ON THE STATIONARY AND UNIFORMLY-ROTATING SOLUTIONS OF ACTIVE SCALAR EQUATIONS

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To my mother Cho Myeongji

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SUMMARY

In this thesis, we study qualitative and quantitative properties of stationary/uniformly-rotating solutions of the 2D incompressible Euler equation and the generalized Surface Quasi-Geostrophic (SQG) equations. The main goal is to establish sufficient and necessary conditions for the stationary/uniformly rotating solutions to be radially symmetric. In addition, we also derive quantitative estimates for non-radial, uniformly-rotating patch solutions for the 2D Euler equation.

In chapter 1, we briefly review basic properties of the 2D incompressible Euler equation and the generalized SQG equations. We also rigorously define stationary and uniformly-rotating solutions to those equations by means of the stream function.

Chapters 2 to 4 describe the joint work with Javier Gómez–Serrano, Jia Shi and Yao Yao [50, 52]. We establish sufficient conditions for stationary/uniformly-rotating solutions for some active scalar equations to be radially symmetric. In short, we prove that for the 2D Euler equation,

i (Patch setting) If a vortex patch $\omega = 1_D$ is uniformly rotating with angular velocity $\Omega \in \left(0, \frac{1}{2}\right)^c$, then D must be radially symmetric up to a translation.

ii (Smooth setting) If a smooth, non-negative compactly supported vorticity ω is uniformly rotating with angular velocity $\Omega \leq 0$, then ω is radially symmetric.

The proof is based on a variational argument that a uniformly-rotating solution can be formally thought of as a critical point of an energy functional. We apply this idea to more general active scalar equations (gSQG) and vortex sheet equation.

In chapter 5, we construct a non-radial vortex sheet with non-constant vortex strength, which is rotating with angular velocity $\Omega > 0$. We obtain a curve of such non-radial solutions, bifurcating from trivial ones. This result comes from the joint work with Javier Gómez–Serrano, Jia Shi and Yao Yao [51].

In chapter 6, we describe the result in [97]. We adapt the variational argument that was used in

chapter 2 to study non-radial rotating vortex patches. It is well known that for $\Omega \in (0, \frac{1}{2})$, there are m-fold symmetric rotating patches. We derive some quantitative estimates for those patches about their angular velocities and the difference with the unit disk.

CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Two dimensional active scalar equations: Euler and gSQG

The two-dimensional incompressible Euler equation in vorticity form reads

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, t) = -\nabla^\perp (-\Delta)^{-1} \omega(\cdot, t) & \text{in } \mathbb{R}^2, \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1.1}$$

where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. Note that we can express u as $u(\cdot, t) = \nabla^\perp(\omega(\cdot, t) * \mathcal{N})$, where $\mathcal{N}(x) := \frac{1}{2\pi} \ln |x|$ is the Newtonian potential in two dimensions. This equation describes the motion of incompressible ideal fluid in two dimensional space. Mathematically, the 2D Euler equation can be seen as an example of active scalar equation, in the sense that the scalar-valued function ω is transported by the velocity u, meanwhile the velocity also depends on ω. Another interesting active scalar equation is the inviscid Surface Quasi-Geostropic equation, where we replace $(-\Delta)^{-1}$ by $(-\Delta)^{-\frac{1}{2}}$ in (Equation 1.1.1). Besides its importance in the context of geophysics and atmosphere science, the inviscid SQG also serves as a toy model for the 3D incompressible Euler equation [32]. More generally, both the 2D Euler equation and the inviscid SQG equation belong to the following family of active scalar equations indexed by a parameter α, ($0 \leq \alpha < 2$), known as the generalized SQG equations:

$$
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\
u(\cdot, t) = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \omega(\cdot, t) & \text{in } \mathbb{R}^2, \\
\omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2.
\end{cases}
\tag{1.1.2}
$$

Here we can also express the Biot–Savart law as

$$
u(\cdot, t) = \nabla^\perp (\omega(\cdot, t) * K_\alpha),
\tag{1.1.3}
$$

where K_α is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, that is,

$$
K_\alpha(x) = \begin{cases}
\frac{1}{2\pi} \ln |x| & \text{for } \alpha = 0, \\
-C_\alpha |x|^{-\alpha} & \text{for } \alpha \in (0, 2),
\end{cases}
\tag{1.1.4}
$$

where $C_\alpha = \frac{1}{2\pi} \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha} \Gamma(1 - \frac{\alpha}{2})}$ is a positive constant only depending on α.

We either work with the *patch* setting, where $\omega(\cdot, t) = 1_{D(t)}$ is an indicator function of a bounded set that moves with the fluid, or the smooth setting, where $\omega(\cdot, t)$ is smooth and compactly-supported in x. For well-posedness results for patch solutions, see the global well-posedness results [9, 26] for (Equation 1.1.1), and local well-posedness results [25, 101, 43, 75, 33] for (Equation 1.1.2) with $\alpha \in (0, 2)$.

1.1.1 Uniformly rotating/Stationary solutions of 2D Euler and gSQG

Let us begin with the definition of a stationary/uniformly-rotating solution in the patch setting. For a bounded domain $D \subset \mathbb{R}^2$ with C^1 boundary, we say $\omega = 1_D$ is a *stationary patch* solution to (Equation 1.1.2) for some $\alpha \in [0, 2)$ if $u(x) \cdot \vec{n}(x) = 0$ on ∂D, with u given by (Equation 1.1.3).

This leads to the integral equation

$$1_D * \mathcal{K}_\alpha \equiv C_i \quad \text{on } \partial D, \tag{1.1.5}$$

where the constant C_i can differ on different connected components of ∂D. And if $\omega(x,t) = 1_D(R_{\Omega t}x)$ is a *uniformly-rotating patch* solution with angular velocity Ω (where $R_{\Omega t}x$ rotates a vector $x \in \mathbb{R}^2$ counter-clockwise by angle Ωt about the origin), then 1_D becomes stationary in the rotating frame with angular velocity Ω, that is, $\left(\nabla^\perp(\omega(\cdot,t) * K_\alpha) - \Omega x^\perp\right) \cdot \vec{n}(x) = 0$ on ∂D. As a result we have

$$1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on } \partial D, \tag{1.1.6}$$

where C_i again can take different values along different connected components of ∂D. Note that a stationary patch D also satisfies (Equation 1.1.6) with $\Omega = 0$, and it can be considered as a special case of uniformly-rotating patch with zero angular velocity.

Likewise, in the smooth setting, if $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a uniformly-rotating solution of (Equation 1.1.2) with angular velocity Ω (which becomes a stationary solution in the $\Omega = 0$ case), then we have $(\nabla^\perp(\omega_0 * K_\alpha) - \Omega x^\perp) \cdot \nabla\omega_0 = 0$. As a result, ω_0 satisfies

$$\omega_0 * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on each connected component of a regular level set of } \omega_0, \tag{1.1.7}$$

where C_i can be different if a regular level set $\{\omega_0 = c\}$ has multiple connected components.

1.1.2 Uniformly rotating/Stationary vortex sheets

A vortex sheet is a weak solution of the 2D Euler equations:

$$v_t + v \cdot \nabla v = -\nabla p, \quad \nabla \cdot v = 0, \tag{1.1.8}$$

whose vorticity $\omega = \text{curl}(v)$ is a delta function supported on a curve or a finite number of curves $\Gamma_i = z_i(\alpha, t)$, i.e.

$$\omega(x,t) = \sum_i \varpi_i(\alpha,t)\delta(x - z_i(\alpha,t)). \tag{1.1.9}$$

Here $\varpi_i(\alpha, t)$ is the vorticity strength on Γ_i with respect to the parametrization z_i, and the above equation is understood in the sense that

$$\int_{\mathbb{R}^2} \varphi(x) d\omega(x,t) = \sum_i \int \varphi(z_i(\alpha,t))\varpi_i(\alpha,t)d\alpha$$

for all test functions $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$.

The motivation of the study of the equation (Equation 1.1.8) with vortex sheet initial data comes from the fact that in fluids with small viscosity, flows separate from rigid walls and corners [87, 102]. To model it, one may think of a solution to (Equation 1.1.8) with one incompressible fluid where the velocity changes sign in a discontinuous (tangential) way across a streamline z. This discontinuity induces vorticity in z.

The equations of motion of ϖ_i and z_i can be derived by means of the Birkhoff-Rott operator ([20, 81, 87, 108]), namely:

$$BR(z,\varpi)(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta,t))^\perp}{|x - z(\beta,t)|^2} \varpi(\beta,t)d\beta, \tag{1.1.10}$$

yielding

$$\partial_t z_i(\alpha,t) = \sum_j BR(z_j,\varpi_i)(z_i(\alpha,t)) + c_i(\alpha,t)\partial_\alpha z_i(\alpha,t) \tag{1.1.11}$$

$$\partial_t \varpi_i(\alpha,t) = \partial_\alpha (c_i(\alpha,t)\varpi_i(\alpha,t)), \tag{1.1.12}$$

where the term $c_i(\alpha, t)$ accounts for the reparametrization freedom of the curves. See the paper [70] by Izosimov–Khesin where they propose geodesic, group-theoretic, and Hamiltonian frameworks for their description.

As in the patch/smooth setting, we first define what we mean by a stationary vortex sheet. Assume the initial data ω_0 of (Equation 1.1.9) is supported on a finite number of curves parametrized by $z_i(\alpha)$, with strength $\varpi_i(\alpha)$ (with respect to the parametrization z_i) respectively. If there exists some reparametrization choice $c_i(\alpha)$ such that the right hand sides of (Equation 1.1.11)–(Equation 1.1.12) are both identically zero for every i, it gives that $\omega(\cdot, t)$ is invariant in time, and we say $\omega(\cdot, t) = \omega_0$ is a *stationary vortex sheet*.

For any $x \in \mathbb{R}^2$ and $\Omega \in \mathbb{R}$, let $R_{\Omega t}x$ denote the rotation of x counter-clockwise by angle Ωt about the origin. We say $\omega(x, t) = \omega_0(R_{\Omega t}x)$ is a *uniformly-rotating vortex sheet* with angular velocity Ω if ω_0 is stationary in the rotating frame with angular velocity Ω. (Note that in the special case $\Omega = 0$, the uniformly-rotating sheet is in fact stationary.) In chapter 4, we will derive the equations satisfied by a stationary/rotating vortex sheet.

1.2 Main results and idea of proofs

In subsection 1.2.1 and subsection 1.2.2, we describe the results of joint work with Javier Gómez–Serrano, Jia Shi and Yao Yao [50, 52, 51], which concern rigidity/flexibility of uniformly-rotating/stationary solutions. The proofs for these results will be contained in chapter 2-chapter 5. In subsection 1.2.3, we describe the results obtained in [97]. The proofs for quantitative estimates will be presented in chapter 6.

1.2.1 Rigidity results for 2D Euler and gSQG [50]

Chapter 2 and Chapter 3 will be devoted to establish rigidity results for 2D Euler and gSQG equations. Clearly, every radially symmetric patch/smooth function automatically satisfies (Equation 1.1.6) or (Equation 1.1.7) for all $\Omega \in \mathbb{R}$. The goal of chapter 2-chapter 3 is to address the complementary question, which can be roughly stated as following:

Question 1. *In the patch or smooth setting, under what condition must a stationary/uniformly-rotating solution be radially symmetric?*

Below we summarize the previous literature related to this question, and state our main results. We will first discuss the 2D Euler equation in the patch and smooth setting respectively, then discuss the gSQG equation with $\alpha \in (0, 2)$.

2D Euler in the patch setting

Let us deal with the patch setting first. So far affirmative answers to Question 1 have only been only obtained for simply-connected patches, for angular velocities $\Omega = 0$, $\Omega < 0$ (under some additional convexity assumptions), and $\Omega = \frac{1}{2}$. For stationary patches ($\Omega = 0$), Fraenkel [41, Chapter 4] proved that if D satisfies (Equation 1.1.6) (where $K_\alpha = \mathcal{N}$) with the *same* constant C on the whole ∂D, then D must be a disk. The idea is that in this case the stream function $\psi = 1_D * \mathcal{N}$ solves a semilinear elliptic equation $\Delta\psi = g(\psi)$ in $\mathbb{R}^2$ with $g(\psi) = 1_{\{\psi<C\}}$, where the monotonicity of the discontinuous function g allows one to apply the moving plane method developed in [105, 48] to obtain the symmetry of ψ. As a direct consequence, every simply-connected stationary patch must be a disk. But if D is not simply-connected, (Equation 1.1.6) gives that $\psi = C_i$ on different connected components of ∂D, thus ψ might not solve a single semilinear elliptic equation in $\mathbb{R}^2$. Even if ψ satisfies $\Delta\psi = g(\psi)$, g might not have the right monotonicity. For these reasons, whether a non-simply-connected stationary patch must be radial still remained an open question.

For $\Omega < 0$, Hmidi [59] used the moving plane method to show that a simply-connected uniformly-rotating patch D satisfies some additional convexity assumption (which is stronger than star-shapedness but weaker than convexity), then D must be a disk. In the special case $\Omega = \frac{1}{2}$, Hmidi [59] also showed that a simply-connected uniformly-rotating patch D must be a disk, using the fact that $1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ becomes a harmonic function in D when $\Omega = \frac{1}{2}$.

On the other hand, it is known that there can be non-radial uniformly-rotating patches for

$\Omega \in (0, \frac{1}{2})$. The first example dates back to the Kirchhoff ellipse [74], where it was shown that any ellipse D with semiaxes a, b is a uniformly-rotating patch with angular velocity $\frac{ab}{(a+b)^2}$. Deem–Zabusky [36] numerically found families of patch solutions of (Equation 1.1.1) with m-fold symmetry by bifurcating from a disk at explicit angular velocities $\Omega_m^0 = \frac{m-1}{2m}$ and coined the term V-states. Further numerics were done in [116, 37, 86, 103]. Burbea gave the first rigorous proof of their existence by using (local) bifurcation theory arguments close to the disk [12]. There have been many recent developments in a series of works by Hmidi–Mateu–Verdera and de la Hoz–Hmidi–Mateu–Verdera [63, 67, 64] in different settings and directions (regularity of the boundary, different topologies, etc.). In particular, [67] showed the existence of m-fold doubly-connected non-radial patches bifurcating at any angular velocity $\Omega \in \left(0, \frac{1}{2}\right)$ from some annulus of radii $b \in (0, 1)$ and 1.

There are many other interesting perspectives of the V-states, which we briefly review below, although they are not directly related to Question 1. Hassainia–Masmoudi–Wheeler [58] were able to perform global bifurcation arguments and study the whole branch of V-states. Other scenarios such as the bifurcation from ellipses instead of disks have also been studied: first numerically by Kamm [71] and later theoretically by Castro–Córdoba–Gómez-Serrano [22] and Hmidi–Mateu [60]. See also the work of Carrillo–Mateu–Mora–Rondi–Scardia–Verdera [19] for variational techniques applied to other anisotropic problems related to vortex patches. Love [82] established linear stability for ellipses of aspect ratio bigger than $\frac{1}{3}$ and linear instability for ellipses of aspect ratio smaller than $\frac{1}{3}$. Most of the efforts have been devoted to establish nonlinear stability and instability in the range predicted by the linear part. Wan [114], and Tang [110] proved the nonlinear stable case, whereas Guo–Hallstrom–Spirn [53] settled the nonlinear unstable one. See also [30]. In [112], Turkington considered N vortex patches rotating around the origin in the variational setting, yielding solutions of the problem which are close to point vortices.

Our first main result is summarized in the following Theorem A, which gives a complete answer to Question 1 for 2D Euler in the patch setting. Note that D is allowed to be disconnected, and each connected component can be non-simply-connected. Figure Figure 1.1 illustrates a comparison of our result (in red color) with the previous results (in black color).

Theorem A (= Corollary 2.1.8, Theorems 2.1.10 and 2.1.12). *Let $D \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary. Assume D is a stationary/uniformly-rotating patch of* (Equation 1.1.1)*, in the sense that D satisfies* (Equation 1.1.6) *(with $K_\alpha = \mathcal{N}$) for some $\Omega \in \mathbb{R}$. Then D must be radially symmetric if $\Omega \in (-\infty, 0) \cup [\frac{1}{2}, \infty)$, and radially symmetric up to a translation if $\Omega = 0$.*

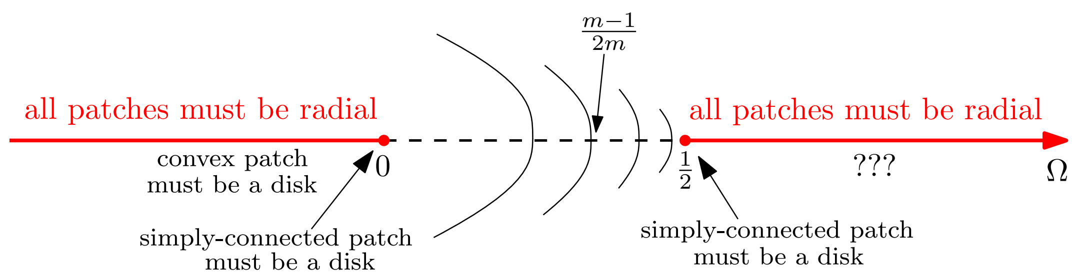


Figure 1.1: For 2D Euler in the patch setting, previous results on Question 1 are summarized in black color. Our results in Theorem A are colored in red.

2D Euler in the smooth setting

One of the main motivations of this work is to find sufficient rigidity conditions in terms of the vorticity, such that the only stationary/uniformly-rotating solutions are radial ones. Heuristically speaking, this belongs to the broader class of "Liouville Theorem" type of results, which show that solutions satisfying certain conditions must have a simpler geometric structure, such as being constant (in one direction, or all directions) or being radial. In the literature we could not find any conditions on 2D Euler that leads to radial symmetry, although several other Liouville-type results have been established for 2D fluid equations: For 2D Euler, Hamel–Nadirashvili [55, 54] proved that any stationary solution without a stagnation point must be a shear flow. (But note that this result does not apply to our setting (Equation 1.1.7), since the velocity field u associated with any compactly-supported ω_0 must have a stagnation point). See also the Liouville theorem by Koch–Nadirashvili–Seregin–Šverák for the 2D Navier–Stokes equations [76].

Let us briefly review some results on the characterization of stationary solutions to 2D Euler, although they are not directly related to Question 1. Nadirashvili [92] studied the geometry and the stability of stationary solutions, following the works of Arnold [2, 3, 4]. Izosimov–Khesin [69] characterized stationary solutions of 2D Euler on surfaces. Choffrut–Šverák [28] showed that locally near each stationary smooth solution there exists a manifold of stationary smooth solutions transversal to the foliation, and Choffrut–Székelyhidi [27] showed that there is an abundant set of stationary weak (L^∞) solutions near a smooth stationary one. Shvydkoy–Luo [84, 85] classified the set of stationary smooth solutions of the form $v = \nabla^\perp(r^\gamma f(\omega))$, where (r, ω) are polar coordinates. In a different direction, Turkington [111] used variational methods to construct stationary vortex patches of a prescribed area in a bounded domain, imposing that the patch is a characteristic function of the set $\{\Psi > 0\}$, and also studied the asymptotic limit of the patches tending to point vortices. Long–Wang–Zeng [80] studied their stability, as well as the regularity in the smooth setting (see also [16]). For other variational constructions close to point vortices, we refer to the work of Cao–Liu–Wei [14], Cao–Peng–Yan [15] and Smets–van Schaftingen [107]. We remark that these results do not rule out that those solutions may be radial. Musso–Pacard–Wei [91] constructed nonradial smooth stationary solutions without compact support in ω. The (nonlinear L^1) stability of circular patches was proved by Wan–Pulvirenti [113] and later Sideris–Vega gave a shorter proof [106]. See also Beichman–Denisov [7] for similar results on the strip.

Lately, Gavrilov [45, 46] provided a remarkable construction of nontrivial stationary solutions of 3D Euler with compactly supported velocity. See also Constantin–La–Vicol for a simplified proof with extensions to other fluid equations [31].

Regarding uniformly-rotating smooth solutions ($\Omega \neq 0$) for 2D Euler, Castro–Córdoba–Gómez-Serrano [24] were able to desingularize a vortex patch to produce a smooth m-fold V-state with $\Omega \sim \frac{m-1}{2m} > 0$ for $m \geq 2$. Recently García–Hmidi–Soler [44] studied the construction of V-states bifurcating from other radial profiles (Gaussians and piecewise quadratic functions).

Our second main result is the following theorem, which gives radial symmetry in the smooth setting for $\Omega \leq 0$, under the additional assumption $\omega_0 \geq 0$:

Theorem B (= Theorem 2.2.5 and Corollary 2.2.6). *Let $\omega_0 \geq 0$ be smooth and compactly-supported. Assume $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating solution of* (Equation 1.1.1) *with $\Omega \leq 0$, in the sense that it satisfies* (Equation 1.1.7). *Then ω_0 must be radially symmetric if $\Omega < 0$, and radially symmetric up to a translation if $\Omega = 0$.*

The gSQG case ($0 < \alpha < 2$)

Recall that in the patch setting, a stationary/uniformly-rotating patch satisfies (Equation 1.1.6) with K_α given in (Equation 1.1.4). Even though the kernels K_α are qualitatively similar for all $\alpha \in [0, 2)$, there is a key difference on the symmetry v.s. non-symmetry results between the cases $\alpha = 0$ and $\alpha > 0$: For the 2D Euler equation ($\alpha = 0$), we proved in Theorem A that any rotating patch D with $\Omega \leq 0$ must be radial, even if D is not simply-connected. However, this result is not true for any $\alpha \in (0, 2)$: de la Hoz–Hassainia–Hmidi–Mateu [68] showed that there exist non-radial patches bifurcating from annuli at $\Omega < 0$ and Gómez-Serrano [49] constructed non-radial, doubly connected stationary patches ($\Omega = 0$). Therefore we cannot expect a non-simply-connected rotating patch D with $\Omega \leq 0$ to be radial for $\alpha \in (0, 2)$.

However, if D is a simply-connected stationary patch, then radial symmetry results were obtained in a series of works for $\alpha \in [0, \frac{5}{3})$, which we review below. These works consider (Equation 1.1.6) in a more general context not limited to dimension 2: Let $K_{\alpha,d}$ be the fundamental solution of $(-\Delta)^{-1+\frac{\alpha}{2}}$ in $\mathbb{R}^d$ for $d \geq 2$, given by

$$K_{\alpha,d} := -C_{\alpha,d}|x|^{-d+2-\alpha} \tag{1.2.1}$$

for some $C_{\alpha,d} > 0$; except that in the special case $-d + 2 - \alpha = 0$ it becomes $K_{\alpha,d} = C_d \ln |x|$ for some $C_d > 0$. Note that $K_{\alpha,d} \in L^1_{loc}(\mathbb{R}^d)$ for all $\alpha < 2$. Consider the following question:

Question 2. *Let $\alpha \in [0, 2)$. Assume $D \subset \mathbb{R}^d$ is a bounded domain such that*

$$K_{\alpha,d} * 1_D - \frac{\Omega}{2}|x|^2 = \text{const} \quad \text{on } \partial D \tag{1.2.2}$$

for some $\Omega \leq 0$, where the constant is the same along all connected components of ∂D. Must D be a ball in $\mathbb{R}^d$?

Positive answers to Question 2 were obtained in the $\Omega = 0$ case for $\alpha < \frac{5}{3}$ in the following works. As we discussed before, Fraenkel [41] proved that D must be a ball for $\alpha = 0$. Also using the moving plane method, Reichel [99, Theorem 2], Lu–Zhu [83] and Han–Lu–Zhu [56] generalized this result to $\alpha \in [0, 1)$. Here [83] also covered generic radially increasing potentials not too singular at the origin (which include all Riesz potentials $K_{\alpha,d}$ with $\alpha \in [0, 1)$). Recently, Choksi–Neumayer–Topaloglu [29, Theorem 1.3] further pushed the range to $\alpha \in [0, \frac{5}{3})$, leaving the range $\alpha \in [\frac{5}{3}, 2)$ an open problem. We point out that in all these results for $\alpha > 0$, ∂D was assumed to be at least C^1. All above results were obtained using the moving plane method.

In our third main result, we use a completely different approach to give an affirmative answer to Question 2 for all $\Omega \leq 0$ and $\alpha \in [0, 2)$, under a weaker assumption on the regularity of ∂D.

Theorem C (= Theorem 3.1.2). *Let D be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary (and if $d = 2$ we only require ∂D to be rectifiable). If D satisfies* (Equation 1.2.2) *for some $\Omega \leq 0$ and $\alpha \in [0, 2)$, then it must be a ball in $\mathbb{R}^d$.*

As a directly consequence, Theorem C implies that for the gSQG equation with $\alpha \in [0, 2)$, any simply-connected rotating patch with $\Omega \leq 0$ must be a disk (see Theorem 3.1.4). In addition, in the smooth setting (Equation 1.1.7), we prove a similar result in Corollary 3.1.7 for uniformly-rotating solutions with $\Omega \leq 0$ for all $\alpha \in [0, 2)$: if the super level-sets $\{\omega_0 > h\}$ are all simply-connected for all $h > 0$, then ω_0 must be radially decreasing.

Next we review the previous literature on uniformly-rotating solutions for the gSQG equation. Note that the case of $\alpha \in (0, 2)$ is more challenging than the 2D Euler case, since the velocity is more singular and this produces obstructions to the bifurcation theory when it comes to the choice of spaces and the regularity of the functionals involved in the construction. Hassainia–Hmidi [57] showed the existence of V-states with C^k boundary regularity in the case $0 < \alpha < 1$, and in [21], Castro–Córdoba–Gómez-Serrano upgraded the result to show existence and C^∞ boundary regularity in the remaining open cases: $\alpha \in [1,2)$ for the existence, $\alpha \in (0,2)$ for the regularity. In that case, the solutions bifurcate at angular velocities given by $\Omega_m^\alpha := 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(m+\frac{\alpha}{2}\right)}{\Gamma\left(m+1-\frac{\alpha}{2}\right)}\right)$. This boundary regularity was subsequently improved to analytic in [22]. See also [62] for another family of rotating solutions, [68, 100] for the doubly connected case and [23] for a construction in the smooth setting.

One can check that Ω_m^α are increasing functions of m for any α, whose limit is a finite number $\Omega^\alpha := 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)}$ for $\alpha \in [0,1)$, and $+\infty$ if $\alpha \geq 1$. It is then a natural question to ask whether there exist V-states (with area π) that rotate with angular velocity faster than Ω_α for $\alpha \in (0,1)$. Our fourth main theorem answers this question among all simply-connected patches:

Theorem D (= Theorem 3.2.1). *For $\alpha \in (0,1)$, let 1_D be a simply connected V-state of area π and let its angular velocity be $\Omega \geq \Omega^\alpha$. Then D must be the unit disk.*

Finally, we illustrate a comparison of our results in Theorem C and D (in red color) with the previous results (in black color) in Figure Figure 1.2.

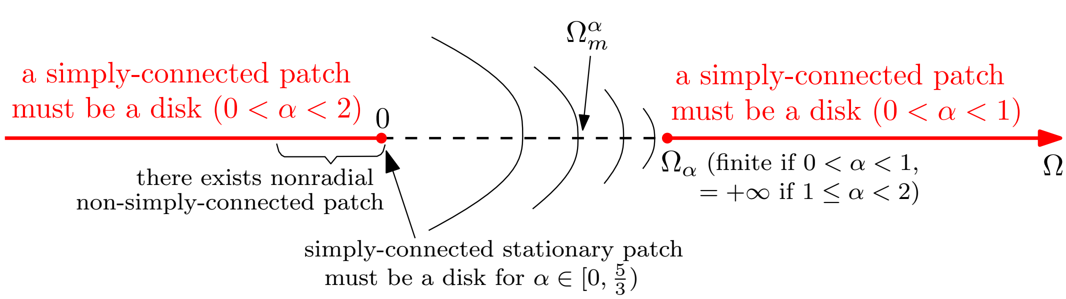


Figure 1.2: For gSQG in the patch setting, previous results on Question 1 are summarized in black color, with our results in Theorem C and D colored in red.

Structure of the proofs: Theorem A-Theorem D

While all the previous symmetry results on Question 1 and Equation 2 [41, 83, 56, 59, 99, 29] are done by moving plane methods, our approaches are completely different, which have a variational flavor.

Theorem A is based on computing the first variation of the energy functional

$$\mathcal{E}[1_D] = -\frac{1}{2}\int_{\mathbb{R}^2} 1_D(x)(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2 1_D(x)dx$$

as we deform D along a carefully chosen vector field. On the one hand, we show the first variation should be 0 if D is a stationary/rotating patch with angular velocity Ω; on the other hand, we show that the first variation must be non-zero if $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$, leading to a contradiction. In the proof, a rearrangement inequality due to Talenti [109] is crucial to get a sign condition. In the case when the patch is non-simply-connected, the choice of the right vector field is more involved since one has to deal with all the connected components at once, and even though the stream function ψ is constant on each of the boundaries, it is not guaranteed that this constant will be the same for each of them, potentially yielding extra boundary terms which are not sign definite, and having to reprove a new version of the aforementioned inequality.

The smooth setting in Theorem B is based on a similar idea, but technically more difficult. The point of view is to approximate a smooth function by step functions and consider the above perturbation in each set where the step function is constant. To do this we need to obtain some quantitative (stability) estimates on our version of Talenti's rearrangement inequality, in particular in terms of the Fraenkel asymmetry of the domain in the spirit of Fusco–Maggi–Pratelli [42].

Theorem C is also based on a variational approach, but we need a different perturbation from the vector field in Theorem A, which heavily relies on the Newtonian potential, and fails for general Riesz potential K_α. The key ingredient to prove Theorem C is to perturb D using the continuous Steiner symmetrization [11], which has been successfully applied in other contexts by Carrillo–

Hittmeir–Volzone–Yao [18] (nonlinear aggregation models) or Morgan [90] (minimizers of the gravitational energy). This method is much more flexible and allows to treat more singular kernels than in the existing papers using moving plane methods. Due to the low regularity of the kernels, instead of computing the derivative of the energy under the perturbation, we work with finite differences instead.

Theorem D uses maximum principles and monotonicity formulas for nonlocal equations. The idea is to find the smallest disk $B(0, R)$ containing D (which intersects ∂D at some x_0), then use two different ways to compute $\nabla(1_{B(0,R)\setminus D} * K_\alpha)$ at x_0, and obtain a contradiction if $\Omega \geq \Omega_\alpha$ and D is not a disk. The proof works for the full range of $\alpha \in [0, 2)$, thus closing the problem raised by Hmidi [59] and de la Hoz–Hassainia–Hmidi–Mateu [66] among all simply-connected patches.

1.2.2 Rigidity/Flexibility results for vortex sheets [51, 52]

In chapter 4 and chapter 5, we will focus on the vortex sheet equation. There are very few known examples of nontrivial steady solutions, and in fact, other than the circle or the line, the list only comprises the segment of length $2a$ and density

$$\gamma(x) = \Omega\sqrt{a^2 - x^2}, \qquad x \in [-a, a], \tag{1.2.3}$$

which is a rotating solution with angular velocity Ω [6] and the family found by Protas–Sakajo [98], made out of segments rotating about a common center of rotation with endpoints at the vertices of a regular polygon. We remark that none of these are supported on a closed curve.

Numerically, O'Neil [93, 94] used point vortices to approximate the vortex sheet and compute uniformly rotating solutions and Elling [40] constructed numerically self-similar vortex sheets forming cusps. O'Neil [95, 96] also found numerically steady solutions which are combinations of point vortices and vortex sheets.

As in the patch/smooth setting, it is easy to see that if the z_i's are concentric circles with constant ϖ_i (with respect to the constant-speed parametrization) for every i, the solution is stationary, and it is also uniformly-rotating with any $\Omega \in \mathbb{R}$. We would like to understand the reverse implication, namely:

Question 3. *Under what conditions must a stationary/uniformly-rotating vortex sheet be radially symmetric?*

The next two theorems are the main results in chapter 4 and chapter 5 regarding the vortex sheet equations:

Theorem E. *Let $\omega(x,t) = \omega_0(R_{\Omega t}x)$ be a stationary/uniformly-rotating vortex sheet with angular velocity Ω. Assume that ω_0 is concentrated on Γ, which is a finite union of smooth curves, and ω_0 has positive vorticity strength on Γ. (See* **(H1)–(H3)** *in section 4.1 for the precise regularity and positivity assumptions.)*

If $\Omega \leq 0$, Γ must be a union of concentric circles, and ω_0 must have constant strength along each circle (with respect to the constant-speed parametrization). In addition, if $\Omega < 0$, all circles must be centered at the origin.

Theorem F (= Theorem 5.1.2). *There exists nontrivial rotating vortex sheets with positive angular velocity $\Omega > 0$, whose vortex strength is strictly positive but not concentrated on a non-radial curve.*

Structure of the proofs: Theorem E-Theorem F

The proof of Theorem E is inspired by our recent rigidity result in the paper [50] on stationary and rotating solutions of the 2D Euler equations both in the smooth and vortex patch settings. To prove it, we constructed an appropriate functional and showed, on one hand, that any stationary solution had to be a critical point, and on the other, for any curve which is not a circle there existed a vector field along which the first variation was non-zero. This vector field is defined in terms of an elliptic equation in the interior of the patch. In the case of the vortex sheet, this is not possible anymore. Instead, we desingularize the problem by considering patches of thickness $\sim \varepsilon$ which are tubular neighborhoods of the sheet. The drawback is that we lose the property that any stationary solution has to be a critical point if $\varepsilon > 0$ and very careful, quantitative estimates need to be done to show that indeed the first variation of a stationary solution tends to 0 as $\varepsilon \to 0$. This setup is also reminiscent of the numerical work by Baker–Shelley [5], where they approximate the motion of a vortex sheet by a vortex patch of very small width. In [8], Benedetto–Pulvirenti proved the stability (for short time) of vortex sheet solutions with respect to solutions to 2D Euler with a thin strip of vorticity around a curve. See also the work by Caflisch–Lombardo–Sammartino [13] for more stability results with a different desingularization.

The main strategy to prove Theorem 5.1.2, is to employ bifurcation theory and try to bifurcate from the simple eigenvalue $b := \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(x,t)dx = 2$. However, the standard methods (Crandall-Rabinowitz [35]) fail since the linearized operator around the circle does not satisfy the transversality condition: in other words, the nontrivial zero set is not transversal to the trivial one (disks with constant vorticity amplitude). This phenomenon is usually known in the literature as a *degenerate bifurcation* [73, 72]. Graphically, this can be seen in Figure Figure 5.1. The problem is that we no longer have a single branch emanating from the disk, but two, and therefore the linearized operator fails to describe the local behaviour at the bifurcation point. To overcome this issue, we first reduce the nonlinear problem to a suitable finite dimensional space by means of a Lyapunov-Schmidt reduction since the restriction of $D\mathcal{F}$ is an isomorphism between Ker$(D\mathcal{F})^{\perp}$ and Im$(D\mathcal{F})$. After having done so, we are left with a finite dimensional system and it is there where we perform a higher order expansion around the bifurcation point, since, as expected by the failure of the transversality condition, the first order approximation is identically zero. We obtain that in suitable coordinates, the zero sets of $\mathcal{F}$ behave as $x^2 - y^2 = 0$ and thus two bifurcation branches emanate from the bifurcation point. The last part of the proof is devoted to handle the higher order terms, which can be controlled if we restrict the bifurcation domain to a suitable small enough neighbourhood. We mention here that this technique had been successfully employed by

Hmidi–Mateu [61] (in the hyperbolic case) and Hmidi–Renault [65] (in the elliptic case).

1.2.3 Quantitative estimates for rotating vortex patches [97]

The goal of chapter 6 is to establish some quantitative estimates for non-radial simply-connected rotating patches, which are known to exist. From now on, we assume that a bounded domain D is simply-connected and has C^2 boundary. If $\omega(x,t) := 1_D(R_{\Omega t}x)$ is a uniformly-rotating patch, then the net velocity in the rotating frame has no contribution to the deformation of the boundary ∂D, namely, $\nabla^{\perp}\left((1_D * \mathcal{N}) - \frac{\Omega}{2}|x|^2\right)\cdot\vec{n} = 0$, where $\vec{n}$ denotes the outer normal vector on ∂D. By integrating this along the boundary, one can derive the following equation for the relative stream function Ψ:

$$\Psi(x) := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = constant \quad \text{for all } x \in \partial D. \tag{1.2.4}$$

In the rest of the paper, we say a pair (D,Ω) is a solution to (Equation 1.2.4) if 1_D and Ω satisfy the equation (Equation 1.2.4).

Small angular velocity Ω

Our first main result is about the outmost point on ∂D when the angular velocity Ω is small. As mentioned earlier, ellipses are uniformly-rotating solutions. More precisely, an ellipse with semi-axes a, b is rotating with angular velocity $\Omega = \frac{ab}{(a+b)^2}$. By imposing $b = \frac{1}{a}$ to keep the area of the patch equal to π, one can easily see that for any $0 < \Omega \le \frac{1}{4}$, there exists an ellipse that is rotating with the given Ω. Moreover, the boundary is stretching as Ω tends to 0 in the sense that the length of the major axis is comparable with $\Omega^{-\frac{1}{2}}$. Note that ellipses are not the only uniformly-rotating solutions for small angular velocities. For example, the existence of secondary bifurcations from ellipses was numerically observed by Kamm in his thesis [71] and theoretically proved in [22, 60].

Thus it is a natural question whether every non-radial simply-connected rotating patch with a fixed area and $0 < \Omega \ll 1$ must have its outmost point very far from the origin (center of rotation). In the next theorem, we prove this is indeed true.

Theorem G. . *Let $D \subset \mathbb{R}^2$ be a simply connected domain such that $|D| = |B| = \pi$, where B is the unit disk centered at the origin. Then there exist positive constants Ω_0 and κ_0 such that if (D, Ω) is a solution to* (Equation 1.2.4) *with $\Omega \in (0, \Omega_0)$, then either $D = B$, or*

$$\sup_{x \in \partial D} |x| > \kappa_0 \Omega^{-\frac{1}{2}}. \tag{1.2.5}$$

Remark 1.2.1. *Note that the power $-\frac{1}{2}$ is sharp since it is achieved by ellipses. Furthermore, one can easily show that* (Equation 1.2.4) *is scaling invariant in the sense that if (D, Ω) is a solution, then (D_a, Ω) is also a solution for any $a > 0$, where $D_a := \{ax \in \mathbb{R}^2 : x \in D\}$. Therefore without the restriction on the size of the patch,* (Equation 1.2.5) *reads as*

$$\frac{1}{\sqrt{|D|}} \sup_{x \in \partial D} |x| > \frac{\kappa_0}{\sqrt{\pi}} \Omega^{-\frac{1}{2}}.$$

m-fold symmetric patches

It has been known since the work of Burbea [12] that there are m-fold symmetric rotating patches for every integer $m \geq 2$. From the numerical results [36, 58], it appears that for $m \gg 1$, the angular velocity Ω along the bifurcation curve is very close to $\frac{1}{2}$ (i.e. $0 < \frac{1}{2} - \Omega \ll 1$ for $m \gg 1$). But there are no such type of quantitative estimates so far. In the next theorem, we will derive a lower bound of the angular velocity by imposing large m.

Theorem H. *There exist $m_0 \geq 2$ and $c > 0$ such that if (Ω, D) is a solution to* (Equation 1.2.4) *and D is simply-connected, non-radial, and m-fold symmetric for some $m \geq m_0$ then*

$$\frac{1}{2} - \Omega \leq \frac{c}{m}.$$

We emphasize that this theorem holds for a general simply-connected patch, which does not need to lie on the bifurcation curve.

For m-fold symmetric solutions on the global bifurcation curves constructed in [58], we will also estimate the difference between a rotating patch and the unit disk. To be precise, we will focus on the curves,

$$\mathscr{C}_m := \left\{ \left(\tilde{u}_m(s), \tilde{\Omega}_m(s)\right) \in C^2(\mathbb{T}) \times \left(0, \frac{1}{2}\right) : s \in [0, \infty) \right\} \quad \text{for } m \geq 2,$$

that satisfy the following properties (see [58, Theorem 1.1] for the details):

(A1) $\tilde{u}_m(s) \in \{u \in C^2(\mathbb{T}) : u(\theta) = \sum_{n=1}^{\infty} a_n \cos(nm\theta) \text{ for some } (a_n)_{n=1}^{\infty} \text{ and } u > -1\}$.

(A2) $(D^{\tilde{u}_m(s)}, \tilde{\Omega}_m(s))$ is a solution for (Equation 1.2.4),
where $D^u := \{r(\cos\theta, \sin\theta) \in \mathbb{R}^2 : 0 \leq r < (1 + u(\theta)),\ \theta \in \mathbb{T}\}$.

(A3) $\partial_\theta u(\theta) < 0$ for all $\theta \in (0, \frac{\pi}{m})$, where $u = \tilde{u}_m(s)$.

(A4) $(\tilde{u}_m(0), \tilde{\Omega}_m(0)) = \left(0, \frac{m-1}{2m}\right)$.

For such curves, we have the following theorem:

Theorem I. *Let $\mathscr{C}_m := \left\{(\tilde{u}_m(s), \tilde{\Omega}_m(s)) \in C^2(\mathbb{T}) \times \left(0, \frac{1}{2}\right) :\ 0 \leq s < \infty\right\}$ be a continuous curve that satisfies the properties (A1)-(A4). Then there exist constants $c > 0$ and $m_0 \geq 3$ such that if $m \geq m_0$, then*

$$\|\tilde{u}_m(s)\|_{L^\infty(\mathbb{T})} \leq \frac{c}{m} \textit{ for all } s \geq 0.$$

Although each curve emanates from the unit disk, the possibility that $\min_{\theta\in\mathbb{T}}(1+\tilde{u}_m(s))$ tends to 0 along the curve has not been completely eliminated ([58, Theorem 4.6, Lemma 6.6]), while it does certainly happen for ellipses ($m = 2$). The significant difference between $m = 2$ and $m \geq 3$ is that if $m \geq 3$, then the stream function ($1_D * \mathcal{N}$) behaves quite nicely, namely, $\frac{|\nabla(1_D*\mathcal{N})|}{|x|}$ is globally bounded (especially near the origin) independently of D (Lemma 6.2.1. See also [38, 39], and the references therein, where global boundedness of gradient of m-fold symmetric stream functions was proved). This will play a crucial role to eliminate the scenario that ∂D almost touches the origin when $\frac{1}{2} - \Omega$ is sufficiently large compared to $\frac{1}{m}$ in Lemma 6.2.2.

We summarize the main results in Figure Figure 1.3

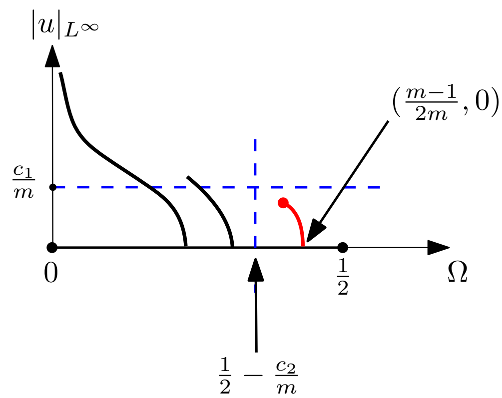


Figure 1.3: Illustration of the main results. a) Any patch with $\Omega \ll 1$ must have its outmost point very far from the origin. b) A bifurcation curve of m-fold symmetric patches cannot be continued beyond the blue dashed lines for large m.

Structure of the proofs: Theorem G - Theorem I

The starting point for Theorem G and Theorem H is the variational formulation of (Equation 1.2.4), used by Gómez-Serrano, Shi, Yao and the author in [50]. Namely, if (D, Ω) is a solution to (Equation 1.2.4) and ∂D is C^2, then formally, 1_D can be thought of as a critical point of the functional,

$$\mathcal{I}(\rho) := \frac{1}{2}\int_{\mathbb{R}^2} \rho * \mathcal{N}(x)\rho(x)dx - \Omega\int_{\mathbb{R}^2} \frac{|x|^2}{2}\rho(x)dx =: \mathcal{I}_1(\rho) - \Omega\mathcal{I}_2(\rho),$$

under measure-preserving perturbations. More precisely, it holds that

$$\int_D \vec{v}\cdot\nabla\Psi(x)dx = 0, \quad \text{for any } v \in C^2(\overline{D}) \text{ such that } \nabla\cdot\vec{v} = 0, \tag{1.2.6}$$

Indeed, (Equation 1.2.6) follows directly from (Equation 1.2.4) and the integration by parts. By choosing a specific vector field $\vec{v} := x + \nabla p$, where p is defined as the solution to the Poisson equation,

$$\begin{cases} \Delta p = -2 & \text{in } D, \\ p = 0 & \text{on } \partial D, \end{cases} \tag{1.2.7}$$

Gómez-Serrano et al. derived the following identity for uniformly-rotating patches:

$$2\Omega\left(\int_D \frac{|x|^2}{2}dx - \frac{|D|^2}{4\pi}\right) = (1-2\Omega)\left(\frac{|D|^2}{4\pi} - \int_D p dx\right). \tag{1.2.8}$$

Note that both parentheses are strictly positive if $D \neq B$, where B is the unit disk centered at the origin. Thanks to the result by Brasco–De Philippis–Velichkov in [10, Proposition 2.1], one can find a lower bound of the right-hand side of (Equation 1.2.8) in terms of $|D\triangle B|$, namely, $\frac{|D|^2}{4\pi} - \int_D pdx \gtrsim |D\triangle B|^2$. Hence (Equation 1.2.8) yields that

$$|D\triangle B|^2 \lesssim \Omega\left(\int_D \frac{|x|^2}{2}dx - \frac{|D|^2}{4\pi}\right) < \Omega \sup_{x\in\partial D}|x|^2,$$

for $\Omega \ll 1$. Therefore we only need to rule out the case where $|D\triangle B|$ is small. Assuming $|D\triangle B|$ and Ω are sufficiently small, we will prove (Lemma 6.1.6) that D is necessarily star-shaped and the boundary can be parametrized by $(1+u(x))x$, for $x \in \partial B_1$ and some $u \in C^2(\partial B)$. However, the difficulty is that we have $|D \triangle B| \sim \|u\|_{L^1(\partial B)}$ and $\int_D \frac{|x|^2}{2} dx - \frac{|D|^2}{4\pi} \sim \|u\|^2_{L^2(\partial B)}$, while L^1 and L^2 are not comparable. The key idea is to use a different vector $\vec{v} := x - 2\nabla(1_D * \mathcal{N})$ in (Equation 1.2.6), which gives another identity for any simply-connected rotating patches,

$$\left(\frac{1}{2} - \Omega\right)\left(\int_D |x|^2 dx - \frac{|D|^2}{2\pi}\right) = \frac{1}{2}\int_D |x - 2\nabla(1_D * \mathcal{N})|^2 dx. \tag{1.2.9}$$

Thanks to the result of Loeper [79, Proposition 3.1], the right-hand side in (Equation 1.2.9) can be estimated in terms of 2-Wasserstein distance between $1_D dx$ and $1_B dx$ (see Proposition 6.1.9). In the proof of Proposition 6.1.8, we will construct an explicit transport map and obtain the bound for the right-hand side: If $\|u\|_{L^\infty(\mathbb{T})} \le \frac{1}{2}$,

$$\int_D |x - 2\nabla(1_D * \mathcal{N})|^2 dx \lesssim \left(a\int_{\mathbb{T}} |u|^2 d\theta + \frac{1}{a}\int_{\mathbb{T}} f(\theta)^2 d\theta\right), \tag{1.2.10}$$

where $f(\theta) := \int_0^\theta u(s)^2 + 2u(s)ds$ and $a \in (2\|u\|_{L^\infty(\mathbb{T})}, 1)$. Since $\|f\|_{L^\infty(\mathbb{T})} \lesssim \|u\|_{L^1(\mathbb{T})}$, (Equation 1.2.9) and (Equation 1.2.10) will give us $\|u\|_{L^1(\mathbb{T})} \sim \|u\|_{L^2(\mathbb{T})}$ for $0 < \Omega \ll 1$, if we can choose a sufficiently small.

The proof of Theorem H also relies on the identity (Equation 1.2.9). By imposing m-fold symmetry on the patch, we can lower the total cost of the transportation, from which we can obtain a suitable upper bound of $\frac{1}{2} - \Omega$ when m is sufficiently large. Indeed, if u is $\frac{2\pi}{m}$ periodic, then f is also $\frac{2\pi}{m}$ periodic as well. Thus by choosing large m, we can lower $\|f\|_{L^\infty(\mathbb{T})}$ on the right-hand side in (Equation 1.2.10) by using Jensen's inequality.

Theorem I will be proved by showing that if $\|\tilde{u}_m(s)\|_{L^\infty}$ is too large, then $\frac{1}{2} - \Omega$ must be large enough to contradict Theorem H. The main difficulty is that $\frac{1}{2} - \Omega$ can be estimated in terms of $\|\tilde{u}_m(s)\|_{L^2(\mathbb{T})}$ by using the identity (Equation 1.2.8) (Lemma 6.2.4), while L^2 and L^∞ are not comparable. We resolve this issue by estimating the gradient of the stream function in a very delicate way (Lemma 6.2.5 and 6.2.6).

CHAPTER 2

RIGIDITY RESULTS FOR 2D EULER

In chapter 2-chapter 3, we study radial symmetry in 2D Euler equation and generalized SQG equations.

Notations Through chapter 2-chapter 3, we use the following notations.

For a simple closed curve Γ, denote int(Γ) by its interior, which is the bounded connected component of $\mathbb{R}^2$ separated by the curve Γ. Note that the Jordan–Schoenflies theorem guarantees that int(Γ) is open and simply connected.

We say that two disjoint simple closed curves Γ_1 and Γ_2 are nested if $\Gamma_1 \subset \text{int}(\Gamma_2)$ or vice versa. We say that two connected domains D_1, D_2 are nested if one is contained in a hole of the other one.

For a bounded connected domain $D \subset \mathbb{R}^2$, we denote by $\partial_{\text{out}} D$ its outer boundary. And if D is doubly-connected, we denote by $\partial_{\text{in}} D$ its inner boundary,

For a set D, we use $1_D(x)$ to denote its indicator function. And for a statement S, we let $\mathbb{1}_S = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false} \end{cases}$. (e.g. $\mathbb{1}_{\pi<3} = 0$).

For a domain $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \vec{n} \cdot \vec{f} d\sigma$, the vector $\vec{n}$ is taken as the outer normal of the domain U in that integral.

2.1 Radial symmetry of steady/rotating patches for 2D Euler equation

Throughout this section we work with the 2D Euler equation (Equation 1.1.1) in the patch setting. For a stationary or uniformly-rotating patch D with angular velocity $\Omega \in \mathbb{R}$, let

$$f_\Omega(x) := (1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2.$$

Recall that in (Equation 1.1.6) we have shown that $f_\Omega \equiv C_i$ on each connected component of ∂D, where the constants can be different on different connected components.

Our goal in this section is to prove Theorem A, which completely answers Question 1 for 2D Euler patches. As we described in the introduction, our proof has a variational flavor, which is done by perturbing D by a carefully chosen vector field, and compute the first variation of an associated energy functional in two different ways. In subsection 2.1.1, we will define the energy functional and the perturbation vector field, and give a one-page proof in Theorem 2.1.2 that answers Question 1 among simply-connected patches. (Note that even among simply-connected patches, it is an open question whether every rotating patch with $\Omega > \frac{1}{2}$ or $\Omega < 0$ must be a disk.) In the following subsections, we further develop this method, and modify our perturbation vector field to cover non-simply-connected patches.

2.1.1 Warm-up: radial symmetry of simply-connected rotating patches

We begin by providing a sketch and some motivations of our approach, and then give a rigorous proof afterwards in Theorem 2.1.2. Suppose that D is a C^1 simply-connected rotating patch with angular velocity Ω that is *not* a disk. We perturb D in "time" (here the "time" t is just a name for our perturbation parameter, and is irrelevant with the actual time in the Euler equation) with a velocity field $\vec{v}(x) \in C^1(D) \cap C(\bar{D})$ that is divergence-free in D, which we will fix later. That is, consider the transport equation

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0$$

with $\rho(\cdot,0)=1_D$. We then investigate how the "energy functional"

$$\mathcal{E}[\rho] := -\int_{\mathbb{R}^2} \frac{1}{2}\rho(x)(\rho*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\rho(x)dx$$

changes in time under the perturbation. Formally, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[\rho]\Big|_{t=0} &= -\int_{\mathbb{R}^2} \rho_t(x,0)\Big((\rho(\cdot,0)*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx \\ &= -\int_D \vec{v}(x)\cdot\nabla\Big((1_D*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx.\end{aligned} \tag{2.1.1}$$

The above transport equation and the energy functional only serve as our motivation, and will not appear in the proof. In the actual proof we only focus on the right hand side of (Equation 2.1.1), which is an integral that is well-defined by itself:

$$\mathcal{I} := -\int_D \vec{v}(x)\cdot\nabla\Big((1_D*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx = -\int_D \vec{v}\cdot\nabla f_\Omega\, dx. \tag{2.1.2}$$

We will use two different ways to compute $\mathcal{I}$, and show that if D is not a disk, the two ways lead to a contradiction for $\Omega\le 0$ or $\Omega\ge\frac{1}{2}$.

On the one hand, since f_Ω is a constant on ∂D (denote it by c), the divergence theorem yields the following for *every* $\vec{v}\in C^1(D)\cap C(\bar{D})$ that is divergence-free in D:

$$\mathcal{I} = -c\int_{\partial D}\vec{n}\cdot\vec{v}d\sigma + \int_D(\nabla\cdot\vec{v})f_\Omega\, dx = -c\int_D\nabla\cdot\vec{v}dx + \int_D(\nabla\cdot\vec{v})f_\Omega\, dx = 0. \tag{2.1.3}$$

On the other hand, we fix $\vec{v}$ as follows, which is at the heart of our proof. Let $\vec{v}(x) := -\nabla\varphi(x)$ in D, where

$$\varphi(x) := \frac{|x|^2}{2} + p(x) \quad \text{ in } D, \tag{2.1.4}$$

with $p(x)$ being the solution to Poisson's equation

$$\begin{cases} \Delta p(x) = -2 & \text{in } D \\ p(x) = 0 & \text{on } \partial D. \end{cases} \tag{2.1.5}$$

Note that φ is harmonic in D, thus $\vec{v}$ is indeed divergence-free in D. This definition of $\vec{v}$ is motivated by the fact that among all divergence-free vector fields in D, such $\vec{v}$ is the closest one to $-\vec{x}$ in the $L^2(D)$ distance. (In fact, such $\vec{v}$ is connected to the gradient flow of $\int_D \frac{|x|^2}{2} dx$ in the metric space endowed by 2-Wasserstein distance, under the constraint that $|D(t)|$ must remain constant [88, 89, 1].) Formally, one expects that D becomes "more symmetric" as we perturb it by $\vec{v}$, which inspires us to consider the first variation of $\mathcal{E}$ under such perturbation.

In the proof we will show that with such choice of $\vec{v}$, we can compute $\mathcal{I}$ in another way and obtain that $\mathcal{I} > 0$ for $\Omega \leq 0$ and $\mathcal{I} < 0$ for $\Omega \geq \frac{1}{2}$. Therefore in both cases, we obtain a contradiction with $\mathcal{I} = 0$ in (Equation 2.1.3).

Our proof makes use of a rearrangement inequality for solutions to elliptic equations, which is due to Talenti [109]. Below is the form that we will use; the original theorem works for a more general class of elliptic equations.

Proposition 2.1.1 ([109], Theorem 1)**.** *Let $D \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary, and let p be defined as in* (Equation 2.1.5)*. Let B be a disk centered at the origin with $|B| = |D|$, and let p_B solve* (Equation 2.1.5) *in B. Then we have $p^* \leq p_B$ pointwise in B, where p^* is the radial decreasing rearrangement of p^*. As a consequence, we have*

$$\int_D p(x)dx \leq \frac{1}{4\pi}|D|^2,$$

and the equality is achieved if and only if D is a disk.

Now we are ready to prove the following theorem, saying that any simply-connected stationary/rotating patch with $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$ must be a disk. Interestingly, the same proof can treat the two disjoint intervals $\Omega \le 0$ and $\Omega \ge \frac{1}{2}$ all at once.

Theorem 2.1.2. *Let D be a simply-connected domain with C^1 boundary. If D is a rotating patch solution with angular velocity Ω, where $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$, then D must be a disk, and it must be centered at the origin unless $\Omega = 0$.*

Proof. Let D be a rotating patch with $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$. As we described above, in this theorem we will use two different ways to compute the integral $\mathcal{I}$ defined in (Equation 2.1.2), where we fix $\vec{v}(x) := -\nabla\varphi(x)$, with φ and p defined as in (Equation 2.1.4) and (Equation 2.1.5).

On the one hand, we have that $\vec{v}$ is divergence free in D, and elliptic regularity theory immediately yields that $\vec{v} \in C^1(D) \cap C(\bar{D})$. Using the assumption that D is a rotating patch, we know f_Ω is a constant on ∂D. (Note that ∂D is a connected closed curve since we assume D is simply-connected). Thus the computation in (Equation 2.1.3) directly gives that $\mathcal{I} = 0$.

On the other hand, we compute $\mathcal{I}$ as follows:

$$\mathcal{I} = -\int_D \vec{v} \cdot \nabla f_\Omega dx = \int_D \nabla\varphi \cdot \nabla f_\Omega dx = \underbrace{\int_D x \cdot \nabla f_\Omega dx}_{=:\mathcal{I}_1} + \underbrace{\int_D \nabla p \cdot \nabla f_\Omega dx}_{=:\mathcal{I}_2}. \tag{2.1.6}$$

For $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_D x \cdot \nabla(1_D * \mathcal{N})dx - \int_D x \cdot \Omega x dx \\
&= \frac{1}{2\pi}\int_D\int_D \frac{x\cdot(x-y)}{|x-y|^2}dydx - \Omega\int_D |x|^2 dx \\
&= \frac{1}{4\pi}\int_D\int_D \frac{x\cdot(x-y) - y\cdot(x-y)}{|x-y|^2}dydx - \Omega\int_D |x|^2 dx \\
&= \frac{1}{4\pi}|D|^2 - \Omega\int_D |x|^2 dx,
\end{aligned} \tag{2.1.7}$$

where the third equality is obtained by exchanging x with y in the first integral, then taking average with the original integral. To compute $\mathcal{I}_2$, using the divergence theorem (and the fact that $p = 0$ on

∂D), we have

$$\mathcal{I}_2 = -\int_D p\Delta f_\Omega dx = (2\Omega - 1)\int_D p dx. \tag{2.1.8}$$

Plugging (Equation 2.1.7) and (Equation 2.1.8) into (Equation 2.1.6) gives

$$\mathcal{I} = \frac{1}{4\pi}|D|^2 - \Omega\int_D |x|^2 dx + (2\Omega - 1)\int_D p dx. \tag{2.1.9}$$

When $\Omega = 0$, Proposition 2.1.1 directly gives that $\mathcal{I} > 0$ if D is not a disk, contradicting $\mathcal{I} = 0$.

When $\Omega \in (-\infty, 0) \cup [\frac{1}{2}, \infty)$, let B be a disk centered at the origin with the same area as D. Towards a contradiction, assume $D \neq B$. Among all sets with the same area as D, the disk B is the unique one that minimizes the second moment, thus we have

$$\int_D |x|^2 dx > \int_B |x|^2 dx = \frac{1}{2\pi}|D|^2,$$

where the last step follows from an elementary computation. Plugging this into (Equation 2.1.9) gives the following inequality for $\Omega \in [\frac{1}{2}, \infty)$:

$$\mathcal{I} < \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p dx\Big) \leq 0.$$

On the other hand, for $\Omega \in (-\infty, 0)$, we have

$$\mathcal{I} > \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p dx\Big) > 0,$$

and we get a contradiction to $\mathcal{I} = 0$ in all the cases, thus the proof is finished. □

Remark 2.1.3. *In fact, one can easily check that Theorem 2.1.2 holds for a bounded disconnected patch $D = \dot\cup_{i=1}^N D_i$ with C^1 boundary, as long as each connected component D_i is simply-*

connected. Here the proof remains the same, except a small change in the $\mathcal{I} = 0$ proof: since now we have $f_\Omega = c_i$ on ∂D_i, (Equation 2.1.3) *should be replaced by*

$$\mathcal{I} = -\sum_{i=1}^{N} \left(c_i \int_{\partial D_i} \vec{n} \cdot \vec{v} d\sigma + \int_{D_i} (\nabla \cdot \vec{v}) f_\Omega dx \right) = 0.$$

Even in the regime $\Omega \in (0, \frac{1}{2})$, where non-radial rotating patches are known to exist (recall that there exist patches bifurcating from a disk at $\Omega_m = \frac{m-1}{2m}$ for all $m \geq 2$), our approach still allows us to obtain the following quantitative estimate, saying that if a simply-connected patch D rotates with angular velocity $\Omega \in (0, \frac{1}{2})$ that is very close to $\frac{1}{2}$, then D must be very close to a disk, in the sense that their symmetric difference must be small.

Corollary 2.1.4. *Let D be a simply-connected domain with C^1 boundary. Assume D is a rotating patch solution with angular velocity Ω, where $\Omega \in (\frac{1}{4}, \frac{1}{2})$. Let $\delta := \frac{1}{2} - \Omega$. Then we have*

$$|D \triangle B| \leq 2\sqrt{2\delta}|D|,$$

where B is the disk centered at the origin with the same area as D.

Proof. In the proof of Theorem 2.1.2, combining the equation $\mathcal{I} = 0$ and (Equation 2.1.9) together, we have that

$$\frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx - (1 - 2\Omega) \int_D p dx = 0.$$

Dividing both sides by Ω and rearranging the terms, we obtain

$$\int_D |x|^2 dx - \frac{1}{2\pi}|D|^2 = \frac{1 - 2\Omega}{\Omega} \left(\frac{1}{4\pi}|D|^2 - \int_D p dx \right) \leq \frac{2\delta|D|^2}{\pi},$$

where in the inequality we used that $2\delta := 1 - 2\Omega$, $\Omega > \frac{1}{4}$, and $\int_D p dx \geq 0$.

Since $\int_B |x|^2 dx = \frac{1}{2\pi}|D|^2$, the above inequality implies that

$$\int_{D\setminus B} |x|^2 dx - \int_{B\setminus D} |x|^2 dx \leq \frac{2\delta|D|^2}{\pi}. \tag{2.1.10}$$

Since D and B has the same area, let us denote $\beta := |D \setminus B| = |B \setminus D|$. Among all sets $U \subset B^c$ with area β, $\int_U |x|^2 dx$ is minimized when U is an annulus with area β and inner circle coinciding with ∂B, thus an elementary computation gives

$$\int_{D\setminus B} |x|^2 dx \geq \inf_{U\subset B^c, |U|=\beta} \int_U |x|^2 dx = \frac{\beta(2|B|+\beta)}{2\pi}.$$

Likewise, among all sets $V \subset B$ with area β, $\int_V |x|^2 dx$ is maximized when V is an annulus with area β and outer circle coinciding with ∂B, thus

$$\int_{B\setminus D} |x|^2 dx \leq \sup_{V\subset B, |V|=\beta} \int_V |x|^2 dx = \frac{\beta(2|B|-\beta)}{2\pi}.$$

Subtracting these two inequalities yields

$$\int_{D\setminus B} |x|^2 dx - \int_{B\setminus D} |x|^2 dx \geq \frac{\beta^2}{\pi},$$

and combining this with (Equation 2.1.10) immediately gives

$$\beta^2 \leq 2\delta|D|^2,$$

thus $|D\triangle B| = 2\beta \leq 2\sqrt{2\delta}|D|$. □

2.1.2 Radial symmetry of non-simply-connected stationary patches

In this subsection, we aim to prove radial symmetry of a connected rotating patch D with $\Omega \leq 0$, where D is allowed to be non-simply-connected. Let $D \subset \mathbb{R}^2$ be a bounded connected domain with C^1 boundary. Assume D has n holes with $n \geq 0$, and then let $h_1, \cdots, h_n \subset \mathbb{R}^2$ denote the n holes of D (each h_i is a bounded open set). Note that ∂D has $n+1$ connected components: they include the outer boundary of D, which we denote by ∂D_0, and the inner boundaries ∂h_i for $i = 1, ..., n$.

To begin with, we point out that even for the steady patch case $\Omega = 0$, the proof of Theorem 2.1.2 cannot be directly adapted to the non-simply-connected patch. If we define $\vec{v}$ in the same way, then the second way to compute $\mathcal{I}$ still goes through (since Proposition 2.1.1 still holds for non-simply-connected D), and leads to $\mathcal{I} > 0$ if D is not a disk. But the first way to compute $\mathcal{I}$ no longer gives $\mathcal{I} = 0$: if D is stationary and not simply-connected, $f(x) := (1_D * \mathcal{N})(x)$ may take different constant values on different connected components of ∂D, thus the identity (Equation 2.1.3) no longer holds.

In order to fix this issue, we still define $\vec{v} = -\nabla\varphi = -\nabla(\frac{|x|^2}{2} + p)$, but modify the definition of p in the following lemma. Compared to the previous definition (Equation 2.1.5), the difference is that p now takes different values $0, c_1, \ldots, c_n$ on each connected component of ∂D. The lemma shows that there exist values of $\{c_i\}_{i=1}^n$, such that $\int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma = -2|h_i|$ along the boundary of each hole. As we will see later, this leads to $\int_{\partial h_i} \vec{v} \cdot \vec{n} d\sigma = 0$ for $i = 1, \ldots, n$, which ensures $\mathcal{I} = 0$. (Of course, with p defined in the new way, the second way of computing $\mathcal{I}$ no longer follows from Proposition 2.1.1, and we will take care of this later in Proposition 2.1.6.)

Lemma 2.1.5. *Let D, h_i and ∂D_0 be given as in the first paragraph of Section subsection 2.1.2. Then there exist positive constants $\{c_i\}_{i=1}^n$, such that the solution $p : \overline{D} \to \mathbb{R}$ to the Poisson equation*

$$\begin{cases} \Delta p = -2 & \text{in } D, \\ p = c_i & \text{on } \partial h_i \text{ for } i = 1, \ldots, n, \\ p = 0 & \text{on } \partial D_0. \end{cases} \tag{2.1.11}$$

satisfies

$$\int_{\partial h_i} \nabla p \cdot \vec{n}\, d\sigma = -2|h_i| \quad \textit{for } i = 1, \dots, n. \tag{2.1.12}$$

Here $|h_i|$ *is the area of the domain* $h_i \subset \mathbb{R}^2$.

Proof. Let u satisfy that

$$\begin{cases} \Delta u = -2 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Furthermore let the function v_j for $j = 1, ..., n$ be the solution to

$$\begin{cases} \Delta v_j = 0 & \text{in } D \\ v_j = 0 & \text{on } \partial D \setminus \partial h_j \\ v_j = 1 & \text{on } \partial h_j. \end{cases}$$

Now we consider the following linear equation,

$$Ax = b, \tag{2.1.13}$$

where $A_{i,j} = \int_{\partial h_i} \nabla v_j \cdot \vec{n}\, d\sigma$ and $b_i = -2|h_i| - \int_{\partial h_i} \nabla u \cdot \vec{n}\, d\sigma$. We argue that (Equation 2.1.13) has a unique solution. Thanks to the divergence theorem, we have

$$0 = \int_D \Delta v_j dx = \int_{\partial D_0} \nabla v_j \cdot \vec{n} d\sigma - \sum_{i=1}^{n} \int_{\partial h_i} \nabla v_j \cdot \vec{n} d\sigma.$$

Therefore,

$$\sum_{i=1}^{n} A_{i,j} = \int_{\partial D_0} \nabla v_j \cdot \vec{n} d\sigma < 0,$$

where the last inequality follows from the Hopf Lemma since v_j attains its minimum value 0 on

∂D_0, and $v_j \not\equiv 0$ on ∂D. A similar argument gives that $A_{i,j} > 0$ for $i \neq j$ and $A_{j,j} < 0$. Thus A is invertible by Gershgorin circle theorem [47], leading to a unique solution of (Equation 2.1.13). Let us denote the solution by $x = (c_1, ..., c_n)^t$. Then the function p defined by

$$p := u + \sum_{i=1}^{n} c_i v_i$$

satisfies the desired properties (Equation 2.1.12).

Now we prove that $c_i > 0$ for $i \geq 1$. Suppose that $c_{i*} := \min_i c_i \leq 0$. Then by the minimum principle, p attains its minimum on ∂h_{i^*}. Therefore,

$$0 \leq \int_{\partial h_{i^*}} \nabla p \cdot \vec{n} d\sigma = -2|h_{i^*}| < 0,$$

which is a contradiction. □

Next we prove a parallel version of Talenti's theorem for the function p constructed in Lemma 2.1.5. We will use this result throughout Section section 4.1–section 4.2.

Proposition 2.1.6. *Let $D \subset \mathbb{R}^2$ be a bounded connected domain with C^1 boundary. Assume D has n holes with $n \geq 0$, and denote by $h_1, \cdots, h_n \subset \mathbb{R}^2$ the holes of D (each h_i is a bounded open set). Let $p : \overline{D} \to \mathbb{R}$ be the function constructed in Lemma 2.1.5. Then the following two estimates hold:*

$$\sup_{\overline{D}} p \leq \frac{|D|}{2\pi} \tag{2.1.14}$$

and

$$\int_D p(x)dx \leq \frac{|D|^2}{4\pi} \tag{2.1.15}$$

Furthermore, for each of the two inequalities above, the equality is achieved if and only if D is either a disk or an annulus.

Proof. The proof is divided into two parts: In step 1 we prove the two inequalities

(Equation 2.1.14) and (Equation 2.1.15), and in step 2 we show that equality can be achieved if and only if D is a disk or an annulus.

Step 1. When D is simply-connected, (Equation 2.1.14) and (Equation 2.1.15) directly follow from Talenti's theorem Proposition 2.1.1. Next we consider a non-simply-connected domain D, and prove that these inequalities also hold when $p : \overline{D} \to \mathbb{R}$ is defined as in Lemma 2.1.5.

For $k \in \mathbb{R}^+$, let us denote $D_k := \{x \in D : \ p(x) > k\}$, $g(k) := |D_k|$ and $\tilde{D}_k := D_k \dot\cup (\dot\cup_{\{i:c_i>k\}} \overline{h_i})$. Elliptic regularity theory gives that $p \in C^\infty(D)$, thus by Sard's theorem, k is a regular value for almost every $k \in (0, \sup_D p)$, that is, $|\nabla p(x)| > 0$ on $\{x \in D : \ p(x) = k\}$. Thus $\{x \in D : \ p(x) = k\}$ is a union of smooth simply closed curves and equal to $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$.

Since $\partial D_k = \partial \tilde{D}_k \dot\cup (\dot\cup_{\{i:c_i>k\}} \partial h_i)$ for $k \notin \{c_1, ..., c_n\}$, we compute

$$\begin{aligned} g(k) &= -\frac{1}{2}\int_{D_k} \Delta p(x) dx \ = -\frac{1}{2}\int_{\partial D_k} \nabla p \cdot \vec{n} d\sigma \\ &= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma + \frac{1}{2}\sum_{\{i:c_i>k\}} \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma \\ &= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma - \sum_{\{i:c_i>k\}} |h_i|, \end{aligned}$$

where the last identity is due to (Equation 2.1.12). Therefore, it follows that

$$g(k) + \sum_{\{i:c_i>k\}} |h_i| = -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma = \frac{1}{2}\int_{\partial \tilde{D}_k} |\nabla p| d\sigma, \tag{2.1.16}$$

where the last equality follows from the fact that ∇p is perpendicular to the tangent vector on the level set.

On the other hand, the coarea formula yields that

$$g(k) = \int_{\mathbb{R}} \int_{\partial \tilde{D}_s} 1_{D_k} \frac{1}{|\nabla p|} d\sigma ds = \int_k^\infty \int_{\partial \tilde{D}_s} \frac{1}{|\nabla p|} d\sigma ds.$$

Therefore, it follows that for almost every $k \in (0, \sup_D p)$,

$$g'(k) = -\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|} d\sigma. \tag{2.1.17}$$

Thus it follows from (Equation 2.1.16) and (Equation 2.1.17) that

$$g'(k)\left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right) = -\frac{1}{2}\left(\int_{\partial \tilde{D}_k} |\nabla p| d\sigma\right)\left(\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|} d\sigma\right) \leq -\frac{1}{2} P(\tilde{D}_k)^2, \tag{2.1.18}$$

where $P(E)$ denotes the perimeter of a rectifiable curve ∂E. Note that the last inequality becomes equality if and only if $|\nabla p|$ is a constant on $\partial \tilde{D}_k$. Also, the isoperimetric inequality gives that

$$P(\tilde{D}_k)^2 \geq 4\pi |\tilde{D}_k|, \tag{2.1.19}$$

where equality holds if and only if $\tilde{D}_k$ is a disk. This yields that

$$g'(k)\left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right) \leq -2\pi |\tilde{D}_k| = -2\pi \left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right). \tag{2.1.20}$$

Therefore, $g'(k) \leq -2\pi$ for almost every $k \in (0, \sup_D p)$. Combining it with the fact that $g(0) = |D|$, we have

$$g(k) \leq (g(0) - 2\pi k)_+ = (|D| - 2\pi k)_+ \text{ for almost every } k \geq 0.$$

This proves that $\sup_{\bar{D}} p \leq \frac{|D|}{2\pi}$. It follows that

$$\int_D p(x)dx = \int_D \int_0^{\frac{|D|}{2\pi}} 1_{\{k<p(x)\}} dk dx = \int_0^{\frac{|D|}{2\pi}} g(k)dk \leq \int_0^{\frac{|D|}{2\pi}} (|D| - 2\pi k)_+ dx = \frac{|D|^2}{4\pi}.$$

Step 2. Now we show that for the two inequalities (Equation 2.1.14) and (Equation 2.1.15), the equality is achieved if and only if D is either a disk or an annulus. First, if D is either a disk or annulus centered at some $x_0 \in \mathbb{R}^2$, then uniqueness of solution to Poisson's equation gives that p is radially symmetric about x_0. Since we have $\Delta p = -2$ in D and $p = 0$ on the outer boundary of D, this gives an explicit formula $p(x) = -\frac{|x-x_0|^2}{2} + \frac{R^2}{2}$ for $x \in D$, where R is the outer radius of D. For either a disk or an annulus, one can explicitly compute $\sup_D p$ and $\int_D p dx$ to check that equalities in (Equation 2.1.14) and (Equation 2.1.15) are achieved.

To prove the converse, assume that either (Equation 2.1.14) or (Equation 2.1.15) achieves equality, and we aim to show that D is either a disk or an annulus. In order for either equality to be achieved, (Equation 2.1.20) needs to achieve equality at almost every $k \in (0, \sup_D p)$. In addition, $g(k)$ needs to be continuous in k since $g(k)$ is decreasing. Since (Equation 2.1.20) follows from a combination of the Cauchy-Schwarz inequality in (Equation 2.1.18) and the isoperimetric inequality in (Equation 2.1.19), we need to have all the three conditions below in order for either (Equation 2.1.14) or (Equation 2.1.15) to achieve equality:

(1) $|\nabla p|$ is a constant on each level set $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$;

(2) $\tilde{D}_k$ is a disk for almost every $k \in (0, \sup_D p)$.

(3) $g(k) = |D_k|$ is continuous in k. As a result, $|\tilde{D}_k|$ is continuous in k at all $k \neq c_i$, with $c_i > 0$ defined in (Equation 2.1.11).

Next we will show that if all these three conditions are satisfied, then D must be an annulus or disk. First, note that by sending $k \searrow 0$ in condition (2), and combining it with the continuity of $|\tilde{D}_k|$ as $k \searrow 0$, it already gives that the outer boundary of D must be a circle. Therefore if D is simply-connected, it must be a disk.

If D is non-simply-connected, using condition (2) and (3), we claim that D can have only one hole, which must be a disk, and p must achieve its maximum value in $\bar{D}$ on the boundary of the hole. To see this, let h_i be any hole of D, and recall that $p|_{\partial h_i} = c_i$. As we consider the set limit of

$\tilde{D}_k$ as k approaches c_i from below and above, by definition of $\tilde{D}_k$ we have

$$\lim_{k \nearrow c_i} \tilde{D}_k = \lim_{k \searrow c_i} \tilde{D}_k \dot{\cup} (\dot{\cup}_{\{j: c_j = c_i\}} \overline{h_j}).$$

By (2) and (3), the left hand side $\lim_{k \nearrow c_i} \tilde{D}_k$ is a disk, and the set $\lim_{k \searrow c_i} \tilde{D}_k$ on the right hand side is also a disk (if the limit is non-empty). But after taking union with the holes $\{h_j : c_j = c_i\}$ (each is a simply-connected set), the right hand side will be a disk if and only if $\lim_{k \searrow c_i} \tilde{D}_k$ is empty, $\dot{\cup}_{\{j: c_j = c_i\}} \overline{h_j} = \overline{h_i}$, and h_i is a disk. This implies $c_i = \sup_D p$ and $c_j < c_i$ for all $j \neq i$. But since h_i is chosen to be any hole of D, we know D can have only one hole (call it h), which is a disk, and $\sup_D p = p|_{\partial h}$. Finally, note that condition (1) gives that all the disks $\{\tilde{D}_k\}$ are concentric, and as a result we have D is an annulus, finishing the proof. $\square$

Finally, we are ready to show that every connected stationary patch D with C^1 boundary must be either a disk or an annulus.

Theorem 2.1.7. *Let $D \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary. Suppose that $\omega(x) := 1_D(x)$ is a stationary patch solution to the 2D Euler equation in the sense of* (Equation 1.1.5). *Then D is either a disk or an annulus.*

Proof. If D has n holes (where $n \geq 0$), denote them by $h_1, \ldots, h_n$. By (Equation 1.1.5), the function $f := 1_D * \mathcal{N}$ is constant on each of connected component of ∂D, and let us denote

$$f(x) = \begin{cases} a_i & \text{on } \partial h_i \\ a_0 & \text{on } \partial D_0. \end{cases} \tag{2.1.21}$$

Let $p : \overline{D} \to \mathbb{R}$ be defined as in Lemma 2.1.5, and let $\varphi := \frac{|x|^2}{2} + p$. Similar to the proof of Theorem 2.1.2, we calculate $\mathcal{I} := \int_D \nabla\varphi \cdot \nabla f dx$ in two different ways. Note that $\nabla f = \nabla(f - a_0)$ in D. Applying the divergence theorem to $\mathcal{I}$ and using (Equation 2.1.21) and $\Delta\varphi = 0$ in D, it follows that

$$\mathcal{I} = \int_{\partial D} (\nabla\varphi \cdot \vec{n})(f - a_0) d\sigma - \int_D \Delta\varphi (f - a_0) dx = -\sum_{i=1}^{n} (a_i - a_0) \int_{\partial h_i} \nabla\varphi \cdot \vec{n} d\sigma. \quad (2.1.22)$$

By definition of φ, and combining it with the property of p in (Equation 2.1.12), we have

$$\begin{aligned} \int_{\partial h_i} \nabla\varphi \cdot \vec{n} d\sigma &= \int_{\partial h_i} \nabla \left(\frac{|x|^2}{2} \right) \cdot \vec{n} d\sigma + \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma \\ &= \int_{h_i} 2 dx + \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma = 0. \end{aligned} \quad (2.1.23)$$

Plugging this into (Equation 2.1.22) gives $\mathcal{I} = 0$. On the other hand, we also have

$$\mathcal{I} = \int_D x \cdot \nabla f dx + \int_D \nabla p \cdot \nabla f dx =: E_1 + E_2.$$

We compute

$$E_1 = \int_D x \cdot (1_D * \nabla \mathcal{N}) dx = \int_D \int_D \frac{1}{2\pi} \frac{x \cdot (x - y)}{|x - y|^2} dy dx = \frac{|D|^2}{4\pi}, \quad (2.1.24)$$

where the last equality is obtained by exchanging x with y and taking the average with the original integral. For E_2, the divergence theorem yields that

$$E_2 = \int_{\partial D} p \nabla f \cdot \vec{n} d\sigma - \int_D p \Delta f dx = \int_{\partial D} p \nabla f \cdot \vec{n} d\sigma - \int_D p dx.$$

Using the property of p in (Equation 2.1.11) and the fact that $\Delta f = 0$ in h_i, the divergence theorem yields

$$\int_{\partial D} p \nabla f \cdot \vec{n} d\sigma = -\sum_{i=1}^{n} \int_{\partial h_i} p \nabla f \cdot \vec{n} d\sigma = -\sum_{i=1}^{n} c_i \int_{h_i} \Delta f dx = 0, \quad (2.1.25)$$

As a result, we have $E_2 = -\int_D p dx$. If D is neither a disk nor an annulus, Proposition 2.1.6 gives

$$\mathcal{I} = E_1 + E_2 = \frac{|D|^2}{4\pi} - \int_D p dx > 0,$$

contradicting $\mathcal{I} = 0$. $\square$

In the next corollary, we generalize the above result to a nonnegative stationary patch with multiple (disjoint) patches.

Corollary 2.1.8. *Let $\omega(x) := \sum_{i=1}^n \alpha_i 1_{D_i}$, where $\alpha_i > 0$, each D_i is a bounded connected domain with C^1 boundary, and $D_i \cap D_j = \emptyset$ if $i \neq j$. Assume that ω is a stationary patch solution, that is, the function $f(x) := \omega * \mathcal{N}$ satisfies $\nabla^\perp f \cdot \vec{n} = 0$ on ∂D_i for all $i = 1, \dots, n$. Then ω is radially symmetric up to a translation.*

Proof. Following similar notations as the beginning of Section subsection 2.1.2, we denote the outer boundary of D_i by ∂D_{i0}, and the holes of each D_i (if any) by h_{ik} for $k = 1, \dots, N_i$. Let $p_i : \overline{D_i} \to \mathbb{R}$ be defined as in Lemma 2.1.5, that is, p_i satisfies

$$\begin{cases} \Delta p_i = -2 & \text{in } D_i \\ p_i = c_{ik} & \text{on } \partial h_{ik} \\ p_i = 0 & \text{on } \partial D_{i0}, \end{cases}$$

where c_{ik} is chosen such that $\int_{\partial h_{ik}} \nabla p_i \cdot \vec{n} d\sigma = -2|h_{ik}|$. We then define $\varphi : \cup_{i=1}^n \overline{D}_i \to \mathbb{R}$, such that in each $\overline{D}_i$ we have $\varphi = \varphi_i := \frac{|x|^2}{2} + p_i$.

Similar to Theorem 2.1.7, we compute

$$\mathcal{I} := \int_{\mathbb{R}^2} \omega \nabla\varphi \cdot \nabla f dx = \sum_{i=1}^n \int_{D_i} \alpha_i \nabla\varphi_i \cdot \nabla f dx$$

in two different ways. On the one hand, since $f = \omega * \mathcal{N}$ is a constant on each connected component of ∂D_i, the same computation of Theorem 2.1.7 yields that $\int_{D_i} \nabla\varphi_i \cdot \nabla f dx = 0$, therefore $\mathcal{I} = 0$. On the other hand, since $\nabla\varphi = x + \nabla p_i$ in each D_i, we break $\mathcal{I}$ into

$$\mathcal{I} = \sum_{i,j=1}^{n} \alpha_i\alpha_j \int_{D_i} x \cdot \nabla(1_{D_j} * \mathcal{N}) dx + \sum_{i,j=1}^{n} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) dx =: \mathcal{I}_1 + \mathcal{I}_2.$$

For $\mathcal{I}_1$, we compute

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2} \left(\int_{D_i} x \cdot \nabla(1_{D_j} * \mathcal{N}) dx + \int_{D_j} x \cdot \nabla(1_{D_i} * \mathcal{N}) dx \right) \\
&= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2} \left(\int_{D_i}\int_{D_j} \frac{x \cdot (x-y)}{2\pi|x-y|^2} dy dx + \int_{D_j}\int_{D_i} \frac{x \cdot (x-y)}{2\pi|x-y|^2} dy dx \right) \\
&= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{4\pi} |D_i||D_j|, \qquad (2.1.26)
\end{aligned}$$

where we exchanged i with j to get the first equality. For $\mathcal{I}_2$, we have

$$\mathcal{I}_2 = \sum_{i=1}^{n} \alpha_i^2 \int_{D_i} \nabla p_i \cdot \nabla(1_{D_i} * \mathcal{N}) dx + \sum_{i \neq j} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) dx =: I_{21} + I_{22}.$$

By the same computation for E_2 in the proof of Theorem 2.1.7, we have

$$I_{21} = -\sum_{i=1}^{n} \alpha_i^2 \int_{D_i} p_i dx. \qquad (2.1.27)$$

For $i \neq j$, we denote $j \prec i$ if D_j is contained in a hole of D_i. (And if D_j is not contained in any hole of D_i, we say $j \not\prec i$.) Using this notation, the divergence theorem directly yields that

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma = -\sum_{k=1}^{N_i} \int_{\partial h_{ik}} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma = 0 \quad \text{if } j \not\prec i. \qquad (2.1.28)$$

And if $j \prec i$, then the divergence theorem and (Equation 2.1.14) in Proposition 2.1.6 yield

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma \geq -\sup_{\partial D_i} p_i |D_j| \geq -\frac{1}{2\pi}|D_i||D_j| \quad \text{if } j \prec i. \tag{2.1.29}$$

Hence it follows that

$$I_{22} \geq -\sum_{i \neq j} \mathbb{1}_{j \prec i} \frac{\alpha_i \alpha_j}{2\pi} |D_i||D_j| = -\sum_{i \neq j} (\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j}) \frac{\alpha_i \alpha_j}{4\pi} |D_i||D_j|, \tag{2.1.30}$$

where the last step is obtained by exchanging i, j and taking average with the original sum. Note that we have $\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j} \leq 1$ for any $i \neq j$. From (Equation 2.1.26), (Equation 2.1.27) and (Equation 2.1.30), we obtain

$$\mathcal{I} \geq \sum_{i=1}^{n} \alpha_i^2 \left(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i dx \right) + \sum_{\substack{j \not\prec i \text{ and } i \not\prec j \\ i \neq j}} \alpha_i \alpha_j \frac{|D_i||D_j|}{4\pi}. \tag{2.1.31}$$

Since we already know that $\mathcal{I} = 0$ and all the summands in (Equation 2.1.31) are nonnegative, it follows that

$$\frac{|D_i|^2}{4\pi} = \int_{D_i} p_i dx \text{ for all } i = 1, \ldots, n \text{ and } \{(i,j) : i \neq j, i \not\prec j \text{ and } j \not\prec i\} = \emptyset.$$

Therefore every D_i is either a disk or an annulus by Proposition 2.1.6 and they are nested. By relabeling the indices, we can assume that $i \prec i+1$ for $i = 1, \ldots, n-1$.

Next we prove that all D_i's are concentric by induction. For $k \geq 1$, suppose $D_1, \ldots, D_k$ are known to be concentric about some $o \in \mathbb{R}^2$. To show D_{k+1} is also centered at o, we break f into

$$f = \sum_{i=1}^{k} (\alpha_i 1_{D_i}) * \mathcal{N} + \sum_{i=k+1}^{n} (\alpha_i 1_{D_i}) * \mathcal{N}.$$

In the first sum, each D_i is centered at o for $i \leq k$, thus Lemma 2.1.9(a) (which we prove right after this theorem) yields that $\sum_{i=1}^k (\alpha_i 1_{D_i})*\mathcal{N} = \frac{C}{2\pi}\ln|x-o|$ on $\partial_{\text{in}} D_{k+1}$, where $C = \sum_{i=1}^k \alpha_i |D_i| > 0$. In the second sum, for each $i \geq k+1$, since each D_i is an annulus with $\partial_{\text{in}} D_{k+1}$ in its hole, Lemma 2.1.9(b) gives that $1_{D_i} * \mathcal{N} \equiv$ const on $\partial_{\text{in}} D_{k+1}$ for all $i \geq k+1$. Thus overall we have $f = \frac{C}{2\pi}\ln|x-o| + C_2$ on $\partial_{\text{in}} D_{k+1}$ for $C > 0$. Combining it with the assumption that f is a constant on $\partial_{\text{in}} D_{k+1}$, we know D_{k+1} must also be centered at o, finishing the induction step. □

Now we state and prove the lemma used in the proof of Corollary 2.1.8, which follows from standard properties of the Newtonian potential.

Lemma 2.1.9. *Assume $g \in L^\infty(\mathbb{R}^2)$ is radially symmetric about some $o \in \mathbb{R}^2$, and is compactly supported in $B(o,R)$. Then $\eta := g * \mathcal{N}$ satisfies the following:*

(a) $\eta(x) = \dfrac{\int_{\mathbb{R}^2} g dx}{2\pi} \ln|x-o|$ *for all* $x \in B(0,R)^c$.

(b) If in addition we have $g \equiv 0$ in $B(o,r)$ for some $r \in (0,R)$, then $\eta = const$ in $B(o,r)$.

Proof. To show (a), we take any $x \in B(o,R)^c$ and consider the circle $\Gamma \ni x$ centered at o. By radial symmetry of η about o and the divergence theorem, we have

$$\nabla\eta \cdot \frac{x}{|x|} = \frac{1}{|\Gamma|}\int_\Gamma \nabla\eta\cdot\vec{n}d\sigma = \frac{1}{|\Gamma|}\int_{\text{int}(\Gamma)} \Delta\eta dx = \frac{\int_{\mathbb{R}^2} g(x)dx}{2\pi|x-o|},$$

which implies $\eta(x) = \frac{\int g dx}{2\pi}\ln|x-o| + C$. To show that $C = 0$, for $|x|$ sufficiently large we have

$$|C| = \Big|\int_{B(o,R)} g(x)(\mathcal{N}(x-y) - \mathcal{N}(x-o))dy\Big| \leq \|g\|_{L^\infty(\mathbb{R}^2)} \sup_{y\in B(o,R)} |\mathcal{N}(x-o) - \mathcal{N}(x-y)|,$$

and by sending $|x| \to \infty$ we have $C = 0$, which gives (a). To show (b), it suffices to prove that $\nabla\eta \equiv 0$ in $B(o,r)$. Take any $x \in B(o,r)$, and consider the circle $\Gamma_2 \ni x$ centered at o. Again, symmetry and the divergence theorem yield that

$$|\nabla\eta(x)| = \frac{1}{|\Gamma_2|}\int_{\Gamma_2} \nabla\eta\cdot\vec{n}d\sigma = \frac{1}{|\Gamma_2|}\int_{\text{int}(\Gamma_2)} \Delta\eta dx = \frac{\int_{\text{int}(\Gamma)} g(x)dx}{|\Gamma|} = 0,$$

finishing the proof of (b). $\square$

2.1.3 Radial symmetry of non-simply-connected rotating patches with $\Omega < 0$

In this subsection, we show that a nonnegative uniformly rotating patch solution (with multiple disjoint patches) must be radially symmetric if the angular velocity $\Omega < 0$.

Theorem 2.1.10. *For $i = 1, \dots, n$, let D_i be a connected domain with C^1 boundary, and assume $D_i \cap D_j = \emptyset$ for $i \neq j$. If $\omega = \sum_{i=1}^n \alpha_i 1_{D_i}$ is a nonnegative rotating patch solution with $\alpha_i > 0$ and angular velocity $\Omega < 0$, then ω must be radially symmetric.*

Proof. In this proof, let

$$f_\Omega(x) := \omega * \mathcal{N} - \frac{\Omega}{2}|x|^2.$$

In each D_i, let us define p_i as in Lemma 2.1.5. Let $\varphi_i := \frac{|x|^2}{2} + p_i$ in each D_i. As in Theorem 2.1.10, we compute $\mathcal{I} := \sum_{i=1}^n \alpha_i \int_{D_i} \nabla\varphi_i \cdot \nabla f_\Omega dx$ in two different ways. Since f_Ω is a constant on each connected component of ∂D_i and $\nabla\varphi_i$ is divergence free in D_i, we still have $\mathcal{I} = 0$ as in the proof of Theorem 2.1.7.

On the other hand, we have

$$\begin{aligned}\mathcal{I} &= \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot \nabla\left(\omega * \mathcal{N}\right) dx + \underbrace{(-\Omega)}_{\geq 0} \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot x dx \\ &=: \mathcal{I}_1 + (-\Omega)\mathcal{I}_2.\end{aligned}$$

As in the proof of Corollary 2.1.8, we have

$$\mathcal{I}_1 = \sum_{i=1}^n \alpha_i^2 \left(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i dx\right) + \sum_{\substack{j \not\prec i \text{ and } i \not\prec j \\ i \neq j}} \alpha_i \alpha_j \frac{|D_i||D_j|}{4\pi} \geq 0. \tag{2.1.32}$$

Note that $\mathcal{I}_1 = 0$ as long as all D_i's are nested annuli/disk, even if they are not concentric. For $\mathcal{I}_2$, using Cauchy-Schwarz inequality in the second step, and Lemma 2.1.11 in the third step (which we will prove right after this theorem), we have

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i=1}^{n} \alpha_i \left(\int_{D_i} |x|^2 dx + \int_{D_i} \nabla p_i \cdot x dx \right) \\ &\geq \sum_{i=1}^{n} \alpha_i \left(\int_{D_i} |x|^2 dx - \Big(\int_{D_i} |\nabla p|^2 dx \Big)^{1/2} \Big(\int_{D_i} |x|^2 dx \Big)^{1/2} \right) \geq 0.\end{aligned} \tag{2.1.33}$$

Combining (Equation 2.1.32) and (Equation 2.1.33) gives us $\mathcal{I} \geq 0$. If there is any D_i that is not a disk or annulus centered at the origin, Lemma 2.1.11 would give a strict inequality in the last step of (Equation 2.1.33), which leads to $\mathcal{I} > 0$ and thus contradicts with $\mathcal{I} = 0$. □

Now we state and prove the lemma that is used in the proof of Theorem 2.1.10.

Lemma 2.1.11. *Let D be a connected domain with C^1 boundary, and let p be as in Lemma 2.1.5. Then we have*

$$-\int_D \nabla p \cdot x dx = \int_D |\nabla p|^2 dx \leq \int_D |x|^2 dx. \tag{2.1.34}$$

Furthermore, in the inequality, "=" is achieved if and only if D is a disk or annulus centered at the origin.

Proof. We compute

$$\begin{aligned}\int_D |\nabla p|^2 dx &= \int_{\partial D} p \nabla p \cdot \vec{n} d\sigma + \int_D 2p dx \\ &= -\int_{\partial D} p x \cdot \vec{n} d\sigma + \int_D 2p dx,\end{aligned}$$

where in the last equality we use that p is constant along each ∂h_i, as well as the following identity due to (Equation 2.1.12) and the divergence theorem (here $\vec{n}$ is the outer normal of h_i):

$$\int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma = -2|h_i| = -\int_{h_i} \Delta \frac{|x|^2}{2} dx = -\int_{\partial h_i} x \cdot \vec{n} d\sigma.$$

On the other hand, the divergence theorem yields

$$-\int_D \nabla p \cdot x dx = -\int_{\partial D} px \cdot \vec{n} d\sigma + \int_D 2p dx.$$

Therefore using Young's inequality $-\nabla p \cdot x \leq \frac{1}{2}|\nabla p|^2 + \frac{1}{2}|x|^2$ (where the equality is achieved if and only if $-\nabla p = x$), we have

$$\int_D |\nabla p|^2 dx = -\int_D \nabla p \cdot x dx \leq \frac{1}{2}\int_D |\nabla p|^2 dx + \frac{1}{2}\int_D |x|^2 dx,$$

which proves (Equation 2.1.34). Here the equality is achieved if and only if $-\nabla p = x$ in D, which is equivalent with $p + \frac{|x|^2}{2}$ being a constant in D, and it can be extended to $\bar{D}$ due to continuity of p. By our construction of p in Lemma 2.1.5, p is already a constant on each connected component of ∂D, implying $\frac{|x|^2}{2}$ is constant on each piece of ∂D, hence ∂D must be a family of circles centered at the origin. By the assumption that D is connected, it must be either a disk or annulus centered at the origin. □

2.1.4 Radial symmetry of non-simply-connected rotating patches with $\Omega \geq \frac{1}{2}$

In this final subsection for patches, we consider a bounded domain D with C^1 boundary. D can have multiple connected components, and each connected component can be non-simply-connected. If 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \geq \frac{1}{2}$, we will show D must be radially symmetric and centered at the origin.

To do this, one might be tempted to proceed as in Theorem 2.1.2 and replace $p : D \to \mathbb{R}$ by the function defined in Lemma 2.1.5. Here the first way of computing $\mathcal{I} = \int_D (x + \nabla p) \cdot \nabla f_\Omega dx$ still yields $\mathcal{I} = 0$, but the second way gives some undesired terms caused by the holes h_i:

$$\mathcal{I} = \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx + (2\Omega - 1)\int_D p dx + 2\Omega \sum_{i=1}^{n} p|_{\partial h_i} |h_i|.$$

Due to the last term on the right hand side, we are unable to show $\mathcal{I} \leq 0$ when $\Omega \geq \frac{1}{2}$ as we did before in Theorem 2.1.2. For this reason, we take a different approach in the next theorem. Instead of defining p as a function in D and $\mathcal{I}$ as an integral in D, we want to define them in D^c. But since D^c is unbounded, we define p^R and $\mathcal{I}_R$ in a truncated set $B(0,R) \setminus D^c$, and then use two different ways to compute $\mathcal{I}_R$. By sending $R \to \infty$, we will show that the two ways give a contradiction unless D is radially symmetric.

Theorem 2.1.12. *For a bounded domain D with C^1 boundary, assume that 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \geq \frac{1}{2}$. Then D is radially symmetric and centered at the origin.*

Proof. Since D is bounded, let us choose $R_0 > 0$ such that $B_{R_0} \supset D$. For any $R > R_0$, consider the domain $B_R \setminus \overline{D}$, which may have multiple connected components. We call the component touching ∂B_R as $D_{0,R}$, and name the other connected components by $U_1, \ldots, U_n$. Throughout this proof we assume that $n \geq 1$: if not, then each connected component of D is simply connected, which has been already treated in Theorem 2.1.2 and Remark 2.1.3. We also define $V := B_R \setminus D_{0,R}$, which is the union of D and all its holes. Note that V may have multiple connected components, but each must be simply-connected. See Figure Figure 2.1 for an illustration of $D_{0,R}$, $\{U_i\}_{i=1}^n$ and V.

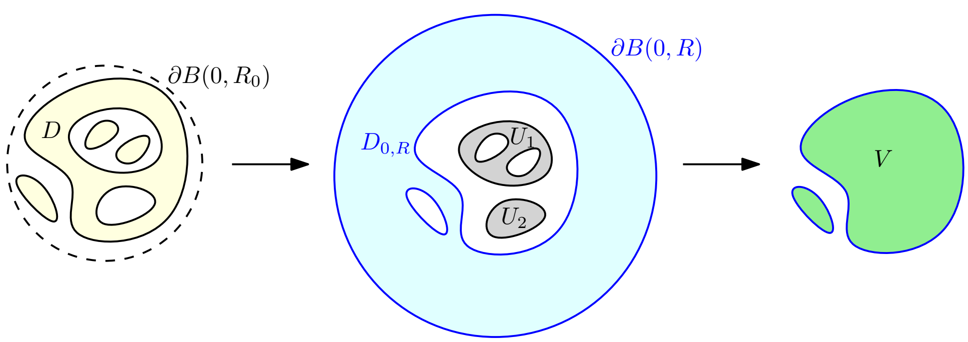


Figure 2.1: For a set $D \subset B(0,R_0)$ (the whole yellow region on the left), the middle figure illustrates the definition of $D_{0,R}$ (the blue region), $\{U_i\}$ (the gray regions), and right right figure illustrate $V = B_R \setminus D_{0,R}$ (the green region).

To prove the theorem, the key idea is to define p^R and $\mathcal{I}_R$ in $B_R \setminus D$, instead of in D. Let $p_{0,R}$ and p_i be defined as in Lemma 2.1.5 in $D_{0,R}$ and U_i respectively, then set $\varphi_{0,R} := p_{0,R} + \frac{|x|^2}{2}$ in $D_{0,R}$, and $\varphi_i := p_i + \frac{|x|^2}{2}$ in U_i for $i = 1, \ldots, n$. Finally, define p^R and $\varphi^R : \mathbb{R}^2 \to \mathbb{R}$ as

$$p^R := p_{0,R} 1_{D_{0,R}} + \sum_{i=1}^{n} p_i 1_{U_i}, \qquad \varphi^R := \varphi_{0,R} 1_{D_{0,R}} + \sum_{i=1}^{n} \varphi_i 1_{U_i}.$$

Since 1_D rotates with angular velocity $\Omega \geq \frac{1}{2}$, we know $f_\Omega := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ is constant on each connected component of ∂D. Next we will compute

$$\mathcal{I}_R := \int_{B_R \setminus \overline{D}} \nabla f_\Omega \cdot \nabla \varphi^R dx \tag{2.1.35}$$

in two different ways. If some connected component of ∂D is not a circle, we will derive a contradiction by sending $R \to \infty$. We point out that as we increase R, the domain $D_{0,R}$ will change, but the domains $\{U_i\}_{i=1}^n$ and V all remain unchanged.

On the one hand, we break $\mathcal{I}_R$ into

$$\mathcal{I}_R = \int_{D_{0,R}} \nabla f_\Omega \cdot \nabla \varphi_{0,R} dx + \sum_{i=1}^{n} \int_{U_i} \nabla f_\Omega \cdot \nabla \varphi_i dx =: \mathcal{I}_R^1 + \mathcal{I}_R^2.$$

Since f_Ω is constant on each connected component of ∂U_i, the same computation as (Equation 2.1.22)–(Equation 2.1.23) gives $\mathcal{I}_R^2 = 0$. For $\mathcal{I}_R^1$, note that although f_Ω is a constant along the boundary of each hole of $D_{0,R}$, it is *not* a constant along $\partial_{\text{out}} D_{0,R} = \partial B_R$. Thus similar computations as (Equation 2.1.22)–(Equation 2.1.23) now give

$$\begin{aligned} \mathcal{I}_R^1 &= \int_{\partial B_R} \big(1_D * \mathcal{N} - \frac{\Omega}{2} R^2\big) \nabla \varphi_{0,R} \cdot \vec{n} d\sigma \\ &= \int_{\partial B_R} \left((1_D * \mathcal{N})(x) - |D| \mathcal{N}(x)\right) \nabla \varphi_{0,R} \cdot \vec{n} d\sigma(x), \end{aligned} \tag{2.1.36}$$

where in the second equality we used $\int_{\partial B_R} \nabla \varphi_{0,R} \cdot \vec{n} d\sigma = 0$ and the fact that $\mathcal{N}(x)$ is constant on

∂B_R. For any $x \in \partial B_R$, since $D \subset B_{R_0}$ and $R > R_0$, we can control $(1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)$ as

$$\Big|(1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)\Big| \le \frac{1}{2\pi}\int_D \Big|\log|x-y| - \log|x|\Big|dy \le \frac{|D|}{2\pi}\left|\log\left(1 - \frac{R_0}{R}\right)\right| \text{ on } \partial B_R. \tag{2.1.37}$$

We introduce the following lemma to control $|\nabla\varphi_{0,R} \cdot \vec{n}|$ on ∂B_R, whose proof is postponed to the end of this subsection.

Lemma 2.1.13. *Let $D \subset B_{R_0}$ be a domain with C^1 boundary. For any $R > R_0$, let $D_{0,R}$, V, $p_{0,R}$ and $\varphi_{0,R}$ be defined as in the proof of Theorem 2.1.12. Then we have*

$$|\nabla\varphi_{0,R} \cdot \vec{n}| \le \frac{NR_0^2}{2R\log(R/R_0)} \quad \textit{on } \partial B_R, \tag{2.1.38}$$

where $N > 0$ is the number of connected components of V (and is independent of R).

Once we have this lemma, plugging (Equation 2.1.38) and (Equation 2.1.37) into (Equation 2.1.36) yields

$$|\mathcal{I}_R^1| \le \frac{N|D|R_0^2}{2}\left|\log\left(1 - \frac{R_0}{R}\right)\right|(\log(R/R_0))^{-1} \le |D|\frac{C(D,R_0)}{R\log R} \to 0 \quad \text{as } R \to \infty.$$

Combining this with $\mathcal{I}_R^2 = 0$ gives

$$\lim_{R\to\infty} \mathcal{I}_R = 0. \tag{2.1.39}$$

Next we compute $\mathcal{I}_R$ in another way. Note that $1_{B_R} * \mathcal{N} - \frac{|x|^2}{4}$ is a radial harmonic function in B_R, thus is equal to some constant C_R in B_R. Using this fact, we can rewrite f_Ω as

$$f_\Omega = 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = (1_D - 1_{B_R}) * \mathcal{N} - \Big(\frac{\Omega}{2} - \frac{1}{4}\Big)|x|^2 + C_R.$$

As a result, $\mathcal{I}_R$ can be rewritten as

$$\begin{aligned}\mathcal{I}_R &= -\int_{B_R\setminus\overline{D}} \nabla\left(1_{B_R\setminus\overline{D}} * \mathcal{N}\right)\cdot\nabla\varphi^R dx - \frac{(2\Omega-1)}{2}\int_{B_R\setminus\overline{D}} x\cdot\nabla\varphi^R dx \\ &=: -\mathcal{J}_R^1 - \frac{(2\Omega-1)}{2}\mathcal{J}_R^2.\end{aligned} \tag{2.1.40}$$

Next we will show $\mathcal{J}_R^1, \mathcal{J}_R^2 \geq 0$, leading to $\mathcal{I}_R \leq 0$. Let us start with $\mathcal{J}_R^2$. Applying Lemma 2.1.11 to each of $D_{0,R}$ and $\{U_i\}_{i=1}^n$ immediately gives

$$\mathcal{J}_R^2 \geq \int_{D_{0,R}} |x|^2 + \nabla p_{0,R}\cdot x dx + \sum_{i=1}^n \int_{U_i} |x|^2 + \nabla p_i \cdot x dx =: T_{0,R} + \sum_{i=1}^n T_i \geq 0. \tag{2.1.41}$$

Note that the T_i's are independent of R for $i = 1,\ldots,n$, and we know $T_i \geq 0$ with equality achieved if and only if U_i is an annulus or a disk centered at the origin. This will be used later to show all $\{U_i\}_{i=1}^n$ are centered at the origin in the $\Omega > \frac{1}{2}$ case. (When $\Omega = \frac{1}{2}$, the coefficient of $\mathcal{J}_R^2$ becomes 0 in (Equation 2.1.40), thus a different argument is needed in this case.)

We now move on to $\mathcal{J}_R^1$. We first break it as

$$\mathcal{J}_R^1 = \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot x dx + \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot\nabla p^R dx =: J_{11} + J_{12}.$$

An identical computation as (Equation 2.1.24) gives $J_{11} = \dfrac{1}{4\pi}\Big(|D_{0,R}| + \sum_{i=1}^n |U_i|\Big)^2$. For J_{12}, the same computation as (Equation 2.1.27)–(Equation 2.1.29) gives the following (where we used that each U_i lies in a hole of $D_{0,R}$ for $i = 1,\ldots,n$):

$$J_{12} \geq -\int_{D_{0,R}} p_{0,R} dx - \sum_{i=1}^n \int_{U_i} p_i dx - \sum_{1\leq i\leq n} |U_i| \sup_{D_{0,R}} p_{0,R} - \sum_{\substack{1\leq i,j\leq n \\ j\prec i}} \frac{|U_i||U_j|}{2\pi}.$$

Adding up the estimates for J_{11} and J_{12}, we get

$$\begin{aligned}\mathcal{J}_R^1 \geq \left(\frac{1}{4\pi}|D_{0,R}|^2 - \int_{D_{0,R}} p_{0,R}dx\right) + \left(\sum_{1\leq i\leq n}|U_i|\right)\left(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\right) \\ + \sum_{i=1}^n \left(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i dx\right) + \sum_{\substack{j\not\prec i \text{ and } i\not\prec j\\ i\neq j}} \frac{1}{4\pi}|U_i||U_j|.\end{aligned} \tag{2.1.42}$$

By Proposition 2.1.6, all terms on the right hand side are nonnegative. But note that only the two terms in the second line are independent of R. Plugging (Equation 2.1.42) and (Equation 2.1.41) into (Equation 2.1.40) gives the following (where we only keep the terms independent of R on the right hand side):

$$\liminf_{R\to\infty}(-\mathcal{I}_R) \geq \sum_{i=1}^n \left(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i dx\right) + \sum_{\substack{j\not\prec i \text{ and } i\not\prec j\\ i\neq j}} \frac{1}{4\pi}|U_i||U_j| + \frac{2\Omega - 1}{2}\sum_{i=1}^n T_i \geq 0.$$

Combining this with the previous limit (Equation 2.1.39), we know U_i must be an annulus or a disk for $i = 1, \dots, n$, and they must be nested in each other. In addition, if $\Omega > \frac{1}{2}$, we have $T_i = 0$ for $i = 1, \dots, n$, implying that each U_i is centered at the origin.

The radial symmetry of $D_{0,R}$ is more difficult to obtain. Although the first two terms on the right hand side of (Equation 2.1.42) are both strictly positive if $D_{0,R}$ is not an annulus, we need some uniform-in-R lower bound to get a contradiction in the $R \to \infty$ limit. Between these two terms, it turns out the second term is easier to control. This is done in the next lemma, whose proof we postpone to the end of this subsection.

Lemma 2.1.14. *Let $D \subset B_{R_0}$ be a domain with C^1 boundary. For any $R > R_0$, let $D_{0,R}$, V and $p_{0,R}$ be given as in the proof of Theorem 2.1.12. If V is not a single disk, there exists some constant*

$C(V) > 0$ *only depending on* V*, such that*

$$\liminf_{R\to\infty}\left(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\right) \geq C(V) > 0.$$

If V is not a disk, Lemma 2.1.14 gives $\liminf_{R\to\infty} \mathcal{J}_R^1 > \left(\sum_{1\leq i\leq n}|U_i|\right) C(V) > 0$. (Recall that in the beginning of this proof we assume $\sum_{1\leq i\leq n}|U_i| > 0$, and it is independent of R.) This implies $\liminf_{R\to\infty}(-\mathcal{I}_R) \geq C(V) > 0$, contradicting (Equation 2.1.39).

So far we have shown that ∂D is a union of nested circles, and it remains to show that they are all centered at 0. For the $\Omega > \frac{1}{2}$ case, we already showed all $\{U_i\}_{i=1}^n$ are centered at 0, so it suffices to show the outmost circle ∂V (denote by $B(\tilde{o},\tilde{r})$) is also centered at 0. By definition of $\{U_i\}_{i=1}^n$, we have $D = B(\tilde{o},\tilde{r}) \setminus \left(\dot{\cup}_{i=1}^n U_i\right)$. Note that $1_{B(\tilde{o},\tilde{r})} * \mathcal{N} = \frac{|x-\tilde{o}|^2}{4} + C$ for some constant C, and $1_{\dot{\cup}_{i=1}^n U_i} * \mathcal{N}$ is radially increasing. Therefore f_Ω can be written as

$$f_\Omega = 1_{B(\tilde{o},\tilde{r})} * \mathcal{N} - 1_{\dot{\cup}_{i=1}^n U_i} * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \frac{|x-\tilde{o}|^2}{4} - g(x),$$

where g is radially symmetric, and strictly increasing in the radial variable. Since both f_Ω and $\frac{|x-\tilde{o}|^2}{4}$ are known to take constant values on $\partial B(\tilde{o},\tilde{r})$, it implies g must be constant on $\partial B(\tilde{o},\tilde{r})$ too, and the fact that g is a radially increasing function gives that $\tilde{o} = 0$. This finishes the proof for $\Omega > \frac{1}{2}$.

For $\Omega = \frac{1}{2}$, we do not know whether $\{U_i\}_{i=1}^n$ are centered at 0 yet. Denote by U_1 be the innermost one. Then we have

$$f_\Omega(x) = \frac{|x-\tilde{o}|^2}{4} - 1_{\dot{\cup}_{i=1}^n U_i} * \mathcal{N} - \frac{|x|^2}{4} = \frac{\tilde{o}\cdot x}{2} + \text{const} \quad \text{for } x \in \partial_{\text{out}} U_1, \tag{2.1.43}$$

where the second equality follows from Lemma 2.1.9(b), where we used that $1 \prec j$ for all $2 \leq j \leq n$. Combining (Equation 2.1.43) with the fact that $f_\Omega = \text{const}$ on $\partial_{\text{out}} U_1$ gives $\tilde{o} = 0$, that is, the outmost circle must be centered at 0. This leads to $f_\Omega = -\sum_{i=1}^n 1_{U_i} * \mathcal{N}$. Since $f_\Omega = \text{const}$

on each connected component of ∂U_i, we can apply the last part in the proof of Corollary 2.1.8 to show that all $\{U_i\}_{i=1}^n$ are all concentric. Denoting their center by o_1, we can show that $o_1 = 0$: Lemma 2.1.9(a) gives $f_\Omega(x) = C \ln|x - o_1|$ for some $C < 0$ on $\partial B(\tilde{o}, \tilde{r})$, and since we have $f = \text{const}$ on $\partial B(\tilde{o}, \tilde{r})$ and $\tilde{o} = 0$, it implies that $o_1 = 0$, finishing the proof. □

Proof of Lemma 2.1.13. For notational simplicity, we shorten $p_{0,R}$, $D_{0,R}$ and $\varphi_{0,R}$ into p_R, D_R and φ_R thoughout this proof. Recall that $\partial D_R = \partial B_R \cup \partial V$. Clearly we have $\varphi_R = \frac{R^2}{2}$ on ∂B_R, due to $p_R = 0$ on $\partial_{\text{out}} D_R = \partial B_R$. We claim that

$$-\frac{NR_0^2}{2} \le \varphi_R - \frac{R^2}{2} \le \frac{R_0^2}{2} \quad \text{on } \partial V, \tag{2.1.44}$$

where $N \ge 1$ is the number of connected components of V. Once it is proved, we apply the comparison principle to the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$, where

$$g(x) := \frac{NR_0^2}{2\log(R/R_0)} \log \frac{R}{|x|}.$$

Note that $g \equiv 0$ on ∂B_R, and $g \ge \frac{NR_0^2}{2}$ on ∂V since $\partial V \subset B_{R_0}$. If $0 \notin \overline{D_R}$, then the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$ are all harmonic in D_R, their values on ∂B_R are all 0, and their boundary values on ∂V are ordered due to (Equation 2.1.44). The comparison principle in D_R then yields

$$-g(x) \le \varphi_R(x) - \frac{R^2}{2} \le g(x) \quad \text{in } D_R. \tag{2.1.45}$$

Since $\varphi_R - \frac{R^2}{2} \equiv g \equiv 0$ on ∂B_R, (Equation 2.1.45) gives $|\nabla \varphi_R \cdot \vec{n}| \le |\nabla g \cdot \vec{n}| = \frac{NR_0^2}{2R\log(R/R_0)}$ on ∂B_R, which is the desired estimate (Equation 2.1.38). And if $0 \in \overline{D_R}$, then (Equation 2.1.45) still holds in $D_R \setminus B_\epsilon$ for all sufficiently small $\epsilon > 0$ by applying the comparison principle in this set, and (Equation 2.1.38) again follows as a consequence.

In the rest of the proof we will show (Equation 2.1.44). Its second inequality is straightforward:

$$\varphi_R - \frac{R^2}{2} \le \frac{R_0^2}{2} + \sup_{\overline{D_R}} p_R - \frac{R^2}{2} \le \frac{R_0^2}{2} \quad \text{on } \partial V,$$

here the first inequality follows from the definition of φ_R and the fact that $V \subset B_{R_0}$, and the second inequality is due to $\sup_{\overline{D_R}} p_R \le \frac{|D_R|}{2\pi} \le \frac{R^2}{2}$ in Proposition 2.1.6.

It remains to prove the first inequality of (Equation 2.1.44). Let us fix any $R > R_0$. Denote the N connected components of ∂V by $\{\Gamma_i\}_{i=1}^N$, and let $\Gamma_0 := \partial B_R$. These notations lead to $\partial D_R = \cup_{i=0}^N \Gamma_i$. For $i = 0, \dots, N$, let $L_i \subset \mathbb{R}$ be the range of $\varphi_R - \frac{R^2}{2}$ on Γ_i. By continuity of φ_R, each L_i is a closed bounded interval, which can be a single point. Clearly, $L_0 = \{0\}$ due to $\varphi_R|_{\partial B_R} = \frac{R^2}{2}$. Towards a contradiction, suppose

$$v_{\min} := \min_{1\le i\le N} \inf L_i = \inf_{\partial V} \left(\varphi_R - \frac{R^2}{2}\right) =: -\frac{N|R_0|^2}{2} - \delta \quad \text{for some } \delta > 0. \tag{2.1.46}$$

As for the maximum value, since $L_0 = \{0\}$ we have

$$v_{\max} := \max_{0\le i\le N} \sup L_i \ge 0. \tag{2.1.47}$$

For $i = 1, \dots, N$, using $p_R|_{\Gamma_i} = \text{const}$, $\varphi_R = p_R + \frac{|x|^2}{2}$ and $\Gamma_i \subset B_{R_0}$, the length of each interval L_i satisfies

$$\big|L_i\big| = \operatorname{osc}_{\Gamma_i} \frac{|x|^2}{2} \le \frac{R_0^2}{2} \quad \text{for } i = 1, \dots, N. \tag{2.1.48}$$

Comparing (Equation 2.1.48) with (Equation 2.1.46)–(Equation 2.1.47), we know the union of $\{L_i\}_{i=0}^N$ cannot fully cover the interval $[v_{\min}, v_{\max}]$, thus they can be separated in the following sense: there exists a nonempty proper subset $S \subset \{0, \dots, N\}$, such that the range of L_i for indices in S and $S^c := \{0, \dots, N\} \setminus S$ are strictly separated by at least δ, i.e. $\min_{i\in S} \inf L_i \ge$

$\max_{i\in S^c}\sup L_i + \delta$. In terms of φ_R, we have

$$\min_{i\in S}\inf_{\Gamma_i}\varphi_R \ge \max_{i\in S^c}\max_{\Gamma_i}\varphi_R + \delta. \tag{2.1.49}$$

Since φ_R is harmonic in D_R, whose boundary is $\cup_{i=0}^N \Gamma_i$, it is a standard comparison principle exercise to show that (Equation 2.1.49) implies

$$\sum_{i\in S}\int_{\Gamma_i}\nabla\varphi_R\cdot\vec{n}d\sigma > 0, \tag{2.1.50}$$

where $\vec{n}$ denotes the outer normal of D_R. But on the other hand, we have

$$\int_{\Gamma_i}\nabla\varphi_R\cdot\vec{n}d\sigma = 0 \quad \text{ for } i = 0,\ldots,N. \tag{2.1.51}$$

To see this, the cases $i = 1,\ldots,N$ can be done by an identical computation as (Equation 2.1.23), and the $i = 0$ case follows from $\int_{\partial D_R}\varphi_R\cdot\vec{n}d\sigma = \int_{D_R}\Delta\varphi_R dx = 0$ and the fact that $\partial D_R = \cup_{i=0}^N\Gamma_i$. Thus we have obtained a contradiction between (Equation 2.1.50) and (Equation 2.1.51), completing the proof. □

Proof of Lemma 2.1.14. Assume V has N connected components $\{V_j\}_{j=1}^N$ for $N \ge 1$. For notational simplicity, we shorten $D_{0,R}$, $p_{0,R}$ and $\varphi_{0,R}$ into D_R, p_R and φ_R in this proof. Let $\epsilon_R := \frac{1}{2\pi}|D_R| - \sup_{D_R} p_R$, which is nonnegative by Proposition 2.1.6. Towards a contradiction, assume there exists a diverging subsequence $\{R_i\}_{i=1}^\infty$ such that $\lim_{i\to 0}\epsilon_{R_i} = 0$.

Define $\tilde\varphi_{R_i} := \varphi_{R_i} - \frac{R_i^2}{2}$. We claim that $\{\tilde\varphi_{R_i}\}_{i=1}^\infty$ has a subsequence that converges locally uniformly to some bounded harmonic function φ_∞ in $\mathbb{R}^2\setminus V$.

To show this, we will first obtain a uniform bound of $\{\tilde\varphi_{R_i}\}_{i=1}^\infty$. Note that (Equation 2.1.44) gives that $\sup_{\partial V}|\tilde\varphi_{R_i}| \le \frac{NR_0^2}{2}$ for all $i\in\mathbb{N}^+$. Since $\tilde\varphi_{R_i}\equiv 0$ on ∂B_{R_i} for all $i\in\mathbb{N}^+$, the maximum principle for harmonic function gives $\sup_{D_{R_i}}|\tilde\varphi_{R_i}| \le \frac{NR_0^2}{2}$ for all $i\in\mathbb{N}^+$.

For any $R > 2R_0$, we will obtain a uniform gradient estimate for $\{\tilde\varphi_{R_i}\}$ in D_R for all $R_i > 2R$.

First note that since ∂B_R is in the interior of D_{R_i} (due to $R_i > 2R$), interior estimate for harmonic function (together with the above uniform bound) gives that $\|\tilde{\varphi}_{R_i}\|_{C^2(\partial B_R)} \leq C(N, R_0)$. On the other boundary ∂V, recall that $\tilde{\varphi}_{R_i}|_{\partial V_j} = \frac{|x|^2}{2} + c_{i,j}$, with $|c_{i,j}| \leq \frac{(N+1)R_0^2}{2}$. Thus $\|\tilde{\varphi}_{R_i}\|_{C^2(\partial D_R)} \leq C(N, R_0)$ for all $R_i > 2R$, and the standard elliptic regularity theory gives the uniform gradient estimate $\sup_{D_R} |\nabla\tilde{\varphi}_{R_i}| \leq C(V)$. This allows us to take a further subsequence (which we still denote by $\{\tilde{\varphi}_{R_i}\}$) that converges uniformly in $\bar{D}_R$ to some harmonic function $\tilde{\varphi}_\infty \in C(\bar{D}_R)$. Since $R > 2R_0$ is arbitrary, we can repeat this procedure (for countably many times) to obtain a subsequence that converges locally uniformly to a harmonic function $\tilde{\varphi}_\infty$ in $\mathbb{R}^2 \setminus V$, where $\tilde{\varphi}_\infty|_{\partial V_j} = \frac{|x|^2}{2} + c_j$ with $|c_j| \leq \frac{(N+1)R_0^2}{2}$. This finishes the proof of the claim.

Now define

$$\tilde{p}_{R_i} := p_{R_i} - \frac{R_i^2}{2} = \tilde{\varphi}_{R_i} - \frac{|x|^2}{2},$$

which is known to converge locally uniformly to $\tilde{p}_\infty := \tilde{\varphi}_\infty - \frac{|x|^2}{2}$ in $\mathbb{R}^2 \setminus V$. Note that $\tilde{p}_\infty$ is *not* radially symmetric up to any translation: To see this, recall that $\tilde{p}_\infty|_{\partial V_j} \equiv c_j$. If $\tilde{p}_\infty$ is radial about some x_0, it must be of the form $-\frac{|x-x_0|^2}{2} + c$ due to $\Delta\tilde{p}_\infty = -2$. As a result, the level sets of $\tilde{p}_\infty$ are all nested circles, thus V must be a single disk (where we used that each connected component of V is simply-connected).

Next we will show that $\lim_{i\to 0} \epsilon_{R_i} = 0$ implies $\tilde{p}_\infty$ is radial up to a translation, leading to a contradiction. For $k \in \mathbb{R}$, let $g_i(k) := |\{x \in D_{R_i} : p_{R_i}(x) > k\}|$. In the proof of Proposition 2.1.6, we have shown that $g_i(0) = |D_{R_i}|$, g_i is decreasing in k, with $g_i'(k) \leq -2\pi$ for almost every $k \in (0, \sup_{D_{R_i}} p_{R_i})$. Since $\sup_{D_{R_i}} p_{R_i} = \frac{1}{2\pi}|D_{R_i}| - \epsilon_{R_i}$, we can control $g_i(k)$ from below and above as follows:

$$(|D_{R_i}| - 2\pi k - 2\pi\epsilon_{R_i})_+ \leq g_i(k) \leq (|D_{R_i}| - 2\pi k)_+ \quad \text{for all } k \geq 0. \tag{2.1.52}$$

Likewise, define $\tilde{g}_i(k) := |\{x \in D_{R_i} : \tilde{p}_{R_i}(x) > k\}|$, and $\tilde{g}_\infty(k) := |\{x \in D_{R_i} : \tilde{p}_\infty(x) > k\}|$. Since $\tilde{p}_{R_i} = p_{R_i} - \frac{R_i^2}{2}$, we have $\tilde{g}_i(k) = g_i(k + \frac{R_i^2}{2})$ for all $k \geq -\frac{R_i^2}{2}$, thus (Equation 2.1.52) is equivalent to

$$(-|V| - 2\pi k - 2\pi\epsilon_{R_i})_+ \le \tilde{g}_i(k) \le (-|V| - 2\pi k)_+ \quad \text{for all } k \ge -\frac{R_i^2}{2}.$$

The locally uniform convergence of p_{R_i} gives $\lim_{i\to\infty} \tilde{g}_i = \tilde{g}_0$, and since we assume $\lim_{i\to\infty} \epsilon_{R_i} = 0$, we take the $i \to \infty$ limit in the above inequality and obtain

$$\tilde{g}_\infty(k) = (-2\pi k - |V|)_+ \quad \text{for all } k \in \mathbb{R},$$

which implies

$$\tilde{g}'_\infty(k) = -2\pi \quad \text{for all } k \in (-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty). \tag{2.1.53}$$

Applying the proof of Proposition 2.1.6 to $\tilde{p}_\infty$ (note that the proof still goes through even though $\tilde{p}_\infty$ takes negative values, and is defined in an unbounded domain), we have that (Equation 2.1.53) can happen only if $\tilde{D}_k := \{\tilde{p}_\infty > k\}\dot\cup(\dot\cup_{\{j:c_j>k\}}\overline{V_j})$ is a disk for almost every $k \in (-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty)$, and $|\nabla\tilde{p}_\infty|$ is a constant on almost every $\partial\tilde{D}_k$. These two conditions imply that all regular sets of $\tilde{p}_\infty$ are concentric circles, thus $\tilde{p}_\infty$ is radial up to a translation, and we have obtained a contradiction. □

2.2 Radial symmetry of nonnegative smooth stationary solutions

Let ω be a nonnegative compactly supported smooth stationary solution of the 2D Euler equation. Note that ω being stationary is equivalent with

$$\nabla^\perp\omega \cdot \nabla(\omega * \mathcal{N}) = 0 \quad \text{in } \mathbb{R}^2.$$

As a result, along every regular level set of ω, we have $f := \omega * \mathcal{N}$ is a constant.

In this section, we prove that such ω must be radially symmetric up to a translation. In the proof, the two key steps are to show that every regular level set of ω is a circle, and these circles are concentric. These are done by approximating ω by a step function $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$ such that the sets $\{D_i\}$ are disjoint, and $\|\omega - \omega_n\|_{L^\infty} = O(1/n)$. We then define $\varphi^n = \frac{|x|^2}{2} + \sum_{i=1}^{M_n} 1_{D_i} p_i$ corresponding to this step function ω_n, and compute $\mathcal{I}_n = \int \omega_n \nabla\varphi^n \cdot \nabla f dx$ in two ways.

Due to the $O(1/n)$ error in the approximation, the qualitative statement in Proposition 2.1.6 that "the equality is achieved if and only if D is a disk or annulus" is no longer good enough for us. We need to obtain various quantitative versions of (Equation 2.1.14) for doubly-connected domains, and two such versions are stated below.

In Lemma 2.2.2, the quantitative constant $c_0 > 0$ depends on the Fraenkel asymmetry of the outer boundary defined in Definition 2.2.1.

Definition 2.2.1 (c.f. [42, Section 1.2]). *For a bounded domain $E \subseteq \mathbb{R}^2$, we define the* Fraenkel asymmetry $\mathcal{A}(E) \in [0,1)$ *as*

$$\mathcal{A}(E) := \inf_{x_0} \left\{ \frac{|E\Delta(x_0 + rB)|}{|E|} : x_0 \in \mathbb{R}^2, \pi r^2 = |E| \right\},$$

where B is a unit disk in $\mathbb{R}^2$.

Lemma 2.2.2. *Let D be a doubly connected set. Let us denote the hole of D by an open set h, and let $\tilde{D} := D \cup \bar{h}$. We define p in D as in Lemma 2.1.5. Then if $\mathcal{A}(\tilde{D}) > 0$, there is a constant c_0 that only depends on $\mathcal{A}(\tilde{D})$, such that*

$$p|_{\partial h} \le \frac{|D|}{2\pi}(1 - c_0),$$

Lemma 2.2.2 will be used in the main theorem to show that all level sets of ω are circles. To obtain radial symmetry of ω, we also need to show all these level sets are concentric. To do this, we need to obtain some quantitative lemmas for a region between two non-concentric disks. In Lemma 2.2.3 we consider a thin tubular region between two non-concentric disks whose radii are close to each other, and obtain a quantitative version of (Equation 2.1.14) for such domain.

Lemma 2.2.3. *For $\epsilon > 0$, consider two open disks $B_1 := B(o_1, 1)$ and $B_2 = B(o_2, 1+\epsilon)$ such that $B_1 \subset B_2$. Suppose $|o_1 - o_2| = a\epsilon$ with $a \in (0,1)$, and let p be defined as in Lemma 2.1.5 in $D := \overline{B_2} \setminus B_1$. Then if ϵ and a satisfy that $0 < \epsilon < \frac{a^2}{64}$, we have*

$$p\big|_{\partial B_1} \leq \frac{|D|}{2\pi}\left(1 - \frac{a^2}{16}\right). \tag{2.2.1}$$

In Lemma 2.2.4 we consider a region between two non-concentric disks (that is not necessarily a thin tubular region), and obtain a quantitative version of (Equation 2.1.14) for such domain.

Lemma 2.2.4. *Consider two open disks $B_r := B(o_1, r)$ and $B_R = B(o_2, R)$ such that $B_r \subset B_R$. Let p be defined as in (2.1.5) in $D := B_R \backslash \overline{B_r}$. Suppose $l := |o_1 - o_2| > 0$ and there exist $\delta_1 > 0$, and $\delta_2 > 0$ such that $\delta_1 < r < R < \delta_2$. Then there exist a constant c_0 that only depends on δ_1, δ_2 and l such that*

$$p|_{\partial B_r} \leq \frac{|D|}{2\pi}(1 - c_0).$$

The proofs of the above quantitative lemmas will be postponed to Section subsection 2.2.1. Now we are ready to prove the main theorem.

Theorem 2.2.5. *Let ω be a compactly supported smooth nonnegative stationary solution to the 2D Euler equation. Then ω is radially symmetric up to a translation.*

Proof. Note that as mentioned in step 1 of Proposition 2.1.6, we have that for almost every $k \in (0, \|\omega\|_{L^\infty})$, $\omega^{-1}(\{k\})$ is a smooth 1-manifold. Furthermore, since ω is compactly supported, each such level set is a disjoint union of finite number of simply closed curves. For any such closed curve, we call it a "level set component" in this proof.

We split the proof into several steps. Throughout step 1, 2 and 3, we prove that all level set components of ω must be circles. In step 4, we will prove that any two level set components are nested, i.e. one is contained in the other. Lastly we present the proof that all level set components are concentric in step 5 and 6.

Step 1. Towards a contradiction, suppose there is $k > 0$ that is a regular value of ω, and $\omega^{-1}(\{k\})$ has a connected component Γ that differs from a circle. Recall that $\text{int}(\Gamma)$ denotes the interior of Γ, which is open and simply connected. Since Γ is not a circle, we have $\mathcal{A}(\text{int}(\Gamma)) > 0$, with $\mathcal{A}$ as in Definition 2.2.1.

In this step, we investigate level set components near Γ. Since k is a regular value, we can find an open neighborhood U of Γ and a constant $\eta > 0$ such that $|\nabla\omega| > \eta$ in U. For any $x \in \Gamma$, consider the flow map $\Phi_t(x)$ originating from x, given by

$$\frac{d}{dt}\Phi_t(x) = \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2}$$

with initial condition $\Phi_0(x) = x$. Since $\frac{\nabla\omega}{|\nabla\omega|^2}$ is smooth and bounded in U, we can choose $\delta_1 > 0$ so that $\Phi_t(\Gamma) := \{\Phi_t(x) : x \in \Gamma\}$ lies in U for any $t \in (-\delta_1, \delta_1)$. Note that Φ_t is a 1-parameter group of diffeomorphisms, thus $\Phi_t(\Gamma)$ is also a smooth simply closed curve for $t \in (-\delta_1, \delta_1)$. Then we observe that

$$\frac{d}{dt}\omega(\Phi_t(x)) = \nabla\omega(\Phi_t(x)) \cdot \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2} = 1 \quad \text{for } (t, x) \in (-\delta_1, \delta_1) \times \Gamma. \tag{2.2.2}$$

Hence for each $t \in (-\delta_1, \delta_1)$, $\Phi_t(\Gamma)$ is a level set component and

$$\omega(\Phi_{t_1}(\Gamma)) \neq \omega(\Phi_{t_2}(\Gamma)) \text{ if } t_1 \neq t_2. \tag{2.2.3}$$

By continuity of the map $(t, x) \mapsto \Phi_t(x)$, we can find $\delta_2 \in (0, \delta_1)$ such that

$$\mathcal{A}(\text{int}(\Phi_t(\Gamma))) > \frac{1}{2}\mathcal{A}(\text{int}(\Gamma)) \text{ for any } t \in (-\delta_2, \delta_2). \tag{2.2.4}$$

Since two different level sets cannot intersect, we can assume without loss of generality that $\text{int}(\Phi_{-\delta_2}(\Gamma)) \subset \text{int}(\Phi_{\delta_2}(\Gamma))$. Then it follows from the intermediate value theorem and (Equation 2.2.2) that

$$\text{int}(\Phi_{-\delta_2}(\Gamma)) \subset \Phi_t(\Gamma) \subset \text{int}(\Phi_{\delta_2}(\Gamma)), \text{ for any } t \in (-\delta_2, \delta_2). \tag{2.2.5}$$

Lastly we denote $V := \text{int}(\Phi_{\delta_2}(\Gamma)) \setminus \overline{\text{int}(\Phi_{-\delta_2}(\Gamma))}$ which is a nonempty open doubly connected set, therefore $|V| > 0$.

Step 2. For any integer $n > 1$, we claim that we can approximate ω by a step function ω_n of the form $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$, which satisfies all the following properties.

(a) Each D_i is a connected open domain with smooth boundary and possibly has a finite number of holes.

(b) Each connected component of ∂D_i is a level set component of ω.

(c) $D_i \cap D_j = \emptyset$ if $i \neq j$.

(d) $\|\omega_n - \omega\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2}{n}\|\omega\|_{L^\infty(\mathbb{R}^n)}$.

To construct such ω_n for a fixed $n > 1$, let $r_0 = 0$ and $r_{n+1} = \|\omega\|_{L^\infty}$. We pick $r_1, \ldots, r_n$ to be regular values of ω, such that $0 < r_1 < \cdots < r_n < \|\omega\|_{L^\infty}$, and $r_{i+1} - r_i < \frac{2}{n}\|\omega\|_{L^\infty}$ for $i = 0, \ldots, n$. We denote $D_i := \{x \in \mathbb{R}^2 : r_i < \omega(x) < r_{i+1}\}$ for $i = 1, \ldots, n-1$, and let $D_n := \{x \in \mathbb{R}^2 : \omega(x) > r_n\}$. Thus for each $i = 1, \cdots, n$, D_i is a bounded domain with smooth boundary. We can then write it as $D_i = \dot{\bigcup}_{l=1}^{m_i} D_i^l$ for some $m_i \in \mathbb{N}$ where D_i^l's are connected components of D_i. Then let $\omega_n(x) := \sum_{i=1}^n r_i \sum_{l=1}^{m_i} 1_{D_i^l}$. By relabeling the indices, we rewrite $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$, where $M_n = \sum_{i=1}^n m_i$, and each $\alpha_i \in \{r_1, \ldots, r_n\}$. One can easily check that such ω_n satisfies properties (a)–(d).

Of course, there are many ways to choose the values $r_1, \cdots, r_n$, with each choice leading to a different ω_n. From now on, for any $n > 1$, we fix $\omega_n(x) := \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ as any function constructed in the above way. (Note that α_i and D_i all depend on n, but we omit their n dependence for notational simplicity.)

Finally, let us point out that for $\omega_n(x)$ constructed above, if $D_i \subset V$ for some i, then D_i must be doubly connected, since step 1 shows that all level set components in V are nested curves. We will use this in step 3 and 5.

Step 3. For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed in Step 2. For each D_i, let we define p_i^n in D_i as in Lemma 2.1.5. We set

$$\begin{cases} p^n := \sum_{i=1}^{M_n} p_i^n 1_{D_i} \\ \varphi_i^n := p_i + \frac{|x|^2}{2} \text{ in } D_i \\ \varphi^n := \sum_{i=1}^{M_n} \varphi_i^n 1_{D_i}. \end{cases} \tag{2.2.6}$$

As in Theorem 2.1.7, let $f := \omega * \mathcal{N}$, and we will compute

$$\mathcal{I}^n := \int_{\mathbb{R}^2} \omega_n(x) \nabla \varphi^n(x) \cdot \nabla f(x) dx \tag{2.2.7}$$

in two different ways and derive a contradiction by taking the $n \to \infty$ limit.

On the one hand, the same computation as in (Equation 2.1.22)–(Equation 2.1.23) yields that

$$\mathcal{I}^n = \sum_{i=1}^{M_n} \alpha_i \left(\int_{\partial D_i} f(x) \nabla \varphi_i^n(x) \cdot \vec{n} d\sigma - \int_{D_i} f(x) \Delta \varphi_i^n(x) dx \right) = 0. \tag{2.2.8}$$

On the other hand,

$$
\begin{aligned}
\mathcal{I}^n &= \int_{\mathbb{R}^2} \omega_n(x) x \cdot \nabla f(x) dx + \int_{\mathbb{R}^2} \omega_n(x) \nabla p^n(x) \cdot \nabla f(x) dx \\
&=: \mathcal{I}_1^n + \mathcal{I}_2^n.
\end{aligned}
$$

Since ω_n satisfies property (d) in step 2, it follows that

$$
\lim_{n\to\infty} \mathcal{I}_1^n = \int_{\mathbb{R}^2} \omega(x) x \cdot \nabla f(x) dx.
$$

A similar computation as in (Equation 2.1.24) yields that

$$
\begin{aligned}
\int_{\mathbb{R}^2} \omega(x) x \cdot \nabla f(x) dx &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x)\omega(y) \frac{x \cdot (x-y)}{|x-y|^2} dx dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x)\omega(y) dx dy \\
&= \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2, \qquad (2.2.9)
\end{aligned}
$$

where we used the symmetry of the integration domain to get the second equality.

Now we estimate the limit of $\mathcal{I}_2^n$. By Lemma 2.1.11, we have $\int_{D_i} |\nabla p_i^n|^2 dx \leq \int_{D_i} |x|^2 dx$, hence $\|\omega_n \nabla p\|_{L^2(\mathbb{R}^2)}$ is uniformly bounded. Since $\omega_n \to \omega$ in L^∞, the bounded convergence theorem yields that

$$
\lim_{n\to\infty} \int_{\mathbb{R}^2} \omega_n \nabla p^n \cdot \nabla \left((\omega_n - \omega) * \mathcal{N} \right)(x) dx = 0,
$$

therefore

$$
\liminf_{n\to\infty} \mathcal{I}_2^n = \liminf_{n\to\infty} \int_{\mathbb{R}^2} \omega_n(x) \nabla p^n(x) \cdot \nabla(\omega_n * \mathcal{N}) dx.
$$

From now on, we will omit the n dependence in p_i^n for notational simplicity. Let us break the integral in the right hand side as

$$\begin{aligned}\int_{\mathbb{R}^2} \omega_n(x)\nabla p^n(x)\cdot\nabla(\omega_n * \mathcal{N})dx &= \sum_{i,j=1}^{M_n} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N})dx \\ &= \sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} \nabla p_i \cdot \nabla(1_{D_i} * \mathcal{N})dx + \sum_{i\neq j} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N})dx \\ &=: F_1 + F_2.\end{aligned} \tag{2.2.10}$$

For F_1, the divergence theorem yields

$$F_1 = \sum_{i=1}^{M_n} \alpha_i^2 \left(\int_{\partial D_i} p_i \nabla(1_{D_i} * \mathcal{N}) \cdot \vec{n} d\sigma - \int_{D_i} p_i dx \right) = -\sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} p_i dx, \tag{2.2.11}$$

where the second equality follows from an identical computation as in (Equation 2.1.25). Then by Proposition 2.1.6, we have

$$F_1 \geq -\frac{1}{4\pi} \sum_{i=1}^{M_n} \alpha_i^2 |D_i|^2. \tag{2.2.12}$$

For F_2, the divergence theorem yields

$$F_2 = \sum_{i\neq j} \alpha_i\alpha_j \left(\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma - \int_{D_i} p_i 1_{D_j} dx \right) = \sum_{i\neq j} \alpha_i\alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma,$$

where we use property (c) in step 2 to get the last equality.

For $i \neq j$, recall that as in the proof of Corollary 2.1.8, we denote $j \prec i$ if D_j is contained in a hole of D_i. Then divergence theorem gives

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\sup_{D_i} p_i |D_j| & \text{if } j \prec i. \end{cases} \tag{2.2.13}$$

Next we will improve this inequality for $j \prec i$ and $i \in L$, where $L := \{i : D_i \subset V\}$. (Note that L depends on ω_n, where we omit this dependence for notational simplicity.) From the discussion at the end of step 2, we know that D_i has exactly one hole for all $i \in L$. Using the divergence theorem together with this observation, (Equation 2.2.13) becomes

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\sup_{D_i} p_i |D_j| & \text{if } j \prec i \text{ and } i \notin L, \\ = -p_i|_{\partial_{\text{in}} D_i} |D_j| & \text{if } j \prec i \text{ and } i \in L. \end{cases} \tag{2.2.14}$$

For the second case on the right hand side, we simply use the crude bound $\sup_{D_i} p_i \leq \frac{|D_i|}{2\pi}$ from Proposition 2.1.6. For the third case we can have a better bound: for any $i \in L$, by Lemma 2.2.2 and (Equation 2.2.4), there exists an $\epsilon > 0$ that only depends on $\mathcal{A}(\text{int}(\Gamma))$ (and in particular is independent of n), such that $p_i|_{\partial_{\text{in}} D_i} \leq (\frac{1}{2\pi} - \epsilon)|D_i|$. Thus (Equation 2.2.14) now becomes

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\frac{1}{2\pi}|D_i||D_j| & \text{if } j \prec i \text{ and } i \notin L, \\ \geq -(\frac{1}{2\pi} - \epsilon)|D_i||D_j| & \text{if } j \prec i \text{ and } i \in L. \end{cases} \tag{2.2.15}$$

Now we are ready to estimate F_2. Let us break it into

$$
\begin{aligned}
F_2 &= \sum_{\substack{j \prec i \\ (i,j)\notin L\times L}} \alpha_i\alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N})\cdot \vec{n} d\sigma + \sum_{\substack{j \prec i \\ (i,j)\in L\times L}} \alpha_i\alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N})\cdot \vec{n} d\sigma \\
&\geq - \sum_{\substack{j \prec i \\ (i,j)\notin L\times L}} \alpha_i\alpha_j \frac{1}{2\pi}|D_i||D_j| - \sum_{\substack{j \prec i \\ (i,j)\in L\times L}} \alpha_i\alpha_j \Big(\frac{1}{2\pi} - \epsilon\Big)|D_i||D_j|
\end{aligned}
$$

where the first equality follows from case 1 of (Equation 2.2.15), and the second inequality follows from case 2,3 of (Equation 2.2.15). Finally, by exchanging i with j and taking average with the original inequality, we have

$$
\begin{aligned}
F_2 &\geq -\frac{1}{4\pi} \sum_{\substack{i\neq j \\ (i,j)\notin L\times L}} (\mathbb{1}_{i\prec j} + \mathbb{1}_{j\prec i})\alpha_i\alpha_j|D_i||D_j| - \frac{1}{2} \sum_{\substack{i\neq j \\ (i,j)\in L\times L}} (\mathbb{1}_{i\prec j} + \mathbb{1}_{j\prec i})\alpha_i\alpha_j\Big(\frac{1}{2\pi} - \epsilon\Big)|D_i||D_j| \\
&\geq -\frac{1}{4\pi} \sum_{\substack{i\neq j \\ (i,j)\notin L\times L}} \alpha_i\alpha_j|D_i||D_j| - \frac{1}{2} \sum_{\substack{i\neq j \\ (i,j)\in L\times L}} \alpha_i\alpha_j\Big(\frac{1}{2\pi} - \epsilon\Big)|D_i||D_j| \\
&= -\frac{1}{4\pi}\sum_{i\neq j}\alpha_i\alpha_j|D_i||D_j| + \frac{\epsilon}{2} \sum_{\substack{i\neq j \\ (i,j)\in L\times L}} \alpha_i\alpha_j|D_i||D_j|,
\end{aligned}
\tag{2.2.16}
$$

where the second inequality is due to the fact that for any $i \neq j$, at most one of $i \prec j$ and $j \prec i$ can be true, thus we always have $\mathbb{1}_{i\prec j} + \mathbb{1}_{j\prec i} \leq 1$.

Therefore, from (Equation 2.2.12) and (Equation 2.2.16) it follows that

$$
\begin{aligned}
F_1 + F_2 &\geq -\frac{1}{4\pi}\sum_{i,j=1}^{M_n}\alpha_i\alpha_j|D_i||D_j| + \frac{\epsilon}{2}\sum_{(i,j)\in L\times L}\alpha_i\alpha_j|D_i||D_j| - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2 \\
&= -\frac{1}{4\pi}\left(\sum_{i=1}^{M_n}\alpha_i|D_i|\right)^2 + \frac{\epsilon}{2}\left(\sum_{i\in L}\alpha_i|D_i|\right)^2 - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2.
\end{aligned}
\tag{2.2.17}
$$

Since we will send $n \to \infty$, in the rest of step 3 we will denote L by L^n to emphasize that L depends on ω_n. (In fact α_i and D_i depend on n as well, and we omit the n dependence for them to avoid overcomplicating the notations.)

Note that $\sum_{i\in L^n} \alpha_i 1_{D_i}$ converges to $\omega 1_V$ in $L^1(\mathbb{R}^2)$. Also if $i \in L^n$, then the nondegeneracy of $|\nabla\omega|$ on V yields that $\lim_{n\to\infty}\sup_{i\in L^n}|D_i| = 0$, consequently

$$\lim_{n\to\infty}\sum_{i\in L^n}\alpha_i^2|D_i|^2 \le \|\omega\|_{L^\infty}\lim_{n\to\infty}\sup_{i\in L^n}|D_i|\int_{\mathbb{R}^2}\omega dx = 0.$$

Therefore it follows that

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{I}_2^n &= \liminf_{n\to\infty}(F_1+F_2)\\ &\ge -\lim_{n\to\infty}\frac{1}{4\pi}\left(\sum_{i=1}^{M_n}\alpha_i|D_i|\right)^2 + \lim_{n\to\infty}\frac{\epsilon}{2}\left(\sum_{i\in L^n}\alpha_i|D_i|\right)^2\\ &= -\frac{1}{4\pi}\left(\int_{\mathbb{R}^2}\omega(x)dx\right)^2 + \frac{\epsilon}{2}\left(\int_V\omega(x)dx\right)^2.\end{aligned} \tag{2.2.18}$$

Note that ω is strictly positive in V, due to $|\nabla\omega| > 0$ in V and $\omega \ge 0$ in $\mathbb{R}^2$. Thus from (Equation 2.2.8), (Equation 2.2.9) and (Equation 2.2.18), it follows that

$$0 = \lim_{n\to\infty}\mathcal{I}^n \ge \lim_{n\to\infty}\mathcal{I}_1^n + \liminf_{n\to\infty}\mathcal{I}_2^n \ge \frac{\epsilon}{2}\left(\int_V\omega(x)dx\right)^2 > 0, \tag{2.2.19}$$

which is a contradiction and we have proved that any connected component of a regular level set is a circle.

Step 4. In this step we show that every pair of disjoint level set components are nested. Towards a contradiction, assume that there exist Γ_1 and Γ_2 that are connected components of level sets of regular values of ω, such that Γ_1 and Γ_2 are not nested.

From step 3, we know that Γ_1 and Γ_2 are circles. Then the disks int(Γ_1) and int(Γ_2) are disjoint, and they must be separated by a positive distance since Γ_1 and Γ_2 are level sets of regular values of ω. As in step 1, using the flow map Φ_t originating from the two circles, we can find disjoint open annuli V_1 and V_2 such that $\Gamma_i \subset V_i$ for $i = 1, 2$, and both $\partial_{\text{out}} V_i$ and $\partial_{\text{in}} V_i$ are level set components of ω.

For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed in step 2, and let

$$L_1^n := \{i : D_i \subset V_1\} \quad \text{and} \quad L_2^n := \{i : D_i \subset V_2\}.$$

Let p_i be defined in (Equation 2.2.6) of step 3, and $\mathcal{I}^n$ defined in (Equation 2.2.7). Then on the one hand, the same computations in step 3 give

$$\lim_{n\to\infty} \mathcal{I}^n = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{I}_1^n = \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2. \tag{2.2.20}$$

Let F_1 and F_2 be given by (Equation 2.2.10). For F_1, the estimate (Equation 2.2.12) still holds. For F_2, using (Equation 2.2.13) and Proposition 2.1.6, we have

$$F_2 \geq -\frac{1}{4\pi} \sum_{i \prec j \text{ or } j \prec i} \alpha_i \alpha_j |D_i||D_j|.$$

Since V_1 and V_2 are assumed to be not nested, if $(i, j) \in L_1^n \times L_2^n$ then neither $i \prec j$ nor $j \prec i$. Therefore it follows that

$$F_2 \geq -\frac{1}{4\pi} \sum_{i \neq j} \alpha_i \alpha_j |D_i||D_j| + \frac{1}{4\pi} \sum_{(i,j) \in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j| + \frac{1}{4\pi} \sum_{(j,i) \in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j|.$$

Combining the estimates for F_1 and F_2 yields

$$F_1 + F_2 \geq -\frac{1}{4\pi} \sum_{i,j=1}^{M_n} \alpha_i \alpha_j |D_i||D_i| + \frac{1}{2\pi} \left(\sum_{i \in L_1^n} \alpha_i |D_i| \right) \left(\sum_{i \in L_2^n} \alpha_i |D_i| \right).$$

As $n \to \infty$, since $\sum_{i \in L_1^n} \alpha_i 1_{D_i}$ and $\sum_{i \in L_2^n} \alpha_i 1_{D_i}$ converge to $\omega 1_{V_1}$ and $\omega 1_{V_2}$ respectively in

$L^1(\mathbb{R}^2)$, we have

$$\liminf_{n\to\infty} \mathcal{I}_2^n \geq \lim_{n\to\infty} (F_1 + F_2) = -\frac{1}{4\pi}\left(\int_{\mathbb{R}^2} \omega(x)dx\right)^2 + \frac{1}{2\pi}\left(\int_{V_1} \omega(x)dx\right)\left(\int_{V_2} \omega(x)dx\right). \tag{2.2.21}$$

Combining (Equation 2.2.20) and (Equation 2.2.21) gives us a similar contradiction as in (Equation 2.2.19), except that $\frac{\epsilon}{2}\left(\int_V \omega(x)dx\right)^2$ is now replaced by $\frac{1}{2\pi}\left(\int_{V_1} \omega(x)dx\right)\left(\int_{V_2} \omega(x)dx\right)$. Thus we complete the proof that level sets are nested.

Step 5. In this step, we aim to show that all level set components are concentric within the *same connected component* of $\operatorname{supp}\omega$. This immediately implies that each connected component of $\operatorname{supp}\omega$ is an annulus or a disk, and ω is radially symmetric about its center.

Towards a contradiction, suppose that there are two level set components Γ_{in} and Γ_{out} in the same connected component of $\operatorname{supp}\omega$, such that they are nested circles, but their centers O_{in} and O_{out} do not coincide. We denote their radii by r_{in} and r_{out}, and define

$$U := \overline{\text{int}(\Gamma_{\text{out}})} \setminus \text{int}(\Gamma_{\text{in}}).$$

For an illustration of Γ_{in} and Γ_{out} and U, see Figure Figure 2.2(a).

We claim that ω is uniformly positive in U. Recall that all level set components of ω are nested by step 4. Thus if ω achieves zero in U, the zero level set must be also nested between Γ_{in} and Γ_{out}, since it can be taken as a limit of level set components whose value approaches 0; but this contradicts with the assumption that Γ_{in} and Γ_{out} lie in the same connected component of $\operatorname{supp}\omega$. As a result, we have $\omega_{\min} := \inf_U \omega > 0$.

For a sufficiently large n, let $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be given in step 2, where we further require both Γ_{in} and Γ_{out} coincide with some boundary of D_i. (This is allowed in our construction of ω_n in

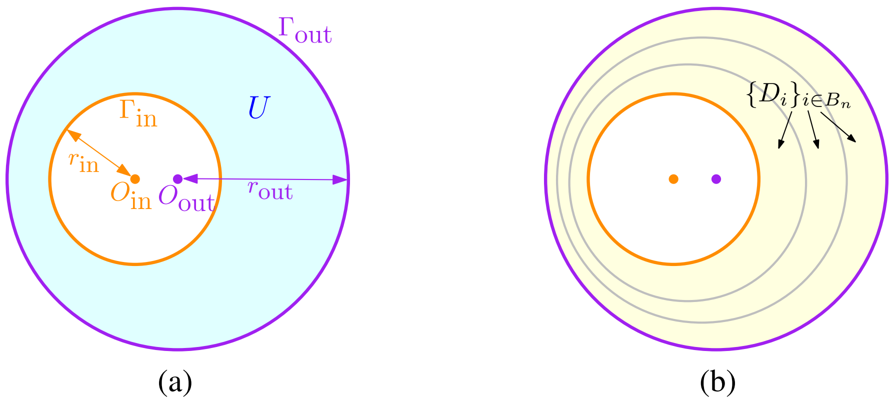


Figure 2.2: (a) Illustration of the circles Γ_{in} and Γ_{out}, whose centers are O_{in} and O_{out}. The set U is colored in blue. (b) For a fixed n, each open set $\{D_i\}_{i\in B_n}$ is colored in yellow. Note that their union gives exactly the set U.

step 2, since ω is regular along both Γ_{in} and Γ_{out}.) Let us denote

$$B_n := \{1 \le i \le M_n : D_i \subset U\},$$

and note that $U := \cup_{i\in B_n} \overline{D}_i$. See Figure Figure 2.2(b) for an illustration of $\{D_i\}_{i\in B_n}$.

As before, we denote $i \prec j$ if D_i is nested in D_j. For the integral $\mathcal{I}^n$ in (Equation 2.2.7), on the one hand, we have $\mathcal{I}^n = 0$ for all $n > 1$ by (Equation 2.2.8). On the other hand, following the same argument as in step 3 up to (Equation 2.2.13) (where we also use that each D_i is already known to be doubly-connected, thus $\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma = -p_i|_{\partial_{\text{in}} D_i}|D_j|$ if $j \prec i$), we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{I}^n &= \liminf_{n\to\infty} \left(\frac{1}{4\pi} \Big(\sum_{i=1}^{M_n} \alpha_i |D_i| \Big)^2 - \sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} p_i dx - \sum_{1\le i,j\le M_n, j\prec i} \alpha_i \alpha_j p_i\big|_{\partial_{\text{in}} D_i} |D_j| \right) \\
&\ge \liminf_{n\to\infty} \underbrace{\left(\sum_{1\le i,j\le M_n,\, j\prec i} \alpha_i \alpha_j \Big(\frac{1}{2\pi} |D_i| - p_i\big|_{\partial_{\text{in}} D_i} \Big) |D_j| \right)}_{=:T_n},
\end{aligned}$$

where in the last step we used Proposition 2.1.6.

Note that Proposition 2.1.6 gives $T_n > 0$, where we have strict positivity, since $O_{\text{in}} \neq O_{\text{out}}$ implies that some $\{D_i\}_{i\in B_n}$ must be non-radial. But since the area of these D_i's may approach 0 as

$n \to \infty$, in order to derive a contradiction after taking $\liminf_{n\to\infty}$, we need to obtain a quantitative estimate for Proposition 2.1.6 for a thin tubular region D_i between two circles, which is done in Lemma 2.2.3.

Next we show that the sets $\{D_i\}_{i\in B_n}$ that are "non-radial to some extent" must occupy a certain portion of U. For $i \in B_n$, denote by o^i_{in} and r^i_{in} the center and radius of $\partial_{\text{in}} D_i$, and likewise o^i_{out} and r^i_{out} the center and radius of $\partial_{\text{out}} D_i$. Note that if D_i is the inner-most set in $\{D_i\}_{i\in B_n}$, then we have $o^i_{\text{in}} = O_{\text{in}}$, and the outmost D_i satisfies $o^i_{\text{out}} = O_{\text{out}}$. In addition, if $\partial_{\text{out}} D_i = \partial_{\text{in}} D_j$ for some $i, j \in B_n$, then $o^i_{\text{out}} = o^j_{\text{in}}$. Thus triangle inequality gives

$$\sum_{i\in B_n} |o^i_{\text{in}} - o^i_{\text{out}}| \geq |O_{\text{in}} - O_{\text{out}}| =: c_0 > 0 \quad \text{for all } n \geq n_0. \tag{2.2.22}$$

In order to apply Lemma 2.2.3 (which requires the region to have inner radius 1), for each $i \in B_n$, consider the scaling

$$\tilde{p}_i(x) := (r^i_{\text{in}})^{-2} p_i(r^i_{\text{in}} x).$$

Then $\tilde{p}_i$ is defined in $\tilde{D}_i := (r^i_{\text{in}})^{-1} D_i$. Due to the scaling, $\tilde{D}_i$ has inner radius 1 (denote the hole by $\tilde{h}_i$), and outer radius $1 + \epsilon_i$, where $\epsilon_i := \frac{r^i_{\text{out}} - r^i_{\text{in}}}{r^i_{\text{in}}} > 0$. In addition, the distance between the centers of $\partial_{\text{in}} \tilde{D}_i$ and $\partial_{\text{out}} \tilde{D}_i$ is $a_i \epsilon_i$, where

$$a_i := \frac{|o^i_{\text{in}} - o^i_{\text{out}}|}{|r^i_{\text{in}} - r^i_{\text{out}}|}.$$

One can also easily check that $\tilde{p}_i$ satisfies $\Delta \tilde{p}_i = -2$ in $\tilde{D}_i$, and $\int_{\partial \tilde{h}_i} \nabla \tilde{p} \cdot \vec{n} d\sigma = -2|\tilde{h}_i| = -2\pi$. By Lemma 2.2.3, if $0 < \epsilon_i < \frac{(a_i)^2}{64}$, then $\tilde{p}_i|_{\partial \tilde{h}_i} \leq \frac{|\tilde{D}_i|}{2\pi}(1 - \frac{a_i^2}{16})$. Thus in terms of p_i, we have

$$p_i\big|_{\partial_{\text{in}} D_i} \leq \frac{1}{2\pi}|D_i| - c_1 a_i^2 |D_i| \quad \text{if } i \in B_n \text{ satisfies } r^i_{\text{out}} - r^i_{\text{in}} \leq c_2 a_i^2, \tag{2.2.23}$$

where $c_1 := \frac{1}{32\pi}$ and $c_2 := \frac{r_{\text{in}}}{64}$ are independent of n and i, due to the fact that $r^i_{\text{in}} \geq r_{\text{in}} > 0$ for all

$i \in B_n$. Using the definition of a_i, (Equation 2.2.22) can be written as

$$\sum_{i \in B_n} a_i |r_{\text{in}}^i - r_{\text{out}}^i| > c_0. \tag{2.2.24}$$

Note that $\sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i|$ satisfies the upper bound

$$\sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i| \leq \frac{|U|}{2\pi r_{\text{in}}} =: M, \tag{2.2.25}$$

which follows from

$$|U| = \sum_{i \in B_n} |D_i| = \pi \sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i| \underbrace{(r_{\text{in}}^i + r_{\text{out}}^i)}_{>2r_{\text{in}}}.$$

Combining (Equation 2.2.24) and (Equation 2.2.25) gives

$$\sum_{i \in B_n} \mathbb{1}_{a_i > \frac{c_0}{2M}} |r_{\text{in}}^i - r_{\text{out}}^i| \geq \sum_{i \in B_n} \Big(a_i - \frac{c_0}{2M}\Big) |r_{\text{in}}^i - r_{\text{out}}^i| \geq \frac{c_0}{2}, \tag{2.2.26}$$

where the first inequality follows from $\mathbb{1}_{a_i > \frac{c_0}{2M}} \geq a_i - \frac{c_0}{2M}$ (recall that $a_i \in (0,1)$), and the second inequality follows from subtracting $\frac{c_0}{2M}$ times (Equation 2.2.25) from (Equation 2.2.24).

Let

$$K_n := \Big\{ i \in B_n : a_i > \frac{c_0}{2M} \Big\}.$$

Using this definition and the fact that $|D_i| > 2\pi r_{\text{in}} |r_{\text{in}}^i - r_{\text{out}}^i|$, (Equation 2.2.26) can be rewritten as

$$\sum_{i \in K_n} |D_i| > 2\pi r_{\text{in}} \sum_{i \in K_n} |r_{\text{in}}^i - r_{\text{out}}^i| \geq \pi r_{\text{in}} c_0. \tag{2.2.27}$$

Now we take a sufficiently large n, and discuss two cases (note that different n may lead to different cases):

Case 1. Every $i \in K_n$ satisfies $r_{\text{out}}^i - r_{\text{in}}^i \leq \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}} c_0}{4 r_{\text{out}}}\}$. By definition of K_n, we have $r_{\text{out}}^i - r_{\text{in}}^i \leq c_2(\frac{c_0}{2M})^2 \leq c_2 a_i^2$ for $i \in K_n$. (The motivation of the second term in the min function will be made clear later.) Then by (Equation 2.2.23), we have

$$\frac{1}{2\pi}|D_i| - p_i\big|_{\partial_{\text{in}} D_i} \geq c_1 a_i^2 |D_i| \geq \frac{c_1 c_0^2}{4M^2}|D_i| \quad \text{for all } i \in K_n.$$

Since K_n is a subset of B_n (and recall that $\alpha_i \geq \omega_{\min} > 0$ for all $i \in B_n$), we have the following lower bound for T_n:

$$\begin{aligned} T_n &\geq \omega_{\min}^2 \sum_{i,j\in K_n,\, j\prec i} \Big(\frac{1}{2\pi}|D_i| - p_i\big|_{\partial_{\text{in}} D_i}\Big)|D_j| \\ &\geq \omega_{\min}^2 \sum_{i,j\in K_n,\, j\prec i} \frac{c_1 c_0^2}{4M^2}|D_i||D_j| = \omega_{\min}^2 \sum_{i,j\in K_n,\, i\neq j} \frac{c_1 c_0^2}{8M^2}|D_i||D_j| \\ &= \omega_{\min}^2 \frac{c_1 c_0^2}{16M^2}\Bigg(\Big(\sum_{i\in K_n}|D_i|\Big)^2 - \sum_{i\in K_n}|D_i|^2\Bigg) \end{aligned} \tag{2.2.28}$$

Note that the second term in the min function in the assumption gives

$$\max_{i\in K_n}|D_i| < 2\pi r_{\text{out}}(r_{\text{out}}^i - r_{\text{in}}^i) \leq \frac{\pi r_{\text{in}} c_0}{2} \leq \frac{1}{2}\sum_{i\in K_n}|D_i|,$$

where we use (Equation 2.2.27) in the last inequality. Applying this to the right hand side of (Equation 2.2.28) gives

$$T_n \geq \omega_{\min}^2 \frac{c_1 c_0^2}{16M^2}\cdot\frac{1}{2}\Big(\sum_{i\in K_n}|D_i|\Big)^2 \geq \omega_{\min}^2\frac{c_1 c_0^2}{32M^2}(\pi r_{\text{in}} c_0)^2.$$

Case 2. If Case 1 is not true, then there must be some $i_0 \in K_n$ satisfying $r_{\text{out}}^{i_0} - r_{\text{in}}^{i_0} > \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}} c_0}{4 r_{\text{out}}}\} =: c_3$, which leads to

$$|o_{\text{in}}^{i_0} - o_{\text{out}}^{i_0}| = a_{i_0}(r_{\text{out}}^{i_0} - r_{\text{in}}^{i_0}) > \frac{c_0 c_3}{2M} =: l.$$

Although this set D_{i_0} is likely not thin enough for us to apply Lemma 2.2.3, since $|o_{\text{in}}^{i_0} - o_{\text{out}}^{i_0}|$ is bounded below by a positive constant independent of n, we can still use Lemma 2.2.4 to conclude that $\frac{1}{2\pi}|D_{i_0}| - p_{i_0}\big|_{\partial_{\text{in}} D_{i_0}} \geq c_4$ for some $c_4 > 0$ only depending on $r_{\text{in}}, r_{\text{out}}$ and l. This leads to

$$T_n \geq \sum_{i_0 \succ j} \omega_{\min} \alpha_j c_4 |D_j| \geq \omega_{\min} c_4 \sum_{D_j \subset \text{int}(\Gamma_{\text{in}})} \alpha_j |D_j| \geq \omega_{\min} c_4 \cdot \frac{1}{2} \int_{\text{int}(\Gamma_{\text{in}})} \omega dx,$$

where the last inequality follows from the fact that for all sufficiently large n, the definition of ω_n gives $\sum_{D_j \subset \text{int}(\Gamma_{\text{in}})} \alpha_j |D_j| = \int_{\text{int}(\Gamma_{\text{in}})} \omega_n dx \geq \frac{1}{2} \int_{\text{int}(\Gamma_{\text{in}})} \omega dx$. Note that the last integral is positive since $\omega > 0$ on Γ_{in}, and it is clearly independent of n.

From the above discussion, for all sufficiently large n, regardless of whether we are in Case 1 or 2 for this n, we always have that T_n is bounded below by some uniformly positive constant independent of n. Therefore taking the $n \to \infty$ limit gives

$$\liminf_{n \to \infty} \mathcal{I}^n \geq \liminf_{n \to \infty} T_n > 0.$$

This contradicts $\mathcal{I}^n = 0$, therefore finishing the proof of step 5.

Step 6. It remains to show that all connected components of $\text{supp}\,\omega$ are concentric. If $\text{supp}\,\omega$ has finitely many connected components, we could proceed similarly as the end of the proof of Corollary 2.1.8. But since $\text{supp}\,\omega$ may have countably many connected components, we need to use a different argument.

Let us denote the connected components of $\text{supp}\,\omega$ by $\{U_i\}_{i \in I}$, where I may have countably many elements. Denote their centers by $\{o_i\}_{i \in I}$, their radii by $\{R_i\}_{i \in I}$, and their outer boundaries by $\{\partial_{\text{out}} U_i\}_{i \in I}$. Without loss of generality, suppose the x-coordinates of their centers $\{o_i^1\}_{i \in I}$ are not all identical.

Among the (possibly infinitely many) collection of circles $\{\partial_{\text{out}} U_i\}_{i \in I}$, let Γ_r be the "circle with rightmost center" among them, in the following sense:

- If there exists some $i_0 \in I$ such that $o_{i_0}^1 = \sup_{i \in I} o_i^1$, we define $\Gamma_r := \partial_{\text{out}} U_{i_0}$. (If the supremum is achieved at more than one indices, we set i_0 to be any of them.)

• Otherwise, take any subsequence $\{i_k\}_{k\in\mathbb{N}} \subset I$ such that $\lim_{k\to\infty} o^1_{i_k} = \sup_{i\in I} o^1_i$. Since ω has compact support, we can extract a further subsequence (which we still denote by $\{i_k\}_{k\in\mathbb{N}}$), such that both o_{i_k} and r_{i_k} converge as $k \to \infty$, and denote their limit by $O_r \in \mathbb{R}^2$ and $R_r \in \mathbb{R}$. Finally let $\Gamma_r := \partial B(O_r, R_r)$.

With the above definition, we point out that $f := \omega * \mathcal{N} = \text{const}$ on Γ_r. Note that in both cases above, we can find a sequence of level set components of ω that converges to Γ_r, in the sense that the Hausdorff distance between the two sets goes to 0. Since $f = \text{const}$ on each level set component of ω, continuity of f gives that $f = \text{const}$ on Γ_r.

Let $f_i(x) := (\omega 1_{U_i}) * \mathcal{N}$ for $i \in I$; note that by definition we have $f = \sum_{i\in I} f_i$. Lemma 2.1.9 gives the following:

(a) For all $x \in (\text{int}(\partial_{\text{out}} U_i))^c$, we have $f_i(x) = \frac{1}{2\pi}\left(\int_{U_i} \omega dx\right) \ln|x - o_i|$.

(b) If U_i is doubly-connected, then $f_i = \text{const}$ in $\text{int}(\partial_{\text{in}} U_i)$, where the constants are different for different i.

Note that for any $i \in I$, U_i must be either nested inside Γ_r, or have Γ_r nested in its hole. (By a slight abuse of notation, we use $i \prec \Gamma_r$ and $i \succ \Gamma_r$ to denote these two relations.) Let $\Gamma^R_r := (O^1_r + R_r, O^2_r)$ and $\Gamma^L_r := (O^1_r - R_r, O^2_r)$ be the rightmost/leftmost point of the circle Γ_r. Note that (b) implies $f_i(\Gamma^R_r) = f_i(\Gamma^L_r)$ for all $i \succ \Gamma_r$, whereas (a) gives the following for all $i \prec \Gamma_r$:

$$f_i(\Gamma^R_r) = \frac{\int_{U_i} \omega dx}{2\pi} \ln|\Gamma^R_r - o_i| \geq \frac{\int_{U_i} \omega dx}{2\pi} \ln|\Gamma^L_r - o_i| = f_i(\Gamma^L_r),$$

where the inequality follows from that $|O^1_r + R_r - o^1_i| \geq |O^1_r - R_r - o^1_i|$, which is a consequence of $o^1_i \leq O^1_r$ due to our choice of O_r. (Also note that Γ^R_r and Γ^L_r have the same y-coordinate.)

As a result, summing over all $i \in I$ gives $f(\Gamma^R_r) \geq f(\Gamma^L_r)$, where the equality is achieved if and only if $o^1_i = O_r$ for all $i \prec \Gamma_r$. Now we discuss two cases:

Case 1. There is some $i \prec \Gamma_r$ with $o^1_i < O_r$. In this case the above discussion gives $f(\Gamma^R_r) >$

$f(\Gamma_r^L)$, which directly leads to a contradiction to $f = \text{const}$ on Γ_r.

Case 2. If case 1 does not hold, then let us define Γ_l as a "circle with leftmost center" among $\{\partial_{\text{out}} U_i\}_{i\in I}$ in the same way as Γ_r. Then we must have $O_l^1 < O_r^1$, and since case 1 does not hold (i.e. all $i \prec \Gamma_r$ satisfy that $o_i^1 = O_r$), we must have $\Gamma_l \succ \Gamma_r$. By definition of Γ_r, there exists some U_{i_0} whose outer boundary is sufficiently close to Γ_r and center sufficiently close to O_r. As a result, $i_0 \prec \Gamma_l$ and $o_{i_0}^1 > O_l^1$.

Let Γ_l^L and Γ_l^R be the leftmost/rightmost point of Γ_l. A parallel argument as above then gives that $f_i(\Gamma_l^L) \geq f_i(\Gamma_l^R)$ for all $i \in I$. Since we have found an $i_0 \prec \Gamma_l$ with $o_{i_0}^1 > O_l^1$, we have $f_{i_0}(\Gamma_l^L) > f_{i_0}(\Gamma_l^R)$, thus summing over all $i \in I$ gives the strict inequality $f(\Gamma_l^L) > f(\Gamma_l^R)$, contradicting with $f = \text{const}$ on Γ_l.

In both cases above we have obtained a contradiction, thus $\{o_i\}_{i\in I}$ must have the same x-coordinate. An identical argument shows that their y-coordinate must also be identical, thus $\{U_i\}_{i\in I}$ are concentric. Since ω is known to be radial within each U_i (about its own center) in step 1–5, the proof is now finished. $\square$

In the next corollary, we show that the above proof for stationary smooth solutions can be extended (with some modifications) to show radial symmetry for rotating patches with $\Omega < 0$.

Corollary 2.2.6. *Let ω be a compactly supported smooth nonnegative rotating solution to the 2D Euler equation, with angular velocity $\Omega < 0$. Then ω is radially symmetric about the origin.*

Proof. The proof is very similar to the proof of Theorem 2.2.5, and we only highlight the differences. Let $\{\omega_n\}$ be the same approximation for ω as in step 2 of Theorem 2.2.5. We consider the same setting as in (Equation 2.2.6) and (Equation 2.2.7), except with $f(x)$ replaced by $f_\Omega(x) := \omega * \mathcal{N} - \frac{\Omega}{2}|x|^2$. From the assumption on ω, we have that f_Ω is a constant on each regular level set component of ω. Thus the same computations in (Equation 2.2.8) give $\mathcal{I}^n = 0$ for all $n > 1$.

On the other hand, we have

$$\mathcal{I}^n = \int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot \nabla\left(\omega * \mathcal{N}\right) dx + (-\Omega) \int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot x dx =: \mathcal{I}_1^n + \underbrace{(-\Omega)}_{\geq 0} \mathcal{I}_2^n. \tag{2.2.29}$$

The same argument as in (Equation 2.1.33) of Theorem 2.1.10 gives that $\mathcal{I}_2^n \geq 0$. As for $\mathcal{I}_1^n$, in step 3 – step 5 of the proof of Theorem 2.2.5, we have already shown that $\liminf_{n\to\infty} \mathcal{I}_1^n \geq 0$, and the equality is achieved if and only if each connected component of $\{\omega > 0\}$ is radially symmetric up to a translation, and they are all nested.

Let us decompose $\operatorname{supp}\omega$ into (possibly infinitely many) connected components $\cup_{i\in I} U_i$, with centers $\{o_i\}_{i\in I}$. Our goal is to show $o_i \equiv (0,0)$ for $i \in I$. Note that it suffices to show that their x-coordinates satisfy $\sup_{i\in I} o_i^1 \leq 0$. Once we prove this, a parallel argument gives $\inf_{i\in I} o_i^1 \geq 0$, which implies $o_i^1 \equiv 0$ for $i \in I$, and the same can be done for the y-coordinate.

Towards a contradiction, suppose $\sup_{i\in I} o_i^1 > 0$. We can then define Γ_r in the same way as step 6 of the proof of Theorem 2.2.5, i.e. it is the "circle with rightmost center" among $\{\partial_{\text{out}} U_i\}_{i\in I}$, and its center O_r satisfies $O_r^1 = \sup_{i\in I} o_i^1 > 0$. Since the new f function takes constant values along each level set component of ω, we again have that $f = \text{const}$ on Γ_r. Let Γ_r^R and Γ_r^L be the rightmost/leftmost points on Γ_r. Note that their distances to the origin satisfy $|\Gamma_r^R| > |\Gamma_r^L|$, where the strict inequality is due to the assumption $O_r^1 > 0$.

Let us define $f_i(x) = (\omega 1_{U_i}) * \mathcal{N}$ for $i \in I$, and note that $f_\Omega = (\sum_{i\in I} f_i) - \Omega|x|^2$. The properties (a,b) in step 6 of Theorem 2.2.5 still hold for f_i, thus we have $f_i(\Gamma_r^R) \geq f_i(\Gamma_r^L)$ for all $i \in I$. This leads to

$$f_\Omega(\Gamma_r^R) = \Big(\sum_{i\in I} f_i(\Gamma_r^R)\Big) + \underbrace{(-\Omega)}_{>0} \big|\Gamma_r^R\big|^2 > \Big(\sum_{i\in I} f_i(\Gamma_r^L)\Big) + (-\Omega)\big|\Gamma_r^L\big|^2 = f_\Omega(\Gamma_r^L),$$

contradicting the fact that $f_\Omega \equiv \text{const}$ on Γ_r. □

2.2.1 Proofs of the quantitative lemmas

Let us start with the proof of Lemma 2.2.2. Let us begin by stating two lemmas that we will use in the proof. The first one is a quantitative version of the isoperimetric inequality obtained by Fusco, Maggi and Pratelli [42].

Lemma 2.2.7 (c.f. [42, Section 1.2])**.** *Let $E \subseteq \mathbb{R}^2$ be a bounded domain. Then there is some constant $c \in (0,1)$, such that*

$$P(E) \geq 2\sqrt{\pi}\sqrt{|E|}\left(1 + c\mathcal{A}(E)^2\right),$$

where $P(E) = \mathcal{H}^1(\partial E)$ denotes the perimeter of E.

The second lemma is a simple result relating the Fraenkel asymmetry of a set E with its subsets U.

Lemma 2.2.8 (c.f. [34, Lemma 4.4])**.** *Let $E \subseteq \mathbb{R}^2$ be a bounded domain. For all $U \subseteq E$ satisfying $|U| \geq |E|(1 - \frac{\mathcal{A}(E)}{4})$, we have*

$$\mathcal{A}(U) \geq \frac{\mathcal{A}(E)}{4}.$$

Proof of Lemma 2.2.2. The proof of the Lemma 2.2.2 is similar to [34, Proposition 4.5] obtained by Craig, Kim and the last author. For the sake of completeness, we give a proof below. Let $g(k)$, D_k and $\tilde{D}_k$ be defined as in Proposition 2.1.6, let $\tilde{D} = D \cup \overline{h}$ and define $p_h := p|_{\partial h}$. We start by following the proof of Proposition 2.1.6, except that after obtaining (Equation 2.1.18), instead of using the isoperimetric inequality, we use the stability version in Lemma 2.2.7 to control $P(\tilde{D}_k)$.

This gives

$$\begin{aligned}
g'(k)\big(g(k)+|h|\mathbb{1}_{p_h>k}\big) &\le -\frac{1}{2}P(\tilde{D}_k)^2 \\
&\le -2\pi|\tilde{D}_k|\left(1+c\mathcal{A}(\tilde{D}_k)^2\right)^2 \\
&\le -2\pi\big(g(k)+|h|\mathbb{1}_{p_h>k}\big)\left(1+c\mathcal{A}(\tilde{D}_k)^2\right).
\end{aligned}$$

Hence it follows from Lemma 2.2.8 that

$$g'(k) \le -2\pi\Big(1+c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big) \text{ for all } k \text{ such that } |\tilde{D}_k| \ge |\tilde{D}|\Big(1-\frac{\mathcal{A}(\tilde{D})}{4}\Big). \tag{2.2.30}$$

We claim that

$$g(k) \le |D| - 2\pi\Big(1+c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k \quad \text{ for } k < \min\Big\{p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\Big\}. \tag{2.2.31}$$

Towards a contradiction, suppose there is $k_0 \le \min\left(p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\right)$ such that (Equation 2.2.31) is violated. Since $1+c\frac{\mathcal{A}(\tilde{D})^2}{16} \le 2$, we have

$$g(k_0) > |D| - 4\pi k_0 \ge |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4},$$

therefore

$$\begin{aligned}
|\tilde{D}_{k_0}| &= g(k_0)+|h| \\
&> |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} + |h| \\
&= |\tilde{D}| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} = |\tilde{D}|\Big(1-\frac{\mathcal{A}(\tilde{D})}{4}\Big).
\end{aligned}$$

Hence for all $k \leq k_0$, $g'(k)$ satisfies the inequality (Equation 2.2.30). Thus we have

$$g(k_0) \leq \int_0^{k_0} -2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)dk + |D|$$
$$= |D| - 2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k_0,$$

contradicting our assumption on k_0.

Finally, to control p_h, we discuss two cases below, depending on which one in the minimum function in (Equation 2.2.31) is smaller. For simplicity, we denote $A := \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}$ and $B := c\frac{\mathcal{A}(\tilde{D})^2}{16}$.

Case 1: $p_h \leq A$. In this case (Equation 2.2.31) holds for all $k \leq p_h$, thus

$$0 \leq g(p_h) \leq |D| - 2\pi\,(1 + B)\,p_h,$$

implying

$$p_h \leq \frac{|D|}{2\pi\,(1+B)} \leq \frac{|D|}{2\pi}(1 - c_0),$$

for some constant c_0 which only depends on $\mathcal{A}(\tilde{D})$.

Case 2: $p_h > A$. In this case (Equation 2.2.31) gives $g(A) \leq |D| - 2\pi(1 + B)A$ and we use a crude bound for $k \geq A$ that is $g'(k) \leq -2\pi$. Therefore for $k > A$,

$$g(k) = \int_A^k g'(k)dk + g(A) \leq -2\pi(k - A) + |D| - 2\pi(1 + B)A$$
$$= |D| - 2\pi k - 2\pi AB$$
$$\leq |D|(1 - \frac{\mathcal{A}(\tilde{D})}{8}B) - 2\pi k,$$

where the last inequality follows from $A > \frac{\mathcal{A}(\tilde{D})|D|}{16\pi}$. Plugging in $k = p_h$ gives

$$0 \le g(p_h) \le |D|(1 - \frac{\mathcal{A}(\tilde{D})}{8}B) - 2\pi p_h$$

leading to

$$p_h \le \frac{|D|}{2\pi}(1 - c_0),$$

again c_0 only depends on $\mathcal{A}(\tilde{D})$. □

Next we prove Lemma 2.2.3.

Proof of Lemma 2.2.3. Without loss of generality, we can assume that $o_1 = (0,0)$ and $o_2 = (a\epsilon, 0)$. To estimate $p|_{\partial B_1}$, we decompose p into

$$p = p\big|_{\partial B_1} g + u,$$

where g satisfies

$$\begin{cases} \Delta g = 0 & \text{in } D \\ g = 1 & \text{on } \partial B_1 \\ g = 0 & \text{on } \partial B_2, \end{cases} \tag{2.2.32}$$

and u satisfies

$$\begin{cases} \Delta u = -2 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{2.2.33}$$

Using this decomposition as well as the definition of p, we have

$$-2|B_1| = \int_{\partial B_1} \nabla p \cdot \vec{n} d\sigma = p\big|_{\partial B_1} \int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma + \int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma,$$

where $\vec{n}$ is the outer-normal of B_1 throughout this proof. Thus

$$p\big|_{\partial B_1} = \frac{1}{\int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma} \left(-2\pi - \int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma \right). \tag{2.2.34}$$

To estimate $p\big|_{\partial B_1}$, it remains to estimate the two integrals in (Equation 2.2.34).

The function g can be explicitly constructed using the conformal mapping from D to a perfect annulus centered at 0. Consider the Möbius map $h : \mathbb{C} \to \mathbb{C}$ given by

$$h(z) := \frac{z+b}{1+bz},$$

where $b \in \mathbb{R}$ will be fixed soon. Note that the unit circle and the real line are both invariant under h, and ∂B_2 is mapped to some circle centered on the real line. In order to make $h(\partial B_2)$ centered at 0, since the left/right endpoints of ∂B_2 are $\pm(1+\epsilon) + a\epsilon$, we look for $b \in \mathbb{R}$ that solves

$$h(1+\epsilon+a\epsilon) = -h(-1-\epsilon+a\epsilon). \tag{2.2.35}$$

Plugging the definition of h into the above equation, we know that b is a root of the quadratic polynomial

$$f(b) := b^2 - \frac{2+(1-a^2)\epsilon}{a} b + 1.$$

Clearly, for $0 < a < 1$, f has two positive roots whose product is 1, thus one is in $(0,1)$ and the other in $(1, +\infty)$. We define b to be the root in $(0,1)$. One can easily check that $f(a) < 0$, and $f(\frac{a}{2}) > 0$ if $a^2 > 2(1-a^2)\epsilon$, which is true due to our assumption $a^2 > 64\epsilon$. Thus for all $\epsilon \in (0, \frac{a^2}{64})$ we have

$$0 < \frac{a}{2} < b < a < 1.$$

Note that h is holomorphic in $\mathbb{C}$ except at the two singularity points $-b$ and $-\frac{1}{b}$. We have already shown that $-b \in B_1$, thus it is outside of D. Next we will show that $-\frac{1}{b} \in B_2^c$, thus is also outside of D. To see this, note that

$$\frac{-1-\epsilon+a\epsilon+b}{1+b(-1-\epsilon+a\epsilon)} = h(-1-\epsilon+a\epsilon) = -h(1+\epsilon+a\epsilon) = -\frac{1+\epsilon+a\epsilon+b}{1+b(1+\epsilon+a\epsilon)} < 0,$$

where the inequality follows from the fact that $a, b, \epsilon > 0$. Since the numerator of the left hand side is already known to be negative due to $a, b \in (0, 1)$, its denominator must be positive, leading to $-\frac{1}{b} < -1-\epsilon+a\epsilon$, i.e. $-\frac{1}{b} \in B_2^c$.

Now we define $g : \mathbb{R}^2 \setminus \{(-b, 0) \cup (-1/b, 0)\} \to \mathbb{R}$ as

$$g(x) := -\frac{1}{\log|h(1+\epsilon+a\epsilon)|} \log|h(z)| + 1 \quad \text{for } z = x_1 + ix_2.$$

Let us check that g indeed satisfies (Equation 2.2.32): first note that g satisfies the boundary conditions in (Equation 2.2.32), since h maps D to a perfect annulus centered at the origin, whose inner boundary is ∂B_1. In addition, g is harmonic in $\mathbb{R}^2 \setminus \{(-b, 0) \cup (-1/b, 0)\}$, thus harmonic in D.

Using the explicit formula of g, we have

$$\Delta g(x) = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|} \Big(\delta_{(-b,0)}(x) - \delta_{(-\frac{1}{b},0)}(x)\Big)$$

in the distribution sense. We can then apply the divergence theorem to g in B_1, and compute the integral containing g in (Equation 2.2.34) explicitly as

$$\int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|}. \tag{2.2.36}$$

As for the integral containing u in (Equation 2.2.34), we compare u with a radial barrier function

$$w(x) := -2(|x|-1)(|x|-1-2\epsilon),$$

which satisfies $w = 0$ on ∂B_1 and $w > 0$ on ∂B_2. Note that

$$\Delta w = \left(\partial_{rr} + \frac{1}{r}\partial_r\right) w = -8 + \frac{4+4\epsilon}{r} \le -2 \quad \text{in } D,$$

where we used that $\epsilon \in (0, \frac{1}{2})$ and $r > 1$ in D in the last inequality. Thus $w - u$ is superharmonic in D and nonnegative on ∂D, which allows us to apply the classical maximum principle to obtain $u \le w$ in $\overline{D}$. Combining this with the fact that $u = w = 0$ on ∂B_1, we have

$$\nabla u(x) \cdot \vec{n}(x) \le \nabla w(x) \cdot \vec{n}(x) = \frac{d}{dr} w(r)\Big|_{r=1} = 4\epsilon \quad \text{for all } x \in \partial B_1,$$

hence

$$\int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma \le 8\pi\epsilon. \tag{2.2.37}$$

Plugging (Equation 2.2.36) and (Equation 2.2.37) into (Equation 2.2.34), we obtain

$$p\big|_{\partial B_1} \le \log(|h(1+\epsilon+a\epsilon)|)(1+4\epsilon).$$

Since $\log s \le s - 1$ for $s > 1$, it follows that

$$\begin{aligned}
\log|h(1+\epsilon+a\epsilon)| \le h(1+\epsilon+a\epsilon) - 1 &= \frac{1+\epsilon+a\epsilon+b}{1+b(1+\epsilon+a\epsilon)} - 1 \\
&= \epsilon\left(1 + \frac{a - 2b - ab - b\epsilon - ab\epsilon}{1+b(1+\epsilon+a\epsilon)}\right) \\
&\le \epsilon\left(1 - \frac{ab}{4}\right) \le \epsilon\left(1 - \frac{a^2}{8}\right),
\end{aligned}$$

where we used $b > \frac{a}{2}$ to obtain the last two inequalities. Finally, using that $\epsilon < \frac{a^2}{64}$, we have

$$p\big|_{\partial B_1} \le \epsilon\left(1 - \frac{a^2}{8}\right)\left(1 + \frac{a^2}{16}\right) \le \epsilon\left(1 - \frac{1}{16}a^2\right) < \frac{|D|}{2\pi}\left(1 - \frac{1}{16}a^2\right),$$

where in the last step we use that $|D| = \pi(1+\epsilon)^2 - \pi > 2\pi\epsilon$. This finishes the proof of the lemma. $\square$

Finally we give the proof of Lemma 2.2.4.

Proof of Lemma 2.2.4. Without loss of generality, we can assume that o_2 is the origin. Let $\beta := p|_{\partial B_r}$. From the proof of Proposition 2.1.6, we already know that $g'(k) \leq -2\pi$, where $g(k) := |\{x \in D : p(x) > k\}|$. This implies that $g(k) \geq -2\pi(k-\beta)$. Therefore we have

$$\int_D p dx = \int_0^\beta g(k) dk \geq \int_0^\beta -2\pi(k-\beta) dk = \pi\beta^2.$$

On the other hand, the same computation in the proof of Lemma 2.1.11 gives

$$\beta|B_r| + \int_D p dx = \frac{1}{2}\int_D |\nabla p|^2 dx \leq \frac{1}{2}\int_D |x|^2 dx.$$

Since

$$\begin{aligned}
\frac{1}{2}\int_D |x|^2 dx &= \frac{1}{2}\left(\int_{B_R} |x|^2 dx - \int_{B_r} |x|^2 dx\right) \\
&= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} + \frac{|B_r|^2}{4\pi} - \frac{1}{2}\int_{B_r} |x|^2 dx \\
&= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2},
\end{aligned}$$

it follows that

$$\pi\beta^2 + \beta|B_r| \leq \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2}. \tag{2.2.38}$$

By solving the quadratic inequality (Equation 2.2.38), we find that

$$\beta \leq \frac{|D|}{2\pi}(1-c_0),$$

for some constant c_0 which only depends on δ_1, δ_2 and l. $\square$

CHAPTER 3

RIGIDITY RESULTS FOR GENERALIZED SQG

3.1 Radial symmetry for stationary/rotating gSQG solutions with $\Omega \le 0$

In this section, we consider the family of gSQG equations with $0 < \alpha < 2$, and study the symmetry property for rotating patch/smooth solutions with angular velocity $\Omega \le 0$.

Let us deal with patch solutions first. As we have discussed in the introduction, we cannot expect a non-simply-connected patch D with $\Omega \le 0$ to be radial, due to the non-radial examples in [68, 49] for $\alpha \in (0, 2)$. For a simply-connected patch D, the constant on the right hand side of (Equation 1.1.6) is the same on ∂D, which motivates us to consider Question 2 in the introduction. The goal of this section is to prove Theorem C, which gives an affirmative answer to Question 2 for the whole range $\alpha \in [0, 2)$.

Our results are not limited to the Riesz potentials $K_{\alpha,d}$ in (Equation 1.1.4); in fact, we only need the potential being radially increasing and not too singular at the origin. Below we state our assumption on the potential K, which covers the whole range of $K_{\alpha,d}$ with $\alpha \in [0, 2)$.

(HK) Let $K \in C^1(\mathbb{R}^d \setminus \{0\})$ be radially symmetric with $K'(r) > 0$ for all $r > 0$. (Here we denote $K(x) = K(r)$ by a slight abuse of notation.) Also assume there is some $\delta > 0$ such that $K'(r) \le r^{-d-1+\delta}$ for all $0 < r \le 1$.

Our proof is done by a variational approach, which relies on a continuous Steiner symmetrization argument in a similar spirit as [18].

3.1.1 Definition and properties of continuous Steiner symmetrization

Below we define the continuous Steiner symmetrization for a bounded open set $D \subset \mathbb{R}^d$ with respect to the direction $e_1 = (1, 0, \ldots, 0)$, which can be easily adapted to any other direction in $\mathbb{R}^d$.

The definition is the same as [18, Section 2.2.1], which we briefly outline below for completeness.

For a *one-dimensional* open set $U \subset \mathbb{R}$, we define its continuous Steiner symmetrization $M^\tau[U]$ as follows. If $U = (a,b)$ is an open interval, then $M^\tau[U]$ shifts the midpoint of this interval towards the origin with velocity 1, while preserving the length of interval. That is,

$$M^\tau[U] := \begin{cases} \big(a - \tau\,\mathrm{sgn}(\frac{a+b}{2}), b - \tau\,\mathrm{sgn}(\frac{a+b}{2})\big) & \text{for } 0 \le \tau < \frac{|a+b|}{2}, \\ (-\frac{b-a}{2}, \frac{b-a}{2}) & \text{for } \tau \ge \frac{|a+b|}{2}. \end{cases}$$

If $U = \cup_{i=1}^N U_i$ is a finite union of open intervals, then $M^\tau[U]$ is defined by $\cup_{i=1}^N M^\tau[U_i]$, and as soon as two intervals touch each other, we merge them into one interval as in [18, Definition 2.10(2)]. Finally, if $U = \cup_{i=1}^\infty U_i$ is a countable union of open intervals, we define $M^\tau[U]$ as a limit of $M^\tau[\cup_{i=1}^N U_i]$ as $N \to \infty$ as in [18, Definition 2.10(3)]. See [18, Figure 1] for an illustration of $M^\tau[U]$.

Next we move on to higher dimensions. We denote a point $x \in \mathbb{R}^d$ by (x_1, x'), where $x' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. For a bounded domain $D \subset \mathbb{R}^d$ and any $x' \in \mathbb{R}^{d-1}$, we define the *section* of D with respect to the direction x_1 as

$$D_{x'} := \{x_1 \in \mathbb{R} : (x_1, x') \in D\},$$

which is an open set in $\mathbb{R}$. If the section $D_{x'}$ is a single open interval centered at 0 for all $x' \in \mathbb{R}^{d-1}$, then we say the set D is *Steiner symmetric* about the hyperplane $\{x_1 = 0\}$. Note that this definition is stronger than being symmetric about $\{x_1 = 0\}$. For example, an annulus in $\mathbb{R}^2$ is symmetric about $\{x_1 = 0\}$, but not Steiner symmetric about it.

Finally, for any $\tau > 0$, the *continuous Steiner symmetrization* of $D \subset \mathbb{R}^d$ is defined as

$$S^\tau[D] := \{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}]\},$$

with M^τ given above being the continuous Steiner symmetrization for one-dimensional open sets. See Figure Figure 3.1 for a comparison of the sets D and $S^\tau[D]$ for small $\tau > 0$.

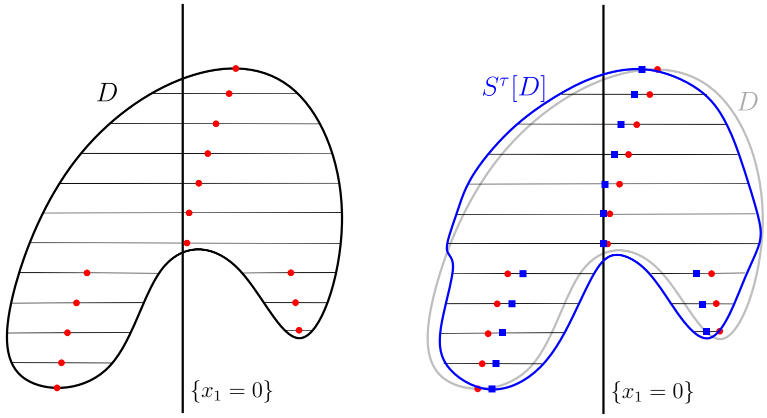


Figure 3.1: Illustration of the continuous Steiner symmetrization $S^\tau[D]$ for a set $D \subset \mathbb{R}^2$. The left figure is the set D, with the midpoints of all subintervals of its 1D section highlighted in red circles. The right figure shows the set $S^\tau[D]$ for some small $\tau > 0$, with the new midpoints denoted by blue squares.

One can easily check that $S^\tau[D]$ satisfies the following properties.

Lemma 3.1.1. *For any bounded open set $D \subset \mathbb{R}^d$, its continuous Steiner symmetrization $S^\tau[D]$ satisfies the following properties:*

(a) $|S^\tau[D]| = |D|$ for any $\tau > 0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$.

(b) $(S^\tau[D]) \triangle D \subset B^\tau[D]$ for any $\tau > 0$, where $\triangle$ is the symmetric difference between the two sets, and $B^\tau[D]$ is the τ-neighborhood of ∂D, given by

$$B^\tau[D] := \{x \in \mathbb{R}^d : \text{dist}(x, \partial D) \leq \tau\}. \tag{3.1.1}$$

Proof. (a) is a direct consequence of the fact that $|M^\tau[U]| = |U|$ for any open set $U \subset \mathbb{R}$ and $\tau > 0$ [18, Lemma 2.11(b)]. To prove (b), one can start with the one-dimensional version: For any bounded open set $U \subset \mathbb{R}$, we have $M^\tau[U] \triangle U \subset \{x \in \mathbb{R} : \text{dist}(x, \partial U) \leq \tau\}$, which follows from the fact that the intervals move with velocity at most 1. Thus for any bounded open set $D \subset \mathbb{R}^d$,

$$
\begin{aligned}
S^\tau[D] \triangle D &= \{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}] \triangle D_{x'}\} \\
&\subset \{(x_1, x') : \text{dist}(x_1, \partial(D_{x'})) \leq \tau\} \\
&\subset B^\tau[D],
\end{aligned}
$$

finishing the proof. □

3.1.2 Simply-connected patch solutions with $\Omega \leq 0$

We assume that $D \subset \mathbb{R}^d$ satisfies the following condition.

(HD) $D \subset \mathbb{R}^d$ is a bounded domain, and there exists some $M > 0$ depending on D, such that $|B^\tau[D]| \leq M\tau$ for all sufficiently small $\tau > 0$, where $B^\tau[D]$ is given in (item 3.1.1).

It can be easily checked that for $d \geq 2$, any bounded domain D with Lipschitz continuous boundary satisfies condition **(HD)**. In fact, for $d = 2$, we will show any domain $D \subset \mathbb{R}^2$ with a rectifiable boundary satisfies **(HD)**, with a precise bound

$$
|B^\tau[D]| \leq 2|\partial D|\tau \quad \text{for all } \tau \geq 0, \tag{3.1.2}
$$

where $|\partial D|$ is the total length of ∂D. Let us first prove (Equation 3.1.2) holds for any polygon $P \subset \mathbb{R}^2$. Erect two polygons at distance τ from P and the transversal sides being bisectors of the inner angles of P (see Figure Figure 3.2). It is clear that $B^\tau[P]$ is contained in the trapezoidal region, which has area no more than $2|\partial P|\tau$. Finally, this can be extended to the general case by approximating any rectifiable curve by polygons.

Below we state our main theorem of this section, which is slightly more general than Theorem Theorem C.

Theorem 3.1.2. *Let $D \subset \mathbb{R}^d$ and $K \in C^1(\mathbb{R}^d \setminus \{0\})$ satisfy the conditions* **(HD)** and **(HK)**

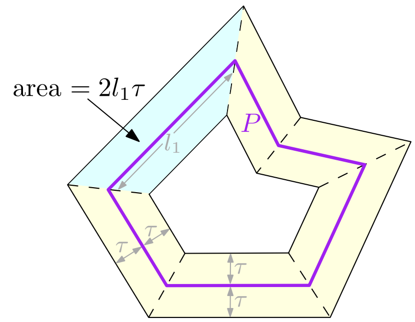


Figure 3.2: Illustration of the polygon P and the underlying trapezoidal region (the whole colored region). Here the blue trapezoid has area $2l_1\tau$ (l_1 is the corresponding side length in P), and summing over all edges gives a total area $2|\partial P|\tau$. Since the trapezoids may intersect for large τ, the whole trapezoidal region has area no more than $2|\partial P|\tau$.

respectively. Let $g \in C^1(\mathbb{R}^d)$ be a radial function with $g'(r) > 0$ for all $r > 0$.

If D satisfies that

$$1_D * K - \frac{\Omega}{2} g(x) = const \quad \text{on } \partial D \tag{3.1.3}$$

for some $\Omega \leq 0$ (where the constant is the same on all connected components of ∂D), then D is a ball. Moreover, the ball is centered at the origin if $\Omega < 0$.

Remark 3.1.3. *(1) Note that D does not need to be simply-connected in Theorem 3.1.2. However, since the constant on the right hand side of* (Equation 3.1.3) *is assumed to be the same on all connected components of ∂D, comparing with* (Equation 1.1.6)*, Theorem 3.1.2 only implies that all simply-connected patches with $\Omega \leq 0$ must be a disk.*

(2) In the case $\Omega = 0$, the problem is translation invariant, so in the proof we assume without loss of generality that the center of mass of D is at the origin.

Proof. We prove it by contradiction. Without loss of generality, we assume D is not Steiner symmetric about the hyperplane $\{x_1 = 0\}$. Let $D^\tau := S^\tau[D]$ be the continuous Steiner symmetrization of D at time $\tau > 0$. By Lemma 3.1.1(b), we have

$$D^\tau \triangle D \subset B^\tau[D], \tag{3.1.4}$$

where B^τ is defined in (item 3.1.1). Let us consider the functional

$$\mathcal{E}[D] := \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_D(x)1_D(y)K(x-y)dxdy}_{=:\mathcal{I}[D]} + \underbrace{(-\Omega)\int_{\mathbb{R}^d} g(x)1_D(x)dx}_{=:\mathcal{V}[D]}.$$

We will use two different ways to compute $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0}$, where $\frac{d^+}{d\tau}$ denotes the right derivative. On the one hand, using the equation (Equation 3.1.3) and the regularity assumptions on D, K and g, we aim to show that

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} = 0. \tag{3.1.5}$$

Instead of directly taking the derivative, we consider the finite difference

$$\mathcal{E}[D^\tau] - \mathcal{E}[D] = \underbrace{\int_{\mathbb{R}^d} 2(1_{D^\tau} - 1_D)\left(1_D * K - \frac{\Omega}{2}g(x)\right)dx}_{=:I_1} + \underbrace{\int_{\mathbb{R}^d}(1_{D^\tau} - 1_D)((1_{D^\tau} - 1_D) * K)dx}_{=:I_2},$$

where in the equality we used that $\int 1_D(1_{D^\tau} * K)dx = \int 1_{D^\tau}(1_D * K)dx$ for any radial kernel K.

Let us control the term I_1 first. First note that (Equation 3.1.4) implies that the integrand is supported in $B_\tau[D]$. Next we claim that **(HK)** implies $1_D * K - \frac{\Omega}{2}g \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' := \min\{\delta, 1\}$, where $C^{0,1}$ stands for Lipschitz continuity. The proof is a simple potential theory estimate, which we provide below for completeness. For any $x, z \in \mathbb{R}^d$,

$$\begin{aligned}
|(1_D * K)(x+z) - (1_D * K)(x)| &= \left|\int_{\mathbb{R}^d} 1_D(x-y)(K(y+z) - K(y))dy\right| \\
&\le \int_{|y|<2|z|} |K(y+z) - K(y)|dy + \int_{|y|>2|z|} 1_D(x-y)|K(y+z) - K(y)|dy \\
&=: J_1 + J_2.
\end{aligned}$$

For J_1, a crude estimate gives

$$J_1 \le \int_{|y|<2|z|} |K(y+z)| + |K(y)|dy \le 2\int_{|y|<3|z|} |K(y)|dy \le C(d)|z|^\delta,$$

where in the last step we used that **(HK)** implies $|K(y)| \leq C|y|^{-d+\delta}$ for $|y| \leq 1$. For J_2, note that **(HK)** and the mean-value theorem gives

$$|K(y+z) - K(y)| \leq C|y|^{-d-1+\delta}|z| \quad \text{for all } |y| > 2|z|,$$

and plugging it into the integral gives $J_2 \leq C(d, |D|)|z|^\delta$. Putting the estimates for J_1 and J_2 together gives that $1_D * K \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' = \min\{\delta, 1\}$, and combining this with the assumption $g \in C^1(\mathbb{R}^d)$ gives $1_D * K - \frac{\Omega}{2} g \in C^{0,\delta'}(\mathbb{R}^d)$.

In addition, by (Equation 3.1.3), We have $1_D * K - \frac{\Omega}{2} g(x) \equiv C_0$ on ∂D for some constant C_0. Thus we have

$$\left| 1_D * K - \frac{\Omega}{2} g(x) - C_0 \right| \leq C(\delta', d, |D|)\tau^{\delta'} \quad \text{in } B^\tau[D]$$

for some constant $C > 0$, where we used the Hölder continuity of $1_D * K - \frac{\Omega}{2} g$ and the definition of $B^\tau[D]$. This leads to

$$|I_1| \leq 2|B^\tau[D]| \sup_{x \in B^\tau[D]} \left| 1_D * K - \frac{\Omega}{2} g(x) - C_0 \right| \leq C(\delta', d, |D|) M \tau^{1+\delta'},$$

where in the first inequality we used that $\int_{B_\tau} (1_D - 1_{D^\tau}) C_0 dx = 0$, which follows from Lemma 3.1.1(a); and in the second inequality we used **(HD)**.

Next we control I_2 by the crude bound

$$|I_2| \leq \int_{\mathbb{R}^d} 1_{B^\tau[D]} |(1_{B^\tau[D]} * K)| dx \leq |B^\tau[D]| \, \|1_{B^\tau[D]} * K\|_\infty \leq M\tau \|(1_{B^\tau[D]})^* * K\|_\infty,$$

where the last step follows from the Hardy–Littlewood inequality, where $(1_{B^\tau[D]})^*$ is the radial decreasing rearrangement of $1_{B^\tau[D]}$. By **(HD)**, $(1_{B^\tau[D]})^*$ is a characteristic function of a ball whose radius is bounded by $C(d)(M\tau)^{1/d}$, thus

$$\|(1_{B^\tau[D]})^* * K\|_\infty \leq \int_0^{C(d)(M\tau)^{1/d}} |K(r)|\,\omega_d r^{d-1}dr \leq \int_0^{C(d)(M\tau)^{1/d}} \omega_d r^{-1+\delta}dx \leq C(d)(M\tau)^{\frac{\delta}{d}},$$

and plugging it into the I_2 estimate gives

$$|I_2| \leq C(d)M^{\frac{d+\delta}{d}}\tau^{1+\frac{\delta}{d}}.$$

Putting the estimates of I_1 and I_2 together directly yields

$$\frac{|\mathcal{E}[D^\tau]-\mathcal{E}[D]|}{\tau} \leq C(\delta', d, M, |D|)\tau^{\min\{\frac{\delta}{d},\delta'\}},$$

and since $\delta > 0$ we have $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} = 0$.

Now, we use another way to calculate $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0}$. Let us deal with the $\Omega < 0$ case first. Since K is radial and increasing in r, it has been shown in [11, Corollary 2] and [78, Theorem 3.7] that the interaction energy $\mathcal{I}[D^\tau] = \int_{D^\tau}\int_{D^\tau} K(x-y)dxdy$ is non-increasing along the continuous Steiner symmetrization, leading to

$$\frac{d^+}{d\tau}\mathcal{I}[D^\tau] \leq 0 \quad \text{ for all } \tau \geq 0.$$

For the other term $\mathcal{V}[D^\tau] = (-\Omega)\int_{D^\tau} g(x)dx$, by the assumptions that $g'(r) > 0$ for all $r > 0$ and D is not Steiner symmetric about $\{x_1 = 0\}$, we can use [18, Lemma 2.22] to show, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[D^\tau]\Big|_{\tau=0} = (-\Omega)\frac{d^+}{d\tau}\int_{D^\tau} g(x)dx\Big|_{\tau=0} < 0.$$

Adding them together gives

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} < 0$$

leading to a contradiction with (Equation 3.1.5).

In the $\Omega = 0$ case, recall that we assume that the center of mass of D is at the origin. Thus if D is not Steiner symmetric about $\{x_1 = 0\}$, the same proof as [18, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time, leading to

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\bigg|_{\tau=0} = \frac{d^+}{d\tau}\int_{D^\tau}\int_{D^\tau} K(|x-y|)dxdy\bigg|_{\tau=0} < 0,$$

again contradicting (Equation 3.1.5). We point out that although the proposition was stated for continuous densities, the same proof works for the patch setting. In addition, although [18] only dealt with the kernels no more singular than Newtonian potential, the proof indeed holds for all kernels K satisfying **(HK)**: see [17, Theorem 6] for an extension to all Riesz potentials $K_{\alpha,d}$ with $\alpha \in (0,2)$. $\square$

The above theorem immediately leads to the following result concerning simply-connected stationary/rotating patch solution with $\Omega \leq 0$.

Theorem 3.1.4. *Let $D \subset \mathbb{R}^2$ be a bounded, simply-connected domain with rectifiable boundary. If 1_D is a V-state for* (Equation 1.1.2) *for some $\alpha \in [0,2)$ with angular velocity $\Omega \leq 0$, then D must be a disk. In addition, the disk must be centered at the origin if $\Omega < 0$.*

Proof. We have $1_D * K - \frac{\Omega}{2}|x|^2 = C$ for some constant C on ∂D. For the Euler equation, $K = \frac{1}{2\pi}\ln|x|$. For the g-SQG equation. $K = -C_\alpha|x|^{-\alpha}$. In both cases, the proof follows from Theorem 3.1.2. $\square$

Remark 3.1.5. *As we discussed in the beginning of this subsection, in the case of gSQG with $\alpha \in (0,2)$, Theorem 3.1.4 is not true if the simply connected assumption is dropped, due to the non-radial patches in [68, 49] bifurcating from annuli.*

3.1.3 Smooth solutions with simply-connected level sets with $\Omega \leq 0$

The rest of this section is devoted to the smooth setting. We will show that any nonnegative smooth rotating solution of the Euler or gSQG equation with angular velocity $\Omega \leq 0$ must be radial, under the additional assumption that all the super level-sets U^h

$$U^h := \{x \in \mathbb{R}^d : \omega(x) > h\} \tag{3.1.6}$$

are simply-connected for any $h > 0$. We believe that the simply-connected assumption is necessary, since it is likely that the bifurcation arguments from annuli in [68, 49] can be extended to the smooth setting as well, using a similar argument as in [24] or [23].

Theorem 3.1.6. *Let $\omega(x) \in C^1(\mathbb{R}^2)$ be nonnegative and compactly supported. In addition, assume the super level-set U^h as in* (Equation 3.1.6) *is simply connected for all $h \in (0, \sup \omega)$. Assume K satisfies* **(HK)**. *If for some $\Omega \le 0$, we have*

$$\omega * K - \frac{\Omega}{2}|x|^2 = C_0(h) \quad \text{on } \partial U^h \text{ for all } h \in (0, \sup \omega), \tag{3.1.7}$$

then ω is radially decreasing up to a translation. Moreover, it is centered at the origin if $\Omega < 0$.

Proof. We prove it by contradiction. For the $\Omega < 0$ case, without loss of generality, we assume ω is not symmetric decreasing about the line $x_1 = 0$. For the $\Omega = 0$ case, similar to Remark 3.1.3, without loss of generality we assume the center of mass is at the origin, and then we assume ω is not symmetric decreasing about the line $x_1 = 0$.

For any $\tau \ge 0$, we define the continuous Steiner symmetrization $\omega^\tau(x)$ in the same way as [18, Definition 2.12]:

$$\omega^\tau(x) := \int_0^{h_0} 1_{S^\tau[U^h]}(x)\, dh,$$

where $h_0 := \sup \omega$, and $S^\tau[U^h]$ is the continuous Steiner symmetrization of the super level set U^h at time $\tau \ge 0$.

Consider the energy functional

$$\mathcal{E}[\omega] := \underbrace{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(x)\omega(y)K(x-y)dxdy}_{=:\mathcal{I}[\omega]} + \underbrace{(-\Omega)\int_{\mathbb{R}^2} \omega(x)|x|^2dx}_{=:\mathcal{V}[\omega]}.$$

We proceed similarly as in Theorem 3.1.2 to compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]$ in two different ways. We first rewrite the finite difference $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega]$ as

$$\begin{aligned}
&\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \\
=\ & \int_{\mathbb{R}^2} 2(\omega^\tau(x) - \omega(x))\left(\omega * K - \frac{\Omega}{2}|x|^2\right)dx + \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\omega^\tau(x)-\omega(x))(\omega^\tau(y)-\omega(y))K(x-y)dxdy \\
=:\ & I_1 + I_2. \qquad (3.1.8)
\end{aligned}$$

Since $\omega \in C_c^1(\mathbb{R}^2)$ and K satisfies **(HK)** (hence is locally integrable), one can easily check that $\omega * K - \frac{\Omega}{2}|x|^2$ is Lipschitz in $\tilde{D} := \{x \in \mathbb{R}^2 : \text{dist}(x, \operatorname{supp}\omega) \le 1\}$. Note that we have $\operatorname{supp}\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Combining this fact with the assumption (Equation 3.1.7), there exists $C_1 > 0$ independent of h, such that

$$\left|(\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\right| \le C_1\tau \quad \text{on } S^\tau[U^h]\triangle U^h \quad \text{for all } h \in (0, h_0). \qquad (3.1.9)$$

Let us first rewrite I_1 as

$$I_1 = 2\int_0^{h_0}\int_{\mathbb{R}^2}\left(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\right)\left((\omega * K)(x) - \frac{\Omega}{2}|x|^2\right)dx\,dh.$$

By Lemma 3.1.1(a), we have $\int_{\mathbb{R}^2}(1_{S^\tau[U^h]}(x) - 1_{U^h}(x))dx = 0$ for all $h \in (0, h_0)$. Thus we can control I_1 as

$$\begin{aligned}
|I_1| &= \left|2\int_0^{h_0}\int_{\mathbb{R}^2}\Big(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\Big)\Big((\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\Big)dxdh\right| \\
&\le 2C_1\tau\int_0^{h_0}\Big|(S^\tau[U^h])\triangle U^h\Big|dh \\
&\le 2C_1\tau\int_0^{h_0} 2|\partial U^h|\tau\, dh \\
&= 4C_1\tau^2\int_{\text{supp}\,\omega}|\nabla\omega|dx \le C(\omega)\tau^2.
\end{aligned} \tag{3.1.10}$$

Here in the second line we used (Equation 3.1.9); in the third line we used Lemma 3.1.1(b) and the property (Equation 3.1.2) in two dimensions; and in the fourth line we used the co-area formula and the fact that $\omega \in C_c^1$.

We next move on to I_2. Since $|\nabla\omega|$ is bounded, Lemma 3.1.1(b) leads to

$$\begin{aligned}
|\omega^\tau(x) - \omega(x)| &= \left|\int_0^\infty 1_{S^\tau[U^h]}(x) - 1_{U^h}(x)dh\right| \\
&\le \|\nabla\omega\|_{L^\infty}\tau \quad \text{ for all } x \in \mathbb{R}^2,
\end{aligned}$$

and supp $\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Thus

$$\begin{aligned}
|I_2| &\le \|\omega^\tau - \omega\|_{L^1}\|(\omega^\tau - \omega) * K\|_{L^\infty} \\
&\le \|\omega^\tau - \omega\|_{L^1}\|\omega^\tau - \omega\|_{L^\infty}\int_{\tilde{D}}|K(x)|dx \\
&\le C(\omega)\tau^2.
\end{aligned}$$

Combining the estimates for I_1 and I_2 gives $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \le C(\omega)\tau^2$ for all $\tau \in [0,1]$, thus

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\Big|_{\tau=0} = \frac{d^+}{d\tau}(I_1 + I_2)\Big|_{\tau=0} = 0. \tag{3.1.11}$$

On the other hand, we compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\big|_{\tau=0}$ in a different way as

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\bigg|_{\tau=0} = \frac{d^+}{d\tau}(\mathcal{I}[\omega^\tau] + \mathcal{V}[\omega^\tau])\bigg|_{\tau=0}.$$

In the $\Omega < 0$ case, similarly as in Theorem 3.1.2, we have $\mathcal{I}[\omega^\tau]$ is non-increasing along the continuous Steiner symmetrization by [11, Corollary 2] and [78, Theorem 3.7], thus

$$\frac{d^+}{d\tau}\mathcal{I}[\omega^\tau] \leq 0 \quad \text{ for all } \tau > 0.$$

For $\mathcal{V}[\omega^\tau]$, by the assumption that ω is not symmetric decreasing about $\{x_1 = 0\}$, we again use [18, Lemma 2.22] to show, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[\omega^\tau] = (-\Omega)\frac{d^+}{d\tau}\int_{\mathbb{R}^2}\omega(x)|x|^2dx\bigg|_{\tau=0} < 0.$$

Adding them together gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0} < 0$, contradicting (Equation 3.1.11).

In the $\Omega = 0$ case, we assume that the center of mass of ω is at the origin. Thus if ω is not symmetric decreasing about $\{x_1 = 0\}$, the same proof as [18, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time (again, the proof holds for all kernels K satisfying **(HK)**; see [17, Theorem 6] for extensions to Riesz kernels $K_{\alpha,d}$ with $\alpha \in (0, 2)$). This gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0} < 0$, again contradicting (Equation 3.1.11). $\square$

The above theorem immediately gives the following corollary concerning the V-states for the Euler and gSQG equations.

Corollary 3.1.7. *Assume $\omega(x) \in C^1(\mathbb{R}^2)$ is a nonnegative, compactly supported V-state satisfying the Euler equation or the gSQG equation for some $\alpha \in (0, 2)$ with $\Omega \leq 0$. In addition, assume the super level-set U^h as in* (Equation 3.1.6) *is simply connected for all $h \in (0, \sup\omega)$. Then ω must be radially decreasing if $\Omega < 0$, and radially decreasing up to a translation if $\Omega = 0$.*

Proof. For the Euler equation, $K = \frac{1}{2\pi}\ln|x|$. For the gSQG equation. $K = -C_\alpha|x|^{-\alpha}$. In both cases, the proof follows from Theorem 3.1.6. $\square$

3.2 Radial symmetry of rotating gSQG solutions with $\Omega > \Omega_\alpha$

In this section, we focus on rotating gSQG patches with area π and $\alpha \neq 0$. As we discussed in the introduction, for $\alpha \in [0,2)$, there exist rotating patches bifurcating from the unit disk at angular velocities $\Omega_m^\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(m+\frac{\alpha}{2}\right)}{\Gamma\left(m+1-\frac{\alpha}{2}\right)}\right)$, where Ω_m^α is increasing in m for any fixed $\alpha \in [0,2)$. Let us denote $\Omega_\alpha := \lim_{m\to\infty}\Omega_m^\alpha$. If $\alpha \in (0,1)$ we have that

$$\Omega_\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)}. \tag{3.2.1}$$

Note that Ω_α is a continuous function of α for $\alpha \in (0,1)$, with $\Omega_0 = \frac{1}{2}$, and $\Omega_\alpha = +\infty$ for all $\alpha \in [1,2)$.

A natural question is whether there can be rotating patches with area π with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$. Note that the area constraint is necessary for all $\alpha > 0$: if D is a rotating gSQG patch for $\alpha \in (0,2)$ with angular velocity Ω, then one can easily check that its scaling $\lambda D = \{\lambda x : x \in D\}$ is a rotating patch with angular velocity $\lambda^{-\alpha}\Omega$.

In Theorem 2.1.12, we showed that for the 2D Euler case ($\alpha = 0$), all rotating patches with $\Omega \geq \Omega_0 = \frac{1}{2}$ must be a disk. In this section, our goal is to show that all *simply-connected* rotating patches with area π with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$ must be a disk. Whether there exist non-simply-connected or disconnected rotating patches with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$ is still an open question.

Below is the main theorem of this section. Recall that for $\alpha \in (0,2)$, $K_\alpha = -C_\alpha|x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, where $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$.

Theorem 3.2.1. *Let $D \subset \mathbb{R}^2$ be a bounded, simply-connected patch with C^1 boundary. Let us denote $R := \max_{x\in D}|x|$. Assume that D is a uniformly rotating patch with angular velocity Ω of* *the gSQG equation with $\alpha \in (0,1)$, i.e.,*

$$1_D * K_\alpha - \frac{\Omega}{2}|x|^2 = C \quad \text{on } \partial D. \tag{3.2.2}$$

Let $\Omega_c(R) := R^{-\alpha}\Omega_\alpha$. If $\Omega \geq \Omega_c(R)$, then D must coincide with $B(0,R)$.

Remark 3.2.2. *(a) Note that all sets $D \subset \mathbb{R}^2$ with area π must have $R \geq 1$. In this case we have $\Omega_c(R) \leq \Omega_\alpha$, thus Theorem 3.2.1 immediately implies that all simply-connected rotating patches with area π and $\Omega \geq \Omega_\alpha$ must be a disk.*

(b) Note that the constant Ω_α is sharp, since there exist patches bifurcating from a disk of radius 1 at velocities Ω_m^α, which can get arbitrarily close to Ω_α as $m \to \infty$ [57, Theorem 1.4].

Proof. Towards a contradiction, assume that $D \neq B(0,R)$. Let $x_0 \in \partial D$ be the farthest point from 0. Then we have that $D \subset B(0,R)$, and let $U := B(0,R) \setminus D$. See Figure Figure 3.3 for an illustration of U and x_0. Then (Equation 3.2.2) can be rewritten as

$$1_U * K_\alpha = 1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2 - C \quad \text{on } \partial D. \tag{3.2.3}$$

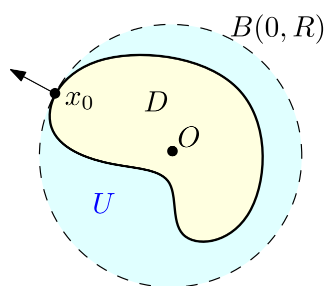


Figure 3.3: Illustration of the set U and the point x_0.

The key idea of this proof is to use two different ways to compute $\nabla(1_U * K_\alpha)(x_0) \cdot x_0$, and obtain a contradiction if $\Omega \geq \Omega_c(R)$. On the one hand,

$$\nabla(1_U * K_\alpha)(x_0) \cdot x_0 = \alpha C_\alpha \int_U \frac{(x_0 - y) \cdot x_0}{|x_0 - y|^{\alpha+2}} dy > 0, \tag{3.2.4}$$

where we used the fact that $(x_0 - y) \cdot x_0 > 0$ for all $y \in U \subset B(0, R)$ since the two vectors point to the same halfplane.

On the other hand, we claim the following properties hold for $1_U * K_\alpha$:

1. $\Delta(1_U * K_\alpha) < 0$ in D.

2. Along ∂D, the minimum of $1_U * K_\alpha$ is achieved at x_0.

To show property 1, using the fact that $K_\alpha = -C_\alpha |x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, we have $1_U * K_\alpha = -(-\Delta)^{-1+\frac{\alpha}{2}} 1_U$, thus $\Delta(1_U * K_\alpha) = (-\Delta)^{\alpha/2} 1_U$. Thus for any $x \in D$, using the singular integral definition of the fractional Laplacian [77, Theorem 1.1, Definition (e)] and the fact that $1_U \equiv 0$ in D, we have

$$(-\Delta)^{\alpha/2} 1_U(x) = C_1(\alpha) \int_{\mathbb{R}^2} \frac{1_U(x) - 1_U(y)}{|x - y|^{2+\alpha}} dy = C_1(\alpha) \int_{\mathbb{R}^2} \frac{0 - 1_U(y)}{|x - y|^{2+\alpha}} dy < 0 \quad \text{for } x \in D$$

for some constant $C_1(\alpha) > 0$. Note that despite the denominator being singular, the integral indeed converges for all $x \in D$, due to the fact that D is open and the integrand is identically zero in D which yields

$$\Delta(1_U * K_\alpha)(x) = (-\Delta)^{\alpha/2} 1_U(x) < 0 \text{ in } D.$$

We now move on to property 2. Due to (Equation 3.2.3) and the fact that x_0 is the outmost point on ∂D, it suffices to show that the radial function $1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2$ is non-increasing in $|x|$ for all $\Omega \geq \Omega_c(R)$. We prove this in Proposition 3.2.3 right after this theorem.

The above claims allow us to apply the maximum principle to $1_U * K_\alpha$, which yields that the minimum of $1_U * K_\alpha$ in $\overline{D}$ is also achieved at x_0, thus

$$\nabla(1_U * K_\alpha)(x_0) \cdot \vec{n}(x_0) \leq 0,$$

where $\vec{n}(x_0)$ is the outer normal of D at x_0. Since $\vec{n}(x_0) = x_0/|x_0|$, the above inequality contradicts with (Equation 3.2.4). As a result, D must coincide with $B(0, R)$. $\square$

Now we prove the proposition that was used in the proof of the above theorem.

Proposition 3.2.3. *For a fixed $\alpha \in (0, 1)$ and $R > 0$, let $\Omega_c(R)$ be the smallest number such that*

$$g_R(x) := 1_{B(0,R)} * K_\alpha - \frac{\Omega_c}{2}|x|^2$$

is non-increasing in $|x|$. Then we have $\Omega_c(R) = R^{-\alpha}\Omega_\alpha$, with Ω_α given in (Equation 3.2.1).

Proof. Recall that $K_\alpha = -C_\alpha |x|^{-\alpha}$ with $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$. Since $|x|^2$ and $1_{B(0,R)} * K_\alpha$ are both radially symmetric and increasing in $|x|$, we have

$$\Omega_c(R) = 2C_\alpha \sup_{|x_1| \neq |x_2|} \frac{\displaystyle\int_{B(0,R)} |x_2 - y|^{-\alpha} dy - \int_{B(0,R)} |x_1 - y|^{-\alpha} dy}{|x_1|^2 - |x_2|^2}.$$

Let us denote the fraction above by $F(x_1, x_2)$. We claim that the $\sup_{|x_1| \neq |x_2|} F(x_1, x_2)$ is attained when $|x_1| = R$, and $|x_2| \to R$.

To prove the claim, we first compute $I(x) := \int_{B(0,R)} |x - y|^{-\alpha} dy$. Taking the Fourier transform:

$$I(x) = CR^{2-\alpha} \int_0^\infty r^{a-2} J_1(r) J_0\left(\frac{|x|r}{R}\right) dr,$$

where C is some positive constant. By Sonine-Schafheitlin's formula [115, p. 401] and by continuity, we obtain

$$I(x) = \begin{cases} CR^{2-\alpha}2^{\alpha-2}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)}\,{}_2F_1\left(\frac{\alpha}{2}-1,\frac{\alpha}{2},1,\frac{|x|^2}{R^2}\right) & \text{if } |x| \leq R \\ CR^{2-\alpha}2^{\alpha-2}|x|^{-\alpha}R^{\alpha}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\,{}_2F_1\left(\frac{\alpha}{2},\frac{\alpha}{2},2,\frac{R^2}{|x|^2}\right) & \text{if } |x| > R. \end{cases}$$

By the mean value theorem, it is enough to check that $\min J(z) = J(R^2)$, where

$$J(z) = \begin{cases} \frac{d}{dz}\left({}_2F_1\left(\frac{\alpha}{2}-1,\frac{\alpha}{2},1,\frac{z}{R^2}\right)\right) & \text{if } z \leq R^2 \\ \frac{d}{dz}\left(\left(1-\frac{\alpha}{2}\right)z^{-\frac{\alpha}{2}}R^{\alpha}\,{}_2F_1\left(\frac{\alpha}{2},\frac{\alpha}{2},2,\frac{R^2}{z}\right)\right) & \text{if } z > R^2 \end{cases}$$
$$= \begin{cases} \frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}\,{}_2F_1\left(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{z}{R^2}\right) & \text{if } z \leq R^2 \\ \frac{\alpha(\alpha-2)}{4}z^{-1-\frac{\alpha}{2}}R^{\alpha}\,{}_2F_1\left(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{R^2}{z}\right) & \text{if } z > R^2 \end{cases}$$

Writing the series expansion (respectively at $z=0$ and $z=\infty$) of the hypergeometric series:

$$\frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}\,{}_2F_1\left(\frac{\alpha}{2},1+\frac{\alpha}{2},2,z\right) = \frac{1}{R^2}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(n+1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma(1+n)\Gamma(2+n)}\left(\frac{z}{R^2}\right)^n$$
$$\frac{\alpha(\alpha-2)}{4}z^{-1-\frac{\alpha}{2}}R^{\alpha}\,{}_2F_1\left(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{1}{z}\right) = \left(\frac{1}{z}\right)^{\frac{\alpha}{2}}R^{\alpha-2}\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(n-1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma(n)\Gamma(n+1)}\left(\frac{R^2}{z}\right)^n,$$

which are both minimized at $z = R^2$ since every coefficient is negative. This proves the claim.

The claim immediately implies

$$\Omega_c(R) = -\frac{C_\alpha}{R}\frac{d}{d|x|}\int_{B(0,R)}|x-y|^{-\alpha}dy\bigg|_{|x|=R}, \tag{3.2.5}$$

Where $\frac{d}{d|x|}$ denotes the derivative in the radial variable (recall that $\int_{B(0,R)}|x-y|^{-\alpha}dy$ is radially symmetric). To compute the derivative at $|x| = R$, we can simply compute the partial derivative in the x_1 direction at the point $(R,0)$:

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\int_{B(0,R)}|x-y|^{-\alpha}dy\bigg|_{x=(R,0)} = -\alpha\int_{B(0,R)}\left((R-y_1)^2+y_2^2\right)^{-\frac{\alpha}{2}-1}(R-y_1)dy_1dy_2\\
&= -2\int_0^R\left((R-y_1)^2+{y_2}^2\right)^{-\frac{\alpha}{2}}\Big|_{y_1=-\sqrt{R^2-{y_2}^2}}^{y_1=\sqrt{R^2-{y_2}^2}}dy_2\\
&= -2^{1-\frac{\alpha}{2}}R^{1-\alpha}\left(\int_0^1\left(1-\sqrt{1-u^2}\right)^{-\frac{\alpha}{2}}du-\int_0^1\left(1+\sqrt{1-u^2}\right)^{-\frac{\alpha}{2}}du\right) \qquad (3.2.6)\\
&= -2^{1-\frac{\alpha}{2}}R^{1-\alpha}\left(\int_0^{\frac{\pi}{2}}(1-\cos\theta)^{-\frac{\alpha}{2}}\cos\theta\,d\theta-\int_0^{\frac{\pi}{2}}(1+\cos\theta)^{-\frac{\alpha}{2}}\cos\theta\,d\theta\right)\\
&=: -2^{1-\frac{\alpha}{2}}R^{1-\alpha}(I_1-I_2),
\end{aligned}$$

where in the third line we used the identity $(R\pm\sqrt{R^2-y_2^2})^2+y_2^2=2R^2(1\pm\sqrt{1-(R^{-1}y_2)^2})$, as well as the substitution $u=R^{-1}y_2$.

Using a substitution $\theta=2\beta$, we rewrite I_1 as

$$I_1=2\int_0^{\frac{\pi}{4}}(1-\cos(2\beta))^{-\frac{\alpha}{2}}\cos(2\beta)\,d\beta=2^{1-\frac{\alpha}{2}}\int_0^{\frac{\pi}{4}}(\sin\beta)^{-\alpha}(1-2\sin^2\beta)\,d\beta.$$

Likewise, the substitution $\theta=\pi-2\beta$ allows us to rewrite $-I_2$ as

$$-I_2=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\cos(2\beta))^{-\frac{\alpha}{2}}\cos(2\beta)\,d\beta=2^{1-\frac{\alpha}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin\beta)^{-\alpha}(1-2\sin^2\beta)\,d\beta.$$

Adding the above two identities for I_1 and $-I_2$ together gives

$$\begin{aligned}
I_1-I_2&=2^{1-\frac{\alpha}{2}}\int_0^{\frac{\pi}{2}}(\sin\beta)^{-\alpha}(1-2\sin^2\beta)\,d\beta\\
&=2^{1-\frac{\alpha}{2}}\left(\frac{1}{2}B\Big(\frac{1-\alpha}{2},\frac{1}{2}\Big)-B\Big(\frac{3-\alpha}{2},\frac{1}{2}\Big)\right)\\
&=2^{-\frac{\alpha}{2}}\frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(1-\frac{\alpha}{2})}-2^{1-\frac{\alpha}{2}}\frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(2-\frac{\alpha}{2})},
\end{aligned}$$

where B stands for the beta function. Here the second identity follows from the property that

$B(x,y) = 2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta$, and the third line follows from the property that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. According to the properties of the gamma function $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, we have

$$I_1 - I_2 = 2^{-1+\frac{\alpha}{2}}\frac{\alpha}{2-\alpha}\cdot\frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}. \tag{3.2.7}$$

Finally, plugging this into (Equation 3.2.6) and (Equation 3.2.5) gives

$$\begin{aligned}\Omega_c(R) &= R^{-\alpha}C_\alpha 2^{1-\frac{\alpha}{2}}(I_1 - I_2)\\ &= R^{-\alpha}\frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}\frac{\alpha}{2-\alpha}\left(\frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}\right)\\ &= R^{-\alpha}\frac{2^{\alpha-1}\Gamma(1-\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(1-\frac{\alpha}{2})^2\Gamma(2-\frac{\alpha}{2})} = R^{-\alpha}\Omega_\alpha,\end{aligned}$$

finishing the proof. □

At the end of this section, we point out that Theorem 3.2.1 directly gives the following quantitative estimate: if a simply-connected patch D rotates with angular velocity $\Omega \in (0,\Omega_\alpha)$ that is very close to Ω_α, then D must be very close to a disk in terms of symmetric difference.

Corollary 3.2.4. *Assume* $0 < \alpha < 1$. *Let* D *be a rotating patch with area* π *and angular velocity* $\Omega \in (0,\Omega_\alpha)$, *and let* B *be the unit disk. Then we have*

$$|D\triangle B| \le 2\pi\left(\left(\frac{\Omega_\alpha}{\Omega}\right)^{2/\alpha} - 1\right).$$

Note that for a fixed $\alpha \in (0,1)$, *the right hand side goes to 0 as* $\Omega \nearrow \Omega_\alpha$.

Proof. Denote $R := \max_{x\in D}|x|$. If D is a rotating patch with angular velocity Ω and is not a disk, Theorem 3.2.1 gives that $\Omega \le R^{-\alpha}\Omega_\alpha$, which gives that $R \le (\frac{\Omega_\alpha}{\Omega})^{1/\alpha}$. Thus $D \subset B(0,(\frac{\Omega_\alpha}{\Omega})^{1/\alpha})$, which implies that the symmetric difference $D \triangle B$ satisfies

$$|D \triangle B| = 2|D \setminus B| \leq 2 \left| B\Big(0, (\frac{\Omega_\alpha}{\Omega})^{1/\alpha}\Big) \setminus B \right| = 2\pi \left(\Big(\frac{\Omega_\alpha}{\Omega}\Big)^{2/\alpha} - 1 \right).$$

$\square$

CHAPTER 4
RIGIDITY RESULTS FOR VORTEX SHEETS

In this chapter, we study radial symmetry of stationary/uniformly-rotating vortex sheets. We fix the following notations in this chapter.

Notations For a bounded domain $D \subset \mathbb{R}^2$, we denote $|D|$ by its area (i.e. its Lebesgue measure). For $x \in \mathbb{R}^2$ and $r > 0$, denote by $B(x, r)$ or $B_r(x)$ the open ball centered at x with radius r.

Through section 4.2-section 4.3, we will desingularize the vortex sheet into a vortex layer with width $\sim \epsilon$, and obtain various quantitative estimates. In all these estimates, we say a term f is $O(g(\epsilon))$ if $|f| \leq Cg(\epsilon)$ for some constant C independent of ϵ.

For a domain $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \vec{f} \cdot n d\sigma$, n denotes the outer normal of the domain U.

4.1 Equations for a stationary/rotating vortex sheet

Let $\omega(\cdot, t) = \omega_0(R_{\Omega t})$ be a stationary/rotating vortex sheet solution to the incompressible 2D Euler equation, where $\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ is a Radon measure. Here $\Omega = 0$ corresponds to a stationary solution, and $\Omega \neq 0$ corresponds to a rotating solution. Assume ω_0 is concentrated on Γ, which is a finite disjoint union of curves. Throughout this paper we assume Γ satisfies the following:

(H1) Each connected component of Γ is smooth and with finite length, and it is either a simple closed curve (denote them by $\Gamma_1, \dots, \Gamma_n$), or a non-self-intersecting curve with two endpoints (denote them by $\Gamma_{n+1}, \dots, \Gamma_{n+m}$). Here we require $n + m \geq 1$, but allow either n or m to be 0.

Let us denote

$$d_\Gamma := \min_{k \neq i} \operatorname{dist}(\Gamma_i, \Gamma_k), \tag{4.1.1}$$

which is strictly positive since we assume the curves $\{\Gamma_i\}_{i=1}^{n+m}$ are disjoint. For $i = 1, \dots, n+m$, denote by L_i the length of Γ_i. Let $z_i : S_i \to \Gamma_i$ denote a constant-speed parameterization of Γ_i (in counter-clockwise direction if Γ_i is a closed curve), where the parameter domain S_i is given by

$$S_i := \begin{cases} \mathbb{R}/\mathbb{Z} & \text{for } i = 1, \dots, n, \\ [0,1] & \text{for } i = n+1, \dots, n+m. \end{cases}$$

Note that this gives $|z_i'| \equiv L_i$, and the arc-chord constant

$$F_\Gamma := \max_{i=1,\dots,n+m} \sup_{\alpha \neq \beta \in S_i} \frac{|\alpha - \beta|}{|z_i(\alpha) - z_i(\beta)|} \tag{4.1.2}$$

is finite, since Γ is non-self-intersecting. Let $\mathbf{s} : \Gamma \to \mathbb{R}^2$ be the unit tangential vector on Γ, given by $\mathbf{s}(z_i(\alpha)) := \frac{z_i'(\alpha)}{|z_i'(\alpha)|} = \frac{z_i'(\alpha)}{L_i}$, and $\mathbf{n} : \Gamma \to \mathbb{R}^2$ be the unit normal vector, given by $\mathbf{n} = \mathbf{s}^\perp$. See Figure Figure 4.1 for an illustration.

For $i = 1, \dots, n+m$, let us denote by $\gamma_i(\alpha)$ the vorticity strength at $z_i(\alpha)$ with respect to the arclength parametrization, which is related to $\varpi_i(\alpha)$ by

$$\gamma_i(\alpha) = \frac{\varpi_i(\alpha)}{|z_i'(\alpha)|} \quad \text{for } \alpha \in S_i. \tag{4.1.3}$$

Throughout this paper we will be working with γ_i, instead of ϖ_i. We impose the following regularity and positivity assumptions on γ_i:

(H2) Assume that $\gamma_i \in C^2(S_i)$ for $i = 1, \dots, n$ and $\gamma_i \in C^b(S_i) \cap C^1(S_i^\circ)$ for some $b \in (0,1)$ for $i = n+1, \dots, n+m$.[1]

(H3) For $i = 1, \dots, n$, assume $\gamma_i > 0$ in S_i. And for $i = n+1, \dots, n+m$, assume $\gamma_i > 0$ in S_i°, and $\gamma_i(0) = \gamma_i(1) = 0$.

[1]For an open curve $i = n+1, \dots, n+m$, note that **(H2)** does not require γ_i to be C^1 up to the boundary of S_i, and its derivative is allowed to blow up at the endpoints. This is motivated by the fact that in the explicit uniformly-rotating solution (Equation 1.2.3), its strength γ is Hölder continuous in $[-a, a]$ and smooth in the interior, but its derivative blows up at the endpoints.

Note that for a closed curve, **(H3)** implies that γ_i is uniformly positive; whereas for an open curve, γ_i is positive in the interior of S_i but vanishes at its endpoints. This is because any stationary/rotating vortex sheet with continuous γ_i must have it vanishing at the two endpoints of any open curve: if not, one can easily check that $|BR(z_i(\alpha)) \cdot \mathbf{n}(z_i(\alpha))| \to \infty$ as α approaches the endpoint, thus such a vortex sheet cannot be stationary in the rotating frame.

With the above notations of z_i and γ_i, the Birkhoff-Rott integral (Equation 1.1.10) along the sheet can now be expressed as

$$BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha)) := \sum_{k=1}^{n+m} PV \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha'))\, \gamma_k(\alpha')|z_k'(\alpha')|\, d\alpha', \quad (4.1.4)$$

with the kernel K_2 given by

$$K_2(x) := (2\pi)^{-1}\nabla^\perp \log|x| = \frac{x^\perp}{2\pi|x|^2}, \quad (4.1.5)$$

and the principal value in (Equation 4.1.4) is only needed for the integral with $k = i$.

Let $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field generated by ω_0, given by $\mathbf{v} := \nabla^\perp(\omega_0 * \mathcal{N})$. Note that $\mathbf{v} \in C^\infty(\mathbb{R}^2 \setminus \Gamma)$, but $\mathbf{v}$ is discontinuous across Γ. Let $\mathbf{v}^+, \mathbf{v}^- : \Gamma \to \mathbb{R}^2$ denote the two limits of $\mathbf{v}$ on the two sides of Γ (with $\mathbf{v}^+$ being the limit on the side that $\mathbf{n}$ points into – see Figure Figure 4.1 for an illustration), and $[\mathbf{v}] := \mathbf{v}^- - \mathbf{v}^+$ the jump in $\mathbf{v}$ across the sheets. $[\mathbf{v}]$ is related to the vortex-sheet strength γ as follows (see [87, Eq. (9.8)] for a derivation): $[\mathbf{v}] \cdot \mathbf{n} = 0$, and

$$[\mathbf{v}] \times \mathbf{n} = [\mathbf{v}] \cdot \mathbf{s} = \gamma.$$

In addition, the Birkhoff-Rott integral (Equation 4.1.4) is the the average of $\mathbf{v}^+$ and $\mathbf{v}^-$, namely

$$BR(z_i(\alpha)) = \frac{1}{2}(\mathbf{v}^+(z_i(\alpha)) + \mathbf{v}^-(z_i(\alpha))) \quad \text{for all } \alpha \in S_i, i = 1, \dots, n+m.$$

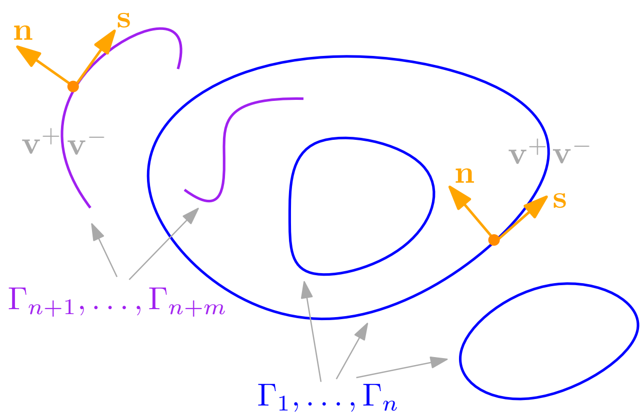


Figure 4.1: Illustration of the closed curves $\Gamma_1, \ldots, \Gamma_n$ and the open curves $\Gamma_{n+1}, \ldots, \Gamma_{n+m}$, and the definitions of $\mathbf{n}$, $\mathbf{s}$, $\mathbf{v}^+$ and $\mathbf{v}^-$.

In the following lemma, we derive the equation that the Birkhoff-Rott integral satisfies for a stationary/rotating vortex sheet.

Lemma 4.1.1. *Assume $\omega(\cdot, t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, and ω_0 is concentrated on $\cup_{i=1}^{n+m}\Gamma_i$, with z_i and γ_i defined as above. Then the Birkhoff-Rott integral BR* (Equation 4.1.4) *and the strength γ_i satisfy the following two equations:*

$$(BR - \Omega x^\perp) \cdot \mathbf{n} = \mathbf{v}^+ \cdot \mathbf{n} = \mathbf{v}^- \cdot \mathbf{n} = 0 \quad \textit{on } \Gamma, \tag{4.1.6}$$

and

$$(BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\, \gamma_i(\alpha) = \begin{cases} C_i & \textit{on } S_i \textit{ for } i = 1, \ldots, n, \\ 0 & \textit{on } S_i \textit{ for } i = n+1, \ldots, n+m. \end{cases} \tag{4.1.7}$$

In particular, the above two equations imply that $BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha) \equiv \mathbf{0}$ for $i = n+1, \ldots, n+m$.

Proof. By definition of the stationary/uniformly-rotating solutions, ω_0 is a stationary vortex sheet in the rotating frame with angular velocity Ω. In this rotating frame, an extra velocity $-\Omega z_i^\perp$

should be added to the right hand side of (Equation 1.1.11). Therefore the evolution equations (Equation 1.1.11)–(Equation 1.1.12) become the following in the rotating frame (where we also use (Equation 4.1.4)):

$$\partial_t z_i(\alpha,t) = BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t) + c_i(\alpha,t)\partial_\alpha z_i(\alpha,t) \tag{4.1.8}$$

$$\partial_t \varpi_i(\alpha,t) = \partial_\alpha (c_i(\alpha,t)\varpi_i(\alpha,t)), \tag{4.1.9}$$

where the term $c_i(\alpha,t)$ accounts for the reparametrization freedom of the curves. Since ω_0 is stationary in the rotating frame, $z_i(\cdot,t)$ parametrizes the same curve as $z_i(\cdot,0)$. Therefore $\partial_t z_i(\alpha,t)$ is tangent to the curve Γ_i, and multiplying $\mathbf{n}(z_i(\alpha,t))$ to (Equation 4.1.8) gives

$$0 = \partial_t z_i(\alpha,t)\cdot\mathbf{n}(z_i(\alpha,t)) = (BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t))\cdot\mathbf{n}(z_i(\alpha,t)), \tag{4.1.10}$$

where we use that $\mathbf{n}(z_i(\alpha,t))\cdot\partial_\alpha z_i(\alpha,t) = 0$. This proves (Equation 5.1.1).

Now we prove (Equation 5.1.2). Towards this end, let us choose

$$c_i(\alpha,t) := -\frac{(BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t))\cdot\mathbf{s}(z_i(\alpha,t))}{|\partial_\alpha z_i(\alpha,t)|},$$

so that multiplying $\mathbf{s}(z_i(\alpha,t))$ to (Equation 4.1.8) gives $\partial_t z_i(\alpha,t)\cdot\mathbf{s}(z_i(\alpha,t)) = 0$, and combining it with (Equation 4.1.10) gives $\partial_t z_i(\alpha,t) = 0$. In other words, with such choice of c_i, the parametrization $z_i(\alpha,t)$ remains fixed in time. Since ω_0 is stationary in the rotating frame, we know that with a fixed parametrization $z_i(\alpha,t) = z_i(\alpha,0)$, the strength $\varpi_i(\alpha,t)$ must also remain invariant in time. Thus (Equation 4.1.9) becomes

$$c_i(\alpha,t)\varpi_i(\alpha,t) \equiv C_i.$$

Plugging the definition of c_i into the equation above and using the fact that z_i is invariant in t, we have

$$\frac{(BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\varpi_i(\alpha)}{|\partial_\alpha z_i(\alpha)|} \equiv -C_i \quad \text{for all } \alpha \in S_i,$$

and finally the relationship between γ_i and ϖ_i in (Equation 4.1.3) yields (Equation 5.1.2) for $i = 1, \ldots, n$.

And for the open curves $i = n+1, \ldots, n+m$, note that we do not have any reparametrization freedom at the two endpoints $\alpha = 0, 1$, therefore the endpoint velocity $BR(z_i(0,t)) - \Omega z_i^\perp(0,t)$ must be 0 to ensure that ω_0 is stationary in the rotating frame. This immediately leads to $C_i = 0$ for $i = n+1, \ldots, n+m$, finishing the proof of (Equation 5.1.2). $\square$

4.2 Approximation by a thin vortex layer

Our aim in this section is to desingularize the vortex sheet ω_0. Namely, for $0 < \epsilon \ll 1$, we will construct a vorticity $\omega^\epsilon \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ that only takes values 0 and ϵ^{-1}, and is supported in an $O(\epsilon)$ neighborhood of Γ, such that ω^ϵ weakly converges to ω_0 as $\epsilon \to 0^+$.

For each $i = 1, \ldots, n+m$, we will describe a neighborhood of Γ_i using the following change of coordinates: let $R_i^\epsilon : S_i \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$R_i^\epsilon(\alpha, \eta) := z_i(\alpha) + \epsilon\gamma_i(\alpha)\mathbf{n}(z_i(\alpha))\eta, \tag{4.2.1}$$

and let

$$D_i^\epsilon := \left\{R_i^\epsilon(\alpha, \eta) : \alpha \in S_i^\circ, \eta \in (-1, 0)\right\}.$$

Note that each D_i^ϵ is a connected open set, and for all $\epsilon > 0$ sufficiently small, the sets $(D_i^\epsilon)_{i=1}^{n+m}$ are disjoint. For $i = 1, \ldots, n$, the domains D_i^ϵ are doubly-connected with smooth boundary, and its inner boundary coincides with Γ_i; see the left of Figure Figure 4.2 for an illustration. And for $i = n+1, \ldots, n+m$, the domains D_i^ϵ are simply-connected, and its boundary is smooth except at at most two points; see the right of Figure Figure 4.2 for an illustration.

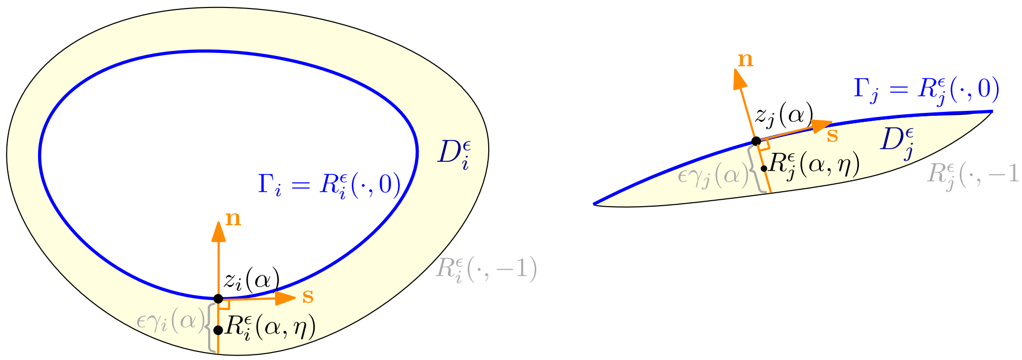


Figure 4.2: Illustration of the definitions of R_i^ϵ and D_i^ϵ for a closed curve (left) and an open curve (right).

In addition, for $\epsilon > 0$ that is sufficiently small, one can check that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is a diffeomorphism. Since $\gamma_i \in C^1(S_i)$ and $z_i \in C^2(S_i)$, we only need to show $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective. Below we prove this fact in a stronger quantitative version, which will be used later.

Lemma 4.2.1. *For any $i = 1, \ldots, n+m$, assume Γ_i and γ_i satisfy* **(H1)–(H2)**. *Then the map $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ given by* (Equation 4.2.1) *is injective. In addition, there exist some $c_0, \epsilon_0 > 0$ depending on $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{L^\infty(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_0)$ we have*

$$|R_i^\epsilon(\alpha', \eta') - R_i^\epsilon(\alpha, \eta)| \geq c_0\big(|\alpha' - \alpha| + \epsilon|\gamma_i(\alpha)\eta - \gamma_i(\alpha')\eta'|\big), \tag{4.2.2}$$

for all $\alpha, \alpha' \in S_i^\circ$, $\eta, \eta' \in (-1, 0)$.[2]

Proof. To begin with, note that (Equation 4.2.2) immediately implies that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective, where we used the positivity assumption $\gamma_i > 0$ in S_i° in **(H2)**. Thus it suffices to prove (Equation 4.2.2). Throughout the proof, we fix any $i \in \{1, \ldots, n+m\}$, and we will omit the subscript i for notational simplicity. Using the definition (Equation 4.2.1), let us break

[2]In fact, (Equation 4.2.2) also holds (with a slightly smaller ϵ_0 and c_0) for $\eta, \eta' \in (-2, 2)$, even though such R_i^ϵ may not belong to D_i^ϵ. We will use this fact later in the proof of Lemma 4.2.5.

$R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta)$ into

$$R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta) = \underbrace{z(\alpha') - z(\alpha)}_{=:T_1} + \underbrace{\epsilon\left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right)\mathbf{n}(z(\alpha'))}_{=:T_2} + \underbrace{\epsilon\gamma(\alpha)\eta\left(\mathbf{n}(z(\alpha')) - \mathbf{n}(z(\alpha))\right)}_{=:T_3}. \tag{4.2.3}$$

For T_1 and T_3, we have

$$\begin{aligned} |T_1 - z'(\alpha')(\alpha' - \alpha)| \leq & \|z\|_{C^2(S)}|\alpha - \alpha'|^2, \\ |T_3| \leq & \epsilon\gamma(\alpha)\|z\|_{C^2(S)}|\alpha - \alpha'|. \end{aligned} \tag{4.2.4}$$

Also, using that $z'(\alpha') = L\mathbf{s}(z(\alpha'))$ is perpendicular to $\mathbf{n}(z(\alpha'))$, we have

$$\begin{aligned} |z'(\alpha')(\alpha' - \alpha) + T_2| &= |L(\alpha' - \alpha)\mathbf{s}(z(\alpha')) + \epsilon\left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right)\mathbf{n}(z(\alpha'))| \\ &\geq \frac{1}{2}L|\alpha' - \alpha| + \frac{1}{2}\epsilon\left|\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right|, \end{aligned}$$

where we use that $\sqrt{x^2 + y^2} \geq \frac{1}{2}(|x| + |y|)$. Combining this with (Equation 4.2.4) gives

$$|T_1 + T_2 + T_3| \geq |\alpha - \alpha'|\left(\frac{L}{2} - \|z\|_{C^2(S)}\left(|\alpha - \alpha'| + \epsilon\gamma(\alpha)\right)\right) + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta|,$$

thus

$$|R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta)| \geq \frac{L}{4}|\alpha - \alpha'| + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{4.2.5}$$

for all $0 < \epsilon < L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$ and $|\alpha - \alpha'| \leq \frac{L}{8\|z\|_{C^2}}$.

For $|\alpha - \alpha'| > \frac{L}{8\|z\|_{C^2}}$, recall that the definition of F_Γ in (Equation 4.1.2) gives $|z(\alpha') - z(\alpha)| \geq F_\Gamma^{-1}|\alpha' - \alpha|$. Thus a crude estimate gives

$$|R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta)| \geq |z(\alpha') - z(\alpha)| - 2\epsilon\|\gamma\|_{L^\infty(S)} \geq \frac{1}{2F_\Gamma}|\alpha' - \alpha| + \epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{4.2.6}$$

for $0 < \epsilon < L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$. (Note that for such ϵ we have $4\epsilon\|\gamma\|_{L^\infty} \leq \frac{1}{2F_\Gamma}|\alpha' - \alpha|$ due to our assumption that $|\alpha - \alpha'| > \frac{L}{8\|z\|_{C^2}}$).

Finally, combining (Equation 4.2.5) and (Equation 4.2.6), it follows that (Equation 4.2.2) holds for $c_0 = \min\{\frac{L}{4}, \frac{1}{2F_\Gamma}, \frac{1}{2}\}$ and $\epsilon_0 = \min\{L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}, L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}\}$. This finishes the proof. $\square$

In the next lemma we compute the partial derivatives and Jacobian of $R_i^\epsilon(\alpha, \eta)$, which will be useful later.

Lemma 4.2.2. *For any $i = 1, \ldots, n+m$, let z_i be a constant-speed parameterization of the curve Γ_i (with length L_i), and let R_i^ϵ be given by* (Equation 4.2.1). *Then its partial derivatives are*

$$\begin{aligned} \partial_\alpha R_i^\epsilon(\alpha, \eta) &= z_i'(\alpha) + \epsilon \left(\gamma_i'(\alpha) \frac{z_i'(\alpha)^\perp}{L_i} \eta + \gamma_i(\alpha) \frac{z_i''(\alpha)^\perp}{L_i} \eta \right), \\ \partial_\eta R_i^\epsilon(\alpha, \eta) &= \epsilon \gamma_i(\alpha) \frac{z_i'(\alpha)^\perp}{L_i}. \end{aligned} \tag{4.2.7}$$

Moreover, its Jacobian is given by

$$\det(\nabla_{\alpha,\eta} R_i^\epsilon) = \epsilon L_i \gamma_i(\alpha) - \epsilon^2 L_i \gamma_i^2(\alpha) \kappa_i(\alpha) \eta, \tag{4.2.8}$$

where $\kappa_i(\alpha)$ denotes the signed curvature of Γ_i at $z_i(\alpha)$.

Proof. Since z_i is the constant-speed parameterization of Γ_i (which has length L_i), we have $|z_i'| \equiv L_i$ and $\mathbf{n}(z_i(\alpha)) = z_i'(\alpha)^\perp / L_i$. Taking the α and η partial derivatives of (Equation 4.2.1) directly yields (Equation 4.2.7).

Putting the two partial derivatives into columns of a 2×2 matrix and computing the determinant, we have

$$\begin{aligned} \det(\nabla_{\alpha,\eta} R_i^\epsilon) &= \epsilon \gamma_i(\alpha) \frac{|z_i'(\alpha)|^2}{L_i} + \epsilon^2 \gamma_i^2(\alpha) \frac{z_i''(\alpha)^\perp \cdot z_i'(\alpha)}{L_i^2} \eta \\ &= \epsilon L_i \gamma_i(\alpha) - \epsilon^2 L_i \gamma_i^2(\alpha) \kappa_i(\alpha) \eta, \end{aligned}$$

where in the second equality we used that $z_i''(\alpha) = \kappa_i(\alpha)\mathbf{n}(z_i(\alpha))L_i^2$ (recall that z_i has constant speed L_i). This finishes the proof. $\square$

Remark 4.2.3. *We point out that for each $i = 1, \ldots, n+m$, the determinant formula* (Equation 4.2.8) *immediately gives the following approximation of $|D_i^\epsilon|$, which will be helpful in the proofs later:*

$$\frac{|D_i^\epsilon|}{\epsilon} = \frac{1}{\epsilon}\int_{D_i^\epsilon} 1dx = \frac{1}{\epsilon}\int_{S_i}\int_{-1}^{0} \det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))\, d\eta d\alpha = L_i \int_{S_i}\gamma_i(\alpha)d\alpha + O(\epsilon), \tag{4.2.9}$$

where the $O(\epsilon)$ error term has its absolute value bounded by $C\epsilon$, with C only depending on $\|z_i\|_{C^2(S_i)}$ and $\|\gamma_i\|_{L^\infty(S_i)}$.

Finally, let $D^\epsilon := \cup_{i=1}^{n+m} D_i^\epsilon$, and $\omega^\epsilon : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\omega^\epsilon(x) := \epsilon^{-1}1_{D^\epsilon}(x) = \epsilon^{-1}\sum_{i=1}^{n+m} 1_{D_i^\epsilon}(x),$$

and let

$$\mathbf{v}^\epsilon = \nabla^\perp(\omega^\epsilon * \mathcal{N}) \tag{4.2.10}$$

be the velocity field generated by ω^ϵ.

In the next lemma we aim to obtain some fine estimate of $\mathbf{v}^\epsilon$ in the thin vortex layer D^ϵ. Our goal is to show that along each cross section of the thin layer (i.e. fix i and α, and let η vary in $[-1, 0]$), the function $\eta \mapsto \mathbf{v}^\epsilon(R_i^\epsilon(\alpha,\eta))$ is almost a linear function in η, with the endpoint values (at $\eta = -1$ and 0) being almost $\mathbf{v}^-(z_i(\alpha))$ and $\mathbf{v}^+(z_i(\alpha))$ respectively.

Lemma 4.2.4. *For $i = 1, \ldots, n+m$, assume Γ_i and γ_i satisfy* **(H1)–(H3)**. *Let*

$$g_i(\alpha,\eta) := BR(z_i(\alpha)) - \Big(\eta + \frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) \quad \textit{for } \alpha \in S_i,$$

and note that $g_i(\alpha, 0) = \mathbf{v}^+(z_i(\alpha))$ and $g_i(\alpha,-1) = \mathbf{v}^-(z_i(\alpha))$ (see Figure Figure 4.3 for an *illustration of $g_i(\alpha,\eta)$). Then for all sufficiently small $\epsilon > 0$, for all $i = 1, \ldots, n+m$ we have*

$$|\mathbf{v}^\epsilon(R_i^\epsilon(\alpha,\eta)) - g_i(\alpha,\eta)| \le C\epsilon^b |\log \epsilon| \quad \textit{for all } \alpha \in S_i, \eta \in [-1,0], \tag{4.2.11}$$

where $b \in (0,1)$ is as in **(H2)***, and C depends on b, $\max_i \|z_i\|_{C^2(S_i)}$, $\max_i \|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ.*

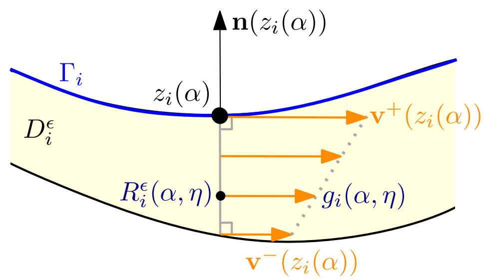


Figure 4.3: Illustration of the definition of $g_i(\alpha,\cdot)$ (the orange arrows).

Proof. Let i be any fixed index in $1,\ldots,n+m$. We begin with breaking $\mathbf{v}^\epsilon$ into contributions from different components $\{D_k^\epsilon\}_{k=1}^{n+m}$, namely

$$\mathbf{v}^\epsilon(x) = \sum_{k=1}^{n+m} \mathbf{v}_k^\epsilon(x) := \sum_{k=1}^{n+m} \epsilon^{-1} \int_{D_i^\epsilon} K_2(x-y)dy,$$

where the kernel K_2 is given by (Equation 4.1.5). Similarly, we can break $BR(z_i(\alpha))$ into $BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha))$, where BR_k is the contribution from the k-th integral in (Equation 4.1.4), and note that the PV symbol is only needed for $k = i$.

- *Estimates for $k \neq i$ terms.* For any $k \neq i$, we aim to show that

$$|\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) - BR_k(z_i(\alpha))| \le C\epsilon, \tag{4.2.12}$$

where C depends on $d_\Gamma, \max_k \|z_k\|_{C^2}$ and $\max_k \|\gamma_k\|_{L^\infty}$. Applying a change of variable $y =$

$R_k^\epsilon(\alpha', \eta')$, we can rewrite $\mathbf{v}_k^\epsilon$ as

$$\begin{aligned} \mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) &= \epsilon^{-1} \int_{D_k^\epsilon} K_2(R_i^\epsilon(\alpha,\eta) - y)\, dy \\ &= \int_{S_k} \int_{-1}^{0} \underbrace{K_2(R_i^\epsilon(\alpha,\eta) - R_k^\epsilon(\alpha',\eta'))}_{=:T_1} \underbrace{\epsilon^{-1} \det(\nabla_{\alpha',\eta'} R_k^\epsilon(\alpha',\eta'))}_{=:T_2}\, d\eta' d\alpha'. \end{aligned} \tag{4.2.13}$$

Using the facts that $R_i^\epsilon(\alpha,\eta) - R_k^\epsilon(\alpha',\eta') = z_i(\alpha) - z_k(\alpha') + O(\epsilon)$ as well as $|z_i(\alpha) - z_k(\alpha')| \geq d_\Gamma > 0$ (recall that d_Γ is as given in (Equation 4.1.1)), for all sufficiently small $\epsilon > 0$ we have $T_1 = K_2(z_i(\alpha) - z_k(\alpha')) + O(\epsilon)$. For T_2, the explicit formula (Equation 4.2.8) for the determinant gives $T_2 = L_k \gamma_k(\alpha') + O(\epsilon)$. Plugging these into the above integral yields

$$\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) = \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha')) L_k \gamma_k(\alpha')\, d\alpha' + O(\epsilon) = BR_k(z_i(\alpha)) + O(\epsilon),$$

finishing the proof of (Equation 4.2.12).

• *Estimates for the $k = i$ term.* It will be more involved to control the $k = i$ term, and our goal is to show that

$$\left| \mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) - BR_i(z_i(\alpha)) + \Big(\eta + \frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) \right| \leq C\epsilon^b |\log \epsilon|. \tag{4.2.14}$$

To begin with, we again rewrite $\mathbf{v}_i^\epsilon$ as in (Equation 4.2.13) with $k = i$, and plug in the formula (Equation 4.2.8) for the determinant. This leads to

$$\begin{aligned} \mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) &= \int_{S_k} \int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta')) \left(L_i \gamma_i(\alpha') - \epsilon L_i \gamma_i^2(\alpha') \kappa_i(\alpha') \eta' \right)\, d\eta' d\alpha' \\ &=: I_1 + I_2, \end{aligned}$$

where I_1, I_2 are the contributions from the two terms in the last parenthesis respectively. Let us control I_2 first, and we claim that

$$|I_2| \leq C\epsilon |\log \epsilon|. \tag{4.2.15}$$

Using (Equation 4.2.2) of Lemma 4.2.1 and the fact that $|K_2(x)| \le |x|^{-1}$, we can bound I_2 as

$$
\begin{aligned}
|I_2| &= \left| \int_{S_k} \int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))\, \epsilon L_i \gamma_i^2(\alpha')\kappa_i(\alpha')\eta'\, d\eta' d\alpha' \right| \\
&\le C\epsilon \int_{S_k} \int_{-1}^{0} \frac{\gamma_i(\alpha')}{|\alpha'-\alpha| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|}\, d\eta' d\alpha' \\
&\le C\epsilon \int_{S_k} \int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\alpha'-\alpha| + \epsilon|\theta'|}\, d\theta' d\alpha' \quad (\theta' := \gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta) \qquad (4.2.16)\\
&\le C\epsilon \int_{-1/\epsilon}^{1/\epsilon} \int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\beta'| + |\theta'|}\, d\theta' d\beta' \quad (\beta' := \epsilon^{-1}(\alpha'-\alpha)) \\
&\le C\epsilon|\log\epsilon|
\end{aligned}
$$

where C depends on $\|z_i\|_{C^2}$ and $\|\gamma_i\|_{L^\infty}$.

In the rest of the proof we focus on estimating $I_1 = \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))L_i\gamma_i(\alpha')\, d\eta' d\alpha'$. For $t \in [0,1]$, let us define

$$
\begin{aligned}
f(\alpha,\alpha',\eta,\eta';t) &:= R_i^\epsilon(\alpha,\eta - t\eta') - R_i^\epsilon(\alpha',\eta' - t\eta'), \\
J(t) &:= \int_{S_k}\int_{-1}^{0} K_2(f(\alpha,\alpha',\eta,\eta';t))L_i\gamma_i(\alpha')\, d\eta' d\alpha'. \qquad (4.2.17)
\end{aligned}
$$

Note that in the definition of f, the argument $\eta - t\eta'$ of R_i^ϵ belongs to $[-1,1]$, instead of $[-1,0]$ as in the original definition of (Equation 4.2.1). Here $R_i^\epsilon(\alpha,\eta - t\eta')$ is defined as in the formula (Equation 4.2.1), even though it might not belong to D_i^ϵ. Clearly, $J(0) = I_1$. The motivation for us to define such f and $J(t)$ is that at $t = 1$, we have

$$
J(1) = \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta-\eta') - z_i(\alpha'))L_i\gamma_i(\alpha')\, d\eta' d\alpha' = \int_{-1}^{0} \mathbf{v}_i(R_i^\epsilon(\alpha,\eta-\eta'))\, d\eta', \qquad (4.2.18)
$$

where $\mathbf{v}_i$ is the velocity field generated by the sheet Γ_i. Recall that $\mathbf{v}_i$ has a jump across Γ_i, where we denote its limits on two sides by $\mathbf{v}_i^\pm$. Using Lemma 4.2.5, which we will prove momentarily, we have

$$\mathbf{v}_i(R_i^\epsilon(\alpha,\eta-\eta')) = \begin{cases} \mathbf{v}_i^+(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta-\eta' \in (0,2), \\ \mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta-\eta' \in (-2,0). \end{cases} \tag{4.2.19}$$

We can then split the integration domain on the right hand side of (Equation 6.2.56) into $\eta' \in (-1,\eta)$ and $\eta' \in (\eta,0)$, and use (Equation 4.2.19) to approximate the integrand in each interval. This gives

$$\begin{aligned} J(1) &= (\eta+1)\mathbf{v}_i^+(z_i(\alpha)) - \eta\mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) \\ &= BR_i(z_i(\alpha)) - \Big(\eta+\frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) + O(\epsilon^b|\log\epsilon|), \end{aligned} \tag{4.2.20}$$

where in the last step we used that $[\mathbf{v}](z_i(\alpha)) = [\mathbf{v}_i](z_i(\alpha))$, since all other $\mathbf{v}_k$ with $k \neq i$ are continuous across Γ_i.

Finally, it remains to control $|J(0)-J(1)|$. Note that by (Equation 4.2.2), we have

$$f(\alpha,\alpha',\eta,\eta';t) \geq c_0\big(|\alpha-\alpha'| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\big).$$

In addition, we have

$$\left|\frac{\partial}{\partial t} f(\alpha,\alpha',\eta,\eta';t)\right| = \big|\epsilon\big(\gamma_i(\alpha)\mathbf{n}(z_i(\alpha)) - \gamma_i(\alpha')\mathbf{n}(z_i(\alpha'))\big)\eta'\big| \leq C\epsilon|\alpha-\alpha'|^b,$$

where the last inequality follows from **(H2)** and the fact that $\mathbf{n}(z_i(\alpha)) \in C^1(S_i)$. Therefore, for any $t \in (0,1)$, taking the t derivative of (Equation 4.2.17) and using that $|\nabla K_2(x)| \leq |x|^{-2}$, we have

$$
\begin{aligned}
|J'(t)| &\le C \int_{S_k}\int_{-1}^{0} \frac{\epsilon|\alpha-\alpha'|^b\gamma_i(\alpha')}{\big(|\alpha-\alpha'|+\epsilon|\gamma_i(\alpha')\eta'-\gamma_i(\alpha)\eta|\big)^2} d\eta' d\alpha' \\
&\le C\epsilon \int_{S_k}\int_{-1}^{0} \frac{\gamma_i(\alpha')}{|\alpha-\alpha'|^{1-b}\left(|\alpha-\alpha'|+\epsilon|\gamma_i(\alpha')\eta'-\gamma_i(\alpha)\eta|\right)} d\eta' d\alpha' \\
&\le C\epsilon^b \int_{-1/\epsilon}^{1/\epsilon}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\beta'|^{1-b}(|\beta'|+|\theta'|)}\, d\theta' d\beta' \quad (\theta' := \gamma_i(\alpha')\eta'-\gamma_i(\alpha)\eta,\ \beta' := \epsilon^{-1}(\alpha'-\alpha)) \\
&\le C\epsilon^b \int_{-1/\epsilon}^{1/\epsilon} |\beta'|^{b-1} \log\left(1+\frac{\|\gamma_i\|_{L^\infty}}{|\beta'|}\right) d\beta' \\
&\le C\epsilon^b,
\end{aligned}
$$

where C depends on b, $\|\gamma_i\|_{C^b(S_i)}$, $\|z_i\|_{C^2(S_i)}$ and F_Γ. This leads to

$$
|J(1)-I_1| = |J(1)-J(0)| \le C\epsilon^b|\log\epsilon|.
$$

Finally, combining this with (Equation 4.2.20) and (Equation 4.2.15) yields (Equation 4.2.14), finishing the proof of the $k=i$ case. We can then conclude the proof by taking the sum of this estimate with all the $k \neq i$ estimates in (Equation 4.2.12). □

The following lemma proves (Equation 4.2.19). Let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, which is smooth in $\mathbb{R}^2\setminus\Gamma_i$, and has a discontinuity across Γ_i. It is known that $\mathbf{v}_i$ converges to $\mathbf{v}_i^\pm$ respectively on the two sides of Γ_i [87]. However, we were unable to find a quantitative convergence rate (in terms of the distance from the point to Γ_i) in the literature, especially under the assumption that γ_i is only in $C^b(S_i)$ for the open curves. Below we prove such an estimate.

Lemma 4.2.5. *For $i = 1,\dots,n+m$, let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, given by*

$$
\mathbf{v}_i(x) := \int_{S_i} K_2(x-z_i(\alpha'))\,\gamma_i(\alpha')|z_i'(\alpha')|\,d\alpha' \quad \textit{for } x\in\mathbb{R}^2\setminus\Gamma_i.
$$

Then there exist constants $C, \epsilon_0 > 0$ depending on on b (as in **(H2)***), $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{C^b(S_i)}$ and F_Γ,* *such that for all* $\epsilon \in (0, \epsilon_0)$ *and* $\eta \in (-2, 2)$ *we have*

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) - \mathbf{v}_i^+(z_i(\alpha))\right| \le C\epsilon^b |\log \epsilon| \quad \text{if } \eta \in (0,2), \tag{4.2.21}$$

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) - \mathbf{v}_i^-(z_i(\alpha))\right| \le C\epsilon^b |\log \epsilon| \quad \text{if } \eta \in (-2,0), \tag{4.2.22}$$

where

$$\mathbf{v}_i^+ = BR_i(z_i(\alpha)) + \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2}, \quad \mathbf{v}_i^- = BR_i(z_i(\alpha)) - \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2},$$

and BR_i *is the contribution from the* i*-th integral in* (Equation 4.1.4).

Proof. We will show (Equation 4.2.21) only since (Equation 4.2.22) can be treated in the same way. From the definition of R_i^ϵ in (Equation 4.2.1), we have

$$\begin{aligned}
\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) &= \frac{L_i}{2\pi}\int_{S_i} \frac{(z_i(\alpha)-z_i(\alpha'))^\perp \gamma_i(\alpha')}{|z_i(\alpha)-z_i(\alpha')+\epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\,d\alpha' \\
&\quad + \frac{L_i}{2\pi}\int_{S_i} \frac{\epsilon\eta\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)\gamma_i(\alpha')}{|z_i(\alpha)-z_i(\alpha')+\epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\,d\alpha' \\
&=: A_1 + A_2.
\end{aligned}$$

We claim that for all $\epsilon > 0$ sufficiently small and $\eta \in [0, 2)$, we have

$$|A_1 - BR_i(z(\alpha))| \le C\epsilon^b |\log \epsilon|, \tag{4.2.23}$$

$$\left|A_2 - \frac{\mathbf{n}(z(\alpha))^\perp \gamma(\alpha)}{2}\right| \le C\epsilon^b, \tag{4.2.24}$$

and note that these two claims immediately yield (Equation 4.2.21). From now on, let us fix $i \in \{1, \dots, n+m\}$ and omit it in the notation for simplicity. Throughout this proof, let us denote

$$\mathbf{y}(\alpha,\alpha') := z(\alpha) - z(\alpha') \quad \text{and} \quad \mathbf{c}(\alpha) := \epsilon\eta\mathbf{n}(z(\alpha))\gamma(\alpha),$$

so that

$$A_1 = \frac{L}{2\pi}\int_S \frac{\mathbf{y}^\perp(\alpha,\alpha')\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha', \quad A_2 = \frac{L}{2\pi}\int_S \frac{\mathbf{c}^\perp(\alpha)\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha'.$$

Note that

$$F_\Gamma^{-1}|\alpha-\alpha'| \le |\mathbf{y}(\alpha,\alpha')| \le \|z\|_{C^1}|\alpha-\alpha'|. \tag{4.2.25}$$

For the closed curves with $i = 1,\dots,n$, since z has period 1, we can always set $\alpha-\alpha' \in [-\frac{1}{2},\frac{1}{2})$ in this proof.

Applying (Equation 4.2.2) (with $\eta' = 0$), we have

$$|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2 \ge c_0(|\alpha-\alpha'|^2+\epsilon^2\eta^2\gamma^2(\alpha)) = c_0(|\alpha-\alpha'|^2+|\mathbf{c}(\alpha)|^2). \tag{4.2.26}$$

Since $z'(\alpha) = L\mathbf{s}(z(\alpha))$, let us define

$$\tilde{\mathbf{y}}(\alpha,\alpha') := L\mathbf{s}(z(\alpha))(\alpha-\alpha'),$$

which is a close approximation of $\mathbf{y}$ in the sense that

$$|\mathbf{y}(\alpha,\alpha')-\tilde{\mathbf{y}}(\alpha,\alpha')| \le \|z\|_{C^2}(\alpha-\alpha')^2. \tag{4.2.27}$$

Using $\mathbf{s}(z(\alpha)) \perp \mathbf{n}(z(\alpha))$, we have

$$|\tilde{\mathbf{y}}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2 = L^2|\alpha-\alpha'|^2+\epsilon^2\eta^2\gamma^2(\alpha) = L^2|\alpha-\alpha'|^2+|\mathbf{c}(\alpha)|^2. \tag{4.2.28}$$

From now on, for notational simplicity, we compress the dependence of $\mathbf{y}(\alpha,\alpha'), \tilde{\mathbf{y}}(\alpha,\alpha'), \mathbf{c}(\alpha)$ on α and α' in the rest of the proof.

- *Estimate* (Equation 4.2.23). Note that $BR_i(z(\alpha))$ can also be written using the above notations as

$$BR_i(z(\alpha)) = \frac{L}{2\pi} PV \int_S \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2} \gamma(\alpha') d\alpha,$$

thus $A_1 - BR_i(z(\alpha))$ can be written as follows:

$$\begin{aligned}
A_1 - BR_i(z(\alpha)) &= \frac{L}{2\pi} PV \int_S \underbrace{\left(\frac{\mathbf{y}^\perp}{|\mathbf{y}+\mathbf{c}|^2} - \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2} \right)}_{=:\mathbf{f}(\mathbf{y},\mathbf{c})} \gamma(\alpha') d\alpha' \\
&= \frac{L}{2\pi} \int_S \mathbf{f}(\mathbf{y},\mathbf{c})(\gamma(\alpha') - \gamma(\alpha)) d\alpha' + \frac{L\gamma(\alpha)}{2\pi} PV \int_S \mathbf{f}(\mathbf{y},\mathbf{c}) d\alpha' \\
&=: A_{11} + A_{12}.
\end{aligned}$$

A direct computation gives

$$\mathbf{f}(\mathbf{y},\mathbf{c}) = -\frac{\mathbf{y}^\perp}{|\mathbf{y}|^2} \frac{2\mathbf{y}\cdot\mathbf{c} + |\mathbf{c}|^2}{|\mathbf{y}+\mathbf{c}|^2}. \tag{4.2.29}$$

Since $\mathbf{y}\cdot\mathbf{c} = (\mathbf{y} - \tilde{\mathbf{y}})\cdot\mathbf{c} \le C|\alpha - \alpha'|^2|\mathbf{c}|$, (where we use $\tilde{\mathbf{y}} \perp \mathbf{n}(z(\alpha))$ and (Equation 4.2.27)), combining this with (Equation 4.2.25) and (Equation 4.2.26) gives a crude bound

$$|\mathbf{f}(\mathbf{y},\mathbf{c})| \lesssim \frac{|\alpha - \alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha - \alpha'|(|\alpha - \alpha'|^2 + |\mathbf{c}|^2)}.$$

Plugging this into A_{11} and using the Hölder continuity of γ, we have

$$\begin{aligned}
|A_{11}| &\lesssim \int_S \frac{|\alpha - \alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha - \alpha'|(|\alpha - \alpha'|^2 + |\mathbf{c}|^2)} |\alpha - \alpha'|^b d\alpha' \\
&\lesssim \int_{|\theta|<|\mathbf{c}|} (|\theta|^{1+b}|\mathbf{c}|^{-1} + |\theta|^{b-1}) d\theta + \int_{|\mathbf{c}|\le|\theta|\le 1} (|\mathbf{c}||\theta|^{b-1} + |\mathbf{c}|^2|\theta|^{b-3}) d\theta \quad (\theta := \alpha' - \alpha) \\
&\lesssim |\mathbf{c}|^b \le C\epsilon^b,
\end{aligned}$$

where the last step follows from the fact that $|\mathbf{c}| \le 2\epsilon\|\gamma\|_\infty$. Now let us turn to A_{12}, which requires a more delicate estimate of $\mathbf{f}(\mathbf{y},\mathbf{c})$. Let us break A_{12} as

$$A_{12} = \frac{L\gamma(\alpha)}{2\pi} \int_S (\mathbf{f}(\mathbf{y},\mathbf{c}) - \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c})) d\alpha' + \frac{L\gamma(\alpha)}{2\pi} PV \int_S \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c}) d\alpha' =: B_1 + B_2.$$

For B_1, let us take the gradient of $\mathbf{f}(\mathbf{y}, \mathbf{c})$ (as in (Equation 4.2.29)) in the first variable. An elementary computation yields that

$$|\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{c})| \le C|\mathbf{x}|^{-2} \min\left\{1, \frac{|\mathbf{c}|}{|\mathbf{x}|}\right\} \tag{4.2.30}$$

as long as $\mathbf{x}$ satisfies

$$|\mathbf{x} + \mathbf{c}|^2 \ge c_0(|\mathbf{x}|^2 + |\mathbf{c}|^2). \tag{4.2.31}$$

We point out that $\mathbf{x} = \xi \mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$ indeed satisfies (Equation 4.2.31) for all $\xi \in [0, 1]$: to see this, in the proof of Lemma 4.2.1, if we replace T_1 in (Equation 4.2.3) by $\xi \mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$, one can easily check the proof still goes through for $\xi \in [0, 1]$. In addition, for any $\xi \in [0, 1]$ we also have

$$|\xi \mathbf{y} + (1-\xi)\tilde{\mathbf{y}}| \ge c_0|\alpha - \alpha'|. \tag{4.2.32}$$

Thus the gradient estimate (Equation 4.2.30) together with (Equation 4.2.27) and (Equation 4.2.32) yields

$$|f(\mathbf{y}, \mathbf{c}) - f(\tilde{\mathbf{y}}, \mathbf{c})| \lesssim \min\{1, |\mathbf{c}||\alpha - \alpha'|^{-1}\} \lesssim \min\{1, \epsilon|\alpha - \alpha'|^{-1}\},$$

and plugging this into B_1 gives

$$|B_1| \lesssim \epsilon + \int_{\epsilon < |\alpha - \alpha'| < 1} \epsilon|\alpha - \alpha'|^{-1} d\alpha' \lesssim \epsilon|\log \epsilon|.$$

As for B_2, using the definition of $\tilde{\mathbf{y}}$, the identity (Equation 4.2.28) and the fact that $\tilde{\mathbf{y}} \cdot \mathbf{c} = 0$, we have

$$\begin{aligned} B_2 &= \frac{L\gamma(\alpha)}{2\pi} PV \int_S -\frac{\tilde{\mathbf{y}}^\perp}{|\tilde{\mathbf{y}}|^2} \frac{|\mathbf{c}|^2}{|\tilde{\mathbf{y}} + \mathbf{c}|^2} d\alpha' \\ &= \frac{L\gamma(\alpha)|\mathbf{c}|^2 \mathbf{n}(z(\alpha))}{2\pi L} PV \int_S \frac{\alpha' - \alpha}{|\alpha' - \alpha|^2 (L^2|\alpha' - \alpha|^2 + |\mathbf{c}|^2)} d\alpha'. \end{aligned}$$

For the closed curves $i = 1, \ldots, n$, we immediately have $B_2 = 0$ since $\alpha - \alpha' \in [-\frac{1}{2}, \frac{1}{2})$, and the integrand is an odd function of $\alpha' - \alpha$.

For the open curves $i = n+1, \ldots, n+m$, the above integral becomes

$$\begin{aligned} B_2 &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} PV \int_{-\alpha}^{1-\alpha} \frac{\theta}{|\theta|^2(L^2|\theta|^2 + |\mathbf{c}|^2)} d\theta \quad (\theta := \alpha' - \alpha) \\ &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} \int_{\alpha}^{1-\alpha} \frac{\theta}{\theta^2(L^2\theta^2 + |\mathbf{c}|^2)} d\theta, \end{aligned}$$

where in the second inequality we used that the integral in $[-\alpha, \alpha]$ gives zero contribution to the principal value, since the integrand is odd.

Next we discuss two cases. If $\alpha > |\mathbf{c}|$, we bound the integrand by $C\theta^{-3}$, which gives

$$|B_2| \le C\gamma(\alpha)|\mathbf{c}|^2\alpha^{-2} \le C|\mathbf{c}|^2\alpha^{b-2} \le C|\mathbf{c}|^b \le C\epsilon^b.$$

where the second inequality follows from the assumption $\gamma(0) = 0$ for an open curve in **(H3)**, as well as the Hölder continuity of γ. And if $0 < \alpha \le |\mathbf{c}|$, the integrand can be bounded above by $\theta^{-1}|\mathbf{c}|^{-2}$, which immediately leads to

$$|B_2| \le C\gamma(\alpha)|\log\alpha| \le C|\mathbf{c}|^b|\log|\mathbf{c}|| \le C\epsilon^b|\log\epsilon|.$$

In both cases we have $|B_2| \le C\epsilon^b|\log\epsilon|$, and combining it with the B_1 and A_{11} estimates gives (Equation 4.2.23).

• *Estimate* (Equation 4.2.24). We break A_2 into

$$\begin{aligned} A_2 = \frac{L\mathbf{c}^\perp}{2\pi} \int_S \frac{\gamma(\alpha') - \gamma(\alpha)}{|\mathbf{y} + \mathbf{c}|^2} d\alpha' + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi} \int_S \left(\frac{1}{|\mathbf{y} + \mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}} + \mathbf{c}|^2} \right) d\alpha' + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi} \int_S \frac{1}{|\tilde{\mathbf{y}} + \mathbf{c}|^2} d\alpha' \\ =: A_{21} + A_{22} + A_{23}. \end{aligned}$$

For A_{21}, (Equation 4.2.26) and the Hölder continuity of γ immediately lead to

$$|A_{21}| \leq C|\mathbf{c}| \int_S \frac{|\alpha-\alpha'|^b}{|\alpha-\alpha'|^2+|\mathbf{c}|^2} d\alpha' \leq |\mathbf{c}|^b \leq C\epsilon^b. \tag{4.2.33}$$

For A_{22}, its integrand can be controlled as

$$\left|\frac{1}{|\mathbf{y}+\mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2}\right| \leq \frac{|\mathbf{y}-\tilde{\mathbf{y}}|(|\mathbf{y}+\mathbf{c}|+|\tilde{\mathbf{y}}+\mathbf{c}|)}{|\mathbf{y}+\mathbf{c}|^2|\tilde{\mathbf{y}}+\mathbf{c}|^2} \leq \frac{C|\alpha-\alpha'|^2}{(|\alpha-\alpha'|^2+|\mathbf{c}|^2)^{3/2}},$$

where the last step follows from (Equation 4.2.26), (Equation 4.2.27) and (Equation 4.2.28). This allows us to control A_{22} as

$$|A_{22}| \leq C|\mathbf{c}| \int_{-1}^{1} \frac{\theta^2}{(\theta^2+|\mathbf{c}|^2)^{3/2}} d\theta \leq C|\mathbf{c}| \left|\log|\mathbf{c}|\right| \leq C\epsilon|\log\epsilon|. \tag{4.2.34}$$

Finally, for the A_{23} term, (Equation 4.2.28) gives

$$A_{23} = \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi}\int_S \frac{1}{L^2|\alpha'-\alpha|^2+|\mathbf{c}|^2}d\alpha' = \frac{\mathbf{n}^\perp(\alpha)\gamma(\alpha)}{2\pi}\int_I \frac{1}{\theta^2+1}d\theta \quad (\text{set } \theta := \frac{L(\alpha'-\alpha)}{|\mathbf{c}|}),$$

where the integration interval $I = (-\frac{L}{2|\mathbf{c}|}, \frac{L}{2|\mathbf{c}|})$ for $i = 1, \ldots, n$, and $I = (-\frac{L\alpha}{|\mathbf{c}|}, \frac{L(1-\alpha)}{|\mathbf{c}|})$ for $i = n+1, \ldots, n+m$, and in the last equality we also used that $\frac{\mathbf{c}^\perp}{|\mathbf{c}|} = \mathbf{n}^\perp$. For $i = 1, \ldots, n$, one can easily check that

$$\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| = 2\int_{\frac{L}{2|\mathbf{c}|}}^{\infty} \frac{1}{\theta^2+1}d\theta \leq C|\mathbf{c}| \leq C\epsilon,$$

which immediately leads to

$$\left|A_{23} - \frac{\mathbf{n}(z(\alpha))^\perp\gamma(\alpha)}{2}\right| = \left|\frac{\mathbf{n}^\perp(\alpha)\gamma(\alpha)}{2\pi}\left(\int_I \frac{1}{\theta^2+1}d\theta - \pi\right)\right| \leq C\epsilon$$

for $i = 1, \ldots, n$. Next we turn to the open curves $i = n+1, \ldots, n+m$, and let us assume $\alpha \in [0, \frac{1}{2}]$

without loss of generality. In this case we have

$$\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| = \int_{-\infty}^{-\frac{L\alpha}{|\mathbf{c}|}} \frac{1}{\theta^2+1}d\theta + \int_{\frac{L(1-\alpha)}{|\mathbf{c}|}}^{\infty} \frac{1}{\theta^2+1}d\theta \le \min\left\{C\frac{|\mathbf{c}|}{\alpha}, \frac{\pi}{2}\right\} + C\epsilon.$$

where we used $1-\alpha > \frac{1}{2}$ to control the second integral by $C\epsilon$. Using the above inequality as well as the fact that $\gamma(\alpha) \le C\alpha^b$ due to **(H3)**, we have

$$\left|A_{23} - \frac{\mathbf{n}(z(\alpha))^\perp \gamma(\alpha)}{2}\right| = \frac{\gamma(\alpha)}{2\pi}\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| \le C\alpha^b \min\left\{\frac{|\mathbf{c}|}{\alpha}, 1\right\} + C\epsilon \le C(|\mathbf{c}|^b + \epsilon) \le C\epsilon^b$$

for $i = n+1, \ldots, n+m$. Finally, combining the A_{23} estimates together with (Equation 4.2.33) and (Equation 4.2.34) yields (Equation 4.2.24). □

4.3 Constructing a divergence-free perturbation

In this section, we aim to construct a divergence-free velocity field $\mathbf{u}^\epsilon : D^\epsilon \to \mathbb{R}^2$, such that $-\mathbf{u}^\epsilon$ tends to make each D_i^ϵ "more symmetric". Let $\mathbf{u}^\epsilon : D^\epsilon \to \mathbb{R}^2$ be given by

$$\mathbf{u}^\epsilon := x + \nabla p^\epsilon \quad \text{ in } D^\epsilon, \tag{4.3.1}$$

where the function $p^\epsilon : \overline{D^\epsilon} \to \mathbb{R}$ is chosen such that

$$\nabla \cdot \mathbf{u}^\epsilon = 0 \quad \text{ in } D^\epsilon, \tag{4.3.2}$$

and on each connected component l of ∂D^ϵ, u^ϵ satisfies

$$\int_l \mathbf{u}^\epsilon \cdot n \, d\sigma = 0, \tag{4.3.3}$$

where n is the unit normal of l pointing outwards of D^ϵ. Note that ∂D^ϵ has a total of $2n + m$ connected components: D_i^ϵ is doubly-connected for $i = 1, \ldots, n$ (denote its outer and inner boundaries by $\partial D_{i,\text{out}}^\epsilon$ and $\partial D_{i,\text{in}}^\epsilon$; note that $\partial D_{i,\text{in}}^\epsilon$ coincides with Γ_i), whereas it is simply-connected for $i = n+1, \ldots, n+m$ (denote its boundary by ∂D_i^ϵ).

Next we show that there indeed exists a function p^ϵ so that $\mathbf{u}^\epsilon$ satisfies (Equation 4.3.2)–(Equation 4.3.3). Clearly, (Equation 4.3.2) requires that p^ϵ satisfies

$$\Delta p^\epsilon = -2 \quad \text{in } D^\epsilon. \tag{4.3.4}$$

As for the boundary conditions, we let

$$p^\epsilon|_{\partial D_i^\epsilon} = 0 \quad \text{for } i = n+1, \ldots, n+m, \tag{4.3.5}$$

so the divergence theorem yields that (Equation 4.3.3) is satisfied for each $l = \partial D_i^\epsilon$ for $i = n + 1, \ldots, n+m$. As for $i = 1, \ldots, n$, we define

$$p^\epsilon = \begin{cases} 0 & \text{on } \partial D_{i,\text{out}}^\epsilon \\ c_i^\epsilon & \text{on } \partial D_{i,\text{in}}^\epsilon = \Gamma_i \end{cases} \qquad \text{for } i = 1, \ldots, n, \tag{4.3.6}$$

where $c_i^\epsilon > 0$ is the unique constant such that

$$\int_{\partial U_i} \nabla p^\epsilon \cdot n d\sigma = -2|U_i| \quad \text{for } i = 1, \ldots, n, \tag{4.3.7}$$

where U_i is the domain enclosed by $\partial D_{i,\text{in}}^\epsilon = \Gamma_i$ (thus U_i is independent of ϵ), and n is the outer normal of U_i (thus the inner normal of D_i^ϵ). The existence of c_i^ϵ is guaranteed by [50, Lemma 2.5]. One can then check that $\int_{\partial U_i} \mathbf{u}^\epsilon \cdot n d\sigma = 0$. Applying the divergence theorem in D_i^ϵ then gives us that $\int_{\partial D_{i,\text{out}}^\epsilon} \mathbf{u}^\epsilon \cdot n d\sigma = 0$ as well.

In [50] we proved a rearrangement inequality for such p^ϵ in a similar spirit of Talenti's rearrangement inequality for elliptic equations [109], which we state below.

Lemma 4.3.1 ([50, Proposition 2.6])**.** *The function* $p^\epsilon : \overline{D^\epsilon} \to \mathbb{R}$ *defined in* (Equation 4.3.4)–(Equation 4.3.7) *satisfies the following in each* D_i^ϵ *for* $i = 1, \dots, n+m$*:*

$$\sup_{D_i^\epsilon} p^\epsilon \le \frac{|D_i^\epsilon|}{2\pi}, \tag{4.3.8}$$

and

$$\int_{D_i^\epsilon} p^\epsilon(x)dx \le \frac{|D_i^\epsilon|^2}{4\pi}. \tag{4.3.9}$$

Moreover, each inequality above achieves equality if and only D_i^ϵ *is either a disk or an annulus.*

Note that the inequalities (Equation 4.3.8)–(Equation 4.3.9) hold for any domain with $C^{1,\alpha}$ boundary. Even though the inequalities are strict when D_i^ϵ is non-radial, they are not strong enough to rule out non-radial vortex sheets, as we need quantitative versions of strict inequalities that are still valid in the $\epsilon \to 0^+$ limit. As we will see in the proof of Proposition 4.4.2, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the following quantitative version of (Equation 4.3.9) holds: $\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \ge c_0 > 0$, where c_0 is independent of ϵ.

In order to upgrade (Equation 4.3.9) into a quantitative version, we need to obtain some fine estimates for p^ϵ that take into account the shape of the thin domains D_i^ϵ. For $i = n+1, \dots, n+m$, since $p^\epsilon = 0$ on ∂D_i^ϵ, and the domain D_i^ϵ is a thin simply-connected domain with width $\epsilon \ll 1$, intuitively one would expect that $|p^\epsilon| \le C\epsilon^2$. The next proposition shows that this crude estimate is indeed true, and its proof is postponed to Section subsection 4.3.1.

Proposition 4.3.2. *For any* $i = n+1, \dots, n+m$*, let* $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ *be given by* (Equation 4.3.4)–(Equation 4.3.5)*. Then there exist* ϵ_1 *and* C *only depending on* $\|z_i\|_{C^2(S_i)}, \|\gamma_i\|_{L^\infty(S_i)}$ *and* F_Γ*, such* *that*

$$|p^\epsilon| \le C\epsilon^2 \quad \textit{in } D_i^\epsilon$$

for all $\epsilon \in (0, \epsilon_1)$.

For $i = 1, \ldots, n$, the estimate is more involved, since p^ϵ takes different values c_i^ϵ and 0 on the inner and outer boundaries of D_i^ϵ. Heuristically speaking, since D_i^ϵ is a doubly-connected thin tubular domain with width $\sim \epsilon$, we would expect that p_i^ϵ (in α, η coordinate) changes almost linearly from 0 to c_i^ϵ as η goes from -1 (outer boundary) to 0 (inner boundary). Next we will show that the error between $p^\epsilon(R_i^\epsilon(\alpha, \eta))$ and the linear-in-η function $c_i^\epsilon(1+\eta)$ is indeed controlled by $O(\epsilon^2)$. We will also obtain fine estimates of the gradient of the function $c_i^\epsilon(1+\eta)$, as well as the boundary value c_i^ϵ. Again, its proof is postponed to Section subsection 4.3.1.

Proposition 4.3.3. *For any* $i = 1, \ldots, n$*, let* $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ *and* $c_i^\epsilon \in \mathbb{R}$ *be given by* (Equation 4.3.4) *and* (Equation 4.3.6)–(Equation 4.3.7). *For such* p^ϵ*, let us define* $\tilde{p}^\epsilon, q^\epsilon : \overline{D_i^\epsilon} \mapsto \mathbb{R}$ *as follows:*

$$\begin{aligned} \tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) &:= c_i^\epsilon(1+\eta) \quad \textit{for } \alpha \in S_i, \eta \in [0,-1], \\ q^\epsilon &:= p^\epsilon - \tilde{p}^\epsilon \qquad \textit{in } \overline{D_i^\epsilon}. \end{aligned} \tag{4.3.10}$$

Also let

$$\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha)d\alpha}. \tag{4.3.11}$$

Then there exist ϵ_1 *and* C *only depending on* $\|z_i\|_{C^3(S_i)}, \|\gamma_i\|_{C^2(S_i)}$ *and* F_Γ*, such that for all* $\epsilon \in$

$(0, \epsilon_1)$ *we have the following:*

$$\begin{cases} |q^\epsilon| \le C\epsilon^2 & \text{in } D_i^\epsilon, \\ q^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases} \tag{4.3.12}$$

$$\left| \frac{c_i^\epsilon}{\epsilon} - \beta_i \right| \le C\epsilon, \tag{4.3.13}$$

$$\left| \nabla \tilde{p}^\epsilon(R_i^\epsilon(\alpha, \eta)) - \frac{\beta_i}{\gamma_i(\alpha)} \mathbf{n}(z_i(\alpha)) \right| \le C\epsilon \quad \text{for } \alpha \in S_i, \eta \in [0, -1]. \tag{4.3.14}$$

4.3.1 Proof of the quantitative lemmas for p^ϵ

In this subsection we aim to prove Propositions 4.3.2 and 4.3.3. We start with a technical lemma on estimating the solution of Poisson's equation (with zero boundary condition) in the domain D_i^ϵ.

Lemma 4.3.4. *For any $i = 1, \ldots, n+m$, assume Γ_i and γ_i satisfy* **(H1)–(H3)**. *Let $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ solve the Poisson's equation with zero boundary condition:*

$$\begin{cases} \Delta v^\epsilon = -1 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon. \end{cases} \tag{4.3.15}$$

Then there exist positive constants $\epsilon_0 = C(\|z_i\|_{C^2(S_i)}, \|\gamma_i\|_{L^\infty(S_i)}, F_\Gamma)$ and $C_1, C_2 = C(\|\gamma_i\|_{L^\infty(S_i)})$, such that for all $\epsilon \in (0, \epsilon_0)$ we have

$$0 \le v^\epsilon \le C_1\epsilon^2 \quad \text{in } D_i^\epsilon \tag{4.3.16}$$

and

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_2\epsilon \quad \text{for } i = 1, \ldots, n. \tag{4.3.17}$$

Proof. Throughout the proof, let $i \in \{1, \ldots, n+m\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i in R_i^ϵ, D_i^ϵ, S_i, z_i and γ_i.

Step 1. We start with a simple geometric result that D^ϵ is "flat" in a small neighborhood of any $z(\alpha)$. For any $\alpha \in S$, let $V^\epsilon(\alpha) := D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$, where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(S)}$. We will show that any $y \in V^\epsilon(\alpha)$ satisfies

$$\big|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\big| \leq 2\epsilon\|\gamma\|_\infty \tag{4.3.18}$$

for all sufficiently small $\epsilon > 0$ (to be quantified in (Equation 4.3.23)). See Figure Figure 4.4(a) for an illustration.

Since $y \in V^\epsilon(\alpha) \subset D^\epsilon$, there exist $\beta \in S$ and $\eta \in (-1, 0)$ such that $y = R^\epsilon(\beta, \eta) = z(\beta) + \epsilon\gamma(\beta)\mathbf{n}(z(\beta))\eta$. It follows that

$$\begin{aligned} \big|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\big| &\leq |(z(\alpha) - z(\beta)) \cdot \mathbf{n}(z(\alpha))| + \epsilon\|\gamma\|_\infty \\ &\leq \|z''\|_\infty(\alpha - \beta)^2 + \epsilon\|\gamma\|_\infty, \end{aligned} \tag{4.3.19}$$

where in the second inequality we used

$$|(z(\alpha) - z(\beta)) - z'(\alpha)(\alpha - \beta)| \leq \|z''\|_\infty(\alpha - \beta)^2 \tag{4.3.20}$$

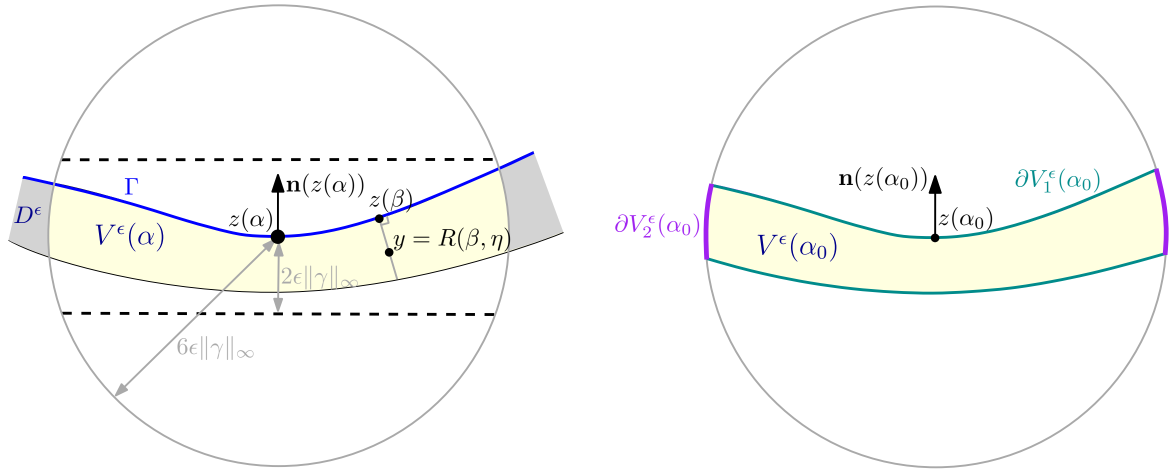
and $z'(\alpha) \cdot \mathbf{n}(z(\alpha)) = 0$. To bound $\alpha - \beta$ on the right hand side of (Equation 4.3.19), the fact that $y \in B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ gives

$$6\epsilon\|\gamma\|_\infty \geq |z(\alpha) - y| \geq |z(\alpha) - z(\beta)| - \epsilon\gamma(\beta), \tag{4.3.21}$$

which implies $|z(\alpha) - z(\beta)| \leq 7\epsilon\|\gamma\|_\infty$. Since the arc-chord constant F_Γ given in (Equation 4.1.2) is finite, this implies

$$|\alpha - \beta| \leq 7F_\Gamma\|\gamma\|_\infty\epsilon. \tag{4.3.22}$$

Plugging this into the right hand side of (Equation 4.3.19), we know (Equation 4.3.18) holds for



(a) (b)

Figure 4.4: (a) In Step 1, $V^\epsilon(\alpha)$ (the yellow set) must lie between the two dashed lines for small ϵ. (b) In Step 2, $\partial V^\epsilon(\alpha_0)$ is decomposed into $\partial V_1^\epsilon(\alpha_0)$ (in dark green) and $\partial V_2^\epsilon(\alpha_0)$ (in purple).

all

$$0 < \epsilon \leq (49\|z''\|_\infty F_\Gamma^2 \|\gamma\|_\infty)^{-1}. \tag{4.3.23}$$

Step 2. Next we prove (Equation 4.3.16). Note that v^ϵ is superharmonic in D^ϵ and vanishes on the boundary, thus it follows from the maximum principle that $v^\epsilon \geq 0$ in D^ϵ. Denote $M := \max_{x \in D^\epsilon} v(x)$, and pick $x_0 = R(\alpha_0, \eta_0) \in D^\epsilon$ such that $v(x_0) = M$. Without loss of generality, we can assume that $z(\alpha_0) = (0,0)$ and $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1 := (1,0)$, so that $\mathbf{n}(z(\alpha_0)) = (0,1)$ and $x_0 = (0, \epsilon\gamma(\alpha_0)\eta_0)$. Let us consider a barrier function $b_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$b_1(x_1, x_2) = x_2^2 - \frac{x_1^2}{2}.$$

Clearly $\Delta b_1 = 1$, so $v^\epsilon + b_1$ is harmonic in D^ϵ. It then follows from the maximum principle that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ is achieved at some boundary point $\tilde{x}_0 \in \partial V^\epsilon(\alpha_0)$. Let us break $\partial V^\epsilon(\alpha_0)$ into $\partial V_1^\epsilon(\alpha_0) \cup \partial V_2^\epsilon(\alpha_0)$ (see Figure Figure 4.4(b) for an illustration), given by

$$\partial V_1^\epsilon(\alpha_0) := \partial D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)), \quad \partial V_2^\epsilon(\alpha_0) := \overline{D^\epsilon} \cap \partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)). \tag{4.3.24}$$

We claim that $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0)$. To see this, note that any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha_0)$ satisfies $|y| = 6\epsilon\|\gamma\|_\infty$ and $|y_2| \leq 2\epsilon\|\gamma\|_\infty$, where the latter follows from (Equation 4.3.18) and our assumption that $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1$. This implies that $|y_1| \geq 4\epsilon\|\gamma\|_\infty > |y_2|$, thus $b_1(y) < 0$. Using that $v^\epsilon(x_0) = M \geq v^\epsilon(y)$ and $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \geq 0$, we have $(v^\epsilon + b_1)(y) < (v^\epsilon + b_1)(x_0)$. This shows that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ cannot be achieved on $\partial V_2^\epsilon(\alpha_0)$, finishing the proof of the claim.

Since $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0) \subset \partial D^\epsilon$, the boundary condition in (Equation 4.3.15) yields that $v^\epsilon(\tilde{x}_0) = 0$. Thus

$$M + b_1(x_0) = v^\epsilon(x_0) + b_1(x_0) \leq v^\epsilon(\tilde{x}_0) + b_1(\tilde{x}_0) = b_1(\tilde{x}_0).$$

Using $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \geq 0$, the above inequality becomes

$$M \leq b_1(\tilde{x}_0) \leq |\tilde{x}_0|^2 \leq 36\|\gamma\|_\infty^2 \epsilon^2, \tag{4.3.25}$$

where the second inequality follows from the definition of b_1. This proves (Equation 4.3.16) for $C_1 = 36\|\gamma\|_\infty^2$.

Step 3. It remains to prove (Equation 4.3.17). First note that for $i \in \{1, \ldots, n\}$, the assumptions **(H1)–(H3)** yield that D_i^ϵ has C^2 boundary, therefore $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$. Let us fix $i \in \{1, \ldots, n\}$ and any $\alpha \in S$, and we aim to show that $|\nabla v^\epsilon(z(\alpha))| \leq C_2\epsilon$. Again, without loss of generality we can assume that $z(\alpha) = (0, 0)$ and $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$. Let us consider a new barrier function $b_2 : \mathbb{R}^2 \to \mathbb{R}$

$$b_2(x_1, x_2) := x_2^2 + 4\epsilon\|\gamma\|_\infty x_2 - \frac{x_1^2}{2}, \tag{4.3.26}$$

which satisfies $b_2(0, 0) = 0$, and one can easily check that its zero level set has horizontal tangent at $(0, 0)$ (thus tangent to ∂D^ϵ at $z(\alpha)$).

Again, let us decompose $\partial V^\epsilon(\alpha)$ as $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$ as in (Equation 4.3.24) (except that α_0

now becomes α). We claim that for all sufficiently small $\epsilon > 0$, the new barrier function b_2 satisfies

$$\Delta b_2 = 1 \quad \text{in } V^\epsilon(\alpha), \tag{4.3.27}$$

$$b_2 \leq 0 \qquad \text{on } \partial V_1^\epsilon, \tag{4.3.28}$$

$$b_2 \leq -\epsilon^2 \quad \text{on } \partial V_2^\epsilon. \tag{4.3.29}$$

Let us assume for a moment that (Equation 4.3.27)–(Equation 4.3.29) are true. Then it follows that

$$v^\epsilon + C_2 b_2 \leq 0 \text{ in } V^\epsilon(\alpha), \tag{4.3.30}$$

where $C_2 := \max\{1, C_1\}$ and C_1 is as in (Equation 4.3.16) (in the end of step 2 we have $C_1 = 36\|\gamma\|_\infty^2$). To show (Equation 4.3.30), note that $v^\epsilon + C_2 b_2$ is subharmonic in $V^\epsilon(\alpha)$ due to (Equation 4.3.27) and the definition of C_2, thus its maximum is attained on its boundary. The boundary conditions in (Equation 4.3.15) and (Equation 4.3.28) yield that $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_1^\epsilon(\alpha)$; whereas (Equation 4.3.16), (Equation 4.3.29) and the definition of C_2 yield that $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_2^\epsilon(\alpha)$. Thus $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$, implying (Equation 4.3.30).

However, $v^\epsilon + C_2 b_2$ is actually zero at $z(\alpha) \in \partial V^\epsilon(\alpha)$, therefore Hopf's Lemma implies that $\nabla \left(v^\epsilon + C_2 b_2\right)(z(\alpha)) \cdot \vec{n}(z(\alpha)) > 0$, where $\vec{n}(z(\alpha))$ is the outer normal of ∂D^ϵ at $z(\alpha)$. Hence

$$|\nabla v^\epsilon(z(\alpha))| = -\nabla v^\epsilon(z(\alpha)) \cdot \vec{n}(z(\alpha)) < C_2 \nabla b_2(z(\alpha)) \cdot \vec{n}(z(\alpha)) = 4C_2\|\gamma\|_\infty \epsilon, \tag{4.3.31}$$

where the first equality follows from the fact that v^ϵ is superharmonic in D^ϵ and constant on ∂D^ϵ, and the second equality is a direct computation of ∇b_2. Thus (Equation 4.3.31) proves (Equation 4.3.17).

To complete the proof, we only need to prove (Equation 4.3.27)–(Equation 4.3.29) for small $\epsilon > 0$. Note that (Equation 4.3.27) follows immediately from computing the Laplacian of b_2. For (Equation 4.3.28), let us pick $y \in \partial V_1^\epsilon(\alpha)$, and we aim to show that $b_2(y) \leq 0$. Note that

$y = R^\epsilon(\beta, 0)$ or $R^\epsilon(\beta, -1)$ for some $\beta \in S$. We first deal with the first case.

Let us denote $y = (y_1, y_2)$. Rewriting (Equation 4.3.20) into two inequalities for the two components, and using that $z(\alpha) = (0, 0)$ and $z'(\alpha) = L\mathbf{e}_1$ (L is the length of the curve Γ_i), we have

$$|0 - y_1 - L(\alpha - \beta)| \leq \|z''\|_\infty (\alpha - \beta)^2 \tag{4.3.32}$$

$$|y_2| = |0 - y_2| \leq \|z''\|_\infty (\alpha - \beta)^2. \tag{4.3.33}$$

Also, (Equation 4.3.22) gives $|\alpha - \beta| \leq 7F_\Gamma \|\gamma\|_\infty \epsilon$. Applying it to (Equation 4.3.32), for all $\epsilon > 0$ sufficiently small we have that

$$|y_1| \geq \frac{L}{2}|\alpha - \beta|. \tag{4.3.34}$$

Plugging (Equation 4.3.34) and (Equation 4.3.33) into $b_2(y) = -\frac{1}{2}y_1^2 + y_2^2 + 4\epsilon\|\gamma\|_\infty y_2$, we have

$$\begin{aligned} b_2(y) &\leq -\frac{L^2}{8}(\alpha - \beta)^2 + \|z''\|_\infty^2 (\alpha - \beta)^4 + 4\epsilon\|\gamma\|_\infty \|z''\|_\infty (\alpha - \beta)^2 \\ &\leq \left(-\frac{L^2}{8} + C\epsilon^2 + C\epsilon \right)(\alpha - \beta)^2 \leq 0, \end{aligned}$$

for all $\epsilon > 0$ sufficiently small, where the second inequality follows from (Equation 4.3.22). This finishes the proof of (Equation 4.3.28) for the case $y = R^\epsilon(\beta, 0)$.

Before we deal with the case $y = R^\epsilon(\beta, -1)$, let us prove (Equation 4.3.29) first. For any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha)$, (Equation 4.3.18) gives $|y_2| \leq 2\epsilon\|\gamma\|_\infty$. Combining this with $|y| = 6\epsilon\|\gamma\|_\infty$ yields $|y_1| \geq \sqrt{32}\epsilon\|\gamma\|_\infty$. Thus

$$b_2(y) \leq (2\epsilon\|\gamma\|_\infty)^2 + 4\epsilon\|\gamma\|_\infty(2\epsilon\|\gamma\|_\infty) - \frac{(\sqrt{32}\epsilon\|\gamma\|_\infty)^2}{2} \leq -4\epsilon^2\|\gamma\|_\infty^2.$$

Finally we turn to the proof of (Equation 4.3.28) for the case $y = R^\epsilon(\beta, -1)$. Note that the curve $\{R^\epsilon(\beta, -1) : \beta \in S\} \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ lies in the interior of the region bounded by $\Gamma \cap$

$B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the top, $\partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the sides, and $y_2 = -2\epsilon\|\gamma\|_\infty$ on the bottom. (The last one follows from (Equation 4.3.18) and our assumption that $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$). We have already shown $b_2 \leq 0$ on $\Gamma \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ and the lateral boundaries, and it is easy to check that $b_2 \leq 0$ on $y_2 = -2\epsilon\|\gamma\|_\infty$. Since the set $\{b_2 \leq 0\}$ is simply-connected, it implies that $b_2 \leq 0$ in the interior of this region, finishing the proof. $\square$

Note that (Equation 4.3.16) of Lemma 4.3.4 immediately implies Proposition 4.3.2. (The only difference is that $\Delta v^\epsilon = -1$ in Lemma 4.3.4 whereas $\Delta p^\epsilon = -2$ in Proposition 4.3.2, so the constant C in Proposition 4.3.2 is twice of that in (Equation 4.3.16)). The lemma also implies the following corollary, which will be helpful in the proof of Proposition 4.3.3.

Corollary 4.3.5. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy* **(H1)–(H3)**. *Assume $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ satisfies that*

$$\begin{cases} |\Delta v^\epsilon| \leq C_0 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases}$$

for some constant $C_0 > 0$. Then for the same constants ϵ_0, C_1, C_2 as in Lemma 4.3.4, the following holds for all $\epsilon \in (0, \epsilon_0)$:

$$|v^\epsilon| \leq C_0 C_1 \epsilon^2 \quad \text{in } D_i^\epsilon, \tag{4.3.35}$$

and if $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$, we have

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \leq C_0 C_2 \epsilon \quad \text{for} \quad i = 1, \dots, n. \tag{4.3.36}$$

Proof. Let $\tilde{v}$ be a solution to

$$\begin{cases} \Delta \tilde{v} = -C_0 & \text{in } D_i^\epsilon, \\ \tilde{v} = 0 & \text{on } \partial D_i^\epsilon. \end{cases}$$

It is clear that $v^\epsilon + \tilde{v}$ is super-harmonic and $v^\epsilon - \tilde{v}$ is sub-harmonic in D_i^ϵ, and they both vanish on the boundary. Thus the maximum principle implies that

$$-\tilde{v} \leq v^\epsilon \leq \tilde{v} \quad \text{ in } D_i^\epsilon. \tag{4.3.37}$$

Applying (Equation 4.3.16) of Lemma 4.3.4 to $\frac{\tilde{v}}{C_0}$, we obtain $0 \leq \tilde{v} \leq C_0 C_1 \epsilon^2$ in D_i^ϵ for all $\epsilon \in (0, \epsilon_0)$, leading to (Equation 4.3.35). Furthermore, (Equation 4.3.37) and the fact that v^ϵ and v both have zero boundary condition imply that

$$|\nabla v^\epsilon| \leq |\nabla \tilde{v}| \quad \text{ on } \partial D_i^\epsilon.$$

We then apply (Equation 4.3.17) of Lemma 4.3.4 to $\frac{\tilde{v}}{C_0}$ and obtain $\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \leq C_0 C_2 \epsilon$, which proves (Equation 4.3.36). □

Now we are ready to prove Proposition 4.3.3.

Proof of Proposition 4.3.3. Throughout the proof, let $i \in \{1, \ldots, n\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i from all terms.

We claim that

$$\left| \nabla \tilde{p}^\epsilon (R^\epsilon(\alpha, \eta)) - \frac{c^\epsilon}{\epsilon \gamma(\alpha)} \mathbf{n}(z(\alpha)) \right| \leq C\epsilon \quad \text{ for all } \alpha \in S, \eta \in [0, -1], \tag{4.3.38}$$

$$\|\Delta q^\epsilon\|_{L^\infty(D^\epsilon)} \leq C \tag{4.3.39}$$

for some constant $C > 0$ only depending on $\|z_i\|_{C^3(S_i)}, \|\gamma_i\|_{C^2(S_i)}$ and F_Γ. Assuming these are true, let us explain how they lead to (Equation 4.3.12)–(Equation 4.3.14). By (Equation 4.3.6) and (Equation 4.3.10), p^ϵ and $\tilde{p}^\epsilon$ have the same boundary condition, thus $q^\epsilon = 0$ on ∂D^ϵ. This and (Equation 4.3.39) allow us to apply Corollary 4.3.5 to q^ϵ to obtain the estimate (Equation 4.3.35), implying (Equation 4.3.12).

Due to (Equation 4.3.36) of Corollary 4.3.5, we also have

$$\|\nabla q^\epsilon\|_{L^\infty(\Gamma)} \leq C\epsilon. \tag{4.3.40}$$

Using (Equation 4.3.7) and $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$, we have

$$\begin{aligned} -2|U| &= \int_{\partial U} \nabla \tilde{p}^\epsilon \cdot n d\sigma + \int_{\partial U} \nabla q^\epsilon \cdot n d\sigma \\ &= -\frac{c^\epsilon L}{\epsilon} \int_S \gamma^{-1}(\alpha) d\alpha + O(\epsilon), \end{aligned}$$

where the second equality follows from (Equation 4.3.38) for $\eta = 0$, $n(z(\alpha)) = -\mathbf{n}(z(\alpha))$ and $d\sigma = Ld\alpha$, as well as (Equation 4.3.40). Rearranging the terms and using the definition of β in (Equation 4.3.11) yields (Equation 4.3.13).

Finally, note that (Equation 4.3.13) and (Equation 4.3.38) directly lead to (Equation 4.3.14), where we are using the fact that γ_i is uniformly positive for $i = 1, \ldots, n$, due to **(H3)**.

The rest of the proof is devoted to proving the claims (Equation 4.3.38) and (Equation 4.3.39). For (Equation 4.3.38), we compute the gradient of $\tilde{p}^\epsilon$. Differentiating (Equation 4.3.10) with respect to α and η, we obtain

$$(\nabla_{\alpha,\eta} R^\epsilon(\alpha,\eta))^t \nabla \tilde{p}(R^\epsilon(\alpha,\eta)) = \begin{pmatrix} 0 \\ c^\epsilon \end{pmatrix}, \tag{4.3.41}$$

where $(\nabla_{\alpha,\eta} R^\epsilon)^t$ denotes the transpose of the Jacobian matrix of R^ϵ. Since $\nabla_{\alpha,\eta} R^\epsilon = (\partial_\alpha R^\epsilon, \partial_\eta R^\epsilon)$, using the formula for inverses of 2×2 matrices, we have

$$\big((\nabla_{\alpha,\eta} R^\epsilon)^t\big)^{-1} = \frac{1}{J(\alpha,\eta)} \big(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp\big). \tag{4.3.42}$$

where $J(\alpha,\eta) := \det(\nabla_{\alpha,\eta} R^\epsilon)$. Multiplying the inverse matrix on both sides of (Equation 4.3.41), we have

$$\nabla\tilde{p}^\epsilon(R^\epsilon(\alpha,\eta)) = \frac{1}{J}\big(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp\big)\begin{pmatrix}0\\ c^\epsilon\end{pmatrix} = \frac{c^\epsilon}{J}(\partial_\alpha R^\epsilon)^\perp. \tag{4.3.43}$$

Recall that Lemma 4.2.2 gives $(\partial_\alpha R^\epsilon)^\perp = z'(\alpha)^\perp + O(\epsilon) = L\mathbf{n}(z(\alpha)) + O(\epsilon)$, and $J = \epsilon L\gamma + O(\epsilon^2)$. Plugging these into (Equation 4.3.43) gives

$$\nabla\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{\epsilon}\left(\frac{\mathbf{n}(z(\alpha))}{\gamma} + O(\epsilon)\right). \tag{4.3.44}$$

Note that it follows from (Equation 4.3.8) that $c^\epsilon \le \frac{|D^\epsilon|}{2\pi}$, where $|D^\epsilon| \le C\epsilon$ due to (Equation 4.2.9). These imply

$$\frac{c_\epsilon}{\epsilon} \le C, \tag{4.3.45}$$

and applying it to (Equation 4.3.44) yields (Equation 4.3.38).

To prove (Equation 4.3.39), since $q^\epsilon = p^\epsilon - \tilde{p}^\epsilon$ and $\Delta p^\epsilon = -2$ in D^ϵ, it suffices to show that

$$|\Delta\tilde{p}^\epsilon| \le C \quad \text{in } D^\epsilon, \tag{4.3.46}$$

and we will begin with an explicit computation of $\partial_{x_1x_1}\tilde{p}^\epsilon$ and $\partial_{x_2x_2}\tilde{p}^\epsilon$. Let us denote $R^\epsilon =: (R^1, R^2)$. For notational simplicity, in the rest of the proof we will use subscripts on R^ϵ, R^1, R^2 and J to denote their partial derivative, e.g. $R^1_\alpha := \partial_\alpha R^1$.

From (Equation 4.3.43), it follows that

$$\partial_{x_1}\tilde{p}^\epsilon(R^\epsilon(\alpha,\eta)) = -\frac{c^\epsilon}{J}R^2_\alpha.$$

Differentiating in α and η, we get

$$\begin{aligned}\nabla\left(\partial_{x_1}\tilde{p}^\epsilon\right)\left(R^\epsilon(\alpha,\eta)\right) &= \left((\nabla_{\alpha,\eta}R^\epsilon)^t\right)^{-1}\nabla_{\alpha,\eta}\left(-\frac{c^\epsilon}{J}R^2_\alpha\right)\\ &= \frac{c^\epsilon}{J}\begin{pmatrix} R^2_\eta & -R^2_\alpha \\ -R^1_\eta & R^1_\alpha\end{pmatrix}\begin{pmatrix}\frac{J_\alpha}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\alpha}\\ \frac{J_\eta}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\eta}\end{pmatrix},\end{aligned}$$

thus

$$\partial_{x_1x_1}\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^2_\eta R^2_\alpha - \frac{1}{J}R^2_\eta R^2_{\alpha\alpha} - \frac{J_\eta}{J^2}(R^2_\alpha)^2 + \frac{1}{J}R^2_\alpha R^2_{\alpha\eta}\right).$$

Likewise, $\partial_{x_2x_2}\tilde{p}(R(\alpha,\eta))$ takes the same expression except every R^2 is changed into R^1. Adding them together gives

$$\Delta\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^\epsilon_\eta\cdot R^\epsilon_\alpha - \frac{1}{J}R^\epsilon_\eta\cdot R^\epsilon_{\alpha\alpha} - \frac{J_\eta}{J^2}R^\epsilon_\alpha\cdot R^\epsilon_\alpha + \frac{1}{J}R^\epsilon_\alpha\cdot R^\epsilon_{\alpha\eta}\right). \tag{4.3.47}$$

Using the explicit formulae of R_α, R_η and J in Lemma 4.2.2, we directly obtain $|R^\epsilon_\alpha|, |R^\epsilon_{\alpha\alpha}| \le C$; $|R^\epsilon_\eta|, |R^\epsilon_{\alpha\eta}|, |J_\alpha| \le C\epsilon$; $|J_\eta| \le C\epsilon^2$; and $J^{-1} \le C\epsilon^{-1}$ when ϵ is sufficiently small, where C depends on $\|z_i\|_{C^3(S_i)}$ and $\|\gamma_i\|_{C^2(S_i)}$. As a result, all the four terms in the parenthesis of (Equation 4.3.47) are bounded by some constant C independent of ϵ. Finally, (Equation 4.3.45) yields $\frac{c_\epsilon}{J} \le C$ as well, thus $|\Delta\tilde{p}^\epsilon| \le C$, and this proves the second claim (Equation 4.3.39). $\square$

4.4 Proof of the symmetry result

In this section we prove that a stationary vortex sheet with positive vorticity must be radially symmetric up to a translation, and a rotating vortex sheet with positive vorticity and angular velocity $\Omega < 0$ must be radially symmetric. The key idea of the proof is to define the integral

$$\begin{aligned} I^\epsilon &:= \int_{D^\epsilon}\epsilon^{-1}\mathbf{u}^\epsilon\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx\\ &= \int_{D^\epsilon}\epsilon^{-1}(x+\nabla p^\epsilon)\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx,\end{aligned} \tag{4.4.1}$$

and compute it in two different ways. The motivation of the definition is as follows. As discussed in [50, Section 2.1], I^ϵ can be thought of as a first variation of an "energy functional"

$$\mathcal{E}[\omega^\epsilon] := \int \frac{1}{2}\omega^\epsilon(\omega^\epsilon * \mathcal{N}) - \frac{\Omega}{2}\omega^\epsilon |x|^2 \, dx$$

when we perturb ω^ϵ by a divergence free vector $\mathbf{u}^\epsilon$ in D^ϵ. (This functional $\mathcal{E}$ only serves as our motivation, and will not appear in the proof.) On the one hand, using that ω_0 is stationary in the rotating frame with angular velocity Ω and ω^ϵ is a close approximation of ω_0, we will show in Proposition 4.4.1 that I^ϵ is of order $O(\epsilon|\log\epsilon|)$, thus goes to zero as $\epsilon \to 0$. On the other hand, using the particular $\mathbf{u}^\epsilon$ that we constructed in Section section 4.3, we will prove in Proposition 4.4.2 that if $\Omega = 0$, I^ϵ is strictly positive independently of ϵ unless all the vortex sheets are nested circles with constant density; and also prove a similar result in Corollary 4.4.3 for $\Omega < 0$.

Proposition 4.4.1. *Assume $\omega(\cdot, t) = \omega_0(R_{\Omega t}\cdot)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, where ω_0 satisfies* **(H1)–(H3)**. *Then there exists some $C > 0$ only depending on b (as in* **(H2)***), $\max_i \|z_i\|_{C^3(S_i)}$, $\max_{i\le n}\|\gamma_i\|_{C^2(S_i)}$, $\max_{i>n}\|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ, such that $|I^\epsilon| < C\epsilon^b|\log\epsilon|$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose $I^\epsilon =: \sum_{i=1}^{n+m} I_i^\epsilon$, where $I_i^\epsilon := \int_{D_i^\epsilon} \epsilon^{-1}(x + \nabla p^\epsilon)\cdot\nabla(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2)dx$.

We start with showing that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1, \dots, n+m$. For such i, $p^\epsilon = 0$ on ∂D_i^ϵ, thus the divergence theorem (and the fact that $\omega^\epsilon = \epsilon^{-1}$ in D_i^ϵ) gives

$$I_i^\epsilon = \underbrace{\int_{D_i^\epsilon} \epsilon^{-1} x \cdot \nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right) dx}_{=:T_i^\epsilon} - \int_{D_i^\epsilon} \epsilon^{-1}(\epsilon^{-1} - 2\Omega)p^\epsilon(x)dx.$$

Using the estimate $|p^\epsilon| \le C\epsilon^2$ in Proposition 4.3.2 and the fact that $|D_i^\epsilon| \le C\epsilon$ from (Equation 4.2.9), we easily bound the second integral by $C\epsilon$. To control the first integral T_i^ϵ, we rewrite it using the change of variables $x = R_i^\epsilon(\alpha,\eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ in (Equation 4.2.10): (also note that on the right hand side we group ϵ^{-1} with the determinant)

$$T_i^\epsilon = \int_{S_i}\int_{-1}^0 R_i^\epsilon(\alpha,\eta)\cdot \underbrace{\Big(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta)\Big)}_{=:J_i^\epsilon} \underbrace{\epsilon^{-1}\det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon}\, d\eta d\alpha.$$

Let us take a closer look at the integrand, which is a product of 3 terms. Clearly, the definition of R_i^ϵ gives $R_i^\epsilon(\alpha,\eta) = z_i(\alpha) + O(\epsilon)$. As for the middle term J_i^ϵ, Lemma 4.2.4 yields

$$J_i^\epsilon(\alpha,\eta) = -BR^\perp(z_i(\alpha)) + \left(\eta+\frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) - \Omega z_i(\alpha) + O(\epsilon^b|\log\epsilon|). \tag{4.4.2}$$

Using the fact that $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$ for $i = n+1,\ldots,n+m$ (which follows from (Equation 5.1.1) and (Equation 5.1.2)), it becomes

$$J_i^\epsilon(\alpha,\eta) = \left(\eta+\frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|). \tag{4.4.3}$$

Also it follows from (Equation 4.2.8) that $K_i^\epsilon(\alpha,\eta) = L_i\gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the above integral gives

$$T_i^\epsilon = \int_{S_i}\int_{-1}^0 z_i(\alpha)\cdot\left(\eta+\frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha))L_i\gamma_i(\alpha)d\eta d\alpha + O(\epsilon^b|\log\epsilon|) = O(\epsilon^b|\log\epsilon|),$$

where the last step follows from the fact that $\int_{-1}^0(\eta+\frac{1}{2})d\eta = 0$. This finishes the proof that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1,\ldots,n+m$, where C depends on b, $\max_i\|z_i\|_{C^2(S_i)}$, $\max_i\|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ.

In the rest of the proof we aim to show $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = 1,\ldots,n$, which is slightly more involved. Recall that in Proposition 4.3.3 we defined $\tilde{p}^\epsilon$ and q^ϵ in D_i^ϵ for $i = 1,\ldots,n$, where they satisfy $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$ in D_i^ϵ, and $q^\epsilon = 0$ on ∂D_i^ϵ. This allows us to apply the divergence theorem (to the q^ϵ term only) and decompose I_i^ϵ as

$$I_i^\epsilon = \int_{D_i^\epsilon} \epsilon^{-1}(x + \nabla \tilde{p}_\epsilon) \cdot \nabla \left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2 \right) dx - \int_{D_i^\epsilon} \epsilon^{-1}(\epsilon^{-1} - 2\Omega) q^\epsilon(x) dx =: I_{i,1}^\epsilon + I_{i,2}^\epsilon.$$

We can easily show that $I_{i,2}^\epsilon = O(\epsilon)$: (Equation 4.3.12) of Proposition 4.3.3 gives $|q^\epsilon| \leq C\epsilon^2$, and combining it with $|D_i^\epsilon| \leq C\epsilon$ in (Equation 4.2.9) immediately yields the desired estimate.

Next we turn to $I_{i,1}^\epsilon$. Again, the change of variables $x = R_i^\epsilon(\alpha, \eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ gives

$$I_{i,1}^\epsilon = \int_{S_i} \int_{-1}^0 \big(R_i^\epsilon(\alpha,\eta) + \nabla \tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta))\big) \cdot \underbrace{\Big(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta) \Big)}_{=:J_i^\epsilon} \underbrace{\epsilon^{-1} \det(\nabla_{\alpha,\eta} R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon} \, d\eta d\alpha.$$

For the three terms in the product of the integrand, we will approximate the first term using the definition of R_i^ϵ and (Equation 4.3.14) of Proposition 4.3.3:

$$R_i^\epsilon(\alpha,\eta) + \nabla \tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) = z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)} \mathbf{n}(z_i(\alpha)) + O(\epsilon),$$

where $\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha) d\alpha}$ is given by (Equation 4.3.11). Lemma 4.2.4 allows us to approximate the middle term J_i^ϵ as (Equation 4.4.2), however (Equation 4.4.3) no longer holds since for $i = 1, \ldots, n$ we do not have $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$. As for K_i^ϵ, we again use (Equation 4.2.8) to approximate it by $K_i^\epsilon(\alpha,\eta) = L_i \gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the integrand of $I_{i,1}^\epsilon$ gives

$$I_{i,1}^\epsilon = \int_{S_i} \left(z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)} \mathbf{n}(z_i(\alpha)) \right) \cdot \Big(-BR^\perp(z_i(\alpha)) - \Omega z_i(\alpha) \Big) L_i \gamma_i(\alpha) d\alpha + O(\epsilon^b |\log \epsilon|),$$

where we again use the fact that the $(\eta + \frac{1}{2})$ term gives zero contribution since $\int_{-1}^0 (\eta + \frac{1}{2}) d\eta = 0$. Next we will show the integral on the right hand side is in fact 0. Since ω is a rotating solution with angular velocity Ω, the conditions (Equation 5.1.1) and (Equation 5.1.2) yield that

$$-BR^{\perp}(z_i(\alpha)) - \Omega z_i(\alpha) = C_i\gamma_i^{-1}(\alpha)\mathbf{n}(z_i(\alpha)),$$

for some constant C_i. Plugging this into the above integral gives

$$\begin{aligned} I_{i,1}^{\epsilon} &= C_i L_i \int_{S_i} \left(z_i(\alpha)\cdot \mathbf{n}(z_i(\alpha)) + \frac{\beta_i}{\gamma_i(\alpha)} \right) d\alpha + O(\epsilon^b |\log \epsilon|) \\ &= C_i L_i \left(\int_{S_i} z_i(\alpha)\cdot \mathbf{n}(z_i(\alpha)) d\alpha + \frac{2|U_i|}{L_i} \right) + O(\epsilon^b |\log \epsilon|), \end{aligned}$$

where the second step follows from the definition of β_i in (Equation 4.3.11). Let us compute the integral on the right hand side by changing to arclength parametrization and applying the divergence theorem:

$$\int_{S_i} z_i(\alpha)\cdot \mathbf{n}(z_i(\alpha))d\alpha = -\frac{1}{L_i}\int_{\partial U_i} x\cdot n d\sigma = -\frac{2|U_i|}{L_i},$$

which yields $I_{i,1}^{\epsilon} = O(\epsilon^b|\log\epsilon|)$, and finishes the proof that $|I_i^{\epsilon}| \le C\epsilon^b|\log\epsilon|$ for $i = 1,\dots,n$, where C depends on b, $\|z_i\|_{C^3(S_i)}$, $\|\gamma_i\|_{C^2(S_i)}$, d_Γ and F_Γ.

Finally, summing the I_i^{ϵ} estimates for $i = 1,\dots,n+m$ gives $|I^{\epsilon}| \le C\epsilon^b|\log\epsilon|$ for all sufficiently small $\epsilon > 0$, thus we can conclude. $\square$

Now we will use a different way to compute I^{ϵ}. Let us first define a new integral $\tilde{I}^{\epsilon}$ that is the same as I^{ϵ} except with Ω set to zero:

$$\tilde{I}^{\epsilon} := \int_{D^{\epsilon}} \epsilon^{-1}(x + \nabla p^{\epsilon})\cdot \nabla\left(\omega^{\epsilon} * \mathcal{N}\right) dx. \tag{4.4.4}$$

Next we will prove that $\tilde{I}^{\epsilon}$ is strictly positive independently of ϵ unless all the vortex sheets are nested circles with constant density. As we will see in the proof, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the estimates on p^{ϵ} in Propositions 4.3.2– 4.3.3 lead to the following quantitative version of (Equation 4.3.9):

$\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \geq c_0 > 0$, where c_0 is independent of ϵ.

Proposition 4.4.2. *Let $\tilde{I}^\epsilon$ be defined as in* (Equation 4.4.4). *Assume that Γ_i and γ_i satisfy* **(H1)–(H3)** *for $i = 1, \dots, n+m$. Then we have $\tilde{I}^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$.*

*In addition, if Γ is **not** a union of nested circles with constant γ_i's on each connected component, there exists some $c_0 > 0$ independent of ϵ, such that $\tilde{I}^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. We start by decomposing $\tilde{I}^\epsilon$ as

$$\tilde{I}^\epsilon = \int_{D^\epsilon} \epsilon^{-1} x \cdot \nabla(\omega^\epsilon * \mathcal{N}) dx + \int_{D^\epsilon} \epsilon^{-1} \nabla p^\epsilon \cdot \nabla(\omega^\epsilon * \mathcal{N}) dx =: I_1^\epsilon + I_2^\epsilon.$$

I_1^ϵ can be easily computed as

$$I_1^\epsilon = \frac{1}{2\pi\epsilon^2} \int_{D^\epsilon} \int_{D^\epsilon} \frac{x \cdot (x-y)}{|x-y|^2} dxdy = \frac{|D^\epsilon|^2}{4\pi\epsilon^2} = \frac{1}{4\pi\epsilon^2} \left(\sum_{i=1}^{n+m} |D_i^\epsilon| \right)^2 \tag{4.4.5}$$

where the second equality is obtained by exchanging x with y and taking the average with the original integral. As for I_2^ϵ, we have

$$\begin{aligned} I_2^\epsilon &= \frac{1}{\epsilon} \int_{\partial D^\epsilon} p^\epsilon \nabla(\omega^\epsilon * \mathcal{N}) \cdot n d\sigma - \frac{1}{\epsilon} \int_{D^\epsilon} p^\epsilon \omega^\epsilon dx \\ &= -\frac{1}{\epsilon} \sum_{i=1}^{n} c_i^\epsilon \int_{\partial U_i} \nabla(\omega^\epsilon * \mathcal{N}) \cdot n d\sigma - \frac{1}{\epsilon^2} \int_{D^\epsilon} p^\epsilon dx \\ &\geq -\frac{1}{\epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n+m} \frac{|D_i^\epsilon|}{2\pi} \int_{U_i} 1_{D_j^\epsilon} dx - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p^\epsilon dx, \end{aligned} \tag{4.4.6}$$

where the first equality follows from the divergence theorem, the second equality follows from the boundary conditions (Equation 4.3.5) and (Equation 4.3.6) for p^ϵ (as well as the fact that ∂U_i and ∂D_i^ϵ have opposite outer normals), and the last inequality follows from the divergence theorem as well as the inequality $c_i^\epsilon \leq \sup_{D_i^\epsilon} p \leq \frac{|D_i^\epsilon|}{2\pi}$ due to (Equation 4.3.8).

Let us denote $j \prec i$ if $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n+m\}$, $j \neq i$ and Γ_j lies in the interior of the domain enclosed by Γ_i (that is, $\Gamma_j \subset U_i$). If not, we denote $j \not\prec i$. Note that for sufficiently small $\epsilon > 0$, we have

$$\int_{U_i} 1_{D_j^\epsilon} dx = \begin{cases} |D_j^\epsilon| & \text{if } j \prec i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.7}$$

Applying this to (Equation 4.4.6) yields

$$\begin{aligned} I_2^\epsilon &\geq -\frac{1}{2\pi\epsilon^2} \sum_{i,j=1}^{n+m} \mathbb{1}_{j \prec i} |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \\ &= -\frac{1}{4\pi\epsilon^2} \sum_{i,j=1}^{n+m} (\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j}) |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \end{aligned} \tag{4.4.8}$$

where in the first step we used that the $i = n+1, \dots, n+m$ terms have zero contribution in the first sum, due to the definition of $j \prec i$.

Adding (Equation 4.4.5) and (Equation 4.4.8) together, we obtain

$$\tilde{I}^\epsilon \geq \sum_{i=1}^{n+m} \underbrace{\frac{1}{\epsilon^2} \left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p_i^\epsilon dx \right)}_{=:A_i^\epsilon} + \sum_{i,j=1}^{n+m} \underbrace{\frac{1}{\epsilon^2} \left(\mathbb{1}_{i \neq j} - (\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j}) \right) \frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi}}_{=:B_{i,j}^\epsilon}, \tag{4.4.9}$$

From (Equation 4.3.9), it follows that $A_i^\epsilon \geq 0$ for all $i = 1, \dots, n+m$, with equality achieved if and only if each D_i^ϵ is a disk or an annulus. Note that $B_{i,j}^\epsilon \geq 0$ as well for all i and j, since for any $i \neq j$, at most one of $i \prec j$ and $j \prec i$ can hold. Putting these together yields that $\tilde{I}^\epsilon \geq 0$ for any sufficiently small $\epsilon > 0$.

In the rest of the proof, we assume Γ is **not** a union of nested circles with constant γ_i's on each connected component. Therefore at least one of the following 3 cases must be true. In each case we aim to show that $\tilde{I}_\epsilon \geq c_0 > 0$, where c_0 is independent of ϵ for all sufficiently small $\epsilon > 0$.

Case 1. There exists some open curve Γ_i that is not a loop. In this case D_i^ϵ is simply-connected, and $p^\epsilon = 0$ on ∂D_i^ϵ by (Equation 4.3.5). Applying Proposition 4.3.2 to p^ϵ in D_i^ϵ, we have $\sup_{D_i^\epsilon} p^\epsilon \leq C\epsilon^2$, where C is independent of ϵ. This leads to $\int_{D_i^\epsilon} p^\epsilon dx \leq C\epsilon^3$, since $|D_i^\epsilon| = O(\epsilon)$ by (Equation 4.2.9). As a result, for the index i we have

$$A_i^\epsilon = \frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} - \epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx \geq \frac{L_i^2}{4\pi}\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)^2 - C\epsilon,$$

where we again used (Equation 4.2.9) in the second inequality. This gives that $A_i^\epsilon \geq \frac{L_i^2}{8\pi}(\int_{S_i}\gamma_i(\alpha)d\alpha)^2 > 0$ for all sufficiently small $\epsilon > 0$.

Case 2. There exists some closed curve Γ_i that is either not a circle, or γ_i is not a constant. In this case we aim to show that $A_i^\epsilon \geq c_0 > 0$, and this will be done by finding good approximations (independent of ϵ) for both terms in A_i^ϵ. For the first term $\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2}$, using (Equation 4.2.9) we again have

$$\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} \geq \frac{L_i^2}{4\pi}\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)^2 - C\epsilon =: J_i - C\epsilon, \tag{4.4.10}$$

where $J_i > 0$ is independent of ϵ. For the second term $\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx$, rewriting the integral using the change of variables $x = R_i^\epsilon(a,\eta)$ gives

$$\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx = \int_{S_i}\int_{-1}^{0}\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon}\frac{\det(\nabla_{\alpha,\eta}R_i^\epsilon)}{\epsilon}d\eta d\alpha.$$

Recall that in Proposition 4.3.3 we defined $\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) := c_i^\epsilon(1+\eta)$ and q_ϵ such that $p^\epsilon - \tilde{p}_\epsilon = q_\epsilon$. By (Equation 4.3.12) and (Equation 4.3.13), for all $\alpha \in S_i$ and $\eta \in (-1,0)$ we have

$$\left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \beta_i(1+\eta)\right| \leq \left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \frac{c_i^\epsilon}{\epsilon}(1+\eta)\right| + \left|\frac{c_i^\epsilon}{\epsilon} - \beta_i\right| \leq C\epsilon,$$

where $\beta_i := \frac{2|U_i|}{L_i\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}$ is defined in (Equation 4.3.11). Combining this with the expression of the determinant in (Equation 4.2.8), we have

$$
\begin{aligned}
\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx &= \int_{S_i}\int_{-1}^{0} (\beta_i(1+\eta)+O(\epsilon))(L_i\gamma_i(\alpha)+O(\epsilon))d\eta d\alpha \\
&\leq \frac{|U_i|}{\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}\int_{S_i}\gamma_i(\alpha)d\alpha + C\epsilon =: K_i + C\epsilon,
\end{aligned}
$$

where K_i is independent of ϵ. Putting this together with (Equation 4.4.10) yields the following:

$$
\begin{aligned}
A_i^\epsilon &\geq J_i - K_i - C\epsilon \\
&= \frac{L_i^2}{4\pi}\frac{\int_{S_i}\gamma_i(\alpha)d\alpha}{\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}\left(\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha\int_{S_i}\gamma_i(\alpha)d\alpha - \frac{4\pi|U_i|}{L_i^2}\right) - C\epsilon.
\end{aligned}
\tag{4.4.11}
$$

Let us take a closer look at the two terms inside the parenthesis. For the first term, Cauchy-Schwarz inequality gives

$$
\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha\int_{S_i}\gamma_i(\alpha)d\alpha \geq 1,
$$

with equality achieved if and only if γ_i is a constant. For the second term, the isoperimetric inequality yields

$$
\frac{4\pi|U_i|}{L_i^2} \leq 1,
$$

(recall that $L_i = |\partial U_i|$), with equality achieved if and only U_i is a disk. By the assumption of Case 2, at least one of the inequalities must be strict, thus the parenthesis on the right hand side of (Equation 4.4.11) is strictly positive (and independent of ϵ). Therefore there exists some constant $c_0 > 0$ such that $\tilde{I}^\epsilon \geq A_i^\epsilon \geq c_0$ for all sufficiently small ϵ.

Case 3. There exist $i \neq j$ such that $i \not\prec j$ and $j \not\prec i$. Then it is clear that for such i, j, $B_{i,j}^\epsilon$ in (Equation 4.4.9) is given by $B_{i,j}^\epsilon = \frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi\epsilon^2}$. Hence (Equation 4.2.9) gives

$$
B_{i,j}^\epsilon \geq L_iL_j\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)\left(\int_{S_j}\gamma_j(\alpha)d\alpha\right) - C\epsilon,
$$

which yields $\tilde{I}^\epsilon \geq \frac{1}{2}L_iL_j(\int_{S_i}\gamma_i d\alpha)(\int_{S_j}\gamma_j d\alpha) > 0$ for all sufficiently small $\epsilon > 0$.

This finishes our discussion on all 3 cases. To conclude, since Γ is not a union of nested circles with constant γ_i's on each connected component, at least one of the 3 cases must hold, and all of them lead to $\tilde{I}^\epsilon \geq c_0 > 0$. □

The above proposition immediately leads to the following corollary for the $\Omega < 0$ case.

Corollary 4.4.3. *Assume that Γ_i and γ_i satisfy* **(H1)–(H3)** *for $i = 1, \ldots, n+m$. Let I^ϵ be defined as in* (Equation 4.4.1)*, and assume $\Omega < 0$. Then we have $I^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$. In addition, if Γ is **not** a union of concentric circles all centered at the origin with constant γ_i's, there exists some $c_0 > 0$ independent of ϵ, such that $I^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose I^ϵ as follows (recall the definition of $\tilde{I}^\epsilon$ in (Equation 4.4.4))

$$I^\epsilon = \tilde{I}^\epsilon + (-\Omega)\left(\epsilon^{-1}\int_{D^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx\right) =: \tilde{I}^\epsilon + \underbrace{(-\Omega)}_{>0} J^\epsilon. \tag{4.4.12}$$

Recall that Proposition 4.4.2 gives $\tilde{I}_\epsilon \geq c_0 > 0$ as long as Γ is not a union of nested circles with constant γ_i's. By [50, Lemma 2.11], we have

$$\int_{D_i^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx \geq 0 \quad \text{ for any } i = 1, \ldots, n+m,$$

thus $J^\epsilon \geq 0$. Putting them together, and using the fact that $\Omega < 0$, we know $I^\epsilon \geq c_0 > 0$ if Γ is not a union of nested circles with constant γ_i's.

To finish the proof, we only need to focus on the case that the Γ_i's are nested circles with constant γ_i's, but not all of them are centered at the origin. Assume that there exists $k \in \{1, \ldots, n\}$ such that Γ_k is a circle with radius r_k centered at $x_k \neq 0$. Since γ_k is a constant, D_k^ϵ is an annulus given by $B(x_k, r_k + \epsilon\gamma_k) \setminus B(x_k, r_k)$. The symmetry of D_k^ϵ about x_k immediately leads to

$p^\epsilon|_{D_k^\epsilon} = -\frac{1}{2}|x - x_k|^2 + \frac{1}{2}(r_k + \epsilon\gamma_k)^2$. An elementary computation gives

$$\epsilon^{-1}\int_{D_k^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx = \epsilon^{-1}\int_{D_k^\epsilon}|x|^2 - (x - x_k)\cdot x dx = \epsilon^{-1}|x_k|^2|D_k^\epsilon| \geq 2\pi r_k\gamma_k|x_k|^2 > 0,$$

where in the second-to-last step we used that $|D_k^\epsilon| = 2\pi\epsilon r_k\gamma_k + \pi\epsilon^2\gamma_k^2$. Setting $c_0 := 2\pi r_k\gamma_k|x_k|^2$ gives $I^\epsilon \geq c_0 > 0$, thus we can conclude. $\square$

Now we are ready to prove Theorem A. Note that for $\Omega < 0$, the symmetry result immediately follows from Proposition 4.4.1 and Corollary 4.4.3. For $\Omega = 0$, Proposition 4.4.1– 4.4.2 already imply that a stationary vortex sheet with positive strength must be a union of nested circles with constant strength on each of them. To finish the proof, we only need to show that these nested circles must be concentric.

Proof of Theorem A. For a uniformly-rotating vortex sheet with $\Omega < 0$, the symmetry result for $\Omega < 0$ is a direct consequence of Proposition 4.4.1 and Corollary 4.4.3. Next we focus on the stationary (i.e. $\Omega = 0$) case.

Combining Propsitions 4.4.1– 4.4.2, we obtain that Γ is a union of nested circles, and γ_i is constant on Γ_i for all $i = 1\ldots, n$. It remains to show that all Γ_i's are concentric. Let us denote by $\mathbf{v}_i$ the contribution to the velocity field by Γ_i. Since Γ_i is a circle with constant strength γ_i, a quick application of the divergence theorem yields that $\mathbf{v}_i \equiv 0$ in the open disk enclosed by Γ_i, whereas $\mathbf{v}_i(x) = \dfrac{\gamma_i L_i(x - x_i^0)^\perp}{2\pi|x - x_i^0|^2}$ in the open set outside Γ_i, where x_i^0 is the center of the circle Γ_i.

Without loss of generality, let us reorder the indices such that Γ_i is nested inside Γ_j for $i < j$. Towards a contradiction, let $k > 1$ be such that Γ_k is the first circle that is not concentric with Γ_1. From the above discussion, we know that $\mathbf{v}_i = 0$ on Γ_k for $i = k+1, \ldots, n$ (since Γ_k is nested inside Γ_i), whereas for $i = 1, \ldots, k-1$ we have $\mathbf{v}_i = \dfrac{\gamma_i L_i(x - x_1^0)^\perp}{2\pi|x - x_1^0|^2}$ on Γ_k, since all these Γ_i's have the same center x_1^0 and are nested inside Γ_k. Summing them up (and also using the fact that

Γ_k contributes zero normal velocity on itself, since it is a circle with constant strength), we have

$$BR(x) \cdot \mathbf{n} = \sum_{i=1}^{n} \mathbf{v}_i(x) \cdot \mathbf{n} = \left(\sum_{i=1}^{k-1} \gamma_i L_i \right) \frac{(x - x_1^0)^\perp \cdot \mathbf{n}}{2\pi |x - x_1^0|^2} \quad \text{on } \Gamma_k,$$

where the right hand side is not a zero function since Γ_k has a different center from x_1^0. This causes a contradiction with the fact that $\omega = \omega_0$ is stationary. As a result, all $\Gamma_1, \ldots, \Gamma_n$ must be concentric circles, finishing the proof. □

CHAPTER 5

FLEXIBILITY RESULTS FOR VORTEX SHEETS

5.1 The equations and the functional spaces

Let $\omega(\cdot, t) = \omega_0(R_{\Omega t})$ be a stationary/rotating vortex sheet solution to the incompressible 2D Euler equation, where $\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$. Here $\Omega = 0$ corresponds to a stationary solution, and $\Omega \neq 0$ corresponds to a rotating solution. Assume that ω_0 is concentrated on Γ. Throughout this paper we will assume that Γ is a simple closed curve and $\Omega > 0$. Following [52, Lemma 2.1], we have that:

Lemma 5.1.1. *Assume $\omega(\cdot, t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, and ω_0 is concentrated on Γ, with z and γ defined as above. Then the Birkhoff-Rott integral BR* (Equation 4.1.4) *and the strength γ satisfy the following two equations:*

$$(BR - \Omega x^{\perp}) \cdot \mathbf{n} = \mathbf{v}^{+} \cdot \mathbf{n} = \mathbf{v}^{-} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{5.1.1}$$

and

$$(BR(z(\alpha)) - \Omega z^{\perp}(\alpha)) \cdot \mathbf{s}(z(\alpha)) \frac{\gamma(\alpha)}{|z'(\alpha)|} = C. \tag{5.1.2}$$

Note that (Equation 5.1.2) can be written as

$$(I - P_0) \left[\left(BR(z, \Gamma)(z(\theta)) - \Omega z(\theta)^{\perp} \right) \cdot \frac{z'(\theta)\gamma(\theta)}{|z'(\theta)|^2} \right] = 0, \tag{5.1.3}$$

where P_0 is a projection to the mean, that is, $P_0 f := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$. For simplicity, we also denote $\fint f(\theta) d\theta := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$. Now plugging $z(\theta) = (1 + r(\theta))(\cos(\theta), \sin(\theta))$ and $\gamma(\theta) := b + g(\theta)$

into (Equation 5.1.1), (Equation 5.1.2) and (Equation 5.1.3) yields that

$$\mathcal{F}(b,g,r) := (\mathcal{F}_1, \mathcal{F}_2) = (0,0), \tag{5.1.4}$$

where

$$\begin{aligned}
\mathcal{F}_1(b,g,r) &:= \fint_{-\pi}^{\pi} (b+g(\eta)) \frac{\left(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta)\right)(1+r(\eta)) - (1+r(\theta))r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} d\eta \\
&\quad + \Omega r'(\theta)(1+r(\theta)), \\
\mathcal{F}_2(b,g,r) &:= (I-P_0)\,\tilde{\mathcal{F}}_2(b,g,r), \\
\tilde{\mathcal{F}}_2(b,g,r) &:= \fint_{-\pi}^{\pi} (b+g(\eta)) \frac{(1+r(\theta))^2 - (r'(\theta)\sin(\theta-\eta) + (1+r(\theta))\cos(\theta-\eta))(1+r(\eta))}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} d\eta \\
&\quad \times \frac{(b+g(\theta))}{r'(\theta)^2 + (1+r(\theta))^2} - \Omega(1+r(\theta))^2 \frac{b+g(\theta)}{r'(\theta)^2 + (1+r(\theta))^2}.
\end{aligned}$$

Throughout the paper we will work with the following analytic function spaces. Let $c > 0$ be a sufficiently small parameter and let $\mathcal{C}_w(c)$ be the space of analytic functions in the strip $|\Im(z)| \le c$. For $k \in \mathbb{N}$, denote

$$\begin{aligned}
X_c^k &:= \left\{ f(\theta) \in \mathcal{C}_w(c), \quad f(\theta) = \sum_{n=1}^{\infty} a_n \cos(2n\theta), \quad \sum_{\pm} \int_{-\pi}^{\pi} |f(\theta \pm ic)|^2 + |\partial^k f(\theta \pm ic)|^2 d\theta < \infty \right\} \\
Y_c^k &:= \left\{ f(\theta) \in \mathcal{C}_w(c), \quad f(\theta) = \sum_{n=1}^{\infty} a_n \sin(2n\theta), \quad \sum_{\pm} \int_{-\pi}^{\pi} |f(\theta \pm ic)|^2 + |\partial^k f(\theta \pm ic)|^2 d\theta < \infty \right\},
\end{aligned}$$

From now on, due to scaling considerations, we will fix $\Omega = 1$ and b will play the role as bifurcation parameter. It is clear that $\mathcal{F}(b,0,0) = (0,0)$ for all $b \in \mathbb{R}$ since $\mathcal{F}_1(b,0,0) = 0$ and $\tilde{\mathcal{F}}_2(b,0,0)$ is constant. Our main theorem in this paper is the following:

Theorem 5.1.2. *Let $k \ge 3$, and let $c > 0$ be sufficiently small. Then, there exists a curve of solutions (b,g,r) of $\mathcal{F} = (0,0)$, belonging to $\mathbb{R} \times X_c^k \times X_c^{k+1}$ and a neighbourhood of $(b,g,r) = (2,0,0)$, bifurcating from $(b,g,r) = (2,0,0)$ such that $(g,r) \neq (0,0)$.*

5.2 Proof of Theorem 5.1.2

The goal of this section is to prove the existence of non-radial uniformly-rotating vortex sheets. To do so, we will split the proof into the following steps: first we will prove that the functional $\mathcal{F}$ is C^3, next we will study $D\mathcal{F}$ to show that, as mentioned in the introduction, it is a Fredholm operator of index 0, with dim(Ker($D\mathcal{F}$)) $= 1$. The next step is to apply Lyapunov-Schmidt theory and reduce the problem to a finite (2) dimensional one. In those coordinates, linear expansions fail to be conclusive (all the linear terms vanish) since 2 nontrivial branches emanate from the bifurcation point (as opposed to 1). Instead, we perform a quadratic expansion to determine that locally the bifurcation branches look like two pairs of straight lines (specifically as $x^2 - y^2 = 0$ in some well-chosen coordinates) and hence the bifurcation does not trivialize (as if it had been of the type $x^2 + y^2 = 0$). We conclude the proof by handling the higher order terms and showing that they don't alter the quadratic behaviour in a sufficiently small neighbourhood of the bifurcation point.

5.2.1 Continuity of the functional

In this subsection, we will check the regularity of $\mathcal{F}$. As explained above, we will reduce the infinite dimensional problem to a finite dimensional problem and investigate its Taylor expansion up to quadratic order. Hence, we need to check if the functional is regular enough to do so. To this end, we have the following proposition:

Proposition 5.2.1. *Let $k \geq 3$. Then there exists a neighborhood U of $(2,0,0) \in \mathbb{R} \times X_c^k \times X_c^{k+1}$ such that $\mathcal{F} \in C^3\left(U; Y_c^k \times X_c^k\right)$.*

Proof. Since the stream function, $\omega * \mathcal{N}$, is invariant under rotations, it follows immediately that $\mathcal{F}$ is also invariant under rotation by π-radians, hence $\mathcal{F}$ has only even Fourier modes. Also the oddness of $\mathcal{F}_1$ and evenness of $\mathcal{F}_2$ follow from the invariance under reflection.

To prove the regularity, we briefly sketch the idea. We impose $k \geq 3$ to ensure that H^k is a Banach algebra. It is clear that $\mathcal{F}$ is smooth in b. It is also straightforward that, for example, for all (g, r) near $(0,0) \in X_c^k \times X_c^{k+1}$,

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}_1(b, g+th_1, r+th_2)\bigg|_{t=0} \\
&\quad = PV \fint h_1(\eta) \frac{(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} \\
&\qquad + (b+g(\eta)) \Bigg[\frac{(h_2'(\theta)\cos(\theta-\eta) - h_2(\theta)\sin(\theta-\eta))(1+r(\eta)) - h_2(\theta)r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} \\
&\qquad + \frac{(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,h_2(\eta) - (1+r(\theta))h_2'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} \\
&\qquad - [(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta)] \\
&\qquad \times \frac{[2(1+r(\theta)h_2(\theta) + 2(1+r(\eta)h_2(\eta) - 2\cos(\theta-\eta)(h_2(\theta)(1+r(\eta)) + h_2(\eta)(1+r(\theta)))]}{((1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta))^2} \Bigg] d\eta \\
&\qquad + \Omega h_2'(\theta) \\
&\quad =: D\mathcal{F}_1(b, g, r)[h_1, h_2],
\end{aligned}
$$

and $D\mathcal{F}_1 : \mathbb{R} \times X_c^k \times X_c^{k+1} \mapsto \mathcal{L}(X_c^k \times X_c^{k+1}; Y_c^k \times X_c^k)$ is continuous. A similar derivation can be performed for $D\mathcal{F}_2$. For the higher derivatives, we refer to [21, 22, 49, 63, 100] for the method to deal with the singular integrals arising throughout the calculations. $\square$

5.2.2 Fredholm index of the linearized operator $D\mathcal{F}$

This subsection is devoted to show that $D\mathcal{F}$ is Fredholm of index zero. We can make all the calculations explicit, moreover the operator diagonalizes in Fourier modes. We have the following lemmas:

Lemma 5.2.2. *Let* $g(\theta) = \sum_{n=1}^{\infty} a_n \cos(2n\theta)$ *and* $r(\theta) = \sum_{n=1}^{\infty} b_n \cos(2n\theta)$*. Then we have that*

$$D\mathcal{F}(b,0,0)\,[g,r] = \begin{pmatrix} \hat{g}(\theta) \\ \hat{r}(\theta) \end{pmatrix},$$

where

$$\hat{g}(\theta) = \sum_{n=1}^{\infty} \hat{a}_n \sin(2n\theta), \quad \hat{r}(\theta) = \sum_{n=1}^{\infty} \hat{b}_n \cos(2n\theta),$$

and the coefficients satisfy, for any $n \geq 1$*:*

$$M_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} := \begin{pmatrix} -\frac{1}{2} & -2n\left(\Omega - \frac{b}{2}\right) \\ \frac{b}{2} - \Omega & b^2(n-1) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \end{pmatrix}.$$

Proof. We use (Equation A.1.3) in Lemma A.1.2 and obtain

$$\begin{aligned} \hat{g}(\theta) = \frac{d}{dt}\mathcal{F}_1(b,tg,tr) &= -\fint \frac{g(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \left(\Omega - \frac{b}{2}\right) r'(\theta) \\ &= -\sum_{n=1}^{\infty} a_n \fint \frac{\cos(2n\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \sum_{n=1}^{\infty}(-2n)\left(\Omega - \frac{b}{2}\right) b_n \sin(2n\theta) \\ &= \sum_{n=1}^{\infty}\left(-\frac{a_n}{2} + (-2n)\left(\Omega - \frac{b}{2}\right) b_n\right)\sin(2n\theta), \end{aligned}$$

where the last equality follows from (Equation A.1.29). Similarly, we apply (Equation A.1.4) in Lemma A.1.2 and (Equation A.1.30) to obtain

$$\begin{aligned} \hat{r}(\theta) &= \left(\frac{b}{2} - \Omega\right) g(\theta) + b^2\left(\fint \frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)}d\eta - r(\theta)\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{b}{2} - \Omega\right) a_n \cos(2n\theta) + b^2 \sum_{n=1}^{\infty} b_n \left(\fint \frac{\cos(2n\theta)-\cos(2n\eta)}{2-2\cos(\theta-\eta)}d\eta - \cos(2n\theta)\right) \\ &= \sum_{n=1}^{\infty}\left(\left(\frac{b}{2} - \Omega\right) a_n + b^2\,(n-1)\,b_n\right)\cos(2n\theta). \end{aligned}$$

This proves the lemma. $\square$

Lemma 5.2.3. *Let us fix $b = 2$ and $\Omega = 1$. We also denote $v := (0, \cos(2\theta)) \in X_c^k \times X_c^{k+1}$ and $w := (0, \cos(2\theta)) \in Y_c^k \times X_c^k$. Then it holds that*

$$\textit{Ker}\,(D\mathcal{F}(2,0,0)) = \textit{span}\,\{v\} \subset X_c^k \times X_c^{k+1},$$
$$\textit{Im}\,(D\mathcal{F}(2,0,0))^{\perp} = \textit{span}\,\{w\} \subset Y_c^k \times X_c^k.$$

Proof. From Lemma 5.2.2, we have

$$M_n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 4(n-1) \end{pmatrix},$$

for all $n \geq 1$. For all $n \geq 2$, M_n is clearly an isomorphism, while Ker$(M_1) = $ Im$(M_1)^{\perp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By orthogonality of Fourier modes, this proves the lemma. $\square$

5.2.3 Lyapunov-Schmidt reduction

In this subsection, we will aim to derive a finite dimensional system which is equivalent to (Equation 5.1.4). From Lemma 5.2.3, we have the following orthogonal decompositions of the function spaces:

$$X := X_c^k \times X_c^{k+1} = \text{span}\,\{v\} \oplus \text{Ker}\,(D\mathcal{F}(2,0,0))^{\perp} =: \text{span}\,\{v\} \oplus \mathcal{X}, \quad v \in \text{Ker}\,(D\mathcal{F}(2,0,0))\,,$$
$$Y := Y_c^k \times X_c^k = \text{span}\,\{w\} \oplus \text{Im}\,(D\mathcal{F}(2,0,0)) =: \text{span}\,\{w\} \oplus \mathcal{Y}, \quad w \in \text{Im}\,(D\mathcal{F}(2,0,0))^{\perp}\,,$$

where v and w are as defined in Lemma 5.2.3. Let us consider the orthogonal projections

$$P : X \to \text{span}\,\{v\}\,, \quad Q : Y \to \text{span}\,\{w\}\,.$$

More precisely, we have

$$P(g(\theta), r(\theta)) = \left(0, \left(\frac{1}{\pi}\int r(\eta)\cos(2\eta)d\eta\right)\cos(2\theta)\right) \qquad \text{for all } (g, r) \in X_c^k \times X_c^{k+1}, \tag{5.2.1}$$

$$Q(G(\theta), R(\theta)) = \left(0, \left(\frac{1}{\pi}\int R(\eta)\cos(2\eta)d\eta\right)\cos(2\theta)\right) \qquad \text{for all } (G, R) \in Y_c^k \times X_c^k. \tag{5.2.2}$$

We remark that we will sometimes abuse notation and identify $\mathcal{F}(b, g, r)$ with $\mathcal{F}(b, (g, r))$, where $(g, r) \in X$. Let us define $G : \mathbb{R} \times \text{span}\{v\} \times \mathcal{X} \mapsto Y$ as follows:

$$G(b, f, x) := \mathcal{F}(b, f + x), \quad \text{for} \quad b \in \mathbb{R}, \quad f \in \text{span}\{v\}, \quad x \in \mathcal{X}.$$

Then (Equation 5.1.4) is equivalent to (for $(g, r) = f + x$)

$$QG(b, f, x) = 0 \quad \text{and} \quad (I - Q)\, G(b, f, x) = 0. \tag{5.2.3}$$

However, it follows from Lemma 5.2.3 that

$$D_x\left((I - Q)G\right)(b, 0, 0) = (I - Q)\, D\mathcal{F}(b, 0)P : \mathcal{X} \mapsto \mathcal{Y} \tag{5.2.4}$$

is an isomorphism, consequently, the implicit function theorem yields that there is an open set $U \subset \mathbb{R} \times \text{span}\{v\}$ near $(b, 0)$ and a function $\varphi : U \mapsto \mathcal{X}$ such that

$$(I - Q)\, G(b, f, \varphi(b, f)) = (I - Q)\, \mathcal{F}(b, f + \varphi(b, f)) = 0.$$

Note that from $\mathcal{F}(b, 0) = (0, 0)$ for any $b \in \mathbb{R}$, we have

$$\varphi(b, 0) = 0, \tag{5.2.5}$$

and thus (Equation 5.2.3) is equivalent to

$$0 = QG(b, f, \varphi(b, f)) = Q\mathcal{F}(b, f + \varphi(b, f)), \quad (b, f) \in U. \tag{5.2.6}$$

Since span $\{v\}$ is one dimensional, we have $f = tv$ for some $t \in \mathbb{R}$, therefore the system (Equation 5.2.6) can be written in terms of the variables b and t as

$$\begin{aligned} 0 = Q\mathcal{F}(b, tv + \varphi(b, tv)) &= \int_0^1 \frac{d}{ds}\left(Q\mathcal{F}(b, stv + \varphi(b, stv))\right) ds \\ &= \int_0^1 QD\mathcal{F}(b, stv + \varphi(b, stv))(tv + t\partial_f \varphi(b, stv)v) ds, \end{aligned}$$

where we used (Equation 5.2.5) to obtain the second equality. Dividing the right-hand side by t to get rid of the trivial solutions, we are led to solve the following two dimensional problem:

$$0 = F_{red}(b, t) := \int_0^1 QD\mathcal{F}(b, stv + \varphi(b, stv))(v + \partial_f \varphi(b, stv)v) ds, \quad (b, tv) \in U. \tag{5.2.7}$$

5.2.4 Quadratic expansion of the reduced functional

The main idea is to expand the reduced functional F_{red} up to quadratic terms. To this end, we recall the following proposition for the derivatives of F_{red}.

Proposition 5.2.4. *([61, Proposition 3], [65, Proposition 3.1]) Let F_{red} be defined as in* (Equation 5.2.7). *Then the following hold:*

(a) First derivatives:

$$\begin{aligned} \partial_b F_{red}(2, 0) &= Q\partial_b D\mathcal{F}(2, 0)v, \\ \partial_t F_{red}(2, 0) &= \frac{1}{2}\frac{d^2}{dt^2} Q\mathcal{F}(2, tv)\bigg|_{t=0}. \end{aligned}$$

(b) Second derivatives:

$$
\begin{aligned}
\partial_{bb} F_{red}(2,0) &= 2Q\partial_b D\mathcal{F}(2,0)\tilde{v}, \\
\partial_{tt} F_{red}(2,0) &= \frac{1}{3}\frac{d^3}{dt^3}\left[Q\mathcal{F}(2,tv)\right]\Big|_{t=0} + Q\frac{d^2}{dtds}\mathcal{F}(2,tv+s\hat{v})\Big|_{t=s=0}, \\
\partial_{tb} F_{red}(2,0) &= \frac{1}{2}\partial_b Q\frac{d^2}{dt^2}\mathcal{F}(b,tv)\Big|_{b=2,t=0} + \frac{1}{2}Q\partial_b D\mathcal{F}(2,0)\hat{v} + Q\frac{d^2}{dtds}\mathcal{F}(2,tv+s\tilde{v})\Big|_{t=s=0},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{v} &:= -\left[D\mathcal{F}(2,0)\right]^{-1}\frac{d^2}{dt^2}\left[(I-Q)\mathcal{F}(2,tv)\right]\Big|_{t=0}, \\
\tilde{v} &:= -\left[D\mathcal{F}(2,0)\right]^{-1}(I-Q)\partial_b D\mathcal{F}(2,0)v.
\end{aligned}
$$

Now using the values found in Lemma A.1.7, we can obtain the derivatives of F_{red}.

Proposition 5.2.5. *Let F_{red} be defined as in* (Equation 5.2.7). *Then it holds that*

$$\partial_b F_{red}(2,0) = 0, \tag{5.2.8}$$

$$\partial_t F_{red}(2,0) = 0. \tag{5.2.9}$$

$$\partial_{bb} F_{red}(2,0) = 2w, \tag{5.2.10}$$

$$\partial_{tt} F_{red}(2,0) = -8w, \tag{5.2.11}$$

$$\partial_{tb} F_{red}(2,0) = 0. \tag{5.2.12}$$

Proof. (Equation 5.2.8) follows immediately from (Equation 5.2.2) and (Equation A.1.14). For (Equation 5.2.9), we use (Equation A.1.15) and the orthogonality of the Fourier modes. (Equation 5.2.10) follows from (Equation A.1.23). (Equation 5.2.11) follows from (Equation A.1.19) and (Equation A.1.20). Lastly, (Equation 5.2.12) follows from (Equation A.1.16), (Equation A.1.21) and (Equation A.1.22). □

5.2.5 Proof of Theorem 5.1.2

Now we are ready to prove the main theorem of this section.

Proof. From (Equation 5.2.7), it suffices to show that there exist (b,t) such that $t \neq 0$ and $F_{red}(b,t) = 0$. To do so, we expand F_{red} up to quadratic terms and obtain that for all (b,t) near $(2,0)$,

$$\begin{aligned}F_{red}(b,t) = \Big[&F_{red}(2,0) + \partial_b F_{red}(2,0)(b-2) + \partial_t F_{red}(2,0)t \\ &+\frac{1}{2}\partial_{bb}F_{red}(2,0)(b-2)^2 + \frac{1}{2}\partial_{tt}F_{red}(2,0)t^2 + \partial_{tb}F_{red}(2,0)(b-2)t + \left((b-2)^2+t^2\right)\epsilon(b,t)\Big]\, w,\end{aligned}$$

where $\epsilon(b,t)$ is a continuous function such that $\lim_{(b,t)\to(2,0)}\epsilon(b,t) = \epsilon(2,0) = 0$. From Proposition 5.2.5, it follows that (we drop w for simplicity)

$$F_{red}(b,t) = (b-2)^2 - 4t^2 + \left((b-2)^2 + t^2\right)\epsilon(b,t).$$

Now we use the change of variables $b := x+2$ and $t = xy$, so that

$$\hat{F}(x,y) := \frac{F_{red}(x+2,xy)}{x^2} = \left(1-4y^2\right) + (1+y^2)\epsilon(x+2,xy). \tag{5.2.13}$$

Clearly, $\hat{F}\left(0,\frac{1}{2}\right) = 0$ and $\partial_y\hat{F}\left(0,\frac{1}{2}\right) = -4 \neq 0$. Therefore the implicit function theorem implies that there exists a continuous function ϕ near 0 such that $\hat{F}(x,\phi(x)) = 0$ and $\phi(0) = \frac{1}{2}$. Therefore it follows from (Equation 5.2.13) that there exists a pair (b,t) such that $t \neq 0$ and $F_{red}(b,t) = 0$. This finishes the proof. $\square$

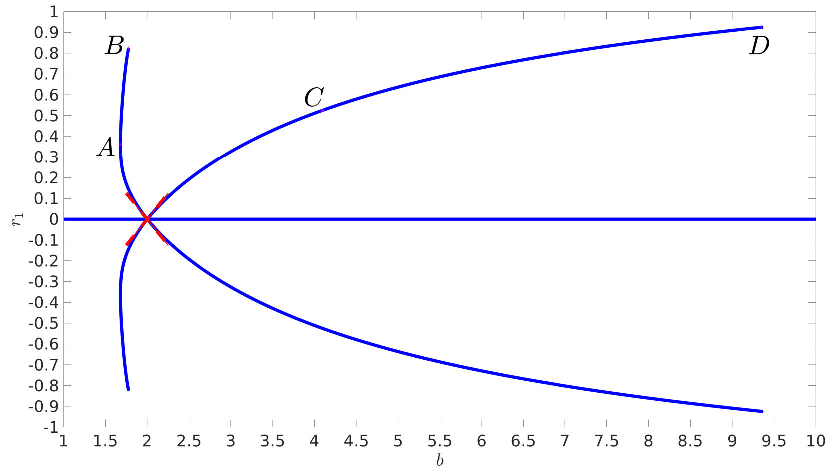


Figure 5.1: Plot of the bifurcation diagram of the solutions given by Theorem 5.1.2. The dotted red lines correspond to the linear expansion (Equation 5.2.13) around the bifurcation point $(2, 0)$. See Figure Figure 5.2 for a numerical plot of the solutions at the points A, B, C, D. The branches continue beyond what is calculated.

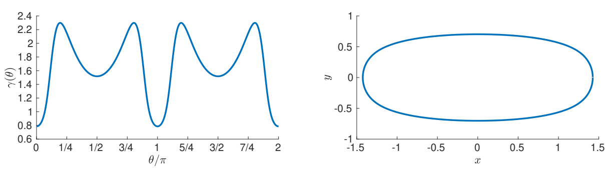
5.3 Numerical results

In this section, we describe how to compute numerically the branches of solutions emanating from the disk, previously proved (locally) in Theorem 5.1.2. See Figure Figure 5.1. To do so, we calculate solutions of the form

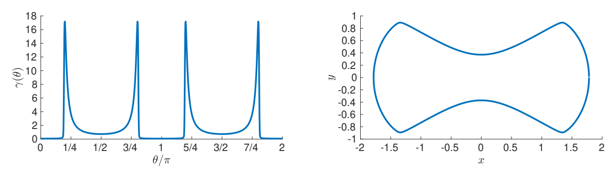
$$R(\theta) = 1 + \sum_{k=1}^{N} r_k \cos(2k\theta), \quad \gamma(\theta) = \sum_{k=0}^{N} \gamma_k \cos(2k\theta)$$

with $\gamma_0 = b$. We first employed continuation in b, in increments of $\Delta b = 0.001$, starting from $b = 1.8$ and $b = 2.1$ and using as initial guess for the starting b the solution given by the linear theory and for the subsequent b the solution found in the previous iteration. After discovering a fold at approximately $b \sim 1.68$, we switched variables and instead we recalculated using continuation in r_1, which appears to be monotonic along the branches. As before, we start at $r_1 = \pm 0.125$ and take an increment $\Delta r_1 = 0.001$.

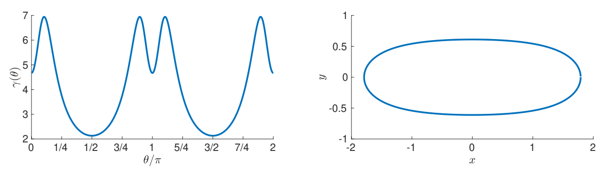
To compute a solution for a fixed r_1 we use the Levenberg-Marquardt algorithm. We aim to find a zero of the system of equations $\mathcal{F}(b,g,r)(\theta_j)$, with $\theta_j = \frac{j\pi}{N_\theta}, \quad j = 1,\ldots,N_\theta$ and $N_\theta = 1024$ with variables $r_k,\ k \neq 1$ (recall that r_1 is fixed at each iteration since it is the continuation parameter) and γ_k. We take $N = 160$. In order to perform the integration in space, we desingularize the principal value at $\eta = \theta$ by subtracting $\frac{1}{2}\mathcal{H}(\gamma)$ to $\mathcal{F}_1$, where $\mathcal{H}$ denotes the Hilbert transform, computed explicitly since we have the Fourier expansion of γ, and perform a trapezoidal integration on the rest (for which the integrand is smooth), with step $h = \frac{2\pi}{N_\theta}$. We remark that the integrand of $\mathcal{F}_2$ has a removable singularity (thus no principal value integration is needed) and can be integrated using the trapezoidal integration if the limit at $\eta = \theta$ is taken properly.



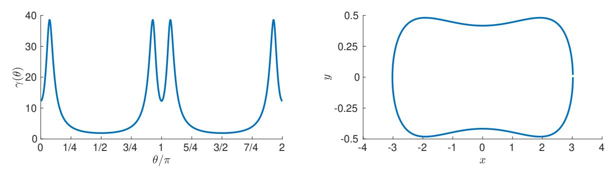
(a) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.362, b \sim 1.6799$



(b) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.825, b \sim 1.7779$



(c) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.525, b \sim 4.0954$



(d) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.925, b \sim 9.3439$

Figure 5.2: Panels (a)-(d): $\gamma(\theta)$ and $z(\theta)$ at the points A–D highlighted in Figure Figure 5.1. In panel (b), γ appears to tend to be concentrated only on the horizontal parts of z, leading to a possible solution consisting only of two symmetric curves (cf. [93, Figure 1]) and a change of topology.

CHAPTER 6

QUANTITATIVE ESTIMATES FOR UNIFORMLY ROTATING VORTEX PATCHES

6.1 Quantitative estimates for small Ω

This section is devoted to the proof of Theorem G. Throughout this section, we will always assume that $|D| = |B| = \pi$. We begin this section by proving two identities for simply-connected rotating patches.

Lemma 6.1.1. *Let* (D, Ω) *be a solution to* (Equation 1.2.4). *Then it holds that*

$$\Omega\left(\int_D |x|^2 dx - \frac{|D|^2}{2\pi}\right) = (1-2\Omega)\left(\frac{|D|^2}{4\pi} - \int_D p dx\right), \tag{6.1.1}$$

$$\left(\frac{1}{2}-\Omega\right)\left(\int_D |x|^2 dx - \frac{|D|^2}{2\pi}\right) = \frac{1}{2}\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2 dx. \tag{6.1.2}$$

where p *is the unique solution to* (Equation 1.2.7).

Proof. The proof of (Equation 6.1.1) can be found in [50, Theorem 2.2]. For the sake of completeness, we give a proof below.

In order to prove (Equation 6.1.1), we plug $\vec{v} = x + \nabla p$ into (Equation 1.2.6) to get

$$\begin{aligned} 0 &= \int_D (x+\nabla p)\cdot\nabla\left(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2\right) dx \\ &= \int_D x\cdot\nabla\left(1_D * \mathcal{N}\right) - \Omega\int_D |x|^2 dx + \int_D \nabla p\cdot\nabla\left(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2\right) dx \\ &= \int_D x\cdot\nabla\left(1_D * \mathcal{N}\right) - \Omega\int_D |x|^2 dx - (1-2\Omega)\int_D p dx, \end{aligned} \tag{6.1.3}$$

where we used divergence theorem for the last equality. Note that the first integral can be computed as

$$\int_D x \cdot \nabla\left(1_D * \mathcal{N}\right) dx = \frac{1}{2\pi}\int_D\int_D \frac{x\cdot(x-y)}{|x-y|^2}dydx = \frac{|D|^2}{4\pi}, \tag{6.1.4}$$

where the last equality is obtained by exchanging x and y in the double integral, and then taking the average with the original integral. Therefore (Equation 6.1.3) and (Equation 6.1.4) yield

$$0 = \frac{|D|^2}{4\pi} - \Omega\int_D |x|^2dx - (1-2\Omega)\int_D pdx,$$

which is equivalent to (Equation 6.1.1).

For (Equation 6.1.2), we choose $\vec{v} = x - 2\nabla\left(1_D * \mathcal{N}\right)$ in (Equation 1.2.6) and obtain

$$\begin{aligned} 0 &= \int_D (x - 2\nabla(1_D * \mathcal{N}))\cdot\nabla\left(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx \\ &= (1+2\Omega)\frac{|D|^2}{4\pi} - 2\int_D |\nabla\left(1_D * \mathcal{N}\right)|^2dx - \Omega\int_D |x|^2dx, \end{aligned} \tag{6.1.5}$$

where we used (Equation 6.1.4). Since $\int_D |\nabla\left(1_D * \mathcal{N}\right)|^2dx$ can be computed as

$$\begin{aligned} 2\int_D |\nabla\left(1_D * \mathcal{N}\right)|^2dx &= \frac{1}{2}\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2dx + 2\int_D x\cdot\nabla\left(1_D * \mathcal{N}\right)dx - \frac{1}{2}\int_D |x|^2dx \\ &= \frac{1}{2}\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2dx + \frac{|D|^2}{2\pi} - \frac{1}{2}\int_D |x|^2dx, \end{aligned}$$

plugging it into (Equation 6.1.5) yields (Equation 6.1.2). □

Thanks to Lemma 6.1.1, the angular velocity can be estimated by comparing the quantities, $\int_D |x|^2dx - \frac{|D|^2}{2\pi}$, $\frac{|D|^2}{4\pi} - \int_D pdx$ and $\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2dx$, which vanish if and only if $D = B$. To estimate those quantities for non-radial patches, we use the following notion of asymmetry.

Definition 6.1.2. *[42, Section 1.1] For a bounded domain $D \subseteq \mathbb{R}^2$, the Fraenkel asymmetry $\mathcal{A}(D)$*

is defined by

$$\mathcal{A}(D) := \inf_{x\in\mathbb{R}^2}\left\{\frac{|D\triangle B_r(x)|}{|D|} : \pi r^2 = |D|\right\}.$$

If $\mathcal{A}(D)$ is not small, then we can find a lower bound of the right-hand side in (Equation 6.1.1) by using the following result:

Proposition 6.1.3. *[10, Proposition 2.1] Let* p *be as in* (Equation 1.2.7) *and* $|D| = \pi$*. Then there exists a constant* $\sigma > 0$ *such that*

$$\frac{|D|^2}{4\pi} - \int_D p dx \geq \sigma\mathcal{A}(D)^2. \tag{6.1.6}$$

Using the above proposition and the identity (Equation 6.1.1), one can easily show that $\sup_{x\in\partial D}|x| \gtrsim \sqrt{\mathcal{A}(D)}\Omega^{-\frac{1}{2}}$. Therefore Theorem G can be proved if we can show $\mathcal{A}(D)$ is always bounded below by a strict positive constant. In other words, we will aim to prove in the next lemmas that if $\mathcal{A}(D)$ and Ω are sufficiently small, then D must be a disk.

In the following lemma, we will estimate the boundedness of rotating patches in a crude way but this will be improved later.

Lemma 6.1.4. *There exist positive constants* Ω_1 *and* $\alpha_1 < \frac{1}{2}$ *such that if* $\Omega < \Omega_1$ *and* $\mathcal{A}(D) < \alpha_1$*, then*

$$D \subset B_2(x_0), \tag{6.1.7}$$

$$|x_0| \leq 4\mathcal{A}(D), \tag{6.1.8}$$

where x_0 *is a point such that* $\frac{|D\triangle B(x_0)|}{\pi} = \mathcal{A}(D)$*.*

Proof. Let us pick Ω_1 and α_1 so that for all $\Omega < \Omega_1$ and $\alpha < \alpha_1 < \frac{1}{2}$, it holds that

$$\frac{1}{2}\log 2 \geq \Omega\left(10 + \frac{4\alpha}{\left(1-\sqrt{2\alpha}\right)^2}\left(1+\sqrt{\frac{1}{4\Omega}}\right)^2\right) + 3\sqrt{2\alpha}. \tag{6.1.9}$$

We will first show that if (D, Ω) satisfies $\Omega < \Omega_1$ and $\mathcal{A}(D) < \alpha_1$, then (Equation 6.1.7) holds.

Note that the center of mass of D is necessarily the origin ([64, Proposition 3]). Therefore we have

$$0 = \frac{1}{\pi}\int x 1_D(x)dx = \frac{1}{\pi}\int x\left(1_D(x) - 1_{B(x_0)}(x)\right)dx + x_0. \tag{6.1.10}$$

Hence it follows from Cauchy-Schwarz inequality that

$$\begin{aligned}|x_0| \leq \frac{1}{\pi}\int |x|\left|1_D(x) - 1_{B(x_0)}(x)\right|dx &\leq \frac{1}{\pi}\left(\sqrt{|D\triangle B(x_0)|}\sqrt{\int_D |x|^2dx + \int_{B(x_0)}|x|^2dx}\right)\\ &\leq \sqrt{\frac{\mathcal{A}(D)}{\pi}}\sqrt{\int_D |x|^2dx} + \sqrt{\frac{\mathcal{A}(D)}{\pi}}\sqrt{\int_{B(x_0)}|x|^2dx}.\end{aligned} \tag{6.1.11}$$

Since $\sqrt{\int_{B(x_0)}|x|^2dx} \leq \sqrt{\int_{B(x_0)} 2|x-x_0|^2dx + 2\int_{B(x_0)}|x_0|^2dx} = \sqrt{\pi + 2|x_0|^2\pi} \leq \sqrt{\pi} + \sqrt{2\pi}|x_0|$, (Equation 6.1.11) yields that

$$\left(1-\sqrt{2\mathcal{A}(D)}\right)|x_0| < \sqrt{\mathcal{A}(D)} + \sqrt{\frac{\mathcal{A}(D)}{\pi}}\sqrt{\int_D |x|^2dx}. \tag{6.1.12}$$

In addition, it follows from (Equation 6.1.1) that

$$\Omega\int_D |x|^2dx = \frac{|D|^2}{4\pi} - (1-2\Omega)\int_D p\,dx < \frac{\pi}{4},$$

where we used $\Omega < \frac{1}{2}$, $p \geq 0$ in D and $|D| = \pi$ to get the last inequality. Plugging this into

(Equation 6.1.12), we obtain

$$\left(1-\sqrt{2\mathcal{A}(D)}\right)|x_0| < \sqrt{\mathcal{A}(D)}\left(1+\sqrt{\frac{1}{4\Omega}}\right),$$

hence,

$$|x_0| < \frac{\sqrt{\mathcal{A}(D)}}{1-\sqrt{2\mathcal{A}(D)}}\left(1+\sqrt{\frac{1}{4\Omega}}\right). \tag{6.1.13}$$

To prove (Equation 6.1.7), let us suppose to the contrary that there exist $x_1 \in \partial B(x_0) \cap \partial D$ and $x_2 \in \partial B_2(x_0) \cap \partial D$. Then it follows from (Equation 1.2.4) that $0 = \Psi(x_1) - \Psi(x_2)$, therefore

$$1_{B(x_0)} * \mathcal{N}(x_2) - 1_{B(x_0)} * \mathcal{N}(x_1) = \Omega(|x_2|^2 - |x_1|^2) + h(x_1) - h(x_2), \tag{6.1.14}$$

where $h(x) := \left(1_D - 1_{B(x_0)}\right) * \mathcal{N}$. For the left-hand side, we use

$$1_{B(x_0)} * \mathcal{N} = \begin{cases} \frac{1}{4}|x-x_0|^2 - \frac{1}{4} & \text{if } |x-x_0| < 1, \\ \frac{1}{2}\log|x-x_0| & \text{otherwise,} \end{cases}$$

and obtain

$$1_{B(x_0)} * \mathcal{N}(x_2) - 1_{B(x_0)} * \mathcal{N}(x_1) = \frac{1}{2}\log 2. \tag{6.1.15}$$

For $\Omega\left(|x_2|^2 - |x_1|^2\right)$ in the right-hand side of (Equation 6.1.14), we use the triangular inequality and (Equation 6.1.13) to obtain

$$|x_1|^2 < 2|x_1 - x_0|^2 + 2|x_0|^2 < 2 + \frac{2\mathcal{A}(D)}{\left(1 - \sqrt{2\mathcal{A}(D)}\right)^2}\left(1 + \sqrt{\frac{1}{4\Omega}}\right)^2, \tag{6.1.16}$$

$$|x_2|^2 < 2|x_2 - x_0|^2 + 2|x_0|^2 < 8 + \frac{2\mathcal{A}(D)}{\left(1 - \sqrt{2\mathcal{A}(D)}\right)^2}\left(1 + \sqrt{\frac{1}{4\Omega}}\right)^2. \tag{6.1.17}$$

To estimate $h(x_1) - h(x_2)$, we use the fact that

$$\|\nabla f * \mathcal{N}\|_{L^\infty} \leq \sqrt{\frac{2}{\pi}}\sqrt{\|f\|_{L^1}\|f\|_{L^\infty}} \quad \text{for any } f \in L^1 \cap L^\infty(\mathbb{R}^2). \tag{6.1.18}$$

Indeed, one can compute with $a := \sqrt{\frac{\|f\|_{L^1(\mathbb{R}^2)}}{2\pi\|f\|_{L^\infty(\mathbb{R}^2)}}}$,

$$|\nabla f * \mathcal{N}(x)| \leq \frac{1}{2\pi}\int_{|x-y|>a} \frac{1}{|x-y|}\,|f(y)|\,dy + \frac{1}{2\pi}\int_{|x-y|\leq a} \frac{1}{|x-y|}\,|f(y)|\,dy \leq \frac{\|f\|_{L^1(\mathbb{R}^2)}}{2\pi a} + a\|f\|_{L^\infty(\mathbb{R}^2)},$$

which yields (Equation 6.1.18). Thus we have

$$|h(x_1) - h(x_2)| < |\nabla h|_{L^\infty}|x_1 - x_2| < \sqrt{\frac{2}{\pi}}\sqrt{|D \triangle B(x_0)|}|x_1 - x_2| = 3\sqrt{2\mathcal{A}(D)}. \tag{6.1.19}$$

Hence it follows from (Equation 6.1.14), (Equation 6.1.15), (Equation 6.1.16), (Equation 6.1.17) and (Equation 6.1.19) that

$$\frac{1}{2}\log 2 < \Omega\left(10 + \frac{4\mathcal{A}(D)}{\left(1 - \sqrt{2\mathcal{A}(D)}\right)^2}\left(1 + \sqrt{\frac{1}{4\Omega}}\right)^2\right) + 3\sqrt{2\mathcal{A}(D)}, \tag{6.1.20}$$

which contradicts our choice of Ω_1 and α_1 for (Equation 6.1.9). This proves the claim (Equation 6.1.7).

To prove (Equation 6.1.8), let us fix Ω_1 and $\alpha_1 < \frac{1}{2}$ so that the claim (Equation 6.1.7) holds. Then for all uniformly-rotating patches with $\Omega < \Omega_1$ and $\mathcal{A}(D) < \alpha_1$, it follows from (Equation 6.1.10) that

$$|x_0| \leq \frac{1}{\pi}|D \triangle B(x_0)| \sup_{x \in D \cup B(x_0)} |x| \leq \mathcal{A}(D)\left(|x_0| + 2\right),$$

where we used (Equation 6.1.7) to get the last inequality. Therefore

$$|x_0| < \frac{2\mathcal{A}(D)}{1 - \mathcal{A}(D)} < 4\mathcal{A}(D),$$

where we used $\mathcal{A}(D) \leq \alpha_1 < \frac{1}{2}$ for the last inequality. Hence (Equation 6.1.8) is proved. □

Since we are interested in patches that rotate about the origin, let us consider the asymmetry between D and the unit disk centered at the origin:

$$\mathcal{A}_0(D) := \frac{|D \triangle B|}{|D|} = \frac{|D \triangle B|}{\pi}.$$

Tautologically, it holds that $\mathcal{A}(D) \leq \mathcal{A}_0(D)$. For rotating patches, we have the following lemma:

Lemma 6.1.5. *There exist positive constants Ω_1 and c_1 such that if (D, Ω) is a solution to* (Equation 1.2.4) *with $\Omega < \Omega_1$, then*

$$\mathcal{A}_0(D) \leq c_1 \mathcal{A}(D). \tag{6.1.21}$$

Proof. Let x_0 be a point in $\mathbb{R}^2$ such that $\frac{|D \cap B(x_0)|}{\pi} = \mathcal{A}(D)$. By Lemma 6.1.4, we can pick Ω_1 and $\alpha_1 < \frac{1}{2}$ such that if $\Omega < \Omega_1$ and $\mathcal{A}(D) < \alpha_1$, then

$$|x_0| \leq 4\mathcal{A}(D). \tag{6.1.22}$$

Let us assume for a moment that the claim is true. If $\mathcal{A}(D) \geq \alpha_1$, then it follows from the definition of $\mathcal{A}_0$ that

$$\mathcal{A}_0(D) \leq \frac{|D| + |B|}{\pi} = 2 \leq \frac{2}{\alpha_1}\mathcal{A}(D). \tag{6.1.23}$$

Now let us assume that $\mathcal{A}(D) < \alpha_1$. For a constant $c > 0$ such that $|B \triangle B(x_0)| \leq c|x_0|$, we can compute

$$\mathcal{A}_0(D) = \frac{|D \triangle B|}{\pi} \leq \frac{|D \triangle B(x_0)| + |B \triangle B(x_0)|}{\pi} \leq \mathcal{A}(D) + \frac{c|x_0|}{\pi} \leq \left(1 + \frac{4c}{\pi}\right)\mathcal{A}(D), \tag{6.1.24}$$

where the last inequality follows from (Equation 6.1.22). Therefore (Equation 6.1.21) follows from (Equation 6.1.23) and (Equation 6.1.24) by choosing $c_1 := \max\left\{\frac{2}{\alpha_1}, \left(1 + \frac{4c}{\pi}\right)\right\}$. □

In the next lemma, we will prove that if $\mathcal{A}_0(D)$ is sufficiently small, then D is necessarily star-shaped.

Lemma 6.1.6. *There exist positive constants Ω_2, α_2 and c_2 such that if (D, Ω) is a solution to* (Equation 1.2.4) *with $\Omega < \Omega_2$ and $\mathcal{A}_0(D) < \alpha_2$, then there exists $u \in C^1(\mathbb{T})$ such that*

$$\partial D = \{(1 + u(\theta))(\cos\theta, \sin\theta) : \theta \in \mathbb{T}\}, \tag{6.1.25}$$

and

$$\|u\|_{L^\infty} \leq c_2 \mathcal{A}_0(D) \left|\log \mathcal{A}_0(D)\right|. \tag{6.1.26}$$

Proof. Without loss of generality, we assume that $D \neq B$. The key observation is that if Ω and $\mathcal{A}_0(D)$ are sufficiently small, then the radial derivative of the relative stream function Ψ is strictly positive near ∂B, while ∂D is a connected level set of Ψ.

To prove the lemma, let us consider the following decomposition of Ψ:

$$\Psi(x) := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \underbrace{1_B * \mathcal{N} - \frac{\Omega}{2}|x|^2}_{=:\Psi^{rad}(x)} + \underbrace{(1_D - 1_B) * \mathcal{N}(x)}_{=:\Psi^{e}(x)}.$$

We claim that there exist positive constants Ω_1 and α_1 such that if (D, Ω) is a solution to (Equation 1.2.4) with $\Omega < \Omega_1$ and $\mathcal{A}_0(D) < \alpha_1$, then it holds for some $c, C > 0$ that

$$\partial_r \Psi^{rad}(x) > c \text{ for } |x| \in \left(\frac{7}{8}, \frac{9}{8}\right) \tag{6.1.27}$$

$$|\Psi^e(x)| < C\mathcal{A}_0(D)\left|\log \mathcal{A}_0(D)\right| \text{ for } x \in \mathbb{R}^2. \tag{6.1.28}$$

Let us assume for a moment that (Equation 6.1.27) and (Equation 6.1.28) are true. Then we set

$$\Omega_2 := \Omega_1 \quad \text{and} \quad \alpha_2 := \{\alpha_1, \alpha^*\},$$

where $\alpha^* = \min\left\{\alpha > 0 : \frac{C}{c}\alpha \log \frac{1}{\alpha} = \frac{1}{16}\right\}$. If $\Omega < \Omega_2$ and $\mathcal{A}_0(D) < \alpha_2$, then for any x_1 and x_2 such that

$$x_1 \in \partial D \cap \partial B, \quad \text{and} \quad |x_2| = 1 - \frac{2C}{c}\mathcal{A}_0(D)\left|\log \mathcal{A}_0(D)\right| > \frac{7}{8},$$

we have

$$\begin{aligned}\Psi(x_1) - \Psi(x_2) &= \left(\Psi^{rad}(x_1) - \Psi^{rad}(x_2)\right) + \left(\Psi^e(x_1) - \Psi^e(x_2)\right)\\ &> c\left(|x_1| - |x_2|\right) + \left(\Psi^e(x_2) - \Psi^e(x_1)\right)\\ &> 0,\end{aligned} \tag{6.1.29}$$

where the first and the second inequalities follow from (Equation 6.1.27) and (Equation 6.1.28) respectively. In the same way, one can easily show that for any x_3 such that $|x_3| = 1 + \frac{2C}{c}\mathcal{A}_0(D)\log\frac{1}{\mathcal{A}_0(D)} < \frac{9}{8}$, we have $\Psi(x_3) - \Psi(x_1) > 0$. Since ∂D is a connected level set of Ψ and $\partial B \cap \partial D \neq \emptyset$, we get

$$\partial D \subset \left\{ x \in \mathbb{R}^2 : 1 - \frac{2C}{c}\mathcal{A}_0(D)\log\frac{1}{\mathcal{A}_0(D)} < |x| < 1 + \frac{2C}{c}\mathcal{A}_0(D)\log\frac{1}{\mathcal{A}_0(D)} \right\}. \quad (6.1.30)$$

Hence the implicit function theorem with (Equation 6.1.27) and (Equation 6.1.30) yields that there exists $u \in C^1(\mathbb{T})$ such that (Equation 6.1.25) holds. Furthermore, (Equation 6.1.30) immediately implies (Equation 6.1.26).

To complete the proof, we need to prove the claims. To prove (Equation 6.1.27), note that $\partial_r \Psi^{rad}(r)$ is explicit and given by

$$\partial_r \Psi^r(r) = \begin{cases} \left(\frac{1}{2} - \Omega\right) r & \text{if } r \leq 1 \\ \frac{1}{2r} - \Omega r & \text{if } r > 1. \end{cases}$$

Then (Equation 6.1.27) follows immediately by choosing sufficiently small Ω_1. For (Equation 6.1.28), note that Lemma 6.1.4 implies that we can choose Ω_1 and α_1 so that $D \subset B_3$. Then we have for any $x \in \mathbb{R}^2$ that

$$\begin{aligned}
|\Psi^e(x)| &= \left| \int_{y \in B_3} (1_D(y) - 1_B(y)) \log|x-y| dy \right| \\
&\lesssim \left| \int_{y \in B_3, |x-y|<10} (1_D(y) - 1_B(y)) \log|x-y| dy \right| + \left| \int_{y \in B_3, |x-y|>10} (1_D(y) - 1_B(y)) \log|x-y| dy \right| \\
&\lesssim \int_{y \in B_3, |x-y|<10} |1_D(y) - 1_B(y)| \Big| \log|x-y| \Big| dy + \left| \int_{y \in B_3, |x-y|>10} (1_D(y) - 1_B(y)) \frac{\log|x-y|}{\log x} dy \right| \\
&\lesssim |D \triangle B| \Big| \log|D \triangle B| \Big| \\
&\lesssim \mathcal{A}_0(D) \Big| \log \mathcal{A}_0(D) \Big|,
\end{aligned}$$

where we used $\int 1_D(y) - 1_B(y)dx = 0$ to get the second inequality. This proves (Equation 6.1.28).

□

The proof of the following proposition will be postponed to the next subsection.

Proposition 6.1.7. *There exist positive constants Ω_3 and α_3 such that if (D, Ω) is a solution to* (Equation 1.2.4) *with $\Omega < \Omega_3$ and D is a star-shaped domain with $\mathcal{A}_0(D) \le \alpha_3$, then $D = B$.*

Now we are ready to prove Theorem G.

Proof of Theorem G: We will choose Ω_0 and α_0 so small that all the previous lemmas are applicable. Let us set $\Omega_0 := \min\left\{\Omega_1, \Omega_2, \Omega_3, \frac{1}{4}\right\}$ and $\alpha_0 := \min\left\{\frac{\alpha_2}{c_1}, \frac{\alpha_3}{c_1}\right\}$, where $\alpha_i' s$ and c_1 are as in Lemma 6.1.5, 6.1.6 and Proposition 6.1.7. Moreover, let σ be as described in Proposition 6.1.3. We assume that (D, Ω) is a solution to (Equation 1.2.4) with $\Omega < \Omega_0$ and $D \neq B$. Then we will prove

$$\sqrt{\frac{\sigma}{2\pi}}\alpha_0\Omega^{-\frac{1}{2}} < \sup_{x\in\partial D} |x|. \tag{6.1.31}$$

Since $D \neq B$, we have $\mathcal{A}(D) \ge \alpha_0$. Indeed, if $\mathcal{A}(D) < \alpha_0$, then Lemma 6.1.5 and Lemma 6.1.6 imply that D is star-shaped and $\mathcal{A}_0(D) < c_1\alpha_0 < \alpha_3$. Therefore, Proposition 6.1.7 yields that $D = B$, which is a contradiction. Thus it follows from (Equation 6.1.1) and (Equation 6.1.6) that

$$\Omega \int_D |x|^2 dx \ge (1 - 2\Omega)\left(\frac{|D|^2}{4\pi} - \int_D p dx\right) > \frac{1}{2}\sigma\alpha_0^2,$$

where we used $\Omega < \frac{1}{4}$. It is clear that $\Omega \int_D |x|^2 dx \le \pi\Omega \left(\sup_{x\in\partial D} |x|\right)^2$, hence the above inequality yields (Equation 6.1.31). □

6.1.1 Proof of Proposition 6.1.7

In this subsection, we aim to prove Proposition 6.1.7. We say a simply-connected bounded domain is star-shaped if there exist $u : \mathbb{T} \mapsto (-1, \infty)$ such that

$$\partial D = \left\{ (1 + u(\theta))(\cos\theta, \sin\theta) \in \mathbb{R}^2 : \theta \in \mathbb{T} \right\}.$$

If $|D| = \pi$, we have that

$$\pi = \int_{\mathbb{R}^2} 1_D(x)dx = \int_{\mathbb{T}} \int_0^{(1+u(\theta))} r dr d\theta = \pi + \frac{1}{2} \int_{\mathbb{T}} u(\theta)^2 + 2u(\theta)d\theta,$$

thus

$$\int_{\mathbb{T}} u(\theta)^2 d\theta = -\int_{\mathbb{T}} 2u(\theta)d\theta. \tag{6.1.32}$$

Furthermore $\mathcal{A}_0(D)$ and the difference of second moments of $1_D dx$ and $1_B dx$ can be written in terms of u as

$$\mathcal{A}_0(D) = \frac{1}{\pi} \int_{\mathbb{T}} |u(\theta)| + sgn(u(\theta)) \frac{u(\theta)^2}{2} d\theta, \tag{6.1.33}$$

$$\int_D |x|^2 dx - \frac{|D|^2}{2\pi} = \int_{\mathbb{T}} |u(\theta)|^2 + u(\theta)^3 + \frac{1}{4} u(\theta)^4 d\theta, \tag{6.1.34}$$

where

$$sgn(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Note that if $\|u\|_{L^\infty(\mathbb{T})} < \frac{1}{2}$, then (Equation 6.1.33) and (Equation 6.1.34) imply that there exists $c_3 > 0$ such that

$$\frac{1}{c_3}\int_{\mathbb{T}} |u(\theta)| d\theta \leq \mathcal{A}_0(D) \leq c_3 \int_{\mathbb{T}} |u(\theta)| d\theta, \tag{6.1.35}$$

$$\frac{1}{c_3}\int_{\mathbb{T}} |u(\theta)|^2 d\theta \leq \int_D \frac{|x|^2}{2} dx - \frac{|D|^2}{4\pi} \leq c_3 \int_{\mathbb{T}} |u(\theta)|^2 d\theta. \tag{6.1.36}$$

The proof of Proposition 6.1.7 is based on the identity (Equation 6.1.2). We will estimate the right-hand side of (Equation 6.1.2) in the following proposition.

Proposition 6.1.8. *Let D be a star-shaped domain parametrized by $u : \mathbb{T} \mapsto \mathbb{R}$ with $\|u\|_{L^\infty} < \frac{1}{2}$. Then there exists $\delta > 0$ such that for any $a \in \left(2\|u\|_{L^\infty(\mathbb{T})}, 1\right)$, it holds that*

$$\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2 dx \leq \delta \left(a \int_{\mathbb{T}} |u|^2 d\theta + \frac{1}{a}\int_{\mathbb{T}} f(\theta)^2 d\theta \right), \tag{6.1.37}$$

where $f(\theta) := \int_0^\theta u(s)^2 + 2u(s) ds$.

The above proposition will play a key role in the proofs of Proposition 6.1.7 and Theorem Theorem H. In the proof of Proposition 6.1.7, we simply use $|f(\theta)| \lesssim \|u\|_{L^1(\mathbb{T})}$, so that the left-hand side can be almost bounded by L^1-norm of u. Note that if we can choose a small enough, then the proposition, together with (Equation 6.1.1) and (Equation 6.1.36) will give $\|u\|_{L^2(\mathbb{T})} \lesssim \|u\|_{L^1(\mathbb{T})}$.

In section section 6.2, we will use the fact that if $u(\theta)$ is $\frac{2\pi}{m}$ periodic, then $f(\theta)$ is also $\frac{2\pi}{m}$-periodic, which follows from (Equation 6.1.32). This will be used for the proof of Theorem Theorem H.

Proof. Using Cauchy-Schwarz inequality, we obtain that

$$\int_D |x - 2\nabla (1_D * \mathcal{N})|^2 dx \lesssim \int_D |x - 2\nabla (1_B * \mathcal{N})|^2 dx + \int_D |\nabla \mathcal{N} * (1_B - 1_D)|^2 dx =: H_1 + H_2 \tag{6.1.38}$$

To estimate H_1, note that

$$\nabla (1_B * \mathcal{N}) = \begin{cases} \frac{x}{2} & \text{if } |x| \le 1 \\ \frac{x}{2|x|^2} & \text{if } |x| > 1. \end{cases}$$

Therefore we can compute

$$\begin{aligned} \int_D |x - 2\nabla (1_B * \mathcal{N})|^2 dx &= \int_{D \setminus B} \left| x - \frac{x}{|x|^2} \right|^2 dx \\ &= \int_{D \setminus B} |x|^2 - 2 + \frac{1}{|x|^2} dx \\ &= \int_{\mathbb{T} \cap \{u > 0\}} \int_1^{1+u(\theta)} \cdot \left(r^2 - 2 + \frac{1}{r^2} \right) r dr d\theta \end{aligned}$$

However, we have that for $u(\theta) > 0$,

$$\begin{aligned} \int_1^{1+u(\theta)} r^3 - 2r + \frac{1}{r} dr &= \frac{1}{4} u(\theta)^4 + u(\theta)^3 + \frac{1}{2} u(\theta)^2 - u(\theta) + \log(1 + u(\theta)) \\ &\le \frac{1}{4} u(\theta)^4 + \frac{4}{3} u(\theta)^3 \\ &\lesssim \|u\|_{L^\infty(\mathbb{T})} |u(\theta)|^2 \\ &\lesssim a |u(\theta)|^2, \end{aligned}$$

where we used $\log(1 + x) \le x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ for $x \ge 0$ and $0 \le u(\theta) < \frac{1}{2}$. Hence it follows that

$$H_1 \lesssim a \int_{\mathbb{T}} |u|^2 d\theta. \tag{6.1.39}$$

In order to estimate H_2, we recall the following result:

Proposition 6.1.9. *[79, Proposition 3.1] Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^d$ with L^∞ densities with respect to Lebesgue measure. Then*

$$\|\nabla(\mathcal{N} * (\rho_1 - \rho_2))\|^2_{L^2(\mathbb{R}^d)} \leq \max(\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}) W_2^2(\rho_1, \rho_2),$$

where $W_2(\rho_1, \rho_2)$ denotes 2-Wasserstein distance between ρ_1 and ρ_2 defined by

$$W_2^2(\rho_1, \rho_2) := \inf \left\{ \int |T(x) - x|^2 d\rho_1(x) : T_{\#}\rho_1 = \rho_2 \right\}.$$

Thanks to Proposition 6.1.9, it follows that

$$H_2 = \|\nabla(\mathcal{N} * (1_D - 1_B))\|^2_{L^2(\mathbb{R}^2)} \leq \int_D |T(x) - x|^2 dx, \tag{6.1.40}$$

for any $T : D \mapsto B$ such that

$$T_{\#}\left(1_D(x)dx\right) = 1_B(x)dx, \tag{6.1.41}$$

where $T_{\#}\rho$ denotes the pushforward measure of ρ by T. Note that in polar coordinates, (Equation 6.1.41) is equivalent to

$$T_{\#}\left(1_{\tilde{D}}(r,\theta) r dr d\theta\right) = 1_{\tilde{B}}(r,\theta) r dr d\theta, \tag{6.1.42}$$

where $\tilde{D} := \{(r,\theta) \in [0,1) \times \mathbb{T} : 0 \leq r < 1 + u(\theta)\}$ and $\tilde{B} := \{(r,\theta) \in [0,1) \times \mathbb{T} : 0 \leq r < 1\}$. Hence it suffices to find a transport map T which gives the desired estimate.

Let us define $T : \tilde{D} \mapsto \tilde{B}$ by,

$$T(r,\theta) := \big(T^r(r,\theta), T^\theta(\theta)\big) := \begin{cases} \Big(\sqrt{\frac{a(2-a)(r^2-(1+u(\theta))^2)}{(u(\theta)+a)(u(\theta)+2-a)}+1}, \frac{f(\theta)}{a(2-a)}+\theta\Big) & \text{if } r > 1-a \\ (r,\theta) & \text{if } r \leq 1-a. \end{cases} \tag{6.1.43}$$

for $a \in (2\|u\|_{L^\infty}, 1)$, where $f(\theta) := \int_0^\theta u(\eta)^2 + 2u(\eta)d\eta$.

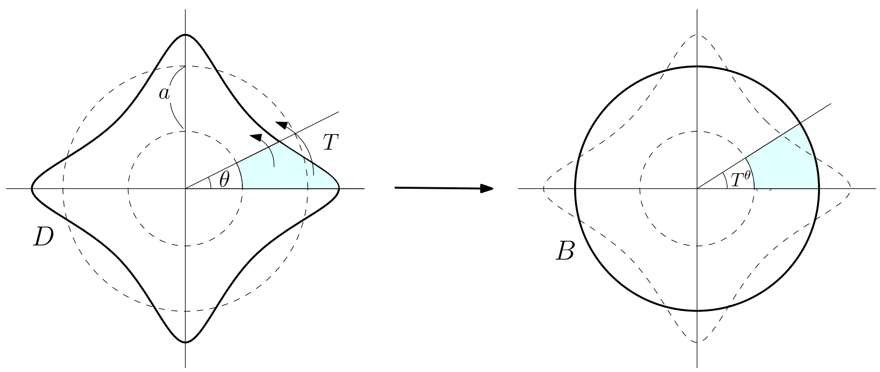


Figure 6.1: Illustration of the transport map T that pushes forwards D to B.

Our motivation for the transport map T is the following: We first choose T^θ so that T^θ is independent of r and preserves the area in the sense that (see Figure Figure 6.1 for the illustration)

$$\int_0^\theta \int_{1-a}^{1+u(s)} rdrds = \int_0^{T^\theta(\theta)} \int_{1-a}^{1} rdrds.$$

And then, we choose $T^r(r,\theta)$ so that (Equation 6.1.42) is satisfied. Note that in order to check the condition (Equation 6.1.42) for T, it suffices to show that

$$1_{\tilde{D}}(r,\theta)r = 1_{\tilde{B}}(T(r,\theta))T^r(r,\theta)|\text{det}(\nabla T)|, \tag{6.1.44}$$

almost everywhere with respect to the measure $1_{\tilde{D}}rdrd\theta$ (see [104]). Then it is clear that $\theta \mapsto$

$T^\theta(\theta)$ and $r \mapsto T^r(r,\theta)$ are increasing for fixed r and θ respectively. Indeed,

$$\frac{d}{d\theta}T^\theta(\theta) = 1 + \frac{u(\theta)^2 + 2u(\theta)}{a(2-a)} \geq \frac{2a - a^2 + \frac{a^2}{4} - a}{a(2-a)} \geq \frac{a - \frac{3}{4}a^2}{a(2-a)} > 0,$$

where the first inequality follows from that $\|u\|_{L^\infty(\mathbb{T})} < \frac{1}{2}a$ and $x \mapsto x^2 + 2x$ is increasing for $x \geq -1$ thus $u(\theta)^2 + 2u(\theta) \geq \frac{a^2}{4} - a$. Since T maps $\{(r,\theta) : r = 1-a \text{ or } 1+u(\theta)\}$ to $\{(r,\theta) : r = 1-a \text{ or } r = 1\}$ continuously, T is bijective and therefore $1_{\tilde{D}}(r,\theta) = 1_{\tilde{B}} \circ T(r,\theta)$. Furthermore, the Jacobian matrix of T can be computed as

$$\nabla T(r,\theta) = \begin{cases} \begin{pmatrix} \frac{1}{T^r(r,\theta)} \frac{a(2-a)r}{(u(\theta)+a)(u(\theta)+2-a)} & \partial_\theta T^r(r,\theta) \\ 0 & \frac{(u(\theta)+a)(u(\theta)+2-a)}{a(2-a)} \end{pmatrix} & \text{if } 1-a < r < 1+u(\theta), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise}, \end{cases}$$

therefore

$$T^r(r,\theta)|\det(\nabla T)| = r,$$

almost everywhere. This implies that T satisfies (Equation 6.1.44) and thus (Equation 6.1.42) holds. Then it follows from (Equation 6.1.40) that

$$H_2 \leq \int_{\mathbb{T}} \int_{1-a}^{1+u(\theta)} |T^r(r,\theta)\cos(T^\theta) - r\cos\theta|^2 + |T^r(r,\theta)\sin(T^\theta) - r\sin\theta|^2 r dr d\theta.$$

The cosine term in the integrand can be estimated as

$$
\begin{aligned}
|T^r(r,\theta)\cos(T^\theta) - r\cos\theta|^2 &= |(T^r(r,\theta)-r)\cos(T^\theta(\theta)) + r(\cos(T^\theta(\theta)) - \cos\theta)|^2 \\
&\lesssim |T^r(r,\theta)-r|^2 + |\cos(T^\theta(\theta)) - \cos\theta|^2 \\
&\lesssim |T^r(r,\theta)-r|^2 + |T^\theta(\theta)-\theta|^2.
\end{aligned}
$$

In the same way, the sine term can be bounded as $|T^r\sin(T^\theta) - r\sin\theta|^2 \lesssim |T^r - r|^2 + |T^\theta - \theta|^2$, thus we have

$$
H_2 \lesssim \int_{\mathbb{T}}\int_{1-a}^{1+u(\theta)} |T^r(r,\theta)-r|^2 r dr d\theta + \int_{\mathbb{T}}\int_{1-a}^{1+u(\theta)} |T^\theta(\theta)-\theta|^2 r dr d\theta =: A_1 + A_2. \quad (6.1.45)
$$

A_2 is bounded by

$$
A_2 \leq \int_{\mathbb{T}}\int_{1-a}^{1+u(\theta)} \frac{f(\theta)^2}{a^2} r dr d\theta \lesssim \int_{\mathbb{T}} f(\theta)^2 \frac{|u(\theta)|+a}{a^2} \lesssim \frac{1}{a}\int_{\mathbb{T}} f(\theta)^2 d\theta, \quad (6.1.46)
$$

where we used $\|u\|_{L^\infty(\mathbb{T})} < a$ to get the first and the last inequalities.

For A_1, we assume for a momoent that for $r \in (1-a, 1+u(\theta))$,

$$
|T^r(r,\theta) - r| \lesssim |u(\theta)|. \quad (6.1.47)
$$

From (Equation 6.1.47), we obtain

$$
A_1 \lesssim \int_{\mathbb{T}}\int_{1-a}^{1+u(\theta)} |u(\theta)|^2 r dr d\theta = \int_{\mathbb{T}} |u(\theta)|^2 \int_{1-a}^{1+u(\theta)} r dr d\theta \lesssim a \int_{\mathbb{T}} |u(\theta)|^2 d\theta, \quad (6.1.48)
$$

where the last inequality follows from $a > \|u\|_{L^\infty(\mathbb{T})}$. Therefore, it follows from (Equation 6.1.45), (Equation 6.1.46) and (Equation 6.1.48) that

$$H_2 \lesssim a \int_{\mathbb{T}} |u|^2 d\theta + \frac{1}{a} \int_{\mathbb{T}} |f|^2 d\theta. \tag{6.1.49}$$

Thus (Equation 6.1.37) follows from (Equation 6.1.38), (Equation 6.1.39) and (Equation 6.1.49).

To check (Equation 6.1.47), let $g(a,r,x) := \frac{\sqrt{\frac{a(2-a)(r^2-(1+x)^2)}{(x+a)(x+2-a)}+1}-r}{x}$ so that $\frac{T^r(r,\theta)-r}{u(\theta)} = g(a,r,u(\theta))$. Then it suffices to show that $|g(a,r,x)| \lesssim 1$ in $\{(a,r,x) : (1-a) < r < 1+x,\ 2|x| < a < 1\}$. Since $g(a,r,x)$ is continuous everywhere except for $x = 0$, we only need to check $|g(a,r,x)| \lesssim 1$ when $0 < x \ll 1$. Taking the limit, we obtain

$$\lim_{x\to 0^+} g(a,r,x) = \frac{\frac{\partial}{\partial x}\left(\sqrt{\frac{a(2-a)(r^2-(1+x)^2)}{(x+a)(x+2-a)}+1}-r\right)\Big|_{x=0}}{1} = \frac{(1-r^2)-a(2-a)}{ra(2-a)}.$$

If $r < \frac{1}{2}$, then $a > \frac{1}{2}$ therefore it follows from $r > 1-a > 0$ that

$$\lim_{x\to 0} |g(a,r,x)| = \frac{|r^2-(a-1)^2|}{ra(2-a)} \leq \frac{r}{a(2-a)} + \frac{(a-1)^2}{ra(2-a)} \leq \frac{2r}{a(2-a)} \lesssim 1,$$

where the second inequality follows from $(1-a) < r$. If $r > \frac{1}{2}$, then it follows from $|r-1| < a$ that

$$\lim_{x\to 0} |g(a,r,x)| \leq \frac{|1-r^2|}{ra(2-a)} + \frac{(2-a)}{r(2-a)} < \frac{1+r}{r(2-a)} + \frac{2-a}{r(2-a)} \lesssim 1.$$

This proves (Equation 6.1.47) and finishes the proof. $\square$

Now we are ready to prove Proposition 6.1.7.

Proof of Proposition 6.1.7: We will fix Ω_3 and α_3 so small that all the lemmas are applicable. To do so, let us denote $h(x) := -x\log x$. Also we denote by $\alpha^* > 0$ the smallest positive number such that

$$(h(\alpha^*))^3 > (\alpha^*)^2. \tag{6.1.50}$$

Furthermore, let $\Omega_i's$, $\alpha_i's$ and $c_i's$ for $i = 1, 2$ be as in Lemma 6.1.5 and Lemma 6.1.6, let c_3 be as in (Equation 6.1.35) and (Equation 6.1.36) and let δ be as in Proposition 6.1.8. Lastly, let $c_4 := 18\pi c_3^3 \delta^2$. Then let us fix

$$\Omega_3 := \min\left\{\Omega_1, \Omega_2, \frac{1}{4}, \frac{\sigma}{16c_1^2c_3^2c_4}\right\} \quad and \quad \alpha_3 := \left\{\alpha^*, \alpha_1, \alpha_2, \left(\frac{1}{2c_2}\right)^{\frac{3}{2}}, \left(\frac{1}{4c_2c_3\delta}\right)^{\frac{3}{2}}\right\}. \tag{6.1.51}$$

Then our goal is to show that if (D, Ω) is a solution to (Equation 1.2.4) with $\Omega < \Omega_3$ and $\mathcal{A}_0(D) < \alpha_3$, then $D = B$.

Step 1. Let us claim that

$$\mathcal{A}_0(D) \le c_1 \mathcal{A}(D). \tag{6.1.52}$$

$$\|u\|_{L^\infty(\mathbb{T})} \le c_2 \mathcal{A}_0(D)^{\frac{2}{3}} \le \frac{1}{2}. \tag{6.1.53}$$

Since $\Omega_3 < \Omega_1$ and $\mathcal{A}(D) \le \mathcal{A}_0(D) < \alpha_3 \le \alpha_1$, it follows from Lemma 6.1.5 that $\mathcal{A}_0(D) < c_1\mathcal{A}(D)$. In addition, $\Omega_3 < \Omega_2$, $\mathcal{A}_0(D) < \alpha_3 \le \alpha_2$ and Lemma 6.1.6 imply that

$$\|u\|_{L^\infty(\mathbb{T})} \le c_2 h(\mathcal{A}_0(D)) \le c_2 \mathcal{A}_0(D)^{\frac{2}{3}},$$

where the last inequality follows from $\alpha_3 \le \alpha^*$. Since $\mathcal{A}_0(D) < \alpha_3 \le \left(\frac{1}{2c_2}\right)^{\frac{3}{2}}$, we have $c_2\mathcal{A}_0(D)^{\frac{2}{3}} \le \frac{1}{2}$, which proves (Equation 6.1.53).

Step 2. In this step, we will show that

$$\frac{1}{2}\int_D |x-2\nabla\left(1_D*\mathcal{N}\right)|^2 dx \le \frac{1}{4c_3}\int_{\mathbb{T}} |u|^2 d\theta + c_4\mathcal{A}_0(D)^2, \tag{6.1.54}$$

where $c_4 := 18\pi c_3^3\delta^2$. Since $\|u\|_{L^\infty((T))} < \frac{1}{2}$, we will apply Proposition 6.1.8 with $a := \frac{1}{2c_3\delta}$. Note that

$$2\|u\|_{L^\infty(\mathbb{T})} \le 2c_2\mathcal{A}_0(D)^{\frac{2}{3}} < 2c_2\alpha_3^{\frac{2}{3}} \le a,$$

where the first inequality follows from (Equation 6.1.53), the second follows from the assumption that $\mathcal{A}_0(D) < \alpha_3$ and the last inequality follows from (Equation 6.1.51), which says $\alpha_3 \le \left(\frac{1}{4c_2c_3\delta}\right)^{\frac{3}{2}}$. Thus we can obtain by using Proposition 6.1.8 that

$$\frac{1}{2}\int_D |x-2\nabla\left(1_D*\mathcal{N}\right)|^2 dx \le \frac{1}{4c_3}\int_{\mathbb{T}} |u|^2 d\theta + c_3\delta^2\int_{\mathbb{T}} f(\theta)^2 d\theta, \tag{6.1.55}$$

where $f(\theta) = \int_0^\theta u(s)^2 + 2u(s)ds$. Moreover, we have

$$|f(\theta)| \le \int_0^\theta 3|u(s)|ds < \int_{\mathbb{T}} 3|u(s)|ds \le 3c_3\mathcal{A}_0(D), \tag{6.1.56}$$

where the last inequality follows from (Equation 6.1.35). Therefore it follows from (Equation 6.1.55) and (Equation 6.1.56) that

$$\frac{1}{2}\int_D |x-2\nabla\left(1_D*\mathcal{N}\right)|^2 dx \le \frac{1}{4c_3}\int_{\mathbb{T}} |u|^2 d\theta + 18\pi c_3^3\delta^2\mathcal{A}_0(D)^2,$$

which proves the claim (Equation 6.1.54).

Step 3. Now we will prove that

$$\int_{\mathbb{T}} |u(\theta)|^2 d\theta \leq 4c_3c_4\mathcal{A}_0(D)^2. \tag{6.1.57}$$

Since $\Omega < \Omega_3 \leq \frac{1}{4}$, it follows from (Equation 6.1.36) that

$$(1-2\Omega)\left(\int_D \frac{|x|^2}{2}dx - \frac{|D|^2}{4\pi}\right) > \frac{1}{2c_3}\int_{\mathbb{T}} |u(\theta)|^2 d\theta. \tag{6.1.58}$$

Thus it follows from (Equation 6.1.2) and (Equation 6.1.54) that

$$\frac{1}{2c_3}\int_{\mathbb{T}} |u(\theta)|^2 d\theta < \frac{1}{4c_3}\int_{\mathbb{T}} |u(\theta)|^2 d\theta + c_4\mathcal{A}_0(D)^2,$$

which proves (Equation 6.1.57).

Step 4. Finally, we will prove $D = B$ by showing that $\mathcal{A}_0(D) = 0$. This will be done by estimating the left/right-hand side in (Equation 6.1.1). It follows from Proposition 6.1.3 and (Equation 6.1.52) that

$$(1-2\Omega)\left(\frac{|D|^2}{4\pi} - \int_D p\,dx\right) \geq \frac{1}{2}\sigma\mathcal{A}(D)^2 \geq \frac{1}{2c_1^2}\sigma\mathcal{A}_0(D)^2, \tag{6.1.59}$$

where we used $\Omega < \frac{1}{4}$. Moreover, it follows from (Equation 6.1.36) and (Equation 6.1.57) that

$$2\Omega\left(\int_D \frac{|x|^2}{2}dx - \frac{|D|^2}{4\pi}\right) \leq 2\Omega c_3\int_{\mathbb{T}} |u(\theta)|^2 d\theta \leq 8\Omega c_3^2 c_4\mathcal{A}_0(D)^2. \tag{6.1.60}$$

Therefore (Equation 6.1.1) yields that

$$\left(8\Omega c_3^2 c_4 - \frac{\sigma}{2c_1^2}\right)\mathcal{A}_0(D)^2 \geq 0.$$

This implies $\mathcal{A}_0(D) = 0$, since $8\Omega c_3^2 c_4 - \frac{\sigma}{2c_1^2} < 8\Omega_3 c_3^2 c_4 - \frac{\sigma}{2c_1^2} \leq 0$, which follows from

(Equation 6.1.51) and $\Omega < \Omega_3$. This proves that $D = B$.

$\square$

6.2 Rotating patches with m-fold symmetry

We now move on to the quantitative estimates for m-fold symmetric rotating patches. We say a domain D is m-fold symmetric, if D is invariant under rotation by $\frac{2\pi}{m}$. We divide this section into two subsections: The first subsection is devoted to the proof of Theorem H and the second subsection is devoted to the proof of Theorem I.

6.2.1 Proof of Theorem H

The goal of this subsection is to prove Theorem H. As explained in Remark 1.2.1, angular velocity Ω is independent of radial dilation, thus we will assume that $|D| = |B| = \pi$ throughout this subsection.

For a simply-connected and m-fold symmetric patch D, we denote $r_{min} := \inf_{x\in\partial D}|x|$, and $r_{max} := \sup_{x\in\partial D}|x|$. Note that the origin is necessarily contained in D since D is simply-connected and m-fold symmetric, therefore $r_{min} > 0$. Furthermore, since we are assuming $|D| = \pi$, it is necessarily $r_{min} < 1$ and $r_{max} > 1$ if D is not a disk.

We will prove the theorem by contrapositive. We suppose to the contrary that (D, Ω) is an m-fold symmetric solution with sufficiently large m and $\lambda := \frac{1}{2} - \Omega$ is sufficient large compared to $\frac{1}{m}$. Then Lemma 6.2.2 tells us that the patch is necessarily star-shaped and the polar graph that parametrizes ∂D must be small. With this fact, we will apply the identity (Equation 6.1.2) and Proposition 6.1.8 to derive an upper bound of λ, which we expect to contradict our initial assumption on λ.

Now we introduce a decomposition of the stream function $1_D * \mathcal{N}$. We define a radial function

$g : \mathbb{R}^2 \mapsto \mathbb{R}$ as follows (where we denote it by $g(r)$ by slight abuse of nontation):

$$g(r) := \frac{1}{2\pi r}\mathcal{H}^1\left(\partial B_r \cap D\right),$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Then we shall write, in polar coordinates,

$$\left(1_D * \mathcal{N}\right)(r,\theta) = \left(g * \mathcal{N}\right)(r) + \left(1_D - g\right) * \mathcal{N}(r,\theta) =: \varphi^r(r) + \varphi_m(r,\theta). \tag{6.2.1}$$

Therefore the relative stream function can be written as $\Psi(r,\theta) = \varphi^r(r) - \frac{\Omega}{2}r^2 + \varphi_m(r,\theta)$.

Note that g is a radial function with the same integral as 1_D on each ∂B_r. If D is m-fold symmetric for large m, we would expect that the velocity field generated by the vorticity 1_D must be very close to the velocity field generated by g, that is, we expect that $|\nabla\varphi_m| \ll 1$ if $m \gg 1$. Below we will give a quantitative proof of this fact in Lemma 6.2.1.

Lemma 6.2.1. *Let D be an m-fold symmetric bounded domain for $m \geq 3$. Then*

$$\partial_r\varphi^r(r) = \frac{|D \cap B_r|}{2\pi r}, \tag{6.2.2}$$

$$|\nabla\varphi_m(r,\theta)| \lesssim \frac{r}{m}. \tag{6.2.3}$$

Proof. Let us prove (Equation 6.2.2) first. Obviously, (Equation 6.2.2) is equivalent to

$$2\pi r\partial_r\varphi^r(r) = |D \cap B_r|. \tag{6.2.4}$$

Clearly both sides of (Equation 6.2.4) are zero at $r = 0$. Also we have that

$$\partial_r\left(|D \cap B_r|\right) = \mathcal{H}^1\left(D \cap \partial B_r\right) = 2\pi r g(r) = 2\pi r\Delta\left(\varphi^r(r)\right) = \partial_r\left(2\pi r\partial_r\varphi^r(r)\right),$$

where we used $\Delta = \frac{1}{r}\partial_r\left(r\partial_r\right) + \frac{1}{r^2}\partial_{\theta\theta}$. This proves (Equation 6.2.4), thus (Equation 6.2.2).

We will prove (Equation 6.2.3) by using the formula for the stream function given in Lemma B.1.2. Let $h(r,\theta) := 1_D(r\cos\theta, r\sin\theta) - g(r)$. We apply (Equation B.2.3) and (Equation B.2.4) to (Equation B.1.2) and (Equation B.1.3) respectively, and obtain

$$\begin{aligned}
\partial_r\varphi_m(r,\theta) &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\frac{\left(\frac{\rho}{r}\right)^{m+1}\left(\cos(m\eta)-\left(\frac{\rho}{r}\right)^m\right)}{\left(1-\left(\frac{\rho}{r}\right)^m\right)^2+2\left(\frac{\rho}{r}\right)^m\left(1-\cos\left(m\eta\right)\right)}\right)d\rho d\eta \\
&\quad -\frac{1}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\frac{\left(\frac{r}{\rho}\right)^{m-1}\left(\cos(m\eta)-\left(\frac{r}{\rho}\right)^m\right)}{\left(1-\left(\frac{r}{\rho}\right)^m\right)^2+2\left(\frac{r}{\rho}\right)^m\left(1-\cos\left(m\eta\right)\right)}\right)d\rho d\eta \\
&=: A_1 - A_2, \qquad (6.2.5)\\
\partial_\theta\varphi_m(r,\theta) &= -r\frac{1}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\frac{\left(\frac{\rho}{r}\right)^{m+1}\sin(m\eta)}{\left(1-\left(\frac{\rho}{r}\right)^m\right)^2+2\left(\frac{\rho}{r}\right)^m\left(1-\cos(m\eta)\right)}\right)d\rho d\eta \\
&\quad -r\frac{1}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\frac{\left(\frac{r}{\rho}\right)^{m-1}\sin(m\eta)}{\left(1-\left(\frac{r}{\rho}\right)^m\right)^2+2\left(\frac{r}{\rho}\right)^m\left(1-\cos(m\eta)\right)}\right)d\rho d\eta \\
&=: -rA_3 - rA_4 \qquad (6.2.6)
\end{aligned}$$

We claim that

$$|A_i| \lesssim \frac{r}{m} \text{ for } i = 1,2,3 \text{ and } 4. \qquad (6.2.7)$$

Let us assume for a moment that the claim is true. Then (Equation 6.2.5) and (Equation 6.2.6) yield that $|\nabla\varphi_m(r,\theta)| \sim |\partial_r\varphi_m(r,\theta)| + |\frac{\partial_\theta\varphi_m(r,\theta)}{r}| \lesssim \frac{r}{m}$, which finishes the proof. We give a proof of (Equation 6.2.7) for only A_2 since the other terms can be proved in the same way. Note that in the proof, we will see that the assumption $m \geq 3$ is crucial to estimate A_2 and A_4.

From the change of the variables, $\left(\frac{r}{\rho}\right)^m \mapsto x$ and $\frac{2\pi}{m}$-periodicity of the integrand in the angular variable, it follows that

$$
\begin{aligned}
|A_2| &\leq \int_{\mathbb{T}}\int_0^1 \left| h(rx^{-\frac{1}{m}}, \eta+\theta)\left(\frac{x^{1-\frac{1}{m}}(\cos(m\eta)-x)}{(1-x)^2+2x(1-\cos(m\eta))}\right)\frac{r}{m}x^{-1-\frac{1}{m}}\right| dxd\eta \\
&\leq \frac{r}{m}\int_{\mathbb{T}}\int_0^1 \left|\frac{x^{-\frac{2}{m}}(\cos(m\eta)-x)}{(1-x)^2+2x(1-\cos(m\eta))}\right| dxd\eta \\
&\leq r\int_0^{\frac{2\pi}{m}}\int_0^1 \frac{x^{-\frac{2}{m}}((1-x)+(1-\cos(m\eta)))}{(1-x)^2+2x(1-\cos(m\eta))}dxd\eta \\
&= \frac{r}{m}\int_{\mathbb{T}}\int_0^1 \frac{x^{-\frac{2}{m}}((1-x)+(1-\cos\eta))}{(1-x)^2+2x(1-\cos\eta)}dxd\eta \\
&= \frac{2r}{m}\int_0^{\pi}\int_0^1 \frac{x^{-\frac{2}{m}}((1-x)+(1-\cos\eta))}{(1-x)^2+2x(1-\cos\eta)}dxd\eta \\
&= \frac{2r}{m}\left(\int_0^{\pi}\int_0^{\frac{1}{2}} \frac{x^{-\frac{2}{m}}((1-x)+(1-\cos\eta))}{(1-x)^2+2x(1-\cos\eta)}dxd\eta + \int_0^{\pi}\int_{\frac{1}{2}}^1 \frac{x^{-\frac{2}{m}}((1-x)+(1-\cos\eta))}{(1-x)^2+2x(1-\cos\eta)}dxd\eta\right) \\
&= \frac{2r}{m}(A_{21}+A_{22})
\end{aligned}
$$

where we used $\frac{2\pi}{m}$-periodicity of the integrand to get the third inequality, the change of variables, $\eta \mapsto \frac{1}{m}\eta$ to get the first equality, and the evenness of the integrand in η to get the second equality. Note that the denominator of the integrand A_{21} is bounded from below by a strictly positive number, therefore

$$
A_{21} \lesssim \int_0^{\pi}\int_0^{\frac{1}{2}} x^{-\frac{2}{m}}dxd\eta \lesssim \frac{m}{m-2} \lesssim 1,
$$

for $m \geq 3$. For A_{22}, we use that $(1-\cos\eta) \sim \eta^2$ for $\eta \in (0,\pi)$ and the change of variables, $x \mapsto 1-x$, to obtain

$$
\begin{aligned}
A_{22} \lesssim \int_0^{\pi}\int_0^{\frac{1}{2}} \frac{x+\eta^2}{x^2+\eta^2}dxd\eta &= \int_0^{\pi}\int_0^{\frac{1}{2}} 1_{\{x<\eta\}}\frac{x+\eta^2}{x^2+\eta^2}dxd\eta + \int_0^{\pi}\int_0^{\frac{1}{2}} 1_{\{x\geq\eta\}}\frac{x+\eta^2}{x^2+\eta^2}dxd\eta \\
&\leq \int_0^{\pi}\int_0^{\eta}\frac{\eta+\eta^2}{\eta^2}dxd\eta + \int_0^{\frac{1}{2}}\int_0^{x}\frac{x+x^2}{x^2}d\eta dx \\
&\lesssim 1.
\end{aligned}
$$

This proves $|A_2| \lesssim \frac{r}{m}$. As mentioned, the same argument applies to A_1, A_3 and A_4 to prove (Equation 6.2.7). This completes the proof.

$\square$

From (Equation 6.2.2) and (Equation 6.2.3) in the above lemma, it is clear that $\partial_r \Psi(r_{min}, \theta) = r_{min}\left(\frac{1}{2} - \Omega - \frac{\partial_r \varphi_m(r_{min},\theta)}{r_{min}}\right)$ and $\left|\frac{\partial_r \varphi_m(r_{min},\theta)}{r_{min}}\right| \sim \frac{1}{m}$. Thus one can expect that if $\frac{1}{2} - \Omega$ is sufficiently large compared to $\frac{1}{m}$, then the level set ∂D cannot be too far from the a circle. We give a detailed proof for this in the following lemma.

Lemma 6.2.2. *Assume that (D, Ω) is a solution to* (Equation 1.2.4)*. Then there exist constants $c_1, c_2 > 0$ and $m_1 \geq 3$ such that if D is m-fold symmetric for some $m \geq m_1$ and $\lambda = \frac{1}{2} - \Omega > \frac{c_1}{m}$, then D is star-shaped and $|r_{max} - r_{min}| < \frac{c_2}{m}$. Hence there exist $u \in C^1(\mathbb{T})$ such that*

$$\partial D = \{(1 + u(\theta))(\cos\theta, \sin\theta) : \theta \in \mathbb{T}\} \quad \textit{and} \quad \|u\|_{L^\infty(\mathbb{T})} < \frac{c_2}{m}.$$

Proof. Thanks to (Equation 6.2.3) in Lemma 6.2.1, we can find a constant $C > 0$ (which we can also assume to be larger then 1) such that

$$|\nabla_x \varphi_m(r, \theta)| < C\frac{r}{m}, \tag{6.2.8}$$

where ∇_x denotes the gradient in Cartesian coordinates, that is, $\nabla_x := \partial_r + \frac{1}{r}\partial_\theta$. We will first prove the bound for $r_{max} - r_{min}$ and show star-shapeness of ∂D afterwards. Let

$$c_1 := \max\left\{6C, \sqrt{48\pi C}\right\} + 1, \quad c_2 := \frac{c_1}{4} \quad \text{and} \quad m_1 := \max\left\{\frac{3C}{2}, \frac{c_1}{4}, 3\right\} + 1. \tag{6.2.9}$$

We will show that if $\lambda > \frac{c_1}{m}$ and $m \geq m_1$, then $r_{max} - r_{min} < \frac{c_2}{m}$.

Let $q(r) := \frac{|D \cap B_r|}{2\pi r^2} - \Omega$. Since $\frac{1}{r^2} > \frac{1}{r_{min}^2} - \frac{2}{r_{min}^3}(r - r_{min})$ for $r > r_{min}$, and $|D \cap B_r|$ is increasing in r, we have that

$$q(r) > \frac{1}{2}\lambda, \text{ for } r \in \left(r_{min}, r_{min}\left(1+\frac{1}{4}\lambda\right)\right),$$

which implies that

$$\partial_r\left(\varphi^r(r) - \frac{\Omega}{2}r^2\right) = rq(r) > \frac{r_{min}}{2}\lambda, \text{ for } r \in \left(r_{min}, r_{min}\left(1+\frac{1}{4}\lambda\right)\right), \tag{6.2.10}$$

where the equality follows from (Equation 6.2.2) in Lemma 6.2.1. Let $\varepsilon := \frac{r_{min}c_1}{4m}$. By the assumption $\lambda > \frac{c_1}{m}$, we have

$$\varepsilon < \frac{r_{min}}{4}\lambda. \tag{6.2.11}$$

We choose $x_1, x_2 \in \mathbb{R}^2$ such that for some $\theta_1, \theta_2 \in \mathbb{T}$,

$$x_1 = r_{min}(\cos(\theta_1), \sin(\theta_1)), \quad x_2 = (r_{min}+\varepsilon)(\cos(\theta_2), \sin(\theta_2)) \quad \text{and} \quad |\theta_1 - \theta_2| \leq \frac{2\pi}{m}.$$

We claim that

$$\Psi(x_2) - \Psi(x_1) > 0. \tag{6.2.12}$$

Let us assume that the claim is true for a moment. Then from m-fold symmetry of D and the fact that ∂D is a level set of Ψ, it follows that $r_{max} \leq r_{min} + \varepsilon$. Thus it follows from (Equation 6.2.9), (Equation 6.2.11) and $r_{min} < 1$ that

$$r_{max} - r_{min} \leq \varepsilon = \frac{c_2 r_{min}}{m} < \frac{c_2}{m}. \tag{6.2.13}$$

Furthermore, for all $x \in \partial D$, it follows from (Equation 6.2.8) and (Equation 6.2.10) that $\partial_r \Psi(x) > \frac{r_{min}}{2}\lambda - \frac{Cr_{max}}{m}$. Hence

$$\partial_r \Psi(x) > \frac{r_{min}c_1}{2m} - \frac{Cr_{max}}{m} \geq \frac{c_1}{2m}(r_{max} - \frac{c_2}{m}) - \frac{Cr_{max}}{m} = \left(\frac{c_1 r_{max}}{4m} - \frac{Cr_{max}}{m}\right) + \frac{c_1}{2m}\left(\frac{r_{max}}{2} - \frac{c_2}{m}\right) > 0,$$

where the first inequality follows from $\lambda > \frac{c_1}{m}$, the second inequality follows from (Equation 6.2.13) and the last inequality follows from (Equation 6.2.9) and $r_{max} \geq 1$, which say $\frac{c_1}{4} > C$ and $\frac{r_{max}}{2} > \frac{1}{2} > \frac{c_2}{m}$. Therefore the implicit function theorem yields that there exists $u \in C^1(\mathbb{T})$ such that $\partial D = \{(1 + u(\theta))(\cos\theta, \sin\theta) : \theta \in \mathbb{T}\}$. This proved star-shapeness of D and the desired L^∞-norm bound for u.

Now it suffices to prove (Equation 6.2.12). We compute

$$\Psi(x_2) - \Psi(x_1) = \underbrace{\left(\varphi^r(|x_2|) - \frac{\Omega}{2}|x_2|^2\right) - \left(\varphi^r(|x_1|) - \frac{\Omega}{2}|x_1|^2\right)}_{=:L_1} + \underbrace{\varphi_m(x_2) - \varphi_m(x_1)}_{L_2}.$$

Thanks to (Equation 6.2.10), we have

$$L_1 > \frac{r_{min}}{2}\lambda\left(|x_2| - |x_1|\right) = \frac{r_{min}\lambda\varepsilon}{2}. \tag{6.2.14}$$

To estimate L_2, let us pick $x_1' = r_{min}(\cos(\theta_2), \sin(\theta_2))$. Then it follows from (Equation 6.2.8) that

$$\begin{aligned} L_2 &= (\varphi_m(x_2) - \varphi_m(x_1')) + (\varphi_m(x_1') - \varphi_m(x_1)) \\ &> -C\frac{|x_2|}{m}(|x_2| - |x_1|) - C\frac{r_{min}^2}{m}\frac{2\pi}{m} \\ &= -\frac{Cr_{min}\varepsilon}{m} - \frac{C\varepsilon^2}{m} - \frac{2\pi C r_{min}^2}{m^2}. \end{aligned} \tag{6.2.15}$$

Hence it follows from (Equation 6.2.14) and (Equation 6.2.15) that (we split $\frac{r_{min}\lambda\epsilon}{2}$ into three pieces evenly)

$$\Psi(x_2) - \Psi(x_1) > \left(\frac{r_{min}\lambda\varepsilon}{6} - \frac{Cr_{min}\varepsilon}{m}\right) + \left(\frac{r_{min}\lambda\varepsilon}{6} - \frac{C\varepsilon^2}{m}\right) + \left(\frac{r_{min}\lambda\varepsilon}{6} - \frac{2\pi C r_{min}^2}{m^2}\right) =: L_3 + L_4 + L_5. \tag{6.2.16}$$

From $\lambda > \frac{c_1}{m}$ and (Equation 6.2.9), which says $c_1 \geq 6C$, we have $L_3 = \frac{r_{min}\varepsilon}{6}\left(\lambda - \frac{6C}{m}\right) \geq 0$. For L_4, it follows from that

$$L_4 = \varepsilon\left(\frac{r_{min}\lambda}{6} - \frac{C\varepsilon}{m}\right) > \varepsilon\left(\frac{r_{min}\lambda}{6} - \frac{Cr_{min}\lambda}{4m}\right) = \varepsilon\frac{r_{min}\lambda}{6}\left(1 - \frac{3C}{2m}\right) > 0,$$

where the first inequality follows from (Equation 6.2.11) and the last inequality follows from (Equation 6.2.9), which says $m \geq m_1 > \frac{3C}{2}$. Finally,

$$L_5 = \frac{r_{min}^2 c_1\lambda}{24m} - \frac{2\pi C r_{min}^2}{m^2} = \frac{r_{min}^2}{24m}\left(c_1\lambda - \frac{48\pi C}{m}\right) > \frac{r_{min}^2}{24m}\left(\frac{c_1^2}{m} - \frac{48\pi C}{m}\right) > 0,$$

where the first equality follows from the definition of ε, the first inequality follows from $\lambda > \frac{c_1}{m}$ and the last inequality follows from (Equation 6.2.9), which says $c_1 \geq \sqrt{48\pi C}$. Therefore it follows from (Equation 6.2.16) that

$$\Psi(x_2) - \Psi(x_1) > 0, \tag{6.2.17}$$

which finishes the proof. $\square$

Now we are ready to prove Theorem H.

Proof of Theorem H: Let c_1, c_2 and m_1 be constants in Lemma 6.2.2 and δ be as in Proposition 6.1.8. Lastly, let c_3 be the constant in (Equation 6.1.36). Now we set

$$c := \max\left\{c_1, \frac{c_3}{2}\left(c_2\delta + \frac{9\pi^2\delta}{c_2}\right)\right\} \quad \text{and} \quad m_0 := \max\{2c_2, m_1\} + 1. \tag{6.2.18}$$

We will prove that if (D, Ω) is a solution to (Equation 1.2.4) such that D is m-fold symmetric for $m \geq m_0$ and simply-connected, then

$$\lambda := \frac{1}{2} - \Omega \leq \frac{c}{m}. \tag{6.2.19}$$

Towards a contradiction, let us suppose that there exists (D, Ω) such that

$$\lambda > \frac{c}{m}. \tag{6.2.20}$$

It is clear that (Equation 6.2.18) implies $\lambda > \frac{c_1}{m}$ and $m \geq m_1$. Thus Lemma 6.2.2 implies that there exists $u \in C^1(\mathbb{T})$ such that

$$\partial D = \{(1 + u(\theta))(\cos\theta, \sin\theta) : \theta \in \mathbb{T}\} \quad \text{and} \quad \|u\|_{L^\infty(\mathbb{T})} < \frac{c_2}{m}. \tag{6.2.21}$$

Since $m \geq m_0 > 2c_2$, which follows from (Equation 6.2.18), we have that $\|u\|_{L^\infty(\mathbb{T})} < \frac{1}{2}$.

To derive a contradiction, we will use the identity (Equation 6.1.2). To estimate the right-hand side of it, we apply Proposition 6.1.8 with $a := \frac{2c_2}{m} \in \left(2\|u\|_{L^\infty(\mathbb{T})}, 1\right)$ and obtain

$$\begin{aligned}\int_D |x - 2\nabla\left(1_D * \mathcal{N}\right)|^2 dx &\leq \frac{2c_2\delta}{m}\int_{\mathbb{T}} |u|^2 d\theta + \frac{\delta m}{2c_2}\int_{\mathbb{T}} f(\theta)^2 d\theta \\ &\leq \frac{2c_2\delta}{m}\int_{\mathbb{T}} |u|^2 d\theta + \frac{\pi\delta m}{c_2}\|f\|^2_{L^\infty(\mathbb{T})},\end{aligned} \tag{6.2.22}$$

where $f(\theta) = \int_0^\theta u(s)^2 + 2u(s)ds$. Using (Equation 6.1.32) and $\frac{2\pi}{m}$-periodicity of u, it is clear that

f is also $\frac{2\pi}{m}$-periodic. Furthermore, for $\theta \in (0, \frac{2\pi}{m})$, we have that (recall that $\|u\|_{L^\infty(\mathbb{T})} < \frac{1}{2}$),

$$|f(\theta)| = \left| \int_0^\theta u(s)^2 + 2u(s)ds \right| \leq \int_0^\theta 3|u(s)|ds \leq 3\sqrt{\int_0^\theta |u(s)|^2 ds}\sqrt{\theta} < 3\sqrt{\int_0^{\frac{2\pi}{m}} |u(s)|^2 ds}\sqrt{\frac{2\pi}{m}}$$
$$\leq 3\frac{\sqrt{2\pi}}{m}\sqrt{\int_{\mathbb{T}} |u(s)|^2 ds}.$$

Thus, (Equation 6.2.22) yields that

$$\int_D |x - 2\nabla (1_D * \mathcal{N})|^2 dx \leq \frac{2c_2\delta}{m}\int_{\mathbb{T}} |u|^2 d\theta + \frac{18\pi^2\delta}{c_2 m}\int_{\mathbb{T}} |u|^2 d\theta = \frac{1}{m}\left(2c_2\delta + \frac{18\pi^2\delta}{c_2}\right)\int_{\mathbb{T}} |u|^2 d\theta. \tag{6.2.23}$$

For the left-hand side of (Equation 6.1.2), we use (Equation 6.1.36) to obtain

$$\frac{2}{c_3}\int_{\mathbb{T}} |u|^2 d\theta \leq \int_D |x|^2 dx - \frac{|D|^2}{2\pi}. \tag{6.2.24}$$

Hence it follows from (Equation 6.2.23), (Equation 6.2.24) and (Equation 6.1.2) that

$$\frac{2\lambda}{c_3}\int_{\mathbb{T}} |u|^2 d\theta \leq \lambda\left(\int_D |x|^2 dx - \frac{|D|^2}{2\pi}\right) = \frac{1}{2}\int_D |x - 2\nabla (1_D * \mathcal{N})|^2 dx \leq \frac{1}{m}\left(c_2\delta + \frac{9\pi^2\delta}{c_2}\right)\int_{\mathbb{T}} |u|^2 d\theta.$$

Therefore we have

$$\lambda \leq \frac{c_3}{2}\left(c_2\delta + \frac{9\pi^2\delta}{c_2}\right)\frac{1}{m} \leq \frac{c}{m},$$

where the last inequality follows from our choice for c in (Equation 6.2.18). This contradicts our assumption (Equation 6.2.20), thus completes the proof. $\square$

By simple maximum principle type argument, Theorem H gives a upper bound for r_{max}.

Corollary 6.2.3. *There exist constants $c > 0$ and $m_1 \geq 3$ such that if (D, Ω) is a solution to*

(Equation 1.2.4) *that is simply-connected, m-fold symmetric for some $m \geq m_1$ and $|D| = \pi$, then* $r_{max} - 1 \leq \frac{c}{m}$.

Proof. Thanks to Theorem H, we can pick a constants C_1 and m_0 such that if $m \geq m_0$, then $\lambda < \frac{C_1}{m}$. Moreover, it follows from Lemma 6.2.1 that there exists $C_2 > 0$ such that $|\nabla \varphi_m(r, \theta)| \leq \frac{C_2 r}{m}$. Now, let us choose

$$m_1 := \max\left\{m_0, 2\left(C_1 + C_2\right)\right\} + 1.$$

Since $\Delta\Psi = 2\lambda > 0$ in D, the maximum principle for subharmonic functions implies that $\partial_r \Psi(r_{max}, 0) > 0$. Thus it follows from Lemma 6.2.1 that

$$\begin{aligned}
0 &< \partial_r \varphi^r(r_{max}) + C_2 \frac{r_{max}}{m} - \Omega r_{max} \\
&= \frac{|D \cap B_{r_{max}}|}{2\pi r_{max}} + C_2 \frac{r_{max}}{m} - \Omega r_{max} \\
&= \frac{1}{2r_{max}} + \left(\frac{C_2}{m} - \frac{1}{2} + \lambda\right) r_{max} \\
&\leq \frac{1}{2r_{max}} + \left(\frac{C_1 + C_2}{m} - \frac{1}{2}\right) r_{max},
\end{aligned}$$

where we used $|D \cap B_{r_{max}}| = |D| = \pi$ to get the second equality and the last inequality follows from $\lambda \leq \frac{C_1}{m}$. Since $\frac{C_1 + C_2}{m} < \frac{1}{2}$, we obtain,

$$r_{max} - 1 \leq \sqrt{\frac{\frac{1}{2}}{\frac{1}{2} - \frac{C_1 + C_2}{m}}} - 1 \lesssim \frac{1}{m}.$$

□

6.2.2 Patches along bifurcation curves

This subsection is devoted to the proof of Theorem I. Since we are interested in a curve $\mathscr{C}_m$ that satisfies (A1)-(A4), we will make the following assumptions for the patches throughout this subsection.

(a) D is star-shaped, that is $\partial D = \{(1+u(\theta))(\cos\theta, \sin\theta) : \theta \in \mathbb{T}\}$ for some $u \in C^2(\mathbb{T})$.

(b) u is even and $\frac{2\pi}{m}$-periodic for some $m \geq 3$, that is, $u(-\theta) = u(\theta)$ and $u(\theta + \frac{2\pi}{m}) = u(\theta)$.

(c) $\partial_\theta u(\theta) < 0$ for all $\theta \in (0, \frac{\pi}{m})$.

For such a patch, we denote $r_{min} := \min_{x\in\partial D} |x| = u(\frac{\pi}{m})$ and $r_{max} := \max_{x\in\partial D} |x| = u(0)$. Furthermore, we denote $\eta := (1+u)^{-1} : (r_{min}, r_{max}) \mapsto (0, \frac{\pi}{m})$. By the symmetry, we only need to focus on the fundamental sector $S := \left\{(r,\theta) : r \geq 0,\ \theta \in (0, \frac{\pi}{m})\right\}$. See Figure Figure 6.2 for an illustration of these definitions.

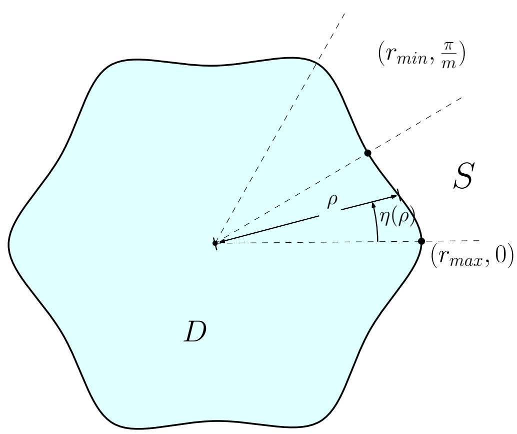


Figure 6.2: Illustration of the definitions of r_{min}, r_{max}, $\eta(\rho)$ and S on a 6-fold vortex patch

Note that we will establish several lemmas with assuming $|D| = |B| = \pi$. Certainly this is not satisfied by the solutions on the curve $\mathscr{C}_m$ but we will resolve this issue in the proof of the theorem

.

Our proof for Theorem I relies on Theorem H. Roughly speaking, we will show that if $\|u\|_{L^\infty(\mathbb{T})}$ is large compared to $\frac{1}{m}$, then λ $(= \frac{1}{2} - \Omega)$ must be large enough to contradict Theorem H. However, the main difficulty comes from the fact that lower bounds for λ that we can derive from the identities (Equation 1.2.8) and (Equation 1.2.9) are not comparable with $\|u\|_{L^\infty}$ (Lemma 6.2.4).

Thus, the scenario that we want to rule out is that for large m, ∂D is so spiky that $\int_{\mathbb{T}} |u|^2 d\theta$ is small while $\|u\|_{L^\infty}$ is relatively large.

Since $r_{\max}$ can be estimated as in Corollary 6.2.3, we will mainly focus on estimating r_{min}. Using the identity (Equation 6.1.1), we derive a lower bound for λ in the next lemma.

Lemma 6.2.4. *If* (D, Ω) *is a solution to* (Equation 1.2.4) *with* $|D| = \pi$ *then*

$$\lambda \gtrsim \frac{\int_{\mathbb{T}} |u|^2 d\theta}{r_{max}\|u\|_{L^\infty(\mathbb{T})}}.$$

Proof. we use (Equation 6.1.1) in Lemma 6.1.1 to obtain

$$\lambda \left(\int_D |x|^2 - 2p(x)dx \right) = \left(\int_D \frac{|x|^2}{2} dx - \frac{|D|^2}{4\pi} \right).$$

For a moment, let us assume that

$$\int_D |x|^2 - 2p(x)dx \lesssim r_{max}\|u\|_{L^\infty(\mathbb{T})}. \tag{6.2.25}$$

Then it follows from (Equation 6.1.36) that

$$\lambda = \frac{\int_D \frac{|x|^2}{2} dx - \frac{|D|^2}{4\pi}}{\int_D |x|^2 - 2p(x)dx} \gtrsim \frac{\int_{\mathbb{T}} |u|^2 d\theta}{r_{max}\|u\|_{L^\infty}(\mathbb{T})},$$

which implies the desired result.

Now let us prove (Equation 6.2.25). Note that $p(x) \geq \frac{r_{min}^2 - |x|^2}{2}$ in $B_{r_{min}}$. Indeed, $p - \frac{r_{min}^2 - |x|^2}{2}$ is harmonic in $B_{r_{min}}$ and non-negative on $\partial B_{r_{min}}$ since p is non-negative in D. Therefore the inequality follows from the maximum principle. From this, we obtain

$$\int_D 2p(x)dx \geq \int_{B_{r_{min}}} r_{min}^2 - |x|^2 dx = \frac{|B_{r_{min}}|^2}{2\pi}.$$

Since $\int_D |x|^2 dx < \int_{B_{r_{max}}} |x|^2 dx = \frac{|B_{r_{max}}|^2}{2\pi}$, it follows that

$$\int_D |x|^2 - 2p(x)dx = \frac{1}{2\pi}\left(|B_{r_{max}}|^2 - |B_{r_{min}}|^2\right) \lesssim r_{max}\|u\|_{L^\infty(\mathbb{T})},$$

which proves (Equation 6.2.25). □

Thanks to Lemma 6.2.4, we only need to rule out the case where $\int_{\mathbb{T}} |u|^2 d\theta$ is too small, compared to $\|u\|_{L^\infty}$. To this end, we will pick r_1, and r_2 so that $r_{min} < r_1 < r_2 < 1$ and find a lower bound for $\frac{\pi}{m} - \eta(r_2)$ by showing that $|u'(\theta)|$ is bounded from above for $1 + u(\theta) \in (r_{min}, r_1)$. Since the relative stream function Ψ is constant on ∂D, we have $\frac{d}{d\theta}\left(\Psi((1+u(\theta)), \theta)\right) = 0$. Therefore (Equation 6.2.1) yields that

$$u'(\theta) = -\frac{\partial_\theta \Psi(r,\theta)}{\partial_r \Psi(r,\theta)} = -\frac{\partial_\theta \varphi_m(r,\theta)}{(\partial_r \varphi^r(r) - \Omega r) + \partial_r \varphi_m(r,\theta)}, \quad \text{where } r = 1 + u(\theta). \tag{6.2.26}$$

In the next two lemmas, we will estimate the denominator and numerator in (Equation 6.2.26) but the proofs will be postponed to the end of this subsection.

Lemma 6.2.5. *Let (D, Ω) be a solution to* (Equation 1.2.4) *that satisfies the assumptions (a)-(c) for some $m \geq 3$ and $|D| = \pi$. Let r_1, $r_2 > 0$ be such that $r_{min} < r_1 < r_2 < 1$ and let $\delta := \frac{\pi}{m} - \eta(r_2)$. Then there exist constants $c, C > 0$ such that if $\|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$, it holds that*

$$\partial_r \varphi^r(r) - \Omega r \geq m\delta\left(c\frac{(1-r_2)^2}{\|u\|_{L^\infty(\mathbb{T})}} - C\left(r_1 - r_{min}\right)\right),$$

for all $r \in (r_{min}, r_1)$.

Lemma 6.2.6. *Let D be a patch that satisfies the assumptions (a)-(c) for some $m \geq 3$. Let us pick*

r_1, $r_2 > 0$ so that $r_{min} < r_1 < r_2 < 1$ and $r_1^2 \leq r_{min} r_2$. If $\delta := \frac{\pi}{m} - \eta(r_2) < \frac{\pi}{4m}$, then it holds that

$$\partial_r \varphi_m(r, \eta(r)) \geq -\frac{cr}{1 - \left(\frac{r_1}{r_2}\right)^m} \delta \tag{6.2.27}$$

$$\partial_\theta \varphi_m(r, \eta(r)) \leq \frac{cr^2}{1 - \left(\frac{r_1}{r_2}\right)^m} \delta. \tag{6.2.28}$$

for all $r \in (r_{min}, r_1)$, where c is a universal constant that does not depend on any variables..

Note that the linear dependence on δ in (Equation 6.2.27) and (Equation 6.2.28) is crucial in the proof of the next lemma, since this allows us to bound u' independently of δ when we plug the above bounds into (Equation 6.2.26).

Now we can rule out the scenario that ∂D is too spiky inwards.

Lemma 6.2.7. *There exist $c, C > 0$ and $m_1 \geq 3$ such that if (D, Ω) is a solution to* (Equation 1.2.4) *that satisfies the assumptions (a)-(c) for some $m \geq m_1$, $|D| = \pi$ and $\|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$, then*

$$1 - r_{min} \leq \frac{c}{m} \quad \textit{or} \quad \int_{\mathbb{T}} |u|^2 d\theta \geq C\left(1 - r_{min}\right)^2.$$

Proof. Thanks to Lemma 6.2.5 and 6.2.6, we can choose c_1 and $c_2 > 0$ such that if $r_{min} < r_1 < r_2 < 1$ and $r_1^2 \leq r_{min} r_2$, then

$$\partial_r \varphi_r(r) - \Omega r \geq m\delta \left(c_1 \frac{(1 - r_2)^2}{\|u\|_{L^\infty(\mathbb{T})}} - c_2 \left(r_1 - r_{min}\right) \right), \tag{6.2.29a}$$

$$\partial_r \varphi_m(r, \eta(r)) \geq -\frac{c_2}{1 - \left(\frac{r_1}{r_2}\right)^m} \delta, \tag{6.2.29b}$$

$$\partial_\theta \varphi_m(r, \eta(r)) \leq \frac{c_2}{1 - \left(\frac{r_1}{r_2}\right)^m} \delta, \tag{6.2.29c}$$

for $r \in (r_{min}, r_1)$. We will pick

$$c := \max\left\{12, \frac{24c_2}{c_1}\right\}, \quad C := \frac{\pi}{18} \quad \text{and} \quad m_1 := 2c \tag{6.2.30}$$

Now let us assume that D is a solution to (Equation 1.2.4) that satisfies the assumptions (a)-(c) for some $m \geq m_1$, $|D| = \pi$ and $1 - r_{min} = \|u\|_{L^\infty(\mathbb{T})} < \frac{1}{2}$. If $1 - r_{min} \leq \frac{c}{m}$, then there is nothing to prove, thus let us assume that

$$1 - r_{min} > \frac{c}{m}. \tag{6.2.31}$$

Let $\tilde{c} := m(1 - r_{min})$ so that $1 - r_{min} = \frac{\tilde{c}}{m}$. From (Equation 6.2.31), we have

$$\tilde{c} > c. \tag{6.2.32}$$

We choose

$$r_1 := 1 - \frac{\tilde{c} - 2}{m} \quad \text{and} \quad r_2 := 1 - \frac{\tilde{c}}{3m}. \tag{6.2.33}$$

And we consider two cases: $\frac{\pi}{m} - \eta(r_2) \geq \frac{\pi}{4m}$ and $\frac{\pi}{m} - \eta(r_2) < \frac{\pi}{4m}$.

Case1. Let us assume that $\frac{\pi}{m} - \eta(r_2) \geq \frac{\pi}{4m}$.

Since $\frac{\pi}{m} - \eta(r_2) \geq \frac{\pi}{4m}$, it follows from the monotonicity of u that $u(\theta) < r_2 - 1 = -\frac{\tilde{c}}{3m}$ for $\theta \in \left(\frac{3\pi}{4m}, \frac{\pi}{m}\right)$. Using m-fold symmetry of u, we obtain

$$\int_{\mathbb{T}} |u|^2 d\theta \geq 2m \int_0^{\frac{\pi}{m}} |u|^2 d\theta > 2m \int_{\frac{3\pi}{4m}}^{\frac{\pi}{m}} |u|^2 d\theta \geq \frac{\pi \tilde{c}^2}{18m^2} = C(1 - r_{min})^2, \tag{6.2.34}$$

where the last equality follows from (Equation 6.2.30), which says $C = \frac{\pi}{18}$.

Case2. Now we assume $\frac{\pi}{m} - \eta(r_2) < \frac{\pi}{4m}$.

We first check whether r_1 and r_2 in (Equation 6.2.33) satisfy the hypotheses in Lemma 6.2.6.

Clearly $r_{min} < r_1 < r_2 < 1$, since $\tilde{c} > c \geq 4$, which follows from (Equation 6.2.30) and (Equation 6.2.32). To show $r_1^2 \leq r_{min} r_2$, we compute

$$\begin{aligned} r_1^2 - r_{min} r_2 &= \left(1 - \frac{\tilde{c} - 2}{m}\right)^2 - \left(1 - \frac{\tilde{c}}{m}\right)\left(1 - \frac{\tilde{c}}{3m}\right) \\ &= -\frac{2\tilde{c}}{3m} + \frac{4}{m} + \frac{2\tilde{c}^2 - 12\tilde{c} + 12}{3m^2} \\ &\leq -\frac{\tilde{c}}{3m} + \frac{4}{m} + \frac{-12\tilde{c} + 12}{3m^2} \\ &\leq 0, \end{aligned}$$

where the first ineqiality follows from $\frac{\tilde{c}}{m} = 1 - r_{min} = \|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$, and the last inequality follows from $\tilde{c} > 12$.

Since ∂D is a level set of Ψ, $\Psi(1 + u(\theta), \theta) =$ does not depend on θ. Therefore,

$$-u'(\theta) = \frac{\partial_\theta \Psi(1 + u(\theta), \theta)}{\partial_r \Psi(1 + u(\theta), \theta)} = \frac{\partial_\theta \varphi_m(1 + u(\theta), \theta)}{\partial_r \varphi^r(1 + u(\theta)) - \Omega r + \partial_r \varphi_m(1 + u(\theta), \theta)}.$$

Hence, it follows from (Equation 6.2.29) that

$$-u'(\theta) \leq \frac{\frac{1}{1 - \left(\frac{r_1}{r_2}\right)^m}}{\frac{c_1}{c_2} m \frac{(1 - r_2)^2}{\|u\|_{L^\infty(\mathbb{T})}} - m\left(r_1 - r_{min}\right) - \frac{1}{1 - \left(\frac{r_1}{r_2}\right)^m}} \quad \text{for} \quad (1 + u(\theta)) < r_1. \tag{6.2.35}$$

Now we assume for a moment that

$$\frac{\frac{c_1}{c_2} m (1 - r_2)^2}{3\|u\|_{L^\infty}} \geq \max\left\{ m(r_1 - r_{min}), \frac{1}{1 - \left(\frac{r_1}{r_2}\right)^m} \right\}. \tag{6.2.36}$$

Then (Equation 6.2.35) yields that

$$0 \leq -u'(\theta) \leq 1 \quad \text{for} \quad (1 + u(\theta)) < r_1,$$

which implies that

$$\frac{2}{m} = r_1 - r_{min} = \int_{\eta(r_1)}^{\frac{\pi}{m}} -u'(\theta)d\theta \le \frac{\pi}{m} - \eta(r_1). \tag{6.2.37}$$

Furthermore, monotonicity of u implies that $|u(\theta)| > |r_1 - 1| > |r_2 - 1|$ for $\theta \in \left(\eta(r_1), \frac{\pi}{m}\right)$. Therefore we obtain that

$$\int_{\mathbb{T}} |u|^2 d\theta > 2m \int_{\eta(r_1)}^{\frac{\pi}{m}} |u|^2 d\theta \ge 2m \left(\frac{\pi}{m} - \eta(r_1)\right)(1 - r_2)^2 \ge \frac{4\tilde{c}^2}{9m^2} = \frac{4}{9}(1 - r_{min})^2 \ge C(1 - r_{min})^2, \tag{6.2.38}$$

where the third inequality follows from (Equation 6.2.33) and (Equation 6.2.37) and the last inequality follows from $C = \frac{\pi}{18} < \frac{4}{9}$. Thus the desired result follows from (Equation 6.2.34) and (Equation 6.2.38).

To complete the proof, we need to show (Equation 6.2.36). It follows from (Equation 6.2.33) that

$$\frac{\frac{c_1}{c_2} m (1 - r_2)^2}{3\|u\|_{L^\infty(\mathbb{T})}} = \frac{\frac{c_1}{c_2} m (1 - r_2)^2}{3(1 - r_{min})} = \frac{c_1 \tilde{c}}{12 c_2} \ge 2, \tag{6.2.39}$$

where the last inequality follows from (Equation 6.2.30) and (Equation 6.2.32) which imply that $\tilde{c} \ge \frac{24c_2}{c_1}$. We also have

$$m(r_1 - r_{min}) = 2, \tag{6.2.40}$$

which follows from (Equation 6.2.33). To estimate $\frac{1}{1 - \left(\frac{r_1}{r_2}\right)^m}$, let us use an elementary inequality that for any $0 < b < a < m$, it holds that

$$\left(\frac{1 - \frac{a}{m}}{1 - \frac{b}{m}}\right)^m \le e^{-(a-b)}. \tag{6.2.41}$$

Indeed, by taking logarithm in the left-hand side, we can compute,

$$\log\left(\left(\frac{1-\frac{a}{m}}{1-\frac{b}{m}}\right)^m\right) = m\log\left(1-\frac{a-b}{m-b}\right) \leq m\log\left(1-\frac{a-b}{m}\right) \leq -(a-b),$$

where the last inequality follows from $\log(1-x) < -x$ for all $x > 0$. This proves (Equation 6.2.41). Then we use (Equation 6.2.33) and obtain

$$\frac{1}{1-\left(\frac{r_1}{r_2}\right)^m} = \frac{1}{1-\left(\frac{1-\frac{\tilde{c}-2}{m}}{1-\frac{\tilde{c}}{3m}}\right)^m} \leq \frac{1}{1-e^{-(\frac{2\tilde{c}}{3}-2)}} \leq 2, \tag{6.2.42}$$

where the last inequality follows from (Equation 6.2.30) and (Equation 6.2.32), which imply $\tilde{c} \geq 12$.

Thus (Equation 6.2.39), (Equation 6.2.40) and (Equation 6.2.42) yield

$$\frac{\frac{c_1}{c_2} m\left(1-r_2\right)^2}{3\|u\|_{L^\infty(\mathbb{T})}} \geq 2 \geq \max\left\{m\left(r_1-r_{min}\right), \frac{1}{1-\left(\frac{r_1}{r_2}\right)^m}\right\}. \tag{6.2.43}$$

This proves (Equation 6.2.36) and finishes the proof. □

Now we can estimate $\|u\|_{L^\infty(\mathbb{T})}$ whose corresponding patch has area π, that is, $|D| = \pi$.

Proposition 6.2.8. *There exist constants $c > 0$ and $m_0 \geq 3$ such that if (D, Ω) is a solution to* (Equation 1.2.4) *that satisfies assumptions (a)-(c) for some $m \geq m_0$, $|D| = \pi$ and $\|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$, then*

$$\|u\|_{L^\infty(\mathbb{T})} \leq \frac{c}{m}.$$

Proof. In order to use the previous lemmas, let us fix some constants. We fix constants $c_i's$ and m_1 so that if (D, Ω) is a solution to (Equation 1.2.4) that satisfies the assumptions (a)-(c) for some $m \geq m_1$ and $|D| = \pi$, then

(B1) (From Theorem H) $\lambda \leq \frac{c_1}{m}$.

(B2) (From Corollary 6.2.3) $r_{max} - 1 \leq \frac{c_2}{m}$.

(B3) (From Lemma 6.2.4) $\lambda \geq c_3 \frac{\int_{\mathbb{T}} |u|^2 d\theta}{r_{max} \|u\|_{L^\infty(\mathbb{T})}}$.

(B4) (From Lemma 6.2.7) if $1 - r_{min} = \|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$ then

$$1 - r_{min} \leq \frac{c_4}{m} \quad \text{or} \quad \int_{\mathbb{T}} |u|^2 d\theta \geq c_5 (1 - r_{min})^2.$$

Let us set

$$c := \max \left\{ c_2, c_4, \frac{2c_1}{c_3 c_5} \right\} + 1, \quad \text{and} \quad m_0 := \max \{m_1, 2c\} + 1. \tag{6.2.44}$$

Then we will prove that if (D, Ω) is a solution to (Equation 1.2.4) and satisfies the assumptions (a)-(c) for some $m \geq m_0$, then

$$\|u\|_{L^\infty(\mathbb{T})} \leq \frac{c}{m}. \tag{6.2.45}$$

Let us assume for a contradiction that

$$\|u\|_{L^\infty(\mathbb{T})} > \frac{c}{m}. \tag{6.2.46}$$

Then we have that

$$\int_{\mathbb{T}} |u|^2 d\theta \geq c_5 (1 - r_{min})^2. \tag{6.2.47}$$

Indeed,

$$\|u\|_{L^\infty(\mathbb{T})} > \frac{c}{m} > \frac{c_2}{m} \geq r_{max} - 1,$$

where we used (Equation 6.2.44), (Equation 6.2.46) and (B2), therefore $\|u\|_{L^\infty(\mathbb{T})} = 1 - r_{min}$. Thus (B4) and (Equation 6.2.46) imply (Equation 6.2.47). Furthermore (B2) and (Equation 6.2.44) also imply that

$$r_{max} \leq 1 + \frac{c_2}{m} \leq 2.$$

Thus we use (B3) and (Equation 6.2.47) to obtain

$$\lambda \geq c_3 \frac{c_5(1-r_{min})^2}{2\|u\|_{L^\infty(\mathbb{T})}} \geq \frac{c_3 c_5 \|u\|_{L^\infty(\mathbb{T})}}{2} \geq \frac{c c_3 c_5}{2m} > \frac{c_1}{m},$$

where the third inequality follows from (Equation 6.2.46) and the last inequality follows from (Equation 6.2.44). However this contradicts (B1). $\square$

Now we are ready to prove the main theorem of this subsection

Proof of Theorem I: Thanks to Proposition 6.2.8, we can pick c_1 and m_1 so that $2c_1 < m_1$ and if (D, Ω) is a solution to (Equation 1.2.4), that satisfies the assumptions (a)-(c) for some $m \geq m_1$ and $\|u\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2}$, then

$$\|u\|_{L^\infty(\mathbb{T})} \leq \frac{c_1}{m}. \tag{6.2.48}$$

Now let us consider a curve $\mathscr{C}_m$, that satisfies the properties (A1)-(A4) for some $m \geq m_1$. We will show that

$$\sup_{s\in[0,\infty)} \|\tilde{u}_m(s)\|_{L^\infty(\mathbb{T})} \lesssim \frac{1}{m}. \tag{6.2.49}$$

To do so, let us define $\hat{u}_m(s)$ so that,

$$1+\hat{u}_m(s) = \frac{\sqrt{\pi}}{\sqrt{|D^{\tilde{u}_m(s)}|}}(1+\tilde{u}_m(s)), \tag{6.2.50}$$

where the definition of $D^{\tilde{u}_m(s)}$ is as in (A2). Clearly, $s \mapsto \hat{u}_m(s)$ is a continuous curve in $C^2(\mathbb{T})$ such that $\left|D^{\hat{u}_m(s)}\right| = \pi$. Since $\hat{u}_m(0) = 0$, it follows from the continuity of the curve and (Equation 6.2.48) that

$$\sup_{s\in[0,\infty)} \|\hat{u}_m(s)\|_{L^\infty(\mathbb{T})} \le \frac{c_1}{m}. \tag{6.2.51}$$

Now let us pick an arbitrary $s \in [0,\infty)$ and denote $\tilde{u} := \tilde{u}_m(s)$ and $\hat{u} := \hat{u}_m(s)$. Then it follows from (A1) and (Equation 6.2.50) that

$$\begin{aligned} 0 = \int_{\mathbb{T}} \tilde{u}(\theta)d\theta &= 2\pi\left(\frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}} - 1\right) + \frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}}\int_{\mathbb{T}} \hat{u}(\theta)d\theta \\ &= 2\pi\left(\frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}} - 1\right) - \frac{1}{2}\frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}}\int_{\mathbb{T}} \hat{u}(\theta)^2 d\theta, \end{aligned}$$

where the last equality follows from (Equation 6.1.32). Hence (Equation 6.2.51) implies that $\frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}} = 1 + O\left(\frac{1}{m^2}\right)$. Therefore (Equation 6.2.50) and (Equation 6.2.51) yield that

$$\|\tilde{u}\|_{L^\infty(\mathbb{T})} = \frac{\sqrt{|D^{\tilde{u}}|}}{\sqrt{\pi}}(1+\|\hat{u}\|_{L^\infty(\mathbb{T})}) - 1 = \|\hat{u}\|_{L^\infty(\mathbb{T})} + O\left(\frac{1}{m^2}\right) \lesssim \frac{1}{m},$$

where the last equality follows from (Equation 6.2.51). This proves (Equation 6.2.49) and the theorem. □

Proofs of Lemma 6.2.5 and 6.2.6

Proof of Lemma 6.2.5: From Lemma 6.2.1, it follows that

$$\partial_r \varphi^r(r) - \Omega r = r\left(\frac{1}{2} - \Omega\right) - \frac{|B_r \backslash D|}{2\pi r} = r\lambda - \frac{|B_r \backslash D|}{2\pi r} =: J_1 - J_2, \tag{6.2.52}$$

where we used $\lambda = \frac{1}{2} - \Omega$. Note that we have $\|u\|_{L^\infty(\mathbb{T})} \le \frac{1}{2}$, therefore $J_1 \sim \lambda$ and $J_2 \sim |B_r \backslash D|$. Let us estimate J_2 first. Since u is even and m-periodic, we have that for all $r \in (r_{min}, r_1)$,

$$|B_r \backslash D| = 2m \int_{r_{min}}^{r} \left(\frac{\pi}{m} - \eta(\rho)\right) \rho d\rho \lesssim mr\delta(r - r_{min}) \le mr\delta(r_1 - r_{min}),$$

where we used $\frac{\pi}{m} - \eta(\rho) < \delta$ for $\rho < r < r_2$ to get the first inequality. Hence we obtain

$$J_2 \lesssim m\delta\left(r_1 - r_{min}\right). \tag{6.2.53}$$

To estimate J_1, we use Lemma 6.2.4 and obtain

$$\lambda \gtrsim \frac{\int_{\mathbb{T}} |u|^2 d\theta}{r_{max}\|u\|_{L^\infty}} \gtrsim \frac{\int_{\mathbb{T}} |u|^2 d\theta}{\|u\|_{L^\infty}}, \tag{6.2.54}$$

where we we used $r_{max} \le 1 + \|u\|_{L^\infty(\mathbb{T})} \lesssim 1$ to get the last inequality. From periodicity of u, it follows that

$$\int_{\mathbb{T}} |u|^2 d\theta = 2m \int_0^{\frac{\pi}{m}} |u|^2 d\theta \ge 2m \int_{\eta(r_2)}^{\frac{\pi}{m}} |u|^2 d\theta \ge 2m(1 - r_2)^2 \left(\frac{\pi}{m} - \eta(r_2)\right) = 2m\delta(1 - r_2)^2, \tag{6.2.55}$$

where we used $1 + u(\theta) < r_2$ for $\theta \in (\eta(r_2), \frac{\pi}{m})$ by monotonicity of u to get the second inequality.

Hence it follows from (Equation 6.2.54) and (Equation 6.2.55) that

$$J_1 \gtrsim \lambda \gtrsim \frac{m\delta(1-r_2)^2}{\|u\|_{L^\infty(\mathbb{T})}}. \tag{6.2.56}$$

Thus the desired result follows from (Equation 6.2.52), (Equation 6.2.53) and (Equation 6.2.56). □

Now we prove Lemma 6.2.6. The proof is based on the formulae given in (B.1.3).

Proof of Lemma 6.2.6: Let us assume that $\delta < \frac{\pi}{4m}$. We will prove (Equation 6.2.27) first. By monotonicity of η (assumption (c)), it follows from Lemma B.1.3 that for all $r \in (r_{min}, r_1)$,

$$\begin{aligned}
\partial_r\varphi_m(r,\eta(r)) \geq \frac{1}{2\pi}\int_{r_{min}}^{r} \frac{\rho}{r}\arctan\left(\frac{(\frac{\rho}{r})^m \sin\left(m(\eta(r)+\eta(\rho))\right)}{1-\left(\frac{\rho}{r}\right)^m\cos\left(m(\eta(r)+\eta(\rho))\right)}\right)d\rho \\
+\frac{1}{2\pi}\int_{r}^{r_{max}}\frac{\rho}{r}\left(\arctan\left(\frac{(\frac{r}{\rho})^m\sin\left(m(\eta(r)-\eta(\rho))\right)}{1-\left(\frac{r}{\rho}\right)^m\cos\left(m(\eta(r)-\eta(\rho))\right)}\right)\right. \\
\left. -\arctan\left(\frac{(\frac{r}{\rho})^m\sin\left(m(\eta(r)+\eta(\rho))\right)}{1-\left(\frac{r}{\rho}\right)^m\cos\left(m(\eta(r)+\eta(\rho))\right)}\right)\right)d\rho,
\end{aligned}$$

where we used that $\sin(m(\eta(r)-\eta(\rho))) \leq 0$ for $\rho < r$ so we can drop one of the integrands for free. Note that the integrand in the second integral is positive for $\rho \in (r, r_2)$, since $\sin(m(\eta(r)-\eta(\rho))) > 0$ and $\sin(m(\eta(r)+\eta(\rho))) < 0$ for $r < \rho < r_2$, which follows from $\frac{\pi}{m} - \eta(r_2) < \frac{\pi}{4m}$. We will use the second integrand in the second integral to cancel the first integral, that is, we have that

$$\begin{aligned}
\partial_r\varphi_m(r,\eta(r)) \geq \frac{1}{2\pi}\left(\int_{r_{min}}^{r}\frac{\rho}{r}\arctan\left(\frac{(\frac{\rho}{r})^m\sin\left(m(\eta(r)+\eta(\rho))\right)}{1-\left(\frac{\rho}{r}\right)^m\cos\left(m(\eta(r)+\eta(\rho))\right)}\right)d\rho\right. \\
\left. -\int_{r}^{r_2}\frac{\rho}{r}\arctan\left(\frac{(\frac{r}{\rho})^m\sin\left(m(\eta(r)+\eta(\rho))\right)}{1-\left(\frac{r}{\rho}\right)^m\cos\left(m(\eta(r)+\eta(\rho))\right)}\right)d\rho\right) \\
+\frac{1}{2\pi}\int_{r_2}^{r_{max}}\frac{\rho}{r}\left(\arctan\left(\frac{(\frac{r}{\rho})^m\sin\left(m(\eta(r)-\eta(\rho))\right)}{1-\left(\frac{r}{\rho}\right)^m\cos\left(m(\eta(r)-\eta(\rho))\right)}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(m(\eta(r)+\eta(\rho))\right)}{1-\left(\frac{r}{\rho}\right)^m \cos\left(m(\eta(r)+\eta(\rho))\right)}\right)\Bigg) d\rho \\
&=: \frac{1}{2\pi}K_1 + \frac{1}{2\pi}K_2. \qquad (6.2.57)
\end{aligned}
$$

To estimate K_1, we use that (note that $\frac{\pi}{m} - \eta(r) < \frac{\pi}{4m}$ for $r \leq r_2$)

$$
\begin{cases}
\sin(m(\eta(r)+\eta(\rho))) \geq \sin(2m\eta(r)) \quad \text{and} \quad \cos(m(\eta(r)+\eta(\rho))) \leq 1 & \text{for } \rho \in (r_{min}, r) \\
\sin(m(\eta(r)+\eta(\rho))) \leq \sin(2m\eta(r)) \quad \text{and} \quad \cos(m(\eta(r)+\eta(\rho))) \geq \cos(2m\delta) & \text{for } \rho \in (r, r_2),
\end{cases}
$$

and obtain

$$
K_1 \geq \int_{r_{min}}^{r} \frac{\rho}{r} \arctan\left(\frac{(\frac{\rho}{r})^m \sin\left(2m\eta(r)\right)}{1-\left(\frac{\rho}{r}\right)^m}\right) d\rho - \int_{r}^{r_2} \frac{\rho}{r} \arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(2m\eta(r)\right)}{1-\left(\frac{r}{\rho}\right)^m \cos\left(2m\delta\right)}\right) d\rho.
$$

From the change of variables $\left(\frac{\rho}{r}\right)^m \mapsto x$ for the first integral and $\left(\frac{r}{\rho}\right)^m \mapsto x$ for the second integral, it follows that

$$
\begin{aligned}
K_1 &\geq \frac{r}{m}\left(\int_{(\frac{r_{min}}{r})^m}^{1} x^{-1+\frac{2}{m}} \arctan\left(\frac{x\sin(2m\eta(r))}{1-x}\right) dx - \int_{\left(\frac{r}{r_2}\right)^m}^{1} x^{-1-\frac{2}{m}} \arctan\left(\frac{x\sin(2m\eta(r))}{1-x\cos(2m\delta)}\right) dx\right) \\
&\geq \frac{r}{m}\int_{\left(\frac{r}{r_2}\right)^m}^{1} x^{-1-\frac{2}{m}} \left(\arctan\left(\frac{x\sin(2m\eta(r))}{1-x}\right) - \arctan\left(\frac{x\sin(2m\eta(r))}{1-x\cos(2m\delta)}\right)\right) dx.
\end{aligned}
$$

where we used $\frac{r_{min}}{r} > \frac{r}{r_2}$ for $r \in (r_{min}, r_1)$, which follows from $r_1^2 < r_{min}r_2$, and $x^{-1-\frac{2}{m}} > x^{-1+\frac{2}{m}}$ for $0 < x < 1$ to get the second inequality (note that the first integrand is negative). Therefore it follows from Lemma B.2.2 that (note that $\sin\left(2m\eta(r)\right) < 0$ for $r < r_1$),

$$
K_1 \gtrsim -\frac{r}{m}(1-\cos(2m\delta)) \gtrsim -\frac{r}{m}\sin(2m\delta) \gtrsim -r\delta, \qquad (6.2.58)
$$

where the second inequality is due to $1-\cos(2m\delta) < \sin(2m\delta)$ for $\delta < \frac{\pi}{4m}$.

Now let us estimate K_2. Note that the integrand in K_2 is non-negative if $\eta(\rho) > \frac{\pi}{m} - \eta(r)$. Indeed, by monotonicity of η, we have $\eta(r) > \eta(\rho)$ for any $\rho \geq r_2 > r$, which implies the first integrand in K_2 is positive for all $\rho \in (r_2, r_{max})$. Thus, if we choose $r_3 := \eta^{-1}(\frac{\pi}{m} - \eta(r)) > r_2$ then the integrand in K_2 is strictly positive for $\rho \in (r_2, r_3)$. Hence we have

$$K_2 \geq \int_{r_3}^{r_{max}} \frac{\rho}{r} \left(\arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(m(\eta(r) - \eta(\rho))\right)}{1 - \left(\frac{r}{\rho}\right)^m \cos\left(m(\eta(r) - \eta(\rho))\right)} \right) \right.$$
$$\left. - \arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(m(\eta(r) + \eta(\rho))\right)}{1 - \left(\frac{r}{\rho}\right)^m \cos\left(m(\eta(r) + \eta(\rho))\right)} \right) \right) d\rho.$$

Note that $\left| \partial_\theta \left(\arctan\left(\frac{x \sin(m\theta)}{1 - x\cos(m\theta)} \right) \right) \right| \lesssim \frac{mx}{1-x}$ for all $0 \leq x < 1$ and $\theta \in [-\frac{\pi}{m}, \frac{\pi}{m}]$. Indeed,

$$\partial_\theta \left(\arctan\left(\frac{x \sin(m\theta)}{1 - x\cos(m\theta)} \right) \right) = \frac{mx(\cos(m\theta) - 1 + (1-x))}{(x-1)^2 + 2x(1 - \cos(m\theta))}.$$

Hence either $(1-x) \leq 1 - \cos(m\theta)$ or $(1-x) > 1 - \cos(m\theta)$, one can easily see that $\left| \frac{mx(\cos(m\theta) - 1 + (1-x))}{(x-1)^2 + 2x(1-\cos(m\theta))} \right| \lesssim \frac{mx}{1-x}$. Since we also have $\frac{r}{\rho} < \frac{r_1}{r_3} < \frac{r_1}{r_2}$ for $\rho > r_3$, and it follows from the mean-value theorem that the integrand can be bounded as

$$\left| \arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(m(\eta(r) - \eta(\rho))\right)}{1 - \left(\frac{r}{\rho}\right)^m \cos\left(m(\eta(r) - \eta(\rho))\right)} \right) - \arctan\left(\frac{(\frac{r}{\rho})^m \sin\left(m(\eta(r) + \eta(\rho))\right)}{1 - \left(\frac{r}{\rho}\right)^m \cos\left(m(\eta(r) + \eta(\rho))\right)} \right) \right|$$
$$\lesssim \frac{m\eta(\rho)\left(\frac{r}{\rho}\right)^m}{1 - \left(\frac{r}{\rho}\right)^m}$$
$$< \frac{m\eta(\rho)\left(\frac{r}{\rho}\right)^m}{1 - \left(\frac{r_1}{r_2}\right)^m}.$$

Therefore we obtain

$$K_2 \gtrsim -\frac{1}{1-\left(\frac{r_1}{r_2}\right)^m}\int_{r_3}^{r_{max}}\left(\frac{r}{\rho}\right)^{m-1} m\eta(\rho)d\rho \gtrsim -m\frac{\left(\frac{\pi}{m}-\eta(r)\right)}{1-\left(\frac{r_1}{r_2}\right)^m}\int_{r_3}^{r_{max}}\left(\frac{r}{\rho}\right)^{m-1}d\rho$$
$$\gtrsim -\frac{rm}{(m-2)}\frac{\left(\frac{\pi}{m}-\eta(r)\right)}{1-\left(\frac{r_1}{r_2}\right)^m},$$

where we used $\eta(\rho) < \frac{\pi}{m} - \eta(r)$ for $\rho > r_3$ to get the second inequality. Since $m \geq 3$ and $\frac{\pi}{m} - \eta(r) < \frac{\pi}{m} - \eta(r_2) = \delta$, the above inequality implies

$$K_2 \gtrsim -\frac{r}{1-\left(\frac{r_1}{r_2}\right)^m}\left(\frac{\pi}{m}-\eta(r)\right) \gtrsim -\frac{r}{1-\left(\frac{r_1}{r_2}\right)^m}\delta. \tag{6.2.59}$$

Thus, (Equation 6.2.57), (Equation 6.2.58) and (Equation 6.2.59) yield (Equation 6.2.27).

Now, let us prove (Equation 6.2.28). Since $\sin(m\eta(\rho)) < \sin(m\delta)$ for all $\rho < r_2$, it follows from Lemma B.1.3 that

$$\partial_\theta\varphi_m(r,\eta(r)) < \frac{1}{4\pi}\underbrace{\int_{r_{min}}^{r}\rho\log\left(1+\frac{4(\frac{\rho}{r})^m\sin^2(m\delta)}{(1-(\frac{\rho}{r})^m)^2}\right)d\rho}_{=:K_3} + \frac{1}{4\pi}\underbrace{\int_{r}^{r_2}\rho\log\left(1+\frac{4(\frac{r}{\rho})^m\sin^2(m\delta)}{(1-(\frac{r}{\rho})^m)^2}\right)d\rho}_{=:K_4}$$
$$+\frac{1}{4\pi}\underbrace{\int_{r_2}^{r_{max}}\rho\log\left(1+\frac{4(\frac{r}{\rho})^m\sin(m\eta(\rho))\sin(m\delta)}{(1-(\frac{r}{\rho})^m)^2}\right)d\rho}_{=:K_5}.$$

Again, the change of variables, $\left(\frac{\rho}{r}\right)^m \mapsto x$ and $\left(\frac{r}{\rho}\right)^m \mapsto x$ yields that

$$K_3 \lesssim \frac{r^2}{m}\int_0^1 x^{-1+\frac{2}{m}}\log\left(1+\frac{4x\sin^2(m\delta)}{(1-x)^2}\right)dx,$$
$$K_4 \lesssim \frac{r^2}{m}\int_0^1 x^{-1-\frac{2}{m}}\log\left(1+\frac{4x\sin^2(m\delta)}{(1-x)^2}\right)dx.$$

Thus it follows from Lemma B.2.3 that

$$K_3,\ K_4 \lesssim \frac{r^2}{m}\left|\sin(m\delta)\right| < r^2\delta. \tag{6.2.60}$$

To estimate K_5, recall that $r < r_1$ and $\log(1+x) \leq x$, which yields that

$$\begin{aligned}
K_5 &\lesssim \int_{r_2}^{r_{max}} \rho \left(\frac{\left(\frac{r}{\rho}\right)^m \sin(m\eta(\rho))\sin(m\delta)}{\left(1-\left(\frac{r}{\rho}\right)^m\right)^2} \right) d\rho \\
&= \frac{r^2}{m}\int_{\left(\frac{r}{r_{max}}\right)^m}^{\left(\frac{r}{r_2}\right)^m} x^{-1-\frac{2}{m}} \left(\frac{x\sin(m\eta(\rho))\sin(m\delta))}{(1-x)^2} \right) dx \\
&\lesssim \frac{r^2}{m}\int_0^{\left(\frac{r_1}{r_2}\right)^m} x^{-\frac{2}{m}} \frac{\sin(m\delta)}{(1-x)^2} dx \\
&= \frac{r^2}{m}\left(\int_0^{\min\left\{\frac{1}{2},\left(\frac{r_1}{r_2}\right)^m\right\}} x^{-\frac{2}{m}} \frac{\sin(m\delta)}{(1-x)^2} dx + \int_{\min\left\{\frac{1}{2},\left(\frac{r_1}{r_2}\right)^m\right\}}^{\left(\frac{r_1}{r_2}\right)^m} x^{-\frac{2}{m}} \frac{\sin(m\delta)}{(1-x)^2} dx \right) \\
&\lesssim \frac{r^2}{m}\left(m\delta + \frac{\sin(m\delta)}{\left(1-\left(\frac{r_1}{r_2}\right)^m\right)} \right) \\
&\lesssim \frac{\delta}{1-\left(\frac{r_1}{r_2}\right)^m}
\end{aligned} \tag{6.2.61}$$

Therefore (Equation 6.2.60) and (Equation 6.2.61) yield (Equation 6.2.28). This finishes the proof.

□

Appendices

APPENDIX A

FUNTIONAL DERIVATIVES AND SOME BASIC INTEGRALS

A.1 Derivatives of the Functional

A.1.1 Functional derivatives

Recall that $\mathcal{F}(b, g, r) = (\mathcal{F}_1, \mathcal{F}_2)$ is given in (Equation 5.1.4). For simplicity, we denote

$$
\begin{aligned}
A_1 &:= (b + g(\eta)),\\
A_2 &:= \left(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta)\right)(1+r(\eta)) - (1+r(\theta))r'(\theta),\\
A_3 &:= \frac{1}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)},\\
A_4 &:= r'(\theta)(1+r(\theta)),\\
A_5 &:= (b+g(\eta))(b+g(\theta)),\\
A_6 &:= (1+r(\theta))^2 - (r'(\theta)\sin(\theta-\eta) + (1+r(\theta))\cos(\theta-\eta))(1+r(\eta)),\\
A_7 &:= \frac{1}{r'(\theta)^2 + (1+r(\theta))^2},\\
A_8 &:= \frac{(1+r(\theta))^2(b+g(\theta))}{r'(\theta)^2 + (1+r(\theta))^2}.
\end{aligned}
$$

We also denote the average integral by $\fint f(\theta)d\theta := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$. Therefore the functional $\mathcal{F}$ can be written as

$$\mathcal{F}_1 = \fint A_1 A_2 A_3 d\eta + \Omega A_4, \tag{A.1.1}$$

$$\tilde{\mathcal{F}}_2 = \fint A_5 A_6 A_7 A_3 d\eta - \Omega A_8. \tag{A.1.2}$$

We will expand $A_i(g, \theta, \eta)$ and $A_i(r, \theta, \eta)$ up to quadratic/cubic order in g and r.

Lemma A.1.1. *Let $A_i's$ be as above. We have*

$$
\begin{aligned}
A_1 &= b + g(\eta),\\
A_2 &= -\sin(\theta-\eta) + \left[r'(\theta)(\cos(\theta-\eta)-1) - (r(\theta)+r(\eta))\sin(\theta-\eta)\right]\\
&\quad + \left[r'(\theta)r(\eta)(\cos(\theta-\eta)-1) - r'(\theta)(r(\theta)-r(\eta)) - r(\theta)r(\eta)\sin(\theta-\eta)\right],\\
A_3 &= \frac{1}{2-2\cos(\theta-\eta)} - \frac{r(\theta)+r(\eta)}{2-2\cos(\theta-\eta)}\\
&\quad + \frac{1}{2-2\cos(\theta-\eta)}\left[r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2 - \frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right]\\
&\quad + \frac{1}{2-2\cos(\theta-\eta)}\left[\frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{1-\cos(\theta-\eta)} - \left[r(\theta)^3 + r(\theta)^2 r(\eta) + r(\theta)r(\eta)^2 + r(\eta)^3\right]\right] + O(r^4),\\
A_4 &= r'(\theta) + r(\theta)r'(\theta),\\
A_5 &= b^2 + b(g(\theta)+g(\eta)) + g(\theta)g(\eta),\\
A_6 &= (1-\cos(\theta-\eta)) + \left[(r(\theta)+r(\eta))(1-\cos(\theta-\eta)) + (r(\theta)-r(\eta)) - r'(\theta)\sin(\theta-\eta)\right]\\
&\quad + \left[r(\theta)(r(\theta)-r(\eta)) + r(\theta)r(\eta)(1-\cos(\theta-\eta)) - r'(\theta)r(\eta)\sin(\theta-\eta)\right],\\
A_7 &= 1 - 2r(\theta) + \left[3r(\theta)^2 - r'(\theta)^2\right] + \left[4r(\theta)r'(\theta)^2 - 4r(\theta)^3\right] + O(r^4),\\
A_8 &= b + g(\theta) - br'(\theta)^2 + \left[2br(\theta)r'(\theta)^2 - g(\theta)r'(\theta)^2\right] + O(r^4+g^4).
\end{aligned}
$$

Proof. Straightforward. □

Linear parts

We denote by A_i^j the jth order term in A_i. For example, $A_3^1 = -\frac{r(\theta)+r(\eta)}{2-2\cos(\theta-\eta)}$.

Lemma A.1.2. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$\frac{d}{dt}\mathcal{F}_1(b,tg,tr)\Big|_{t=0} = -\fint \frac{g(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \left(\Omega - \frac{b}{2}\right)r'(\theta), \tag{A.1.3}$$

$$\frac{d}{dt}\tilde{\mathcal{F}}_2(b,tg,tr)\Big|_{t=0} = \left(\frac{b}{2}-\Omega\right)g(\theta) + b^2\left(\fint \frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)}d\eta - r(\theta)\right). \tag{A.1.4}$$

Proof. We compute $\mathcal{F}_1$ first. In view of (Equation A.1.1), we collect the linear terms in $A_1A_2A_3 + \Omega A_4$ from Lemma A.1.1. Hence we have

$$\begin{aligned}
\mathcal{F}_1(b, tg, tr) = &\fint -\frac{tg(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} + bt\frac{r'(\theta)(\cos(\theta-\eta)-1) - (r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} \\
&+ \frac{bt\sin(\theta-\eta)(r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta + t\Omega r'(\theta) + O(t^2) \\
&= -\fint \frac{tg(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + t\left(\Omega - \frac{b}{2}\right) r'(\theta) + O(t^2).
\end{aligned}$$

Thus we obtain (Equation A.1.3) by differentiating with respect to t.

In order to compute the derivative of $\tilde{\mathcal{F}}_2$, we collect the linear terms in (Equation A.1.2) from Lemma A.1.1 and obtain

$$\begin{aligned}
\tilde{\mathcal{F}}_2(b, tg, tr) = &\fint \frac{bt(g(\theta)+g(\eta))}{2} \\
&+ tb^2\left[\frac{(r(\theta)+r(\eta))}{2} + \frac{(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - \frac{r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&- tb^2 r(\theta) - \frac{tb^2}{2}(r(\theta)+r(\eta)) - t\Omega g(\theta) d\eta + O(t^2) \\
= &\, t\left(\frac{b}{2} - \Omega\right) g(\theta) + tb^2\left(\fint \frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)}d\eta - r(\theta)\right) + O(t^2),
\end{aligned}$$

where we used $\fint g(\eta)d\eta = \fint r(\eta)d\eta = 0$ and $\fint \frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = 0$. By differentiating in t, we obtain the desired result (Equation A.1.4). $\square$

Quadratic parts

Now we compute the quadratic expansion of $\mathcal{F}_1$ and $\tilde{\mathcal{F}}_2$.

Lemma A.1.3. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$\mathcal{F}_1 = \left(\frac{b}{2}+\Omega\right) r(\theta)r'(\theta) - b\fint \frac{r'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\fint \frac{(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta$$
$$+ \textit{linear terms} + O(r^3+g^3), \tag{A.1.5}$$
$$\tilde{\mathcal{F}}_2 = \frac{3b^2}{2}r(\theta)^2 - bg(\theta)r(\theta) + b\left(\Omega - \frac{b}{2}\right) r'(\theta)^2$$
$$- \frac{b^2}{2}\fint \frac{(r(\theta)-r(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\fint \frac{(g(\theta)+g(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$- b\fint \frac{g(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \textit{linear terms} + O(r^3+g^3). \tag{A.1.6}$$

Proof. We compute $\mathcal{F}_1$ first. By collecting quadratic terms in (Equation A.1.1) from Lemma A.1.1, we have (we will have, for example, $A_1^2A_2^0A_3^0 + A_1^0A_2^2A_3^0 + A_1^0A_2^0A_3^2 + A_1^1A_2^1A_3^0 + A_1^1A_2^0A_3^1 + A_1^0A_2^1A_3^1 + \Omega A_4^2$).

$$\begin{aligned}
\mathcal{F}_1 &= \fint b\left[-\frac{1}{2}r'(\theta)r(\eta) - \frac{r'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - \frac{r(\theta)r(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&\quad - \frac{b\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\left[r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2 - \frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right] \\
&\quad + \left[-\frac{g(\eta)r'(\theta)}{2} - \frac{g(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] + \frac{g(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} \\
&\quad + \left[\frac{br'(\theta)(r(\theta)+r(\eta))}{2} + \frac{b(r(\theta)+r(\eta))^2\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] + \Omega r(\theta)r'(\theta)d\eta + O(r^3+g^3) \\
&= \fint \left(\frac{b}{2}+\Omega\right) r(\theta)r'(\theta) - \frac{br'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + \frac{b(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta \\
&\quad + \text{linear terms} + O(r^3+g^3),
\end{aligned}$$

which yields (Equation A.1.5). Now we will expand $\tilde{\mathcal{F}}_2$ up to the quadratic order. By collecting all quadratic terms in (Equation A.1.2) from Lemma A.1.1, we obtain (we will have $A_5^2A_6^0A_7^0A_3^0 + A_5^0A_6^2A_7^0A_3^0 + A_5^0A_6^0A_7^2A_3^0 + A_5^0A_6^0A_7^0A_3^2 + A_5^1A_6^1A_7^0A_3^0 + A_5^1A_6^0A_7^1A_3^0 + A_5^1A_6^0A_7^0A_3^1 + A_5^0A_6^1A_7^1A_3^0 +$

$A_5^0 A_6^1 A_7^0 A_3^1 + A_5^0 A_6^0 A_7^1 A_3^1 - \Omega A_8^2),$

$$
\begin{aligned}
\tilde{\mathcal{F}}_2 = &\fint \frac{g(\theta)g(\eta)}{2} + \left[b^2 \frac{r(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + b^2\frac{r(\theta)r(\eta)}{2} - b^2\frac{r'(\theta)r(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&+ b^2\frac{(3r(\theta)^2 - r'(\theta)^2)}{2} + \left[\frac{b^2(r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2)}{2} - \frac{b^2}{2}\frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right] \\
&+ \left[b\frac{(g(\theta)+g(\eta))(r(\theta)+r(\eta))}{2} + b\frac{(g(\theta)+g(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - b\frac{(g(\theta)+g(\eta))r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&- b((g(\theta)+g(\eta)))r(\theta) - b\frac{(g(\theta)+g(\eta))(r(\theta)+r(\eta))}{2} \\
&+ \left[-b^2 r(\theta)(r(\theta)+r(\eta)) - 2b^2\frac{r(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + 2b^2\frac{r(\theta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&+ \left[-b^2\frac{(r(\theta)+r(\eta))^2}{2} - b^2\frac{(r(\theta)^2 - r(\eta)^2)}{2-2\cos(\theta-\eta)} + b^2\frac{r'(\theta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\
&+ b^2 r(\theta)(r(\theta)+r(\eta)) + b\Omega r'(\theta)^2 d\eta + O(r^3+g^3) \\
= &\fint b^2\frac{(3r(\theta)^2 - r'(\theta)^2)}{2} - bg(\theta)r(\theta) + b\Omega r'(\theta)^2 \\
&+ \frac{1}{2-2\cos(\theta-\eta)}\left(-\frac{b^2}{2}(r(\theta)-r(\eta))(5r(\theta)+r(\eta)) + b(g(\theta)+g(\eta))(r(\theta)-r(\eta))\right) \\
&- \frac{bg(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \text{linear terms} + O(r^3+g^3),
\end{aligned}
$$

where we used $\fint g(\eta)d\eta = \fint r(\eta)d\eta = \fint \frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = 0$. This yields the desired result (Equation A.1.6). □

Lemma A.1.4. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$
\begin{aligned}
\frac{d^2}{dtds}\mathcal{F}_1(b, tg+s\tilde{g}, tr+s\tilde{r})\Big|_{t=s=0} = &\left(\frac{b}{2}+\Omega\right)(r(\theta)\tilde{r}'(\theta) + \tilde{r}(\theta)r'(\theta)) \\
&- b\fint \frac{r'(\theta)(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - b\fint\frac{\tilde{r}'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&+ 2b\fint \frac{(r(\theta)-r(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}, \qquad \text{(A.1.7)}
\end{aligned}
$$

and

$$\frac{d^2}{dtds}\tilde{\mathcal{F}}_2(b, tg+s\tilde{g}, tr+s\tilde{r})\bigg|_{t=s=0} = 3b^2 r(\theta)\tilde{r}(\theta) - b(g(\theta)\tilde{r}(\theta)+\tilde{g}(\theta)r(\theta)) + 2b\left(\Omega - \frac{b}{2}\right)r'(\theta)\tilde{r}'(\theta)$$
$$- \frac{b^2}{2}\fint \frac{(r(\theta)-r(\eta))(5\tilde{r}(\theta)+\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - \frac{b^2}{2}\fint \frac{(\tilde{r}(\theta)-\tilde{r}(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$+ b\fint \frac{(g(\theta)+g(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\fint \frac{(\tilde{g}(\theta)+\tilde{g}(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$- b\fint \frac{g(\eta)\tilde{r}'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta - b\fint \frac{\tilde{g}(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta. \qquad \text{(A.1.8)}$$

Proof. We compute $\mathcal{F}_1$ first. From (Equation A.1.5) in Lemma A.1.3, we collect all st terms and obtain

$$\mathcal{F}_1(b, tg+s\tilde{g}, tr+s\tilde{r}) = st\Bigg[\left(\frac{b}{2}+\Omega\right)(r(\theta)\tilde{r}'(\theta)+\tilde{r}(\theta)r'(\theta))$$
$$-b\fint \frac{r'(\theta)(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - b\fint \frac{\tilde{r}'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$+2b\fint \frac{(r(\theta)-r(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{(2-2\cos(\theta-\eta))^2}\Bigg] + \text{linear terms} + O(t^2+s^2).$$

Once we differentiate the above equation with respect to t and s, the desired result (Equation A.1.7) follows immediately. Similarly, we collect all st terms from (Equation A.1.6) and obtain

$$\tilde{\mathcal{F}}_2(b, tg+s\tilde{g}, tr+s\tilde{r}) = st\Bigg[3b^2 r(\theta)\tilde{r}(\theta) - b(g(\theta)\tilde{r}(\theta)+\tilde{g}(\theta)r(\theta)) + 2b\left(\Omega - \frac{b}{2}\right)r'(\theta)\tilde{r}'(\theta)$$
$$-\frac{b^2}{2}\fint \frac{(r(\theta)-r(\eta))(5\tilde{r}(\theta)+\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - \frac{b^2}{2}\fint \frac{(\tilde{r}(\theta)-\tilde{r}(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$+b\fint \frac{(g(\theta)+g(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\fint \frac{(\tilde{g}(\theta)+\tilde{g}(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta$$
$$-b\fint \frac{g(\eta)\tilde{r}'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta - b\fint \frac{\tilde{g}(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta\Bigg]$$
$$+ \text{linear terms} + O(t^2+s^2).$$

Once we differentiate the above equation with respect to t and s, the desired result (Equation A.1.8) follows immediately. $\square$

Cubic parts

We will expand $\tilde{\mathcal{F}}_2$ up to cubic order with respect to the r variable (we fix $g = 0$). We denote $B := A_3 A_6$ so that (Equation A.1.2) can be written as (with $g = 0$)

$$\tilde{\mathcal{F}}_2 = b^2 \fint B d\eta A_7 - \Omega A_8. \tag{A.1.9}$$

We will first expand B up to cubic order.

Lemma A.1.5. *Let $\tilde{\mathcal{F}}_2$, A_i and B be as defined as before. Then*

$$\begin{aligned}
\fint B d\eta = \frac{1}{2} + \fint \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} d\eta - \frac{1}{2} \fint \frac{r(\theta)^2 - r(\eta)^2}{2 - 2\cos(\theta - \eta)} d\eta \\
+ \left[\frac{1}{2} \fint \frac{(r(\theta) - r(\eta))(r(\theta)^2 + r(\eta)^2)}{2 - 2\cos(\theta - \eta)} d\eta - \fint \frac{(r(\theta) - r(\eta))^3}{(2 - 2\cos(\theta - \eta))^2} d\eta \right. \\
\left. + \fint \frac{r'(\theta)(r(\theta) - r(\eta))^2 \sin(\theta - \eta)}{(2 - 2\cos(\theta - \eta))^2} d\eta \right] + O(r^4).
\end{aligned}$$

Proof. Using (A.1.1), we will compute the constant ($=: B^0$), linear ($=: B^1$), quadratic ($=: B^2$) and cubic ($=: B^3$) terms of $B = A_3 A_6$ separately. It is straightforward that

$$\fint B^0 d\eta = \frac{1}{2}. \tag{A.1.10}$$

For B^1, we compute $B^1 (= A_3^1 A_6^0 + A_3^0 A_6^1)$,

$$\fint B^1 d\eta = \fint \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} - \frac{r'(\theta)\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta = \fint \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} d\eta, \tag{A.1.11}$$

where we used $\fint \frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta = 0$. For B^2, we compute $A_3^2 A_6^0 + A_3^0 A_6^2 + A_3^1 A_6^1$, hence

$$\begin{aligned}
\fint B^2 d\eta &= \left[\frac{r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2}{2} - \frac{1}{2}\fint \frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)} d\eta\right] \\
&\quad + \left[\fint \frac{r(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + \frac{r(\theta)r(\eta)}{2} - \frac{r'(\theta)r(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta\right] \\
&\quad + \left[\fint -\frac{1}{2}(r(\theta)+r(\eta))^2 - \frac{r(\theta)^2 - r(\eta)^2}{2-2\cos(\theta-\eta)} + \frac{r'(\theta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta\right] \\
&= -\frac{1}{2}\fint \frac{(r(\theta)^2 - r(\eta)^2)}{2-2\cos(\theta-\eta)} d\eta, \qquad \text{(A.1.12)}
\end{aligned}$$

where we used $\fint \frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta = 0$. For B^3, we compute $B^3 (= A_3^3 A_6^0 + A_3^2 A_6^1 + A_3^1 A_6^2 + A_3^0 A_6^3)$,

$$\begin{aligned}
\fint B^3 d\eta &= \left[\fint \frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)} d\eta - \frac{1}{2}\fint r(\theta)^3 + r(\theta)^2 r(\eta) + r(\theta)r(\eta)^2 + r(\eta)^3 d\eta\right] \\
&\quad + \Bigg[\fint \frac{(r(\theta)+r(\eta))(r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2)}{2} d\eta - \frac{1}{2}\fint \frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)} d\eta \\
&\quad + \fint \frac{r(\theta)^3 - r(\eta)^3}{2-2\cos(\theta-\eta)} d\eta - \fint \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2} d\eta \\
&\quad - \fint \frac{r'(\theta)(r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta + \fint \frac{r'(\theta)(r(\theta)-r(\eta))^2 \sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2} d\eta\Bigg] \\
&\quad + \Bigg[-\fint \frac{r(\theta)(r(\theta)^2 - r(\eta)^2)}{2-2\cos(\theta-\eta)} d\eta - \fint \frac{r(\theta)r(\eta)(r(\theta)+r(\eta))}{2} d\eta \\
&\quad + \fint \frac{r'(\theta)r(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta\Bigg] \\
&= \frac{1}{2}\fint \frac{(r(\theta)-r(\eta))(r(\theta)^2 + r(\eta)^2)}{2-2\cos(\theta-\eta)} d\eta - \fint \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2} d\eta \\
&\quad + \fint \frac{r'(\theta)(r(\theta)-r(\eta))^2 \sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}. \qquad \text{(A.1.13)}
\end{aligned}$$

Thus the desired result follows from (Equation A.1.10), (Equation A.1.11), (Equation A.1.12) and (Equation A.1.13). □

Lemma A.1.6. *Let $\tilde{\mathcal{F}}_2$, A_i's and B be as defined as before. Then*

$$\begin{aligned}\frac{1}{6}\frac{d^3}{dt^3}\tilde{\mathcal{F}}_2(b,0,tr)\Big|_{t=0} &= \frac{b^2}{2}\fint \frac{(r(\theta)-r(\eta))(3r(\theta)^2+2r(\theta)r(\eta)+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - b^2\fint \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \\ &+ b^2\fint \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}\eta + b^2\fint \frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &+ 2b(b-\Omega)r(\theta)r'(\theta)^2 - 2b^2r(\theta)^3\end{aligned}$$

Proof. We first collect all cubic terms of $\tilde{\mathcal{F}}_2(b,0,r)$ in r. From (Equation A.1.9), we have that the cubic terms consist of $b^2(B^3A_7^0 + B^2A_7^1 + B^1A_7^2 + B^0A_7^3) - \Omega A_8^3$. Using Lemma A.1.1 and A.1.5 and the fact that A_7 does not depend on η, we obtain

$$\begin{aligned}\tilde{\mathcal{F}}_2(b,0,r) &= b^2\left[\frac{1}{2}\fint \frac{(r(\theta)-r(\eta))(r(\theta)^2+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - \fint \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \right.\\ &\left.+ \fint \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta\right] + b^2\fint \frac{r(\theta)(r(\theta)^2-r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta \\ &+ b^2\fint \frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b^2\left[2r(\theta)r'(\theta)^2 - 2r(\theta)^3\right] \\ &- 2b\Omega r(\theta)r'(\theta)^2 + \text{lower order terms} + O(r^4) \\ &= \frac{b^2}{2}\fint \frac{(r(\theta)-r(\eta))(3r(\theta)^2+2r(\theta)r(\eta)+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - b^2\fint \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \\ &+ b^2\fint \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}\eta + b^2\fint \frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &+ 2b(b-\Omega)r(\theta)r'(\theta)^2 - 2b^2r(\theta)^3 + \text{lower order terms} + O(r^4).\end{aligned}$$

Therefore, the desired result follows immediately. □

A.1.2 Derivatives of the reduced functional

We denote $v := (0,\cos(2\theta))$. Given a pair of functions (g,r), we denote Q be the projection to the second mode of r, that is, $Q(g,r) := \left(\frac{1}{\pi}\int r(\theta)\cos(2\theta)d\theta\right)\cos(2\theta)$.

Lemma A.1.7. *Let $\mathcal{F}$, v, Q be defined as before. We fix $b = 2$ and $\Omega = 1$. Then,*

$$\partial_b D\mathcal{F}(2,0)v = (\sin(2\theta), 0), \tag{A.1.14}$$

$$\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0} = (-2\sin(4\theta), -3\cos(4\theta)), \tag{A.1.15}$$

$$\partial_b Q\frac{d^2}{dt^2}\mathcal{F}(b,tv)\Big|_{b=2,t=0} = (0,0), \tag{A.1.16}$$

$$\tilde{v} := -\left[D\mathcal{F}(2,0)\right]^{-1}(I-Q)\partial_b D\mathcal{F}(2,0)v = (2\cos(2\theta), 0), \tag{A.1.17}$$

$$\hat{v} := -\left[D\mathcal{F}(2,0)\right]^{-1}\frac{d^2}{dt^2}\left[(I-Q)\mathcal{F}(2,tv)\right]\Big|_{t=0} = \left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta)\right), \tag{A.1.18}$$

$$\frac{d^2}{dtds}Q\mathcal{F}(2,tv+s\hat{v})\Big|_{t=s=0} = (0, -12\cos(2\theta)), \tag{A.1.19}$$

$$\frac{1}{3}\frac{d^3}{dt^3}Q\mathcal{F}(2,tv)\Big|_{t=0} = (0, 4\cos(2\theta)), \tag{A.1.20}$$

$$Q\frac{d^2}{dtds}\mathcal{F}(2,tv+s\tilde{v}) = (0,0), \tag{A.1.21}$$

$$\frac{1}{2}Q\partial_b D\mathcal{F}(2,0)\hat{v} = (0,0), \tag{A.1.22}$$

$$2Q\partial_b D\mathcal{F}(2,0)\tilde{v} = (0, 2\cos(2\theta)). \tag{A.1.23}$$

Proof. To prove (Equation A.1.14), it follows from Lemma A.1.2 that

$$\partial_b D\mathcal{F}(b,0)v = \left(\sin(2\theta), 2b\left(\fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta - \cos(2\theta)\right)\right) = (\sin(2\theta), 0),$$

where the last equality follows from (Equation A.1.30).

To prove (Equation A.1.15), note that $\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0} =$

$\left(\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}_1(2,tv)\Big|_{t=0}, \frac{1}{2}(I-P_0)\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(2,tv)\Big|_{t=0}\right)$. Hence it follows from Lemma A.1.3 that

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}_1(2,tv)\Big|_{t=0} &= -4\cos(2\theta)\sin(2\theta) + 4\sin(2\theta)\fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta \\
&\quad + 2\fint \frac{(\cos(2\theta)-\cos(2\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta \\
&= -4\cos(2\theta)\sin(2\theta) + 4\sin(2\theta)\cos(2\theta) - 2\sin(4\theta) \\
&= -2\sin(4\theta),
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and Lemma A.1.11. Also, Lemma A.1.3 gives

$$\begin{aligned}
&\frac{1}{2}(I-P_0)\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(2,tv)\Big|_{t=0} \\
&\qquad = (I-P_0)\left[6\cos^2(2\theta)\right] - 2(I-P_0)\fint \frac{(\cos(2\theta)-\cos(2\eta))(5\cos(2\theta)+\cos(2\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&\qquad = -4(I-P_0)\cos^2(2\theta) - 2(I-P_0)\fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta \\
&\qquad = -3\cos(4\theta),
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and the last equality follows from Lemma A.1.9. Therefore we obtain (Equation A.1.15).

To prove (Equation A.1.16), we can repeat the above computation and find that $\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(b,tv) \in \text{span}\{\cos(4\theta)\}$, independently of b. By projecting it to the space of the second mode, we obtain (Equation A.1.16).

To prove (Equation A.1.17), note that $(I-Q)\partial_b D\mathcal{F}(2,0)v = (\sin(2\theta),0)$, which follows from (Equation A.1.14). Also, it follows from Lemma A.1.2 and (Equation A.1.29) that

$$D\mathcal{F}(2,0)(-2\cos(2\theta),0) = (\sin(2\theta),0) = (I-Q)\partial_b D\mathcal{F}(2,0)v.$$

This immediately implies (Equation A.1.17).

To prove (Equation A.1.18), we use Lemma A.1.2 and (Equation A.1.30) and (Equation A.1.29) to obtain

$$D\mathcal{F}(2,0)\left(8\cos(4\theta), -\frac{3}{2}\cos(4\theta)\right) = (-4\sin(4\theta), -6\cos(4\theta)) = \frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0},$$

where the last equality follows from (Equation A.1.15). Therefore we obtain

$$-\left[D\mathcal{F}(2,0)\right]^{-1}\frac{d^2}{dt^2}\left[(I-Q)\mathcal{F}(2,tv)\right]\Big|_{t=0} = \left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta)\right),$$

which proves (Equation A.1.18).

To prove (Equation A.1.19), note that $\frac{d^2}{dtds}Q\mathcal{F} = \left(0, P_2\frac{d^2}{dtds}\tilde{\mathcal{F}}_2\right)$. Therefore, it follows from (Equation A.1.18) and (Equation A.1.8) in Lemma A.1.4 that (plugging $g = 0$, $\tilde{g} = -8\cos(4\theta)$, $r = \cos(2\theta)$, and $\tilde{r} = \frac{3}{2}\cos(4\theta)$)

$$\begin{aligned}
\frac{d^2}{dtds}P_2\mathcal{F}_2(2, tv + s\tilde{v}) = P_2\Big[&(18\cos(2\theta)\cos(4\theta) + 16\cos(2\theta)\cos(4\theta)) \\
&-2\fint \frac{(\cos(2\theta)-\cos(2\eta))(\frac{15}{2}\cos(4\theta)+\frac{3}{2}\cos(4\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&-3\fint \frac{(\cos(4\theta)-\cos(4\eta))(5\cos(2\theta)+\cos(2\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&-16\fint \frac{(\cos(4\theta)+\cos(4\eta))(\cos(2\theta)-\cos(2\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&-32\fint \frac{\cos(4\eta)\sin(2\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta\Big] \\
=:{}& P_2K_1 + P_2K_2 + P_2K_3 + P_2K_4 + P_2K_5.
\end{aligned}$$

For K_1, we compute

$$P_2K_1 = P_2(34\cos(2\theta)\cos(4\theta)) = 17P_2(\cos(2\theta)+\cos(6\theta)) = 17\cos(2\theta). \tag{A.1.24}$$

For K_2 we compute

$$\begin{aligned}
P_2K_2 &= P_2\left(-15\cos(4\theta)\fint\frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta - 3\fint\frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2\cos(\theta-\eta)}d\eta\right)\\
&= P_2\left(-15\cos(2\theta)\cos(4\theta)+\frac{3}{2}(\cos(2\theta)-\cos(6\theta))\right)\\
&= P_2(-6\cos(2\theta)-9\cos(6\theta))\\
&= -6\cos(2\theta), \qquad \text{(A.1.25)}
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and Lemma A.1.9. For K_3, we compute

$$\begin{aligned}
P_2K_3 &= P_2\left(-15\cos(2\theta)\fint\frac{\cos(4\theta)-\cos(4\eta)}{2-2\cos(\theta-\eta)}d\eta - 3\fint\frac{(\cos(4\theta)-\cos(4\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta\right)\\
&= P_2\left(-30\cos(2\theta)\cos(4\theta)-3\cos(6\theta)\right)\\
&= P_2\left(-15\cos(2\theta)-18\cos(6\theta)\right)\\
&= -15\cos(2\theta), \qquad \text{(A.1.26)}
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and Lemma A.1.9. For K_4, we compute

$$\begin{aligned}
P_2K_4 &= P_2\left(-16\cos(4\theta)\fint\frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta - 16\fint\frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2-\cos(\theta-\eta)}d\eta\right)\\
&= P_2\left(-16\cos(2\theta)\cos(4\theta)+8\cos(2\theta)-8\cos(6\theta)\right)\\
&= P_2\left(-16\cos(6\theta)\right)\\
&= 0, \qquad \text{(A.1.27)}
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and Lemma A.1.9. For K_5, we compute

$$\begin{aligned}
P_2 K_5 &= -16 P_2(\sin(2\theta)\sin(4\theta)) \\
&= -8 P_2\left(\cos(2\theta) - \cos(6\theta)\right) \\
&= -8\cos(2\theta), \qquad \text{(A.1.28)}
\end{aligned}$$

where the first equality follows from (Equation A.1.29). Hence it follows from (Equation A.1.24), (Equation A.1.25), (Equation A.1.26), (Equation A.1.27) and (Equation A.1.28) that

$$\frac{d^2}{dtds} Q\mathcal{F} = \left(0, P_2 \frac{d^2}{dtds}\tilde{\mathcal{F}}_2\right) = (0, -12\cos(2\theta)),$$

which proves (Equation A.1.19).

To prove (Equation A.1.20), note that $\frac{1}{3} Q \frac{d^3}{dt^3}\mathcal{F} = \left(0, \frac{1}{3} P_2 \frac{d^3}{dt^3}\tilde{\mathcal{F}}_2\right)$. We use Lemma A.1.6 with $r = \cos(2\theta)$ and obtain

$$\begin{aligned}
\frac{1}{3} P_2 \frac{d^3}{dt^3}\tilde{\mathcal{F}}_2(2, tv) =& P_2 \Bigg[4 \fint \frac{(\cos(2\theta) - \cos(2\eta))(3\cos^2(2\theta) + 2\cos(2\theta)\cos(2\eta) + \cos^2(2\eta))}{2 - 2\cos(\theta - \eta)} d\eta \\
& -8 \fint \frac{(\cos(2\theta) - \cos(2\eta))^3}{(2 - 2\cos(\theta - \eta))^2} d\eta \\
& -16\sin(2\theta) \fint \frac{(\cos(2\theta) - \cos(2\eta))^2 \sin(\theta - \eta)}{(2 - 2\cos(\theta - \eta))^2} d\eta \\
& +8 \fint \frac{(3\cos^2(2\theta) - 4\sin^2(2\theta))(\cos(2\theta) - \cos(2\eta))}{2 - 2\cos(\theta - \eta)} d\eta \\
& + \left(32\cos(2\theta)\sin^2(2\theta) - 16\cos^3(2\theta)\right)\Bigg] \\
=& P_2 L_1 + P_2 L_2 + P_2 L_3 + P_2 L_4 + P_2 L_5.
\end{aligned}$$

For L_1, we compute

$$\begin{aligned}
P_2 L_1 &= P_2 \left(12\cos^2(2\theta) \fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)} d\eta + 8\cos(2\theta) \fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)} d\eta \right. \\
&\quad \left. +2 \fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)} d\eta + 2 \fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2\cos(\theta-\eta)} d\eta \right) \\
&= P_2 \left(12\cos^3(2\theta) + 8\cos(2\theta)\left(-\frac{1}{2}+\frac{1}{2}\cos(4\theta)\right) + 2\cos(2\theta) + (-\cos(2\theta)+\cos(6\theta)) \right) \\
&= P_2(8\cos(2\theta) + 6\cos(6\theta)) \\
&= 8\cos(2\theta),
\end{aligned}$$

where the second equality follows from (Equation A.1.30) and A.1.9. For L_2, we use Lemma A.1.10 and obtain

$$\begin{aligned}
P_2 L_2 &= -8P_2\left(\frac{9}{4}\cos(2\theta) - \cos(6\theta)\right) \\
&= -18\cos(2\theta).
\end{aligned}$$

For L_3, we use Lemma A.1.11 and obtain

$$P_2 L_3 = 16 P_2 \sin(2\theta)\sin(4\theta) = 8P_2(\cos(2\theta)-\cos(6\theta)) = 8\cos(2\theta).$$

For L_4, we compute

$$\begin{aligned}
P_2 L_4 &= P_2 \left(24\cos^2(2\theta) \fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)} d\eta - 32\sin^2(2\theta) \fint \frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)} d\eta \right) \\
&= P_2(24\cos^3(2\theta) - 32\sin^2(2\theta)\cos(2\theta)) \\
&= P_2(10\cos(2\theta) + 14\cos(6\theta)) \\
&= 10\cos(2\theta),
\end{aligned}$$

where the second equality follows from (Equation A.1.30). For L_5, it follows immediately that

$$P_2 L_5 = P_2\left(-4\cos(2\theta) - 12\cos(6\theta)\right) = -4\cos(2\theta).$$

Collecting the above results, we obtain

$$\frac{1}{3} P_2 \frac{d^3}{dt^3} \tilde{\mathcal{F}}_2(2, tv) = 4\cos(2\theta),$$

which implies (Equation A.1.20).

To prove (Equation A.1.21), we use (Equation A.1.17) and (Equation A.1.8) in Lemma A.1.4 with $g = 0,\ \tilde{g} = 2\cos(2\theta),\ r = \cos(2\theta)$ and $\tilde{r} = 0$ and obtain

$$\begin{aligned}
P_2 \frac{d^2}{dtds} \tilde{\mathcal{F}}_2(2, tv + s\tilde{v}) &= P_2\left(-4\cos^2(2\theta) + 4 \fint \frac{(\cos(2\theta) + \cos(2\eta))(\cos(2\theta) - \cos(2\eta))}{2 - 2\cos(\theta - \eta)} d\eta\right. \\
&\qquad \left. + 8 \fint \frac{\cos(2\eta)\sin(2\theta)\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta\right) \\
&= P_2\left(-4\cos^2(2\theta) + 4\cos^2(2\theta) + 4\left(-\frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) + 4\sin^2(2\theta)\right) \\
&= 0,
\end{aligned}$$

where the second equality follows from (Equation A.1.30), (Equation A.1.29) and (A.1.9). This implies (Equation A.1.21).

To prove (Equation A.1.22), it follows from (Equation A.1.18) and Lemma A.1.2 that

$$\partial_b D\mathcal{F}(2, 0)\left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta)\right) = (3\sin(4\theta), 2\cos(4\theta)).$$

By projecting it to the image of Q, we obtain (Equation A.1.22).

To prove (Equation A.1.23), we note that $Q\partial_b D\mathcal{F}(2,0)\tilde{v} = \left(0, P_2 \partial_b \frac{d}{dt} \tilde{\mathcal{F}}_2(2, t\tilde{v})\right)\Big|_{t=0}$. Hence it follows from (Equation A.1.4) in Lemma A.1.2 and (Equation A.1.17) that $P_2 \partial_b \frac{d}{dt} \tilde{\mathcal{F}}_2(2, t\tilde{v})\Big|_{t=0} =$

$\cos(2\theta)$. This implies (Equation A.1.23). ☐

A.1.3 Basic Integrals

Lemma A.1.8. *For* $\mathbb{N} \ni m \geq 1$, *it holds that*

$$\fint \frac{\cos(m\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = \frac{1}{2}\sin(m\theta), \tag{A.1.29}$$

$$\fint \frac{\cos(m\theta)-\cos(m\eta)}{2-2\cos(\theta-\eta)}d\eta = \frac{m}{2}\cos(m\theta). \tag{A.1.30}$$

Proof. For (Equation A.1.29), it is clear that $\fint \frac{\cos(m\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = \frac{1}{2}\fint \cos(m\eta)\cot\left(\frac{\theta-\eta}{2}\right)d\eta = \frac{1}{2}H(\cos(m\theta))(\theta)$, where H denotes the Hilbert transform in the periodic domain. Therefore the result follows immediately since $H(\cos(m\theta))(\theta) = \sin(m\theta)$.

For (Equation A.1.30), we recall that $\fint \frac{f(\theta)-f(\eta)}{1-\cos(\theta-\eta)}d\eta = \Lambda f(\theta) =: (-\Delta)^{\frac{1}{2}}f(\theta)$. Thus (Equation A.1.30) follows immediately. ☐

Lemma A.1.9.

$$\fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2\cos(\theta-\eta)}d\eta = -\frac{1}{2}\cos(2\theta)+\frac{1}{2}\cos(6\theta), \tag{A.1.31}$$

$$\fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta = -\frac{1}{2}+\frac{1}{2}\cos(4\theta), \tag{A.1.32}$$

$$\fint \frac{(\cos(4\theta)-\cos(4\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta = \cos(6\theta), \tag{A.1.33}$$

$$\fint \frac{(\cos(2\theta)-\cos(2\eta))\sin(2\eta)}{2-2\cos(\theta-\eta)}d\eta = \frac{1}{2}\sin(4\theta). \tag{A.1.34}$$

Proof. We will show (Equation A.1.31) only. (Equation A.1.32), (Equation A.1.33) and

(Equation A.1.34) can be proved in the same way. For (Equation A.1.31), one can write

$$\begin{aligned}\fint \frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2\cos(\theta-\eta)}d\eta &= \frac{1}{2}\Lambda(\cos(2\theta)\cos(4\theta))(\theta) - \frac{1}{2}\cos(2\theta)\Lambda(\cos(4\theta))(\theta) \\ &= -\frac{1}{2}\cos(2\theta) + \frac{1}{2}\cos(6\theta),\end{aligned}$$

where the last equality follows from (Equation A.1.30). □

Lemma A.1.10.

$$\fint \frac{(\cos(2\theta)-\cos(2\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta = \frac{9}{4}\cos(2\theta) - \cos(6\theta).$$

Proof. We compute

$$\begin{aligned}&\fint \frac{(\cos(2\theta)-\cos(2\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta = \fint \frac{(-2\sin(\theta-\eta)\sin(\theta+\eta))^3}{\left(4\sin^2\left(\frac{\theta-\eta}{2}\right)\right)^2}d\eta = 4\fint -\frac{\cos^3\left(\frac{\theta-\eta}{2}\right)\sin^3(\theta+\eta)}{\sin\left(\frac{\theta-\eta}{2}\right)}d\eta \\ &= 4\fint \frac{\cos^3\frac{\eta}{2}\sin^3(2\theta+\eta)}{\sin\frac{\eta}{2}}d\eta = 4\fint \frac{\cos^3\frac{\eta}{2}}{\sin\frac{\eta}{2}}\left(3\sin^2(2\theta)\cos(2\theta)\cos^2\eta\sin\eta + \cos^3(2\theta)\sin^3\eta\right)d\eta \\ &= 12\sin^2(2\theta)\cos(2\theta)\fint \frac{\cos^3\frac{\eta}{2}\cos^2\eta\sin\eta}{\sin\frac{\eta}{2}}d\eta + 4\cos^3(2\theta)\fint \frac{\cos^3\frac{\eta}{2}\sin^3\eta}{\sin\frac{\eta}{2}}d\eta \\ &= \frac{21}{4}\sin^2(2\theta)\cos(2\theta) + \frac{5}{4}\cos^3(2\theta) \\ &= \frac{9}{4}\cos(2\theta) - \cos(6\theta),\end{aligned}$$

which proves the lemma. □

Lemma A.1.11.

$$\fint \frac{(\cos(2\theta)-\cos(2\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta = -\sin(4\theta).$$

Proof. Using the integration by parts, we compute

$$\fint \frac{(\cos(2\theta) - \cos(2\eta))^2 \sin(\theta - \eta)}{(2 - 2\cos(\theta - \eta))^2} d\eta = -2 \fint \frac{(\cos(2\theta) - \cos(2\eta)) \sin(2\eta)}{2 - 2\cos(\theta - \eta)} d\eta.$$

Therefore the result follows from (Equation A.1.34). $\square$

APPENDIX B

DERIVATIVES OF STREAM FUNCTION

B.1 Derivatives of the stream function

In this appendix, we will derive some formulae for zero-mean stream function by using Fourier series.

Lemma B.1.1. *For $\rho > 0$, let $h \in L^2(\partial B_\rho)$ such that $\int_{|y|=\rho} h(y) d\mathcal{H}^1(y) = 0$. Then it holds that for $x = (r\cos\theta, r\sin\theta)$,*

$$\frac{1}{2\pi}\int_{|y|=\rho} h(y)\log|x-y| d\mathcal{H}^1(y) = \begin{cases} -\sum_{n=-\infty n\neq 0}^{\infty} \frac{\rho}{2|n|}\hat{h}(\rho,n)\left(\frac{r}{\rho}\right)^{|n|} e^{in\theta} & \text{if } \rho \geq r, \\ -\sum_{n=-\infty n\neq 0}^{\infty} \frac{\rho}{2|n|}\hat{h}(\rho,n)\left(\frac{\rho}{r}\right)^{|n|} e^{in\theta} & \text{if } \rho < r, \end{cases}$$

where $\hat{h}(\rho,n) = \frac{1}{2\pi}\int_{\partial B} h(\rho y) e^{-iny} d\mathcal{H}^1(y)$.

Proof. By adapting the abuse of notation $h(y) = h(\rho,\eta)$, for $y = (\rho\cos\eta, \rho\sin\eta)$, we have that

$$\begin{aligned} \frac{1}{2\pi}\int_{|y|=\rho} h(y)\log|x-y| d\mathcal{H}^1(y) &= \frac{\rho}{4\pi}\int_{\mathbb{T}} h(\rho,\eta)\log|(r\cos\theta, r\sin\theta) - (\rho\cos\eta,\rho\sin\eta)|^2 d\eta \\ &= \frac{\rho}{4\pi}\int_{\mathbb{T}} h(\rho,\eta)\log(r^2+\rho^2-2r\rho\cos(\theta-\eta))d\eta. \end{aligned}$$

Using the Fourier expansion $h(\rho,\eta) := \sum_{n=-\infty,n\neq 0}^{\infty} \hat{h}(\rho,n)e^{in\eta}$ where $\hat{h}(\rho,n) := \frac{1}{2\pi}\int_{\mathbb{T}} h(\rho,\eta)e^{-in\eta}d\eta$, we have

$$\begin{aligned} &\frac{\rho}{4\pi}\int_{\mathbb{T}} h(\rho,\eta)\log(r^2+\rho^2-2r\rho\cos(\theta-\eta))d\eta \\ &\qquad = \sum_{n=-\infty,n\neq 0}^{\infty} \frac{\rho}{4\pi}\hat{h}(\rho,n)\underbrace{\int_{\mathbb{T}} e^{in\eta}\log(r^2+\rho^2-2r\rho\cos(\theta-\eta))d\eta}_{=:A_n(r,\rho,\theta)}, \end{aligned}$$

where we used $\hat{h}(\rho,0)=0$ since h has zero mean on ∂B_ρ. To compute A_n, we recall from [24, Lemma A.1] that for $0\le x\le 1$ and $\mathbb{Z}\ni n\neq 0$, it holds that

$$\int_{\mathbb{T}} e^{in\eta}\log(1+x^2-2x\cos(\theta-\eta))d\eta = -\frac{2\pi}{|n|}e^{in\theta}x^{|n|}. \tag{B.1.1}$$

Then it directly follows from (Equation B.1.1) that

$$A_n(r,\rho,\theta)=\begin{cases} -\frac{2\pi}{|n|}e^{in\theta}\left(\frac{r}{\rho}\right)^{|n|} & \text{if } \rho\ge r\\ -\frac{2\pi}{|n|}e^{in\theta}\left(\frac{\rho}{r}\right)^{|n|} & \text{if } \rho< r.\end{cases}$$

Plugging this into the above equation, the desired result follows immediately.

□

Lemma B.1.2. *For a bounded m-fold symmetric domain D in $\mathbb{R}^2$, let us consider a decomposition of $1_D*\mathcal{N}$,*

$$1_D*\mathcal{N}(r,\theta)=g*\mathcal{N}(r)+(1_D-g)*\mathcal{N}(r,\theta)=:\varphi^r(r)+\varphi_m(r,\theta),$$

where $g(r):=\frac{1}{2\pi r}\mathcal{H}^1\left(\partial B_r\cap D\right)$. Then,

$$\begin{aligned}\partial_r\varphi_m(r,\theta)&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n=1}^\infty\left(\frac{\rho}{r}\right)^{nm+1}\cos(nm\eta)\right)d\rho d\eta\\ &\quad-\frac{1}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n=1}^\infty\left(\frac{r}{\rho}\right)^{nm-1}\cos(nm\eta)\right)d\rho d\eta \qquad \text{(B.1.2)}\\ \partial_\theta\varphi_m(r,\theta)&=-\frac{r}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n=1}^\infty\left(\frac{\rho}{r}\right)^{nm+1}\sin(nm\eta)\right)d\rho d\eta\\ &\quad-\frac{r}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n=1}^\infty\left(\frac{r}{\rho}\right)^{nm-1}\sin(nm\eta)\right)d\rho d\eta, \qquad \text{(B.1.3)}\end{aligned}$$

where $h(\rho,\theta):=1_D(\rho(\cos\theta,\sin\theta))-g(\rho)$.

Proof. We compute that for $x := (r\cos\theta, r\sin\theta)$,

$$\begin{aligned}
\varphi_m(r,\theta) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} (1_D(y) - g(|y|))\log|x-y|dy \\
&= \int_0^\infty \left(\frac{1}{2\pi}\int_{|y|=\rho} \underbrace{(1_D(y) - g(|y|))}_{=:h(y)} \log|x-y| d\mathcal{H}^1(y) \right) d\rho.
\end{aligned}$$

By adapting the abuse of notation $h(y) = h(\rho,\eta)$ for $y = (\rho\cos\eta, \rho\sin\eta)$, we have $\int_{\mathbb{T}} h(\rho,\eta)d\eta = 0$ for all $\rho > 0$. Since D is m-fold symmetric, we also have that $\eta \mapsto h(\rho,\eta)$ is $\frac{2\pi}{m}$-periodic function for each fixed ρ. Therefore, it follows from Lemma B.1.1 that

$$\varphi_m(r,\theta) := -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{2|nm|}\left(\int_0^r \rho\hat{h}(\rho,nm)\left(\frac{\rho}{r}\right)^{|nm|} e^{inm\theta}d\rho + \int_r^\infty \rho\hat{h}(\rho,nm)\left(\frac{r}{\rho}\right)^{|nm|} e^{inm\theta}d\rho \right),$$

where $\hat{h}(\rho,nm) := \frac{1}{2\pi}\int_{\mathbb{T}} h(\rho,\eta)e^{-inm\eta}d\eta$. Therefore we have

$$\partial_r\varphi_m(r,\theta) = -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{2}\left(-\int_0^r \hat{h}(\rho,nm)\left(\frac{\rho}{r}\right)^{|nm|+1} e^{inm\theta}d\rho + \int_r^\infty \hat{h}(\rho,nm)\left(\frac{r}{\rho}\right)^{|nm|-1} e^{inm\theta}d\rho \right), \tag{B.1.4}$$

$$\partial_\theta\varphi_m(r,\theta) = -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{in}{2|n|}r\left(\int_0^r \hat{h}(\rho,nm)\left(\frac{\rho}{r}\right)^{|nm|+1} e^{inm\theta}d\rho + \int_r^\infty \hat{h}(\rho,nm)\left(\frac{r}{\rho}\right)^{|nm|-1} e^{inm\theta}d\rho \right). \tag{B.1.5}$$

To simplify the radial derivative, we use the definition of $\hat{h}$ and (Equation B.1.4) to obtain

$$\begin{aligned}
\partial_r\varphi_m(r,\theta) &= \frac{1}{4\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta)\left(\sum_{n\neq 0} \left(\frac{\rho}{r}\right)^{|nm|+1} e^{inm(\theta-\eta)} \right) d\rho d\eta \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta)\left(\sum_{n\neq 0} \left(\frac{r}{\rho}\right)^{|nm|-1} e^{inm(\theta-\eta)} \right) d\rho d\eta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n\neq 0}\left(\frac{\rho}{r}\right)^{|nm|+1} e^{-inm\eta}\right)d\rho d\eta \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n\neq 0}\left(\frac{r}{\rho}\right)^{|nm|-1} e^{-inm\eta}\right)d\rho d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^{nm+1} \cos(nm\eta)\right)d\rho d\eta \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^{nm-1} \cos(nm\eta)\right)d\rho d\eta,
\end{aligned}
$$

where we used the change of variables, $\eta \mapsto \eta + \theta$ to get the second inequality. This proves (Equation B.1.2). In the same way, we use (Equation B.1.5) and the change of variables to obtain

$$
\begin{aligned}
\partial_\theta \varphi_m(r,\theta) &= -\frac{r}{4\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta)\left(\sum_{n\neq 0}\frac{in}{|n|}\left(\frac{\rho}{r}\right)^{|nm|+1} e^{inm(\theta-\eta)}\right)d\rho d\eta \\
&\quad - \frac{r}{4\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta)\left(\sum_{n\neq 0}\frac{in}{|n|}\left(\frac{r}{\rho}\right)^{|nm|-1} e^{inm(\theta-\eta)}\right)d\rho d\eta \\
&= -\frac{r}{4\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n\neq 0}\frac{in}{|n|}\left(\frac{\rho}{r}\right)^{|nm|+1} e^{-inm\eta}\right)d\rho d\eta \\
&\quad - \frac{r}{4\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n\neq 0}\frac{in}{|n|}\left(\frac{r}{\rho}\right)^{|nm|-1} e^{-inm\eta}\right)d\rho d\eta \\
&= -\frac{r}{2\pi}\int_{\mathbb{T}}\int_0^r h(\rho,\eta+\theta)\left(\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^{nm+1} \sin(nm\eta)\right)d\rho d\eta \\
&\quad - \frac{r}{2\pi}\int_{\mathbb{T}}\int_r^\infty h(\rho,\eta+\theta)\left(\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^{nm-1} \sin(nm\eta)\right)d\rho d\eta,
\end{aligned}
$$

which proves (Equation B.1.3).

□

Lemma B.1.3. *For a patch D that satisfies the assumptions (a)-(c) in subsection subsection 6.2.2,* *it holds that for* $r \in (r_{min}, r_{max})$ *and* $\eta := u^{-1}$,

$$\partial_r \varphi_m(r,\theta) = \int_{r_{min}}^{r_{max}} f_1(\rho, r, \theta) d\rho$$
$$\partial_\theta \varphi_m(r,\theta) = \int_{r_{min}}^{r_{max}} f_2(\rho, r, \theta) d\rho,$$

where

$$f_1(\rho,r,\theta) = \begin{cases} \frac{1}{2\pi}\frac{\rho}{r}\left(\arctan\left(\frac{(\frac{r}{\rho})^m \sin(m(\theta-\eta(\rho)))}{1-\left(\frac{r}{\rho}\right)^m \cos(m(\theta-\eta(\rho)))}\right) - \arctan\left(\frac{(\frac{r}{\rho})^m \sin(m(\theta+\eta(\rho)))}{1-\left(\frac{r}{\rho}\right)^m \cos(m(\theta+\eta(\rho)))}\right)\right) & \text{if } \rho \geq r \\ \frac{1}{2\pi}\frac{\rho}{r}\left(\arctan\left(\frac{(\frac{\rho}{r})^m \sin(m(\theta+\eta(\rho)))}{1-\left(\frac{\rho}{r}\right)^m \cos(m(\theta+\eta(\rho)))}\right) - \arctan\left(\frac{(\frac{\rho}{r})^m \sin(m(\theta-\eta(\rho)))}{1-\left(\frac{\rho}{r}\right)^m \cos(m(\theta-\eta(\rho)))}\right)\right) & \text{if } \rho < r, \end{cases} \tag{B.1.6}$$

$$f_2(\rho,r,\theta) = \begin{cases} \frac{\rho}{4\pi}\log\left(1 + \frac{4(\frac{r}{\rho})^m \sin(m\eta(\rho))\sin(m\theta)}{1+(\frac{r}{\rho})^{2m} - 2(\frac{r}{\rho})^m \cos(m(\theta-\eta(\rho)))}\right) & \text{if } \rho \geq r \\ \frac{\rho}{4\pi}\log\left(1 + \frac{4(\frac{\rho}{r})^m \sin(m\eta(\rho))\sin(m\theta)}{1+(\frac{\rho}{r})^{2m} - 2(\frac{\rho}{r})^m \cos(m(\theta-\eta(\rho)))}\right) & \text{if } \rho < r. \end{cases} \tag{B.1.7}$$

Proof. The proof is based on Lemma B.1.2. Using m-fold symmetry and evenness of the patch, we will compute the series.

From Lemma B.1.2 and Fubini theorem, it follows that

$$\begin{aligned}\partial_r \varphi_m(r,\theta) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_0^r \left(\frac{\rho}{r}\right)^{nm+1}\left(\int_{\mathbb{T}} h(\rho, s+\theta)\cos(nms)ds\right)d\rho \\ -\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_r^{\infty} \left(\frac{r}{\rho}\right)^{nm-1}\left(\int_{\mathbb{T}} h(\rho, s+\theta)\cos(nms)ds\right)d\rho,\end{aligned} \tag{B.1.8}$$

and

$$\begin{aligned}\partial_\theta \varphi_m(r,\theta) = -\frac{r}{2\pi}\sum_{n=1}^{\infty}\int_0^r \left(\frac{\rho}{r}\right)^{nm+1}\left(\int_{\mathbb{T}} h(\rho, s+\theta)\sin(nms)ds\right)d\rho \\ -\frac{r}{2\pi}\sum_{n=1}^{\infty}\int_r^{\infty} \left(\frac{r}{\rho}\right)^{nm-1}\left(\int_{\mathbb{T}} h(\rho, s+\theta)\sin(nms)ds\right)d\rho,\end{aligned} \tag{B.1.9}$$

where $h(\rho,s) := 1_D(\rho\cos s, \rho\sin s) - g(\rho)$. Using the definition of $\eta = u^{-1}$, the following holds for $s \in [-\frac{\pi}{m}, \frac{\pi}{m}]$:

$$h(\rho,s) = \begin{cases} 1 - g(\rho) & \text{if } s \in (-\eta(\rho), \eta(\rho)), \\ -g(\rho) & \text{if } s \in [-\frac{\pi}{m}, \frac{\pi}{m}] \backslash (-\eta(\rho), \eta(\rho)). \end{cases}$$

Therefore m-fold symmetry of D yields that

$$\begin{aligned} \int_{\mathbb{T}} h(\rho, s+\theta)\cos(nms)ds &= m\int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} h(\rho,s)\cos(nm(s-\theta))ds \\ &= m\int_{-\eta(\rho)}^{\eta(\rho)} \cos(nm(s-\theta))ds \\ &= \frac{1}{n}\left(\sin(nm(\eta(\rho)-\theta)) + \sin(nm(\eta(\rho)+\theta))\right). \end{aligned} \tag{B.1.10}$$

Similarly, we have

$$\int_{\mathbb{T}} h(\rho, s+\theta)\sin(nms)ds = -\frac{1}{n}\left(\cos(nm(\eta(\rho)-\theta)) - \cos(nm(\eta(\rho)+\theta))\right) \tag{B.1.11}$$

Hence (Equation B.1.8) and (Equation B.1.10) yield that

$$\begin{aligned} &\partial_r\varphi_m(r,\theta) \\ &= \frac{1}{2\pi}\int_0^r \sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\frac{\rho}{r}\right)^{nm+1}\left(\sin(nm(\eta(\rho)-\theta)) + \sin(nm(\eta(\rho)+\theta))\right)\right)d\rho \\ &- \frac{1}{2\pi}\int_r^{\infty} \sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\frac{r}{\rho}\right)^{nm-1}\left(\sin(nm(\eta(\rho)-\theta)) + \sin(nm(\eta(\rho)+\theta))\right)\right)d\rho \\ &= \frac{1}{2\pi}\int_0^r \frac{\rho}{r}\left(\arctan\left(\frac{\left(\frac{\rho}{r}\right)^m\sin(m(\eta(\rho)-\theta))}{1-\left(\frac{\rho}{r}\right)^m\cos(m(\eta(\rho)-\theta))}\right) + \arctan\left(\frac{\left(\frac{\rho}{r}\right)^m\sin(m(\eta(\rho)+\theta))}{1-\left(\frac{\rho}{r}\right)^m\cos(m(\eta(\rho)+\theta))}\right)\right)d\rho \\ &- \frac{1}{2\pi}\int_r^{\infty}\frac{\rho}{r}\left(\arctan\left(\frac{\left(\frac{r}{\rho}\right)^m\sin(m(\eta(\rho)-\theta))}{1-\left(\frac{r}{\rho}\right)^m\cos(m(\eta(\rho)-\theta))}\right)\right. \end{aligned}$$

$$+ \arctan\left(\frac{\left(\frac{r}{\rho}\right)^m \sin(m(\eta(\rho)+\theta))}{1-\left(\frac{r}{\rho}\right)^m \cos(m(\eta(\rho)+\theta))}\right) d\rho,$$

where the last equality follows from (Equation B.2.2) in Lemma B.2.1. Since the integrands in the above integrals are zero if $\rho < r_{min}$ or $\rho > r_{max}$, we can replace 0 and ∞ in integration limits by r_{min} and r_{max}, respectively. This proves (Equation B.1.6). To prove (Equation B.1.7), we use (Equation B.1.9),(Equation B.1.11) and (Equation B.2.1) to obtain

$$\begin{aligned}
&\partial_\theta \varphi_m(r,\theta) \\
&\quad = \frac{r}{2\pi}\int_0^r \sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\frac{\rho}{r}\right)^{nm+1}(\cos(nm(\eta(\rho)-\theta))-\cos(nm(\eta(\rho)+\theta)))\right) d\rho \\
&\quad + \frac{r}{2\pi}\int_r^\infty \sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\frac{r}{\rho}\right)^{nm-1}(\cos(nm(\eta(\rho)-\theta))-\cos(nm(\eta(\rho)+\theta)))\right) d\rho \\
&\quad = \frac{r}{4\pi}\int_0^r \frac{\rho}{r}\log\left(\frac{1+\left(\frac{\rho}{r}\right)^{2m}-2\left(\frac{\rho}{r}\right)^m\cos(m(\eta(\rho)+\theta))}{1+\left(\frac{\rho}{r}\right)^{2m}-2\left(\frac{\rho}{r}\right)^m\cos(m(\eta(\rho)-\theta))}\right) d\rho \\
&\quad + \frac{r}{4\pi}\int_r^\infty \frac{\rho}{r}\log\left(\frac{1+\left(\frac{r}{\rho}\right)^{2m}-2\left(\frac{r}{\rho}\right)^m\cos(m(\eta(\rho)+\theta))}{1+\left(\frac{r}{\rho}\right)^{2m}-2\left(\frac{r}{\rho}\right)^m\cos(m(\eta(\rho)-\theta))}\right) d\rho \\
&\quad = \frac{1}{4\pi}\int_0^r \rho\log\left(1+\frac{2\left(\frac{\rho}{r}\right)^m(\cos(m(\eta(\rho)-\theta))-\cos(m(\eta(\rho)+\theta)))}{1+\left(\frac{\rho}{r}\right)^{2m}-2\left(\frac{\rho}{r}\right)^m\cos(m(\eta(\rho)-\theta))}\right) d\rho \\
&\quad + \frac{1}{4\pi}\int_r^\infty \rho\log\left(1+\frac{2\left(\frac{r}{\rho}\right)^m(\cos(m(\eta(\rho)-\theta))-\cos(m(\eta(\rho)+\theta)))}{1+\left(\frac{r}{\rho}\right)^{2m}-2\left(\frac{r}{\rho}\right)^m\cos(m(\eta(\rho)-\theta))}\right) d\rho \\
&\quad = \frac{1}{4\pi}\int_0^r \rho\log\left(1+\frac{4(\frac{\rho}{r})^m\sin(m\eta(\rho))\sin(m\theta)}{1+(\frac{\rho}{r})^{2m}-2(\frac{\rho}{r})^m\cos(m(\theta-\eta(\rho)))}\right) d\rho \\
&\quad + \frac{1}{4\pi}\int_r^\infty \rho\log\left(1+\frac{4(\frac{r}{\rho})^m\sin(m\eta(\rho))\sin(m\theta)}{1+(\frac{r}{\rho})^{2m}-2(\frac{r}{\rho})^m\cos(m(\theta-\eta(\rho)))}\right) d\rho,
\end{aligned}$$

where the last equality follows from $\cos(x-y)-\cos(x+y) = 2\sin x \sin y$. This proves (Equation B.1.7). $\square$

B.2 Helpful lemmas

Lemma B.2.1. *For* $|x| < 1$ *and* $y \in (-\pi, \pi)$*, it holds that*

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n \cos(ny) = -\frac{1}{2} \log(1 + x^2 - 2x \cos y), \tag{B.2.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n \sin(ny) = \arctan\left(\frac{x \sin y}{1 - x \cos y}\right). \tag{B.2.2}$$

Consequently, we have

$$\sum_{n=1}^{\infty} x^n \cos(ny) = \frac{x(\cos y - x)}{(1-x)^2 + 2x(1 - \cos y)}, \tag{B.2.3}$$

$$\sum_{n=1}^{\infty} x^n \sin(ny) = \frac{x \sin y}{(1-x)^2 + 2x(1 - \cos y)}. \tag{B.2.4}$$

Proof. Let $f(x, y) := \sum_{n=1}^{\infty} \frac{1}{n} x^n e^{iny}$. Then we compute

$$\begin{aligned}
\partial_x f(x, y) &= \frac{1}{x} \sum_{n=1}^{\infty} \left(xe^{iy}\right)^n = \frac{e^{iy}}{1 - xe^{iy}} = \frac{(\cos y - x) + i \sin y}{(1 - x\cos y)^2 + x^2 \sin^2 y} \\
&= \partial_x \left(-\frac{1}{2} \log(1 + x^2 - 2x \cos y) + i \arctan\left(\frac{x \sin y}{1 - x \cos y}\right)\right).
\end{aligned}$$

Since $f(0, y) = 0$, we have $f(x, y) = -\frac{1}{2} \log(1 + x^2 - 2x \cos y) + i \arctan\left(\frac{x \sin y}{1 - x \cos y}\right)$. Equating the real and imaginary parts separately, we can obtain (Equation B.2.1) and (Equation B.2.2). By differentiating (Equation B.2.1) and (Equation B.2.2) and multiplying by x, one can easily obtain (Equation B.2.3) and (Equation B.2.4). □

Lemma B.2.2. *For* $m \geq 3$ *and* $a, b \in (0, 1)$*, it holds that*

$$\int_0^1 x^{-1-\frac{2}{m}} \left(\arctan\left(\frac{ax}{1-x}\right) - \arctan\left(\frac{ax}{1-bx}\right)\right) dx \lesssim 1 - b$$

Proof. By the change of variables, $bx \mapsto x$, we have

$$\begin{aligned}\int_0^1 x^{-1-\frac{2}{m}} \arctan\left(\frac{ax}{1-bx}\right) dx &= \int_0^b b^{\frac{2}{m}} x^{-1-\frac{2}{m}} \arctan\left(\frac{\frac{ax}{b}}{1-x}\right) dx \\ &\geq \int_0^b bx^{-1-\frac{2}{m}} \arctan\left(\frac{ax}{1-x}\right) dx,\end{aligned}$$

where we used $b^{\frac{2}{m}} \geq b$ for $0 < b < 1$ and $m \geq 3$. Therefore it follows that

$$\begin{aligned}\int_0^1 x^{-1-\frac{2}{m}} \left(\arctan\left(\frac{ax}{1-x}\right) - \arctan\left(\frac{ax}{1-bx}\right)\right) dx &\leq \int_b^1 x^{-1-\frac{2}{m}} \arctan\left(\frac{ax}{1-x}\right) dx \\ &\quad + (1-b) \int_0^b x^{-1-\frac{2}{m}} \arctan\left(\frac{ax}{1-x}\right) dx \\ &\lesssim 1-b,\end{aligned}$$

which proves the desired inequality. $\square$

Lemma B.2.3. *For $m \geq 3$ and $a \in (0,1)$, it holds that*

$$\int_0^1 x^{-1-\frac{2}{m}} \log\left(1+\frac{ax}{(1-x)^2}\right) dx \lesssim \sqrt{a}. \tag{B.2.5}$$

Proof. If $x < \frac{1}{2}$, then $\log(1+\frac{ax}{(1-x)^2}) \lesssim ax$. Therefore,

$$\begin{aligned}\int_0^1 x^{-1-\frac{2}{m}} \log\left(1+\frac{ax}{(1-x)^2}\right) dx &\lesssim \int_0^{\frac{1}{2}} ax^{-\frac{2}{m}} dx + \int_{\frac{1}{2}}^1 \log\left(1+\frac{ax}{(1-x)^2}\right) dx \\ &\lesssim a + \int_{\frac{1}{2}}^1 \log\left(1+\frac{a}{(1-x)^2}\right) dx,\end{aligned} \tag{B.2.6}$$

where we used $m \geq 3$ to estimate the first integral and $x \in (\frac{1}{2}, 1)$ for the second integral. To estimate the second integral, we compute

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} \log\left(1+\frac{a}{(1-x)^2}\right) dx &= \int_{\frac{1}{2}}^{1} \frac{d}{dx}(x-1)\log\left(1+\frac{a}{(1-x)^2}\right) dx \\
&= \frac{1}{2}\log(1+4a) + \int_{\frac{1}{2}}^{1} \frac{2a}{(1-x)^2+a} dx \\
&\lesssim a + \int_{\frac{1}{2}}^{1-\sqrt{a}} \frac{a}{(1-x)^2} dx + \int_{1-\sqrt{a}}^{1} 1 dx \\
&\lesssim \sqrt{a}. \qquad \text{(B.2.7)}
\end{aligned}$$

Thus the desired result follows from (Equation B.2.6) and (Equation B.2.7). □

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