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DETERMINATION OF THE HEIGHT OF A TARGET ABOVE A CONDUCTING PLANE BY MEANS OF RADAR MEASUREMENTS

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CHAPTER I

STATEMENT OF THE PROBLEM

Introduction

Techniques for the tracking of a target in free space by radar are well known and are generally satisfactory.[1] Most conventional techniques exploit the directional property of antenna beams or lobes. For example, we may take an antenna system employing two identical lobes symmetrically displaced in elevation about an axis. If we position the axis in elevation to obtain equal signal strengths from both lobes, the angle of elevation of this axis will be the elevation of the target. This "lobe comparison" may be sequential as in a conical scan system or simultaneous as in a monopulse system.

When the target is near a conducting surface, as in the case of an aircraft near the surface of the sea, then the tracking problem becomes much more complicated. If the radar system is also near the conducting surface, the problem of radar tracking is no longer a free-space problem, since the radar receives energy from the target by two paths, the direct path and a second path involving reflection from the surface. Under these circumstances serious tracking errors

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[1] L. N. Ridenour, Radar System Engineering (New York and London: McGraw-Hill Book Company, Inc., 1947).

occur.[2]

Successful tracking can be realized at low target elevations if the directivity of the antenna is increased; that is, if the ability of the antenna to reject surface reflected energy is increased. There is a practical limit, however, on the physical dimensions of a radar antenna, which defines a "low-angle region" in which the energy received by way of the indirect path is not excluded. It is the object of the present analysis to characterize certain radar tracking procedures or techniques through which, ideally at least, the elevation of a target in the low-angle region can be determined.

The Output Voltage of a Radar Antenna in the Low-angle Region

In order to acquire more insight into the source and nature of tracking errors experienced with conventional radars in the low-angle region, we shall obtain mathematical expressions for the output voltage of an antenna in this region. The output voltage will depend on the physical characteristics of the antenna, and also on the geometric configuration of the target, antenna, and conducting surface.

We idealize the physical situation in the following way. We assume that the received field is the same as one which would exist if the radar transmitter were located at the target, i. e., we consider only a "one way" problem. We assume that the target is a stationary

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[2] Donald E. Kerr, Propagation of Short Radio Waves (New York: McGraw-Hill Book Company, Inc., 1951). 5.14.

non-directional point source. We assume further that the received field is the superposition of two plane waves, one being the direct wave and the other being the indirect or surface-reflected wave. The transmitter modulation, including polarization and wavelength, λ , is assumed to be fixed. Finally, we assume that the reflection coefficient is constant. The magnitude of the reflection coefficient, ρ , is defined to be the ratio of the magnitude of the component of the field at the radar receiver due to the indirect wave to the magnitude of the component due to the direct wave. The phase or angle of the reflection coefficient, Γ , is defined to be the phase retardation on reflection from the surface.

Referring to Figure 1, we summarize our hypotheses:

(1) The transmitter modulation is fixed.

(2) A stationary, non-directional point source is located at T.

(3) The electromagnetic field at A is the resultant of two plane waves, a direct wave propagated alone TA and an indirect wave propagated alone TPA.

(4) The reflection coefficient is constant.

(5) The slant range, R, is fixed.

The receiver antenna consists of a uniformly illuminated rectangular aperture of vertical width 2a with center located at A (see Figure 1). We shall be concerned with elevation-angle tracking only. Hence the directional properties of the antenna in azimuth will not concern us in this analysis.

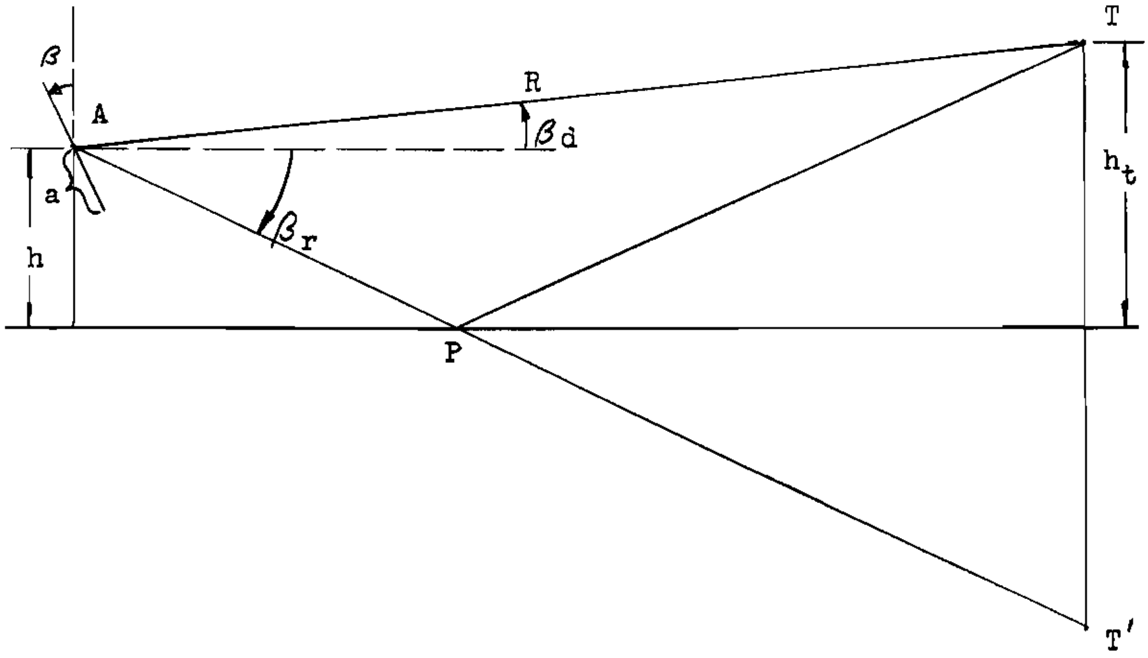

Figure 1. Geometrical Configuration in the Low-angle Problem

Notation:

A: position of receiver

T: position of target or point source

a: one-half aperture width

h: height of aperture center

h_t: target height

β_d: angle of arrival of direct wave

β_r: angle of arrival of reflected wave

R: slant range

β: tilt of aperture.

The antenna is linear, i.e., its response to the resultant of two plane waves is the sum of its responses to the individual waves.

The derivation of the response of a uniformly illuminated rectangular aperture to a plane wave is readily available in the literature[3] and will not be repeated here. Let E_d denote the magnitude of the direct wave at the antenna aperture. Then if we denote the response of the antenna to the direct wave by the vector[4] d, in the notation of Figure 1,

$$(1) \qquad |d| = E_d \alpha \left| \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta - \beta_d)\right]}{\frac{2\pi a}{\lambda} \sin(\beta - \beta_d)} \right| \qquad E_d \alpha \left\{ f(a, \beta, \beta_d) \right\} ,$$

where $|d|$ denotes the magnitude of the vector d, and α is a conversion factor depending on the physical characteristics of the receiving system. Similarly, if we denote the response of the antenna to the indirect wave by the vector r, then

$$|r| = \rho E_d \alpha \left| \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta + \beta_r)\right]}{\frac{2\pi a}{\lambda} \sin(\beta + \beta_r)} \right| \quad \rho E_d \alpha \left\{ g(a, \beta, \beta_r) \right\} .$$

The quantities

$$\left[\frac{2\pi a}{\lambda} \sin(\beta - \beta_d)\right]$$

and

$$\left[\frac{2\pi a}{\lambda} \sin(\beta + \beta_r)\right]$$

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[3] S. Silver, Microwave Antenna Theory and Design (New York: McGraw-Hill Book Company, Inc., 1949) Section 6.5.

[4] We shall avoid the use of special notation for vectors, since we shall employ only three vectors in the analysis. These are d, r, and δ. All other quantities are scalars.

shall be restricted to lie between $-\pi$ and π. Physically this means that we are not concerned with "side lobes".

The target height, h_t, enters equations (1) and (2) through β_d and β_r; thus,

$$(3) \qquad \beta_d = \frac{h_t - h}{R} ,$$

$$(4) \qquad \beta_r = \frac{h_t + h}{R} .$$

Expressions (3) and (4), like (5) below, are derived in Appendix A.

If we take as a phase reference the phase of the direct wave at the center of the aperture, then the angle of the vector d will be zero, and the vector r will lag by an angle α radians, where

$$(5) \qquad \alpha = \Gamma - \frac{4\pi h h_t}{\lambda R} .$$

The angle α is the phase difference between direct and indirect waves. The term $\frac{4\pi h h_t}{\lambda R}$ represents the geometric path difference in radians, and Γ the phase retardation on reflection at the surface.

The resultant voltage in the antenna output is the vector sum of r and d. If we denote this resultant by the vector δ, then

$$|\delta|^2 = (E_d a)^2 \left[f^2 + \rho^2 g^2 + 2\rho f g \cos\alpha \right] .$$

The phase of the resultant output is the angle by which the vector δ

lags the vector d. If we denote this angle by Θ , then

$$\Theta = \tan^{-1}\left\{ \frac{|r| \sin\alpha}{|r| \cos\alpha + |d|} \right\}$$

$$= \tan^{-1}\left\{ \frac{\rho g \sin\alpha}{f + \rho g \cos\alpha} \right\} .$$

Outline of Approach to the Problem

We can now show why conventional tracking procedures fail in the low-angle region. We shall take as an example of a conventional tracking radar the monopulse equipment. For elevation determination, a monopulse employs a single aperture illuminated by two vertically displaced feeds, so as to provide two antenna lobes symmetrically displaced in elevation about a line called the axis of the system. The displacement of each lobe from the axis is approximately one half the half-power beam width. The elevation of the axis is controlled by a servomechanism which balances the magnitudes of the lobe output voltages. In free space this balance would occur when the axis coincides with the line of sight to the target.

Denoting the outputs of the upper and lower lobes by δ_1 and δ_2 respectively, the condition for a balance or "null" is given by the equation

(8) $$|\delta_1|^2 - |\delta_2|^2 = 0 .$$

In the low-angle region the vector δ_1 is the sum of two vectors

d_1 and r_1, where d_1 and r_1 are the components of δ_1 due to the direct and reflected waves respectively. Similarly, δ_2 is the sum of d_2 and r_2. If we denote the magnitude of the displacement in elevation of each lobe from the axis by η , and set, as in equations (1) and (2),

$$f_1 = f(\beta_1, \beta_d) = f(\beta + \eta, \beta_d) ,$$
$$f_2 = f(\beta_2, \beta_d) = f(\beta - \eta, \beta_d) ,$$
$$g_1 = g(\beta_1, \beta_r) = g(\beta + \eta, \beta_r) ,$$

and

$$g_2 = g(\beta_2, \beta_r) = g(\beta - \eta, \beta_r) ,$$

then equation (8) becomes

$$(9) \qquad f_1^2 - f_2^2 + \rho^2(g_1^2 - g_2^2) + 2\rho(f_1 g_1 - f_2 g_2) \cos\alpha = 0,$$

where, as before,

$$\alpha = \Gamma - \frac{4\pi h h_t}{\lambda R} ,$$

and h is the height of the aperture center.

The principle on which the monopulse operates is the following. When a balance is secured, the elevation of the axis β is the elevation of the target. In other words, a value of $\beta = \beta_0$ for which equation (9) is satisfied is the "indicated target elevation". We can show from equation (9) that for a given target height h_t and for fixed values of all the other parameters including ρ and Γ the indicated elevation will differ in general from the true target elevation. If the indicated elevation were the true elevation, β_0 would equal β_d. From equation (1) it follows that $f_1 = f_2 \equiv f \gtreqless 0$. Hence, if $\beta_0 = \beta_d$ is a solution of equation (9), then it is necessary that

$$\rho^2(g_1^2 - g_2^2) + 2\rho f \cos\alpha \, (g_1 - g_2) = 0,$$

or

$$\rho(g_1 - g_2)\left[\rho(g_1 + g_2) + 2f\cos\alpha\right] = 0$$

for all values of α. Since $\beta_r \neq \beta_d$, it can be demonstrated that $g_1 \neq g_2$. Hence

$$\left[\rho(g_1 + g_2) + 2f\cos\alpha\right] = 0$$

for all values of α. But this is false, since $\rho > 0$, $f \gtreqless 0$, $g_1 \gtreqless 0$, and $g_2 \gtreqless 0$.

The above argument shows that indicated target elevation is in general not the true target elevation. If we consider the difference between the indicated and true target elevations as an error, we can plot the error as a function of target position. This has been done by Kerr[5] for a conventional tracking radar essentially of the same type as the monopulse described above.

One approach to the low-angle tracking problem is to fix h_t, determine β_0 such that equation (9) is satisfied, and then study the

[5] Kerr, op. cit., Section 5.14.

error in indicated target elevation with the object of devising means of minimizing this error. On the other hand, the approach we shall take is the inverse of that just mentioned. We shall fix β , examine all the solutions of equation (9) for h_t, and then determine conditions to be imposed so that an unique determination of h_t is secured.

If we regard indicated elevation as a radar measurement, then we have shown that this radar measurement in the case of the monopulse system does not in general yield the true target elevation. Our analysis will be concerned with the development of necessary and sufficient radar measurement procedures for the unique determination of target elevation in the low-angle region.

CHAPTER II

FORMALIZATION

Definitions

Lobe.--By lobe we mean the main beam of a uniformly illuminated aperture with its associated feed, specified by three parameters a, β, h, as illustrated in Figure 1. The parameter a is one half the vertical aperture width, β the tilt of the aperture from the vertical, and h is the height of the center of the aperture above the conducting plane. The output voltage δ of a lobe is a vector which is a single-valued function Δ of the parameters a, β, h, ρ, Γ, λ, E_d, R, and h_t. For our purposes we regard δ as independent of the time t. We obtain two different lobes by assigning different values to the parameters a, β, h in the lobe function Δ. A difference in lobes will be indicated by subscripts on the output voltages; for example,

$$\delta_1 = \Delta(a, \beta, h_1, \ldots)$$

and

$$\delta_2 = \Delta(a, \beta, h_2, \ldots)$$

represent two lobes at different heights. For simplicity, we shall denote a lobe by the output voltage; i.e., if the output is δ, then we shall speak of the lobe δ.

We have chosen the lobe as the fundamental unit so that we may analyze almost all antenna configurations with a single theory. More complicated antenna structures can be thought of as combinations of single-lobe antennas. In this way we avoid such confusion as might arise from the question whether a monopulse is one antenna or two antennas.

Receiver Parameters.--By receiver parameters we mean the quantities a, β, and h.

Radar Measurements.--In a radar system employing several lobes the outputs of the lobes may be added or subtracted, and each output may be attenuated or shifted in phase. We do not observe either the lobe outputs or linear combinations of them, with or without attenuation or phase shift. What we do have access to is the magnitudes of such vectors. It is natural therefore to employ as our basic quantities inner products of two vectors. By radar measurement or an admissible measurement we mean an inner product of two vectors which may be lobe outputs or linear combinations of them, with or without phase shift. We denote phase shift through the angle θ by right multiplication of the vector by a matrix Φ_θ. Formally, our radar measurements may be represented as inner products on a linear vector space[6] whose elements are the lobe outputs δ_i. Phase shifting represents a rotation in our space.

Measurement Procedure.--By a radar measurement procedure we mean a combination of admissible measurements. By the value of a measurement procedure is meant the value of, or number resulting from, such a

[6] Properties of inner products on linear vector spaces are given in Appendix B.

combination. In particular, we shall speak of elementary measurement procedures, by which is meant either the measurement procedure

$$|\delta_1|^2 - |\delta_2|^2,$$

or

$$|\delta_1 \Phi_\theta - \delta_2|^2,$$

where Φ_θ is a rotation, and may be the identity matrix.

It should be noted that the first of these elementary measurements procedures compares the magnitudes of lobe outputs, while the second depends on their phase difference as well as their magnitudes.

<u>Closed-Loop Procedure</u>.--The value of a radar measurement procedure depends in particular on the receiver parameters (a, β, h) in each lobe, on the various phase shifts (Φ_θ) and on the target height h_t. We could calculate the value of a measurement for fixed a, β, h, and Φ_θ, but it will be more convenient to vary these quantities in order to obtain a prescribed value of the measurement. A closed-loop procedure is a procedure whereby we prescribe the value of a measurement procedure and vary the quantities a, β, h, or Φ_θ so as to obtain the fixed value. Physically, closed-loop procedures are effected through servo systems. Generally the prescribed value of a measurement procedure is taken to be zero.

<u>Null</u>.--Suppose that in a closed-loop procedure the receiver parameters or Φ_θ depend on a controlled variable. Then a value of the controlled variable such that the value of the procedure is zero is called a null.

The term "null" as used here may not necessarily coincide with the "null" associated with a free-space antenna pattern.

Mathematical Formulation of the Problem

We shall represent measurement procedures by a real-valued function $F(\delta_1, \ldots, \delta_p)$ of the lobe output vectors, $\delta_1, \ldots, \delta_p$. If a procedure under consideration involves shifting of phase in lobe outputs, this will be indicated by writing $F(\delta_1, \ldots, \delta_p, \Phi_{\Theta_1}, \ldots, \Phi_{\Theta_p})$. The nature of the function F is restricted by our definition of admissible measurements, that is, F is a combination of inner products of the lobe output vectors. We may rotate and add or subtract these vectors before taking inner products.

Example 1. Interferometer.

An interferometer is a device consisting of two lobes at different heights tilted through the same angle. In the case of an incident plane wave the difference in phase between the lobe outputs is proportional to the angle of arrival of the incident wave. Let

$$\delta_1 = \Delta(a, \beta, h_1)$$

and

$$\delta_2 = \Delta(a, \beta, h_2)$$

denote the two lobe outputs. Then a measurement procedure (the one usually employed with interferometers) is represented by

$$F(\delta_1, \delta_2, \Phi_\Theta) = \left|\delta_1\Phi_\Theta - \delta_2\right|^2$$

$$= (\delta_1, \delta_1) + (\delta_2, \delta_2) - 2(\delta_1\Phi_\Theta, \delta_2).$$

Example 2. Monopulse.

The monopulse has already been described in Chapter I. It consists of two lobes at the same height but differing tilt. The difference in tilt of the two lobes is held fixed, while the structure as a whole is tilted, so as to balance the magnitudes of the lobe output voltages. Let

$$\delta_1 = \Delta(a, \beta + \eta, h)$$

and

$$\delta_2 = \Delta(a, \beta - \eta, h)$$

where $\beta + \eta$ is the tilt of the upper lobe and $\beta - \eta$ is the tilt of the lower lobe. Then the usual measurement procedure is represented by

$$F(\delta_1, \delta_2) = |\delta_1|^2 - |\delta_2|^2 = (\delta_1, \delta_1) - (\delta_2, \delta_2) .$$

The function $F(\delta_1, \ldots, \delta_p, \Phi_{\Theta_1}, \ldots, \Phi_{\Theta_p})$ representing a measurement procedure can be written as

$$F(a_1, \ldots, a_p, \beta_1, \ldots, \beta_p, h_1, \ldots, h_p, \Phi_{\Theta_1}, \ldots, \Phi_{\Theta_p}; h_t)$$

since δ_i depends on a_i, β_i, h_i. In a closed-loop procedure the servo system finds a null of F. We can describe a closed-loop procedure, then, by an equation

$$F(a_1, \ldots, a_p, \beta_1, \ldots, \beta_p, h_1, \ldots, h_p, \Phi_{\Theta_1}, \ldots \Phi_{\Theta_p}; h_t) = 0.$$

The problem can now be formulated as follows. It is required to characterize those functions F for which

$$F(a_i, \beta_i, h_i, \Phi_{\Theta_i}; h_t) = 0 \tag{10}$$

determines h_t uniquely in terms of $a_i, \beta_i, h_i, \Phi_{\Theta_i}$. When we say that h_t is determined uniquely in terms of $a_i, \beta_i, h_i, \Phi_{\Theta_i}$ from equation (10), we do not mean necessarily that h_t is a single-valued function of these parameters. It may be that h_t is a multiple-valued function of these parameters, but that limitations on the beam width exclude all but one of these values. In other words, there is one and only one value of h_t determined in the low-angle region.

CHAPTER III

SOLUTION

Necessary Procedures

We have defined admissible measurements and shown that a measurement procedure may be represented by a function F which is a combination of such measurements. A closed-loop procedure determines a null of F. Our problem is to find what measurements are necessary in order that the equation $F = 0$ determine h_t uniquely. After investigating this problem mathematically, we shall give some practical considerations influencing the arrangement and choice of these necessary measurements.

We shall show that in a closed-loop procedure it is necessary to employ more than one elementary measurement. In other words, if the equation $F = 0$ is to determine h_t uniquely, then F must involve at least two elementary measurements. We have already seen in Chapter I that for the monopulse, a special instance of a single elementary measurement procedure, a null of F is not in general the true target elevation. However, this result does not preclude the unique determination of h_t from a null of F. In this chapter we shall show that the equation $F = 0$, where F involves but one elementary measurement, is satisfied by more than one value of h_t in the low-angle region.

One way of proving that it is impossible in general to determine the target height uniquely through a single elementary measurement procedure would be to consider all possible elementary measurements, ascertaining for each if there exists more than one h_t in the low-angle region corresponding to a given null. The values of the target height corresponding to a given null in each case depend among other things on the particular antenna pattern and the values assumed for ρ and Γ. In Appendix C, a particular case is worked out showing the several values of h_t determined by $F = 0$, where F is an elementary measurement procedure. In order to avoid the difficulties of a proof based on examination of every possible case, and to illustrate more clearly the reason for the indeterminacy in h_t, we shall introduce some approximations usually adopted in practice. These approximations will be stated when the proof is given.

We recall that a single elementary measurement procedure involves two lobes. These lobes differ in one of the receiver parameters a, β, h. If we denote the two lobes by δ_1 and δ_2, and the elementary measurement procedures by F_a and F_p, then

$$(11) \qquad F_a = |\delta_1|^2 - |\delta_2|^2 = (\delta_1, \delta_1) - (\delta_2, \delta_2),$$

$$= (d_1, d_1) - (d_2, d_2) + (r_1, r_1) - (r_2, r_2)$$

$$+ 2(d_1, r_1) - 2(d_2, r_2),$$

and

$$(12) \qquad F_p = |\delta_1 - \delta_2|^2 = (\delta_1, \delta_1) + (\delta_2, \delta_2) - 2(\delta_1, \delta_2),$$

$$= (d_1, d_1) + (r_1, r_1) + (d_2, d_2) + (r_2, r_2)$$

$$+ 2(d_1, r_1) + 2(d_2, r_2) - 2(d_1, d_2) - 2(d_1, r_2)$$

$$- 2(r_1, d_2) - 2(r_1, r_2) \quad .$$

We are now ready to prove that if there exists a value of $h_t = h_t'$ such that

$$F_a(h_t') = 0 \qquad \left[F_p(h_t') = 0\right],$$

then there exists at least one other value $h_t = h_t'' \neq h_t'$ in the low-angle region such that

$$F_a(h_t'') = 0 \qquad \left[F_p(h_t'') = 0\right],$$

where the square brackets indicate that the same proposition holds for F_p.

We now introduce the approximations mentioned earlier. Since changes in h_t or h cause variations in β_d and β_r (see page 6), these changes result in variations in $|d|$ and $|r|$. We shall assume as an approximation that changes in h_t or h such that β_d and β_r vary by no more than a small fraction of the beam width result in no change in $|d|$ and $|r|$. In practice it is sufficient to take this fraction to be of the order of 1/20.

We shall divide the proof into two parts.

Case I ($h_1 \neq h_2$)

In this case, under the assumption made above, F_a reduces to

(13) $$F_a = 2|d_1|\,|r_1|\,(\cos\alpha_1 - \cos\alpha_2),$$

where

$$\alpha_i = \Gamma - \frac{4\pi h_i\, h_t}{\lambda R}, \qquad [i = 1, 2] .$$

Let

$$h_t' = \frac{(\Gamma + n\pi)\lambda R}{2\pi(h_1 + h_2)} ,$$

and

$$h_t'' = \frac{(\Gamma + (n-1)\pi)\lambda R}{2\pi(h_1 + h_2)} ,$$ where n is an integer.

Then

$$F_a(h_t') = F_a(h_t'') = 0,$$

and

$$h_t' - h_t'' = \frac{\lambda R}{2(h_1 + h_2)} .$$

In practice R is of the order of 20,000 yards, $\lambda = 0.1$ ft., and $h \doteq 75$ ft. Hence

$$\frac{h_t' - h_t''}{R}$$

is well within a beam width of 1°, which is representative of beam widths normally employed.

F_p reduces to

$$(14)\quad F_p = 2|d_1||r_1|\left[\cos\alpha_1 + \cos\alpha_2 - \cos(\alpha_1 + \gamma_r) - \cos(\alpha_2 - \gamma_r)\right],$$

where γ_r denotes the phase difference between r_2 and r_1, and γ_d the phase difference between d_2 and d_1, and

$$\gamma_r - \gamma_d = \alpha_2 - \alpha_1 .$$

We may write F_p as

$$(15)\quad F_p = |d_1||r_1|\left[2\sin\tfrac{1}{2}(2\alpha_1 + \gamma_r)\sin\tfrac{1}{2}(-\gamma_r) + 2\sin\tfrac{1}{2}(2\alpha_2 - \gamma_r)\sin\tfrac{1}{2}(\gamma_r)\right],$$

$$= 2|d_1||r_1|\sin\frac{\gamma_r}{2}\quad \sin(\alpha_2 - \frac{\gamma_r}{2}) - \sin(\alpha_1 + \frac{\gamma_r}{2}),$$

$$= 2|d_1||r_1|\sin\frac{\gamma_r}{2}\sin\frac{\gamma_d}{2}\left[\cos\left(\frac{\alpha_2 + \alpha_1}{2}\right)\right].$$

Let

$$h_t' = \frac{[2\Gamma - (2n+1)\pi]\lambda R}{4\pi(h_1 + h_2)},$$

where n is an integer, and

$$h_t'' = \frac{[2\Gamma - (2n-1)\pi]\lambda R}{4\pi(h_1 + h_2)} .$$

Then

$$F_p(h_t') = F_p(h_t'') = 0.$$

As in the case of F_a,

$$h_t'' - h_t' = \frac{\lambda R}{2(h_1 + h_2)} .$$

Case II ($h_1 = h_2$)

In this case $\alpha_1 = \alpha_2$. Let this common value be denoted by α. Then F_a reduces to

$$F_a = (d_1, d_1) - (d_2, d_2) + (r_1, r_1) - (r_2, r_2) \tag{16}$$

$$+ 2(|d_1||r_1| - |d_2||r_2|) \cos \alpha .$$

Let h_t' be such that

$$F_a(h_t') = 0 .$$

If we now choose

$$h_t'' = h_t' - \frac{\lambda R}{2h} ,$$

then

$$F_a(h_t'') = 0,$$

since this change in h_t' does not change the value of $\cos \alpha$.

The same argument applies for F_p.

We have shown that at least two elementary measurements are necessary for an unique determination of h_t in the low-angle region through a closed-loop procedure.

Presently, we shall show that two elementary measurements are sufficient. To give insight into the construction of such procedures, we shall first consider some practical aspects of the problem. From an engineering point of view the existence of a solution, or in other words, the fact that a particular equation $F = 0$ determines h_t uniquely, is not sufficient. We must devise a simple and direct method for determining h_t.

The monopulse and the interferometer are simple direct methods for determining the angle of arrival (and hence h_t) of a single plane wave. It is natural therefore to seek an arrangement of elementary measurements in a closed-loop procedure such that the determination of target height is effected through a basic monopulse or interferometer procedure. We shall exhibit two methods by which this is accomplished in this chapter under "Sufficient Procedures".

The monopulse or interferometer is effective in free-space since only a single wave, the direct wave, is involved. One way, therefore, of modifying these methods so that they may apply to the low-angle region is to design a measurement procedure in which at least two elementary measurements are employed such that nulls of F occur if and only if $d_1 = d_2$, or $r_1 = r_2$, or $\alpha_1 = \alpha_2 + 2\pi$. This means that essentially we have effected an isolation of either the direct or indirect wave or the phase difference between the two waves.

If such a measurement procedure can be found, then the target height can be determined simply as in the free-space monopulse or interferometer method. In Chapter I we showed that in the case of a monopulse even when $d_1 = d_2$, a null of F may not occur. However, by use of two monopulses consisting of four lobes, the pair δ_1 and δ_2 and the pair δ_3 and δ_4, and a procedure F involving two elementary measurements, we may obtain a situation where $F = 0$ if and only if $d_1 = d_2$ and $d_3 = d_4$. Such a procedure is exhibited in this chapter.

It should be pointed out that in a measurement procedure F such that $F = 0$ if and only if $d_1 = d_2$ or $r_1 = r_2$ or $\alpha_1 = \alpha_2 + 2\pi$, the nulls of F and hence the target height are independent of ρ and Γ. This is desirable from a practical point of view, since ρ and Γ may vary as propagation conditions change.

The two procedures to be discussed under "Sufficient Procedures" determine the target height uniquely from the conditions $d_1 = d_2$ or $r_1 = r_2$. We do not exhibit a measurement procedure in which the target height is uniquely determined from the conditions $\alpha_1 = \alpha_2 + 2\pi$. Such measurement procedures can be exhibited, but at present are impractical.

Sufficient Procedures

The Dual Interferometer.[7]--Let δ_1, δ_2, and δ_3 be three lobes vertically spaced so that the distance between the uppermost lobe δ_1 and the middle lobe δ_2 is equal to the distance between the middle lobe and the bottom lobe δ_3. Denote this distance by s. Let Φ_+ denote counterclockwise rotation through an angle θ or phase advance, and let Φ_- denote clockwise rotation through the same angle or phase retardation.

We take as the measurement procedure

$$(17)\quad F(\delta_1, \delta_2, \delta_3, \theta) \equiv \left|\delta_2 - \delta_1 \Phi_+\right|^2 - \left|\delta_2 - \delta_3 \Phi_-\right|^2 ,$$

in which θ is the controlled variable. Expanding each term on the right, we have

$$\left|\delta_2 - \delta_1 \Phi_+\right|^2 = (\delta_2, \delta_2) + (\delta_1, \delta_1) - 2(\delta_2, \delta_1 \Phi_+) ,$$

$$= (\delta_2, \delta_2) + (d_1, d_1) + (r_1, r_1) + 2(d_1, r_1)$$

$$- 2(d_2, d_1 \Phi_+) - 2(d_2, r_1 \Phi_+) - 2(r_2, d_1 \Phi_+) - 2(r_2, r_1 \Phi_+) ,$$

and similarly

$$\left|\delta_2 - \delta_3 \Phi_-\right|^2 = (\delta_2, \delta_2) + (d_3, d_3) + (r_3, r_3) + 2(d_3, r_3)$$

$$- 2(d_2, d_3 \Phi_-) - 2(d_2, r_3 \Phi_-) - 2(r_2, d_3 \Phi_-) - 2(r_2, r_3 \Phi_-).$$

---

[7] This is the Brooks three-lobe interferometer. See F. E. Brooks, Jr., The Brooks Antenna System for Measuring Low Elevation Angles, The Electrical Engineering Research Laboratory, University of Texas, Report No. 49, 1 February 1951 (Prepared under U. S. Navy Contract N5ori -136, T.O.1. CONFIDENTIAL.).

Then

$$(18)\quad F(\Theta) = (d_1, d_1) - (d_3, d_3) + (r_1, r_1) - (r_3, r_3) + 2(d_1, r_1)$$

$$- 2(d_3, r_3) - 2(d_2, d_1\Phi_+ - d_3\Phi_-)$$

$$- 2(d_2, r_1\Phi_+) + 2(d_2, r_3\Phi_-) - 2(r_2, d_1\Phi_+ - d_3\Phi_-)$$

$$- 2(r_2, r_1\Phi_+ - r_3\Phi_-) \quad .$$

Since both the direct and indirect waves are plane by hypothesis, and β and a are the same for each lobe, then by the approximation introduced in Chapter III,

$$(d_1, d_1) = (d_2, d_2) = (d_3, d_3) \quad ,$$

$$(r_1, r_1) = (r_2, r_2) = (r_3, r_3) \quad ,$$

$$(d_2, d_1\Phi_+ - d_3\Phi_-) = 0 \quad ,$$

$$(r_2, r_1\Phi_+ - r_3\Phi_-) = 0 \quad ,$$

for all values of Θ . Therefore $F(\Theta)$ reduces to

$$(19)\quad F(\Theta) = - 2(r_2, d_1\Phi_+ - d_3\Phi_-) - 2(r_1\Phi_+, d_2 - d_1\Phi_+)$$

$$- 2(r_3\Phi_-, d_3\Phi_- - d_2).$$

We may rearrange the terms in (19) to obtain

$$(20)\quad F(\Theta) = -2(d_2, r_1\Phi_+ - r_3\Phi_-) - 2(d_1\Phi_+, r_2 - r_1\Phi_+)$$

$$- 2(d_3\Phi_-, r_3\Phi_- - r_2) \quad .$$

Let γ_d be the phase by which d_1 lags d_2. Since the wave is assumed to be a plane wave, and the middle lobe is halfway between the top and bottom lobes, then γ_d will also be the phase by which d_2 will lag d_3. Hence

$$(21)\quad d_1\Phi_+ = d_2 = d_3\Phi_- \quad ,$$

when $\Theta = \gamma_d$.

Similarly, let γ_r be the phase by which r_1 lags r_2. Then if $\Theta = \gamma_r$,

$$(22)\quad r_1\Phi_+ = r_2 = r_3\Phi_- \quad .$$

For these two values of Θ, it is seen from (19) and (20) that $F(\Theta) = 0$.

Further, the target height is determined uniquely from either null. For example, if $\Theta = \gamma_d$, then

$$\Theta = \frac{2\pi s}{\lambda}\sin\beta_d = \frac{2\pi s}{\lambda}\left(\frac{h_t - h}{R}\right) ,$$

or

$$h_t = \frac{\lambda R\Theta}{2\pi s} + h.$$

An ambiguity arises if the null is $\Theta + 2n\pi$ instead of Θ. If we are unable to distinguish between Θ and $\Theta + 2n\pi$, a number of possible target heights will be determined by the procedure. However, the angular separation of these possible target heights is λ/s. In practice s is restricted, so that only one possible target height is determined in the low-angle region, and "uniqueness" as defined in Chapter II holds.

In order to ensure that $F(\Theta) = 0$ only if $\Theta = \gamma_d + 2n\pi$ or $\Theta = \gamma_r + 2n\pi$, we need the following theorem.

Theorem:

If $F(\Theta) \not\equiv 0$, and if there exists a value of Θ such that $F(\Theta) = 0$, then either (i) $\Theta = \gamma_d + 2n\pi$, or (ii) $\Theta = \gamma_r + 2n\pi$.

Proof:

By hypothesis, there exists a value of Θ such that $F(\Theta) = 0$. Hence

$$(23)\quad (r_2, d_1\Phi_+ - d_3\Phi_-) + (r_1\Phi_+, d_2 - d_1\Phi_+) + (r_3\Phi_-, d_3\Phi_- - d_2) = 0.$$

Expanding, we obtain

$$(24)\quad (r_2, d_1\Phi_+) - (r_2, d_3\Phi_-) + (r_1\Phi_+, d_2) - (r_1, d_1) + (r_3, d_3) - (r_3\Phi_-, d_2) = 0.$$

Recalling that α_i denotes the phase lag of r_i behind d_i, we may expand the inner products in equation (24) as follows, denoting $|d_i||r_i|$ by K,

$$
\begin{aligned}
(25)\quad (r_2, d_1\Phi_+) &= K\cos(\gamma_d - \theta - \alpha_2) = K\left[\cos\alpha_2 \cos(\gamma_d - \theta) + \sin\alpha_2 \sin(\gamma_d - \theta)\right], \\
(r_2, d_3\Phi_+) &= K\cos(\alpha_2 + \gamma_d - \theta) = K\left[\cos\alpha_2 \cos(\gamma_d - \theta) - \sin\alpha_2 \sin(\gamma_d - \theta)\right], \\
(r_1\Phi_+, d_2) &= K\cos(\gamma_r - \theta + \alpha_2) = K\left[\cos\alpha_2 \cos(\gamma_r - \theta) - \sin\alpha_2 \sin(\gamma_r - \theta)\right], \\
(r_1, d) &= K\cos(\gamma_r + \alpha_2 - \gamma_d) = K\left[\cos\alpha_2 \cos(\gamma_r - \gamma_d) - \sin\alpha_2 \sin(\gamma_r - \gamma_d)\right], \\
(r_3, d_3) &= K\cos(\gamma_r - \alpha_2 - \gamma_d) = K\left[\cos\alpha_2 \cos(\gamma_r - \gamma_d) + \sin\alpha_2 \sin(\gamma_r - \gamma_d)\right], \\
(r_3\Phi_-, d_2) &= K\cos(\gamma_r - \theta - \alpha_2) = K\left[\cos\alpha_2 \cos(\gamma_r - \theta) + \sin\alpha_2 \sin(\gamma_r - \theta)\right].
\end{aligned}
$$

Since $K \neq 0$, equations (25) together with equation (24) imply

$$(26)\quad \sin\alpha_2\left[\sin(\gamma_d - \theta) - \sin(\gamma_r - \theta) + \sin(\gamma_r - \gamma_d)\right] = 0.$$

The first hypothesis implies that $\sin\alpha_2$ is not zero. We can argue this as follows. Suppose

$$\sin\alpha_2 = 0.$$

Then, since

$$\sin\alpha_2 = \frac{(d_2\Phi_{\pi/2}, r_2)}{|d_2|\,|r_2|},$$

it follows that $d_2 = k\, r_2$, where k is a scalar $\neq 0$. But this implies that

$$(d_2, r_1 \Phi_+ - r_3 \Phi_-) = k(r_2, r_1 \Phi_+ - r_3 \Phi_-) = 0$$

for all values of Θ. Hence, $F(\Theta)$ as given in (20) reduces to

$$F(\Theta) = -2(d_1 \Phi_+, r_2 - r_1 \Phi_+) - 2(d_3 \Phi_-, r_3 \Phi_- - r_2) \quad ,$$

or,

$$(27) \quad F(\Theta) = -2(r_2, d_1 \Phi_+ - d_3 \Phi_-) + 2(d_1, r_1) - 2(d_3, r_3) \quad .$$

The first term on the right is zero for all Θ, since

$$(r_2, d_1 \Phi_+ - d_3 \Phi_-) = \frac{1}{k}(d_2, d_1 \Phi_+ - d_3 \Phi_-) = 0.$$

From equation (25)

$$(d_1, r_1) - (d_3, r_3) = -2K \sin \alpha_2 \sin(\gamma_r - \gamma_d).$$

Since $\sin \alpha_2 = 0$,

$$(d_1, r_1) = (d_3, r_3) \quad ,$$

and, therefore $F(\Theta) \equiv 0$. Hence the first hypothesis implies that $\sin \alpha_2 \neq 0$.

From equation (26) we have

$$(28) \qquad \sin(\gamma_d - \Theta) - \sin(\gamma_r - \Theta) + \sin(\gamma_r - \gamma_d) = 0.$$

On applying trigonometric identities, we obtain

$$\sin(\gamma_r - \gamma_d) - \cos\theta\,(\sin\gamma_r - \cos\gamma_d) + \sin\theta\,(\cos\gamma_r - \cos\gamma_d) = 0,$$

and

$$2\sin\left(\frac{\gamma_r - \gamma_d}{2}\right)\cos\left(\frac{\gamma_r - \gamma_d}{2}\right) - 2\cos\theta\left[\cos\left(\frac{\gamma_r + \gamma_d}{2}\right)\sin\left(\frac{\gamma_r - \gamma_d}{2}\right)\right],$$

$$- 2\sin\theta\left[\sin\left(\frac{\gamma_r + \gamma_d}{2}\right)\sin\left(\frac{\gamma_r + \gamma_d}{2}\right)\right] = 0.$$

Hence,

$$(29)\quad 2\sin\left(\frac{\gamma_r - \gamma_d}{2}\right)\left[\cos\left(\frac{\gamma_r - \gamma_d}{2}\right) - \cos\theta\,\cos\left(\frac{\gamma_r + \gamma_d}{2}\right) - \sin\theta\,\sin\left(\frac{\gamma_r + \gamma_d}{2}\right)\right] = 0.$$

We shall show that the first hypothesis implies

$$\sin\left(\frac{\gamma_r - \gamma_d}{2}\right) \neq 0.$$

Suppose

$$\sin\left(\frac{\gamma_r - \gamma_d}{2}\right) = 0.$$

Then

$$\gamma_r - \gamma_d = 2n\pi.$$

We can write $F(\Theta)$ as given in (20) as

(30) $$F(\theta) = -2(d_2, r_2\Phi_-\gamma_r\Phi_+ - r_2\Phi\gamma_r\Phi_-)$$

$$-2(d_2\Phi_-\gamma_d\Phi_+, r_2 - r_2\Phi_-\gamma_r\Phi_+)$$

$$-2(d_2\Phi\gamma_d\Phi_-, r_2\Phi\gamma_r\Phi_- - r_2).$$

Rearranging terms, we obtain

(31) $$F(\theta) = -2(d_2, (r_2\Phi_-\gamma_r - r_2\Phi_-\gamma_d)\Phi_+)$$

$$-2(r_2, (d_2\Phi_-\gamma_d - d_2\Phi_-\gamma_r)\Phi_+)$$

$$+2(d_2\Phi_-\gamma_d\Phi_+, r_2\Phi_-\gamma_r\Phi_+)$$

$$-2(d_2\Phi\gamma_d\Phi_-, r_2\Phi\gamma_r\Phi_-).$$

Each of the first two terms on the right is identically zero in θ, and the sum of the last two terms is zero for all values of θ. Hence $F(\theta) \equiv 0$, contrary to hypothesis.

Equation (29) by the first hypothesis implies

$$\left[\cos \frac{\gamma_r - \gamma_d}{2} - \cos \left(\theta - \frac{\gamma_r + \gamma_d}{2}\right)\right] = 0. \tag{32}$$

Equation (32) implies either

(i) $$\frac{\gamma_r - \gamma_d}{2} - \theta + \frac{\gamma_r + \gamma_d}{2} = 2n\pi,$$

or

(ii) $$\frac{\gamma_r - \gamma_d}{2} + \theta - \frac{\gamma_r + \gamma_d}{2} = 2n\pi,$$

where n is an integer. Simplifying, we find that equation (32) implies either

(i) $$\gamma_r - \theta = 2n\pi$$

or

(ii) $$-\gamma_d + \theta = 2n\pi.$$

This is the desired conclusion.

The physical significance of the first hypothesis is that the middle antenna is not situated on a maximum or a minimum of the interference pattern, i.e., $d_2 \neq k\, r_2$, and furthermore, the vertical separation of the lobes in the interference pattern does not coincide with that of the antenna lobes.

Dual Monopulse.--Consider two monopulse systems at different heights, h' and h''. Denote the upper and lower lobes of the top system by δ_1 and δ_2 respectively, and the upper and lower lobes of the bottom system by δ_3 and δ_4 respectively. Let β denote the tilt of both systems. We take as the measurement procedure

$$F(\beta) = |\delta_1 - \delta_2|^2 - |\delta_3 - \delta_4|^2 \quad , \tag{33}$$

where the controlled variable is β.

Expanding each term, we obtain

$$|\delta_1 - \delta_2|^2 = (\delta_1, \delta_1) + (\delta_2, \delta_2) - 2(\delta_1, \delta_2)$$

$$= (r_1, r_1) + (d_1, d_1) + 2(r_1, d_1) + (r_2, r_2) + (d_2, d_2)$$

$$+ 2(r_2, d_2) - 2(r_1, r_2) - 2(r_1, d_2) - 2(d_1, d_2) - 2(d_1, r_2),$$

and similarly

$$|\delta_3 - \delta_4|^2 = (r_3, r_3) + (d_3, d_3) + 2(r_3, d_3) + (r_4, r_4) + (d_4, d_4)$$

$$+ 2(r_4, d_4) - 2(r_3, r_4) - 2(r_3, d_4) - 2(d_3, d_4) - 2(d_3, r_4).$$

By the approximation introduced on page (19)

$$(r_1, r_1) = (r_3, r_3) \quad ,$$

$$(d_1, d_1) = (d_3, d_3) \quad ,$$

$$(d_2, d_2) = (d_4, d_4) \quad ,$$

and $$(r_2, r_2) = (r_4, r_4) \quad .$$

Hence $F(\beta)$ reduces to

$$(34) \quad F(\beta) = 2(r_1, d_1) + 2(r_2, d_2) - 2(r_1, d_2) - 2(d_1, r_2) - 2(r_3, d_3) - 2(r_4, d_4) + 2(r_3, d_4) + 2(d_3, r_4).$$

Or

$$F(\beta) = 2(d_1, r_1 - r_2) + 2(d_2, r_2 - r_1) - 2(d_3, r_3 - r_4) - 2(d_4, r_4 - r_3),$$

or

$$F(\beta) = 2(r_1, d_1 - d_2) + 2(r_2, d_2 - d_1) - 2(r_3, d_3 - d_4) - 2(r_4, d_4 - d_3).$$

If $\beta = \beta_d$, then

$$|d_1| = |d_2| \quad \text{and} \quad |d_3| = |d_4| \quad ,$$

by the discussion of the monopulse given in Chapter II. Similarly, if $\beta = \beta_r$, then

$$|r_1| = |r_2| \quad \text{and} \quad |r_3| = |r_4| \quad .$$

Hence either

$$\beta = \beta_d \text{ or } \beta = \beta_r \text{ is a null of } F(\theta).$$

The target elevation is determined uniquely from either of these two nulls. Effectively we have a monopulse determination of the angle of arrival of a single plane wave.

In order to ensure that these are the only nulls, we need the following theorem.

Theorem:

If $F(\beta) \not\equiv 0$, and if there exists a value of β such that $F(\beta) = 0$, then either (i) $\beta = \beta_d$, or, (ii) $\beta = \beta_r$.

Proof:

By hypothesis there exists a value of β such that $F(\beta) = 0$. Hence from (34) we may write

(35) $$(|d_1|\,|r_1| - |d_1|\,|r_2| + |d_2|\,|r_2| - |d_2|\,|r_1|)\cos\alpha'$$

$$- (|d_3|\,|r_3| - |d_3|\,|r_4| + |d_4|\,|r_4| - |d_4|\,|r_3|)\cos\alpha'' = 0,$$

where

$$\alpha' = \Gamma - \frac{4\pi h' h_t}{\lambda R},$$

and

$$\alpha'' = \Gamma - \frac{4\pi h'' h_t}{\lambda R}.$$

Since $|d_1| = |d_3|$, $|r_1| = |r_3|$, $|d_2| = |d_4|$, and $|r_2| = |r_4|$, equation (35) becomes

(36) $$(|d_1|\,|r_1| - |d_1|\,|r_2| + |d_2|\,|r_2| - |d_2|\,|r_1|)(\cos\alpha' - \cos\alpha'')$$

$$= 0.$$

From the first hypothesis it follows that $\cos\alpha' \neq \cos\alpha''$. Hence, equation (36) implies

$$(|d_1| - |d_2|)(|r_1| - |r_2|) = 0,$$

and therefore either

(i) $|d_1| = |d_2|$,

or

(ii) $|r_1| = |r_2|$.

The first relation holds only if $\beta = \beta_d$, and the second only if $\beta = \beta_r$.

APPENDICES

APPENDIX A

Formulae Derived from the Geometry

We shall assume that h/R and h_t/R are extremely small positive quantities. For example, R may be 20,000 yards, while h and h_t may be about 100 feet.

From Figure 1,

$$\beta_d \doteq \sin\beta_d = \frac{h_t - h}{R}$$

and

$$\beta_r \doteq \sin\beta_r = \frac{h_t + h}{R} \quad .$$

Since the angle of incidence of the surface reflected wave is equal to the angle of reflection, the length of the path TPA is equal to that of the path T'PA. The length of the path T'PA is

$$\sqrt{R^2 - (h_t - h)^2 + (h_t + h)^2} \quad ,$$

or

$$\sqrt{R^2 + 4\,h_t\,h} = R\sqrt{1 + \frac{4h_t + h}{R^2}} \quad .$$

Expanding the last expression by the binomial theorem, we obtain

$$R\sqrt{1 + \frac{4h_t\,h}{R^2}} \doteq R + \frac{2h_t\,h}{R} \quad .$$

The terms neglected involve R in the denominator to at least the third power. The difference in the lengths of the paths TPA and TA is

$$\frac{2h_t\, h}{R}$$

or

$$\frac{4\pi h_t\, h}{\lambda R} \qquad \text{radians.}$$

APPENDIX B

Properties of Inner Products and Certain Transformations on Linear Vector Spaces

A postulational definition of a linear vector space in which an inner product is defined is given by P. R. Halmos.[8] We shall not repeat this definition here, but we shall state certain properties of inner (or "dot") products and transformations used in the analysis.

We are dealing with a real finite dimensional vector space. The inner product is a real-valued function (x, y) of the ordered pair of vectors x and y such that

$$(1) \quad (x, y) = (y, x) \quad ,$$

$$(2) \quad (a_1 x_1 + a_2 x_2, y) = a_1(x_1, y) + a_2(x_2, y) \quad ,$$

where a_1 and a_2 are real numbers, and

$$(3) \quad (x, x) \geqq 0; \quad (x, x) = 0 \text{ is equivalent to } x = 0.$$

We denote the norm or length of a vector x by $|x|$, where $|x| = +\sqrt{(x, x)}$.

The rotation or phase shift Φ_θ introduced in Chapter II is the matrix representation in a particular coordinate system of a linear transformation, i.e.,

$$(a_1 x_1 + a_2 x_2)\Phi_\theta = a_1 x_1 \Phi_\theta + a_2 x_2 \Phi_\theta \quad .$$

[8] P. R. Halmos, Finite Dimensional Vector Spaces (Princeton: Princeton University Press, 1948). (Annals of Mathematics Studies, Number 7.)

The transformation Φ_θ is <u>unitary</u>, that is

(1) $|x \Phi_\theta| = |x|$, for all x,

or equivalently,

(2) $(x \Phi_\theta, y \Phi_\theta) = (x, y)$ for all x and y.

The following additional properties hold for rotation transformations

(1) $x \Phi_{\theta_1} \Phi_{\theta_2} = x \Phi_{\theta_2} \Phi_{\theta_1}$

and

(2) $x \Phi_{-\theta} \Phi_{+\theta} = x \Phi_0 = x$, for all x,

where Φ_0 denotes the identity transformation.

APPENDIX C

Proof That a Monopulse Target Height Determination May be Multiple-valued

In this appendix we take a special case of a single elementary measurement procedure, the monopulse, and show in a numerical example that there may be several possible values of target height corresponding to a given null.

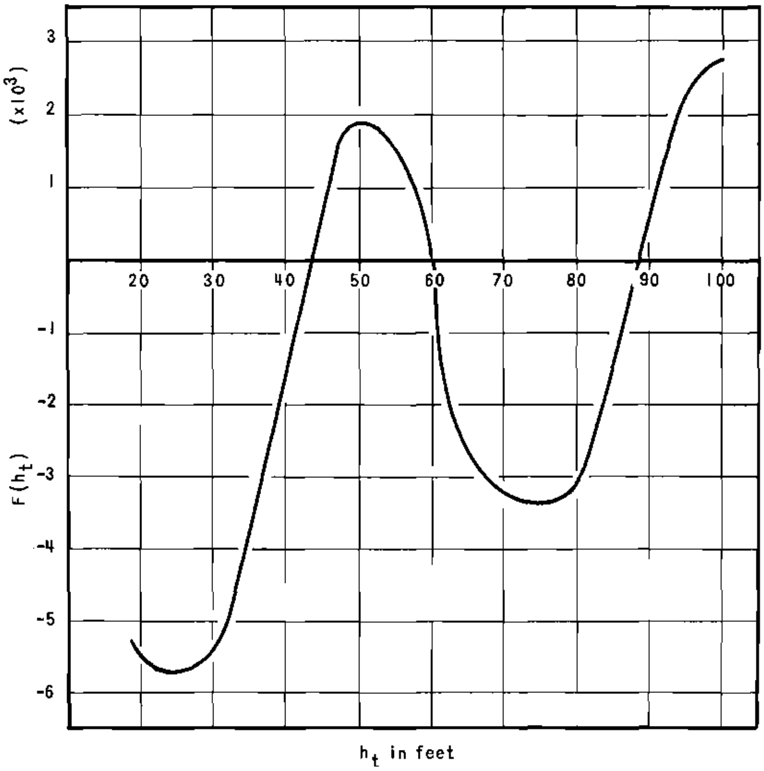

Figure 2. Target Heights Determined by a Monopulse Null

$$F(\beta, h_t) = \left[f_1^{\,2} - f_2^{\,2} - \rho^2(g_2^{\,2} - g_1^{\,2}) + 2\rho(f_1 g_1 - f_2 g_2)\cos\alpha\right]\left(\frac{\lambda}{2\pi a}\right)^2$$

where

$$f_1 = \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta + \eta - \beta_d)\right]}{\frac{2\pi a}{\lambda} \sin(\beta + \eta - \beta_d)} ,$$

$$f_2 = \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta - \eta - \beta_d)\right]}{\frac{2\pi a}{\lambda} \sin(\beta - \eta - \beta_d)} ,$$

$$g_1 = \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta + \eta + \beta_r)\right]}{\frac{2\pi a}{\lambda} \sin(\beta + \eta + \beta_r)} ,$$

and

$$g_2 = \frac{\sin\left[\frac{2\pi a}{\lambda} \sin(\beta - \eta + \beta_r)\right]}{\frac{2\pi a}{\lambda} \sin(\beta - \eta + \beta_r)} .$$

$$\beta_d = \frac{h_t - h}{R} , \quad \beta_r = \frac{h_t + h}{R} ,$$

and

$$\alpha = \Gamma - \frac{4\pi h_t h}{\lambda R} .$$

Numerical values chosen for the various parameters are the following.

λ = 0.1 ft.

a = 2.5 ft.

η = 0.5 degree

h = 60 ft.

ρ = 0.8

Γ = $-\pi$

$$R = 20{,}000 \text{ yards}$$

$$\beta = -(0.55) \text{ milli radian}$$

We note that in Figure 1 solutions of the equation $F(h_t) = 0$ are approximately $h_t = 40$ ft., 60 ft., and 90 ft. The solutions exhibited fall within the low-angle region, since the low-angle region in our example extends to a height of roughly 1000 ft. at 20,000 yards.

[redacted]

ABSTRACT

"Determination of the Height of a Target Above a Conducting Plane by Means of Radar Measurements", by W. W. Wright, a Thesis Presented to the Faculty of the Graduate Division in partial fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics.

Techniques for the tracking of a target in free space by radar are well known and are generally satisfactory.[1] Most conventional techniques exploit the directional property of antenna beams or lobes. For example, we may take an antenna system employing two identical lobes symmetrically displaced in elevation about an axis. If we position the axis in elevation to obtain equal signal strengths from both lobes, the angle of elevation of this axis will be the elevation of the target. This "lobe comparison" may be sequential as in a conical scan system or simultaneous as in a monopulse system.

When the target is near a conducting surface, as in the case of an aircraft near the surface of the sea, then the tracking problem becomes much more complicated. If the radar system is also near the conducting surface, the problem of radar tracking is no longer a free-space problem, since the radar receives energy from the target by two paths, the direct path and a second path involving reflection from the surface. Under these circumstances conventional radar tracking techniques fail.[2]

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[1] L. N. Ridenour, Radar System Engineering, (New York: McGraw-Hill Book Company, Inc., 1947).

[2] Donald E. Kerr, Propagation of Short Radio Waves, (New York: McGraw-Hill Book Company, Inc., 1951) Section 5.14.

[redacted]

When the target is sufficiently near the conducting plane so that the angle of arrival of the surface reflected wave differs from that of the direct wave by less than the antenna beam width, the target is said to be in the "low-angle" region. The problem considered in this thesis is that of investigating the modifications of conventional tracking radars which will be needed in order to determine the height of a target in the low-angle region.

The mathematical formulation is based on the two plane wave model. The concept of radar measurements if formalized, and in particular, an "elementary measurement" is defined. Conventional tracking radars employ only a single elementary measurement. It is shown, subject to an approximation usually adopted in practice, that the height of a low-angle target is not determined uniquely in general by a closed-loop procedure involving only a single elementary measurement. It is then shown that it is possible to determine uniquely the height of a target in the low-angle region by a closed-loop procedure involving two elementary measurements. Two such solutions are exhibited.[3] The result of the analysis is that in a closed-loop procedure two elementary measurements are both necessary and sufficient for the unique determination of the target height in the low-angle region.

Chairman of Thesis Advisory Committee

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[3]One of these is the Brooks three-lobe interferometer. See F. E. Brooks, Jr., The Brooks Antenna System for Measuring Low Elevation Angles, The Electrical Engineering Research Laboratory, University of Texas, Report No. 49, 1 February 1951 (Prepared under U. S. Navy Contract N5ori - 136, T.O. 1. [redacted]).