all states from which the state in question may be reached in a single transition, each multiplied by the product of the mean production rate and the mean proportion of time that the machine is operable which must complete an operation to effect the transition to the state in question.

The probability of remaining in any state is unity minus the sum of the probabilities of all transitions which can be made from it. Assuming z machines in series and a state $E_{n_1,n_2,\ldots n_{z-1}}$ which can be reached by all three types of transitions and from which all three types can originate, the expression for the general form may be written symbolically:

$$[\mu_1(1-R_1) + \mu_2(1-R_2) + \ldots\ldots\ldots + \mu_z(1-R_z)]\ P_{n_1,n_2,\ldots n_{z-1}} =$$

$$\mu_1(1-R_1)P_{n_1-1,n_2,n_3,\ldots n_{z-1}} + \mu_2(1-R_2)P_{n_1+1,n_2-1,n_3,n_4,\ldots n_z-1}$$

$$+ \mu_3(1-R_3)P_{n_1,n_2+1,n_3-1,n_4,\ldots n_z-1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \mu_{z-1}(1-R_{z-1})P_{n_1,n_2,\ldots.n_{z-2}+1,n_{z-1}-1} + \mu_z(1-R_z)P_{n_1,n_2,n_3,\ldots,n_{z-1}+1}$$

For states from which Type I transitions cannot occur, the first term in the coefficient of the left side becomes zero; if Type III transitions cannot occur the last term becomes zero. If it is impossible to leave the state in question by some or all of the Type II transitions, remaining terms corresponding to the impossible transitions become zero. Similarly, the first, last, and some or all of the intermediate terms of the right side become zero if it is impossible to reach the state in question by Type I, Type III, or some or all of the type II transitions, respectively.

Regardless of which procedure is used to obtain the relations between the queue state probabilities, a series of eliminations and substitutions is then necessary to place the expressions in a suitable form for solving them one at a time in terms of any selected queue state probability. The normalizing equation is then applied.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Findings.--It was the original purpose of this study to develop a decision process for determining the optimum amount and location of in-process storage capacity for the general case when any number of machines or work stations are arranged in series. When the amount of work involved in such an undertaking became apparent, this was revised and it was decided instead to attempt to develop a procedure for describing the output of such a series. The procedure could then be used in the development of future decision processes. A general expression was derived to describe the state probabilities, and eventually the steady state average output, when the series is limited to two machines. No general expression could be derived when there are three or more machines in the series, but a method was developed by which the particular expressions for any specific case can be derived and the steady state average output determined. Specific expressions were derived in terms of one selected queue state probability for a three machine series with all possible combinations of five or fewer units of storage capacity.

Limitations and Areas of Applicability.--The findings are not limited to systems of machines in the strict sense; they may be applied to systems of assembly stations or to any other activity where operations are conducted in series. Care must be taken, of course, to insure that the service times and repair times are exponentially distributed and that the arrival times of breakdowns or interruptions are Poisson distributed. The breakdowns may be any type of interruption, and the repair times simply the times taken to restore operation. The requirement for Poisson and exponential distributions is a definite limitation, but these distributions are not rare in practice, and the model should find useful applications. The limitations discussed in Chapter I should also be kept in mind. In particular, this model, like any other mathematical model of an actual system, must ignore many real features of the system modeled, and the results must be used with caution. It should be remembered that only steady state conditions are described, and a system that exhibits transient or time-dependent behavior is not described by this model.

Two other features which limit the usefulness of the model are the relative magnitudes of the numbers involved in actual computation and the labor involved in deriving the expressions for the queue state probabilities. To illustrate the first of these limitations, consider the example in Chapter V where $N_1 = 1$ and $N_2 = 4$. Here the results included $251.7588\ P_{04} = 1.0$ and $P_{14} = 0.00298$. Three places left of the decimal point were used and five to the right, and even then there was a variation of almost three figures in the second column right of the decimal point in the output results because of roundoff error. This limitation becomes more pronounced in more complex systems because there are more possible queue states. The second limitation, the labor in deriving the expressions, becomes even more critical as the complexity of the system increases, although once the expressions have been derived they may be used for any production rates and proportions of time operable. For example, in a six machine series with a capacity of five units in each queue there are $(5)^5$ or 3125 possible queue states. The addition of another unit of capacity anywhere in the system would increase this by 6/5 or 1.20, and there are five possible places where the additional unit could be placed!

Another limitation is that it is necessary that each machine's production rate multiplied by its mean proportion of time operable be equal to or greater than the corresponding product for the machine before it in order that a steady state solution may exist.

Because of these limitations the model as it now stands will probably be limited in application to fairly simple systems. With improvements in the model and improved computational techniques in the future, this basic approach might find more extensive application. At present, even with fairly simple systems the amount of labor involved may often make its use uneconomical where in-process storage facilities are relatively inexpensive to provide in relation to the value of increased production, as in cases where the units produced are small and the volume is high. It could find application in systems where a few large and relatively expensive assemblies are brought together or similar operations are performed, where production is relatively low, and where in-process storage facilities are expensive. It might also be applied to more complex systems which can be broken down into a series of subsystems, if the service rates, interruptions, and operation restoration times for the subsystems follow the assumed distributions. Examples might be found in the military facilities which overhaul routinely a particular aircraft model in production line fashion. A possible limitation here is that the model assumes all blocked or run out time is lost, while in practice the personnel and facilities might be put to some auxiliary or deferrable work.

Recommendations for Further Study.--Since the expressions for queue state probabilities in terms of one of them, once derived, may be used for any values of production rates and proportion of time idle, it would be useful to have these derived and tabulated for more arrangements than were given here. General computer programs could then be developed to determine the average output of each arrangement. It appears likely that this approach could only be applied to fairly simple systems. Because of the labor of derivation and computation, simulation may be a better approach for more complicated arrangements, and an extension of Bedworth's (10) work would be useful.

For the simpler systems, models similar to this one but assuming Poisson, Erlangian, or other inputs rather than the infinite input might be useful, since these distributions also occur frequently in practice.

Other modifications of this model which would be useful would be to allow for deferrable work which might be performed by the operators or even the machines when the normal path is blocked or when normal work is not available, and to allow for scrap losses or rejections at each station instead of assuming that all material goes completely through the system.

The model developed in this study or any of the suggested modifications may prove useful in future studies aimed at the development of economic decision rules or processes for determining optimum amounts and locations of in-process storage space.

APPENDICES

APPENDIX A

SAMPLE CALCULATIONS OF MACHINE CONDITION TRANSITION PROBABILITIES, TWO MACHINE CASE

Starting in queue state E_{n_1} with $(1 < n_1 < N_1 - 1)$:

The probability of a transition from C(1x) to C(11) is $(1 - \Lambda_1\Delta t)(M_2\Delta t) = M_2\Delta t$.

From C(1x) to C(1x): $(1 - \Lambda_1\Delta t)(1 - M_2\Delta t) = 1 - \Lambda_1\Delta t - M_2\Delta t$

From C(1x) to C(x1): $(\Lambda_1\Delta t)(M_2\Delta t) = 0$

From C(1x) to C(xx): $(\Lambda_1\Delta t)(1 - M_2\Delta t) = \Lambda_1\Delta t$

From C(x1) to C(11): $(M_1\Delta t)(1 - \Lambda_2\Delta t) = M_1\Delta t$

From C(x1) to C(1x): $(M_1\Delta t)(\Lambda_2\Delta t) = 0$

From C(x1) to C(x1): $(1 - M_1\Delta t)(1 - \Lambda_2\Delta t) = 1 - M_1\Delta t - \Lambda_2\Delta t$

From C(x1) to C(xx): $(1 - M_1\Delta t)(\Lambda_2\Delta t) = \Lambda_2\Delta t$

From C(xx) to C(11): $(M_1\Delta t)(M_2\Delta t) = 0$

From C(xx) to C(1x): $(M_1\Delta t)(1 - M_2\Delta t) = M_1\Delta t$

From C(xx) to C(x1): $(1 - M_1\Delta t)(M_2\Delta t) = M_2\Delta t$

From C(xx) to C(xx): $(1 - M_1\Delta t)(1 - M_2\Delta t) = 1 - M_1\Delta t - M_2\Delta t$

Starting in queue state E_1:

From C(11) to C(11): $(1 - \Lambda_1\Delta t)(1 - \Lambda_2\Delta t)[\mu_1\Delta t + (1 - \mu_1\Delta t)(1 - \mu_2\Delta t)]$

$= (1 - \Lambda_1\Delta t - \Lambda_2\Delta t)(\mu_1\Delta t + 1 - \mu_1\Delta t - \mu_2\Delta t) = 1 - \Lambda_1\Delta t - \Lambda_2\Delta t - \mu_2\Delta t$

From C(11) to C(10): $(1 - \Lambda_1\Delta t)(1 - \Lambda_2\Delta t)(\mu_2\Delta t)(1 - \mu_1\Delta t) = \mu_2\Delta t$

From C(11) to C(1x): $(1 - \Lambda_1\Delta t)(\Lambda_2\Delta t) = \Lambda_2\Delta t$

From C(11) to C(x1): $(\Lambda_1 \Delta t)(1 - \Lambda_2 \Delta t)[\mu_1 \Delta t + (1 - \mu_2 \Delta t)] = \Lambda_1 \Delta t$

From C(11) to C(x0): $(\Lambda_1 \Delta t)(1 - \Lambda_2 \Delta t)(1 - \mu_1 \Delta t)(\mu_2 \Delta t) = 0$

From C(11) to C(xx): $(\Lambda_1 \Delta t)(\Lambda_2 \Delta t) = 0$

From C(1x) to C(11): $(1 - \Lambda_1 \Delta t)(M_2 \Delta t)[\mu_1 \Delta t + (1 - \mu_2 \Delta t)]$

$= (M_2 \Delta t)(\mu_1 \Delta t + 1 - \mu_2 \Delta t) = M_2 \Delta t$

From C(1x) to C(10): $(1 - \Lambda_1 \Delta t)(M_2 \Delta t)(\mu_1 \Delta t) = 0$

From C(1x) to C(1x): $(1 - \Lambda_1 \Delta t)(1 - M_2 \Delta t) = 1 - \Lambda_1 \Delta t - M_2 \Delta t$

From C(1x) to C(x1): $(\Lambda_1 \Delta t)(M_2 \Delta t)(1 - \mu_2 \Delta t) = 0$

From C(1x) to C(x0): $(\Lambda_1 \Delta t)(M_2 \Delta t)(1 - \mu_2 \Delta t + 1 - \mu_1 \Delta t) = 0$

From C(1x) to C(xx): $(\Lambda_1 \Delta t)(1 - M_2 \Delta t) = \Lambda_1 \Delta t$

APPENDIX B

SAMPLE CALCULATIONS OF MACHINE CONDITION TRANSITION PROBABILITIES, THREE MACHINE CASE

Starting in queue state E_{n_1,n_2} with $1 - n_1 - N_1 - 1$ and $1 < n_2 < N_2 - 1$:

The probability of a transition from C(x1x) to C(x1x) is $(1 - M_1\Delta t)(1 - \Lambda_2\Delta t)(1 - M_3\Delta t) = 1 - M_1\Delta t - \Lambda_2\Delta t - M_3\Delta t$

From C(x1x) to C(xx1): $(1 - M_1\Delta t)(\Lambda_2\Delta t)(M_3\Delta t) = 0$

From C(x1x) to C(xxx): $(1 - M_1\Delta t)(\Lambda_2\Delta t)(1 - M_3\Delta t) = \Lambda_2\Delta t$

From C(x1x) to C(x11): $(1 - M_1\Delta t)(1 - \Lambda_2\Delta t)(M_3\Delta t) = M_3\Delta t$

From C(x1x) to C(11x): $(M_1\Delta t)(1 - \Lambda_2\Delta t)(1 - M_3\Delta t) = M_1\Delta t$

From C(x1x) to C(1x1): $(M_1\Delta t)(\Lambda_2\Delta t)(M_3\Delta t) = 0$

From C(x1x) to C(1xx): $(M_1\Delta t)(\Lambda_2\Delta t)(1 - M_3\Delta t) = 0$

From C(x1x) to C(111): $(M_1\Delta t)(1 - \Lambda_2\Delta t)(M_3\Delta t) = 0$

From C(xx1) to C(x1x): $(1 - M_1\Delta t)(M_2\Delta t)(\Lambda_3\Delta t) = 0$

From C(xx1) to C(xx1): $(1 - M_1\Delta t)(1 - M_2\Delta t)(1 - \Lambda_3\Delta t) = 1 - M_1\Delta t - M_2\Delta t - \Lambda_3\Delta t$

From C(xx1) to C(xxx): $(1 - M_1\Delta t)(1 - M_2\Delta t)(\Lambda_3\Delta t) = \Lambda_3\Delta t$

From C(xx1) to C(x11): $(1 - M_1\Delta t)(M_2\Delta t)(1 - \Lambda_3\Delta t) = M_2\Delta t$

From C(xx1) to C(11x): $(M_1\Delta t)(M_2\Delta t)(\Lambda_3\Delta t) = 0$

From C(xx1) to C(1x1): $(M_1\Delta t)(1 - M_2\Delta t)(1 - \Lambda_3\Delta t) = M_1\Delta t$

From C(xx1) to C(1xx): $(M_1\Delta t)(1 - M_2\Delta t)(\Lambda_3\Delta t) = 0$

From C(xx1) to C(111): $(M_1\Delta t)(M_2\Delta t)(1 - \Lambda_3\Delta t) = 0$

APPENDIX C

SAMPLE CALCULATIONS FOR RESULTANT QUEUE STATE PROBABILITIES WITH INITIAL CONDITIONS E_{n_1,n_2} $(1< n_1 < N_1-1)$, $(1 < n_2 < N_2-1)$ AND C(xx1)

Final machine condition C(xx1):

The probability of this transition is $1 - M_1\Delta t - M_2\Delta t - \Lambda_3\Delta t$, and the possible resultant queue states are E_{n_1,n_2} with probability $1 - \mu_3\Delta t$ and E_{n_1,n_2-1} with probability $\mu_3\Delta t$.

Final Machine Condition C(xxx):

The probability of this transition is $\Lambda_3\Delta t$, and the possible resultant queue states are E_{n_1,n_2} with probability $1 - \mu_3\Delta t$, and E_{n_1,n_2-1} with probability $\mu_3\Delta t$.

Final machine condition C(x11):

The probability of this transition is $M_2\Delta t$, and the possible resultant queue states are E_{n_1,n_2} with probability $1 - \mu_2\Delta t - \mu_3\Delta t$, E_{n_1,n_2-1} with probability $\mu_3\Delta t$, and E_{n_1-1,n_2+1} with probability $\mu_2\Delta t$.

Final machine condition C(1x1):

The probability of this transition is $M_1\Delta t$, and the possible resultant queue states are E_{n_1,n_2} with probability $1 - \mu_1\Delta t - \mu_3\Delta t$, E_{n_1,n_2-1} with probability $\mu_3\Delta t$, and E_{n_1+1,n_2} with probability $\mu_1\Delta t$.

The original probability of $C(xx1)$, given E_{n_1,n_2}, is $R_1R_2 - R_1R_2R_3$, so the probability of the resultant queue state E_{n_1,n_2}, given initial conditions E_{n_1,n_2} and $C(xx1)$, is

$$(R_1R_2 - R_1R_2R_3)[(1 - \mu_3\Delta t)(1 - M_1\Delta t - M_2\Delta t - \Lambda_3\Delta t + \Lambda_3\Delta t$$

$$+ (1 - \mu_2\Delta t - \mu_3\Delta t)(M_2\Delta t) + (1 - \mu_1\Delta t - \mu_3\Delta t)(M_1\Delta t)] ,$$

or

$$(1 - \mu_3\Delta t)(R_1R_2 - R_1R_2R_3) .$$

The probability of the resultant queue state E_{n_1,n_2-1} is

$$(R_1R_2 - R_1R_2R_3)[(\mu_3\Delta t)(1 - M_1\Delta t - M_2\Delta t - \Lambda_3\Delta t + \Lambda_3\Delta t + M_2\Delta t + M_1\Delta t),$$

or

$$\mu_3\Delta t(R_1R_2 - R_1R_2R_3) .$$

The probability resultant queue state E_{n_1-1,n_2+1} is

$$(R_1R_2 - R_1R_2R_3)(\mu_2\Delta t)(M_2\Delta t) = 0 ,$$

and the probability of resultant queue state E_{n_1+1,n_2} is

$$(R_1R_2 - R_1R_2R_3)(\mu_1\Delta t)(M_1\Delta t) = 0 .$$

This process is continued for all possible initial machine conditions in the initial queue state. The resultant queue state probabilities, given the initial queue state and machine condition are multiplied by the probability of the initial machine condition given the initial queue state, and added to give the probability of each resultant queue state given the initial queue state. These are the transition probabilities sought. In this example, it is found that, for initial queue state E_{n_1,n_2} $(1 < n_1 < N_1 - 1)$ and $(1 < n_2 < N_2 - 1)$, the probability of a transition to E_{n_1-1,n_2+1} is $\mu_2 \Delta t(1 - R_2)$, the probability of a transition to E_{n_1,n_2-1} is $\mu_3 \Delta t(1 - R_3)$, the probability of a transition to E_{n_1+1,n_2} is $\mu_1 \Delta t(1 - R_1)$, and the probability of remaining in E_{n_1,n_2} is $1 - \mu_1 \Delta t(1 - R_1) - \mu_2 \Delta t(1 - R_2) - \mu_3 \Delta t(1 - R_3)$.

BIBLIOGRAPHY

(1) Saaty, T. L., Elements of Queueing Theory, New York, McGraw-Hill Book Company, Inc., 1961, "Historical Background," 20-25.

(2) Muther, R., Production-Line Technique, First Edition, New York, McGraw-Hill Book Company, Inc., 1944, 104-132.

(3) Jones, W. D., "Mathematical and Experimental Calculations of Machine Interference Time," The Research Engineer, Georgia Institute of Technology, Atlanta, Georgia, January, 1949, 9-10 and 20-23.

(4) Feller, W., An Introduction to Probability Theory and Its Applications, Second Edition, New York, John Wiley and Sons, Inc., 1957, 416-420.

(5) Benson, F., and Cox, D. R., "The Productivity of Machines Requiring Attention at Random Intervals," Journal of the Royal Statistical Society, Series B, 13, (1951), 65-82.

(6) Naor, P., "Normal Approximation to Machine Interference with Many Repairmen," Journal of the Royal Statistical Society, Series B, 19, (1957), 334-341.

(7) Jackson, R. R. P., "Random Queuing Processes with Phase-Type Service," Journal of the Royal Statistical Society, Series B, 18, (1956), 129-132.

(8) Hunt, Gordon C., "Sequential Arrays of Waiting Lines," Operations Research, 4, (1956), 674-683.

(9) White, Harrison, and Christie, L. S., "Queuing with Preemptive Priorities or with Breakdown," Operations Research, 6, (1958), 79-95.

(10) Bedworth, D. D., Design of an Electro-Mechanical Device for Simulating Product Flow Characteristics in a Manufacturing Plant, Unpublished Ph. D. Thesis, Purdue University, 1961.

(11) Saaty, op. cit., 33-34.