OPEN BOOK DECOMPOSITIONS IN HIGH DIMENSIONAL CONTACT MANIFOLDS

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Master of Science in the
School of Mathematics

Georgia Institute of Technology
May 2016

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Date Approved: 22 February 2016

ACKNOWLEDGEMENTS

First of all, I would like to thank my advisor, Professor John Entyre, for being a great advisor, clarifying my all questions and guiding my studies. After every meeting, I left his office with a great motivation since he presented all the material in the clearest and most understandable way. Our meetings and discussions helped me understand the contact and symplectic topology together. I appreciate his time and effort that made this thesis possible.

I also want to thank to Professor Mohammad Ghomi, for his help and explaining many things in his office hours. His help and his Differential Topology class helped me to have a better understanding. I also want to thank to Professor Manfred Lehn for introducing me the world of topology during my Erasmus Exchange Program semester at Johannes Gutenberg Universität, Germany. I will never forget the lecture in which Professor Lehn explained the two definitions of continuity (the one we know from Calculus and the one he presents in his class) were the same. I also want to thank to my undergraduate studies advisor Professor Ismet Karaca for conducting my first self-study and guiding my first steps in research.

I am extremely grateful to my dear wife, Sevil, for her continuous patience and encouragement. My degree would not be possible without her unconditional support and love. I am also grateful to my 11 months old daughter, Mila. This work definitely has been lighted by her first smiles.

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SUMMARY

In this Masters thesis, we focus on understanding the open book decompositions of high dimensional contact manifolds. Open book decompositions are very crucial figures in three dimensional contact topology thanks to Giroux's theorem but the role of open books in higher dimensions is currently being studied.

In Chapter 1, we give a basic introduction to contact and symplectic topology. Symplectic topology is included in this thesis since open books have pages with symplectic structures on them. In Chapter 1, we also see contact structures as distributions and explain some well known facts in terms of distributions.

Chapter 2 introduces the open book decompositions. We first define them for ordinary manifolds and then explain why they are important for contact manifolds. The correspondences between contact structures and open book decompositions are explained.

Chapter 3 is about the generalization of Murasugi sums and plumbings to higher dimensions. This chapter is mostly based on a paper called "Generalized Plumbings and Murasugi Sums" by Ozbagci and Popescu-Pampu. It describes an operation called embedded sum which can be considered as a generalization of Murasugi sums for manifolds of any dimensions. It explains that Murasugi sum of open books is again an open book. However, the relations between embedded sums and contact structures supported by open books is currently being studied.

Chapter I

ON CONTACT AND SYMPLECTIC MANIFOLDS

The word symplectic topology first appears in a paper called The First Steps in Symplectic Topology written by Vlademir Igorevic Arnold in 1986. But the development of contact topology is even more recent than symplectic topology. Symplectic and contact topology are generally mentioned together. Symplectic topology can be defined on even dimensional manifolds where contact topology can be defined on odd dimensional manifolds.

Although the primary focus of this study is contact manifolds and open book decompositions of contact manifolds, it is useful to start with some basic structures on symplectic manifolds since pages of open books are even dimensional and they have symplectic structures.

1.1 Basics on Symplectic Geometry and Topology

1.1.1 Symplectic Maps and Symplectic Vector Spaces

Let V be an n-dimensional vector space and let $\omega : V \times V \rightarrow R$ be a skew-symmetric bilinear map. Note that ω is skew-symmetric if $\omega(v_1, v_2) = -\omega(v_2, v_1)$ for every $v_1, v_2 \in V$. Under these conditions, naturally we can define a linear map $\overline{\omega} : V \rightarrow V^*$ by $\overline{\omega}(v_1) = \omega_{v_1}$ where V^* denotes the dual space of the vector space V and $\omega_{v_1} : V \rightarrow R$ is defined by $\omega_{v_1}(v_2) = \omega(v_1, v_2)$ for every $v_2 \in V$.

Let us see $\overline{\omega}$ is a linear map. Let V be a vector space over a field F. Then for any $v_1, v_2 \in V$ and $a, b \in F$ we have :

$$\overline{\omega}(av_1 + bv_2) = \omega_{av_1+bv_2}$$

For any $u \in V$ using the bi-linearity of ω, we have

$$\omega_{av_1+bv_2}(u) = \omega(av_1 + bv_2, u) = a\omega(v_1, u) + b\omega(v_1)(u)$$

which proves

$$\overline{\omega}(av_1 + bv_2) = a\overline{\omega}(v_1) + b\overline{\omega}(v_2).$$

Definition 1. *For a vector space V, a skew-symmetric bilinear map $\omega : V \times V \longrightarrow R$ is called a symplectic form if the map $\overline{\omega}$ defined as above is bijective. The pair (V, ω) is called a symplectic vector space.*

There exists a more famous definition of symplectic forms.

Definition 2. *On a vector space V of finite dimension, a bilinear map ω is called non-degenerate if $\omega(u, v) = 0$ for all $v \in V$ then $u = 0$. Moreover ω is called symplectic if it is non-degenerate and skew-symmetric.*

In order to see the equivalence between Definition 1 and Definition 2, assume that we have the map $\overline{\omega}$ is bijective and $\omega(u, v) = 0$ for all $v \in V$ where V is a finite dimensional vector space. Then we would have $\overline{\omega}(u)$ is the zero map. By the linearity and injectivity of $\overline{\omega}$ we would have $u = 0$. In order to prove the reverse implication we need to show $\overline{\omega}$ is bijective when ω is given as in Definition 2. $\overline{\omega}$ is injective because $\overline{\omega}(v_1) = \overline{\omega}(v_2)$ implies $\omega_{v_1} = \omega_{v_2}$, so $\omega(v_1, u) = \omega(v_2, u)$ for all $u \in V$. Since ω is linear we have $\omega(v_1 - v_2, u) = 0$. By the statement in Definition 2, $v_1 - v_2$ has to be zero which gives $v_1 = v_2$. Hence $\overline{\omega}$ is injective. Surjectivity of $\overline{\omega}$ follows from the rank nullity theorem. $\overline{\omega}$ is an injective linear map, so its kernel is the single element 0. Hence the kernel of $\overline{\omega}$ has dimension 0 which gives $dim(Im(\overline{\omega})) = dim(V) = dim(V^*)$. Thus $\overline{\omega}$ is bijective.

We now introduce some examples of symplectic vector spaces.

Example 3. *Let V be a vector space and let V^* denote the dual space of V. Set $W = V \oplus V^*$ and define $\omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2)$. Since*

$$\omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2) = -(\alpha_1(v_2) - \alpha_2(v_1)) = -\omega((v_2, \alpha_2), (v_1, \alpha_1)),$$

we conclude that ω is skew-symmetric. It is also non-degenerate. In order to see this assume that $\omega((v_1, \alpha_1), (v, \alpha)) = 0$ for all $(v, \alpha) \in W$. Then

$$\alpha(v_1) - \alpha_1(v) = 0$$

$$\Rightarrow \alpha(v_1) = \alpha_1(v)$$

which is possible only when $\alpha \in V^*$ is the zero map and $v = 0$.

Definition 4. *Let (V, ω) be a symplectic vector space and let E be a subspace of V. Then the symplectic orthogonal of E is denoted by $E^\perp$ and it is defined as*

$$E^\perp := \{v \in V \mid \omega(u, v) = 0, \forall u \in E\}.$$

Definition 5. *For a symplectic vector space (V, ω) and a subspace $E \subset V$*

1. *E is called isotropic if $E \subset E^\perp$*
2. *E is called Lagrangian if $E = E^\perp$*
3. *E is symplectic if $E \cap E^\perp = \{0\}$.*

Proposition 6. *Let (V, ω) be a symplectic vector space and let E be a symplectic subspace of V. Then $V = E \oplus E^\perp$.*

Proof. We start with showing that for any subspace E of V, $dim(E) + dim(E^\perp) = dim(V)$. Consider the exact sequence

$$0 \longrightarrow E \longrightarrow V \longrightarrow V/E \longrightarrow O$$

which induces the exact sequence

$$0 \longrightarrow (V/E)^* \longrightarrow V^* \longrightarrow E^* \longrightarrow 0.$$

Note that the dual space $(V/E)^*$ is isomorphic to the space of linear maps $\lambda : V \longrightarrow R$ which vanish on E. $(V/E)^*$ is called the annihilator of E in V^* which is denoted by

$AnnE$. By the isomorphism $\overline{\omega} : V \longrightarrow V^*$, $E^{\perp}$ is identified with $AnnE$ so they are of the same dimension. From linear algebra, for any subspace E of V, $V = E \oplus AnnE$ so $dim(E) + dim(E^{\perp}) = dimV$.

Moreover since $E \cap E^{\perp} = \{0\}$, we have $V = W \oplus W^{\perp}$. □

Both the definitions describing symplectic vector spaces we have given do not give a direct reference to the fact that symplectic vector spaces have even dimension. We will have a proposition addressing to this issue.

Proposition 7. *Let (V, ω) be a symplectic vector space. Then there exists a basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ of V such that*

$$\omega(e_i, e_j) = 0$$

$$\omega(f_i, f_j) = 0$$

$$\omega(e_i, f_j) = \delta_{ij}.$$

Proof. Fix a non-zero element $v \in V$. Since ω is non-degenerate there exists an element $u^{'} \in V$ satisfying $\omega(v, u^{'}) = c \neq 0$. Set $u = \frac{u^{'}}{c}$. Then by the bilinearity of ω, $\omega(v, u) = 1$. The bilinearity of ω also implies v and u are linearly independent since if $v = \lambda u$, then $c = \omega(v, u) = \omega(\lambda u, u) = \lambda\omega(u, u) = 0$ which is a contradiction. Let U be the subspace generated by u and v. Then U is of dimension 2 and it is symplectic. The proof is complete if $dim(V) = 2$ as well. If not, we can generate the claimed basis by using induction, Proposition 6 and the fact that for a symplectic subspace E, $E^{\perp}$ is a symplectic subspace as well. □

1.1.2 Symplectic Manifolds

Now it is time to introduce the symplectic forms.

Definition 8. *Let ω be a 2-form on M. Then for each $p \in M$, we have $w_p : T_pM \times T_pM \rightarrow R$ which is skew-symmetric and bilinear on T_pM. We say a 2-form ω is* *symplectic on a even dimensional manifold M if ω is closed and w_p is symplectic on the vector space T_pM for each $p \in M$. The pair (M, ω) is called a symplectic manifold.*

As the definition describes, given a symplectic manifold (M, ω), the following conditions are satisfied;

1. d$\omega = 0$, i.e. ω is closed on M

2. For any $p \in M$, whenever $w \mid_{T_pM} (u, v) = 0$ for all $v \in T_pM$ we have $u = 0$.

Example 9. ***(Standard Symplectic Structure on*** R^{2n})
Take $M = R^{2n}$ with coordinates $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$. Define the 2-form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. Then (R^{2n}, ω) is a symplectic manifold.

The fact that symplectic manifolds are even dimensional manifolds does not promise that every even dimensional manifold can be considered as symplectic manifold. The following proposition is an example of this situation.

Proposition 10. *S^4 does not admit a symplectic structure.*

Proof. Assume that S^4 has a symplectic structure, i.e. there exists a 2-form ω which is closed and non-degenerate on S^4. Since the second de Rham cohomology group of S^4 is trivial, ω is exact on S^4 so there exists a 1-form α such that $\omega = d\alpha$. Now set $\Omega = \omega \wedge \omega$ which is a volume form on S^4 by the non-degenerativity of ω. Note that

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + \omega \wedge d\alpha = \omega \wedge \omega = \Omega$$

which means $\Omega = \omega \wedge \omega$ is an exact form, and thus is zero in cohomology. However,

$$\int_{S^4} \Omega = \int_{S^4} 0 = 0$$

which contradicts the fact that Ω is a volume form. $\square$

Two well-known manifolds of even dimension are tangent bundles and cotangent bundles. The following example shows cotangent bundles are symplectic manifolds.

Example 11. *Let M be a smooth manifold and let T^*M be its cotangent bundle. Any section of $\pi : T^*M \to M$ is a 1-form, then we can define a 1-form on α on T^*M as*

$$\alpha(\xi_x) = \pi^*(\xi_x)$$

where $\pi(\xi_x) = x$ and

$$\pi^* : T_x^*M \to T_{\xi_x}^*(T^*M)$$

is the pull-back. So for a given $v_{\xi_x} \in T_{\xi_x}(T_x^*M)$, $\alpha(v_{\xi_x}) = \xi_x(\pi_* v_{\xi_x})$. After this set-up define the symplectic form on T^*M by

$$\omega = -d\alpha.$$

Definition 12. *For two symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$ a diffeomorphism $\phi : M_1 \to M_2$ is called a symplectomorphism if $\phi^*\omega_2 = \omega_1$ where ϕ^* denotes the pull-back of ϕ.*

Given a smooth manifold of even dimension, M, it is possible to have two symplectic forms ω_1 and ω_2 which make (M, ω_1) and (M, ω_2) two different symplectic manifolds. The following two definitions relate to this situation.

Definition 13. *Let M be a smooth manifold with two symplectic forms ω_1 and ω_2. Then (M, ω_1) and (M, ω_2) are said to be strongly isotopic if there exists an isotopy $\rho_t : M \to M$ satisfying $\rho_1^*\omega_1 = \omega_0$.*

Clearly strongly isotopic symplectic manifolds are symplectomorphic.

Definition 14. *(M, ω_1) and (M, ω_2) are said to be deformation equivalent if there exists a smooth family ω_t of symplectic forms joining ω_1 to ω_2. (M, ω_1) and (M, ω_2) are said to be isotopic if they are deformation equivalent and the de Rham cohomology class $[\omega_t]$ is independent of t.*

An important and useful tool we have on symplectic manifolds is called Moser's trick which can be considered as a result of Cartan's magic formula. Assume that we have two k-forms α_0 and α_1 on a manifold M and we need a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi^*\alpha_1 = \alpha_0$. Now let α_t be a smooth family of k-forms including α_0 and α_1. So we would like a vector field X whose flow ϕ satisfies $\phi_t^*\alpha_t = \alpha_0$. When $\phi_t^*\alpha_t = \alpha_0$ is satisfied we have

$$\frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*(\frac{d\alpha_t}{dt} + L_{X_t}\alpha_t) = 0.$$

By Cartan's formula, $L_X\alpha_t = i_X d\alpha_t + d i_X \alpha_t$, we have,

$$\phi_t^*(\frac{d\alpha_t}{dt} + i_X d\alpha_t + d i_X \alpha_t) = 0$$

so

$$\frac{d\alpha_t}{dt} + i_X d\alpha_t + d i_X \alpha_t = 0.$$

If there exists a vector field X satisfying this equation and M is compact, the existence of ϕ is given by integration. The following theorem describes the Moser's trick, where we can solve the equation for X.

Theorem 15. *Let ω_0 and ω_1 be two symplectic forms on a compact manifold M such that $[\omega_0] = [\omega_1]$ and for every $t \in [0,1]$, $\omega_t = t\omega_1 + (1-t)\omega_0$ is a symplectic form on M. Then there exists a smooth family of diffeomorphisms $\phi_t : M \longrightarrow M$ satisfying $\phi_0 = id_M$ and $\phi_t^*\omega_t = \omega_0$. Moreover (M, ω_0) is symplectomorphic to (M, ω_1).*

Proof. Note that $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ so

$$[\frac{d\omega_t}{dt}] = [\omega_1 - \omega_0] = 0 \in H^2_{dR}(M).$$

Then there exist a 1-form β such that

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\beta.$$

Since ω_t is non-degenerate there exists a vector field X_t such that

$$i_{X_t}\omega_t = -\beta.$$

Now let ϕ_t be a set of diffeomorphisms such that

$$\frac{d\phi_t}{dt} = X_t \circ \phi_t.$$

Then

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^* L_{X_t}\omega_t + \phi_t^*\frac{d\omega_t}{dt} = \phi_t^* d i_{X_t}\omega_t + \phi_t^* d\beta = -\phi_t^* d\beta + \phi_t^* d\beta = 0.$$

So $\phi_t^*\omega_t = \omega_0$. □

There exists another theorem which is actually an relative version of Moser's Theorem. The relative theorem points to the case where we have compact submanifold of a symplectic manifold.

Theorem 16. *Let N be a compact submanifold of a symplectic manifold with symplectic forms ω_0 and ω_1 such that ω_0 and ω_1 agree at all points of N. Then there exist two open sets of M, U_0 and U_1 such that $N \subset U_0$ and $N \subset U_1$ and there exist a diffeomorphism $\varphi : U_0 \longrightarrow U_1$ satisfying $\varphi|_N = id_N$ and $\varphi^*\omega_1 = \omega_0$.*

Proof. Take U_0 as a tubular neighborhood of N. Then $\omega_1 - \omega_0$ is a closed 2-form on U_0. Also for all $p \in N$, $(\omega_1 - \omega_0)_p = 0$. Using the homotopy formula on the tubular neighborhood, we get a 1-form μ on U_0 such that $d\mu = \omega_1 - \omega_0$ and $\mu_p = 0$ for all $p \in N$. Set

$$\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t d\mu$$

which can assumed symplectic for $t \in [0,1]$ since we can shrink U_0 if necessary. Now consider the Moser equation

$$i_{v_t}\omega_t = -\mu.$$

Integrating v_t and shrinking U_0 again we get an isotopy

$$\rho : U_0 \times [0,1] \longrightarrow M$$

satisfying $\rho_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. So we are done by setting $\varphi = \rho_1$ and $U_1 = \rho_1(U_0)$. □

We finish this section by introducing a special type of vector fields of symplectic manifolds.

Definition 17. *A Liouville vector field on a symplectic manifold (M, ω) is a vector field such that $L_X\omega = \omega$ and X is transverse to ∂M.*

1.2 Basics on Contact Geometry and Topology

Before we introduce contact manifolds and their some basic properties, we start with introducing hyperplane fields to have a better understanding of some widely used statements.

Definition 18. *Let ξ be a smooth subbundle of the tangent bundle, TM. Then ξ is called a hyperplane field on M if for every $p \in M$, $\xi_p = \xi \cap T_pM$ is a vector space of codimension 1 in T_pM.*

Let α be a 1-form on a manifold M of dimension 2n+1. Then at each point $p \in M$ we have a linear map $\alpha_p : T_pM \to R$. Assuming α_p is not the zero map, by the rank-nullity theorem on linear map, $\ker(\alpha_p)$ is a hyperplane of dimension 2n. Actually, we have even more than that. Fixing a Riemannian metric on M, one can consider the hyperplane ξ as the kernel of a certain 1-form ω as follows. Let g be a Riemannian metric on M. Consider $\xi^\perp$ as the orthogonal complement of ξ in TM, then $TM \cong \xi \oplus \xi^\perp$ and $TM/\xi \cong \xi^\perp$. Since $\xi_p = \xi \cap T_pM$ has codimension 1 in T_pM for each $p \in M$, ξ is a trivial line bundle over a neighborhood U_p of each p. Taking X_p a non-zero section of $\xi^\perp |_{U_p}$ we can define $\alpha_{U_p} = g(X, -)$. Note that for each $v_p \in \xi_p$, since X_p is orthogonal to each v_p, $\alpha_{U_p}(v_p) = g_p(X_p, v_p) = 0$. Conversely, for a $q \in M$ if $\alpha_{U_q}(v_q) = 0$, then $g_q(X_q, v_q) = 0$ which means $v_q \in (\xi_q^\perp)^\perp = \xi_q$. Thus $\xi |_{U_p} = \ker \alpha_p$.

Actually the fact that ξ can locally be written as a kernel of 1-form can be reached by using other tools. These other tools will include smooth distributions which will be described later.

Definition 19. *Let ξ be a hyperplane field on a manifold M of dimension $2n+1$. Then ξ is called a contact structure if for any 1-form α on M such that $\xi = \ker\alpha$, the 2n+1 form $\alpha \wedge (d\alpha)^n$ never vanishes. The pair (M,ξ) is called a contact manifold.*

Note that for such an $\alpha, \alpha \wedge (d\alpha)^n$ is a 2n+1 form on M which never equals 0, so it is a volume form.

Example 20. ***(The Standard Contact Structure on** R^{2n+1}**)*** *Let us present the coordinates of R^{2n+1} by $(x_1, y_1, ..., x_n, y_n, z)$ and define the 1-form*

$$\alpha = dz + \sum_{j=1}^{n} x_j dy_j.$$

Note that $d\alpha = \sum_{i=1}^{n} dx_i \wedge dy_i$ and clearly $\alpha \wedge (d\alpha)^n$ never vanishes. So $\xi = \ker\alpha$ is a contact structure.

Example 21. *Using the radial coordinates (r_j, φ_j) we have another contact structure on R^{2n+1} by setting*

$$\alpha = dz + \sum_{j=1}^{n} r_j^2 d\varphi_j.$$

Example 22. ***(The Standard Contact Structure on** S^{2n+1}**)*** *Take the standard coordinates $x_1, y_1, ..., x_{n+1}, y_{n+1}$ on R^{2n+2}, and set*

$$\alpha = \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j).$$

If S^{2n+1} is the unit sphere in R^{2n+2}, then $(S^{2n+1}, \ker(\alpha))$ is a contact manifold.

The other tools which give a way of understanding the relation between 1-forms and contact structures can be used here as well. Many sources on contact geometry stating the condition $\alpha \wedge d\alpha^n$ never vanishes say ξ is maximally non-integrable. Understanding why this is true requires some effort.

Definition 23. *Let M be a smooth manifold and let TM be its tangent bundle. A rank-k distribution D on M is a rank-k subbundle of TM. We say D is a smooth distribution if it is a smooth subbundle.*

More explicitly, if we have a rank-k distribution D on M, then for each $p \in M$, D gives $D_p \subseteq T_pM$ where D_p is k-dimensional and $D = \bigcup_{p \in M} D_p$.

Note that D is a smooth k-distribution if and only if each point $p \in M$ has a neighborhood U_p on which there exist k vector fields $X_1, X_2, ..., X_k : U_p \to TM$ such that $X_1|_q, X_2|_q, ..., X_k|_q$ form a basis for D_q at each $q \in U_p$.

Definition 24. *Let D be a smooth distribution on M. A nonempty immersed submanifold $N \subset M$ is an integral manifold of D if $D_p = T_pN$ at each point $p \in N$.*

Given two different vector fields on a manifold M, a useful way of generating a new vector field on the same manifold is the Lie Bracket of two vector fields. Note that given two vector fields X_1 and X_2, their Lie Bracket, $[X_1, X_2]$ is another vector field such that $[X_1, X_2](f) = X_1(X_2)(f) - X_2(X_1)(f)$. The following example shows that distributions do not have to be closed under Lie Brackets.

Example 25. *Let $M = R^3$ and let D be a distribution spanned by*

$$X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \text{ and } X_2 = \frac{\partial}{\partial y}$$

The Lie Bracket of X_1 and X_2 gives another vector field

$$[X_1, X_2] = X_1(X_2) - X_2(X_1) = X_1(1)\frac{\partial}{\partial y} - X_2(1)\frac{\partial}{\partial x} - X_2(y)\frac{\partial}{\partial z} = -\frac{\partial}{\partial z}$$

Since $-\frac{\partial}{\partial z}$ can not be written as a linear combination of X_1 and X_2, this example shows that distributions are not necessarily closed under Lie Brackets.

Definition 26. *A smooth vector field X defined on an open subset U of a manifold M satisfying $X_p \in D_p$ for each $p \in U$ is called a smooth local section of D.*

Definition 27. *A smooth distribution D on a manifold M is called an involutive distribution if for any two smooth local sections of D, their Lie Bracket is also a local section of D.*

Involutive distributions will have a great importance for us while working on contact structures.

Definition 28. *A smooth distribution D on M is called an integrable distribution if for each $p \in M$ there exists an integral manifold N_p of D.*

A very famous and important theorem called the Frobenius Theorem gives us a correspondence between involutive and integrable distributions. The proof of this theorem requires more machinery, but we have enough material to prove the following:

Proposition 29. *Every integrable distribution is involutive.*

Proof. Let D be an integrable distribution on M and let U be an open subset of M. We need to prove that for any two local sections X_1 and X_2, $[X_1, X_2]$ is also a local section of D.

Let p be any point in U. Since D is integrable there exists an integral manifold N_p of D such that $p \in N_p$. So by the existence of N_p, we can write $T_q N_p = D_q, \forall q \in U$. So X_1 and X_2 are tangent to N_p. Since Lie Bracket of smooth vector fields which are tangent to a manifold results in a vector field tangent to the same manifold, we have $[X_1, X_2]_p \in T_p N_p = D_p$. Then we conclude that D is involutive. □

Just after Definition 22, we have discussed a way to see hyperplane fields as a kernel of 1-forms. The following proposition can actually be considered as a generalization of this situation.

Proposition 30. *Let D be a smooth distribution on M. Then D is an involutive distribution if and only if for any smooth 1-form β which annihilates D on an open subset $U \subset M$ then $d\beta$ annihilates D on U as well.*

Before the proof note that for a rank-k distribution on a smooth n-manifold M, any $n-k$ linearly independent 1-forms $\omega_1, ..., \omega_{n-k}$ on a open subset $U \subset M$ such that $D_q = \ker \omega_1|_q \cap ... \cap \ker \omega_{n-k}|_q$ for each $q \in U$ are called local defining forms for

D. Moreover for $0 \leq p \leq n$ and $\omega \in \Omega^p(M)$ if $\omega(X_1, ..., X_p) = 0$ for any local sections $X_1, ..., X_p$ of D then we say ω annihilates D.

Proof. Assume that D is an involutive distribution on M and let β be a smooth form annihilating on $U \subset M$ where U is open. Then

$$d\beta(X_1, X_2) = X_1(\beta(X_2)) - X_2(\beta(X_1)) - \beta([X_1, X_2]) = 0 - 0 - 0 = 0.$$

So $d\beta$ annihilates on U as well.

For the other direction of the proof assume that for any 1-form β if β annihilates D on U then $d\beta$ annihilates D on U. Let X_1, X_2 be two smooth local sections of D. Taking $\omega_1, ..., \omega_{n-k}$ as local defining forms for D, then for any $1 \leq i \leq n-k$

$$\omega_i([X_1, X_2]) = X_1(\omega_i(X_2)) - X_2(\omega_i(X_1)) - d\omega_i(X_1, X_2) = 0$$

So we conclude that $[X_1, X_2]$ is in D. So D is involutive. $\square$

Corollary 31. *Let M be a smooth manifold of dimension n and D be a smooth distribution of rank k on M and let $\omega_1, ...\omega_{n-k}$ be smooth defining forms for D on an open subset $U \subset M$. Then D is involutive if and only if $d\omega_1, ..., d\omega_{n-k}$ annihilates D.*

It is time to explain The Frobenius Theorem which states a very direct correspondence between integrable and involutive distribution. So far we have concluded that every integrable distribution is involutive but Frobenius Theorem will state the other direction works perfectly as well. Before stating the Frobenius Theorem we need some more terminology.

Definition 32. *Let D be a rank-k distribution on M. Then for a smooth coordinate chart (U, φ) of M, if $\varphi(U)$ is a cube in R^n and D is a distribution spanned by $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^k}$ then (U, φ) is flat for D. If there exist a flat chart on a neighborhood of each point of M then D is called a completely integrable distribution. Clearly every completely integrable distribution is integrable.*

The statements about involutive, integrable and completely integrable distributions which we have verified can be summarized as

$$\textit{completely integrable} \Rightarrow \textit{integrable} \Rightarrow \textit{involutive}.$$

The Frobenius Theorem will lead to the fact that we can interchange the order of the statement above in any way, i.e we also have the following relation

$$\textit{integrable} \Leftrightarrow \textit{completely integrable} \Leftrightarrow \textit{involutive}.$$

Now we state the theorem. We skip the proof.

Theorem 33. ***Frobenius Theorem****[8] Every involutive distribution on a manifold M of dimension m is completely integrable.*

In order to make conclusions about contact manifolds by using Frobenius Theorem, we need to introduce the representation of Frobenius Theorem in terms of differential forms. We will use the maximal non–integrability condition and some theorems to conclude that integral manifolds of contact distributions do not exist. Before restating the theorem in terms of differential forms we start with some definitions.

Definition 34. *Let D be a rank-k distribution on a smooth manifold M of dimension n. For any $m \geq 1$, set*

$$I^m(D) = \{\omega \in \Omega^m(M) : \omega(X_1, ..., X_m) = 0 \ for \ X_i \in D\}$$

where $\Omega^m(M)$ denotes the differential m-forms on M. Moreover set

$$I(D) = \bigoplus_{m=1}^{n} I^m(D)$$

which is all the possible differential forms on M which vanish on D.

Proposition 35. *Algebraically $I(D)$ is an ideal of $\Omega^*(M)$, i.e. for any $\omega \in I(D)$ and $\alpha \in \Omega^*(M), \alpha \wedge \omega \in I(D)$.*

Proof. Take $\omega \in I^m(D)$ and $\alpha \in \wedge^k(M)$. Then $\alpha \wedge \omega$ is the $m+k$ form such that

$$\alpha \wedge \omega(X_1, X_2, ..., X_{k+l}) = \tfrac{1}{(k+l)!} \textstyle\sum_{\sigma \in S_{2k+1}} \alpha(X_{\sigma(1)}, X_{\sigma(2)}, ..., X_{\sigma(k)}) \omega(X_{\sigma(k+1)}, ..., X_{\sigma(k+m)}) = 0$$

for $X_i \in D$. Thus $\alpha \wedge \omega \in I(D)$. □

Now we have another proposition which constructs a correspondence between distributions and 1-forms.

Proposition 36. *For a rank-k distribution D on a smooth n-manifold M, for every point $p \in M$ there exist a neighborhood U of p and $(n-k)$ 1-forms $\omega_1, \omega_2, ..., \omega_{n-k}$ which are linearly independent at each point of U such that for any $\alpha \in I(D)$ there exist differential forms β_i such that*

$$\alpha = \sum_{i=1}^{n-k} \beta_i \wedge \omega_i.$$

In other words the ideal $I(D)$ is locally generated by (n-k) linearly independent 1-forms. Moreover for each $q \in U$,

$$D_q = \{X \in T_qM : \omega_1(X) = \omega_2(X) = ... = \omega_{n-k}(X) = 0\}.$$

Proof. For any $p \in M$ take a sufficiently small neighborhood U of p such that there exist vector fields $X_{n-k+1}, X_{n-k+2}, ..., X_n$ which generate D. These vector fields and $X_1, ..., X_{n-k}$ together construct a basis for tangent space of each point of U. Consider the 1-forms $\omega_1, \omega_2, ..., \omega_n$ on U such that $\omega_i(X_j) = \delta_{ij}$. Let ω be an m-form on U. Then we should have ω as the linear combination of m-forms

$$\omega_{i_1} \wedge \omega_{i_2} \wedge ... \wedge \omega_{i_m} \textit{ where } i_1 < i_2 < ... < i_m.$$

and their coefficient functions. If $\omega \in I^m(D)$ then obviously we should have zero coefficients for $i_1, ..., i_m$ when they are all different from $1, ..., n-k$. Hence ω is an element of the ideal generated by the forms $\omega_1, \omega_2, ..., \omega_{n-k}$. □

The following proposition will give a direct relation between the ideal $I(D)$ and involutivity of distributions. An interpretation of this proposition will lead us to make an important remark on the the maximal non-integrability condition ($\alpha \wedge (d\alpha)^n \neq 0$) of contact forms.

Proposition 37. *Let D be a rank-k distribution on a smooth n-manifold M and let $I(D)$ be the ideal whose elements are the forms which vanish on D. Then D is involutive if and only if*

$$dI(D) \subset I(D)$$

where $dI(D)$ consists of the differential forms which are exterior derivatives of the elements of $I(D)$.

Proof. Pick an $\omega \in I^l(D)$. Then for any $X_1, X_2, ..., X_l, X_{l+1} \in D$, by the involutivity of D,

$$[X_i, X_j] \in D.$$

Since $d\omega(X_1, ..., X_{l+1})$ can be written as the sum

$$\frac{1}{l+1}(\textstyle\sum_{i+1}^{l+1}(-1)^{l+1}X_i(\omega(X_1, ..., \hat{X}_i, ..., X_{l+1})) + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{l+1}))$$

we have $d\omega(X_1, X_2, ..., X_{l+1}) = 0$ which means $d\omega \in I^{l+1}(D)$. Thus

$$dI(D) \subset I(D).$$

For the other direction assume that $dI(D) \subset I(D)$ and $X, Y \in D$. Since for any 1-form $\omega \in I(D)$

$$0 = d\omega(X, Y) = \frac{1}{2}(X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

we have $\omega([X, Y]) = 0$. Hence D is involutive. $\square$

Interpreting Proposition 36 and 37 together results in the following corollary.

Corollary 38. *Let D be a rank-k distribution on a smooth n-manifold M. Then D is involutive on an open subset U of M if and only if there exist 1-forms ω_{ij} on U such that for each defining form ω_i of D*

$$d\omega_i = \sum_{j=1}^{n-k} \omega_{ij} \wedge \omega_j.$$

Now we can restate Frobenius Theorem in terms of differential forms.

Theorem 39. ***(Frobenius Theorem in Terms of Differential Forms)*** *Let D be a rank-k distribution on smooth n-manifold M such that*

$$D_q = \{X \in T_qM : \omega_1(X) = ... = \omega_{n-k}(X) = 0\}$$

where ω_i are the linearly independent 1-forms on neighborhoods of each point of M. Then D is completely integrable if and only if for each ω_i the condition

$$d\omega_i = \sum_{j=1}^{n-k} \omega_{ij} \wedge \omega_j$$

from Corollary 38 is satisfied.

Remember that in the definition of contact manifolds, the local condition $\xi_p = \ker \alpha_p$ gives a rank-2n distribution of a smooth manifold of dimension 2n+1. The following proposition will present some remarks on this contact distribution.

Proposition 40. *Let (M, ξ) be a contact manifold of dimension (2n+1) such that locally $\xi = \ker \alpha$ where α is a 1-form satisfying the maximal non-integrability condition*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Then ξ is not completely integrable.

Proof. As a very first argument, we note that by Frobenius Theorem we know that we are done if we prove that ξ is not involutive. Assume that ξ is involutive. By Theorem 39, we would have

$$d\alpha = \beta \wedge \alpha$$

where β is a 1-form on M. But this gives a contradiction on the maximal non-integrability condition since

$$\alpha \wedge (d\alpha)^n = \alpha \wedge (\beta \wedge \alpha)^n = 0.$$

So ξ is not involutive. By Frobenius Theorem it is also not integrable and maximally integrable. □

Now we state the Darboux's Theorem which states contact structures are locally isomorphic. A proof can be found in [5].

Theorem 41. ***(Darboux's Theorem)*** *Let M be a manifold of dimension 2n+1 and and let α be a contact form on M. Taking p as a point in M, there exist coordinates $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n, z$ on a neighborhood U_p of p such that $p = (0, 0, ..., 0)$ and $\alpha \mid_{U_p} = dz + \sum_{j=1}^{n} x_j dy_j$.*

Definition 42. *Assume that (M_1, ξ_1) and (M_2, ξ_2) are two contact manifolds with a diffeomorphism $f : M_1 \to M_2$. We say (M_1, ξ_1) and (M_2, ξ_2) are diffeomorphic if the induced map $f_* : TM \to TM$ satisfies $f_*(\xi_1) = \xi_2$. Also the map $f : M_1 \to M_2$ is called a contactomorphism.*

1.3 A Bridge Between Symplectic and Contact Manifolds

Considering the definitions of contact and symplectic manifolds, it is not quite clear to see the correspondences between them. In contact manifolds, we actually have a 1-form and we locally work on a hyperplane plane which comes as the kernel of the 1-form. For symplectic manifolds, we have a special kind of closed 2-form. A theorem mostly based on linear algebra implies the dimension of the manifold is even.

Now we will introduce two concepts which supplies a transition between symplectic and contact manifolds. These two concepts are called symplectisation and contactization. After symplectisation a contact manifold turns into a symplectic manifold.

However, for the other direction we need more than ordinary symplectic manifolds. We start with symplectisations.

Definition 43. *Let (M, ξ) be a contact manifold. Set $X = R \times M$ and define the symplectic form as $\beta = d(e^t \alpha)$ where t is a coordinate function on R and α is the contact form of M. Then (X, β) is a symplectic manifold called the symplectization of (M, ξ).*

Now we introduce the other process which constructs contact manifolds out of symplectic manifolds. In topology it is always very easy to increase the dimension of a given manifold since we are allowed to consider products. We will use this advantage in this process but we will need to have a exact symplectic manifolds.

Definition 44. *Let (N, ω) be an exact symplectic manifold, i.e. there exists a 1-form β on N such that $\omega = d\beta$. Then set $M = N \times S^1$. Then (M, α) is a contact manifold where $\alpha = dz - \beta$ and z is projection on S^1.*

We will need the following lemma when we work on open books in higher dimensions.

Lemma 45. *Let X be a Liouville vector field on a symplectic manifold (M, ω) where M is of dimension $2n + 2$. Then $\alpha = i_X \omega$ is a contact form on N where N is any codimension-1 submanifold of M transverse to X.*

Proof. We need to show $\alpha \wedge (d(\alpha))^n$ never vanishes.

$$\alpha \wedge (d(\alpha))^n = i_X(\omega) \wedge (d(i_X(\omega)))^n = i_X\omega \wedge (L_X(\omega))^n = i_X\omega \wedge \omega^n = \frac{1}{n+1} i_X \omega^{n+1} \neq 0$$

So we are done. $\square$

Chapter II

OPEN BOOK DECOMPOSITIONS

Another way of understanding contact manifolds is by focusing on their open book decompositions. One thing which makes open books important is the fact that for contact manifolds of dimension 3 we have a very clear relation between contact manifolds and open book decompositions by a very famous theorem of Giroux.

2.1 Open Books

Before focusing on open book decompositions of contact manifolds it is useful to start with the open book decompositions of ordinary manifolds. Actually the basic definition of open books has nothing to do with contact manifolds.

Definition 46. *Let N be a compact n-manifold with boundary and let $\phi : N \to N$ be a diffeomorphism such that $\phi|_U = id_U$ where U is an open subset of N including ∂N. The mapping torus of ϕ on N, denoted N_ϕ is a quotient space of $N \times [0,1]$ defined as*

$$N_\phi = \frac{N \times [0,1]}{(p,0) \sim (\phi(p),1)}.$$

Since a mapping torus is defined using manifolds with boundary it always has boundary. More clearly,

$$\partial N_\phi = \partial(\frac{N \times [0,1]}{(p,0) \sim (\phi(p),1)}).$$

Since $\phi(p) = p$ near the boundary, we can write

$$\partial N_\phi = \partial(\frac{N \times [0,1]}{(p,0) \sim (p,1)}) = \partial N \times S^1.$$

In order to turn N_ϕ into a closed manifold, we glue N_ϕ to $\partial N \times D^2$ and get

$$M_{(N,\phi)} = \frac{N_\phi \cup (\partial N \times D^2)}{(p,\theta) \in N_\phi \sim (\overline{p},\overline{\theta}) \in \partial(\partial N \times D^2) = (\partial N \times S^1)}.$$

Definition 47. *For a closed n-manifold X, we say (N, ϕ) is an open book decomposition of X if X is diffeomorphic to $M_{(N,\phi)}$. Moreover ϕ is called the monodromy of the open book. The pair (N, ϕ) is also called an abstract open book.*

We can present a definition of open book decompositions using a different approach.

Definition 48. *We can see an open book decomposition of a manifold M as a pair (B, π) where*

1. *B is a codimension-2 submanifold of M with trivial normal bundle and it is called the binding of the open book, and*

2. *$\pi : (M - B) \longrightarrow S^1$ is a fibration such that for all $\theta \in S^1$, $\pi^{-1}(\theta)$ is interior of a compact hypersurface $\Sigma_\theta \subset M$ and $\partial\Sigma_\theta = B$. The compact hypersurface Σ_θ is called a page of the open book.*

The relation between these two definitions is not trivial. The main idea behind this is the relation between between fibrations over S^1 and the mapping tori. Any locally trivial bundle with fiber N over S^1 is canonically isomorphic to the fibration $\frac{N\times I}{(x,1)\sim(\psi(x),0)} \longrightarrow I/\partial I \approx S^1$. Moreover, the monodromy ψ is determined up to isotopy and conjugation by the fibration. Conversely given a compact oriented hyperspace N with boundary with a $\psi \in \Gamma_N$ where Γ_N is the mapping class group of N, then the mapping torus can be defined as $\frac{N\times I}{(x,1)\sim(\psi(x),0)}$.

Our study will focus on understanding open books on contact manifolds which are manifolds of odd dimension. We have the following theorem.

Theorem 49. *Any closed oriented manifold of odd dimension has an open book decomposition.*

A similar statement for manifolds of odd dimension more than 7 was proved by Wilkelnkemper(1973) in [15] and by Tamura(1972) in [12]. But in their study they had extra hypotheses. But in 1978, Lawson proved exactly the same statements for manifold of odd dimensions equal or greater than 7 [7]. Quinn gave a proof for manifolds of dimension 5[10]. 3-dimensional case was already proved by Alexander in 1920 [1]. In order to see how we pass from a mapping torus (N, ψ) to a closed manifold $M_{(N,\psi)}$, it is useful to see a very simple example.

Example 50. *Take $N = [0, 1]$ and $\psi = id$. Then*

$$[0, 1]_{id} = \frac{[0, 1] \times [0, 1]}{(x, 0) \sim (x, 1)}$$

which is a cylinder having two boundary components $\{0\} \times S^1$ and $\{1\} \times S^1$.

The identification $\sim$ in the definition of $M_{(N,\psi)}$ implies N_{id} needs to be glued with $\{0, 1\} \times D^2$ by identifying $\partial(N_{id}) = \{0, 1\} \times S^1$ with $\partial(\{0, 1\} \times D^2) = \{0, 1\} \times S^1$. This construction gives us a figure like a cylinder with two lids which is a closed manifold homeomorphic to S^2

Proposition 51. *S^2 is the only oriented 2-manifold that has an open book decomposition.*

Proof. Let M be an oriented 2-manifold with an open book decomposition. Let B be the binding and F be the page of the open book decomposition. Then we have the fibration

$$M - B \longrightarrow S^1.$$

Since $\chi(M - B) = \chi(F).\chi(S^1)$ we have $\chi(M - B) = 0$. Note that B is a zero dimensional manifold, so it is a collection of disjoint points. For each neighborhood of a point taken out from the 2-manifold M, M gets one more boundary component. Considering the Euler characteristic formula for surfaces with boundary, we have $\chi(M - B) = 2 - 2g - b = 0$ where g denotes the genus and b denotes the number of boundary components. The only possible solution to this equation is $b = 2$ and $g = 0$. So M has to be S^2. □

We now give some examples of open book decompositions.

Example 52. *Let* $M = S^3 \subset C^2$ *where any point* $(z_1, z_2) \in C^2$ *is presented as* $(z_1, z_2) = (r_1 e^{i\Theta_1}, r_2 e^{i\Theta_2})$*. Take* $B = \{z_1 = 0\}$*. Then define the fibration*

$$\pi : S^3 - B \to S^1 \textit{ by } \pi(z_1, z_2) = \frac{z_1}{|z_1|}.$$

Then $(\{z_1 = 0\}, \pi)$ *is an open book decomposition on* S^3*.*

A close look at Example 52 will touch to the fact that S^3 can actually be considered as union of two solid tori, one torus is the neighborhood of the binding $B = \{z_1 = 0\} \approx C$ and the other one is the union of the pages.

In the following example we give another decomposition on S^3 with a different binding. But before the example it is useful to introduce Hopf fibrations since the binding of the example is generated by fibers of the Hopf fibration.

Definition 53. *The fibration*

$$\mu : S^3 \longrightarrow S^2 = CP^1 \textit{ where } \mu(z_1, z_2) = (z_1 : z_2)$$

is called the Hopf fibration.

Now we can give another example of open book decomposition on S^3.

Example 54. *Take* $S^3 \subset C^2$ *as* $S^3 = \{(z_1, z_2) \in C^2 : |z_1|^2 + |z_2|^2 = 1\}$ *and let* $H_1 = \{(z_1, z_2) \in S^3 : z_1 = 0\} = \mu^{-1}(0 : z_2)$ *and* $H_2 = \{(z_1, z_2) \in S^3 : z_2 = 0\} = \mu^{-1}(z_1 : 0)$ *be two Hopf fibers and consider the binding as*

$$B = H_1 \cup H_2 = \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}.$$

Then define the fibration

$$\psi : S^3 - B \longrightarrow S^1 \textit{ by } \psi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|}$$

which gives an open book decomposition on S^3*.*

We can actually generalize Example 54. But we need a theorem for that.

Theorem 55. ***(Milnor Fibration Theorem)*** *Let $S_r^{2n-1} \subset C^n$ be a sphere of radius r and let $f : C^n \longrightarrow C$ be a holomorphic map such that $f(0,0,...,0) = 0$ and $(0,0,...,0)$ is an isolated singular point. Then there exists ε such that taking $B = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ the following map is a fibration:*

$$\psi : S_\varepsilon^{n-1} - B \longrightarrow S^1 \text{ where } \psi(z) = \frac{f(z)}{|f(z)|}.$$

So (B, ψ) is an open book for S_ε^{2n-1}.

In dimension 3 we have following family of examples.

Example 56. *Let : $C^2 \longrightarrow C$ be a polynomial such that $f(0,0) = 0$ without any other critical point except $(0,0)$. As the binding take $B = f^{-1}(0) \cap S^3$ and define the fibration*

$$\psi : S^3 - B \to S^1 \text{ by } \psi(z_1, z_2) = \frac{f(z_1, z_2)}{|f(z_1, z_2)|}.$$

We also have the following explicit example.

Example 57. *Let $f : C^2 \to C$ be a map defined as $f(z,w) = z^2 + w^3$. Clearly $f(0,0) = 0$ and $(0,0)$ is critical point. Then $B = f^{-1}(0) \cap S^3$ is a trefoil knot. By Milnor Fibration Theorem the map*

$$\psi : S^3 - B \to S^1 \text{ where } \psi(z,w) = \frac{z^2 + w^3}{|z^2 + w^3|}$$

is a fibration.

Example 58. *Otto van Koert describes a open book on the 5-manifold $S^2 \times S^3$ in [14] as the following. Let $P = \Sigma_k$ be a Stein manifold, the 2-disk bundle over S^2 with Euler number $-k$ where $k \geq 2$ and set $S_k = \partial(\Sigma_k)$. Since $\partial(D^2) = S^1$, S_k is actually a S^1 bundle over S^2 with Euler number $-k$. Consider the pair (Σ_k, id). The mapping torus of this pair is clearly diffeomorphic to $A = \Sigma_k \times S^1$. Consider a neighborhood* *of the binding as $B = S_k \times D^2$ and set $X = A \cup_\partial B$. Now consider the rank 4 disk bundle $\Sigma_k \times D^2$ over S^2. We have*

$$\partial(\Sigma_k \times D^2) = \Sigma_k \times S^1 \cup_\partial S_k \times D^2 = A \cup_\partial B = X$$

When k is even, $\Sigma_k \times D^2 = S^2 \times D^4$ so we have an open book for $S^2 \times S^3$.

2.2 Open Books and Contact Manifolds of Dimension 3

In 2000, Giroux proved the existence of a strong correspondence between contact manifolds of dimension 3 and open book decompositions [6]. In this section we will work on this correspondence and explain why this is important.

Definition 59. *Let (M, ξ) be a contact manifold where M has dimension 3 and let (B, π) be an open book decomposition of M. Then we say (M, ξ) is supported by the open book (B, π) if for a contact form α defining ξ the following are satisfied:*

1. *$d\alpha$ is always positive on each page of (B, π), and*

2. *$\alpha > 0$ on B.*

We will have a theorem which presents equivalent statements to Definition 60. But before the theorem we describe Reeb vector fields.

Definition 60. *Let α be a contact form. Then R_α is Reeb vector field of α is the unique vector field satisfying:*

1. *$i_R d\alpha = 0$,*

2. *$\alpha(R) = 1$.*

Theorem 61. *[2],[6] Let (M, ξ) be a contact 3-manifold and let (π, B) be an open book decomposition of M. Then the following statements are equivalent:*

1. *(M, ξ) is supported by the open book (π, B).*

2. *On the compact subsets of the pages of (π, B), ξ can be isotoped arbitrarily close to the tangent planes of the pages of (π, B) such that after some point the contact planes are transverse to B and are transverse to pages of the open book in a certain neighborhood of B.*

3. *There exits a Reeb vector field X for a contact structure which is isotopic to ξ such that X is positively tangent to B and positively transverse to pages of (π, B).*

Theorem 62. *[13] In dimension 3, every open book decomposition (Σ, ϕ) supports a contact structure ξ_ϕ on M_ϕ.*

Proof. Consider the closed manifold

$$M_{(\Sigma,\phi)} = \Sigma_\phi \cup_\phi (\coprod_{|\partial\Sigma|} S^1 \times D^2).$$

We start with constructing a contact structure on the mapping torus Σ_ϕ. Define the convex set

$$S = \{\alpha | \alpha \; is \; a \; 1-form \; s.t. \; \alpha = (1+s)d\theta \; near \; \partial\Sigma \; and \; d\alpha \; is \; a \; volume \; form \; on \; \Sigma\}$$

where $(s, \theta) \in \Sigma \times [0, 1]$ are coordinates near each boundary component of Σ.

The set S can seen to be not empty as follows. Let α_1 be a 1-form on Σ such that $\alpha_1 = (1+s)d\theta$ near $\partial\Sigma$. Then by Stokes Theorem

$$\int_{\partial\Sigma} d\alpha_1 = \int_\Sigma \alpha_1 = 2\pi|\partial\Sigma|$$

which is always positive.

Now let ω be a volume form on Σ such that

$$\int_\Sigma \omega = 2\pi|\partial\Sigma|$$

and

$$\omega = ds \wedge d\theta \; around \; \partial\Sigma.$$

Then

$$\int_{\Sigma}(\omega - d\alpha_1) = \int_{\Sigma}\omega - \int_{\Sigma} d\alpha_1 = 2\pi|\partial\Sigma| - 2\pi|\partial\Sigma| = 0$$

and

$$\omega - d\alpha_1 = 0 \; around \; \partial\Sigma.$$

By de Rham isomorphism we can find a 1-form β such that $d\beta = \omega - d\alpha_1$ and β is zero around $\partial\Sigma$. Now $\lambda = \alpha_1 + \beta \in S$ so $S \neq \emptyset$. On $\Sigma \times [0,1]$ define the 1-form

$$\lambda_{(t,x)} = t\lambda_x + (1-t)(\phi^*\lambda)_x$$

where $(x,t) \in \Sigma[0,1]$. Also define

$$\alpha_K = \lambda_{(t,x)} + Kdt$$

which is a contact form for a large enough K. On the mapping torus Σ_ϕ, α_K gives a contact form which can be extended to a solid tori neighborhood of the binding. In order to make this extension, let ψ be the function which identifies the solid tori to the mapping torus. We can write

$$\psi(\varphi,(r,\upsilon)) = (r-1+\epsilon, -\varphi, \upsilon)$$

where $(\varphi,(r,\upsilon))$ are coordinates on $S^1 \times D^2$. When we pull-back the contact form α_K by ψ to get

$$\alpha_\psi = Kd\upsilon - (r+\epsilon)d\varphi$$

which can be extended to all over $S^1 \times D^2$ using the 1-form

$$f(r)d\varphi + g(r)d\upsilon.$$

where $g'f - f'g > 0$ and near the boundary we take $f(r) = -(r+\epsilon), g(r) = K$ and near the core of $S^1 \times D^2$ we take $f(r) = 1 \; and \; g(r) = r^2$ which give us the contact form we need. $\square$

The following theorem of Giroux explains the relation between two contact structures supported by the same open book. The proof can be found in [2].

Theorem 63. *[2] Two contact structures on a closed oriented 3-manifold M supported by the same open book are isotopic.*

In 2000, Giroux proved also the other direction, i.e the existence of open books given a contact 3-manifold. A sketch of proof for the following theorem can be found in [2].

Theorem 64. *Every oriented contact structure on an oriented manifold of dimension 3 is supported by an open book decomposition.*

In Chapter 4, we will discuss some operations of open books in high dimensions. But before that, in order to give the Giroux correspondence without losing anything, we should define 3-dimensional version of these operations.

Definition 65. *[3] The oriented surface $R \subset M^3$ is a Murasugi sum of compact oriented surfaces R_1 and R_2 in M^3 if:*

1. $R = R_1 \cup_D R_2, D = 2n - gon,$
2. $R_1 \subset B_1, R_2 \subset B_2$ *where* $B_1 \cap B_2 = S$, S *a 2-sphere* $B_1 \cup B_2 = M^3$ *and* $R_1 \cap S = R_2 \cap S = D.$

Definition 66. *A 4-Murasugi sum of two compact surfaces R_1 and R_2 is also called the plumbing of R_1 and R_2.*

Note that we did not mention about open books in the operations we have introduced so far. In [2], Etnyre explains the plumbing of two abstract books as the following. Given two abstract open books (Σ_1, ϕ_1) and (Σ_2, ϕ_2), let c_1 and c_2 be two properly embedded arcs in Σ_1 and Σ_2 respectively. Set the rectangular neighborhoods $R_1 = c_1 \times [-1, 1] \subset \Sigma_1$ and $R_2 = c_2 \times [-1, 1] \subset \Sigma_2$. Then the plumbing of

$(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)$ is another abstract open book with page $\Sigma_1 * \Sigma_2 = \Sigma_1 \cup_{R_1=R_2} \Sigma_2$ where $c_1 \times \{-1, 1\} \subset \partial R_1$ is identified to $\partial c_2 \times [-1, 1] \subset \partial R_2$ and $c_2 \times \{-1, 1\} \subset \partial R_2$ is identified to $\partial c_1 \times [-1, 1] \subset \partial R_1$. The monodromy is $\phi_1 \circ \phi_2$.

It can be shown that $(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)$ corresponds the 4-Murasugi sum of open books (Σ_1, ϕ_1) and (Σ_2, ϕ_2).

Now we will state a theorem on 4-Murasugi sums. However, the proof will be provided in Chapter 3.

Theorem 67. *Let $(\Sigma_1, \phi_1), (\Sigma_2, \phi_2)$ be two abstract open books. Then $M_{(\Sigma_1,\phi_1)} \sharp M_{(\Sigma_2,\phi_2)}$ is diffeomorphic to $M_{(\Sigma_1,\phi_1)*(\Sigma_2,\phi_2)}$.*

We will also define the stabilizations of open books which is an operation on open books and results in a new open book. But before that we have to present Dehn twists.

Definition 68. *For a simple closed curve c in a closed, orientable surface S, let A be a tubular neighborhood of c. Then clearly A is homeomorphic to $S^1 \times I$. Let (s, t) be coordinates on A such that s is a complex number of the form $e^{i\theta}$. Let f be a map which is zero outside of A but $f(s, t) = (se^{i2\pi t}, t)$ inside A. The f is called a Dehn twist about the curve c.*

Now we define the stabilizations.

Definition 69. *Given an abstract open book (Σ, ϕ) the positive (negative) stabilization of (Σ, ϕ) is the open book (Σ', ϕ')which is defined as*

1. *$\Sigma' = \Sigma \cup D_1$ where D_1 is a 1-handle*

2. *$\phi' = \phi \circ \tau_c$ where τ_c is a right(left)-handed Dehn twist about a curve c in Σ' that intersects the cocore of the 1-handle exactly 1-time.*

Stabilizations are denoted by $S_{(a,\pm)}(\Sigma, \phi)$ where $a = c \cap \Sigma$ and $\pm$ indicates the sign of the stabilization.

In dimension 3, stabilization can be seen as Murasugi sum of Hopf bands. On 3-manifolds plumbing a Hopf band is the same as contact connected sum with S^3.

Theorem 70. *Let* (Σ, ϕ) *be an open book supporting a contact 3-manifold* (M, ξ). *Then for any arc* a *in* Σ, $\xi_{S_{(a,+)}(\Sigma,\phi)}$ *is isotopic to* $\xi_{(\Sigma,\phi)}$.

So now we have a well defined map

$$\Psi : \Gamma \longrightarrow \Delta$$

where $\Delta = \{$*Oriented contact structures on* M *up to isotopy*$\}$ and $\Gamma = \{$*Open book decompositions of* M *up to positive stabilization* $\}$ which is actually a bijection. This bijection results in the famous Giroux theorem [6].

2.3 Open Books and Contact Manifolds of Higher Dimensions

When we let our contact manifolds have dimensions higher than 3, the correspondence between contact structures (up to isotopy) and open books (up to positive stabilization) is more complicated. We start with explaining what we mean when we say an open book supports a contact manifold of higher dimensions.

Definition 71. *Let* (M, ξ) *be a contact manifold of dimension higher than 3 and let* (B, π) *be an open book decomposition of M. Then we say* (M, ξ) *is supported by the open book* (B, π) *if for a contact form* α *defining* ξ *the following are satisfied:*

1. *The restriction of* α *to the binding is a contact form on the binding.*

2. $d\alpha$ *is a symplectic form on the pages* Σ *and the Liouville vector field of* α, X, *points outward along the boundary* $\partial\Sigma$.

An equivalent description is the following.

Definition 72. *A contact manifold* (M, ξ) *is supported by the open book* (B, π) *if there is a contact form* α *defining* ξ *such that following are satisfied:*

1. $(B, \alpha_B = \alpha|_B)$ *is a contact manifold.*

2. *The Reeb vector field* R_α *associated to the contact form* α *is positively transerve to the fibration* π*, in other words* $d\pi(R_\alpha) > 0$*. Also* R_α *is tangent to the binding* B*.*

Let $(\Sigma, d\alpha)$ be the symplectic manifold obtained as the page of an abstract open book (Σ, ϕ). Then by Lemma 45, the 1-form $\gamma = \alpha|_{\partial\Sigma}$ is a contact form on $\partial\Sigma$. Also we can say $\alpha = (2-r)\gamma$ where $r \in [1, 1+\delta]$ is an inwards pointing transverse coordinate which is obtained as a result of changes of the coordinates associated to the Liouville vector field X.

Theorem 73. *Let* (Σ, ϕ) *be an abstract open book and* β *a 1-form such that* $(\Sigma, d\beta)$ *is a symplectic manifold. Let* X *be a vector field such that* $i_X d\beta = \beta$*. Then if* X *points out of* $\partial\Sigma$ *and* $\phi^* d\beta = d\beta$*, then there exists a contact structure supported by the open book* (Σ, ϕ)*.*

Before the proof of Theorem 73 we give a lemma of Giroux which will be needed in the proof.

Lemma 74. *Let* ϕ *be the diffeomorphism given in the statement of Theorem 73. Then* ϕ *can be isotoped to a symplectomorphism* $\overline{\phi}$ *such that* $\overline{\phi}$ *is identity near* $\partial\Sigma$ *and* $\overline{\phi}^* \beta = \beta - dh$ *where* h *is a real valued function on* Σ*.*

Proof. Set the 1-form $\mu = \phi^*\beta - \beta$ which is closed since

$$d\mu = d(\phi^*\beta - \beta) = \phi^* d\beta - d\beta = d\beta - d\beta = 0.$$

Since $\phi^*\beta = \mu + \beta$, we have

$$d(\phi^*\beta) = d(\mu + \beta)$$

$$\Rightarrow \phi^* d\beta = d(\mu + \beta)$$

Note that since $d\beta$ is non-degenerate, there exists a unique vector field v such that $\mu + i_v d\beta = 0$. Also

$$L_v d\beta = d(i_v d\beta) = d(-\mu) = 0.$$

Note that μ and v both vanish near $\partial\Sigma$. Taking ψ_t as the flow of the vector field v, set the symplectomorphism $\overline{\phi} = \phi \circ \psi_1$. Clearly $\overline{\phi}$ is the identity near $\partial\Sigma$. Since $L_X \mu = 0$ we have $\psi_t^* \mu = \mu$ for all t. Note that

$$(\overline{\phi})^* \beta - \beta = \psi_1^*(\mu + \beta) - \beta = \mu + \psi_1^* \beta - \beta.$$

Also

$$\begin{aligned} \psi_1^* \beta - \beta &= \int_0^1 \frac{d}{dt} \psi_t^* \beta dt = \int_0^1 (\psi_t^* L_X \beta) dt \\ &= \int_0^1 \psi_t^*(i_X d\beta + d(i_X \beta)) dt = -\mu + d \int_0^1 \psi_t^*(i_X \beta) dt. \end{aligned}$$

Hence $\psi_1^* \beta - \beta + \mu = \overline{\phi}^* \beta - \beta$ is an exact form. Taking $h = \int_0^1 \psi_t^*(i_X \beta) dt$ we are done.

□

Now we can prove Theorem 73.

Proof of Theorem 73. As a result of Lemma 74, we may assume $\phi^* \beta = \beta - dh$. The form $\alpha = \beta + dy$ is a contact form on $\Sigma \times R$. Now the mapping torus T_ϕ can be expressed as the quotient

$$A_{\Sigma,\phi} = \Sigma \times R / (x, y) \sim \Psi(x, y)$$

where $\Psi(x, y) = (\phi(x), y + h(x))$.

The form $\alpha = \beta + dy$ induces a contact form on the mapping torus $A_{\Sigma,\phi}$ since

$$\Psi^* \alpha = dy + dh + \phi^* \beta = dy + dh + \beta - dh = \alpha.$$

A neighborhood of $\partial A_{\Sigma,\phi}$ is $N = (-\frac{1}{2}, 0] \times \partial\Sigma \times S^1$ with the contact form

$$\alpha = e^r \beta|_{\partial\Sigma} + dy.$$

Now set $A(r,R)$ as the annulus $A(r,R) = \{z \in C : r < |z| < R\}$. Using the map

$$\Upsilon : \partial\Sigma \times A(\frac{1}{2}, 1) \longrightarrow (-\frac{1}{2}, 0] \times \partial\Sigma \times S^1$$

where

$$\Upsilon(x, re^{iy}) = (\frac{1}{2} - r, x, y),$$

$A_{\Sigma,\phi}$ and $\partial\Sigma \times D^2$ can be glued together. Also the 1-form

$$\Upsilon^*\alpha = e^{\frac{1}{2}-r}\beta|_{\partial\Sigma} + dy$$

on $\partial\Sigma \times A(\frac{1}{2}, 1)$ can be extended to a 1-form γ on the interior of $\partial\Sigma \times D^2$ as

$$\gamma = p(r)\beta|_{\partial\Sigma} + k(r)dy$$

where p and k are functions described by the following graphs. Note that

$$\gamma \wedge (d\gamma)^n = p^{n-1}\frac{pk' - kp'}{r}\beta|_{\partial\Sigma} \wedge (d\beta|_{\partial}\Sigma)^{n-1} \wedge dr \wedge dy > 0$$

since

$$\frac{pk' - kp'}{r} > 0$$

is satisfied when we choice the functions p and k as described in the graphs.

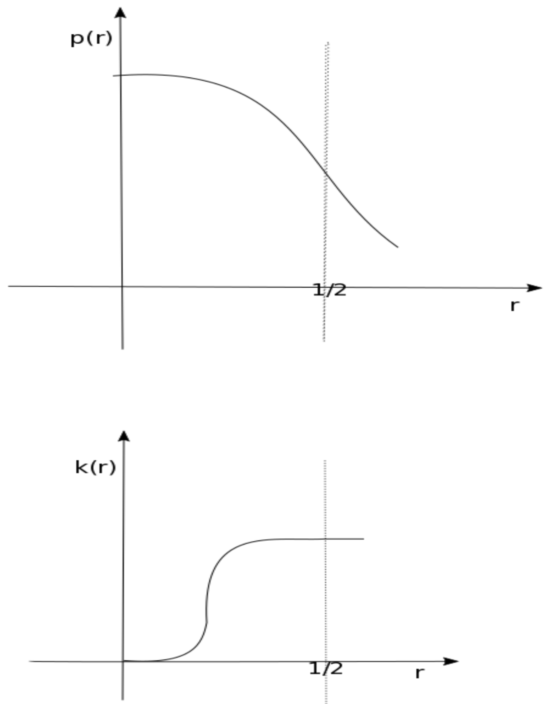

Figure 1: The functions p and k near the binding

$\square$

The other direction also works, i.e. an open book can be constructed for a given contact manifold which supports a contact form. More explicitly we have the following theorem.

Theorem 75. *[6]Given a contact manifold (M, ξ), there exists an open book supporting ξ.*

A proof of Theorem 74 is sketched in [6]. In [6], Giroux constructs the open book using asymptotically holomorphic sections.

Chapter III

OPERATIONS ON OPEN BOOKS

By the theorems of Stalling and Gabai, plumbings and Murasugi sums work in a very nice way in dimension 3. In this chapter, we will work on a generalization of the plumbings and Murasugi sums to higher dimensions in a way that the operation satisfies the Stalling's Theorem. This chapter will be mostly based on Ozbagci and Pampu's paper called "Generalized Plumbings and Murasugi Sums" [9].

Actually, we will construct the definitions of the operations on ordinary manifolds rather than open books. But then we will apply these definitions on open books.

3.1 Plumbings and Murasugi Sums of 3-Manifolds

In Section 2.1 we have already defined the Murasugi sums and plumbings in 3-dimensional open books since we needed them to reach Giroux's famous theorem. And we stated the following theorem without its proof: Let $(\Sigma_1, \phi_1), (\Sigma_2, \phi_2)$ be two abstract open books of 3-manifolds M_1 and M_2. Then $M_{(\Sigma_1,\phi_1)} \sharp M_{(\Sigma_2,\phi_2)}$ is diffeomorphic to $M_{(\Sigma_1,\phi_1)*(\Sigma_2,\phi_2)}$.
Now we prove this theorem.

Proof of Theorem 67. The idea is to construct a sphere S and show that $M_{(\Sigma_1 * \Sigma_2, \phi_1 \circ \phi_2)} - S$ has $M_{(\Sigma_1,\phi_1)} - B_1$ and $M_{(\Sigma_2,\phi_2)} - B_2$ as its components where B_1 and B_2 are 3-balls. Set $\Sigma = \Sigma_1 * \Sigma_2$.

Let c_1 and c_2 be two properly embedded arcs in Σ_1 and Σ_2 respectively. Set $R_1 = c_1 \times [-1, 1]$ and $R_2 = c_2 \times [-1, 1]$ as the rectangular neighborhoods of c_1 and c_2 respectively. Seperate the unit interval $I = [0, 1]$ in two pieces I_1 and I_2 where $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. We see Σ_1 and Σ_2 as the submanifolds of Σ and consider

R_1 and R_2 as one submanifold R of Σ. Now set $s_1 = R \cap \partial\Sigma_1$ and $s_2 = R \cap \partial\Sigma_2$ which consist of two disjoint properly embedded arcs in Σ. Note that ∂s_1 and ∂s_2 consist of four points in $\partial\Sigma$ where $\partial s_1 = \partial s_2$.

Note the surfaces $s_1 \times I_1$ and $s_2 \times I_2$ are actually unions of disks in $\Sigma_{\phi_1 \circ \phi_2}$. Also $R \times \{0\}$ and $R \times \{\frac{1}{2}\}$ are disks in $\Sigma_{\phi_1 \circ \phi_2}$ as well. Gluing $((s_1 \times I_1) \cup (s_2 \times I_2))$ to $((R \times \{0\}) \cup (R \times \{\frac{1}{2}\})$ along $(s_1 \times \{0\}) \cup (s_1 \times \{\frac{1}{2}\}) \cup (s_2 \times \{0\}) \cup (s_2 \times \frac{1}{2})$ we get a surface S' with four boundary components such that $\partial S' = (\partial s_1) \times S^1$.

We get a sphere S out of S' by capping off the boundary components of S' by $\partial s_1 \times D^2$ and $\partial s_2 \times D^2$ in the neighborhood of the binding. Note that the union $((\Sigma \times I_1) \cup (\Sigma \times I_2))$ where ϕ_1 identifies $(\Sigma \times \{\frac{1}{2}\}) \subset (\Sigma \times I_1)$ to $(\Sigma \times \{\frac{1}{2}\}) \subset (\Sigma \times I_2)$ and ϕ_2 identifies $\Sigma \times \{0\} \subset \Sigma \times I_1$ to $\Sigma \times \{1\} \subset \Sigma \times I_2$ is actually equivalent to the mapping torus $\Sigma_{\phi_1 \circ \phi_2}$. Since ϕ_1 is id_{Σ_2} on Σ_2 and ϕ_2 is id_{Σ_1} on Σ_1 removing S' from $\Sigma_{\phi_1 \circ \phi_2}$ gives $(\Sigma_1)_{\phi_1} - (R \times I_1)$ and $(\Sigma_2)_{\phi_2} - (R \times I_2)$.

For the binding note that $D^2 \times \partial\Sigma$ is cut into four pieces along $D^2 \times \partial s_1$. By gluing the binding, $M_{(\Sigma, \phi_1 \circ \phi_2)}$ decomposes into two pieces:

$$(\Sigma_{1_{\phi_1}} \cup (D^2 \times \partial\Sigma_1)) - ((R \times I_1) \cup (D^2 \times s_1)) = M_1 - ((R \times I_1) \cup (D^2 \times s_1))$$

and

$$(\Sigma_{2_{\phi_2}} \cup (D^2 \times \partial\Sigma_2)) - ((R \times I_2) \cup (D^2 \times s_2)) = M_2 - ((R \times I_2) \cup (D^2 \times s_2)).$$

But notice that these are also complements of $B_1 = (R \times I_1) \cup (D^2 \times s_1)$ and $B_2 = (R \times I_2) \cup (D^2 \times s_2)$ in M_1 and M_2 respectively. So we see $M_{(\Sigma, \phi_1 \circ \phi_2)} = M_1 \sharp M_2$. $\square$

Actually Theorem 67 we have just proved is nothing other than a special case of a theorem which was proven by Stallings in 1976 in [11]. In [11], Stallings proves that the Murasugi sum of two pages of two open books is again the page of an open book. Actually the original statement in his paper was "If T_1 and T_2 are fibre surfaces so is T_3" where he defined T_3 is the surface which is obtained as Murasugi sum of T_1 and T_2. He used the fundamental groups in order to prove this statement.

In 1983, David Gabai advanced Stalling's Theorem one step further and proved that the other direction also works in [4]. Before stating his theorem we introduce some more definitions.

Definition 76. *For a link L a surface S is called a Seifert surface for L if $\partial S = L$.*

Definition 77. *We call a link L a fibered link if there exists one-parameter family F_t of Seifert surfaces for L where $t \in S^1$ and when $t \neq s$, $F_t \cap F_s = L$*

A knot in S^3 is fibered if and only if it is a binding of an open book decomposition of S^3.

Now we give Gabai's Theorem. We skip the proof.

Theorem 78. *Let B_1, B_2 be two fibered links in S^3 with fibers F_1, F_2 respectively such that $F = F_1 * F_2$. Then $B = \partial F$ is fibered link. Conversely if B is a fibered link with fiber $F = F_1 * F_2$ then B_1 and B_2 are fibered links.*

3.2 High Dimensional Version of Plumbings and Murasugi Sums

Recall that in dimension 3 we saw that there was an operation, Murasugi sums, that one could perform on a pair of abstract open books and an operation, connected sum, that could be performed on a pair of 3-manifolds. The main result here is that these two operations coincide, that is if one Murasugi sums two open books then the manifold described by the resulting open book will be the connected sum of the manifolds described by the original open books. We will now describe an operations on higher dimensional open books and a operation on higher dimensional manifolds and see that these two operations are similarly related.

We start with defining Seifert hypersurfaces.

Definition 79. *Let W be a manifold with boundary and let M be a compact hypersurface of W such that ∂M is embedded in W. Then if*

1. Each connected component of M has non-empty boundary,

2. M is also embedded in the interior of W,

3. M is co-oriented

are satisfied then M is called a Seifert hypersurface.

Definition 80. *Assume that W is a manifold with boundary and $M \hookrightarrow W$ is a Seifert hypersurface. Consider a tubular neighborhood $D^2 \times \partial M \subset W$ of ∂M. Let $\Pi_{\partial M}(W)$ denote the manifold obtained by gluing $(D^2 - \{0\}) \times \partial M \subset W \backslash \partial M$ to $(0,1] \times S^1 \times \partial M \subset [0,1] \times S^1 \times \partial M$ by the diffeomorphism $\pi_0 \times id_{\partial M}$ where $\pi_0 : [0,1] \times S^1 \to D^2$ is the map defined as $\pi_0(r, \theta) = (rcos\theta, rsin\theta)$. Then the map*

$$\pi_{\partial M} : \Pi_{\partial M}(W) \to W$$

is called radial blow-up of W along ∂M if it is obtained by gluing the inclusion map $i_1 : (W - \partial M) \longrightarrow W$ and $\pi_0 \times id_{\partial M} : [0,1] \times S^1 \times \partial M \longrightarrow D^2 \times \partial M \subset W$. Moreover the manifold $\Pi_{\partial M}(W)$ is called the piercing of W along ∂M.

Definition 81. *Let P be a compact manifold such that $\partial P \neq \emptyset$. If A is a codimensional 0 submanifold with boundary of ∂P then A is called an attaching region. Moreover $\overline{\partial P - A}$ is called the non-attaching region. The pair (P, A) is called an attaching pair.*

Definition 82. *Let M be a compact n-manifold with boundary and let (P, A) be an attaching pair of another compact manifold of dimension n. If P is embedded in M such that $P \cap \partial M = B$ where B is the non-attaching region of (P, A) then the pair (M, P) is called a patched manifold with patch (P, A).*

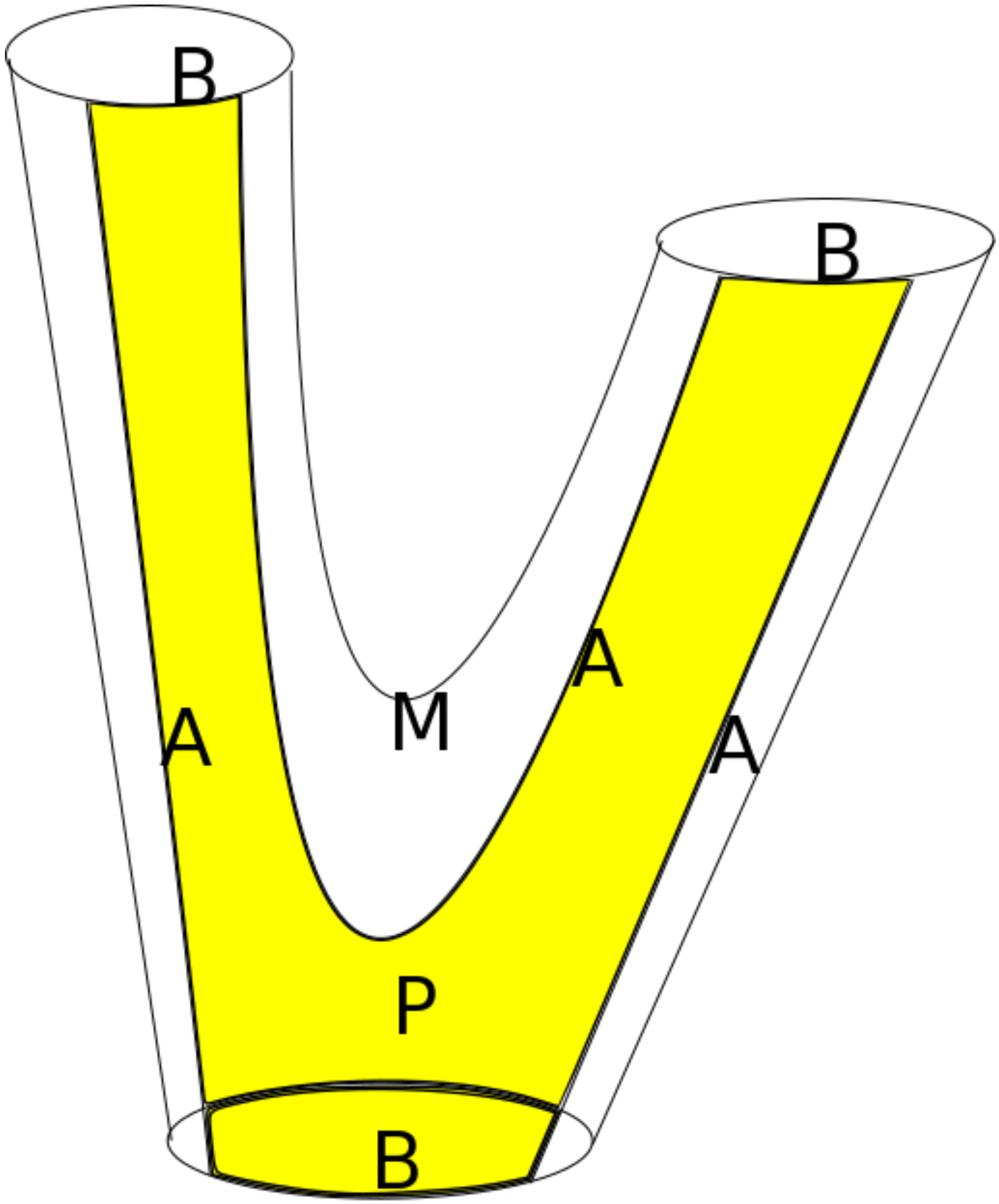

Figure 2: A patched manifold (M,P) with a patch (P,A)

Now we define an operation which can be considered as a generalization of Murasugi Sums to high dimensions.

Definition 83. *Let M_1 and M_2 be two compact n-manifolds with boundary and let P*

be compact n-manifold with boundary such that there are two embeddings $\phi_1 : P \hookrightarrow M_1$ *and* $\phi_2 : P \hookrightarrow M_2$*. If* $(M_1, \phi_1(P))$ *and* $(M_2, \phi_2(P))$ *are patched manifolds with patches* $(\phi_1(P), \phi_1(A_1))$ *and* $(\phi_2(P), \phi_1(A_2))$ *where* $A_1 \cap A_2 = \emptyset$ *respectively, then* $(M_1, \phi_1(P))$ *and* $(M_2, \phi_2(P))$ *are called summable.*

Definition 84. *For two summable manifolds* $(M_1, \phi_1(P))$ *and* $(M_2, \phi_2(P))$*, we define an operation by considering the disjoint union* $M_1 \sqcup M_2$ *and gluing the points of the patch by* $\phi_2 \circ \phi_1^{-1}$*. This operation is called the abstract sum of* M_1 *and* M_2 *along* P *and it is denoted* $M_1 \overset{P}{\uplus} M_2$*. The patch of the obtained manifold* $M_1 \overset{P}{\uplus} M_2$ *is the embedding* $P \hookrightarrow M_1 \overset{P}{\uplus} M_2$ *obtained by identifying two patches via the map* $\phi_2 \circ \phi_1^{-1}$ *.*

In dimension 3 case we have proven a nice property of Murasugi sums in Theorem 67. But talking about a similar relation in high dimensions requires some more work.

Definition 85. *Let* (M, P) *be a patched manifold and let* W *be a compact manifold with boundary such that* $M \hookrightarrow int(W)$ *is an embedding as an hypersurfaces. Then we have a series of embeddings*

$$P \hookrightarrow M \hookrightarrow W.$$

If P *is coorientable in* W *and such a coorientation is chosen then the triple* (W, M, P) *is called a patch-cooriented triple.*

Definition 86. *Let* $(W_1, M_1, \phi_1(P))$ *and* $(W_2, M_2, \phi_2(P))$ *be two patch-cooriented triples where* $\phi_1 : P \hookrightarrow M_1$ *and* $\phi_2 : P \hookrightarrow M_2$ *are embeddings such that* $(M_1, \phi_1(P))$ *and* $(M_2, \phi_2(P))$ *are two summable patched manifolds. Then the triples* $(W_1, M_1, \phi_1(P))$ *and* $(W_2, M_2, \phi_2(P))$ *are called summable.*

A summation operation between two summable patch co-oriented triples can be described as the following. First we consider the positive and negative thick patches. For a patch co-oriented triple (W, M, P), a positive thick patch is a choice of positive side $I^+ \times P \to W$ of the embedding $P \hookrightarrow W$ intersecting M only along P where I^+

is the compact oriented interval $[0,1]$. Similarly we define a negative thick patch as choice $I^- \times P \to W$ of embedding intersecting M only along P where $I^- = [-1,0]$. For the operation consider a positive thick patch $I^+ \times P \to W_1$ of (W_1, M_1, P) and a negative thick patch $I^- \times P \to W_2$ of (W_2, M_2, P) and set $W_1' = W_1 \backslash int(I^+ \times P)$ and $W_2' = W_2 \backslash int(I^- \times P)$ as the complements of the interiors of thick patches. Now let $\sigma : I^+ \to I^-$ be an orientation reversing diffeomorphism, i.e $\sigma(\partial_\pm I^+) = \partial_\mp I^-$, we glue W_1' to W_2' by restricting $\sigma \times id_P : I^+ \times P \to I^- \times P$ to $\partial(I^+ \times P)$. By this restriction we identify $\partial(I^+ \times P)$ to $\partial(I^- \times P)$ to get $(W_1, M_1) \biguplus^P (W_2, M_2)$. Finally we have the canonical embeddings:

$$P \to M_1 \biguplus^P M_2 \to (W_1, M_1) \biguplus^P (W_2, M_2).$$

In topology there exists a very interesting idea which has very nice applications. This idea is called cobordism. Actually cobordism is an equivalence relation and it plays an important role in topology. We will spend some effort on introducing cobordism and then will focus on our real matter again.

We start with giving the definition of cobordism.

Definition 87. *Let W be a compact manifold with boundary. A cobordism with corners structure on W is a structure which consists of two codimension zero compact submanifolds with boundary $\partial_- W$ and $\partial_+ W$ of ∂W where the following the conditions are satisfied;*

1. *$\partial_- W$ is cooriented with the incoming coorientation and it is called the incoming boundary region,*
2. *$\partial_+ W$ is cooriented with the outgoing coorientation and it is called the outgoing boundary region,*

A cobordism with corners structure on W is denoted $W : \partial_- W \Rightarrow \partial_+ W$.

Definition 88. *Let M^- and M^+ be two manifolds such that M^- is diffeomorphic to $\partial_- W$ and M^+ is diffeomorphic to $\partial_+ W$. Then W is called cobordism with corners from M^- to M^+. This case is denoted $W : M^- \Rightarrow M^+$. Given two cobordisms with corners $W_1 : M_1 \Rightarrow M_2$ and $W_2 : M_2 \Rightarrow M_3$, their composition $W_1 \circ W_2 : M_1 \Rightarrow M_3$ is another cobordism with corners which is obtained by gluing W_1 to W_2 along M_2.*

For a cobordism with corners structure on a manifold with boundary W, if $\partial_- W$ and $\partial_+ W$ are diffeomorphic, then the diffeomorphism between $\partial_- W$ and $\partial_+ W$ enables us to identify these two submanifolds with boundary. The following definitions will be based on this idea.

Definition 89. *Assume that there exists $W : \partial_- W \Rightarrow \partial_+ W$ such that $\partial_- W$ is diffeomorphic to $\partial_+ W$ with a fixed diffeomorphism. Then W is called an endobordism of M where M is a manifold diffeomorphic to $\partial_- W$ and $\partial_+ W$.*

Definition 90. *Let W be an endobordism $W : M^- \Rightarrow M^+$. Then the mapping torus of W is the manifold with boundary which is obtained by gluing M^- to M^+ via the fixed diffeomorphism. The mapping torus of W is denoted $T(W)$.*

Now we will define a special type of cobordism.

Definition 91. *For a manifold with boundary M the product $I \times M$ where $I = [0, 1]$ is called a cylinder with base M. A cylinder with base M is a cobordism where $\{0\} \times M$ is the incoming boundary region and $\{1\} \times M$ is the outgoing boundary region. Moreover if M^+ and M^- are two manifolds diffeomorphic to M, we define a cylindrical cobordism with base M as a cobordism $W : M^- \Rightarrow M^+$ if $\partial(I \times M) = \{0, 1\} \times M \cup I \times \partial M$ is diffeomorphic to the union of connected components of ∂W which have non-emtpy intersection with $M^+ \cup M^-$. The union at these components is called the cylindrical boundary and it is denoted $\partial_{cyl} W$. The interval $I = [0, 1]$ is called the directing segment of the cylindrical cobordism.*

Definition 92. *Take a compact manifold with boundary W and consider a cooriented and properly embedded compact hypersurface with boundary $M \hookrightarrow W$. Then $[-1,1] \times M$ is a collar neighborhood of M which can be considered as an cylinder $Z_{[-1,1]}$: $\{-1\} \times M \Rightarrow \{1\} \times M$ such that $Z_{[-1,1]} \simeq Z_{[0,1]} \circ Z_{[-1,0]}$ where $Z_{[0,1]}$ and $Z_{[-1,0]}$ denote the cylinders related to $[0,1]$ and $[-1,0]$ respectively. Let W_M be the closure of $W \backslash ([-1,1] \times M)$ in W which can be considered as an endobordism $W_M : \{-1\} \times M \Rightarrow \{1\} \times M$ which gives $Z_{[-1,0]} \circ W_M \circ Z_{[0,1]}$ as an endobordism of M. This endobordism is denoted by $\Sigma_M(W) : M^- \Rightarrow M^+$ and it is called splitting of W along M where M^- and M^+ are both diffeomorphic to M.*

Definition 93. *For a cylindrical cobordism $W : M^- \Rightarrow M^+$ with base M consider its mapping torus $T(W)$. A new manifold obtained by collapsing each circle $S^1 \times \{m\}$ to $\{0\} \times \{m\}$ as m varies on ∂M is called circle-collapsed mapping torus of W and it is denoted by $T_c(W)$.*

Definition 94. *For a cylindrical cobordism $W : M^- \longrightarrow M^+$ with base M, the embedding $M \longrightarrow T_c(W)$ is called the associated Seifert hypersurface of the cylindrical cobordism.*

We will give a theorem about the mapping tori of the cylindrical cobordisms. This theorem is going to be used in our important theorem which will show up at the end of the section. The proof can be found in [9].

Theorem 95. *Given a cylindrical cobordism $W : M^- \Rightarrow M^+$ with base M let N be the manifold obtained by gluing the mapping torus $T(W)$ to $D^2 \times \partial M$ via the canonical identification diffeomorphism of their boundaries, i.e.*

$$N = T(W) \cup D^2 \times \partial M / \sim$$

such that

$$(\theta, p) \in \partial T(W) \sim (\theta, p) \in \partial(D^2 \times \partial M) \approx S^1 \times \partial M.$$

Then N is diffeomorphic to the circle collapsed mapping torus $T_c(W)$ by a diffeomorphism ψ satisfying $\psi|_{T(W)\setminus V} = id$,where V is any neighborhood of $D^2 \times \partial M$, and $\psi(0 \times \partial M) = \partial M$.

Now we are going to define a special type of cylindrical cobordisms and we will use this specific type in the definition of a new operation.

Definition 96. *Let $W : M^- \Rightarrow M^+$ be a cylindrical cobordism and let V be a neighborhood of $M^- \cup M^+$ in W. If there exist a diffeomorphism of a neighborhood $(I - int(C)) \times M$ of $(\partial I) \times M$ in $I \times M$ extending the restriction to V to the diffeomorphism between $\partial_{cyl} W$ and $\partial(I \times M)$ then the cylindrical cobordism $W : M^- \Rightarrow M^+$ is called stiffened cylindrical cobordism. Here the neighborhood V is called the stiffening of the stiffened cylindrical cobordism $W : M^- \Rightarrow M^+$ and C is a compact subsegment which is embedded in $int(I)$ is called the core.*

Now we can define an operation on stiffened cylindrical cobordisms.

Definition 97. *Let $(W_1 : M_1^- \Rightarrow M_1^+, V_1)$ and $(W_2 : M_2^- \Rightarrow M_2^+, V_2)$ be two stiffened cylindrical cobordisms with the same directing segment I and cores C_1 and C_2 where $(M_1, \phi_1(P))$ and $(M_2, \phi_2(P))$ are two summable patched manifolds with the attaching regions $\phi_1(A_1)$ and $\phi_2(A_2)$ respectively. If $C_1 \cap C_2 = \emptyset$ and C_1 situated after C_2 with respect to the orientation on I then they are called summable where their sum*

$$(W_1, V_1) \overset{P}{\biguplus} (W_2, V_2)$$

is described in the following definition.

Definition 98. *Let $(W_1 : M_1^- \Rightarrow M_1^+, V_1)$ and $(W_2 : M_2^- \Rightarrow M_2^+, V_2)$ be two summable stiffened cylindrical cobordisms as described in Definition 97.*

The summation operation of two summable stiffened cylindrical cobordisms $(W_1 : M_1^- \Rightarrow M_1^+, V_1)$ and $(W_2 : M_2^- \Rightarrow M_2^+, V_2)$ is denoted by $(W_1, V_1) \overset{P}{\biguplus} (W_2, V_2)$ and it can be described by the following steps:

1. *For each t in $(I - int(C_1))$, considering the canonical identifications $\phi_1(A_1) \to \partial\overline{(M_1/\phi_1(P))}$ and $\phi_1(A_1) \to \partial M_2$ glue fiberwise $\overline{(M_1/\phi_1(P))}$ to M_2 where $\phi_1(A_1)$ is the attaching region.*

2. *For each t in $(I - int(C_2))$, similarly considering the canonical identifications $\phi_2(A_2) \to \partial\overline{(M_2/\phi_2(P))}$ and $\phi_2(A_2) \to \partial M_1$ glue fiberwise $\overline{(M_2/\phi_2(P))}$ to M_1 where $\phi_2(A_2)$ is the attaching region.*

3. *For each t in $(I - (int(C_1) \cup int(C_2))$ sum fiberwise $(M_1, \phi_1(P))$ to $(M_2, \phi_2(P))$.*

Definition 99. *Let $(M_1, \phi_1(P))$ and $(M_2, \phi_2(P))$ be two summable patched manifolds with attaching regions $\phi_1(A_1)$ and $\phi_2(A_2)$. Consider the cylindrical cobordisms $W_1 : M_1^- \Rightarrow M_1^+$ and $W_2 : M_2^- \Rightarrow M_2^+$ with directing segments I_1 and I_2 and stiffenings V_1 and V_2 respectively. Take an orientation preserving diffeomorphism $\varphi : I_1 \to I_2$ placing the core segment of I_1 after the core segment of I_2.The sum of W_1 and W_2 is obtained by identifying the directing segments I_1 and I_2 by φ and applying the operation described in Definition 98. The sum of W_1 and W_2 is denoted by $W_1 \overset{P}{\uplus} W_2$.*

Now we can state and prove our big theorem. We will use the definitions and the operations we have introduced in this section for the proof.

Theorem 100. *Given two summable patched Seifert hypersurfaces $(W_1, M_1, \phi_1(P))$ and $(W_2, M_2, \phi_2(P))$, their embedded sum*

$$M_1 \overset{P}{\uplus} M_2 \to (W_1, M_1) \overset{P}{\uplus} (W_2, M_2)$$

as described after Definition 86 is diffeomorphic (up to isotopy) to the circle collapsed mapping torus of the sum of cylindirical cobordisms

$$T_c(\Sigma_{M_1}(W_1) \overset{P}{\uplus} \Sigma_{M_2}(W_2)).$$

Proof. The statement of the theorem includes two constructions. The first one is the circle collapsed mapping torus of the sum of two cylindrical cobordisms $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$ and the second one is the embedded sum of patch cooriented triples $(W_1, M_1, \phi_1(P))$ and $(W_2, M_2, \phi_2(P))$. The proof will be based on showing that the circle collapsed mapping torus of the sum $\Sigma_{M_1}(W_1) \overset{P}{\uplus} \Sigma_{M_2}(W_2)$ actually is the manifold obtained from circle collapsed mapping tori of $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$ by first taking out submanifolds which are diffeomorphic to $[0,1] \times P$ and then identifying the boundaries.

Note that the circle-collapsed mapping tori $T_c(\Sigma_{M_1}(W_1))$ and $T_c(\Sigma_{M_2}(W_2))$ are diffeomorphic to manifolds constructed by filling $\partial T_c(\Sigma_{M_1}(W_1))$ with $D_1 \times \partial M_1$ and filling $\partial T(\Sigma_{M_2}(W_2))$ with $D_2 \times \partial M_2$ respectively. Keeping this fact in our mind, refer to Definition 80 and set

$$\Phi_{\partial M_1}(W_1) = \Pi_{\partial M_1}(W_1) \cup_{S^1 \times \partial M_1} (D^2 \times \partial M_1)$$

and

$$\Phi_{\partial M_2}(W_2) = \Pi_{\partial M_2}(W_2) \cup_{S^1 \times \partial M_2} (D^2 \times \partial M_2)$$

where $\Pi_{\partial M_1}(W_1)$ and $\Pi_{\partial M_2}(W_2)$ denote piercings of W_1 and W_2 along ∂M_1 and ∂M_2 respectively.

In order to make a similar filling of $\partial(T(\Sigma_{M_1}(W_1) \overset{P}{\uplus} \Sigma_{M_2}(W_2)))$ consider the stiffenings V_1 of $\Sigma_{M_1}(W_1)$ and V_2 of $\Sigma_{M_2}(W_2)$ such that the operation described in Definiton 98 can be performed. We need to identify the directing segments of $\Sigma_{W_1}(M_1)$ and $\Sigma_{W_2}(M_2)$ in order to perform the operation described before. Note that this operation has three different processes in three regions. But one of these three regions requires some more attention. Over $(I - (int(C_1) \cup int(C_2)))$ the operation can be performed in many different ways. For that reason we need to search for the description which makes us most comfortable.

Now set $\alpha_+ = \partial_+ I$ and $\alpha_- = \partial_- I$ and take a point β between the cores C_1

and C_2. Now set two new intervals $I_1 = [\alpha_-, \beta]$ and $I_2 = [\beta, \alpha_+]$. Let E_1 and E_2 be the closures of $\partial M_1 - B_1$ in ∂M_1 and $\partial M_2 - B_2$ in ∂M_2 respectively where B_1 is the non-attaching region for $(M_1, \phi_1(P))$ and B_2 is the non-attaching region for $(M_2, \phi_2(P))$. And let K_1 and K_2 be the closures of $\partial(\phi_1(P)) - (\phi_1(A_1) \cup \phi_1(A_2))$ and $\partial(\phi_2(P)) - (\phi_2(A_1) \cup \phi_2(A_2))$ respectively. Over I_1 and I_2 we will perform the gluing

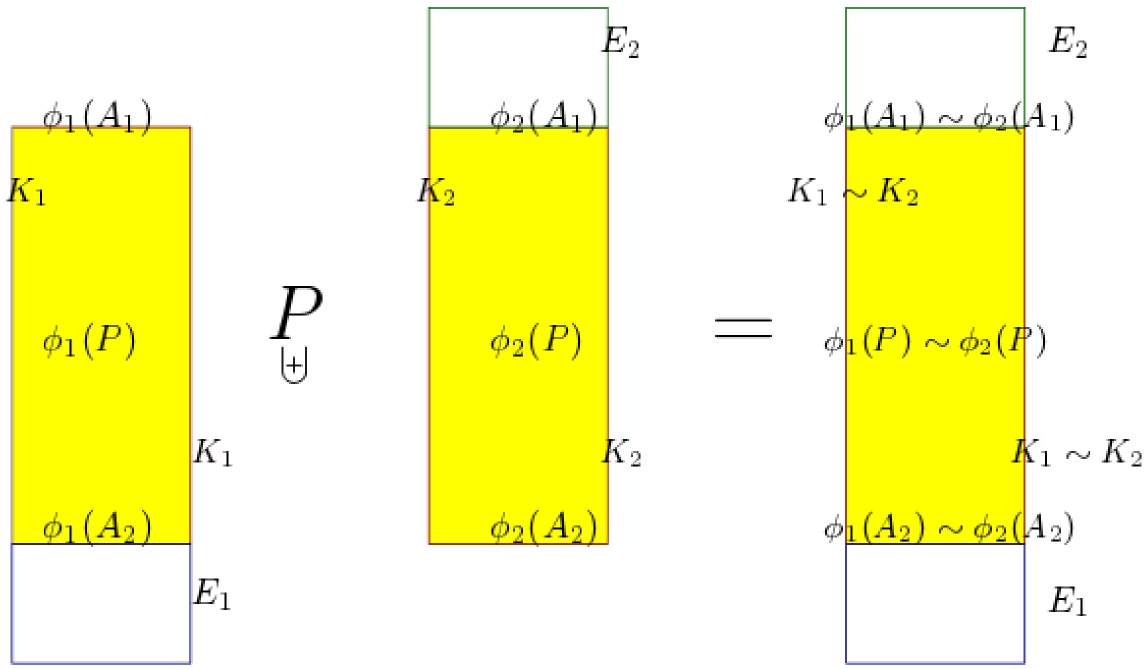

Figure 3: The presentation of E_i and K_i

by removing $\phi_1(P)$ and $\phi_2(P)$ from $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$ respectively.

Now we can perform the following steps:

First, remove $(I_1 \times \phi_1(P)) \cup (D^2 \times \partial M_1)$ from $\Phi_{M_1}(W_1)$ and take the closure. Similarly, remove $(I_2 \times \phi_2(P)) \cup (D^2 \times \partial M_2)$ from $\Phi_{M_2}(W_2)$ and take the closure.

Secondly, identify the boundaries that we have after the first step. Note that the boundaries are isomorphic to

$$(I_1 \times A_1) \cup (I_2 \times A_2) \cup (\{\alpha_-, \alpha_+\} \times \phi_1(P)) \cup (\beta \times \phi_2(P)).$$

Since we identify K_1 and K_2 in $\partial(M_1 \overset{P}{\uplus} M_2)$, we can consider them as one submanifold

K. Now fill the new boundary by

$$D^2 \times \partial(M_1 \overset{P}{\biguplus} M_2) = (D^2 \times E_1) \cup (D^2 \times K) \cup (D^2 \times E_2).$$

Note that while performing the operation the pieces $(D^2 \times E_1)$ and $(D^2 \times E_2)$ are moving twice: they are taken out first and inserted back again. But the middle piece $(D^2 \times K)$ moves three times: It is removed twice (once from M_1 and once form M_2) but inserted back only once. An alternative way is to cut disc D^2 into two pieces, say D_1 and D_2, and remove $D_1 \times K$ from $\Phi_{\partial M_1}(W_1)$ and $D_2 \times K$ from $\Phi_{\partial M_2}(W_2)$.

What we are actually doing here is to remove $(I_1 \times \phi_1(P)) \cup (D^2 \times A_2) \cup (D_1 \times K)$ from $\Phi_{\partial M_1}(W_1)$ and to remove $(I_2 \times \phi_2(P)) \cup (D^2 \times A_1) \cup (D_2 \times K)$ from $\Phi_{\partial M_2}(W_2)$ and then take the closures.

Now referring to the fact that $(I_1 \times \phi_1(P)) \cup (D^2 \times A_2) \cup (D_1 \times K)$ is isomorphic to $I_1 \times \phi_1(P)$ and $(I_2 \times \phi_2(P)) \cup (D^2 \times A_1) \cup (D_2 \times K)$ is isomorphic to $I_2 \times \phi_2(P)$.

In order to complete the proof of Theorem 100, we need to state a new theorem which allows us to take the final step. The proof can be found in [9].

Theorem 101. *Let M be a manifold with boundary and let N be a codimension 1 submanifold with boundary of M embedded in ∂M. Let Z be a manifold obtained by gluing $[0,1] \times N$ to M by identifying $\{0\} \times N$ with N. Then Z is isomorphic to M with an isomorphism which is the identity map outside an arbitrary small neighborhood of N in M.*

Now we turn back to the proof of Theorem 100. Since D_1 and D_2 are both diffeomorphic to $[0,1] \times [0,1]$ applying Theorem 101 gives us the isomorphism which completes the proof. □

We will take one step further and give another theorem which is a generalization of the theorem about stabilizations which we have in dimension 3.

Theorem 102. *Let $(W_1, M_1, \phi_1(P))$ and $(W_2, M_2, \phi_2(P))$ be two summable patched Seifert hypersurfaces being pages of open books on the closed manifolds W_1 and W_2*

respectively. Then the associated Seifert hypersurface of $(W_1, M_1) \biguplus (W_2, M_2)$ *is page of an open book whose monodromy is the composition of monodromies of initial open books.*

Proof. Let $(\partial M_1, \theta_1)$ and $(\partial M_2, \theta_2)$ be open books on W_1 and W_2, respectively, satisfying $M_1 = \theta_1^{-1}(0)$ and $M_2 = \theta_2^{-1}(0)$. The maps $\theta_1 : W_1 \backslash \partial M_1 \to S^1$ and $\theta_2 : W_2 \backslash \partial M_2 \to S^1$ lift to maps $\overline{\theta_1} : \Pi_{\partial M_1} W_1 \to S^1$ and $\overline{\theta_2} : \Pi_{\partial M_2} W_2 \to S^1$ respectively. By splitting S^1 at the point of argument 0 we have the lifts

$$\Sigma(\overline{\theta_1}) : \Sigma_{M_1}(W_1) \to [0, 2\pi] \; and \; \Sigma(\overline{\theta_1}) : \Sigma_{M_1}(W_1) \to [0, 2\pi]$$

Taking $C_1, C_2 \subset (0, 2\pi)$ as arbitrary compact segments with nonempty interiors, the preimages $\Sigma(\overline{\theta_1})^{-1}([0, 2\pi] - int(C_1))$ and $\Sigma(\overline{\theta_2})^{-1}([0, 2\pi] - int(C_2))$ are stiffening of $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$ respectively. In order to perform the operation as described before, we need to assume that $C_1 \cap C_2 = \emptyset$ and we see $\Sigma(\overline{\theta_1})$ and $\Sigma(\overline{\theta_2})$ as the height functions.

Gluing the height functions we have a new height function

$$h : \Sigma_{M_1}(W_1) \overset{P}{\biguplus} \Sigma_{M_2}(W_2) \to [0, 2\pi]$$

which is a fiber bundle projection whose genetic fiber is isomorphic to $M_1 \overset{P}{\biguplus} M_2$.

Then we have the associated Seifert hypersurface as a page of the open book $M_1 \overset{P}{\biguplus} M_2$.

Theorem 101 gives us the isomorphism between this associated Seifert hypersurface and $M_1 \overset{P}{\biguplus} M_2 \to (W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)$

□

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