Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

Controller Synthesis with Guaranteed Closed-Loop Phase Constraints

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***Abstract*— In this paper, we present an analysis and synthesis framework for guaranteeing that the phase of a single-input, single-output closed-loop transfer function is contained in the interval $[-\alpha, \alpha]$ for a given $\alpha > 0$ at all frequencies. Specifically, we first derive a sufficient condition involving a frequency domain inequality for guaranteeing a given phase constraint. Next, we use the Kalman-Yakubovich-Popov theorem to derive an equivalent time domain condition. In the case where $\alpha = \frac{\pi}{2}$, we show that frequency and time domain sufficient conditions specialize to the positivity theorem. Furthermore, using linear matrix inequalities, we develop a controller synthesis framework for guaranteeing a phase constraint on the closed-loop transfer function. Finally, we extend this synthesis framework to address mixed gain and phase constraints on the closed-loop transfer function.**

I. Introduction

The ability to address gain and phase uncertainties is essential for maximizing achievable performance in controlling uncertain dynamical systems. The small gain theorem guarantees robust stability by requiring that the loop gain (including desired weighing functions for loop shaping) be less than unity at all frequencies. The small gain theorem, however, does not make use of phase information in guaranteeing stability. To some extent, phase information is accounted for by means of positivity theory [1–5]. In this theory, a positive real plant and a strictly positive real uncertainty are both assumed to have phase less than $90°$ so that the loop transfer function has less than $180°$ of phase shift, hence guaranteeing robust stability in spite of gain uncertainty. Other notable results addressing phase information include concepts such as principal phases [6], [7], multivariable phase margin [8], phase spread [9], phase envelope [10], phase matching [11–14], phase-sensitive structured singular value [15], [16], and plant uncertainty templates [17–19]. With the exception of positivity theory all of the aforementioned methods are restricted to frequency domain characterizations and are not amenable to state space formulations necessary for developing controller synthesis methods with guaranteed phase constraints.

In this paper, we present an analysis and synthesis framework for guaranteeing that the phase of a single-input, single-output closed-loop transfer function is contained in the interval $[-\alpha, \alpha]$ for a given $\alpha > 0$ at all frequencies. Specifically, we first derive a sufficient condition involving a frequency domain inequality for guaranteeing a given phase constraint. Next, we use the Kalman-Yakubovich-Popov (KYP) theorem to derive an equivalent time domain condition. In the case where $\alpha = \frac{\pi}{2}$, we show that frequency and time domain sufficient conditions specialize to the positivity theorem. Furthermore, using linear matrix inequalities (LMIs), we develop a controller synthesis framework for guaranteeing a phase constraint on the closed-loop transfer function. Finally, we extend this synthesis framework to address mixed gain and phase constraints on the closed-loop transfer function.

This research was supported in part by the Air Force Office of Scientific Research under Grant FA9550-06-1-0240 and the National Science Foundation under Grant ECS-0551947.

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II. Mathematical Preliminaries

In this section, we introduce notation and several key results necessary for developing the main results of this paper. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^{n\times m}$ denote the set of real $n \times m$ matrices, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and let A^{T} and A^* denote the transpose and complex conjugate transpose of A, respectively. We write $\|\cdot\|_2$ to denote the Euclidean vector norm, I_n or I to denote the $n \times n$ identity matrix, and $M \geq 0$ (resp., $M > 0$) to denote the fact that the symmetric matrix M is nonnegative-definite (resp., positive-definite). Furthermore, we write

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

to denote the state space realization of the transfer function $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization. In the case where $G(s)$ is a scalar transfer function, $\angle G(\jmath\omega)$ denotes the phase of $G(\jmath\omega)$.

Let $\mathcal{L}_2$ denote the space of bounded Lebesgue measurable functions on $[0, \infty)$. For a measurable function $v : [0, \infty) \to \mathbb{R}^r$ recall that the $\mathcal{L}_2$ function norm with Euclidean spatial norm is given by

$$|||v(t)|||_{\mathcal{L}_2} \triangleq \left(\int_0^\infty \|v(t)\|_2^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

and the $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ with input u and output y is defined as

$$|||G(s)|||_\infty \triangleq \sup_{u(\cdot)\in\mathcal{L}_2} \frac{|||y(t)|||_{\mathcal{L}_2}}{|||u(t)|||_{\mathcal{L}_2}} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(\jmath\omega)].$$

Next, we state the well-known Kalman-Yacubovich-Popov (KYP) theorem.

Theorem 2.1 ([20]): Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{l\times m}$. Furthermore, let $\hat{Q} \in \mathbb{S}^l$, $\hat{S} \in \mathbb{R}^{l\times m}$, and $\hat{R} \in \mathbb{S}^m$. Then,

$$G^*(\jmath\omega)\hat{Q}G(\jmath\omega) + G^*(\jmath\omega)\hat{S} + \hat{S}^{\mathrm{T}}G(\jmath\omega) + \hat{R} \leq 0, \quad \omega \in \mathbb{R}, \quad (1)$$

if and only if there exists $P \in \mathbb{S}^n$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P+PA-C^{\mathrm{T}}\hat{Q}C & PB-C^{\mathrm{T}}(\hat{Q}D+\hat{S}) \\ B^{\mathrm{T}}P-(\hat{Q}D+\hat{S})^{\mathrm{T}}C & -(\hat{R}+\hat{S}^{\mathrm{T}}D+D^{\mathrm{T}}\hat{S}+D^{\mathrm{T}}\hat{Q}D) \end{bmatrix} \leq 0. \quad (2)$$

Remark 2.1: Note that if in Theorem 2.1 $\hat{Q} \leq 0$ and A is Hurwitz, then $P \geq 0$.

Corollary 2.1: (**Bounded Real Lemma [2]**) Let $\gamma > 0$ and consider the linear dynamical system

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and $D \in \mathbb{R}^{l\times m}$. Then the following statements are equivalent:

i) There exists matrices $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{p\times n}$, and $W \in \mathbb{R}^{p\times m}$, with P positive definite, such that

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L, \quad (3)$$
$$0 = PB + C^{\mathrm{T}}D + L^{\mathrm{T}}W, \quad (4)$$
$$0 = \gamma^2 I_m - D^{\mathrm{T}}D - W^{\mathrm{T}}W. \quad (5)$$

ii) $|||G(s)|||_\infty \leq \gamma$.

Proof. The proof is a direct consequence of Theorem 2.1 and Remark 2.1 with $\hat{Q} = -I_l$, $\hat{S} = 0$, and $\hat{R} = \gamma^2 I_m$. $P > 0$ follows from the fact that (A, C) is observable. □

Remark 2.2: Note that (3)–(5) can be written as

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C & PB + C^{\mathrm{T}}D \\ (PB + C^{\mathrm{T}}D)^{\mathrm{T}} & -\gamma^2 I_m + D^{\mathrm{T}}D \end{bmatrix} \leq 0, \quad (6)$$

and in dual form as

$$\begin{bmatrix} AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} & QC^{\mathrm{T}} + BD^{\mathrm{T}} \\ (QC^{\mathrm{T}} + BD^{\mathrm{T}})^{\mathrm{T}} & -\gamma^2 I_l + DD^{\mathrm{T}} \end{bmatrix} \leq 0, \quad (7)$$

where $Q > 0$.

The following theorem gives sufficient conditions for guaranteeing that the phase of a scalar transfer function $G(s)$ is bounded by $\pm\alpha$, where $\alpha \in (0, \frac{\pi}{2}]$.

Theorem 2.2: Let $\alpha \in (0, \frac{\pi}{2}]$, let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$, and let $\lambda > 0$. Then,

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L, \quad (8)$$
$$0 = B^{\mathrm{T}}P + (D - \lambda)C + W^{\mathrm{T}}L, \quad (9)$$
$$0 = 2\lambda D - D^2 - \lambda^2\cos^2\alpha - W^{\mathrm{T}}W, \quad (10)$$

where $P \in \mathbb{S}^n$, $L \in \mathbb{R}^{p\times n}$, and $W \in \mathbb{R}^{p\times 1}$, if and only if

$$G^*(\jmath\omega)G(\jmath\omega) - \lambda(G^*(\jmath\omega) + G(\jmath\omega)) + \lambda^2\cos^2\alpha \leq 0, \quad \omega \in \mathbb{R}. \quad (11)$$

Furhtermore, if (11) or, equivalently, (8)–(10) hold, then $\angle G(\jmath\omega) \in [-\alpha, \alpha]$, $\omega \in \mathbb{R}$.

Proof. The equivalence of (8)–(10) and (11) is a direct consequence of Theorem 2.1 with $\hat{Q} = -1$, $\hat{S} = \lambda$, and $\hat{R} = -\lambda^2\cos^2\alpha$. To show that $\angle G(\jmath\omega) \in [-\alpha, \alpha]$, where $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$, define $G(\jmath\omega) = \beta e^{\jmath\theta}$, where $\beta > 0$ and $\theta \in \mathbb{R}$. In this case, (11) can be written as

$$\beta^2 - 2\lambda\beta\cos\theta + \lambda^2\cos^2\alpha \leq 0, \quad (12)$$

or, equivalently,

$$(\beta - \lambda\cos\alpha)^2 + 2\lambda\beta(\cos\alpha - \cos\theta) \leq 0, \quad (13)$$

which implies $\cos\alpha \leq \cos\theta$. Hence, $\angle G(\jmath\omega) \in [-\alpha, \alpha]$, where $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$. □

Remark 2.3: Note that if in Theorem 2.2 A is Hurwitz, then $P \geq 0$. If, in addition, (A, C) is observable, then $P > 0$.

Remark 2.4: A dual representation to (8)–(10) is given by

$$0 = AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} + LL^{\mathrm{T}}, \quad (14)$$
$$0 = CQ + (D - \lambda)B^{\mathrm{T}} + WL^{\mathrm{T}}, \quad (15)$$
$$0 = 2\lambda D - D^2 - \lambda^2\cos^2\alpha - WW^{\mathrm{T}}, \quad (16)$$

or, equivalently,

$$\begin{bmatrix} AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} & QC^{\mathrm{T}} + (D - \lambda)B \\ CQ + (D - \lambda)B^{\mathrm{T}} & -2\lambda D + D^2 + \lambda^2\cos^2\alpha \end{bmatrix} \leq 0, \quad (17)$$

where $Q \in \mathbb{S}^n$. The sign definitness of Q can be established using identical assumptions as in Remark 2.3.

Remark 2.5: Note that it follows from Theorem 2.2 that if there exists $P \in \mathbb{S}^n$, $L \in \mathbb{R}^{p\times n}$, and $W \in \mathbb{R}^{p\times 1}$ such that

$$0 = A^{\mathrm{T}}P + PA + \cos\alpha C^{\mathrm{T}}C + L^{\mathrm{T}}L, \quad (18)$$
$$0 = B^{\mathrm{T}}P + \cos\alpha DC - C + W^{\mathrm{T}}L, \quad (19)$$
$$0 = 2D - \cos\alpha D^2 - \cos\alpha - W^{\mathrm{T}}W, \quad (20)$$

or, equivalently,

$$\cos\alpha(G^*(\jmath\omega)G(\jmath\omega)+1) - (G^*(\jmath\omega)+G(\jmath\omega)) \leq 0, \quad \omega \in \mathbb{R}, \quad (21)$$

then $\angle G(\jmath\omega) \in [-\alpha, \alpha]$, $\omega \in \mathbb{R}$. To see this, note that (8)–(10) are identical to (18)–(21) with $\lambda = \sec\alpha$ and P, L, and W replaced by $P\cos\alpha$, $L\sqrt{\cos\alpha}$, and $W\sqrt{\cos\alpha}$, respectively.

The next corollary specializes Theorem 2.2 to the generalized positive real theorem.

Corollary 2.2: (**Generalized Positive Real Theorem**) Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$. Then

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \quad (22)$$
$$0 = B^{\mathrm{T}}P - C + W^{\mathrm{T}}L, \quad (23)$$
$$0 = 2D - W^{\mathrm{T}}W, \quad (24)$$

where $P \in \mathbb{S}^n$, $L \in \mathbb{R}^{p\times n}$, and $W \in \mathbb{R}^{p\times 1}$, if and only if

$$G^*(\jmath\omega) + G(\jmath\omega) \geq 0, \quad \omega \in \mathbb{R}. \quad (25)$$

Furthermore, if (25) or, equivalently, (22)–(24) hold, then $\angle G(\jmath\omega) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\omega \in \mathbb{R}$.

Proof. The proof is a direct consequence of Theorem 2.2 with $\alpha = \pi/2$. □

III. Controller Synthesis with Guaranteed Phase and Gain Constraints

In this section, we present a control design framework for single-input, single-output systems with guaranteed closed-loop phase and gain constraints. We formulate this problem using linear matrix inequalities. First, we present the phase constrained control problem.

Phase Constrained Control Problem. Given the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \quad x(0) = x_0, \quad t \geq 0, \quad (26)$$

with performance variables

$$z(t) = E_1x(t) + E_2u(t) + D_2w(t), \qquad (27)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(\cdot) \in \mathcal{L}_2$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D_1 \in \mathbb{R}^{n\times 1}$, $E_1 \in \mathbb{R}^{1\times l}$, $E_2 \in \mathbb{R}^{1\times m}$, and $D_2 \in \mathbb{R}$, determine a static feedback control law

$$u(t) = Kx(t), \qquad (28)$$

that satisfies the following design criteria:

i) The undisturbed ($w(t) \equiv 0$) closed-loop system given by (26) and (28) is asymptotically stable, that is, $A+BK$ is Hurwitz; and

ii) $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$ for some given $\alpha \in (0, \frac{\pi}{2}]$, where $\tilde{G}(s)$ is the closed-loop system given by

$$\tilde{G}(s) \sim \left[\begin{array}{c|c} A+BK & D_1 \\ \hline E_1+E_2K & D_2 \end{array}\right].$$

Theorem 3.1: Consider the linear dynamical system (26) and (27), and assume that (A, D_1) is stabilizable and (A, E_1) is detectable. Let $\alpha \in (0, \frac{\pi}{2}]$ and $\lambda > 0$. Suppose there exist $Q \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$, with Q positive definite, such that

$$\begin{bmatrix} X & M \\ M^{\mathrm{T}} & Y \end{bmatrix} \leq 0, \qquad (29)$$

where

$$X \triangleq AQ + QA^{\mathrm{T}} + BZ + Z^{\mathrm{T}}B^{\mathrm{T}} + D_1D_1^{\mathrm{T}}, \quad (30)$$

$$M \triangleq QE_1^{\mathrm{T}} + Z^{\mathrm{T}}E_2^{\mathrm{T}} + (D_2 - \lambda)D_1, \quad (31)$$

$$Y \triangleq -2\lambda D_2 + D_2^2 + \lambda^2\cos^2\alpha. \quad (32)$$

Then the feedback controller (28) with $K = ZQ^{-1}$ guarantees that $A+BK$ is Hurwitz and $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$, $\omega \in \mathbb{R}$.

Proof. Note that the closed-loop system (26)–(28) is given by

$$\dot{x}(t) = \tilde{A}x(t) + D_1w(t), \quad x(0) = x_0, \quad t \geq 0, \quad (33)$$

$$z(t) = \tilde{E}x(t) + D_2w(t), \quad (34)$$

where $\tilde{A} \triangleq A + BK$ and $\tilde{E} \triangleq E_1 + E_2K$. Now, it follows from (29) that

$$0 \geq AQ + QA^{\mathrm{T}} + BZ + Z^{\mathrm{T}}B^{\mathrm{T}} + D_1D_1^{\mathrm{T}}, \qquad (35)$$

which can be equivalently written as

$$0 \geq (A+BK)Q + Q(A+BK)^{\mathrm{T}} + D_1D_1^{\mathrm{T}}. \qquad (36)$$

Hence, since (A, D_1) is stabilizable by assumption, $(\tilde{A}, D_1)$ is also stabilizable, and hence, since $Q > 0$, $\tilde{A} = A + BK$ is Hurwitz.

To show that $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$, where $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$, note that with $Z = KQ$, (29) can be written as

$$\begin{bmatrix} \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + D_1D_1^{\mathrm{T}} & Q\tilde{E}^{\mathrm{T}} + (D_2-\lambda)D_1 \\ \tilde{E}Q + (D_2-\lambda)D_1^{\mathrm{T}} & -2\lambda D_2 + D_2^2 + \lambda^2\cos^2\alpha \end{bmatrix} \leq 0, \qquad (37)$$

or, equivalently,

$$\begin{bmatrix} U & N \\ N^{\mathrm{T}} & V \end{bmatrix} \leq 0, \qquad (38)$$

where

$$U \triangleq (A+BK)Q + Q(A+BK)^{\mathrm{T}} + D_1D_1^{\mathrm{T}}, \quad (39)$$

$$N \triangleq Q(E_1+E_2K)^{\mathrm{T}} + (D_2-\lambda)D_1, \quad (40)$$

$$V \triangleq -2\lambda D_2 + D_2^2 + \lambda^2\cos^2\alpha. \quad (41)$$

Now, it follows from Theorem 2.2 and Remark 2.4 that $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$, where $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$. □

Remark 3.1: Note that in the case where $D_2 = 0$, (29) and (32) imply that $\lambda^2\cos^2\alpha \leq 0$, which holds if and only if $\alpha = \frac{\pi}{2}$. Hence, if the direct transmission term $D_2 = 0$ in the performance variable z, then the only feasible value for α is $\frac{\pi}{2}$. Since for $\mathcal{H}_2$ optimal control we require that $D_2 = 0$, it follows that it is impossible to guarantee a closed-loop phase of $\pm\alpha \in (0, \frac{\pi}{2})$ for optimal linear-quadratic regulators.

Mixed Phase and Gain Constrained Problem. Given the linear dynamical system (26) with performance variables (27), determine a static feedback control law (28) that satisfies design criteria *i)*, *ii)*, and

iii) The $\mathcal{H}_\infty$ norm of the closed-loop system satisfies $|||\tilde{G}(s)|||_\infty \leq \gamma$, for some given constant $\gamma > 0$.

Theorem 3.2: Consider the linear dynamical system (26) and (27), and assume that (A, D_1) is stabilizable and (A, E_1) is detectable. Let $\alpha \in (0, \frac{\pi}{2}]$, $\lambda > 0$, and $\gamma > 0$. Suppose there exist $Q \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$, with Q positive definite, such that (29) holds and

$$\begin{bmatrix} X & \hat{M} \\ \hat{M}^{\mathrm{T}} & -\gamma^2 + D_2^2 \end{bmatrix} \leq 0, \qquad (42)$$

where X is given by (30) and

$$\hat{M} \triangleq QE_1^{\mathrm{T}} + Z^{\mathrm{T}}E_2^{\mathrm{T}} + D_1D_2. \qquad (43)$$

Then the feedback controller (28) with $K = ZQ^{-1}$ guarantees that $A+BK$ is Hurwitz, $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$, $\omega \in \mathbb{R}$, and $|||\tilde{G}(s)|||_\infty \leq \gamma$.

Proof. Asymptotic stability of the closed-loop system (26)–(28) and the phase constraint $\angle\tilde{G}(\jmath\omega) \in [-\alpha, \alpha]$, where $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$, follow as a direct consequence of (29) using Theorem 3.1. Next, it follows from Corollary 2.1 and Remark 2.2 that $|||\tilde{G}(s)|||_\infty \leq \gamma$ if and only if

$$\begin{bmatrix} \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + D_1D_1^{\mathrm{T}} & Q\tilde{E}^{\mathrm{T}} + D_1D_2 \\ (Q\tilde{E}^{\mathrm{T}} + D_1D_2)^{\mathrm{T}} & -\gamma^2 + D_2^2 \end{bmatrix} \leq 0, \qquad (44)$$

or, equivalently,

$$\begin{bmatrix} U & \hat{N} \\ \hat{N}^{\mathrm{T}} & -\gamma^2 + D_2^2 \end{bmatrix} \leq 0, \qquad (45)$$

where U is given by (39) and

$$\hat{N} \triangleq Q(E_1+E_2K)^{\mathrm{T}} + D_1D_2. \qquad (46)$$

Hence, since (45) is equivalent to (42) with $Z = KQ$, the result follows. □

IV. ILLUSTRATIVE NUMERICAL EXAMPLE

Consider a mass-spring-damper system with mass $m = 1$, damping coefficient $c = 1$, and spring stiffness $k = 1$. Suppose that the inputs to the system consists of a control force $u(t)$ exerted by an actuator and an external disturbance

$w(t)$, where $w(\cdot) \in \mathcal{L}_2$. The dynamic equations of the system are given by (26) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (47)$$

Let the performance variable $z(t)$ be given by (27) with $E_1 = [0, 1]$, $E_2 = 1$, and $D_2 = 1$. Here, we consider two designs. First, we impose a phase constraint on the closed-loop system. Specifically, let $\alpha = 10$ degrees and $\lambda = 1.18$. The YALMIP [21] and SeDuMi [22] MATLAB toolboxes are used to solve the LMI feasibility problem given by Theorem 3.1. The feasible value of the controller gain was found to be $K^* = [0.0719 - 0.7245]$. For the second design, we add a gain constraint of $\gamma = 1.1$ and solve the LMI feasibility problem given by Theorem 3.2. The feasible value of the controller gain was found to be $K^* = [0.0051 - 0.9762]$. The magnitude and phase plots for both closed-loop designs along with a standard $\mathcal{H}_\infty$ control design are given in Figure 1 and 2, respectively.

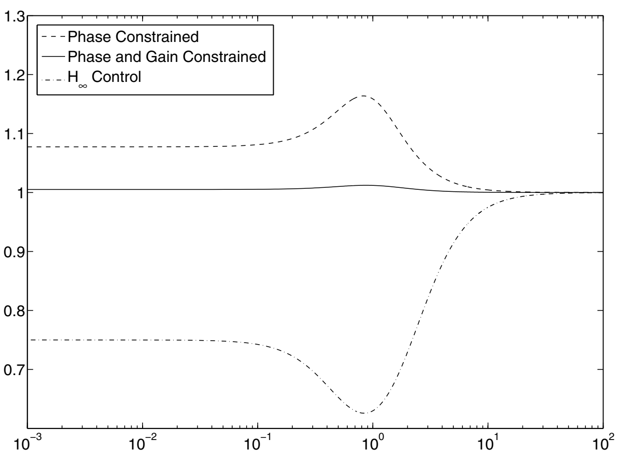

Fig. 1. Magnitude plot of closed-loop system

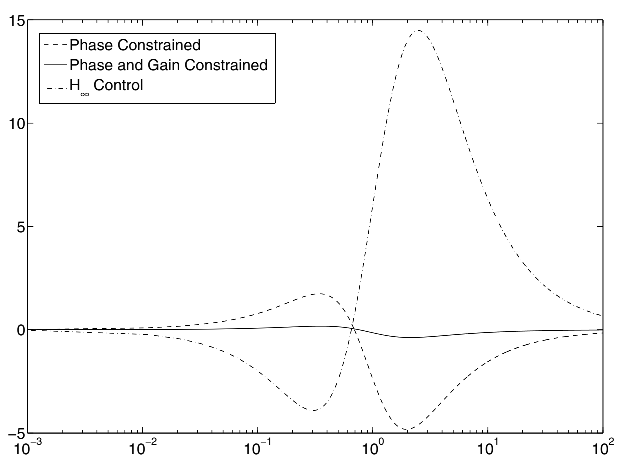

Fig. 2. Phase plot of closed-loop system

V. Conclusion

In this paper, we developed a controller design framework for guaranteeing closed-loop phase constraints of single-input, single-output systems. The framework can be easily extended to ensure that the phase of the loop-gain transfer function is well behaved in frequency regimes in which the loop transfer function has gain greater than unity. In particular, phase stabilization can be used to allow high loop gains, and hence, achieve high system performance in frequency regimes in which sufficient phase information is available thereby avoiding gain stabilization (e.g., rolloff) needed to ensure stability where the phase of the system is poorly known. Finally, using the recently developed notion of the structured phase margin [23], future research will concentrate on multivariable extensions of the proposed phase stabilization approach.

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