Eigensystem multiscale analysis for Anderson localization in energy intervals II

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Eigensystem multiscale analysis

- We consider the usual Anderson model.

- General strategy: Information about eigensystems at a given scale is used to derive information about eigensystems at larger scales.

- Need to carry over deterministic and probabilistic information since the system is random. The probabilistic part is close to the one in the standard MSA, will not be discussed here.

Level spacing and localization

Definition

A box $\Lambda_L = [-L, L]^d + x_0$ is called L-level spacing for H if all eigenvalues of H_{Λ_L} are simple, and

$$|\lambda - \lambda'| \geq \mathrm{e}^{-L^\beta} \quad \text{for all} \quad \lambda, \lambda' \in \sigma(H_{\Lambda_L}),\ \lambda \neq \lambda'.$$

Definition

Let Λ_L be a box, $x \in \Lambda_L$, and $m \geq 0$. Then $\varphi \in \ell^2(\Lambda_L)$ is said to be (x, m)-localized if $\|\varphi\| = 1$ and

$$|\varphi(y)| \leq \mathrm{e}^{-m\|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L^\tau.$$

Interval localization (naïve)

Definition (naïve)

Let I be a bounded interval and let $m > 0$. A box Λ_L will be called (m, I)-localizing for H if

1. Λ_L is level spacing for H.
2. There exists an (m, I)-localized eigensystem for H_{Λ_L}, that is, an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ for H_{Λ_L} such that for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that φ_ν is (x_ν, m)-localized.

▶ Level spacing helps to overcome the small denominator problem (resonances), replaces the Wegner estimate.

Failure of naïve approach to EMSA

Consider $\ell \ll L$ and suppose that

- A box Λ_L is L-level spacing for H;
- *Any* box $\Lambda_\ell \subset \Lambda_L$ is (m, I)-localizing for H (in a naïve sense as above).

Can we show that Λ_L is $(\hat{m}, \hat{I})$-localizing for H (allowing for *small* losses in m and I)?

The answer is NO.

Failure of naïve approach to EMSA

We don't know anything about the structure of eigenvectors for H_{Λ_ℓ} outside I. In particular, the quantum tunneling between localized states just inside I for one box Λ_ℓ and the delocalized states just outside another box Λ'_ℓ is possible (when we consider H_{Λ_L} as perturbation of decoupled boxes of size ℓ).

- This indicates that on the new scale L, localization on I is no longer uniform (as far as localization length is concerned): As we approach the edges of I, the mass m goes to zero.
- Deep inside I we expect localization to survive, since the quantum tunneling between energetically separated states is suppressed by locality of H (Combes-Thomas estimate).

Correct approach to EMSA (simplified)

- We replace the naive definition with

Definition

Let $E \in \mathbb{R}$, $I = (E - A, E + A)$, and $m > 0$. A box Λ_L will be called (m, I)-localizing for H if

1. Λ_L is level spacing for H.
2. There exists an (m, I)-localized eigensystem for H_{Λ_L}, i.e. an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ for H_{Λ_L} such that for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that φ_ν is $(x_\nu, m h_I(\nu))$-localized.

- The modulating function h_I satisfies $h_I(E) = 1$ and $h_I(E \pm A) = 0$.

Key step (simplified version)

Consider $\ell \ll L$ and suppose that

- A box Λ_L is L-level spacing for H;
- *Any* box $\Lambda_\ell \subset \Lambda_L$ is (m, I)-localizing for H.

Can we show that Λ_L is $(\hat{m}, \hat{I})$-localizing for H for some choice of the modulating function h_I, and allowing for *small* losses in m and I?

The answer now is YES.

- Tricky part: Choice of h_I and control over the decay rate.

EMSA on intervals implies MSA

- The general startegy of going from scale ℓ to scale L concerns the expansion of a true eigenfunction of H_{Λ_L} in terms of eigenfunctions of Hamiltonians H_ℓ.
- Although the process itself is very natural, preparations can take some time to explain.
- Instead, we will illustrate some ideas of the proof by showing how the eigensystem MSA for energy intervals implies the exponential localization of the Green function (the key player in the usual MSA).
- It will also reveal our top secret way of choosing the modulating function h_I mentioned earlier ☺.

EMSA on intervals implies MSA

- Let $I = (E - A, E + A)$ with $E \in \mathbb{R}$ and $A > 0$.
- Suppose that Λ_L is (m, I)-localizing for H.
- Let $\lambda \in I_L$ with $\operatorname{dist}\{\lambda, \sigma(H_{\Lambda_L})\} \geq \mathrm{e}^{-L^\beta}$.
- WTS: For m not too small and not too large,

$$|G_{\Lambda_L}(\lambda; x, y)| \leq e^{-\hat{m} h_I(\lambda)\|x-y\|} \text{ whenever } \|x - y\| \geq L^{\tau'}.$$

- Losses in m should be (controllably) small:

$$\hat{m} \geq m\left(1 - CL^{-\gamma}\right) \text{ for some } \gamma > 0.$$

Analyticity and localization

- We can try to split $(H_{\Lambda_L}-\lambda)^{-1}$ into $(H_{\Lambda_L}-\lambda)^{-1}P_I+(H_{\Lambda_L}-\lambda)^{-1}\bar{P}_I$
- P_I is the spectral projection of H_{Λ_L} onto I, $\bar{P}_I=1-P_I$.
- We have no information on φ_ν for $\nu\notin I$, though, and the decay rate of φ_ν for $\nu\in I$ is not uniform. Not good!
- Gentler approach: Filter out eigenvalues outside I using an analytic function $F_I(H_{\Lambda_L})$ instead of P_I: $(H_{\Lambda_L}-\lambda)^{-1}=(H_{\Lambda_L}-\lambda)^{-1}F_I(H_{\Lambda_L})+(H_{\Lambda_L}-\lambda)^{-1}\bar{F}_I(H_{\Lambda_L})$.
- Want (a) F_I to be exponentially small outside I and (b) $(z-\lambda)^{-1}\bar{F}_I(z)$ to be analytic in a strip that contains real axis (then Combes-Thomas estimate will kick in, and the corresponding term will be exponentially small too).

Analyticity and localization

To summarize:

1. $\left\langle \varphi_\nu, (H_{\Lambda_L} - \lambda)^{-1} F_I(H_{\Lambda_L}) \varphi_\nu \right\rangle = (\nu - \lambda)^{-1} F_I(\nu)$;
2. $|\varphi_\nu(x)\varphi_\nu(y)| \le \mathrm{e}^{-m h_I(\nu)\|x-y\|}$;
3. If $K(z) = (z-\lambda)^{-1}\bar{F}_I(z)$ is analytic and bounded in the strip $|Im\, z| < \eta$ by $\|K\|_\infty$, then (Aizenman-Graf)

$$|\langle \delta_x, K(H_\Lambda)\delta_y\rangle| \le C\|K\|_\infty \mathrm{e}^{-\left(\log\left(1+\frac{\eta}{4d}\right)\right)\|x-y\|}.$$

Analyticity and localization

Let's start tying up loose ends:

- Combining (1) – (2), we get the uniform exponential decay for $\left|\left\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1} F_I(H_{\Lambda_L})\delta_y \right\rangle\right|$ *as long as*

$$(*)\quad F_I(\nu)\mathrm{e}^{-mh_I(\nu)\|x-y\|} \le \mathrm{e}^{-mh_I(\lambda)\|x-y\|}$$

for *all* $\nu \in \sigma(H_\Lambda)$.

- (3) yields exponential decay for $|\langle \delta_x, K(H_\Lambda)\delta_y\rangle|$ *as long as*

$$(**)\quad \|K\|_\infty \le \mathrm{e}^{\left(\log\left(1+\frac{\eta}{4d}\right)/2\right)\|x-y\|}.$$

- Are there a filter F_I and a modulating function h_I out there that satisfy both (*) and (**)?

End game

- The choice

$$F_I(z) = \mathrm{e}^{-t\left((z-E)^2-(\lambda-E)^2\right)}; \quad t = \frac{m\|x-y\|}{A^2},$$

and

$$h_I(t) = h\left(\frac{t-E}{A}\right)$$

with

$$h(s) = \begin{cases} 1-s^2 & \text{if } s \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

does the trick! In fact, it turns Eq. (*) into the identity for $v \in I$.

THANKS!