SOLUTION OF INITIAL-VALUE PROBLEMS
FOR SOME HALF-INFINITE RL LADDER NETWORKS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

Michael Scott West

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mathematics

Georgia Institute of Technology

November 1978

SOLUTION OF INITIAL-VALUE PROBLEMS

FOR SOME HALF-INFINITE RL LADDER NETWORKS

Approved:

Chairman

Co-Chairman

Date approved by Chairman: Nov. 28, 1978

ACKNOWLEDGMENTS

The results presented here were derived from an investigation which was undertaken in order to answer some questions concerning initial-value problems for half-infinite RL ladder networks. These questions were posed by Dr. M. B. Sledd and Dr. A. L. Mullikin, both of whom encouraged and stimulated me while giving unstintingly of their own time to the completion of this project. I am grateful to them for all they have taught me and for the opportunity of working under their guidance.

I also thank Dr. J. M. Tanner for serving on the reading committee.

Ms. Kathy Massett is an excellent typist, and I thank her for her service.

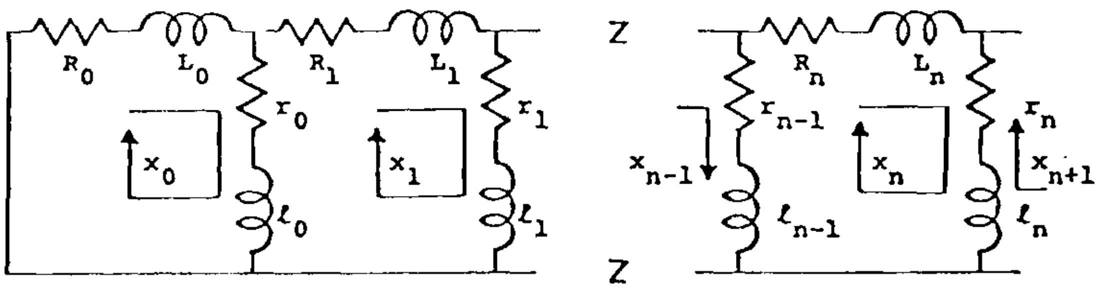
My parents have supported me totally in my efforts to further my education, and it is to them that I dedicate this dissertation.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

An RL ladder network is an electric circuit of the form shown in the accompanying sketch, where R_n and r_n denote resistances and L_n and ℓ_n denote inductances. Such a network is called half-infinite if it has



a first loop but no last loop. If no voltage nor current sources are present, the differential equations which apply to the initial-value problem are a countable system of first-order ordinary differential equations with constant coefficients:

$$[(\ell_o+L_o)D + r_o + R_o]x_o - (\ell_o D + r_o)x_1 = 0 \quad ,$$

$$-(\ell_{n-1}D+r_{n-1})x_{n-1} + [(\ell_{n-1}+L_n+\ell_n)D + (r_{n-1}+R_n+r_n)]x_n$$

$$-(\ell_n D+r_n)x_{n+1} = 0 \quad , \quad n\geq 1 \quad ,$$

where $D \sim d/dt$. A significant feature of these equations is that each (except the first) contains the derivatives of three successive loop currents; thus the equations are coupled both through the loop currents and through their derivatives.

The object of this investigation is to solve these differential equations (subject to a finite number of non-zero initial conditions) for various values of the circuit parameters. A solution is defined as a sequence $\{x_n(t)\}_{n=o}^{\infty}$ of differentiable functions which satisfy the prescribed initial conditions and reduce the differential equations to identities in t on t>0. For prescribed values of the circuit parameters and for finitely many prescribed non-zero initial conditions, there are usually many solutions; hence, additional conditions (suggested by the behavior of finite systems) are imposed which are sufficient to guarantee uniqueness.

The main procedure used to obtain a solution is an extension of the method of generalized eigenfunctions applicable to the finite system consisting of the first N loops of the corresponding infinite system. The procedure involves two steps. First, the coefficients of the differential system are used to generate a sequence of rational fractions $\{\Psi_n(x)/(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}\}_{n=o}^{\infty}$ such that $\{e^{xt}\cdot\Psi_n(x)/(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}\}_{n=o}^{\infty}$ reduces the differential equations to identities in t for each real x. Thus, the sequence $\{\Psi_n(x)/(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}\}_{n=o}^{\infty}$ may be thought of as an eigenvector (corresponding to the eigenvalue x) of the associated algebraic problem. Secondly, an additional sequence $\{W_n(x)\}_{n=o}^{\infty}$ of rational fractions is generated; and by using superposition and the biorthogonality property of the two sequences, a solution of the differential equations

$$x_n(t) = \gamma_j(\ell_{j-1}+L_j+\ell_j)\int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}} e^{xt} d\beta(x) \quad ,$$

$$n\geq 0 \quad ,$$

is constructed which satisfies the initial conditions $x_n(0)=0$, $n\neq j$, $x_j(0)=\gamma_j$. Here $\beta(x)$ is a suitable integrator, the determination of which is the key to solving a particular problem. Finitely many non-zero initial conditions may be accommodated by a finite sum on j.

Some techniques for finding integrators are described and are applied to several examples. These techniques are especially effective for systems with periodic coefficients, to which parts of Chapters III and IV are devoted. Questions concerning the existence of an integrator are considered.

Many of the calculations involved in various stages of the work are relegated to appendices.

CHAPTER I

INTRODUCTION

An RL ladder network is an electric circuit of the form shown in Figure 1. Such a network is called half-infinite if it has a first loop but no last loop. The differential equations which apply to the initial-value problem are a countable system of first-order ordinary differential equations with constant coefficients:

$$[(\ell_o+L_o)D + r_o + R_o]x_o(t) - (\ell_o D+r_o)x_1(t) = 0,$$

$$-(\ell_{n-1}D+r_{n-1})x_{n-1}(t) + [(\ell_{n-1}+L_n+\ell_n)D + (r_{n-1}+R_n+r_n)]x_n(t) \tag{1.1}$$

$$-(\ell_n D+r_n)x_{n+1}(t) = 0 \quad , \quad n\geq 1 \quad , \quad (D \sim d/dt) \quad ,$$

where $x_n(t)$, $n\geq 0$, denotes the n^{th} loop current. The object of this investigation is to solve these differential equations (subject to a finite number of non-zero initial conditions) for various values of the circuit parameters. A solution is defined as a sequence $\{x_n(t)\}_{n=o}^{\infty}$ of differentiable functions which satisfy the prescribed initial conditions and reduce the differential equations (1.1) to identities in t on t>0.

Previous workers [3,8,10,11] in infinite differential systems have used a technique in which a solution of the form

$$x_n(t) = y_n(x)u(x,t) \quad , \quad n\geq 0 \quad , \tag{1.2}$$

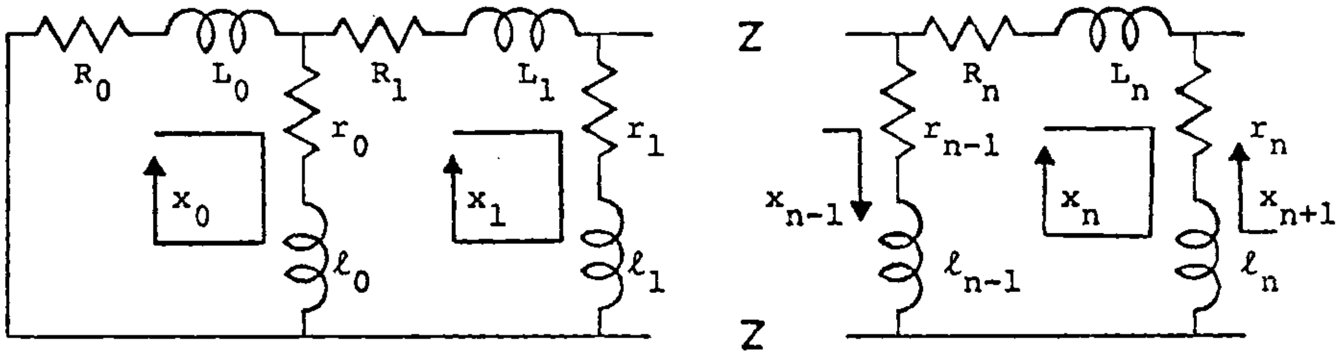


Figure 1. A Half-Infinite RL Network

is sought, where $y_n(x)$ is a polynomial of degree n. The system (1.1) differs in an important way from those investigated in [3,8,10,11]: the derivative coupling between successive equations requires that the functions $y_n(x)$, $n\geq 0$, be, in general, rational fractions instead of polynomials. More specifically, it is shown in Chapter II, when the expression (1.2) is substituted in the system (1.1), that for $y_n(x) = [(L_o+\ell_o)/(\ell_{n-1}+L_n+\ell_n)]^{\frac{1}{2}}\,\Psi_n(x)$, $n\geq 0$, $\ell_{-1}=0$, the $\Psi_n(x)$ satisfy the relations

$$b_o(x-d_o)\Psi_1(x) = (x-a_o)\Psi_o(x) \quad , \tag{1.3a}$$

$$b_n(x-d_n)\Psi_{n+1}(x) = (x-a_n)\Psi_n(x) - b_{n-1}(x-d_{n-1})\Psi_{n-1}(x) \quad , \quad n\geq 1 \quad , \tag{1.3b}$$

where, for convenience, $\Psi_o(x)=1$, and where a_n, b_n, and d_n, $n\geq 0$, are given in (2.2). If d_n is independent of n, (1.3a) and (1.3b) reduce to the more familiar relations for polynomials (see [10,11])

$$B_oP_1(z) = (z+A_o)P_o(z) \quad , \tag{1.4a}$$

$$B_nP_{n+1}(z) = (z+A_n)P_n(z) - B_{n-1}P_{n-1}(z) \quad , \quad n\geq 1 \quad , \tag{1.4b}$$

where $z = 1/(x-d_o)$, $\Psi_n(x) = [(d_o-a_o)/(d_o-a_n)]^{\frac{1}{2}}\,P_n(z)$, $n\geq 0$, and where A_n and B_n, $n\geq 0$, are obtained from (2.3). The expressions in (1.3b) and (1.4b) are called <u>three-term recurrence relations</u>, and $\Psi_n(x)$, $n\geq 2$, and $P_n(z)$, $n\geq 2$, are said to be <u>generated</u> by these relations.

By substituting (1.2) into (1.1), the dependence of $x_n(t)$ on n and on t is separated. The result is that u(x,t) satisfies one of the equations

$$u_t(x,t) = xu(x,t) \quad , \tag{1.5a}$$

$$u_t(z,t) = (\frac{1}{z} + d_o)u(z,t) \quad , \tag{1.5b}$$

and that the functions $\Psi_n(x)$, $n\geq 1$, or $P_n(z)$, $n\geq 1$, are generated by (1.3) or (1.4), respectively. In [10,11], the <u>orthonormality</u> of the polynomials (see Definition 2.2) $P_n(z)$, $n\geq 0$, in (1.4) is used to satisfy initial conditions on the differential system. Thus, if solutions are sought in the form

$$x_n(t) = P_n(z)u(z,t) \quad , \quad n\geq 0 \quad , \tag{1.6}$$

satisfying the initial conditions

$$\begin{aligned} x_j(0) &= \gamma_j \quad , \\ x_k(0) &= 0 \quad , \quad k\neq j \quad , \end{aligned} \tag{1.7}$$

then a solution to the initial value problem is

$$x_n(t) = \gamma_j(R_j+d_oL_j) \int_{-\infty}^{\infty} \frac{P_n(z)}{(R_n+d_oL_n)^{\frac{1}{2}}} \; \frac{P_j(z)}{(R_j+d_oL_j)^{\frac{1}{2}}} \cdot u(z,t)d\alpha(z) \quad , \quad n\geq 0 \quad , \tag{1.8}$$

where $\alpha(z)$ is an <u>integrator</u> (see Definition 2.1) for the polynomials

$P_n(z)$, $n\geq 0$. Since the rational fractions $\Psi_n(x)$, $n\geq 0$, used in this study do not in general have the type of orthonormality exhibited by polynomials, initial conditions on the system (1.1) must often be satisfied differently. In such circumstances (to be described later), a second system of rational fractions $\{W_n(x)\}_{n=o}^{\infty}$ defined by

$$W_o(x) = \Psi_o(x) - b_o\Psi_1(x) \quad , \tag{1.9a}$$

$$W_n(x) = -b_{n-1}\Psi_{n-1}(x) + \Psi_n(x) - b_n\Psi_{n+1}(x) \quad , \quad n\geq 1 \quad , \tag{1.9b}$$

is used. The system $\{W_n(x)\}_{n=o}^{\infty}$ is <u>biorthonormal</u> with respect to the system $\{\Psi_n(x)\}_{n=o}^{\infty}$ (see Definition 2.4), which means that

$$\int_{-\infty}^{\infty} W_j(x)\Psi_n(x)d\beta(x) = \delta_{nj} \quad , \quad n\geq 0,\ j\geq 0 \quad , \tag{1.10}$$

for some integrator (see Definition 2.3); and now a solution to (1.1) with the initial conditions given in (1.7) is

$$x_n(t) = \gamma_j(\ell_{j-1}+L_j+\ell_j) \int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \; \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}}$$

$$\cdot\ u(x,t)d\beta(x) \quad , \quad n\geq 0 \quad . \tag{1.11}$$

Chapter II is concerned with three of the basic areas mentioned above: reduction of the differential system (1.1) by means of the substitution (1.2) into the separated equations (1.3) and (1.5a) or (1.4) and (1.5b), results on finding integrators for systems of polynomials, and results on finding integrators for systems of rational fractions.

In Chapter III, the results of Chapter II are applied to the solution of some networks in which the coupling branches all have the same R/L ratio. In such cases the functions $\Psi_n(x)$, $n\geq 0$, of (1.3) reduce to the polynomials of (1.4); so much in this chapter is similar to work done in [10,11]. The analysis of those systems which have a constant R/L ratio in the coupling branches but a periodic R/L ratio in the carrier branches has its mathematical foundations in [4,6]. It is of interest that the integrators for some of the sequences of polynomials in this chapter are not increasing. While this fact does not complicate the task of exhibiting a solution, it does cause some difficulties in proving uniqueness, which is considered in Chapter VI.

The results of Chapter II are again applied in Chapter IV, this time to obtain the solutions of networks in which the coupling branches have periodic R/L ratios. In these cases, the rational fractions $\Psi_n(x)$, $n\geq 0$, appear.

Chapter V investigates what happens when a necessary condition for the existence of an integrator for the rational fractions $\Psi_n(x)$, $n\geq 0$, of (1.3) is not fulfilled. A complete resolution of the problem is not accomplished. Some examples are given.

One difficulty with the system (1.1) is that it fails to have a unique solution. In fact, any function in $C_1[0,\infty)$ satisfying the initial condition on $x_o(t)$ can be used to generate a solution to the problem by solving the successive equations of the system for the remaining functions $x_n(t)$, $n\geq 1$.

Chapter VI provides conditions under which a solution is unique.

These conditions are that $x_n(t)$, $n \geq 0$, be expansible in a Maclaurin series and that each sequence $\{x_n^{(k)}(0)\}_{n=o}^{\infty}$, $k \geq 1$, contains a subsequence which converges to zero. The bulk of this chapter is concerned with showing that the solutions generated in the preceding chapters satisfy these conditions.

CHAPTER II

INTEGRATORS FOR POLYNOMIALS AND RATIONAL FRACTIONS GENERATED BY SOME THREE-TERM RECURRENCE RELATIONS

Introduction

In Chapter I it was stated that the system (1.1) could be separated into the equations (1.3) and (1.5a) or, in some cases, into the equations (1.4) and (1.5b). The separation procedure is carried out in this chapter and the parameters $a_n, b_n, d_n, A_n, B_n, n\geq 0$, of (1.3) and (1.4) are defined in terms of the R's and L's of the circuit in Figure 1. Recall that the reason for this separation is that the biorthonormality (see Definition 2.4) properties of (1.3) or the orthonormality (see Definition 2.2) properties of (1.4) can be used to satisfy the initial conditions on the differential system (1.1). Various theorems regarding the existence of these properties are also presented in this chapter. For (1.4) this is a matter of presenting known results. For (1.3), the presentation is considerably more complicated. In order to keep the main lines of thought as uncluttered as possible, the proofs of many of the lemmas and theorems stated in this chapter are relegated to Appendix A.

Separation of the Differential System

If the substitution $x_o(t) = h_o(t)$, $x_n(t) = [(L_o+\ell_o)/(\ell_{n-1}+L_n+\ell_n)]^{\frac{1}{2}}$ $\cdot\, h_n(t)$, $n\geq 1$, is made in (1.1) under the assumption that $\ell_n>0$, $n\geq 0$, then

$$[D + \frac{r_o+R_o}{\ell_o+L_o}]h_o(t) - \frac{\ell_o[D + \frac{r_o}{\ell_o}]h_1(t)}{[(\ell_o+L_o)(\ell_o+L_1+\ell_1)]^{\frac{1}{2}}} = 0 \quad ,$$

$$- \frac{\ell_{n-1}[D + \frac{r_{n-1}}{\ell_{n-1}}]h_{n-1}(t)}{[(\ell_{n-2}+L_{n-1}+\ell_{n-1})(\ell_{n-1}+L_n+\ell_n)]^{\frac{1}{2}}} + [D + \frac{r_{n-1}+R_n+r_n}{\ell_{n-1}+L_n+\ell_n}]h_n(t) \tag{2.1}$$

$$- \frac{\ell_n(D + \frac{r_n}{\ell_n})h_{n+1}(t)}{[(\ell_{n-1}+L_n+\ell_n)(\ell_n+L_{n+1}+\ell_{n+1})]^{\frac{1}{2}}} = 0 \quad , \quad n\geq 1 \quad ,$$

where $\ell_{-1} = 0$. Now for $n\geq 0$ let

$$a_n = -(\frac{r_{n-1}+R_n+r_n}{\ell_{n-1}+L_n+\ell_n}) \quad , \quad b_n = \frac{\ell_n}{[(\ell_{n-1}+L_n+\ell_n)(\ell_n+L_{n+1}+\ell_{n+1})]^{\frac{1}{2}}} \quad ,$$

$$d_n = - \frac{r_n}{\ell_n} \quad , \tag{2.2}$$

where $r_{-1} = 0$. Furthermore, let $h_n(t) = \Psi_n(x)u(x,t)$ where x is a separation parameter to be defined shortly. Then (2.1) becomes

$$[u_t-a_ou]\Psi_o(x) - b_o[u_t-d_ou]\Psi_1(x) = 0 \quad ,$$

$$- b_{n-1}[u_t-d_{n-1}u]\Psi_{n-1}(x) + [u_t-a_nu]\Psi_n(x)$$

$$- b_n[u_t-d_nu]\Psi_{n+1}(x) = 0 \quad , \quad n\geq 1 \quad .$$

Divide by u and set $x = u_t/u$ to obtain (1.3a) and (1.3b). If in (1.3) $d_n = d_o$, $n \geq 0$, then for $z = 1/(x-d_o)$ and $\Psi_n(x) = [(d_o-a_o)/(d_o-a_n)]^{\frac{1}{2}} P_n(z)$, $n \geq 0$, it follows (since $\Psi_o(x) = 1$) that if $a_n (n \geq 0) \neq d_o$, then

$$P_o(z) = 1 \quad ,$$

$$\frac{b_o P_1(z)}{[(d_o-a_o)(d_o-a_1)]^{\frac{1}{2}}} = (z + \frac{1}{d_o-a_o}) P_o(z) \quad , \tag{2.3}$$

$$\frac{b_n P_{n+1}(z)}{[(d_o-a_n)(d_o-a_{n+1})]^{\frac{1}{2}}} = (z + \frac{1}{d_o-a_n}) P_n(z) - \frac{b_{n-1} P_{n-1}(z)}{[(d_o-a_{n-1})(d_o-a_n)]^{\frac{1}{2}}} \quad , \quad n \geq 1 \quad .$$

Upon letting $A_n = 1/(d_o-a_n)$, $B_n = b_n/[(d_o-a_n)(d_o-a_{n+1})]^{\frac{1}{2}}$, $n \geq 0$, (1.4) follows. Equation (1.5) follows from the substitution $x = u_t/u$.

Integrators--Some Basic Terminology

The next three chapters are devoted to defining and generating integrators for the sequences of (1.3) and (1.4). The problem of definition and generation contains three parts: explaining what is meant by the term integrator, proving the existence of an integrator, and exhibiting an integrator. The remainder of the present chapter concerns itself primarily with the first two of these subjects, the third being examined in subsequent chapters by use of the ideas now to be developed.

First some basic terminology is needed.

Definition 2.1. A sequence of functions $\{P_n(z)\}_{n=o}^{\infty}$ is said to be orthonormalizable with respect to α, a function of bounded variation on $(-\infty,\infty)$ if

$$\int_{-\infty}^{\infty} P_i(z)P_j(z)d\alpha(z) = 0 \quad , \quad i\neq j \quad ,$$
$$\int_{-\infty}^{\infty} P_i^2(z)d\alpha(z) = c_i\neq 0 \quad , \quad \text{for all } i,j\geq 0 \quad .\blacksquare \tag{2.4}$$

The function $\alpha(z)$ is called an integrator for the sequence $\{P_n(z)\}_{n=o}^{\infty}$, and it may be assumed without loss of generality that $\alpha(-\infty) = 0$ and $\int_{-\infty}^{\infty} P_o^2(z)\,d\alpha(z) = 1$.

Definition 2.2. If in Definition 2.1, $c_i=1$, $i\geq 0$, then the sequence of functions $\{P_n(z)\}_{n=o}^{\infty}$ is said to be orthonormal with respect to α.■

Definition 2.3. The sequences of functions $\{\Psi_n(x)\}_{n=o}^{\infty}$, $\{W_n(x)\}_{n=o}^{\infty}$ are said to be biorthonormalizable with respect to β, a function of bounded variation on $(-\infty,\infty)$, if

$$\int_{-\infty}^{\infty} \Psi_i(x)W_j(x)d\beta(x) = 0 \quad , \quad i\neq j \quad ,$$
$$\int_{-\infty}^{\infty} \Psi_i(x)W_i(x)d\beta(x) = c_i\neq 0 \ , \ \text{for all } i,j\geq 0 \ .\blacksquare \tag{2.5}$$

The function $\beta(x)$ is called an integrator for the sequences $\{\Psi_n(x)\}_{n=o}^{\infty}$, $\{W_n(x)\}_{n=o}^{\infty}$, and it may be assumed without loss of generality that $\beta(-\infty) = 0$ and $\int_{-\infty}^{\infty} W_o(x)\Psi_o(x)d\beta(x) = 1$.

Definition 2.4. If in Definition 2.3, $c_i = 1$, $i\geq 0$, then the sequences of functions are said to be biorthonormal with respect to β.■

Integrators for Systems of Polynomials

Conditions guaranteeing the existence of an integrator for the polynomials generated by the recurrence relation (1.4) are given in the following theorem.

Theorem 2.1. For A_n real, B_n real or pure imaginary and $B_n \neq 0$, $n \geq 0$, there exists an integrator $\alpha(z)$ such that the sequence $\{P_n(z)\}_{n=0}^{\infty}$ of (1.4) satisfies Definition 2.1.■

This rather general theorem is stated without proof in a paper by Shohat [15]. The proof is based on a theorem by Boas [2].

Given that the integrator $\alpha(z)$ exists, it can be profitably investigated by means of the continued fraction

$$\cfrac{1}{z+A_0 - \cfrac{B_0^2}{z+A_1 - \cfrac{B_1^2}{z+A_2 - \cfrac{B_2^2}{z+A_3 - \cdots}}}} \quad . \tag{2.6}$$

Under conditions which will be stated in Theorem 2.3, the continued fraction (2.6) converges to the Stieltjes transform of the integrator. The integrator can then be recovered from the transform by the Stieltjes inversion formula. To make this statement more specific, some addi-tional definitions and theorems are needed.

Definition 2.5. Let $\gamma(x)$ be a function of bounded variation on $(-\infty,\infty)$. Then

$$I(z) = \int_{-\infty}^{\infty} \frac{d\gamma(x)}{z-x} \tag{2.7}$$

defines the Stieltjes transform of γ for $\mathrm{Im}(z) \neq 0$. For $\mathrm{Im}(z)=0$, the following limit, if it exists, defines the Stieltjes transform of $\gamma(x)$:

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{z-\epsilon} + \int_{z+\epsilon}^{\infty}\right] \frac{d\gamma(x)}{z-x} \quad , \quad \epsilon>0 \quad . \blacksquare \tag{2.8}$$

Theorem 2.2. Given I(z) in Definition 2.5, $\gamma(x)$ may be recovered through the formula

$$\frac{1}{2}[\gamma(x_1+0) + \gamma(x_1-0)] - \frac{1}{2}[\gamma(x_o+0) + \gamma(x_o-0)] =$$

$$\lim_{\epsilon\to 0} \left(-\frac{1}{2\pi i}\int_{x_o}^{x_1} [I(x+i\epsilon) - I(x-i\epsilon)]dx\right) \quad , \quad \epsilon>0 \quad . \blacksquare \tag{2.9}$$

For a proof of this theorem, see Wall [18].

In order to explain what is meant by the convergence of the continued fraction, it is necessary first to generate a sequence of approximations to (2.6). For w, a real parameter, z complex, define an approximation to (2.6) by

$$C(n;z,w) = \frac{1|}{|z+A_o} - \frac{B_o^2|}{|z+A_1} - \cdot\cdot\cdot - \frac{B_{n-1}^2|}{|z+A_n-w} \quad , \quad n\geq 1 \quad , \tag{2.10}$$

where the notation of Pringsheim [14] has been used to write this finite continued fraction.

Definition 2.6. The continued fraction (2.6) is said to converge completely at a given point z, if at this point the sequence $\{C(n;z,w)\}_{n=1}^{\infty}$ converges to the same limit for all real w (including $w=\infty$) and uniformly in w.$\blacksquare$

Theorem 2.3. Let $p(z) = 1/\sum_{n=o}^{\infty} |P_n(z)|^2$ for the $P_n(z)$ of (1.4) and further suppose A_n, B_n real, $B_n\neq 0$, $n\geq 0$. Then, if there exists a finite z with $p(z) = 0$, (2.6) converges completely to I(z), the Stieltjes transform of the function $\alpha(z)$ in Definition 2.1. Furthermore, $p(z)=0$ at all points of continuity of $\alpha(z)$ and equals the jump of $\alpha(z)$ at a point of discontinuity.■

Proofs of these assertions are available in Shohat and Tamarkin [16].

Integrators for Systems of Rational Fractions

As in the polynomial case, integrators for rational fractions generated by the recurrence relation (1.3) may also be profitably investigated by means of a continued fraction, now of the form

$$\frac{1|}{|x-a_o} - \frac{b_o^2(x-d_o)^2|}{|x-a_1} - \frac{b_1^2(x-d_1)^2|}{|x-a_2} - \cdots \quad . \tag{2.11}$$

A sequence of approximations to (2.11) is generated by defining $\{F_n(x)\}_{n=o}^{\infty}$, $\{G_n(x)\}_{n=o}^{\infty}$ where $F_o(x) = 0$, $F_1(x) = 1$, $G_o(x) = 1$, $G_1(x) = x-a_o$, and

$$\begin{aligned} F_{n+1}(x) &= (x-a_n)F_n(x) - b_{n-1}^2(x-d_{n-1})^2F_{n-1}(x) \quad , \\ G_{n+1}(x) &= (x-a_n)G_n(x) - b_{n-1}^2(x-d_{n-1})^2G_{n-1}(x) \quad , \quad n\geq 1 \quad . \end{aligned} \tag{2.12}$$

Khovanskii [9] has shown that

$$\frac{F_{n+1}(x)}{G_{n+1}(x)} = \frac{1|}{|x-a_o} - \frac{b_o^2(x-d_o)^2|}{|x-a_1} - \cdots - \frac{b_{n-1}^2(x-d_{n-1})^2|}{|x-a_n} \quad .$$

As an aside, note that if $\Phi_n(x)$, $n\geq 0$, is defined by $\Phi_o(x) = 0$,

$\Phi_{n+1}(x) = F_{n+1}(x)/[b_o(x-d_o)b_1(x-d_1) \; . \; . \; . \; b_n(x-d_n)]$, $n\geq 0$, then it also follows that $F_{n+1}(x)/G_{n+1}(x) = \Phi_{n+1}(x)/\Psi_{n+1}(x)$, $n\geq 0$, since $\Psi_o(x) = G_o(x)$ and

$$\Psi_{n+1}(x) = \frac{G_{n+1}(x)}{b_o(x-d_o)b_1(x-d_1) \; \ldots \; b_n(x-d_n)} \quad , \quad n\geq 0 \quad , \tag{2.13}$$

by substitution in (1.3). The sequence $\{\Phi_n(x)\}_{n=o}^{\infty}$ is again encountered in Chapter V.

Now the verification of the existence of an integrator (see Definition 2.3) for the sequences of (1.3) and (1.9) employs the functions $F_n(x)/G_n(x)$, $n\geq 0$, and consists of three basic steps: first, it is necessary to show that each function $F_n(z)/G_n(z)$, $n\geq 0$, is the Stieltjes transform of some function $\beta_n(x)$ which is bounded and increasing; second, it is necessary to extract a convergent subsequence from the sequence of functions $\beta_n(x)$, $n\geq 0$; third, it is necessary to show that the limit of this subsequence is the desired integrator.

In showing that each function $F_n(z)/G_n(z)$ is the Stieltjes transform of some function, the following representation theorem is useful.

<u>Theorem 2.4.</u> If $F_n(z)/G_n(z)$ is analytic in the half-plane $y>0$ and $\mathrm{Im}(F_n(z)/G_n(z)) \leq 0$ in this half-plane, and if, in addition, there exists a finite limit $\Lambda_n = \lim_{z\to\infty} zF_n(z)/G_n(z)$, then for $y>0$, $F_n(z)/G_n(z) = \int_{-\infty}^{\infty} d\beta_n(x)/[z-x]$, where $\beta_n(x)$ is bounded and increasing, Λ_n is real, and $\int_{-\infty}^{\infty} d\beta_n(x) = \Lambda_n$.■

The proof of this theorem depends on several other similar representation theorems and may be found in [16] or [17].

To use the theorem, it must be shown that the hypotheses are satisfied--that is, that $F_n(z)/G_n(z)$, $n\geq 0$, satisfies three conditions:

$$F_n(z)/G_n(z) \text{ is analytic for } \mathrm{Im}(z)>0 \quad ;$$

$$\mathrm{Im}(F_n(z)/G_n(z)) \le 0 \quad \text{for } \mathrm{Im}(z)>0 \quad ;$$

$$\lim_{z\to\infty} zF_n(z)/G_n(z) \quad \text{is finite} \quad .$$

First, since $F_n(z)$ and $G_n(z)$ are polynomials, the analyticity of $F_n(z)/G_n(z)$ for $\mathrm{Im}(z)>0$ is established if it is shown that $G_n(z)$ has no zeros in the upper half-plane. The following theorem provides conditions sufficient to guarantee this result.

Theorem 2.5. Suppose there exist g_i with $0<g_{i-1}<1$, $i\ge 0$, such that $b_i^2 \le g_i(1-g_{i-1})$. Further suppose $G_{i+1}(d_i) \ne 0$, $i\ge 0$, and that a_i, d_i are real for $i\ge 0$. Then $G_n(x)$ has n real distinct zeros for $n\ge 1$.■

The proof of this theorem is in Appendix A.

The second condition is not so easily established. The key is to examine, for fixed z, the composition map $t(n;z,w) = t_{-1}t_o \ldots t_{n-2}t_{n-1}(z,w)$ where $t_n(z,w)$ is defined by

$$t_n(z,w) = (z-a_n) - \frac{b_n^2(z-d_n)^2}{w} \quad , \quad n\ge 0 \quad , \tag{2.14}$$

and $t_{-1}(z,w) = 1/w$. Thus

$$t(n;z,w) = \frac{1|}{|z-a_o} - \frac{b_o^2(z-d_o)^2|}{|z-a_1} - \cdots - \frac{b_{n-1}^2(z-d_{n-1})^2|}{|w} \; , \; n\ge 1 \quad . \tag{2.15}$$

Note that $t(n;z,\infty) = F_n(z)/G_n(z)$; so showing that $\mathrm{Im}(t(n;z,\infty)) \le 0$ for $\mathrm{Im}(z) > 0$ verifies that the second condition of Theorem 2.4 is satisfied. The following theorem is helpful in establishing this fact.

Theorem 2.6. Suppose there exist g_i with $0<g_{i-1}<1$, $i\geq 0$, such that $b_i^2 \leq g_i(1-g_{i-1})$; and suppose that a_i, d_i are real for $i\geq 0$. Let $y = \mathrm{Im}(z)$; let $t_n(z,w)$ be defined by (2.14). Then for $\mathrm{Im}(w) \geq g_n y$, $n\geq 0$, it follows that $\mathrm{Im}(t_n(z,w)) \geq g_{n-1}y$.■

The proof of this theorem may be found in Appendix A.

By employing an inductive argument on n, Theorem 2.6 is used to show that $\mathrm{Im}(t(n;z,\infty)) \leq 0$ for $\mathrm{Im}(z) > 0$ as follows.

First examine the region $\mathrm{Im}(w) \geq g_{-1}y$. From the theory of conformal mapping or from a simple calculation, it can be shown that in this region $t(0;z,w)$ $(=t_{-1}(z,w) = 1/w)$ satisfies

$$\left|t + \frac{i}{2yg_{-1}}\right| \leq \frac{1}{2yg_{-1}} , \tag{2.16}$$

for $y = \mathrm{Im}(z) > 0$. So $\mathrm{Im}(t(0;z,w)) \leq 0$ for $\mathrm{Im}(w) \geq g_{-1}y$ and $y = \mathrm{Im}(z) > 0$.

Now examine the region $\mathrm{Im}(w) \geq g_{n-1}y$. By induction and Theorem 2.6, it follows that if $\mathrm{Im}(w) \geq g_{n-1}y$, then $\mathrm{Im}(t_o(t_1(t_2 \ldots (t_{n-1}(z,w))\ldots)))$ $\geq g_{-1}y$. If w in the previous paragraph is replaced by $t_o t_1 t_2 \ldots t_{n-1}(z,w)$, it can be shown that $t(n;z,w)$ $(=t_{-1}t_o t_1 \ldots t_{n-1}(z,w) = 1/(t_o t_1 t_2 \ldots t_{n-1}(z,w)))$ satisfies (2.16) for $y = \mathrm{Im}(z) > 0$. So $\mathrm{Im}(t(n;z,w)) \leq 0$ for $\mathrm{Im}(w) \geq g_{n-1}y$ and $y = \mathrm{Im}(z) > 0$.

To establish the second condition of Theorem 2.4, let w go to infinity through those values for which $\mathrm{Im}(w) \geq y$; then $\mathrm{Im}(t(n;z,\infty)) = \mathrm{Im}(F_n(z)/G_n(z)) \leq 0$ for $y = \mathrm{Im}(z) > 0$.

The third and final hypothesis of the representation theorem is established by means of the following lemma.

Lemma 2.1. Suppose there exist g_i with $0<g_{i-1}<1$, $i\geq 0$, such that

$b_i^2 \le g_i(1-g_{i-1})$; and suppose that a_i, d_i are real for $i \ge 0$. Then $F_n(x)$ and $G_n(x)$, as defined in (2.12), are polynomials of degree n-1 and n, respectively, and $\lim_{z\to\infty} (zF_n(z)/G_n(z))$ exists and is uniformly bounded for all $n \ge 2$.■

A proof is given in Appendix A.

Thus the first of the three parts necessary in the proof of the existence of an integrator for the sequences in (1.3) and (1.9) has been concluded.

It is now possible to write

$$\frac{F_n(z)}{G_n(z)} = \int_{-\infty}^{\infty} \frac{d\beta_n(x)}{z-x} \tag{2.17}$$

for some $\beta_n(x)$ which is bounded and increasing. Actually a great deal more is known about $\beta_n(x)$. In fact, each $\beta_n(x)$ is, by Theorem 2.5, a sum of step functions, each of which begins at a zero of $G_n(x)$ and has a magnitude equal to the residue of $F_n(x)/G_n(x)$ at that zero. Although some of this information about $\beta_n(x)$ is not needed in the proof of the existence of an integrator, it does provide a clearer picture of the form of each $\beta_n(x)$. The part which is needed later is formalized in the following lemma.

Lemma 2.2. Let G denote the closure of the set of all zeros of the $G_n(x)$, $n \ge 0$. Then the closure of the set of all points of increase of the $\beta_n(x)$, $n \ge 1$, is contained in G.■

The second part in the proof of the existence of the integrator for the sequences in (1.3) and (1.9) is the extraction of a convergent subsequence from $\{\beta_n(x)\}_{n=1}^{\infty}$.

Theorem 2.7. Let $\beta_n(x)$, $n\geq 2$, be a sequence of non-decreasing functions on $(-\infty,\infty)$ such that $\int_{-\infty}^{\infty} d\beta_n(x) \leq \Lambda$, $n\geq 2$, where Λ is finite. Then there exists a non-decreasing function $\beta(x)$ such that $\int_{-\infty}^{\infty} d\beta(x) = \Lambda$ and a sequence of indices $(n,1) < (n,2) < \ldots < (n,j) < \ldots$, such that for all x, $\lim_{k\to\infty} \beta_{n,k}(x) = \beta(x)$.■

A proof can be found in [18]. This concludes the second part.

That $\beta(x)$ is an integrator for the sequences of (1.3) and (1.9) is now established in three basic steps: first, it is shown that

$$\int_{-\infty}^{\infty} W_m(x)d\beta(x) = \delta_{om} \quad , \quad m\geq 0 \quad , \tag{2.18}$$

$$\int_{-\infty}^{\infty} \frac{\Psi_{m+1}(x)d\beta(x)}{(x-d_m)^j} = 0 \quad , \quad 1\leq j\leq e_m \quad , \quad m\geq 0 \quad , \tag{2.19}$$

where e_m measures how many times the number d_m appears in the product $(x-d_o)(x-d_1) \ldots (x-d_m)$; next, equation (2.19) is used to show that

$$\int_{-\infty}^{\infty} \frac{\Psi_{n+m}(x)d\beta(x)}{(x-d_m)^{e_m}} = 0 \quad , \quad m\geq 0,\ n\geq 1 \quad , \tag{2.20}$$

(see Lemma A.4); finally, (2.18) and (2.20) are used together with the partial fraction expansions of $W_j(x)$ and $\Psi_j(x)$ to prove Theorem 2.9, the main existence result. Only the first of these three steps is undertaken in this chapter. The proofs of the remaining steps are given in Appendix A.

The basic approach in the first step is to show that for each $\beta_n(x)$, $n\geq 1$,

$$\int_{-\infty}^{\infty} W_m(x)\,d\beta_n(x) = \delta_{om} \quad , \quad 0\le m\le n-1 \quad , \tag{2.21}$$

$$\int_{-\infty}^{\infty} \frac{\Psi_{m+1}(x)\,d\beta_n(x)}{(x-d_m)^j} = 0 \quad , \quad 0\le m\le n-1,\ 1\le j\le e_m \quad , \tag{2.22}$$

and then to take a limit, over the subsequence of Theorem 2.7, in each of these equations to obtain (2.18) and (2.19). The conditions governing the validity of (2.21) and (2.22) are given in Theorem 2.8, while a general discussion of taking limits underneath integral signs is given in Lemma 2.3. This is followed by the application of this lemma to (2.21) and (2.22).

Theorem 2.8. Let D denote the closure of the set of all d_n, $n\ge 0$. If the intersection of D with G (see Lemma 2.2) is empty, so that (2.21) and (2.22) are well-defined, and if the hypotheses of Theorem 2.5 are satisfied, then (2.21) and (2.22) are valid for every $n\ge 1$.■

A proof is available in Appendix A.

Lemma 2.3. Let C_∞ denote the class of all continuous functions $f(x)$ defined on $(-\infty,\infty)$ and having the property that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)=0$. Then under the hypotheses of Theorem 2.7,

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(x)\,d\beta_{n,k}(x) = \int_{-\infty}^{\infty} f(x)\,d\beta(x) \quad .\blacksquare \tag{2.23}$$

This lemma is proved in [13]. When G and D have an empty intersection, there is a function in C_∞ with the same values as $W_m(x)$ (see (A.25)) on G and similarly for $\Psi_{m+1}(x)/(x-d_m)^j$, $1\le j\le e_m$ (see (2.13)). Thus (2.18) and (2.19) are obtained by applying Lemma 2.3 to (2.21) and

(2.22). Note that Lemma 2.2 is needed here to insure that the contributions to the integrals of (2.21) and (2.22) outside of G are zero.

Conditions sufficient to prove that $\beta(x)$ is an integrator for the sequences of (1.3) and (1.9) are given below in Theorem 2.9. The proof of this theorem employs (2.18) and (2.20) and may be found in Appendix A.

Theorem 2.9. Under the conditions that

(i) a_j, b_j, d_j are real, $j\geq 0$,

(ii) there exist g_j with $0<g_{j-1}<1$, $j\geq 0$, and $b_j^2 \leq g_j(1-g_{j-1})$,

(iii) G and D are disjoint,

$\beta(x)$ is an integrator (see Definition 2.3) for the sequences of (1.3) and (1.9).■

CHAPTER III

NETWORKS WITH THE SAME R/L RATIO IN EACH COUPLING BRANCH

This chapter examines solutions to (1.1) when the ratio r_n/ℓ_n, $n\geq 0$, is a constant. Under this condition, the separation equations that arise from (1.1) are (1.4) and (1.5), where the coefficients A_n and B_n of (1.4) are given by

$$A_n = \frac{\ell_{n-1}+L_n+\ell_n}{R_n+d_oL_n} \quad , \quad B_n = \frac{\ell_n}{[(R_n+d_oL_n)(R_{n+1}+d_oL_{n+1})]^{\frac{1}{2}}} \quad , \quad n\geq 0 \tag{3.1}$$

(see (2.3)). It is clear from Theorem 2.1 that an integrator for the sequence $\{P_n(z)\}_{n=o}^{\infty}$ of (1.4) exists if $R_n+d_oL_n \neq 0$, $n\geq 0$, and $\ell_n>0$, $n\geq 0$. Then a solution to (1.1) satisfying the initial conditions of (1.7) is

$$x_n(t) = \gamma_j(R_j+d_oL_j)\int_{-\infty}^{\infty} \frac{P_n(z)}{(R_n+d_oL_n)^{\frac{1}{2}}}\,\frac{P_j(z)}{(R_j+d_oL_j)^{\frac{1}{2}}}\, e^{(\frac{1}{z}+d_o)t}\, d\alpha(z) \quad , \ n\geq 0 \ , \tag{3.2}$$

provided:

(i) $x_n(t)$, as given in (3.2), exists for all $t>0$,

(ii) $dx_n(t)/dt$ exists and is obtained by differentiating (3.2) under the integral sign.

With these provisions, substitution directly verifies that (3.2) is a solution to (1.1). A derivation of (3.2) can be based on the results of [3]. The question of the permissibility of differentiating (3.2) under the integral sign may involve a considerable pathology if the closure of the set on which $\alpha(z)$ is non-constant includes $z=0$ (this trouble does not occur in either of the examples given below). But if $\alpha(z)$ is non-constant only on the union of two closed half-infinite intervals not containing zero and is monotone on each of them separately (for example, increasing on one and decreasing on the other), then (3.2) is a solution to (1.1) (see Apostol [1]). Solutions to two systems are developed in this chapter, one which generates classical polynomials and one which, in general, does not.

The first example is based on the Laguerre polynomials. Let $\ell_n = n+1$, $r_n = 4(n+1)$, $L_n = 1$, $R_n = 2$, $n \geq 0$, so that $A_n = -(n+1)$, $B_n = (n+1)/2$, $n \geq 0$. Then it is easily shown that $P_n(z) = (-1)^n L_n(2z-1)$ where $L_n(x)$ is the n^{th} Laguerre polynomial. By using this fact, it follows that the integrator $\alpha(z)$ of (3.2) is given by

$$d\alpha(z) = \begin{cases} 0 & , \quad -\infty < z < \frac{1}{2} \; , \\ 2e^{-2z+1}dz & , \quad \frac{1}{2} \leq z < \infty \; , \end{cases} \tag{3.3}$$

To obtain other physical systems involving classical polynomials, a method [7] may be used whereby it is possible to determine whether a system of polynomials generated by a given recurrence relation is classical to within a linear transformation of the argument and to within a function of n. A method for finding the integrator in such cases is also given.

Now an example involving non-classical polynomials is presented. Suppose $r_{2n}=r_o$, $\ell_{2n}=\ell_o$, $r_{2n-1}=r_1$, $\ell_{2n-1}=\ell_1$, $R_{2n}=R_2$, $L_{2n}=L_2$, $R_{2n-1}=R_1$, $L_{2n-1}=L_1$, $n\geq 1$, and in addition $R_o+d_oL_o=R_2+d_oL_2$, $r_1/\ell_1=r_o/\ell_o$. Then $B_{2n}=\ell_o/[(R_o+d_oL_o)(R_1+d_oL_1)]^{\frac{1}{2}}$, $B_{2n+1}=\ell_1/[(R_o+d_oL_o)(R_1+d_oL_1)]^{\frac{1}{2}}$, $n\geq 0$, $A_{2n}=(\ell_1+L_2+\ell_o)/(R_o+d_oL_o)$, $A_{2n-1}=(\ell_o+L_1+\ell_1)/(R_1+d_oL_1)$, $n\geq 1$, and $A_o=(\ell_o+L_o)/(R_o+d_oL_o)$ (recall $d_o=-r_o/\ell_o$). Thus $P_o=1$, $B_oP_1(z)=z+A_o$, and

$$
\begin{aligned}
B_1P_{2n}(z) &= (z+A_1)P_{2n-1}(z) - B_oP_{2n-2}(z) \quad , \quad n\geq 1 \quad , \\
B_oP_{2n+1}(z) &= (z+A_2)P_{2n}(z) - B_1P_{2n-1}(z) \quad , \quad n\geq 1 \quad .
\end{aligned}
\tag{3.4}
$$

Let $x=z+A_2$, $a=A_o-A_2$, $b=A_1-A_2$, $c=B_o^2$, $d=B_1^2$, $B_o^nB_1^nP_{2n}(z)=S_{2n}(x)$, $B_o^{n+1}B_1^nP_{2n+1}(z)=S_{2n+1}(x)$, $n\geq 0$. Then $S_o=1$, $S_1=x+a$, and

$$
\begin{aligned}
S_{2n}(x) &= (x+b)S_{2n-1}(x) - cS_{2n-2}(x) \quad , \quad n\geq 1 \quad , \\
S_{2n+1}(x) &= xS_{2n}(x) - dS_{2n-1}(x) \quad , \quad n\geq 1 \quad .
\end{aligned}
\tag{3.5}
$$

The remainder of this chapter concerns the derivation of an integrator for the polynomials of (3.5). The coefficients of the single recurrence relation equivalent to (3.5) are periodic of period two and a great deal of work has been done on such systems [4,6]. The most complete effort is due to Geronimus [5]. In the cases when c and d are positive (assuring the existence of a non-decreasing integrator), Geronimus has shown that the derivative of an integrator for (3.5) is given by $1/\pi$ times the imaginary part of the continued fraction

$$F(x) = \frac{1|}{|x+a} - \frac{c|}{|x+b} - \frac{d|}{|x} - \frac{c|}{|x+b} - \cdots$$

plus possibly some impulse functions.

The periodicity of the continued fraction permits a simple evaluation. In this case,

$$\frac{1}{\pi}\operatorname{Im}(F(x)) = \frac{1}{2\pi}\left(\frac{\sqrt{4cd-(x^2+bx-d-c)^2}}{ax^2+x(ab+d+a^2)+(a^2b+ad-ac)}\right) . \tag{3.6}$$

The zeros of the denominator of (3.6), which are the points where the impulse functions may occur, are $x=[(cd)^{\frac{1}{2}}q/a]-a$ and $x=[(cd)^{\frac{1}{2}}p/a]-a$ where $p=[a^2-ab-d+\sqrt{H}]/2(cd)^{\frac{1}{2}}$ and $q=[a^2-ab-d-\sqrt{H}]/2(cd)^{\frac{1}{2}}$, and where $H=(ab+d-a^2)^2+4a^2c$. The cases when c and d are both negative are not covered by Geronimus, but it can be verified directly that often the derivative of an integrator is still $\frac{1}{\pi}\operatorname{Im}(F(x))dx$ plus possibly some impulse functions. These results are presented in the following theorem.

Theorem 3.1. If $cd>0$ (and $p\neq q$ when $p^2>1$), then the system (3.5) (derived from (2.3) with non-negative RL values) is orthonormalizable with respect to the integrator $\mu(x)$ where

$$d\mu(x) = \begin{cases} \frac{1}{\pi}(\operatorname{Im}(F(x)))dx & \text{on } I_1 , \\ -\frac{1}{\pi}(\operatorname{Im}(F(x)))dx & \text{on } I_2 , \end{cases} \tag{3.7}$$

for

$$I_1 = [(-b+\sqrt{4(d+c)+b^2-8(cd)^{\frac{1}{2}}})/2, \quad (-b+\sqrt{4(d+c)+b^2+8(cd)^{\frac{1}{2}}})/2] \tag{3.8}$$

and

$$I_2 = [(-b-\sqrt{4(d+c)+b^2+8(cd)^{\frac{1}{2}}})/2, \quad (-b-\sqrt{4(d+c)+b^2-8(cd)^{\frac{1}{2}}})/2] \tag{3.9}$$

and where $\mu(x)$ has a jump at:

(A) $x = [(cd)^{\frac{1}{2}}q/a] - a$ of magnitude $(p^2-1)(cd)^{\frac{1}{2}}/p\sqrt{H}$ if $p^2 > 1$, $q^2 \le 1$;

(B) $x = [(cd)^{\frac{1}{2}}p/a] - a$ of magnitude $(q^2-1)(cd)^{\frac{1}{2}}/q\sqrt{H}$ if $q^2 > 1$, $p^2 \le 1$;

(C) $x = [(cd)^{\frac{1}{2}}p/a] - a$ and $x = [(cd)^{\frac{1}{2}}q/a] - a$ of magnitudes $(q^2-1)(cd)^{\frac{1}{2}}/q\sqrt{H}$ and $(p^2-1)(cd)^{\frac{1}{2}}/p\sqrt{H}$, respectively if $p^2 > 1$, $q^2 > 1$, $p \ne q$;

(D) $x = [(cd)^{\frac{1}{2}}p/a] - a$ of magnitude 1 if $p^2 = 1$, $p = q$.

If $p^2 \le 1$, $q^2 \le 1$, $p \ne q$ or $p^2 < 1$, $p = q$, there are no jumps in $\mu(x)$.■

An outline of steps to be followed in verifying Theorem 3.1 is available in Appendix B.

Theorem 3.1 excludes the particular combination $cd > 0$, $p^2 > 1$, $p = q$. There are two basic reasons for not including this case in the preceding theorem: first, this case does not occur for non-negative RL values; secondly, the function $\mu(x)$ obtained from the continued fraction in this case contains an impulse or delta function at $x = [(cd)^{\frac{1}{2}}p/a] - a$ and hence possibly should not be referred to as an integrator (if integrators are restricted to be functions of bounded variation). Both of these results are contained in the following theorem.

Theorem 3.2. For non-negative RL values in (1.1), the case $cd > 0$, $p^2 > 1$, $p = q$ does not occur. The polynomials in (3.5) still satisfy (2.4) provided $\alpha(z)$ is replaced by the generalized function $\eta(z)$ such that

$$\begin{aligned} \langle\phi,\eta\rangle = {} & \frac{1}{\pi}\int_{I_1}\phi(x)(\mathrm{Im}(F(x)))dx \\ & - \frac{1}{\pi}\int_{I_2}\phi(x)(\mathrm{Im}(F(x)))dx \\ & + \frac{p^2+1}{p^2}\,\phi\left(\frac{(cd)^{\frac{1}{2}}p}{a} - a\right) \\ & + \frac{(1-p^2)(cd)^{\frac{1}{2}}}{ap}\;\frac{d\phi(x)}{dx}\Bigg/_{x = \frac{(cd)^{\frac{1}{2}}p}{a} - a} \end{aligned} \tag{3.10}$$

(see (3.8) and (3.9) for the definitions of I_1 and I_2, respectively).∎

An outline of steps to be followed in verifying Theorem 3.2 is available in Appendix B.

If the endpoints of the intervals I_1 and I_2 are no longer real, then I_1 and I_2 must be replaced by contours in the complex plane. These contours are obtained by applying the transformations $x = (-b\pm\sqrt{4(d+c)+b^2+4z})/2$ to z in the interval $[\mp 2\sqrt{cd}, \pm 2\sqrt{cd}]$ (upper signs for I_1, lower for I_2). This result applies to Theorem 3.2, and it also applies to Theorem 3.1 should it be desirable to admit other than non-negative RL values.

CHAPTER IV

PERIODIC NETWORKS IN WHICH THE R/L RATIO OF THE COUPLING BRANCHES IS NOT CONSTANT

This chapter examines solutions to (1.1) when the ratio r_n/ℓ_n, $n\geq 0$, varies with n. Under this condition, the separation equations that arise from (1.1) are (1.3) and (1.5). In Chapter II sufficient conditions are given so that a solution to (1.1) (for the initial conditions of (1.7)) can be written as (1.11).

These sufficient conditions are:

(i) a_n, b_n, d_n are real, $n\geq 0$;

(ii) there exist g_n with $0 < g_{n-1} < 1$, $n\geq 0$ and $b_n^2 \leq g_n(1-g_{n-1})$;

(iii) G and D are disjoint.

Some simplification of these conditions is possible. For example, when interpreted in terms of the R's and L's of the circuit, the first condition is clearly satisfied, as seen from (2.2). A simplification of the requirements of the second condition is also obtained by translating these requirements into the R's and L's of the circuit. This step occupies the first part of the chapter. For particular examples, the third condition is by far the most troublesome to verify. Although it can be weakened, some knowledge of the zeros of the G_n, $n\geq 0$, is still required. The weakening of condition (iii) is accomplished by introducing a theorem which shows that a necessary condition for the existence of an integrator is that $G_n(d_{n-1}) \neq 0$, $n \geq 1$. All of this information is then combined in Theorem 4.2 to provide a formal statement on the existence of a solution to (1.1) of the form in (1.11) (for the initial conditions of (1.7)). However, when no conditions other than (i), (ii), (iii), above, are imposed on the circuit parameters, the difficulty of exhibiting the integrator whose existence is guaranteed by Theorem 4.2 is substantial. This difficulty is eased by the additional restriction that the R's and L's be periodic. The chapter ends with an example of this type. Because some of the details of the example are still tedious despite the simplifications resulting from periodicity, they are relegated to Appendix C.

By using Definition A.1 and Lemma A.2, the second condition above can be written as

> "(ii) $\{b_n^2\}_{n=o}^{\infty}$ is a chain sequence with at least one sequence of positive parameters."

For the RL network of (1.1), it is shown in Lemma 4.1 that $\{b_n^2\}_{n=o}^{\infty}$ is indeed a chain sequence provided $\ell_n > 0$, $n \geq 0$; also, the requirement that at least one sequence of parameters be positive is interpreted below in terms of a simpler condition on the L's of the circuit.

<u>Lemma 4.1.</u> The sequence $\{b_n^2 = \ell_n^2/[(\ell_{n-1}+L_n+\ell_n)(\ell_n+L_{n+1}+\ell_{n+1})]\}_{n=o}^{\infty}$, $\ell_{-1} = 0$, is a chain sequence (Definition A.1) for $\ell_n > 0$, $n \geq 0$.∎

<u>Proof.</u> By Lemma A.2, it suffices to show that $\{\ell_n^2/[(\ell_{n-1}+\ell_n)(\ell_n+\ell_{n+1})]\}_{n=o}^{\infty}$, $\ell_{-1} = 0$, is a chain sequence. But, in fact, this sequence has minimal parameters, as defined in Lemma A.1, given by $\gamma_n = \ell_n/(\ell_n+\ell_{n+1})$, $n \geq -1$.∎

So now $\{b_n^2\}_{n=o}^{\infty}$ is a chain sequence, but it remains to show that it has at least one sequence of positive parameters g_{n-1}, $n\geq 0$. Lemma A.2 can be used to determine simple conditions under which, in particular, the maximal parameters M_n, $n\geq -1$, of the chain sequence $\{b_n^2\}_{n=o}^{\infty}$ are positive. In Lemma A.2, let $\sigma_i^2 = \ell_i^2/[(\ell_{i-1}+\ell_i)(\ell_i+\ell_{i+1})]$, $i\geq 0$, and let $\tau_i^2 = b_i^2$, $i\geq 0$. Then by Lemma A.2 each maximal parameter M_i, $i\geq 0$, of the chain sequence $\{\tau_i^2\}_{i=o}^{\infty}$ is bounded below by the parameters $\gamma_i = \ell_i/(\ell_i+\ell_{i+1})$, $i\geq 0$, of the chain sequence $\{\sigma_i^2\}_{i=o}^{\infty}$. Notice that each such γ_i is positive, provided $\ell_i > 0$, $i\geq 0$, and thus $M_i > 0$, $i\geq 0$. This leaves only M_{-1} to consider. Unfortunately, M_{-1} is not always positive for $\ell_i > 0$, $i\geq 0$, as the following lemma illustrates.

Lemma 4.2. Let $\{\sigma_i^2\}_{i=o}^{\infty}$ be a chain sequence with minimal parameters γ_{i-1}, $i\geq 0$. The sequence $\{\sigma_i^2\}_{i=o}^{\infty}$ determines its parameters uniquely if, and only if, the series $\sum_{i=o}^{\infty} \prod_{n=o}^{i} \gamma_n/(1-\gamma_n)$ diverges.■

For a proof see [18].

If $\sigma_i^2 = \ell_i^2/[(\ell_{i-1}+\ell_i)(\ell_i+\ell_{i+1})]$, $i\geq 0$, $\ell_{-1} = 0$, it is easily calculated that $\sum_{i=o}^{\infty} \prod_{n=o}^{i} \gamma_n/(1-\gamma_n) = \ell_o \sum_{i=1}^{\infty} 1/\ell_i$. If this series converges, then by Lemmas 4.2 and A.2, the sequence $\{b_i^2\}_{i=o}^{\infty}$ has $M_{-1} > 0$. But if this series diverges, then in order that M_{-1} be positive, it follows from Lemma A.2 that b_o^2 must be less than σ_o^2 or

$$\frac{\ell_o^2}{(\ell_o+L_o)(\ell_o+L_1+\ell_1)} < \frac{\ell_o^2}{(\ell_o)(\ell_o+\ell_1)} \quad . \tag{4.1}$$

It is clear that (4.1) holds if, and only if, $L_o+L_1 > 0$ (recall that all the ℓ's and L's are non-negative). Thus the second condition listed above can be replaced by "(ii) $\ell_n > 0$, $n \geq 0$, $L_o + L_1 > 0$."

Now as has been mentioned previously, the third sufficient condition (G and D are disjoint) for the existence of an integrator is very difficult to verify. Since it is not a necessary condition for the existence of an integrator, it can be weakened by introducing a necessary condition (unfortunately, almost as difficult to verify) together with some additional conditions which yield sufficiency. The necessary condition is the subject of the following theorem.

Theorem 4.1. A necessary condition that an integrator exist for the sequences of (1.3) and (1.9) is that $G_{n+1}(d_n) \neq 0$, $n \geq 0$.■

Proof. Assume for the sake of contradiction that $G_{n+1}(d_n) = 0$ for some $n \geq 0$. Let j be the smallest such n. It follows from the expression for r_m^m in the proof of Lemma A.5 that $r_j^j = 0$ and so there exist constants t_k^j such that $W_j(x) = \sum_{k=o}^{j-1} t_k^j W_k(x)$. From Definition 2.3, upon substituting for $W_j(x)$, $c_j = \int_{-\infty}^{\infty} W_j(x)\Psi_j(x)d\beta(x) = 0$--a contradiction.■

With the addition of the above necessary condition, the disjointness of G and D can be relaxed a little. By referring to the proofs of the basic premises in Chapter II, it can be seen that if there is a sequence of $F_{n,i}/G_{n,i}$, with n,i going to infinity, such that the closure of the set of all zeros of these $G_{n,i}$, GN, does not intersect the closure of the set of all d_i, D, then this condition plus Theorem 4.2 can replace the disjointness of G and D. Thus condition (iii) becomes

"(iii) $G_n(d_{n-1}) \neq 0$, $n \geq 1$, and there exist $(n,1) < (n,2) < \ldots$ going to infinity such that GN and D are disjoint."

In other words, given $G_n(d_{n-1}) \neq 0$, $n\geq 1$, it is sufficient to have only an infinite number, rather than all, of the G_n with zeros different from the d_n. The above results are formalized in the following theorem.

Theorem 4.2. Let $\{n,i\}_{i=0}^{\infty}$ be an increasing sequence of integers; let GN denote the closure of the set of all zeros of the $G_{n,i}(x)$, $i\geq 0$ (see (2.12)); let D be as in Theorem 2.8; let $\Psi_n(x)$, $W_n(x)$, $n\geq 0$, be defined by (2.2), (1.3), and (1.9). Then if

(i) $\ell_n > 0$, $n\geq 0$ and $L_o+L_1 > 0$,

(ii) $G_{n+1}(d_n) \neq 0$, $n\geq 0$ and $GN \wedge D = \phi$,

a solution to (1.1) with initial conditions $x_j(0) = \gamma_j$, $x_k(0) = 0$, $j \neq k$, is

$$x_n(t) = \gamma_j(\ell_{j-1}+L_j+\ell_j) \int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}} e^{xt} d\beta(x) \quad , \quad n\geq 0 \ , \tag{4.2}$$

for some $\beta(x)$ which is non-decreasing with $\int_{-\infty}^{\infty} d\beta(x) < \infty$.∎

Some further discussion of this theorem can be found in Appendix D.

The remainder of this chapter is a discussion of an example in which the circuit parameters all have period two, that is $r_{2n} = r_o$, $r_{2n+1} = r_1$, $\ell_{2n} = \ell_o$, $\ell_{2n+1} = \ell_1$, $R_{2n+2} = R_2$, $R_{2n+1} = R_1$, $L_{2n+2} = L_2$, $L_{2n+1} = L_1$, $n\geq 0$, where all parameters are assumed positive. In addition, the choices $R_o = R_2+r_o$, $L_o = L_2+\ell_o$ are made to keep the mathematics simple. With these choices, it follows that $a_{2n} = a_o$, $a_{2n+1} = a_1$, $b_{2n} = b_o$, $b_{2n+1} = b_1$, $d_{2n} = d_o$, $d_{2n+1} = d_1$, $n\geq 0$. Since a translation in the variable of the recurrence relation (1.3) can make a_1, a_o, d_1 or d_o equal to zero, it may be assumed without loss of generality that $a_o = 0$. So with $\Psi_{-1} = 0$, $\Psi_o = 1$, (1.3) becomes

$$b_o(x-d_o)\Psi_{2n+1}(x) = (x)\Psi_{2n}(x) - b_1(x-d_1)\Psi_{2n-1}(x) \quad , \quad n\geq 0 \quad ,$$
$$b_1(x-d_1)\Psi_{2n+2}(x) = (x-a_1)\Psi_{2n+1}(x) - b_o(x-d_o)\Psi_{2n}(x) \quad , \quad n\geq 0 \quad . \tag{4.3}$$

The associated polynomials from (2.12) are given by $G_o(x) = 1$, $G_1(x) = x$ and

$$G_{2n+2}(x) = (x-a_1)G_{2n+1}(x) - b_o^2(x-d_o)^2 G_{2n}(x) \quad , \quad n\geq 0 \quad ,$$
$$G_{2n+3}(x) = (x)G_{2n+2}(x) - b_1^2(x-d_1)^2 G_{2n+1}(x) \quad , \quad n\geq 0 \quad . \tag{4.4}$$

The question of the existence of an integrator is now investigated. Condition (i) of Theorem 4.2 is satisfied by requiring $\ell_o > 0$, $\ell_1 > 0$. This leaves condition (ii) of Theorem 4.2. First, conditions so that $G_n(d_{n-1}) \neq 0$, $n\geq 1$, are derived. For $K = [d_o(d_o-a_1)-b_1^2(d_o-d_1)^2]$, it can be shown that $G_{2n}(d_o) = d_o(d_o-a_1)K^{n-1}$, $G_{2n+1}(d_o) = d_oK^n$; so it is necessary to require that $d_o \neq 0$, $K \neq 0$. Similarly for $L = [d_1(d_1-a_1) - b_o^2(d_1-d_o)^2]$, it can be shown that $G_{2n}(d_1) = L^n$, $G_{2n+1}(d_1) = d_1L^n$. So it is necessary that $L \neq 0$ and it is useful to require that $d_1 \neq 0$. Thus the conditions $L \neq 0$, $K \neq 0$, $d_o \neq 0$, $d_1 \neq 0$ are sufficient to guarantee $G_n(d_{n-1}) \neq 0$, $n\geq 1$. Now it is shown that the closure of the set containing the zeros of $G_{2n+1}(x)$, $n\geq 1$, does not contain d_o or d_1 (recall that

$d_{2n} = d_o$, $d_{2n+1} = d_1$). In order to find the zeros of $G_{2n+1}(x)$, $n \geq 0$, let

$$A = \frac{x}{b_o(x-d_o)} \quad ; \quad B = \frac{x-a_1}{b_1(x-d_1)} \quad ; \quad C = \frac{b_1(x-d_1)}{b_o(x-d_o)} \quad ; \quad z = AB-C-\frac{1}{C} \quad . \tag{4.5}$$

Then for $R_o(z) = 1$, $R_1(z) = z$,

$$R_{n+1}(z) = zR_n(z) - R_{n-1}(z) \quad , \quad n \geq 1 \quad , \tag{4.6}$$

it can be shown that

$$\Psi_{2n}(x) = R_n(z)+CR_{n-1}(z) \quad ; \quad \Psi_{2n+1}(x) = AR_n(z) \quad ,$$

$$n \geq 0, \; R_{-1} = 0 \quad . \tag{4.7}$$

The zeros of $R_n(z)$, $n \geq 0$, are well-known (R_n is a Tchebichef polynomial of the second kind), and in fact, $R_n(z)$ has n zeros on the interval $(-2,2)$. Now let

$$m = a_1-2(b_o^2d_o+b_1^2d_1)-b_ob_1z(d_o+d_1) \quad ; \quad s = 2(1-b_o^2-b_1^2-b_ob_1z) \quad ;$$

$$r = 2(-b_o^2d_o^2-b_1^2d_1^2-b_ob_1d_od_1z) \quad ; \quad q = \sqrt{m^2-sr} \quad . \tag{4.8}$$

It follows that for each value of z, $x = (m+q)/s$ or $x = (m-q)/s$. It is easy to see that $q > 0$ on $(-2,2)$ since $-sr > 0$ on $(-2,2)$[1]. Thus there

---

[1] $s>0$ on $(-2,2)$ since $s=2(1-(b_o+b_1)^2+(2-z)b_ob_1)$ and $(b_o+b_1)^2=(\ell_o+\ell_1)^2/[(\ell_o+\ell_1+L_1)(\ell_o+\ell_1+L_2)]\leq 1$. Also $r<0$ on $(-2,2)$ since $r=-2[(b_od_o-b_1d_1)^2 + (2-z)b_ob_1d_od_1]$ $d_od_1>0$ and $r=-2[(b_od_o+b_1d_1)^2+(-2-z)b_ob_1d_od_1]$ for $d_od_1<0$.

are two values of x for each value of z in $(-2,2)$. So by using the fact that $\Psi_{2n+1}(x)$ and $G_{2n+1}(x)$ have the same zeros, it is clear that $G_{2n+1}(x)$ has 2n zeros in the intervals corresponding to z in $(-2,2)$ plus an additional zero at $x=0$ due to the presence of the factor A. Thus in the x-domain, $G_{2n+1}(x)$ has n zeros in the interval with endpoints

$$\frac{a_1-2(b_o+b_1)(b_od_o+b_1d_1)+\sqrt{(a_1-2(b_o+b_1)(b_od_o+b_1d_1))^2+4(1-(b_o+b_1)^2)(b_od_o+b_1d_1)^2}}{2(1-(b_o+b_1)^2)}, \tag{4.9a}$$

$$\frac{a_1-2(b_o-b_1)(b_od_o-b_1d_1)+\sqrt{(a_1-2(b_o-b_1)(b_od_o-b_1d_1))^2+4(1-(b_o-b_1)^2)(b_od_o-b_1d_1)^2}}{2(1-(b_o-b_1)^2)}, \tag{4.9b}$$

and n zeros in the interval with endpoints

$$\frac{a_1-2(b_o+b_1)(b_od_o+b_1d_1)-\sqrt{(a_1-2(b_o+b_1)(b_od_o+b_1d_1))^2+4(1-(b_o+b_1)^2)(b_od_o+b_1d_1)^2}}{2(1-(b_o+b_1)^2)}, \tag{4.10a}$$

$$\frac{a_1-2(b_o-b_1)(b_od_o-b_1d_1)-\sqrt{(a_1-2(b_o-b_1)(b_od_o-b_1d_1))^2+4(1-(b_o-b_1)^2)(b_od_o-b_1d_1)^2}}{2(1-(b_o-b_1)^2)}, \tag{4.10b}$$

plus a zero at $x=0$. The interior of the closed interval corresponding to (4.9) consists of positive values in x while that of (4.10) has negative values. Finally, to see that d_o and d_1 are not in these intervals, note that from (4.5),

$$z = \frac{x(x-a_1) - b_o^2(x-d_o)^2 - b_1^2(x-d_1)^2}{b_o(x-d_o)b_1(x-d_1)} \quad . \tag{4.11}$$

and for $x = d_o$, z is not in the interval $[-2,2]$ unless possibly $d_o(d_o-a_1) - b_1^2(d_o-d_1)^2 = 0$. But this is the same thing as $K = 0$ and it is assumed $K \neq 0$. Similarly $L \neq 0$ guarantees that $x = d_1$ does not yield a z in the interval $[-2,2]$. All requirements for the existence of an integrator have now been met.

By using the results in Chapters II and III concerning periodic continued fractions and integrators, the following theorem results.

<u>Theorem 4.3.</u> Suppose that in (1.1) $r_{2n} = r_o$, $\ell_{2n} = \ell_o$, $r_{2n+1} = r_1$, $\ell_{2n+1} = \ell_1$, $R_{2n+2} = R_2$, $L_{2n+2} = L_2$, $R_{2n+1} = R_1$, $L_{2n+1} = L_1$, for $n \geq 0$ and $R_o = R_2 + r_o$, $L_o = L_2 + \ell_o$. Let b_i, $i \geq 0$, be as in (2.2); let $f = -(R_o+r_1)/(L_o+\ell_1)$, $a_{2n} = 0$, $a_{2n+1} = [-(r_o+R_1+r_1)/(\ell_o+L_1+\ell_1)]-f$, $d_{2n} = (-r_o/\ell_o)-f$, $d_{2n+1} = (-r_1/\ell_1)-f$, $n \geq 0$. Then for $p = -b_1d_1/b_od_o$, $d_o \neq 0$, $d_1 \neq 0$, $d_o(d_o-a_1) \neq b_1^2(d_o-d_1)^2$, $d_1(d_1-a_1) \neq b_o^2(d_1-d_o)^2$, a solution to (1.1) for $x_j(0) = \gamma_j$, $x_k(0) = 0$, $K \neq j$ is (4.2) (with the $\Psi_n(x)$ of (1.3) replaced by those in (4.3), the $W_n(x)$ of (1.3), (1.9) replaced by those in (4.3), (1.9), and e^{xt} replaced by $e^{(x+f)t}$) where $d\beta(x)$ is given by

$$\frac{(4b_o^2(x-d_o)^2 b_1^2(x-d_1)^2 - (x(x-a_1)-b_o^2(x-d_o)^2-b_1^2(x-d_1)^2)^2)^{\frac{1}{2}}dx}{2\pi|x|b_1^2(x-d_1)^2} \tag{4.12}$$

on the intervals determined by the endpoints given in (4.9) and (4.10).

Elsewhere $\beta(x)$ is constant unless $p^2 > 1$, in which case $\beta(x)$ has a jump of strength $(1-(1/p^2))$ at $x = 0$.■

An outline of a proof to Theorem 4.3 appears in Appendix C.

CHAPTER V

EXAMPLES WITHOUT INTEGRATORS (INCLUDING A NON-PERIODIC NETWORK)

This chapter consists primarily of two examples of systems of rational fractions $(\{\Psi_n(x)\}_{n=0}^{\infty}, \{W_n(x)\}_{n=0}^{\infty})$ for which no integrator exists. In each example, the condition that $G_n(d_{n-1}) \neq 0$, $n \geq 1$, is violated. Both examples are treated by the same approach, which consists basically of the following steps:

1. truncate the infinite system (2.1) as in (5.1) below;
2. find the Laplace transforms of $h_i(t)$, $0 \leq i \leq N$;
3. invert these transforms, and separate each of the results into two sums, one which depends on the size of the finite system and one which does not;
4. in the expressions for $h_i(t)$, $0 \leq i \leq N$, found in 3., let N approach infinity;
5. if the $h_i(t)$, $0 \leq i \leq N$, approach limits as N approaches infinity, test to see if these limits are a solution of the infinite system.

Consider the following finite system of differential equations

$$(D-a_o)h_o(t)-b_o(D-d_o)h_1(t)=0 \quad ,$$

$$-b_{n-1}(D-d_{n-1})h_{n-1}(t)+(D-a_n)h_n(t)-b_n(D-d_n)h_{n+1}(t)=0 \ , \ 1\le n\le N-1 \quad , \tag{5.1}$$

$$-b_{N-1}(D-d_{N-1})h_{N-1}(t)+(D-a_N)h_N(t)=0 \quad ,$$

subject to the initial conditions $h_k(0) = \lambda_k$, $h_n(0) = 0$, $n \neq k$. Let $H_n(s)$, $n\ge 0$, denote the Laplace transform of $h_n(t)$. Then it can be shown (solve for $H_o(s)$; use (A.10) extensively in the process; then find $H_n(s)$, $n\ge 1$, iteratively) that

$$\frac{H_n(s)}{\lambda_k} = \Psi_n(s)\{\frac{F_{N+1}(s)}{G_{N+1}(s)} W_k(s) - V_k(s)\} \quad , \quad 0\le n\le k-1 \quad , \tag{5.2a}$$

$$\frac{H_k(s)}{\lambda_k} = \Psi_k(s)\{\frac{F_{N+1}(s)}{G_{N+1}(s)} W_k(s) - V_k(s)\} + b_{k-1}[\Phi_k(s)\Psi_{k-1}(s) - \Phi_{k-1}(s)\Psi_k(s)] \quad , \tag{5.2b}$$

$$= W_k(s)\{\frac{F_{N+1}(s)}{G_{N+1}(s)} \Psi_k(s) - \Phi_k(s)\} + b_k[\Psi_k(s)\Phi_{k+1}(s) - \Psi_{k+1}(s)\Phi_k(s)] \quad , \tag{5.2c}$$

$$\frac{H_n(s)}{\lambda_k} = W_k(s)\{\frac{F_{N+1}(s)}{G_{N+1}(s)} \Psi_n(s) - \Phi_n(s)\} \quad , \quad k+1\le n\le N \quad , \tag{5.2d}$$

where F_n and G_n are given in (2.12), W_n in (1.9), Ψ_n in (1.3),

$$\Phi_o(s) = 0, \quad \Phi_n(s) = \frac{F_n(s)}{b_o(s-d_o)b_1(s-d_1)\ \ldots\ b_{n-1}(s-d_{n-1})} \quad , \quad n\geq 1 \quad , \tag{5.3}$$

and

$$V_o(s) = -b_o\Phi_1(s)+\Phi_o(s), \quad V_n(s) = -b_n\Phi_{n+1}(s)+\Phi_n(s)-b_{n-1}\Phi_{n-1}(s) \quad , \quad n\geq 1 \ . \tag{5.4}$$

The first two steps of the procedure outlined above are now complete. As an aside it is interesting to see what $h_n(t)$ becomes, from these equations, when it is assumed that $G_{N+1}(d_i) \neq 0$, $0\leq i\leq N$. Let the zeros of $G_{N+1}(s)$ be u_i, $0\leq i\leq N$. Then the solution to (5.1) can be shown to be

$$h_n(t) = \lambda_k \sum_{i=o}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} [\Psi_n(u_i)W_k(u_i)]e^{u_i t} \quad , \quad 0\leq n\leq N \ . \tag{5.5}$$

Not surprisingly, if $\beta_{N+1}(x)$ is chosen so that

$$\frac{F_{N+1}(z)}{G_{N+1}(z)} = \int_{-\infty}^{\infty} \frac{d\beta_{N+1}(x)}{z-x} \quad , \tag{5.6}$$

then (5.5) can be written as

$$h_n(t) = \lambda_k \int_{-\infty}^{\infty} \Psi_n(x)W_k(x)e^{xt}d\beta_{N+1}(x) \quad , \quad 0\leq n\leq N \ . \tag{5.7}$$

Finally note that under the conditions of Theorem 2.9, this solution to the finite system (5.1) is the same as the solution to the infinite system (2.1), for the same initial conditions, except that in the solution to (2.1), $\beta_{N+1}(x)$ is replaced by $\beta(x)$ which is obtained as a limit of a subsequence of the sequence $\{\beta_{N+1}(x)\}_{N=0}^{\infty}$ (see Theorem 2.7).

Now the following example indicates how (5.2) can be employed to suggest a solution to an infinite system for which no integrator exists. Consider the system of (2.1) with $R_o = 2$, $L_o = 4$, $R_{n+1} = L_{n+1} = r_n = 1$, $\ell_n = 2$, $n \geq 0$. Then $b_o = \sqrt{2/15}$, $b_n = 2/5$, $d_{n-1} = -1/2$, $a_o = -1/2$, $a_{n-1} = -3/5$, $n \geq 1$. The G_n's (see (2.12)) obtained for this system are given by

$$\begin{aligned} G_1(x) &= (x+\tfrac{1}{2})G_o(x) \quad , \\ G_2(x) &= (x+\tfrac{3}{5})G_1(x) - \tfrac{2}{15}(\tfrac{1}{2}+x)^2 G_o(x) \quad , \\ G_{n+1}(x) &= (x+\tfrac{3}{5})G_n(x) - \tfrac{4}{25}(\tfrac{1}{2}+x)^2 G_{n-1}(x) \quad , \quad n \geq 2 \quad . \end{aligned} \tag{5.8}$$

Note that $d_n = -1/2$, $n \geq 0$, and $G_n(d_{n-1}) = 0$ for all $n \geq 1$. There are three cases to consider in determining the solution to this system, namely $k=0$ ($h_o(0) = \lambda_o$), $k=1$, and $k \geq 2$.

<u>Case 1: k=0.</u>

If $k=0$, it is easily calculated that $W_o(s) = 0$. If the infinite system is truncated in the fashion of (5.1), it is easy to see from (5.2) that a solution to the finite system for any N is

$$\begin{aligned} h_o(t) &= \lambda_o e^{-t/2} \quad , \\ h_n(t) &= 0 \quad , \quad 1 \leq n \leq N \quad . \end{aligned} \tag{5.9}$$

Note that no part of (5.9) depends on N; hence, if a limit is taken on

N, it might be suspected that the solution to the infinite system is also (5.9). Direct substitution verifies that this is the case.

Case 2: k=1.

First find $h_o(t)$ for the system truncated as above. Note that (5.2a) for n=0 becomes

$$\frac{H_o(s)}{\lambda_k} = -b_1\{\frac{F_{N+1}(s)G_2(s)-F_2(s)G_{N+1}(s)}{G_{N+1}(s)}\} \frac{1}{b_o(s-d_o)b_1(s-d_1)}$$
$$+ \{\frac{F_{N+1}(s)G_1(s)-F_1(s)G_{N+1}(s)}{G_{N+1}(s)}\} \frac{1}{b_o(s-d_o)} \tag{5.10}$$
$$-b_o\{\frac{F_{N+1}(s)G_o(s)-F_o(s)G_{N+1}(s)}{G_{N+1}(s)}\} \quad .$$

But from (A.10), $[F_{N+1}(s)G_j(s)-F_j(s)G_{N+1}(s)]/[G_{N+1}(s)b_o(s-d_o)b_1(s-d_1)\ldots b_{j-1}(s-d_{j-1})] = b_o(s-d_o)b_1(s-d_1)\ldots b_{j-1}(s-d_{j-1})X_{j,N+1-j}(s)/G_{N+1}(s)$. Thus the Heaviside inversion formula can be applied to (5.10) over the zeros of $G_{N+1}(s)$, one of which is $s=d_N=d_o$. Let u_i, $1\le i\le N$, denote the zeros of $G_{N+1}(s)$ with $u_o=d_o=d_N$. Under these conditions, it follows that

$$\frac{h_o(t)}{\lambda_k} = \sum_{i=1}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} (-b_1G_2(u_i)+G_1(u_i)-b_oG_o(u_i))e^{u_it}$$
$$-b_o \frac{X_{o,N+1}(u_o)e^{-t/2}}{G'_{N+1}(u_o)} \tag{5.11}$$

$$= \sum_{i=1}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} [W_1(u_i)]e^{u_it} -b_o \frac{F_{N+1}(d_o)}{G'_{N+1}(d_o)} e^{-t/2} \quad .$$

It is easily calculated, for this example, that $F_{N+1}(d_o)/G'_{N+1}(d_o) = 1$. Thus the terms under the summation sign are dependent on N, but not the term $b_o e^{-t/2}$. If a limit in N is now taken, it might be suspected that

$$\frac{h_o(t)}{\lambda_1} = \int_{-\infty}^{\infty} W_1(x)\Psi_o(x)e^{xt}d\beta(x) - \sqrt{2/15}\, e^{-t/2} \quad , \tag{5.12}$$

where $\beta(x)$ is some, as of yet, unknown function. In order to find $\beta(x)$, consider the sequence $\{\beta_N(x)\}_{N=1}^{\infty}$ formed from defining $\beta_{N+1}(x)$ by

$$\frac{F_{N+1}(z)}{G_{N+1}(z)} = \int_{-\infty}^{\infty} \frac{d\beta_{N+1}(x)}{z-x} \quad . \tag{5.13}$$

Notice in particular that

$$\begin{aligned}\frac{F_{N+1}(z)}{G_{N+1}(z)} - \frac{1}{z-d_o} &= \int_{-\infty}^{\infty} \frac{d\beta_{N+1}(x)}{z-x} - \frac{1}{z-d_o} \\ &= \sum_{i=o}^{N} \int_{-\infty}^{\infty} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} \frac{\delta(x-u_i)dx}{z-x} - \frac{1}{z-d_o} \\ &= \sum_{i=1}^{N} \int_{-\infty}^{\infty} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} \frac{\delta(x-u_i)dx}{z-x}\end{aligned} \tag{5.14}$$

and so

$$\frac{h_o(t)}{\lambda_1} = \int_{-\infty}^{\infty} W_1(x)\Psi_o(x)e^{xt}\left[\sum_{i=1}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)}\,\delta(x-u_i)\right]dx \quad . \tag{5.15}$$

Thus (see Chapter II), it is possible that $\beta(x)$ is given by the inverse Stieltjes transform of $\lim_{N\to\infty}[(F_{N+1}(z)/G_{N+1}(z))-1/(z-d_o)]$. By referring to the techniques of Chapter IV, it can be shown that the $\beta(x)$ obtained in this manner satisfies

$$d\beta(x) = \frac{9\sqrt{(\frac{2}{9})^2 - (x+\frac{7}{9})^2}\,dx}{\pi(5x+1)(2x+1)} \quad , \quad x\epsilon[-1,-\frac{5}{9}] \quad . \tag{5.16}$$

It can also be shown, for the infinite system, that

$$h_o(t) = \lambda_1 \int_{-1}^{-\frac{5}{9}} W_1(x)\Psi_o(x)e^{xt}d\beta(x) - \sqrt{2/15}\, e^{-t/2} \quad . \tag{5.17}$$

For k=1, (5.2b) becomes

$$\frac{H_1(s)}{\lambda_1} = -b_1\Psi_1(s)[\frac{F_{N+1}(s)}{G_{N+1}(s)}\Psi_2(s)-\Phi_2(s)]+\Psi_1(s)[\frac{F_{N+1}(s)}{G_{N+1}(s)}\Psi_1(s)-\Phi_1(s)]$$

$$-b_o\Psi_o(s)[\frac{F_{N+1}(s)}{G_{N+1}(s)}\Psi_1(s)-\Phi_1(s)] \quad ; \tag{5.18}$$

so

$$h_1(t) = \lambda_1 \sum_{i=1}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} e^{u_i t} [-b_1\Psi_1(u_i)\Psi_2(u_i)+\Psi_1^2(u_i)-b_o\Psi_1(u_i)]$$

$$= \lambda_1 \sum_{i=1}^{N} \frac{F_{N+1}(u_i)}{G'_{N+1}(u_i)} e^{u_i t} W_1(u_i)\Psi_1(u_i) \quad . \tag{5.19}$$

Again for the infinite system it can be shown that for $d\beta(x)$ as given in (5.16),

$$h_n(t) = \lambda_1 \int_{-1}^{-\frac{5}{9}} W_1(x)\Psi_n(x)e^{xt}d\beta(x) \quad , \quad n\geq 1 \quad . \tag{5.20}$$

Case 3: k≥2.

By using the same approach as above, it can be shown that

$$h_n(t) = \lambda_k \int_{-1}^{-\frac{5}{9}} W_k(x)\Psi_n(x)e^{xt}d\beta(x) \quad , \quad n\geq 0 \quad . \tag{5.21}$$

This concludes the first example.

Now consider a slightly more complicated example where the d_i, $i\geq 0$, are all distinct, but $G_n(d_{n-1})$ is still zero for all $n\geq 1$. First note that if q_i denotes the leading coefficient of $G_i(x)$ as in the proof of Theorem 2.5, then

$$G_n(x) = q_n(x-d_o)(x-d_1)\ .\ .\ .\ (x-d_{n-1}) \tag{5.22}$$

and thus

$$q_{n+1} = q_n - b_{n-1}^2 q_{n-1} \quad , \quad n\geq 0,\ q_{-1} = 0 \quad . \tag{5.23}$$

Further note then that

$$\Psi_n(x) = \frac{q_n}{b_o b_1 \cdots b_{n-1}} \quad , \quad n\geq 0 \quad , \tag{5.24}$$

and hence

$$W_n(x) = \frac{-b_n q_{n+1}}{b_o b_1 \cdots b_n} + \frac{q_n}{b_o b_1 \cdots b_{n-1}} - \frac{b_{n-1} q_{n-1}}{b_o b_1 \cdots b_{n-2}} = 0 \quad , \quad n\geq 0 \quad , \tag{5.25}$$

from (5.23). Thus (5.2) becomes

$$\frac{H_n(s)}{\lambda_k} = -V_k(s)\Psi_n(s) \quad , \quad 0 \le n \le k-1 \quad , \tag{5.26a}$$

$$\frac{H_k(s)}{\lambda_k} = -V_k(s)\Psi_k(s) + b_{k-1}[\Phi_k(s)\Psi_{k-1}(s) - \Phi_{k-1}(s)\Psi_k(s)] \tag{5.26b}$$

$$= b_k[\Psi_k(s)\Phi_{k+1}(s) - \Psi_{k+1}(s)\Phi_k(s)] \tag{5.26c}$$

$$\frac{H_n(s)}{\lambda_k} = 0 \quad , \quad k+1 \le n \le N. \tag{5.26d}$$

In order to find $h_n(t)$, $0 \le n \le k$, it remains to find $F_n(s)$ for $s = d_i$, $0 \le i \le n-1$. To this end, note that from (1.3) and (5.24),

$$d_n q_{n+1} = a_n q_n - b_{n-1}^2 d_{n-1} q_{n-1} \quad , \quad n \ge 0 \quad , \tag{5.27}$$

or

$$a_n = \frac{d_n q_{n+1} + b_{n-1}^2 d_{n-1} q_{n-1}}{q_n} \quad , \quad n \ge 0 \quad . \tag{5.28}$$

By using (5.28) and (5.23), it can be shown that

$$F_{n+1}(d_j) = b_o^2 b_1^2 \ldots b_{j-1}^2 (d_j - d_o)(d_j - d_1) \ldots (d_j - d_{j-1})(d_j - d_{j+1}) \ldots (d_j - d_n) \cdot \frac{q_{n+1}}{q_j q_{j+1}} \quad , \quad j \le n \quad . \tag{5.29}$$

An example of the calculations leading to (5.29) follows (for $j=n$). From (2.12) $F_{n+1}(d_n) = (d_n - a_n)F_n(d_n) - b_{n-1}^2(d_n - d_{n-1})^2 F_{n-1}(d_n)$ or, by using (5.28) to replace a_n,

$$F_{n+1}(d_n) = \frac{b_{n-1}^2(d_n-d_{n-1})}{q_n}(q_{n-1}F_n(d_n) - q_n(d_n-d_{n-1})F_{n-1}(d_n)) \quad . \tag{5.30}$$

In a similar fashion,

$$q_{n-1}F_n(d_n)-(d_n-d_{n-1})q_nF_{n-1}(d_n) =$$

$$b_{n-2}^2(d_n-d_{n-2})[q_{n-2}F_{n-1}(d_n)-b_{n-2}^2q_{n-1}(d_n-d_{n-2})F_{n-2}(d_n)] \quad . \tag{5.31}$$

By putting (5.31) into (5.30), $F_{n+1}(d_n)$ becomes

$$F_{n+1}(d_n) = \left(\frac{b_{n-1}^2 b_{n-2}^2(d_n-d_{n-1})(d_n-d_{n-2})q_{n+1}}{q_nq_{n+1}}\right)$$

$$\cdot\ [q_{n-2}F_{n-1}(d_n)-b_{n-2}^2q_{n-1}(d_n-d_{n-2})F_{n-2}(d_n)] \quad . \tag{5.32}$$

Induction readily establishes the desired result. $F_{n+2}(d_n)$ is then easily found from $F_{n+1}(d_n)$ and similarly for other values.

Now return to (5.26a). For $k \geq 1$,

$$\frac{H_n(s)}{\lambda_k} = \frac{q_n}{b_ob_1 \cdots b_{n-1}} \left[\frac{b_kF_{k+1}(s)}{b_o(s-d_o)b_1(s-d_1) \cdots b_k(s-d_k)}\right.$$

$$- \frac{F_k(s)}{b_o(s-d_o)b_1(s-d_1) \cdots b_{k-1}(s-d_{k-1})}$$

$$\left. + \frac{b_{k-1}F_{k-1}(s)}{b_o(s-d_o)b_1(s-d_1) \cdots b_{k-2}(s-d_{k-2})}\right] \quad . \tag{5.33}$$

Invert (5.33) by using the Heaviside formula and the fact (from (5.29)) that $F_{k+1}(s)-(s-d_k)F_k(s)-b_{k-1}(s-d_{k-1})(s-d_k)F_{k-1}(s) = 0$ for $s=d_i$, $0\le i\le k-2$. The remaining contributions (from $s=d_{k-1},d_k$) yield

$$\frac{h_n(t)}{\lambda_k} = \frac{q_n}{b_o b_1 \cdots b_{n-1}} \{ [\frac{b_k b_o^2 b_1^2 \cdots b_{k-2}^2 q_{k+1}}{b_o b_1 \cdots b_k q_{k-1} q_k}$$

$$- \frac{b_o^2 b_1^2 \cdots b_{k-2}^2 q_k}{b_o b_1 \cdots b_{k-1} q_{k-1} q_k}] e^{d_{k-1}t} + [\frac{b_k b_o^2 b_1^2 \cdots b_{k-1}^2 q_{k+1}}{b_o b_1 \cdots b_k q_k q_{k+1}}] e^{d_k t} \}$$

$$= \frac{q_n b_o b_1 \cdots b_{k-1}}{q_k b_o b_1 \cdots b_{n-1}} [e^{d_k t} - e^{d_{k-1} t}] \quad , \quad 0\le n\le k-1 \quad . \tag{5.34}$$

For n=k, it is easily seen from (5.26c) that

$$\frac{h_k(t)}{\lambda_k} = e^{d_k t} \quad . \tag{5.35}$$

This concludes the second example. That such an example can arise from Figure 1 with positive circuit parameters is illustrated by the choices $r_n = 4^{n+1}/(4^{n+1}-1)$, $\ell_n = 2$, $L_n = 1$, $R_o = 2/3$, $R_{n+1} = 4^{n+1}/(1+2(4)^{n+1})$, $n\ge 0$. Define B_n, $n\ge 0$, by $B_n = (1+2(4)^n)/3$. Then for $x_k(0) = \gamma_k$, $x_n = 0$, $n\ne k$, a solution to (1.1) is

$$x_n(t) = \gamma_k \frac{\ell_n \ell_{n+1} \cdots \ell_{k-1} B_n}{B_k} (e^{-r_k t/\ell_k} - e^{-r_{k-1} t/\ell_{k-1}}) \quad , \quad n<k ,$$

$$x_k(t) = \gamma_k e^{-r_k t/\ell_k} \quad ,$$

$$x_n(t) = 0 \quad , \quad n>k \quad . \tag{5.36}$$

CHAPTER VI

UNIQUENESS

The solutions (3.2) and (4.2) to the system (1.1) with the initial conditions (1.7) are not unique. In fact, there are an uncountable number of solutions, since any function in $C_1[0,\infty)$ satisfying the initial condition on $x_o(t)$ can be used to generate $x_i(t)$, $i\geq 1$. This chapter is devoted to determining conditions sufficient to guarantee that (4.2) is a unique solution of (1.1) (the techniques employed are similarly applicable to (3.2)). The procedure separates into two parts. In the first part it is shown that if a solution exists satisfying (1.1), (1.7) and the additional restrictions to be derived below, such a solution is unique. The second part determines conditions sufficient to guarantee that (4.2) satisfies the restrictions derived in the first part.

Part I

In line with similar investigations of countable systems [3], it is assumed that each $x_n(t)$ is expansible in a Maclaurin's series. The values $x_n(0)$ are assumed given since they are the initial loop currents. Then, if the value $x_n^{(k)}(0)$, the k^{th} derivative of $x_n(t)$ at $t=0$, is uniquely determined for each k, $x_n(t)$ is unique. It is shown below that two additional restrictions are sufficient to guarantee this uniqueness.

With the intent of eventually solving for $x_n^{(k+1)}(0)$, $n\geq 1$, in terms of $x_i^{(k)}(0)$, $i\geq 0$, and $x_o^{(k+1)}(0)$ (see (6.6)), differentiate (2.1) k times and let $\dot{v}_n(t) = h_n^{(k+1)}(t)$, $v_n(t) = h_n^{(k)}(t)$, $n\geq 0$. For $t=0$, it follows that

$$-b_o\dot{v}_1 + \dot{v}_o = av_o - b_o d_o v_1 \quad ,$$

$$-b_n\dot{v}_{n+1}+\dot{v}_n-b_{n-1}\dot{v}_{n-1} = a_n v_n-b_n d_n v_{n+1}-b_{n-1}d_{n-1}v_{n-1} \quad , \quad n\geq 1 \quad . \tag{6.1}$$

The equations in (6.1) are more amenable to manipulation and hence to solution if the changes $v_o(t) = t_o(t)$, $T_{-1}(t) = -(a_o v_o(t)-b_o d_o v_1(t))$, $v_n(t) = t_n(t)/(b_o b_1 \ldots b_{n-1})$, and $T_{n-1}(t) = -(b_o b_1 \ldots b_{n-1})(-b_n d_n v_{n+1}(t) -b_{n-1}d_{n-1}v_{n-1}(t)+a_n v_n(t))$, $n\geq 1$, are made. Then (6.1) becomes, for $t=0$,

$$\begin{aligned} \dot{t}_1 - \dot{t}_o &= T_{-1} \\ \dot{t}_{n+1}-\dot{t}_n+b_{n-1}^2\dot{t}_{n-1} &= T_{n-1} \quad , \quad n\geq 1 \quad . \end{aligned} \tag{6.2}$$

Further, since $\{b_n^2\}_{n=o}^{\infty}$ is a chain sequence, recall from Lemma A.1 that $0 = m_{-1}$, $b_o^2 = m_o$, $b_{n-1}^2 = m_{n-1}(1-m_{n-2})$, $n\geq 2$, where the m_n are the minimal parameters of Lemma A.1. Then, in the interest of further simplification, if the substitutions $u_o(t) = t_1(t)-t_o(t)$ and $u_n(t) = (t_{n+1}(t) -(1-m_{n-1})t_n(t))$, $n\geq 1$, are made, (6.1) via (6.2) becomes, for $t=0$,

$$\dot{u}_n-m_{n-1}\dot{u}_{n-1} = T_{n-1} \quad , \quad n\geq 0 \quad . \tag{6.3}$$

This is a difference equation amenable to standard solution techniques and in fact has solution

$$\dot{u}_n(0) = \left(\prod_{i=o}^{n-1} m_i\right)\left(T_{-1}(0) + \sum_{i=o}^{n-1} \frac{T_i(0)}{\prod_{j=o}^{i} m_j}\right) \quad , \quad n\geq 1 \quad . \tag{6.4}$$

Now it is necessary to solve the difference equation, for $t=0$; $\dot{t}_{n+1}-(1-m_{n-1})\dot{t}_n = \dot{u}_n$ (with initial condition $\dot{t}_1 = \dot{u}_o+\dot{t}_o$) since $\dot{t}_n(0)$ then gives $\dot{v}_n(0)$ which then gives $h_n^{(k+1)}(0)$ which then gives $x_n^{(k+1)}(0)$. By standard techniques, it follows that

$$\dot{t}_n(0) = \left[\prod_{i=o}^{n-1} (1-m_{i-1})\right]\left[\dot{t}_o(0) + \sum_{i=o}^{n-1} \frac{\dot{u}_i(0)}{\prod_{j=o}^{i} (1-m_{j-1})}\right] \quad , \quad n\geq 1 \quad , \tag{6.5}$$

whence

$$x_n^{(k+1)}(0) = \left(\frac{L_o+\ell_o}{\ell_{n-1}+L_n+\ell_n}\right)^{\frac{1}{2}} \frac{1}{(1-m_{n-1})^{\frac{1}{2}}}\left[\prod_{i=o}^{n-1} \left(\frac{1-m_i}{m_i}\right)^{\frac{1}{2}}\right]\left[x_o^{(k+1)}(0)\right.$$

$$\left. + \sum_{i=o}^{n-1} \frac{\dot{u}_i(0)}{\prod_{j=o}^{i} (1-m_{j-1})}\right] \quad , \quad n\geq 1 \quad . \tag{6.6}$$

Thus, in general, $x_n^{(k+1)}(0)$ for $n\geq 1$, is uniquely determined in terms of a_i, b_i, d_i, $x_i^{(k)}(0)$ for $i\geq 0$, and $x_o^{(k+1)}(0)$. The difficulty is that, as of yet, $x_o^{(k+1)}(0)$ is undetermined. But if $\{x_n^{(k+1)}(0)\}_{n=1}^{\infty}$ contains a subsequence which converges to zero, it is shown below that the existence of such a subsequence is sufficient to determine $x_o^{(k+1)}(0)$ uniquely and hence each $x_n^{(k+1)}(0)$, $n\geq 1$. Before proceeding, however, some preliminary information is required.

<u>Lemma 6.1.</u> Let $\{b_n^2\}_{n=o}^{\infty}$ be the chain sequence of Lemma 4.1. Then

$$m_n \le \frac{\ell_n}{\ell_n+L_{n+1}+\ell_{n+1}} \quad , \quad \frac{1}{1-m_n} \le \frac{\ell_n+L_{n+1}+\ell_{n+1}}{\ell_{n+1}} \quad , \quad n\ge 0 \quad .\blacksquare \tag{6.7}$$

Proof. Follows immediately by induction.■

Theorem 6.1. Let $\{b_n^2\}_{n=o}^{\infty}$ be as in Lemma 4.1, and suppose a solution to (1.1) exists with the properties

(i) $x_n(t)$ is expansible in a Maclaurin's series;

(ii) each sequence $\{x_n^{(k)}(0)\}_{n=1}^{\infty}$, $k\ge 1$, contains a subsequence which converges to zero.

Then the solution to (1.1) is unique.■

Proof. Upon referring to (6.6), it is seen that if the coefficient $[(L_o+\ell_o)/(\ell_{n-1}+L_n+\ell_n)]^{\frac{1}{2}} \cdot 1/(1-m_{n-1})^{\frac{1}{2}} \cdot \prod_{i=o}^{n-1} ((1-m_i)/m_i)^{\frac{1}{2}}$ were bounded away from zero for all $n\ge 1$, it would follow that $x_o^{(k+1)}(0) = -\sum_{i=o}^{\infty} \dot{u}_i(0)/(\prod_{j=o}^{i}(1-m_{j-1}))$. Thus $x_o^{(k+1)}(0)$ would be uniquely determined in terms of a_i, b_i, d_i, and $x_i^{(k)}(0)$, $i\ge 0$; and the theorem would be proved. But by using Lemma 6.1,

$$\left(\frac{L_o+\ell_o}{\ell_{n-1}+L_n+\ell_n}\right)^{\frac{1}{2}} \frac{1}{(1-m_{n-1})^{\frac{1}{2}}} \prod_{i=o}^{n-1} \left(\frac{1-m_i}{m_i}\right)^{\frac{1}{2}} \ge \left(1+\frac{L_o}{\ell_o}\right)^{\frac{1}{2}} > 0 .\blacksquare \tag{6.8}$$

Notice that no assumption concerning the number of non-zero initial loop currents is made.

Part II

In this section sufficient conditions are determined so that the solution (4.2) exhibited in Theorem 4.2 has the desired property that each sequence $\{x_n^{(k)}(0)\}_{n=1}^{\infty}$, $k\geq 1$, contains a subsequence which converges to zero. The elements in the sequence $\{x_n^{(k)}(0)\}_{n=1}^{\infty}$ are obtained by differentiating (4.2) k times and evaluating at $t=0$. Parenthetically, one may observe that successively differentiating under the integral sign and then setting $t=0$ in the resulting integrands yields the required one-sided derivatives at $t=0$ provided, as is assumed in Theorem 6.2, that $\left|\int_{-\infty}^{\infty} x^r d\beta(x)\right| < \infty$, $r\geq 1$. See Appendix D for a further discussion. The result of these operations is

$$x_n^{(k)}(0) = \gamma_j(\ell_{j-1}+L_j+\ell_j)\int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}} x^k d\beta(x) \quad ,$$

$$n\geq 0 \quad . \tag{6.9}$$

The approach to be followed from this point is fairly simple: first, orthonormalize the sequence $\{W_n(x)\}_{n=o}^{\infty}$ to obtain a new sequence $\{\phi_n(x)\}_{n=o}^{\infty}$; then, expand $\Psi_n(x)$ in terms of the $\phi_i(x)$, $i\geq 0$; verify that the coefficients in this expansion are square summable; replace $\Psi_n(x)$ in (6.9) by its expansion; interchange the resulting summation and integral signs; finally, apply Bessel's inequality to the function $x^k W_j(x)$ and obtain, after some algebra, either $\lim_{i\to\infty}(x_i^{(k)}(0))[(1-2b_i)(\ell_{i-1}+L_i+\ell_i)^{\frac{1}{2}}] = 0$ or $\lim_{i\to\infty}(x_i^{(k)}(0))(\ell_{i-1}+L_i+\ell_i)^{\frac{1}{2}} = 0$.

The key to this investigation is the orthonormalizing of the sequence $\{W_n(x)\}_{n=o}^{\infty}$. To this end, define a new sequence $\{\phi_n(x)\}_{n=o}^{\infty}$ by $\phi_o(x) = W_o(x)$ and

$$\phi_i(x) = \frac{W_i(x) - \sum_{k=o}^{i-1} \phi_k(x)\int_{-\infty}^{\infty} \phi_k(s)W_i(s)d\beta(s)}{\left(\int_{-\infty}^{\infty} (W_i(t) - \sum_{k=o}^{i-1} \phi_k(t)\int_{-\infty}^{\infty} \phi_k(s)W_i(s)d\beta(s))^2 d\beta(t)\right)^{\frac{1}{2}}} \quad , \quad i\geq 1 \quad . \tag{6.10}$$

Then it is easily verified that $\int_{-\infty}^{\infty} \phi_i(x)\phi_j(x)d\beta(x) = \delta_{ij}$, $i,j\geq 0$. Note that (6.10), together with the definition $\phi_o(x) = W_o(x)$, actually says that $\phi_i(x)$, $i\geq 0$, can be written as a linear combination of $W_k(x)$, $0\leq k\leq i$; that is, $\phi_i(x) = \sum_{k=o}^{i} a_{ik}W_k(x)$, $i\geq 0$. In expanding $\Psi_n(x)$ in terms of the $\phi_i(x)$, some additional information about the coefficients a_{ik} is needed. This information is now obtained.

Lemma 6.2. Let m_i, $i\geq -1$, be the minimal parameter sequence (obtained from Lemma A.1) for the chain sequence $\{b_i^2\}_{i=o}^{\infty}$. Then

$$a_{ik} = \left(\frac{1}{(1-m_{k-1})} \prod_{j=k}^{i-1} \frac{m_j}{(1-m_j)}\right)^{\frac{1}{2}} \quad , \quad 0\leq k\leq i-1 \quad ,$$

$$a_{ii} = \frac{1}{(1-m_{i-1})^{\frac{1}{2}}} \quad , \tag{6.11}$$

$$a_{ik} = 0 \quad , \quad k>i \quad . \blacksquare$$

Proof. Let $s_o=1$ and define

$$s_n = \frac{1|}{|1} - \frac{b_{n-1}^2|}{|1} - \frac{b_{n-2}^2|}{|1} - \cdots - \frac{b_1^2|}{|1} - b_o^2 \quad , \quad n\geq 1 \quad . \tag{6.12}$$

Then it is possible to verify directly that

$$\phi_m(x) = s_m^{\frac{1}{2}}\left[\sum_{j=o}^{m-1}\left(\prod_{k=j}^{m-1} b_k s_k\right)W_j(x)+W_m(x)\right] \quad , \quad m\geq 0 \tag{6.13}$$

(to do so, substitute (6.13) for $0\leq m\leq i-1$ into the right side of (6.10); perform the indicated integrations; show that the result is (6.13) for $m=i$). The above verification is facilitated by noting that the s_n, $n\geq 1$, satisfy the difference equation

$$s_n = \frac{1}{1-b_{n-1}^2 s_{n-1}} \quad , \quad n\geq 1 \quad , \tag{6.14}$$

with initial condition $s_o=1$. With $q_o=q_1=1$, $s_n=q_n/q_{n+1}$, $n\geq 0$, (6.14) becomes

$$q_{n+1} = q_n - b_{n-1}^2 q_{n-1} \quad , \quad n\geq 1 \quad . \tag{6.15}$$

Equation (6.15) is solved in Appendix A in the proof of Theorem 2.5 and in fact $q_n = \prod_{i=-1}^{n-2} (1-m_i)$, $n\geq 1$. The conclusion follows by substitution.■

This completes the first step listed above.

For future reference, denote the matrix of coefficients a_{ik} by $[a]_w$.

Once the sequence $\{\phi_n(x)\}_{n=o}^{\infty}$ is acquired it is fairly easy to write each $\phi_n(x)$, $n\geq 0$, as a linear combination of $\Psi_i(x)$, $i\geq 0$, simply by using (1.9) and (6.13), but the inverse problem is slightly more complicated. The method of attack employed here is first to write each $W_i(x)$, $i\geq 0$, as a linear combination of $\phi_n(x)$, $n\geq 0$, then to use this result to solve for $\phi_n(x)$, $n\geq 0$, in terms of $\Psi_j(x)$, $j\geq 0$, and finally to use the insight developed through this procedure to invert the relation between $\phi_n(x)$ and $\Psi_j(x)$.

Lemma 6.3. Let $\phi_o(x) = W_o(x)$; let $\phi_n(x)$, $n\geq 1$, be as in (6.10); let m_i, $i\geq -1$, be as in Lemma 6.2. Then

$$W_i(x) = (1-m_{i-1})^{\frac{1}{2}}\phi_i(x) - m_{i-1}^{\frac{1}{2}}\phi_{i-1}(x) \quad , \quad i\geq 0 \quad .\blacksquare \tag{6.16}$$

Proof. Follows by direct substitution in (6.13).■

Thus each $W_i(x)$, $i\geq 0$, has now been written as a linear combination of $\phi_n(x)$, $n\geq 0$. Next each $\phi_i(x)$, $i\geq 0$, must be found as a linear combination of $\Psi_j(x)$, $j\geq 0$. For this endeavor, some additional notation is required. Denote the infinite matrix of coefficients in (6.16) by $[b]_w$ with transpose $[b]_w^T$. Let (Ψ), (ϕ), and (W) represent the infinite column of vectors of $\Psi_i(x)$, $\phi_i(x)$, and $W_i(x)$, $i\geq 0$. Then

$$[b]_w[b]_w^T(\Psi) = (W) \tag{6.17}$$

because

$$[b]_w[b]_w^T = \begin{bmatrix} 1 & -b_o & 0 & \cdots \\ -b_o & 1 & -b_1 & \cdots \\ 0 & -b_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} , \tag{6.18}$$

as direct multiplication shows. Since from (6.16), $[b]_w(\phi) = (W)$, it might be suspected from (6.17) that $[b]_w^T(\Psi) = (\phi)$.

Lemma 6.4. $[b]_w^T(\Psi) = (\phi)$.■

Proof. Direct substitution.■

Now employ the insight garnered thus far to invert the relation of Lemma 6.4. Specifically, recall that $[a]_w(W) = (\phi)$ and $[b]_w(\phi) = (W)$ which seems to imply that $[a]_w$ is an inverse for $[b]_w$ or, more importantly, that $[a]_w^T$ is an inverse for $[b]_w^T$.

Lemma 6.5. $[a]_w^T$ is both a left-hand and a right-hand inverse for $[b]_w^T$.■

Proof. By direct multiplication.■

As a result of Lemmas 6.4 and 6.5,

$$\phi_i(x) = (1-m_{i-1})^{\frac{1}{2}}\Psi_i(x) - m_i^{\frac{1}{2}}\Psi_{i+1}(x) \quad , \quad i\geq 0 \quad , \tag{6.19}$$

and $(\Psi) = [a]_w^T(\phi)$ or

$$\Psi_i(x) = \frac{1}{(1-m_{i-1})^{\frac{1}{2}}}[\phi_i(x) + \sum_{j=i+1}^{\infty} \prod_{k=i}^{j-1} (\frac{m_k}{1-m_k})^{\frac{1}{2}} \phi_j(x)] \quad , \quad i\geq 0 \quad . \tag{6.20}$$

This completes the second step.

Now it must be shown that the coefficients in (6.20) are square summable. In fact, the sum of the squares of these coefficients becomes

$$\frac{1}{(1-m_{i-1})}[1 + \sum_{j=i+1}^{\infty} \prod_{k=i}^{j-1} (\frac{m_k}{1-m_k})] = \frac{1}{M_{i-1}-m_{i-1}} \quad , \quad i\geq 0 \quad , \tag{6.21}$$

from [18] where M_{i-1} is a maximal parameter for $\{b_i^2\}_{i=o}^{\infty}$; and since $M_i > m_i$, for all $i\geq -1$ (see Chapter IV), the coefficients are indeed square summable. Thus, the third step is complete. This step is important because it justifies the interchange of an integration and a summation with regard to these coefficients in the proof of the following theorem which encompasses the remaining steps.

Theorem 6.2. Suppose that $\left|\int_{-\infty}^{\infty} x^r d\beta(x)\right| < \infty$, $r\geq 0$, and that no subsequence of either $\{(1-2b_i)\}_{i=0}^{\infty}$ or $\{(\ell_{i-1}+L_i+\ell_i)\}_{i=0}^{\infty}$ has limit zero. Then the solution (4.2) to (1.1), (1.7) is unique.■

Proof. It must be shown that there exists a subsequence of the sequence $\{x_n^{(k)}(0)\}_{n=0}^{\infty}$ which converges to zero. From Naylor and Sell [12], with b_{ij} denoting the entries in a matrix formed from (6.20) (that is, $[a]_w^T$),

$$\int_{-\infty}^{\infty} \Psi_i(x) W_\ell(x) x^k d\beta(x) = \sum_{j=i}^{\infty} b_{ij} \int_{-\infty}^{\infty} \phi_j(x) W_\ell(x) x^k d\beta(x) \quad . \tag{6.22}$$

Let $c_i = \int_{-\infty}^{\infty} \phi_i(x) W_\ell(x) x^k d\beta(x)$. Since it can be shown that $\left|\int_{-\infty}^{\infty} x^r d\beta(x)\right| < \infty$, $r\geq 0$, implies $\int_{-\infty}^{\infty} W_\ell^2(x) x^{2k} d\beta(x) < \infty$, $k\geq 0$, it follows from Bessel's inequality that $\sum_{i=0}^{\infty} c_i^2 < \infty$. Let

$$f_{\ell,i}^{(k)} = \int_{-\infty}^{\infty} \Psi_i(x) W_\ell(x) x^k d\beta(x) \quad . \tag{6.23}$$

Then

$$c_i^2 = \left\{\int_{-\infty}^{\infty} [(1-m_{i-1})^{\frac{1}{2}} \Psi_i(x) - m_i^{\frac{1}{2}} \Psi_{i+1}(x)] W_\ell(x) x^k d\beta(x)\right\}^2$$

$$= (1-m_{i-1})(f_{\ell,i}^{(k)})^2 + m_i (f_{\ell,i+1}^{(k)})^2 - 2b_i f_{\ell,i}^{(k)} f_{\ell,i+1}^{(k)} \ , \tag{6.24}$$

and

$$\sum_{i=o}^{\infty} c_i^2 = \sum_{i=o}^{\infty} [(f_{\ell,i}^{(k)})^2 - 2b_i f_{\ell,i}^{(k)} f_{\ell,i+1}^{(k)}] \quad . \tag{6.25}$$

But $f_{\ell,i+1}^{(k)} = \sum_{j=i+1}^{\infty} b_{ij}c_i = \sum_{j=i}^{\infty} b_{ij}c_i - b_{ii}c_i = f_{\ell,i}^{(k)} - c_i/(1-m_{i-1})^{\frac{1}{2}}$ and so

$$\sum_{i=o}^{\infty} c_i^2 = \sum_{i=o}^{\infty} f_{\ell,i}^{(k)}(f_{\ell,i}^{(k)}(1-2b_i)+2m_i^{\frac{1}{2}}c_i) \quad . \tag{6.26}$$

Thus, since $\sum_{i=o}^{\infty} c_i^2 < \infty$, it follows that either $\{f_{\ell,i}^{(k)}\}_{i=o}^{\infty}$ has a subsequence $\{f_{\ell,i(n)}^{(k)}\}_{n=o}^{\infty}$ which converges to zero or $\{f_{\ell,i}^{(k)}(1-2b_i)\}_{i=o}^{\infty}$ has a subsequence $\{f_{\ell,i(j)}^{(k)}(1-2b_{i(j)})\}_{j=o}^{\infty}$ which converges to zero. Thus (from (6.9)) either $\lim_{n\to\infty}(x_{i(n)}^{(k)}(0))(\ell_{i(n)-1}+L_{i(n)}+\ell_{i(n)})^{\frac{1}{2}} = 0$ or $\lim_{j\to\infty}(x_{i(j)}^{(k)}(0))(1-2b_{i(j)})(\ell_{i(j)-1}+L_{i(j)}+\ell_{i(j)})^{\frac{1}{2}} = 0$. The conclusion follows.■

CHAPTER VII

DISCUSSION AND RESULTS

This chapter briefly discusses some of the more unusual results obtained in the preceding work. Also some of the extensions and applications of these results are mentioned.

Notice first that the sufficient conditions for uniqueness, as given in Theorem 6.2, hold for all of the examples in Chapters III and IV except the Laguerre polynomials. In this case $\lim_{i\to\infty}(1-2b_i)(\ell_{i-1}+L_i+\ell_i)^{\frac{1}{2}} = 0$, so Theorem 6.2 does not apply. For this reason an alternative (but more specialized) uniqueness result is now obtained. This result requires none of the machinery developed in Part II of Chapter VI but it applies only to polynomial systems. More particularly, it applies only to those polynomials in (2.3) for which the integrator of (3.2) is known to be non-decreasing.

<u>Theorem 7.1.</u> The solution to (1.1), (1.7) given by (2.3), (3.2) satisfies Theorem 6.1 provided:

(i) $\inf_n |R_n+d_oL_n| \neq 0$;

(ii) there exists a non-decreasing $\alpha(z)$ which satisfies (2.4);

(iii) $\left|\int_{-\infty}^{\infty}(d_o+1/z)^r d\alpha(z)\right| < \infty$, $r\geq 1$.∎

<u>Proof.</u> Since $\inf_n |R_n+d_oL_n| \neq 0$, it is sufficient to show (see (3.2)) that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} P_n(z)P_j(z)(d_o+1/z)^k d\alpha(z) = 0 \quad , \quad k\geq 1 \quad . \tag{7.1}$$

But if (iii) holds, then $P_j(z)(d_o+1/z)^k$ can be written as $\sum_{n=o}^{\infty} a_n P_n(z)$ where each a_n is given by the integral in (7.1). Since $\alpha(z)$ is non-decreasing, it follows that $\sum_{n=o}^{\infty} a_n^2 < \infty$ and hence (7.1) holds for $k\geq 1$.■

For the examples of Chapter V, that is, those for which no integrator exists, the uniqueness criteria developed in Part I of Chapter VI (see Theorem 6.1) can still be applied directly in each particular example.

In the general case, the question of the form of a solution to (1.1) when no integrator exists for (1.3), (1.9) remains unanswered. The two examples given in Chapter V represent two widely varying cases, one for which an integrator "almost" exists (see (5.9,17,20,21)) and one for which nothing like an integrator exists (see (5.36)). Note, however, that despite their dissimilarities, the same approach, based on (5.2), works in each case.

Another interesting result is that sometimes an integrator for the system of polynomials (1.4) may be only of bounded variation while an integrator for the corresponding system of rational fractions (1.3), (1.9) may be non-decreasing. In particular, consider the second example of Chapter III. From (3.5) and [10], it can be seen that the integrator defined in Theorem 3.1 is of bounded variation whenever c and d are both negative. Yet it can also be shown that when the physical system for this integrator has its solution couched in the form of (4.2), the integrator $\beta(x)$ in (4.2) is non-decreasing. In order to make this more concrete, let $r_n = \ell_n = 1$, $R_{2n+1} = 2$, $L_{2n+1} = 1$, $R_{2n+2} = 1$, $L_{2n+2} = 2$, $n\geq 0$, and let $R_o = 2$, $L_o = 3$. Let

$$\mu(z) = \frac{\sqrt{4-(z^2-z-10)^2}}{2\pi(4-z)}\, dz \quad . \tag{7.2}$$

Then $d\alpha(z)$ is defined by $\mu(z)$ on $[1+\sqrt{33}/2, 4]$ and by $-\mu(z)$ on $[-3, 1-\sqrt{33}/2]$. On the other hand, let

$$\nu(x) = [((x+1)^4-(6(x+\tfrac{3}{4})(x+\tfrac{4}{3})-(x+1)^2)^2)^{\frac{1}{2}}/(\pi(x+1)^2(x+\tfrac{3}{4}))]dx \quad . \tag{7.3}$$

Then $d\beta(x)$ for the same RL values is $-\nu(x)$ on $[(-17-\sqrt{33})/16, -4/3]$ and $\nu(x)$ on $[-3/4, (-17+\sqrt{33}/16]$. Notice that $\beta(x)$ is non-decreasing while $\alpha(x)$ is not.

Some extensions on the above chapters have been made but have not been included since they do not pertain directly to the problem of solving (1.1). Among these are: the problem of determining the coefficients in the recurrence relation for $\Psi_n(x)$, $n\geq 0$, given an integrator $\beta(x)$; the extension of (5.2) to systems of the form $[A]d^2\vec{x}/dt^2 + [B]d\vec{x}/dt + [C]\vec{x} = \vec{0}$ where $[A]$, $[B]$, and $[C]$ are finite tridiagonal matrices, and $\vec{x}$ is a vector function of t; and the evaluation of such integrals as $\int_{-\infty}^{\infty} \Psi_j(x)\Psi_k(x)d\beta(x)$, primarily by means of Chapter VI (for example, $\int_{-\infty}^{\infty} \Psi_o^2(x)d\beta(x) = 1/M_{-1}$).

One result which has been included is the example (in Theorem 3.2) of an "integrator" which is neither non-decreasing nor of bounded variation. In fact, this "integrator" contains an impulse function. Although (1.1) cannot give rise to this case when admitting only non-negative RL values, this result has been included for two basic reasons: first, it completes the consideration of (3.5) for $cd>0$; secondly, it represents a slight extension of the integrators in [10,11]. No attempt has been made to determine whether or not this extension provides solutions which are somehow "better" than the solutions of Theorem 2.1. Comparison with a solution using an integrator of bounded variation (guaranteed to exist by Theorem 2.1) might prove interesting, particularly if uniqueness is to be discussed.

APPENDIX A

This appendix contains the statements and proofs of a number of theorems and lemmas stated in Chapter II. Additional supportive lemmas, definitions, and proofs are given as needed. Theorems 2.5 and 2.6 and Lemma 2.1 are the first results proved here. They deal with the problem of representing $F_n(z)/G_n(z)$, $n\geq 1$, as the Stieltjes transform of some function $\beta_n(x)$. Before these results are proved, however, some information regarding chain sequences is needed.

Definition A.1. The sequence $\{\sigma_i^2\}_{i=o}^{\infty}$ is called a chain sequence if there exist numbers γ_i with $0\leq\gamma_i\leq 1$, $i\geq -1$, such that $\sigma_i^2=\gamma_i(1-\gamma_{i-1})$, $i\geq 0$. The numbers γ_i, $i\geq -1$, are called parameters of the sequence $\{\sigma_i^2\}_{i=o}^{\infty}$.■

Lemma A.1. Every chain sequence $\{\sigma_i^2\}$ has minimal parameters m_i, $i\geq -1$, and maximal parameters M_i, $i\geq -1$, such that $m_i\leq\gamma_i\leq M_i$, $i\geq -1$, for all other parameters γ_i, $i\geq -1$, of the chain sequence. In fact, $m_{-1}=0$ and

$$m_{i+1}=\begin{cases}0 & \text{, if } m_i=1\\ \dfrac{\sigma_{i+1}^2}{(1-m_i)} & \text{, if } m_i<1,\ i\geq -1\end{cases}, \tag{A.1}$$

$$M_i=1-\frac{\sigma_{i+1}^2|}{|1}-\frac{\sigma_{i+2}^2|}{|1}-\ .\ .\ .\ .\ ,\ i\geq -1\quad .■$$

Lemma A.2. If $\{\sigma_i^2\}_{i=o}^{\infty}$ is a chain sequence and $\tau_i^2\leq\sigma_i^2$, $i\geq 0$, then

$\{\tau_i^2\}_{i=o}^{\infty}$ is a chain sequence. Furthermore, if m_i and M_i, $i\geq -1$, are the minimal and maximal parameters, respectively, for $\{\tau_i^2\}_{i=o}^{\infty}$ and if γ_i, $i\geq -1$, are parameters for $\{\sigma_i^2\}_{i=o}^{\infty}$, then $m_i \leq \gamma_i \leq M_i$, $i\geq -1$. In particular, if $\tau_{k+1}^2 < \sigma_{k+1}^2$ for some $k\geq -1$, then $m_{k+1} < \gamma_{k+1}$ and $\gamma_k < M_k$.∎

The proofs of Lemmas A.1 and A.2 can be found in [18].

Theorem 2.5. Suppose there exist g_i with $0<g_{i-1}<1$, $i\geq 0$, such that $b_i^2 \leq g_i(1-g_{i-1})$. Further suppose that $G_{i+1}(d_i) \neq 0$, $i\geq 0$, and that a_i, d_i are real for $i\geq 0$. Then $G_n(x)$ has n real distinct zeros for $n\geq 1$. [The quantities a_n, b_n, d_n, $G_n(x)$ are defined in equations (2.2) and (2.12)].∎

Proof. Since $\{g_i(1-g_{i-1})\}_{i=o}^{\infty}$ is a chain sequence, so is $\{b_i^2\}_{i=o}^{\infty}$ by Lemma A.2. Let $\{m_{i-1}\}_{i=o}^{\infty}$ be the minimal parameter sequence for $\{b_i^2\}_{i=o}^{\infty}$. Note that $0<m_i<1$ for $i\geq 0$, since the physical assumption that $\ell_i > 0$ guarantees that $b_i^2 > 0$, $i\geq 0$. The fact that $b_i^2 = m_i(1-m_{i-1})$ with $0<m_i<1$ is now used to show that the coefficient of x^n in $G_n(x)$ is positive. Let q_n denote this coefficient. With $q_o = q_1 = 1$, q_n satisfies the difference equation $q_{n+1} = q_n - b_{n-1}^2 q_{n-1}$ which becomes $(q_{n+1} - (1-m_{n-1})q_n) = m_{n-1}(q_n - (1-m_{n-2})q_{n-1})$, whose solution is $q_n = \prod_{i=-1}^{n-2} (1-m_i)$, $n\geq 1$. Thus q_n is positive. So $\lim_{x\to\infty} G_n(x) = \infty$, while $\lim_{x\to-\infty} G_{2n}(x) = \infty$ and $\lim_{x\to-\infty} G_{2n-1}(x) = -\infty$, all for $n\geq 1$. Now $G_1(x)$ has one real zero at $x = a_o$. Since $G_2(a_o) = -b_o^2(a_o - d_o)^2$ and $a_o \neq d_o$ by assumption that $G_1(d_o) \neq 0$, it follows that $G_2(a_o) < 0$ and hence that $G_2(x)$ has two real zeros separated by the zero of $G_1(x)$. Now assume that the zeros of $G_i(x)$ separate those of $G_{i+1}(x)$, $1\leq i\leq n-1$ and show that the zeros of $G_n(x)$ separate those of $G_{n+1}(x)$. Let $x_{i,o} < x_{i,1} < \ldots < x_{i,i-1}$ denote the zeros of $G_i(x)$. Then

$$G_{n+1}(x_{n,m}) = -b_{n-1}^2 (x_{n,m} - d_{n-1})^2 G_{n-1}(x_{n,m}) \quad . \tag{A.2}$$

But by hypothesis $x_{n,m} \neq d_{n-1}$ so $G_{n+1}(x_{n,m})$ and $G_{n-1}(x_{n,m})$ have opposite signs. Since in addition $G_{n-1}(x_{n,m})$ and $G_{n-1}(x_{n,m+1})$ have opposite signs for $0 \le m \le n-2$, it follows that $G_{n+1}(x_{n,m})$ and $G_{n+1}(x_{n,m+1})$ have opposite signs for each value $0 \le m \le n-2$. This accounts for n-1 zeros of $G_{n+1}(x)$. Now $G_{n-1}(x)$ and $G_{n+1}(x)$ have the same limit as x tends to $-\infty$ but have opposite signs at $x_{n,o}$ [set m=0 in (A.2)]. Since $G_{n-1}(x)$ does not again cross the x-axis on $(-\infty, x_{n,o})$, $G_{n+1}(x)$ must, which accounts for one additional zero. Similarly $G_{n-1}(x_{n,n-1}) > 0$ while $G_{n+1}(x_{n,n-1}) < 0$ and since $\lim_{x\to\infty} G_{n+1}(x) = \infty$, the last zero lies in $(x_{n,n-1}, \infty)$.■

Although the proof of Theorem 2.6 proceeds along lines similar to the proof of a lemma in [18], the differences make an explicit presentation desirable.

Theorem 2.6. Suppose there exist g_i with $0<g_{i-1}<1$, $i\ge 0$, such that $b_i^2 \le g_i(1-g_{i-1})$; and suppose that a_i, d_i, are real for $i\ge 0$. Let $y = \mathrm{Im}(z) > 0$; let $t_n(z,w)$ be defined by (2.14). Then for $\mathrm{Im}(w) \ge g_n y$, $n\ge 0$, it follows that $\mathrm{Im}(t_n(z,w)) \ge g_{n-1}y$.■

Proof. The proof hinges on the fact that if $z = x+iy$, then with $Q = b_n^2(z-d_n)^2$, it follows that

$$2b_n^2y^2 = |Q| - \mathrm{Re}(Q) \quad , \tag{A.3}$$

as may be verified by expanding the right-hand side. By hypothesis,

$$\mathrm{Im}(w) \ge g_n y = \frac{g_n(1-g_{n-1})y^2}{(1-g_{n-1})y} \ge \frac{b_n^2 y^2}{(1-g_{n-1})y} \quad . \tag{A.4}$$

Now by multiplying (A.4) by two and subsequently replacing $b_n^2y^2$ by its equivalent from (A.3),

$$2\mathrm{Im}(w) + \frac{\mathrm{Re}(Q)}{y(1-g_{n-1})} \geq \frac{|Q|}{y(1-g_{n-1})} \quad ; \tag{A.5}$$

and so, since $|A+iB| \geq \mathrm{Im}(A)+\mathrm{Re}(B)$ for any complex numbers A and B,

$$\left|w + \frac{i(Q)}{2y(1-g_{n-1})}\right| \geq \frac{|Q|}{2y(1-g_{n-1})} \quad . \tag{A.6}$$

Upon squaring both sides of (A.6), it follows that

$$(\mathrm{Re}(w))^2 + \frac{(\mathrm{Im}(Q))^2}{4y^2(1-g_{n-1})^2} - \frac{\mathrm{Re}(w)\,\mathrm{Im}(Q)}{y(1-g_{n-1})} + (\mathrm{Im}(w))^2 +$$

$$\frac{(\mathrm{Re}(Q))^2}{4y^2(1-g_{n-1})^2} + \frac{\mathrm{Im}(w)\,\mathrm{Re}(Q)}{y(1-g_{n-1})} \geq \frac{(\mathrm{Im}(Q))^2+(\mathrm{Re}(Q))^2}{4y^2(1-g_{n-1})^2} \tag{A.7}$$

Now in (A.7) use the fact that

$$\mathrm{Im}(\frac{Q}{w}) = \frac{\mathrm{Re}(w)\,\mathrm{Im}(Q)-\mathrm{Im}(w)\,\mathrm{Re}(Q)}{|w|^2}$$

to obtain

$$y(1-g_{n-1}) \geq \mathrm{Im}(\frac{Q}{w}) = \mathrm{Im}(\frac{b_n^2(z-d_n)^2}{w}) \quad , \tag{A.8}$$

from which the desired conclusion follows by using (2.14).■

<u>Lemma 2.1.</u> Suppose there exist g_i with $0<g_{i-1}<1$, $i\geq 0$, such that $b_i^2 \leq g_i(1-g_{i-1})$; and suppose that a_i, d_i are real for $i\geq 0$. Then $F_n(x)$ and

$G_n(x)$, as defined in (2.12), are polynomials of degree n-1 and n, respectively, and $\lim_{z\to\infty}(zF_n(z)/G_n(z))$, z complex, exists and is uniformly bounded for all $n\geq 2$.■

Proof. In the proof of Theorem 2.5, it is shown that $G_n(x)$ is a polynomial of degree n with leading coefficient $g_n = \prod_{i=o}^{n-2}(1-m_i)$, $n\geq 2$. Similarly, if p_n is the coefficient of x^{n-1} in $F_n(x)$, then $p_1 = 1$, $p_2 = 1$, and $p_{n+1} = p_n - b_{n-1}^2 p_{n-1}$. The solution of this difference equation is $p_n = [\prod_{r=o}^{n-2}(1-m_r)](1+\sum_{j=1}^{n-1}\prod_{i=o}^{j-1}(m_i/(1-m_i)))$, $n\geq 2$, for the m_i appearing in the proof of Theorem 2.5. Clearly, then $\lim_{z\to\infty}(zF_n(z)/G_n(z)) = 1 + \sum_{j=1}^{n-1}\prod_{i=o}^{j-1}(m_i/(1-m_i))$, $n\geq 2$. Wall [18] has shown that if $\{M_{n-1}\}_{n=o}^{\infty}$ is the maximal parameter sequence for $\{b_n^2\}_{n=o}^{\infty}$, then $1/M_{-1} = 1 + \sum_{j=1}^{\infty}\prod_{i=o}^{j-1}(m_i/(1-m_i))$. Since $M_{-1} > g_{-1}$ by Lemma A.2 and $g_{-1} > 0$, the series for $1/M_{-1}$ converges. Thus $\lim_{z\to\infty}(zF_n(z)/G_n(z)) = 1 + \sum_{j=1}^{n-1}\prod_{i=o}^{j-1}(m_i/(1-m_i)) \leq 1 + \sum_{j=1}^{\infty}\prod_{i=o}^{j-1}(m_i/(1-m_i)) = 1/M_{-1} < \infty$.■

The proof of Theorem 2.8 requires the following lemma.

Lemma A.3. Let U_n, V_n, $n\geq 0$, be defined by $U_o=0$, $U_1=1$, $V_o=1$, $V_1=p_o$,

$$U_{n+1} = p_n U_n - q_{n-1} U_{n-1} \quad , \quad n\geq 1 \quad ,$$

$$V_{n+1} = p_n V_n - q_{n-1} V_{n-1} \quad , \quad n\geq 1 \quad ,$$

where p_n and q_n are coefficients depending on n and perhaps on additional parameters. Further, let $X_{m,n}$ be defined by $X_{m,o} = 0$, $X_{m,1} = 1$,

$$X_{m,n+1} = p_{m+n}X_{m,n} - q_{m+n-1}X_{m,n-1} \quad , \quad n\geq 1,\ m\geq 0 \quad . \tag{A.9}$$

Then, for those values of p_n and q_n for which $V_i \neq 0$ $(i\geq 0)$,

$$\frac{U_k}{V_k} - \frac{U_m}{V_m} = \frac{q_{-1}q_o q_1 \cdots q_{m-1}}{V_m V_k} X_{m,k-m} \quad , \quad k-m \geq 0, \; q_{-1} = 1 \quad . \blacksquare \tag{A.10}$$

Proof. Note that (A.10) holds trivially for k-m=0. In [9], it is shown by induction that (A.10) holds for k-m=1. Assume then that (A.10) holds for $0 \leq k-m \leq n$ $(n \geq 1)$ and show that it holds for $k-m = n+1$. By using (A.10) on $(U_{m+n}/V_{m+n}) - (U_m/V_m)$ and $(U_{m+n+1}/V_{m+n+1}) - (U_{m+1}/V_{m+1})$, adding and then using (A.10) for $(U_{m+n}/V_{m+n}) - (U_{m+1}/V_{m+1})$, it can be established that

$$\frac{U_{m+n+1}}{V_{m+n+1}} - \frac{U_m}{V_m} = \frac{q_{-1}q_o \cdots q_{m-1}}{V_m V_{m+n+1}} \left[\frac{X_{m,n}V_{m+1}V_{m+n+1} + q_m V_m V_{m+n} X_{m+1,n}}{V_{m+1}V_{m+n}} - \frac{q_m V_m V_{m+n+1} X_{m+1,n-1}}{V_{m+1}V_{m+n}}\right] . \tag{A.11}$$

So it must be shown that

$$V_{m+1}(V_{m+n}X_{m,n+1} - V_{m+n+1}X_{m,n}) = q_m V_m (V_{m+n}X_{m+1,n} - V_{m+n+1}X_{m+1,n-1}) \quad . \tag{A.12}$$

Direct substitution verifies (A.12) for n=1. For $n \geq 2$, first verify by induction on i (≥ 1) that $(V_{j+i}X_{j,i+1} - V_{j+i+1}X_{j,i}) = q_j q_{j+1} \cdots q_{j+i-1} V_j$, for $j \geq 0$, $i \geq 1$; then (A.12) follows (replace j by m, i by n; replace j by m+1, i by n-1; substitute in (A.12)).■

Theorem 2.8. Let D denote the closure of the set of all d_n, $n \geq 0$. If the intersection of D with G (see Lemma 2.2) is empty, so that (2.21) and (2.22) are well-defined, and if the hypotheses of Theorem 2.5 are satisfied, then

$$\int_{-\infty}^{\infty} W_m(x)\,d\beta_n(x) = \delta_{om} \quad , \quad 0\le m\le n-1 \quad , \tag{2.21}$$

and

$$\int_{-\infty}^{\infty} \frac{\Psi_{m+1}(x)\,d\beta_n(x)}{(x-d_m)^j} = 0 \quad , \quad 0\le m\le n-1,\ 1\le j\le e_m \quad , \tag{2.22}$$

are valid for every $n\ge 1$.■

Proof. First define $\pi_j(x)$ by $\pi_{-1}(x) = 1$ and $\pi_j(x) = b_o(x-d_o)\cdot$ $b_1(x-d_1)\ \ldots\ b_j(x-d_j)$, $j\ge 0$. Now it is desirable to show that for t not in G,

$$\int_{-\infty}^{\infty} \left(\frac{G_j(t)\pi_{j-1}(x)-G_j(x)\pi_{j-1}(t)}{(t-x)\pi_{j-1}(x)}\right) d\beta_n(x) = F_j(t) \quad , \quad 1\le j\le n \quad , \tag{A.13}$$

because the process used to derive (A.13) also yields (2.21); and once (A.13) is obtained, (2.22) follows almost immediately. When j=1,

$$\int_{-\infty}^{\infty} \left[\frac{G_1(t)\pi_o(x)-G_1(x)\pi_o(t)}{(t-x)\pi_o(x)}\right] d\beta_n(x)$$

$$= \int_{-\infty}^{\infty} \left[\frac{(t-a_o)(x-d_o)-(x-a_o)(t-d_o)}{(t-x)(x-d_o)}\right] d\beta_n(x)$$

$$= \int_{-\infty}^{\infty} \frac{(a_o-d_o)(t-x)}{(t-x)(x-d_o)}\, d\beta_n(x)$$

or

$$\int_{-\infty}^{\infty} [\frac{G_1(t)\pi_o(x)-G_1(x)\pi_o(t)}{(t-x)\pi_o(x)}]d\beta_n(x) = \int_{-\infty}^{\infty} W_o(x)d\beta_n(x) \quad . \tag{A.14}$$

Evaluate (A.14) at $t=d_o$ and use (2.17) to obtain $G_1(d_o)F_n(d_o)/G_n(d_o) = \int_{-\infty}^{\infty} W_o(x)d\beta_n(x)$. But from (A.10) with $U=F$, $V=G$, $p_m = x-a_m$, and $q_m = b_m^2(x-d_m)^2$ (see (2.12)), it follows, since $q_o(d_o)=0$, that $F_n(d_o)/G_n(d_o) = F_1(d_o)/G_1(d_o)$; so $\int_{-\infty}^{\infty} W_o(x)d\beta_n(x) = F_1(d_o) = 1$, and (A.13) holds for $j=1$. Induction is now used to prove (A.13) for $j>1$. For this purpose, assume (A.13) holds for $1\le j\le k$ and show that it holds for $j=k+1$. The steps in the procedure are, first, to write the left-hand side of (A.13) with $j=k+1$ and $\pi_{k-1}(t)W_k(x)$ added in and out and then to use the relation $W_k(x) = [(a_k-d_k)G_k(x)+b_{k-1}^2(d_k-d_{k-1})(x-d_{k-1})G_{k-1}(x)]/[\pi_{k-1}(x)(x-d_k)]$ (see (A.25)), and the induction hypothesis. As is shown below, the result is

$$\int_{-\infty}^{\infty} [\frac{G_{k+1}(t)\pi_k(x)-G_{k+1}(x)\pi_k(t)}{(t-x)\pi_k(x)}]d\beta_n(x) = F_{k+1}(t)$$

$$+ \pi_{k-1}(t)\int_{-\infty}^{\infty} W_k(x)d\beta_n(x) \quad , \tag{A.15}$$

which can be manipulated, as is also shown below, to obtain (A.13). Now these steps are actually done. By using (2.12) and the relation for $W_k(x)$, it follows that

$$\int_{-\infty}^{\infty} \{[\frac{G_{k+1}(t)\pi_k(x)-G_{k+1}(x)\pi_k(t)}{(t-x)\pi_k(x)}]+[\pi_{k-1}(t)W_k(x)-\pi_{k-1}(t)W_k(x)]\}d\beta_n(x)$$

$$= \pi_{k-1}(t)\int_{-\infty}^{\infty} W_k(x)d\beta_n(x) + \int_{-\infty}^{\infty} [\frac{(t-a_k)G_k(t)\pi_{k-1}(x)}{(t-x)\pi_{k-1}(x)}$$

$$- \frac{b_{k-1}^2(t-d_{k-1})^2 G_{k-1}(t)\pi_{k-2}(x)}{(t-x)\pi_{k-2}(x)}] d\beta_n(x) - \int_{-\infty}^{\infty} [\frac{(x-a_k)G_k(x)\pi_{k-1}(t)b_k(t-d_k)}{(t-x)\pi_k(x)}$$

$$+ \frac{\pi_{k-1}(t)(a_k-d_k)(t-x)G_k(x)b_k}{(t-x)\pi_k(x)}] d\beta_n(x)$$

$$+ \int_{-\infty}^{\infty} [\frac{b_{k-1}^2(x-d_{k-1})^2 G_{k-1}(x)\pi_{k-1}(t)b_k(t-d_k)}{(t-x)\pi_k(x)}$$

$$- \frac{b_{k-1}^2(d_k-d_{k-1})(x-d_{k-1})(t-x)b_k G_{k-1}(x)\pi_{k-1}(t)}{(t-x)\pi_k(x)}] d\beta_n(x) \quad .$$

The last two integrals on the right-hand side of this expression become

$$\int_{-\infty}^{\infty} \frac{b_k\pi_{k-1}(t)}{(t-x)\pi_k(x)} \; (-G_k(x)(t-a_k)(x-d_k)$$

$$+ b_{k-1}^2(x-d_{k-1})(x-d_x)(t-d_{k-1})G_{k-1}(x))d\beta_n(x) \quad .$$

Thus

$$\int_{-\infty}^{\infty} (\frac{G_{k+1}(t)\pi_k(x)-G_{k+1}(x)\pi_k(t)}{(t-x)\pi_k(x)})d\beta_n(x) = \pi_{k-1}(t)\int_{-\infty}^{\infty} W_k(x)d\beta_n(x)$$

$$+ \; (t-a_k)\int_{-\infty}^{\infty} (\frac{G_k(t)\pi_{k-1}(x)-G_k(x)\pi_{k-1}(t)}{(t-x)\pi_{k-1}(x)})d\beta_n(x)$$

$$- \; b_{k-1}^2(t-d_{k-1})^2\int_{-\infty}^{\infty} (\frac{G_{k-1}(t)\pi_{k-2}(x)-G_{k-1}(x)\pi_{k-2}(t)}{(t-x)\pi_{k-2}(x)})d\beta_n(x)$$

from which (A.15) follows by using the induction hypothesis and (2.12). It remains to show that $\int_{-\infty}^{\infty} W_k(x)\,d\beta_n(x) = 0$, $1\le k\le n-1$. To do so, first divide both sides of (A.15) by $G_{k+1}(t)$ to obtain

$$\frac{F_n(t)}{G_n(t)} - \int_{-\infty}^{\infty} \frac{G_{k+1}(x)\pi_k(t)\,d\beta_n(x)}{G_{k+1}(t)\pi_k(x)(t-x)} = \frac{F_{k+1}(t)}{G_{k+1}(t)} + \frac{\pi_{k-1}(t)}{G_{k+1}(t)} \int_{-\infty}^{\infty} W_k(x)\,d\beta_n(x) \quad . \tag{A.16}$$

Now differentiate both sides of (A.16) e_k-1 times with respect to t, evaluate at $t=d_k$, use the fact (from (A.10)) that

$$\left.\frac{d^m\left(\frac{F_n(t)}{G_n(t)}\right)}{dt^m}\right/_{t=d_k} = \left.\frac{d^m\left(\frac{F_{k+1}(t)}{G_{k+1}(t)}\right)}{dt^m}\right/_{t=dk} , \qquad 0\le m\le 2e_k-1,\ n>k \quad , \tag{A.17}$$

and obtain the result $\int_{-\infty}^{\infty} W_k(x)\,d\beta_n(x) = 0$; so (A.15) becomes (A.13) with $j=k+1$. Parenthetically, note that differentiation under the integral sign is valid because of the form of each $\beta_n(x)$ (see the discussion preceding Lemma 2.2). Hence (A.13) holds for $1\le j\le n$. So from (A.16) and (2.13),

$$\frac{F_n(t)}{G_n(t)} - \frac{F_{k+1}(t)}{G_{k+1}(t)} = \frac{1}{\Psi_{k+1}(t)} \int_{-\infty}^{\infty} \frac{\Psi_{k+1}(x)}{(t-x)}\,d\beta_n(x) \quad , \quad 0\le k\le n \quad . \tag{A.18}$$

To obtain (2.22) for $j=1$, differentiate both sides of (A.18) (with respect to t) e_k times, evaluate at $t=d_k$, and use (A.17). To obtain the remaining equations of (2.22) use induction on j. Assume that $\int_{-\infty}^{\infty}[\Psi_{k+1}(x)/(x-d_k)^j]d\beta_n(x) = 0$, $0\le k<n$, $1\le j\le i<e_k$ and show that this result holds for $j=i+1$. Differentiate both sides of (A.18) (with respect to t) e_k+i times; evaluate at $t=d_k$; use (A.17) and the induction hypothesis to obtain the desired result.■

The proof of Theorem 2.9 depends on the following three lemmas (A.4-A.6).

Lemma A.4. Under the conditions that

(i) a_j, b_j, d_j are real, $j\ge 0$,

(ii) there exist g_j with $0<g_{j-1}<1$, $j\ge 0$ and $b_j^2\le g_j(1-g_{j-1})$,

(iii) G and D are disjoint,

$$\int_{-\infty}^{\infty}\frac{\Psi_{n+m}(x)d\beta(x)}{(x-d_m)^j} = 0 \quad , \quad n\ge 1,\ m\ge 0,\ 1\le j\le e_m \quad .■ \tag{2.20}$$

Proof. The method of proof is by induction on j where at each stage induction on n is used.

Let $j=1$. By (2.19), $\int_{-\infty}^{\infty}[\Psi_{m+1}(x)/(x-d_m)]d\beta(x) = 0$. Now use induction on n, assuming that

$$\int_{-\infty}^{\infty}\frac{\Psi_{n+m}(x)d\beta(x)}{(x-d_m)} = 0 \tag{A.19}$$

holds for $1\le n\le k$ and showing that this result holds for $n=k+1$. In order to show that (A.19) holds for $n=k+1$, it is first verified that

$$\int_{-\infty}^{\infty} \frac{b_{m+n}(d_m-d_{m+n})\Psi_{m+n+1}(x)d\beta(x)}{(x-d_m)} =$$

$$\int_{-\infty}^{\infty} \frac{[(d_m-a_{m+n})\Psi_{m+n}(x)-b_{m+n-1}(d_m-d_{m+n-1})\Psi_{m+n-1}(x)]d\beta(x)}{(x-d_m)} , \tag{A.20}$$

by using the recurrence relation (1.3) for $\Psi_{m+n+1}(x)$, dividing by $(x-d_m)$, writing $(x-d_{m+n})/(x-d_m)$ as $1+(d_m-d_{m+n})/(x-d_m)$ (similarly for $(x-a_{m+n})/(x-d_m)$ and $(x-d_{m+n+1})/(x-d_m)$), collecting the terms which appear in (A.20), integrating, and using (1.9) and (2.18) on the remaining terms. Then from (A.20) with n=k (the right-hand side being zero by the induction hypothesis), it follows that either $d_m = d_{m+k}$ or $\int_{-\infty}^{\infty}(\Psi_{m+k+1}(x)/(x-d_m))d\beta(x)=0$. But if $d_m=d_{m+k}$, then by (2.19) $\int_{-\infty}^{\infty}(\Psi_{m+k+1}(x)/(x-d_{m+k}))d\beta(x)=0$ or $\int_{-\infty}^{\infty}(\Psi_{m+k+1}(x)/(x-d_m))d\beta(x)=0$; so the desired result follows whether $d_m=d_{m+k}$ or not.

Now assume that (2.20) holds for $1\le j\le i<e_m$, $n\ge 1$, and show that (2.20) holds for j=i+1. By (2.19), $\int_{-\infty}^{\infty}(\Psi_{m+1}(x)/(x-d_m)^{i+1})d\beta(x) = 0$. Now use induction on n, assuming that

$$\int_{-\infty}^{\infty} \frac{\Psi_{m+n}(x)d\beta(x)}{(x-d_m)^{i+1}} = 0 \tag{A.21}$$

holds for $1\le n\le k$ and showing that this result holds for n=k+1. In order to show that (A.21) holds for n=k+1, first observe that

$$\int_{-\infty}^{\infty} \frac{b_{m+n}(d_m-d_{m+n})\Psi_{m+n+1}(x)d\beta(x)}{(x-d_m)^{i+1}} =$$

$$\int_{-\infty}^{\infty} \frac{((d_m-a_{m+n})\Psi_{m+n}(x)-b_{m+n-1}(d_m-d_{m+n-1})\Psi_{m+n-1}(x))d\beta(x)}{(x-d_m)^{i+1}} \tag{A.22}$$

as can readily be verified as above except that after collecting the terms in (A.20), divide by $(x-d_m)^i$, integrate, and use the induction hypothesis to eliminate the remaining terms. The rest of the proof proceeds exactly as above.■

Lemma A.5. If $G_m(d_{m-1}) \neq 0$, $m\geq 1$, then for some constants r_k^m, s_k^m, $0\leq k\leq m$,

$$W_m(x) = \sum_{k=0}^{m} \frac{r_k^m}{(x-d_k)^{e_k}} \quad , \quad m\geq 0 \quad , \tag{A.23}$$

where $r_m^m \neq 0$, $m\geq 0$. Also

$$\frac{1}{(x-d_m)^{e_m}} = \sum_{k=0}^{m} s_k^m W_k(x) \quad , \quad m\geq 0 \quad .■ \tag{A.24}$$

Proof. By beginning with (1.9), using (1.3) to eliminate $\Psi_{m+1}(x)$, and then using (2.13), it follows that

$$W_m(x) = \frac{((a_m-d_m)G_m(x)+b_{m-1}^2(x-d_{m-1})(d_m-d_{m-1})G_{m-1}(x))}{b_o(x-d_o)b_1(x-d_1)\ \ldots\ b_{n-1}(x-d_{m-1})(x-d_m)} \quad , \quad m\geq 1 \quad . \tag{A.25}$$

Then a partial fraction expansion of (A.25) yields (A.23) for $m\geq 1$. For $m=0$, $W_o(x) = (a_o-d_o)/(x-d_o)^{e_o}$. It is readily verified that $r_m^m = (-1/b_ob_1 \ldots b_{m-1}) \prod_{i=o,d_m\neq d_i}^{m-1} G_{m+1}(d_m)/(d_m-d_i)$ which is non-zero since $G_{m+1}(d_m)\neq 0$, $m\geq 0$. Equation (A.24) now follows by a matrix inversion.■

Lemma A.6. $\int_{-\infty}^{\infty} \Psi_j(x)W_j(x)d\beta(x) = \int_{-\infty}^{\infty} \Psi_o(x)W_o(x)d\beta(x) = 1$, $j\geq 0$ (see (2.18)), under the same conditions as Lemma A.4.■

Proof. $\int_{-\infty}^{\infty} \Psi_j(x)W_j(x)d\beta(x) = \int_{-\infty}^{\infty} \Psi_j(x)[r_j^j/(x-d_j)^{e_j}]d\beta(x)$ by Lemmas A.4 and A.5. Now perform the following steps on $\int_{-\infty}^{\infty} \Psi_j(x)W_j(x)d\beta(x)$,

$j \geq 1$: replace $\Psi_j(x)$ by its partial fraction expansion; eliminate the constant term in this expansion by (2.18); rewrite $W_j(x)$ by using (1.9); eliminate all possible remaining terms by using (2.20). This together with the first equation above yields

$$\int_{-\infty}^{\infty} \Psi_j(x)W_j(x)d\beta(x) = \int_{-\infty}^{\infty} \Psi_{j-1}(x) \frac{r_{j-1}^{j-1}d\beta(x)}{(x-d_{j-1})^{e_{j-1}}}$$

$$= \int_{-\infty}^{\infty} \Psi_{j-1}(x)W_{j-1}(x)d\beta(x) \quad , \quad j\geq 1 \quad .$$

By repeated use of this result, the conclusion follows.■

<u>Theorem 2.9.</u> Under the conditions that

(i) a_j, b_j, d_j are real, $j\geq 0$,

(ii) there exist g_j with $0<g_{j-1}<1$, $j\geq 0$ and $b_j^2 \leq g_j(1-g_{j-1})$,

(iii) G and D are disjoint,

$\beta(x)$ is the integrator of Definition 2.3 for the sequences (1.3) and (1.9).■

<u>Proof.</u> It must be shown that

$$\int_{-\infty}^{\infty} \Psi_j(x)W_k(x)d\beta(x) = 0 \quad , \quad j\neq k,\ j,k\geq 0 \quad . \tag{A.26}$$

If $k<j$, (A.26) follows immediately from (A.23) and Lemma A.4. If $j<k$, then (A.26) follows by an expansion of $\Psi_j(x)$ into a constant plus a sum

of factors of the form $r/(x-d_i)^{e_i}$ for r a constant and $0\leq i\leq j-1$. Finally $\int_{-\infty}^{\infty} \Psi_j(x)W_j(x)d\beta(x) = 1 \neq 0$, $j\geq 0$, follows from Lemma A.6.■

APPENDIX B

The tedious process of directly verifying that $\mu(x)$ and $\eta(x)$ of Theorems 3.1 and 3.2 work as intended is eased by the following result.

Theorem B.1. Given the system of (1.4) and (3.1), suppose that there exists a generalized function $\gamma(z)$ such that

$$\begin{aligned} \int_{-\infty}^{\infty} P_n(z)\,d\gamma(z) &= 0 \quad , \quad n\geq 1 \quad , \\ \int_{-\infty}^{\infty} d\gamma(z) &\neq 0 \quad . \end{aligned} \tag{B.1}$$

Then (2.4) is satisfied for the sequence $\{P_n(z)\}_{n=o}^{\infty}$ of (1.4) and (3.1), when $\alpha(z)$ (see Definition 2.1) is replaced by $\gamma(z)$.∎

A similar theorem is proved in Appendix C; so the proof of this result is not presented (it is a straightforward use of the fact that $P_n(x)P_m(x) = \sum_{i=o}^{n+m} \alpha_{i(n,m)} P_i(x)$).

Outline for Theorem 3.1. By Theorem B.1, it is sufficient to show that (B.1) holds with $\gamma(x) = \mu(x)$ and $P_n(x) = S_n(x)$. The approach is very similar to that found in Chapter IV and Appendix C. First a change of variable is effected. Let $z = x(x+b)-d-c$, $R_o(z) = 1$, $R_1(z) = z$, and

$$R_{n+1}(z) = zR_n(z) - cdR_{n-1}(z) \quad . \tag{B.2}$$

Then

$$S_{2n}(x) = R_n(z) + (ax+ab+d)R_{n-1}(z)$$
$$S_{2n+1}(x) = (x+a)R_n(z) + acR_{n-1}(z) \tag{B.3}$$

Now let $z = 2(cd)^{\frac{1}{2}}\cos\theta$ and define $A(z)$, $B(z)$ by

$$A(z)-B(z) = \frac{1}{\sqrt{G}}\frac{1}{\sqrt{H}}\left((b-2a)\left(\frac{p(cd)^{-\frac{1}{2}}}{1+p^2-2p\cos\theta} - \frac{q(cd)^{-\frac{1}{2}}}{1+q^2-2q\cos\theta}\right) - \frac{2a}{d}\left(\frac{1}{1+p^2-2p\cos\theta} - \frac{1}{1+q^2-2q\cos\theta}\right)\right), \tag{B.4}$$

and

$$A(z) + B(z) = \frac{(cd)^{-\frac{1}{2}}}{\sqrt{H}}\left(\frac{p}{1+p^2-2p\cos\theta} - \frac{q}{1+q^2-2q\cos\theta}\right), \tag{B.5}$$

where $G = 4d+4c+b^2+4z$. It can be shown by a tedious but straightfoward procedure of substitution that both $\sqrt{4-z^2}\,A(z)dz$ and $\sqrt{4-z^2}\,B(z)dz$ become

$$\omega(x)dx = \frac{\sqrt{4cd-(x^2+bx-d-c)^2}\,dx}{(ax^2+x(ab+d+a^2)+(a^2b+ad-ac))} \tag{B.6}$$

on the intervals of (3.8) and (3.9), respectively (note that (B.6) is (3.6) to within a factor of 2π). To show the above, it is necessary to use the facts that, for $A(z)$, $\sqrt{G} = 2x+b$, $x = \frac{1}{2}(-b+\sqrt{4d+4c+b^2+4z})$, and for $B(z)$, $\sqrt{G} = -(2x+b)$, $x = \frac{1}{2}(-b-\sqrt{4d+4c+b^2+4z})$. It can also be shown that the integral of $S_{2n}(x)\omega(x)$ on the intervals of (3.8) and (3.9) becomes

$$\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \sqrt{4cd-z^2}\ [(A(z)+B(z))(R_n(z)+\frac{ab+2d}{2}R_{n-1}(z)) + (A(z)-B(z))(\frac{a\sqrt{G}}{2}R_{n-1}(z))]dz\ . \tag{B.7}$$

Similarly, the integral of $S_{2n+1}(x)\omega(x)$ on the intervals of (3.8) and (3.9) becomes

$$\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \sqrt{4cd-z^2}\ [(A(z)+B(z))(\frac{2a-b}{2}R_n(z)+acR_{n-1}(z)) + (A(z)-B(z))(\frac{\sqrt{G}}{2}R_n(z))]dz\ . \tag{B.8}$$

By using the fact that

$$\frac{1}{2\pi}\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \frac{R_n(z)(4cd-z^2)^{\frac{1}{2}}}{1+r^2-2r(\cos\theta)}\,dz = \begin{cases} (cd)^{\frac{n+2}{2}}\ r^n\ , & r^2<1\ , \\ (cd)^{\frac{n+2}{2}}\ \frac{1}{r^{n+2}}\ , & r^2>1\ , \end{cases} \tag{B.9}$$

plus the facts $pq=-a^2/d$, $a^2-ab-d=(cd)^{\frac{1}{2}}(p+q)$, a straightfoward calculation will verify that $\omega(x)/2\pi$ is the weight function for the case $p^2<1$, $q^2<1$. For the cases when either $p^2>1$ or $q^2>1$ or both, the fact that for $x=[(cd)^{\frac{1}{2}}q/a]-a$, $S_{2n}=(cd)^{n/2}/p^n$, $S_{2n+1}=(-a/d)(cd)^{(n+1)/2}/p^{n+1}$ (also true for p and q interchanged) helps verify orthonormalizability for (A), (B), and (C) of Theorem 3.1 (when strict inequality holds). In the cases when either $p^2=1$ or $q^2=1$, $p\neq q$, the formula

$$\frac{1}{2\pi}\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \frac{R_n(z)(4cd-z^2)^{\frac{1}{2}}}{2\pm\cos\theta}\,dz = (\mp\sqrt{cd})^{n+2} \tag{B.10}$$

is helpful. Finally for $p=q$, $p^2 \le 1$, first note that (B.4) and (B.5) become

$$A(z)+B(z) = \frac{1}{(cd)}\frac{1}{(1+p^2-2p(\cos\theta))^2}(1-p^2) \tag{B.11}$$

and

$$A(z)-B(z) = \frac{(b-2a)}{(cd)\sqrt{G}}\frac{(1-p^2)}{(1+p^2-2p(\cos\theta))^2} + \frac{4a(p-\cos\theta)}{(cd)^{\frac{1}{2}}d\sqrt{G}(1+p^2-2p(\cos\theta))^2}\ . \tag{B.12}$$

In these cases use (B.10) if $p^2=1$; otherwise use the results that

$$\frac{1}{2\pi}\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \frac{R_n(z)(4cd-z^2)^{\frac{1}{2}}(r-\cos\theta)}{(1+r^2-2r(\cos\theta))^2}\,dz = \begin{cases} -\frac{n}{2}(cd)^{(n+2)/2}r^{n-1}, & r^2<1 \\ \frac{n+2}{2}(cd)^{(n+2)/2}\frac{1}{r^{n+3}}, & r^2>1 \end{cases} \tag{B.13}$$

and

$$\frac{1}{2\pi}\int_{-2\sqrt{cd}}^{2\sqrt{cd}} \frac{R_n(z)(4cd-z^2)^{\frac{1}{2}}}{(1+r^2-2r(\cos\theta))^2}\,dz = \begin{cases} \frac{(n+1)r^n}{(1-r^2)}(cd)^{(n+2)/2}, & r^2<1\ , \\ \frac{(n+1)(cd)^{(n+2)/2}}{(r^2-1)r^{n+2}}, & r^2>1\ . \end{cases} \tag{B.14}$$

In the subsequent evaluations, the fact that $b=[-(cd)^{\frac{1}{2}}(p+q)/a]+a(1+1/pq)$

is helpful.■

<u>Outline for Theorem 3.2.</u> First it is shown that this case does not occur for non-negative RL values. Recall that a, b, c, and d (as defined for (3.5)) are given by $a=(L_o-\ell_1-L_2)/u$, $b=-((\ell_o+L_1+\ell_1)/v)-((\ell_1+L_2+\ell_o)/u)$, $c=-\ell_o^2/(uv)$, and $d=-\ell_1^2/(uv)$ where, for convenience, $u=R_2+d_oL_2$ and $v=-(R_1+d_oL_1)$ (note that uv>0 since c<0 and d<0). The requirement that $p^2>1$ becomes $(L_o-\ell_1-L_2)^2v/(u\ell_1^2)>1$ (since $p^2=-a^2/d$) or

$$(L_o-L_2-2\ell_1)(L_o-L_2) > (\frac{u}{v}-1)\ell_1^2 \quad . \tag{B.15}$$

The requirement that p=q becomes H=0 (see Chapter III) or $(ab+d-a^2) = \pm 2a\sqrt{-c}$ which reduces to

$$-L_1-\ell_1-\frac{L_ov}{u}-\frac{\ell_1^2}{(L_o-\ell_1-L_2)} = \ell_o(1\pm\sqrt{\frac{v}{u}})^2 \tag{B.16}$$

(note that $L_o-\ell_1-L_2=0$ implies a=0 or $p^2 \ngtr 1$, a different case). A necessary condition so that (B.16) holds for non-negative RL values is that $(L_o-\ell_1-L_2)<0$. Let $L_o-\ell_1-L_2=-\epsilon$ for some $\epsilon>0$. Then (B.15) reduces to

$$\epsilon^2 > \frac{u}{v}\ell_1^2 \quad . \tag{B.17}$$

Further, since the left-side of (B.16) must be non-negative,

$$-L_1-L_2\frac{v}{u}+\frac{1}{\epsilon}(\ell_1-\epsilon)(\ell_1-\frac{v}{u}\epsilon) \geq 0 \quad . \tag{B.18}$$

There are three cases to consider in (B.18):

Case 1: $\ell_1 > \epsilon$.

Since from (B.17), $\ell_1 < \sqrt{\frac{v}{u}}\,\epsilon$, it follows that $\sqrt{\frac{v}{u}} > 1$; hence, $\frac{v}{u} > \sqrt{\frac{v}{u}}$. Thus $\ell_1 < \sqrt{\frac{v}{u}}\,\epsilon < \frac{v}{u}\epsilon$ or $\ell_1 - \frac{v}{u}\epsilon < 0$, which contradicts (B.18).

Case 2: $\ell_1 = \epsilon$.

Then from (B.16) $\sqrt{\frac{v}{u}} = 1$ and from (B.17) $\frac{u}{v} < 1$, a contradiction.

Case 3: $\ell_1 < \epsilon$.

Return to (B.16); since $-\ell_1 - \ell_1^2/(L_o - \ell_1 - L_2)$ is now negative, (B.16) is contradicted.

Now Theorem B.1 can be invoked to verify the remainder of Theorem 3.2. The integrals involving (3.10) are discussed in (B.11) through (B.14). The values of S_{2n} and S_{2n+1} at $x = [(cd)^{\frac{1}{2}}p/a]-a$ are also discussed in the proof of Theorem 3.1. It remains to determine the values of the derivatives of S_{2n} and S_{2n+1} at $x = [(cd)^{\frac{1}{2}}p/a]$. These tedious calculations (use the chain rule on (B.3) plus various recurrence relations) are omitted. The results are

$$\left.\frac{dS_{2n-1}(x)}{dx}\right/_{x = \frac{(cd)^{\frac{1}{2}}}{a}p-a} = \frac{(cd)^{\frac{n-1}{2}}(n)}{p^{n-1}}\quad, \quad n\geq 1\quad,$$

$$\left.\frac{dS_{2n+2}(x)}{dx}\right/_{x = \frac{(cd)^{\frac{1}{2}}}{a}p-a} = \frac{a(cd)^{\frac{n}{2}}(n+1)}{p^{n+2}}\quad, \quad n\geq 0\quad. \tag{B.19}$$

This information can now be used to verify that the requirements of Theorem B.1 are satisfied.■

APPENDIX C

The proof of Theorem 4.3 requires a great deal of algebra. Normally, it would be necessary to verify the biorthonormalizability properties of all possible combinations of Ψ's and W's. Given periodicity, however, the task is eased by a result somewhat similar to that given in Appendix B.

Theorem C.1. Let the d_i, $i \geq 0$, of (1.3) be periodic with period n, that is, $d_i = d_{i+n}$, $i \geq 0$. Let $\gamma(x)$ be a function of bounded variation with the following properties:

$$\text{(i)} \qquad \int_{-\infty}^{\infty} W_o(x)d\gamma(x) \neq 0,$$

$$\text{(ii)} \qquad \int_{-\infty}^{\infty} \Psi_i(x)W_o(x)d\gamma(x) = 0 \quad , \quad i \geq 1 \quad .$$

Then if an integrator for the system $\{\Psi_i(x), W_i(x)\}_{i=o}^{\infty}$ exists (see Definition 2.3), $\gamma(x)$ is such an integrator.■

Proof. Assume that there exists a function $\beta(x)$ of bounded variation such that

$$\text{(iii)} \quad \int_{-\infty}^{\infty} \Psi_i(x)W_j(x)d\beta(x) = 0, \; i \neq j \quad ,$$

$$\text{(iv)} \quad \int_{-\infty}^{\infty} \Psi_i(x)W_i(x)d\beta(x) \neq 0, \quad i \geq 0 \quad .$$

Then $\beta(x)$ is an integrator. It is desired to replace $\Psi_i(x)W_j(x)$ by $W_o(x)$ times a sum of Ψ's. To do this, observe that $W_o(x) = (a_o - d_o)/(x - d_o)$ and

$\Psi_i(x)W_j(x) = W_o(x)[\Psi_i(x)W_j(x)/W_o(x)]$. Now write $\Psi_i(x) = \sum_{m=o}^{i} \delta_m/(x-d_m)^{e_m} + \delta_{i+1}$ and $W_j(x)/W_o(x) = \sum_{m=o}^{j} \gamma_m/(x-d_m)^{e_m} + \gamma_{j+1}$, multiply together and perform a partial fraction expansion resulting in terms of the form:

(A) a constant;

(B) $\eta/(x-d_m)^q$, $1 \leq q \leq 2e_m$.

Write (A) as $\Psi_o(x)$ times a constant. Since the d_m are periodic, there is a p such that $e_p = q$ and $d_m = d_p$; so (B) can be written as a sum of Ψ's by using Lemma A.5 and (1.9). Thus $\Psi_i(x)W_j(x)/W_o(x) = \sum_{k=o}^{\ell} \alpha_{k(i,j)}\Psi_k(x)$ for some finite ℓ with i,j arbitrary. Then by using (iii) and (iv) on $\Psi_i(x)W_j(x)$, it can be seen that $\alpha_{o(i,i)} \neq 0$ and $\alpha_{o(i,j)} = 0$, $i \neq j$. With $\alpha_{o(i,i)} \neq 0$ and $\alpha_{o(i,j)} = 0$, it then follows by use of (i) and (ii) that (iii) and (iv) hold with $d\beta(x)$ replaced by $d\gamma(x)$. That is, $\gamma(x)$ is an integrator.■

Proof of Theorem 4.3. It is clear that $x_n(t)$ of (4.2) satisfies the differential system (1.1). It remains to show that $x_n(t)$ satisfies the initial conditions, $x_j(0) = \gamma_j$, $x_k(0) = 0$, $k \neq j$, or, in other words, $\beta(x)$ as given in Theorem 4.3 is in fact the desired integrator (whose existence is assured by the conditions in the theorem).

By Theorem C.1, it is sufficient to show that

$$\int_{-\infty}^{\infty} \Psi_n(x)W_o(x)d\beta(x) = \delta_{on} \quad , \quad n \geq 0 \quad . \tag{C.1}$$

To simplify subsequent arguments, it will be assumed that both d_o and d_1 lie between the intervals whose endpoints are given in (4.9) and (4.10). Under these assumptions, it is possible to find the right-hand endpoint of each interval. For the interval of (4.9), it is known that $G_{2n+1}(x)$ is positive at the right-hand endpoint (see the proof of Theorem 2.5). But $G_{2n+1}(x) = b_o^n(x-d_o)^n b_1^n(x-d_1)^n x R_n(z)$ from (4.7) and (2.13) while $R_n(2) = n+1$ and $R_n(-2) = (-1)^n(n+1)$. Thus (4.9a) is a right-hand endpoint. Similarly, knowing that $G_{2n+1}(x)$ is negative at the left-hand endpoint of the interval given in (4.10) says that (4.10b) is the right-hand endpoint of that interval. In all subsequent discussions, where $\pm$ or $\mp$ appear, the upper sign refers to the interval of (4.9) and the lower to (4.10).

Case 1: $p^2 < 1.$

For the $d\beta(x)$ of (4.12), it can be shown, since $a_o = 0$, that

$$W_o(x)d\beta(x) = \left(\frac{-d_o}{x-d_o}\right) \frac{b_o(x-d_o)b_1(x-d_1)\sqrt{4-z^2}\,dx}{2\pi|x|b_1^2(x-d_1)^2} = \frac{\sqrt{4-z^2}}{2\pi}\left(\frac{-b_od_o(dx)}{b_1(x-d_1)|x|}\right) \quad . \tag{C.2}$$

Now $1/x = s/(m\pm q) = (m\mp q)/r$; so $-dx/x^2 = [(m_z \mp q_z)r - (m\mp q)r_z]dz/r^2$, where m, q, r and s are defined in Chapter IV and m_z, q_z, r_z represent the derivatives of the functions m, q, and r, respectively, with respect to z. With $U = b_o(d_o^2 s - 2d_o m + r)$, which is a constant, it follows that

$$-\frac{dx}{x^2} = \frac{1}{r}\left(-2b_ob_1d_o\left(1-\frac{d_1}{x}\right) \mp \frac{Ub_1(x-d_1)x}{qx(x-d_o)}\right)dz \quad , \tag{C.3}$$

$$\frac{dx}{xb_1(x-d_1)} = \frac{1}{r}\left(2b_od_o \pm \frac{Ux}{q(x-d_o)}\right)dz \quad , \tag{C.4}$$

where the upper sign holds on the interval of (4.9) and the lower on (4.10). Thus

$$W_o d\beta(x) = \frac{\sqrt{4-z^2}}{2\pi}\left(\frac{\mp b_o d_o}{r}\right)\left(2b_o d_o \pm \frac{Ux}{q(x-d_o)}\right)dz \quad . \tag{C.5}$$

Let $T = (-b_o d_o/r)(2b_o d_o + Ux/[q(x-d_o)])$, $V = (-b_o d_o/r)(2b_o d_o - Ux/[q(x-d_o)])$. Then on the interval of (4.9),

$$T = \frac{1}{2}\left[\frac{Iq-Hz-J-p}{Iq(1+p^2-pz)}\right] \quad , \tag{C.6}$$

and on the interval of (4.10),

$$V = \frac{1}{2}\left[\frac{Iq+Hz+J+p}{Iq(1+p^2-pz)}\right] \quad , \tag{C.7}$$

where H, I, and J are coefficients in the expansion $C = Hz \pm Iq+J$ (see (4.5)). Equations (C.6) and (C.7) are verified by working backwards and using the fact $I = b_1(d_1-d_o)/U$. In addition, if A (see (4.5)) is written as $Ez \pm Fq + G$, where E, F, and G are constants, then for $z \in [-2,2]$,

$$T + V = \frac{1}{1+p^2-pz} \quad , \tag{C.8}$$

$$T - V = \frac{-(J+p+Hz)}{Iq(1+p^2-pz)} = \frac{(G+Ez)}{Fq(1+p^2-pz)} \quad . \tag{C.9}$$

Thus (see (4.7))

$$\int_{-\infty}^{\infty} \Psi_{2n+1}(x)W_o(x)d\beta(x) = \int_{-2}^{2} (Ez+Fq+G)R_n(z)\,\frac{\sqrt{4-z^2}}{2\pi}\,T(z)dz$$

$$+\int_{-2}^{2} (Ez-Fq+G)R_n(z)\,\frac{\sqrt{4-z^2}}{2\pi}\,(-V(z))dz \qquad (C.10)$$

$$= 0 \quad ,$$

by using (C.8) and (C.9) after collecting terms on the right side of (C.10). Also from (4.7),

$$\Psi_{2n}(z) = R_n(z) + \left(\frac{b_1 d_1}{b_o d_o} + \frac{b_1(d_o-d_1)x}{b_o d_o(x-d_o)}\right)R_{n-1}(z) = R_n(z) + \left(\frac{b_1(d_o-d_1)}{d_o}\right.$$

$$\left.\cdot A - p\right)R_{n-1}(z) \quad .$$

Thus

$$\int_{-\infty}^{\infty} \Psi_{2n}(x)W_o(x)d\beta(x) = \frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-z^2}\left(\frac{R_n(z)-pR_{n-1}(z)}{1+p^2-pz}\right)dz \quad . \qquad (C.11)$$

By using the fact that

$$\frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-z^2}\,\frac{R_n(z)}{1+p^2-pz}\,dz = \begin{cases} p^n & , \quad p^2<1 \quad , \\ \frac{1}{p^{n+2}} & , \quad p^2>1 \quad , \end{cases} \qquad (C.12)$$

the case $p^2 < 1$ is now concluded.

Case 2: $p^2 > 1$.

It is not hard to see that when $x=0$, $z = (1+p^2)/p$, $R_n(z) = [1/(1-p^2)][1/p^n-p^{n+2}]$, and $\Psi_{2n}(0) = 1/p^n$. So for $p^2 > 1$, (C.11) becomes

$1/p^{n+2} - p/p^{n+1} = 1/p^n(1/p^2 - 1)$. Thus it is necessary to add to $\beta(x)$ a jump of magnitude $(1-1/p^2)$ at $x=0$ in order to make the integral in (C.11) zero; and adding this jump leaves the integral in (C.10) unaffected because $\Psi_{2n+1}(0) = 0$.

Case 3: $p^2 = 1$.

By using the facts

$$\frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-z^2}\,\frac{R_n(z)}{2 \pm z}\,dz = (\mp 1)^n \quad , \tag{C.13}$$

the desired result follows ($\pm$ and $\mp$ here do not refer to (4.8) and (4.9)).∎

APPENDIX D

Several times it has been necessary to differentiate (4.2) underneath the integral sign. This appendix discusses some of the factors which permit this interchange of operations. First it is shown that the integral in (4.2) is well-defined; that is, it is demonstrated that $\Psi_n(x)W_j(x)$ and e^{xt} are both bounded for $x\epsilon S$ (the closure of the set of all points of increase of $\beta(x)$), and for $t\epsilon[0,a]$, "a" finite. This argument is then extended to permit the desired differentiation.

Recall that

$$x_n(t) = \gamma_j(\ell_{j-1}+L_j+\ell_j)\int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}} e^{xt}d\beta(x) \quad ,$$

$$n\geq 0 \quad . \qquad (4.2)$$

By Lemma 2.2, S is contained in G, the closure of the set of all zeros of the $G_n(x)$, $n\geq 0$. Further, in Theorem 2.8, it is required that D, the closure of the set of all d_n, $n\geq 0$, not intersect G. Since the singularities of $\Psi_n(x)$ (see (2.13)) and $W_j(x)$ (see (A.25)) all lie in D, and since the product $\Psi_n(x)W_j(x)$ is finite as $x\rightarrow\pm\infty$($\Psi_n(x)W_j(x)$ is the ratio of a polynomial of degree (j+n) to a polynomial of degree (j+n+1)), it follows that $\Psi_n(x)W_j(x)$ is bounded for $x\epsilon G$ (hence S), $t\epsilon[0,a]$. The function e^{xt} is also bounded for $x\epsilon G$, $t\epsilon[0,a]$ because the set G is contained in the interval $(-\infty,0]$ as is now shown.

$G_o(0)$ and $G_1(0)\geq 0$ (see (2.2) and (2.12)); so by the zero separation properties of the $G_n(x)$, $n\geq 0$, (see the proof of Theorem 2.5) and by the fact that the leading coefficient of $G_n(x)$ is positive (see the proof of Theorem 2.5 again), it follows that it is sufficient to show that $G_n(0) \geq 0$, $n\geq 0$. To show this, first note that if $\beta(x)$ in (4.2) exists, then $a_n \neq 0$, $n\geq 0$ (the condition $a_n = 0$ for some n requires that $r_n = r_{n-1} = 0$ or $d_n = d_{n-1} = 0$ and then from $G_{n+1}(0) = (-a_n)G_n(0) - b^2_{n-1}d^2_{n-1}G_{n-1}(0)$ (see (2.12)) it follows that $G_{n+1}(d_n) = G_{n+1}(0) = 0$ which violates Lemma 4.1). Since $a_n \neq 0$, $n\geq 0$, it is possible to write $G_n(0) = (-a_o)(-a_1) \ldots (-a_{n-1})H_n$ where $H_{n+1} = H_n - r^2_{n-1}H_{n-1}/[(r_{n-2}+R_{n-1}+r_{n-1})(r_{n-1}+R_n+r_n)]$. The coefficient of H_{n-1} in this equation generates a chain sequence and by the techniques of solution developed earlier (see the proof of Theorem 2.5 and the first part of Chapter IV), $H_n \geq 0$, $n\geq 0$, from which it follows that $G_n(0) \geq 0$, $n\geq 0$.

Since $\beta(x)$ is non-decreasing and bounded, it is now clear that (4.2) is well-defined.

Now, in order to verify that (4.2) is a solution of (1.1) for $t>0$, it must be shown that

$$\frac{dx_n(t)}{dt} = \gamma_j(\ell_{j-1}+L_j+\ell_j)\int_{-\infty}^{\infty} \frac{\Psi_n(x)}{(\ell_{n-1}+L_n+\ell_n)^{\frac{1}{2}}} \frac{W_j(x)}{(\ell_{j-1}+L_j+\ell_j)^{\frac{1}{2}}} xe^{xt}d\beta(x) \quad , \quad n\geq 0 . \qquad (D.1)$$

An extension of the argument above shows that $x\Psi_n(x)W_j(x)$ is bounded on G and by the Weierstrass M-test and Theorem 14-24 in Apostol [1], $dx_n(t)/dt$ is indeed given by (D.1) for $t>0$.

In order to apply the Weierstrass M-test and Theorem 14-24 in

Apostol [1] to higher order derivatives, it is sufficient to require that

$$\left|\int_{-\infty}^{\infty} x^r d\beta(x)\right| < \infty \quad , \quad r \geq 0 \quad .$$

Note also that the above results all hold if the set G is replaced by the set GN of Theorem 4.2.

BIBLIOGRAPHY

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Reading, MA, (1960).

2. R. P. Boas, Jr., "The Stieltjes Moment Problem for Functions of Bounded Variation," Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 399-404.

3. W. G. Christian, A. G. Law, W. F. Martens, A. L. Mullikin and M. B. Sledd, "Solution of Initial-Value Problems for Some Infinite Chains of Harmonic Oscillators," Journal of Mathematical Physics, vol. 17 (1976), pp. 146-158.

4. Ya. L. Geronimus, "Sur quelques équations aux différences finies et les systèmes correspondants des polynômes orthogonaux," C. R. (Doklady) Acad. Sci. U.R.S.S (N.S.), vol. 29 (1940), pp. 536-538 (contains errors).

5. Ya. L. Geronimus, "On Some Finite Difference Equations and Corresponding Systems of Orthogonal Polynomials," Memoirs of the Mathematical Section of the Faculty of Mathematics and Physics at Kharkov State University and the Kharkov Mathematical Society, vol. 25 (1957), pp. 87-100.

6. J. Grommer, "Ganze transcendente Funktionen mit lauter reelen Nullstellen," Journal für die reine und angewandte Mathematik, vol. 144 (1914), pp. 114-165.

7. D. V. Ho, J. W. Jayne and M. B. Sledd, "Recursively Generated Sturm-Liouville Polynomial Systems," Duke Mathematical Journal, vol. 33 (1966), pp. 131-140.

8. S. Karlin and J. L. McGregor, "The Differential Equations of Birth and Death Processes and the Stieltjes Moment Problem," Transactions of the American Mathematical Society, vol. 85 (1957), pp. 489-546.

9. A. N. Khovanskii, The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory (translated by P. Wynn), P. Noordhoff, Ltd., Groningen, The Netherlands (1963).

10. A. G. Law, Solutions of Some Countable Systems of Ordinary Differential Equations, Doctoral Dissertation, Georgia Institute of Technology, 1968.

11. W. F. Martens, Solutions of the Differential Equations of Some Infinite Linear Chains and Two Dimensional Arrays, Doctoral Dissertation, Georgia Institute of Technology, 1971.

12. A. W. Naylor and G. R. Sell, Linear Operator Theory in Engineering and Science, Holt, Rinehart and Winston, Inc., Atlanta, GA (1971).

13. I. P. Natanson, Theory of Functions of a Real Variable (translated by L. F. Boron), Frederick Ungar Publishing Co., New York, NY (1960).

14. A. Pringsheim, "Über die Konvergenz unendlichen Kettenbrüche," Sitzungsberichte der mathematisch-physikalischen Classe der Kgl. Bayer. Akademie der Wissenschaften, vol. 28 (1898), pp. 295-334.

15. J. Shohat, "Sur les polynomes orthogonaux généralisés," Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 556-558.

16. J. A. Shohat and J. D. Tamarkin, The Problem of Moments, Mathematical Surveys Number I, published by the American Mathematical Society (1943).

17. M. H. Stone, Linear Transformations in Hilbert Space and Their Applications to Analysis, American Mathematical Society Colloquium Publications, vol. XV (1964).

18. H. S. Wall, Analytic Theory of Continued Fractions, D. Van Nostrand Company, Inc., New York, NY (1948).

19. A. Wouk, "Difference Equations and J-Matrices," Duke Mathematical Journal, vol. 20 (1953), pp. 141-159.

VITA

Michael Scott West was born September 1, 1950, in Newport News, Virginia. He attended the public schools in Newport News until he entered Rensselaer Polytechnic Institute as a freshman in September 1968. In September 1971, he transferred to George Mason University, where he received the degree of Bachelor of Science in Mathematics in August 1972.

In September 1973, after working at the Newport News Shipbuilding Company for a year, he entered the Ph.D. program in mathematics at the Georgia Institute of Technology. After his first year, during which he was supported by a President's Fellowship, he became a graduate teaching assistant in the School of Mathematics.

In June 1977, he began work on a master's degree in electrical engineering. He received the degree of Master of Science in Applied Mathematics in December 1977.