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THEOREMS ON THE REALIZABILITY OF PHYSICAL SYSTEMS HAVING STURM-LIOUVILLE SECULAR POLYNOMIALS

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CHAPTER I

INTRODUCTION

In order that the investigation described in the following pages may appear in its proper perspective, it should be viewed as a part of a more ambitious undertaking whose goals are as follows:

(1) the solution of certain infinite systems of first- and second-order linear ordinary differential equations with constant coefficients;

(2) the investigation of physical phenomena of which such systems are thought to be appropriate mathematical models; and

(3) the use of the solutions of the infinite systems as approximations to the more cumbersome solutions of related finite systems involving many equations (perhaps several hundred thousand).

A method which has proved successful in solving some technically significant infinite systems of ordinary differential equations consists of truncating the infinite system after K equations, solving the resulting finite system, and then allowing K to increase without bound. In solving the finite system, it is necessary to know the zeros of the corresponding secular polynomial. For various physical systems involving only linear nearest-neighbor coupling, the secular polynomials corresponding to increasing values of K are elements of a sequence $\{\varphi_n(x)\}_{n=0}^{\infty}$ of polynomials generated by a three-term recursive relation of the form

$$\varphi_0(x) = 1 , \tag{1}$$

$$\varphi_1(x) = A_0 x + B_0 ,$$

$$\varphi_{n+1}(x) = (A_n x + B_n)\varphi_n(x) - C_n\varphi_{n-1}(x), \quad n \geq 1 ,$$

in which $A_0 \neq 0$ and $A_n C_n \neq 0$ $(n \geq 1)$. Here both x and the coefficients A_i, B_i, C_i may be complex. If the polynomials generated by (1) are Sturm-Liouville polynomials associated with a second-order ordinary differential equation, much is known about their zeros. In order to take advantage of this knowledge in prosecuting the larger project of which the present study is a part, it seemed desirable to investigate Sturm-Liouville polynomials which are recursively generated and then to begin the study of infinite systems of ordinary differential equations by first considering systems which, when truncated, have such polynomials as their secular polynomials.

The results of the study of recursively generated Sturm-Liouville polynomials are contained in the doctoral dissertation of J. W. Jayne [2], of which the present paper makes extensive use. It is assumed that the reader is familiar with Jayne's work; and wherever possible, the notation used here agrees with his.

Jayne showed

(1) that if an infinite sequence of Sturm-Liouville polynomials associated with a linear ordinary differential equation of the second order is generated by the recursive relation (1), then the polynomials of the sequence are of one of four types (Hermite,

extended generalized Laguerre, generalized Bessel, or generalized Jacobi); and

(2) that for physical systems of which the dissipationless mass-spring combination is the prototype, the only infinite sequences of second-order Sturm-Liouville secular polynomials which can be generated by successively increasing the size of the system are of Laguerre or Jacobi type.

He also raised the following question [2, p. 28]:

For an arbitrary preassigned positive integer N, is it possible to construct successively larger spring-mass systems of order up to and including N for each of which the secular polynomial is of Hermite or Bessel type?

Though the question arose initially simply as a matter of intellectual curiosity, from the standpoint of physical applications an answer to it was desirable for the following reason: The study of infinite systems of differential equations which, when truncated, lead to Hermite or Bessel secular polynomials would hold little promise of applicability unless physical systems occur in which the secular polynomial is a Hermite or Bessel polynomial of high degree. Thus, an answer to the question would serve as a guide for future effort.

In the present study this question is answered affirmatively. Various additional questions which arose as the study progressed are outlined in the next few paragraphs.

As a prelude to that outline, the reader is asked to bear in mind that the essential difference between the theory and applications developed by Jayne [2] and those to be presented here is that Jayne dealt with infinite sequences of Sturm-Liouville polynomials generated by the recursive formula (1) while the present study is concerned with finite sequences of Sturm-Liouville polynomials generated by (1). In the sequel such finite sequences are referred to as proper three-term recursive finite Sturm-Liouville polynomial sequences. The adjective proper is prefixed to emphasize the requirement that no A_i nor C_i in (1) may be zero.

In Chapter II theorems and definitions necessary for the later work are stated. The material presented is much like a synopsis of Chapter II of [2]; the only essential difference is that indices have a finite rather than an infinite range. Proofs of the theorems are omitted since they are, except for appropriate changes in the range of the indices, identical to those in [2].

Chapter III answers affirmatively the following question: If a finite sequence of Sturm-Liouville polynomials is recursively generated by (1), can the polynomials be of types other than Hermite, extended generalized Laguerre, extended generalized Bessel, or generalized Jacobi? Since the answer is somewhat detailed, the reader is referred to the last few pages of the chapter. Necessary and sufficient conditions that a proper three-term recursive finite Sturm-Liouville polynomial sequence be -- aside from a possible linear change of variable and nonzero multiplicative factors -- a sequence of Hermite, extended generalized Laguerre, extended generalized Bessel, or generalized Jacobi polynomials are also stated and proved. A complete mathematical classification of all proper three-term recursive finite Sturm-Liouville polynomial sequences is made, but the chapter is not concerned with physical realizability.

Chapter IV is addressed to the simplification of certain theorems appearing in Chapter III. More specifically, a relevant difference equation is solved in order to facilitate application of those theorems. It is pointed out how the theorems may be used to decide whether a given spring-mass system is of Hermite, Laguerre, Bessel, or Jacobi type; and groundwork is laid for Chapter V, in which it is shown that spring-mass systems having N^{th} degree secular polynomials of Hermite, Laguerre, Bessel, or Jacobi type are physically realizable (N being preassigned).

In Chapter VI numerical examples are given, and comments are made on the somewhat subjective question of how "realistic" the physically realizable systems described in Chapter V actually are. Finally, there is a brief discussion of the possibility of devising an efficient numerical procedure for constructing "realistic" physical systems of the types known from Chapter V to be realizable.

Chapter VII is a brief résumé of all major results of preceding chapters. It appears for the convenience of the reader.

CHAPTER II

PRELIMINARY THEOREMS AND DEFINITIONS

Let N be an integer greater than or equal to three and let $\left\{\varphi_n(x)\right\}_{n=o}^{N}$ be a finite polynomial sequence generated as in (1). Definitions and theorems pertaining to this sequence which will be needed in later work are stated in this chapter. The most important theorem gives a necessary and sufficient condition that $\left\{\varphi_n(x)\right\}_{n=o}^{N}$ form a finite Sturm-Liouville system.

<u>Theorem 2.1</u>. Suppose that for $n = 0,1,2,\ldots,N$, $\varphi_n(x)$ is a solution of

$$a_o(x)\,\frac{d^2y}{dx^2} + a_1(x)\,\frac{dy}{dx} + [a_2(x) + \mu_n]y = 0\;,$$

where μ_n is a parameter depending on n but not on x. Then $a_o(x)$, $a_1(x)$, and $a_2(x) + \mu_n$ must be of the form $\gamma x^2 + \beta x + \alpha$, $\varepsilon x + \delta$, and λ_n, respectively, where λ_n is a parameter depending on n but not on x, and α, β, γ, δ, and ε are constants.

<u>Theorem 2.2</u>. If $\varphi_n(x)$ is a solution of

$$(\gamma x^2 + \beta x + \alpha)\,\frac{d^2y}{dx^2} + (\varepsilon x + \delta)\,\frac{dy}{dx} + \lambda_n y \;=\; 0 \tag{2}$$

for $n = 0,1,2,\ldots,N$, then

$$\text{(i)}\quad \gamma x^2 + \beta x + \alpha = (\lambda_1 - \frac{\lambda_2}{2})\,x^2 + [\lambda_1(A_oB_1 + 3A_1B_o)$$

$$- \lambda_2(A_oB_1 + A_1B_o)]\,x/2A_oA_1 + [\lambda_1(B_oB_1 + \frac{A_1B_o^2}{A_o})$$

$$- \lambda_2(B_oB_1 - C_1)]/2A_oA_1\ ,$$

$$\text{(ii)}\quad \varepsilon x + \delta = -\lambda_1(x + \frac{B_o}{A_o})\ , \quad \text{and}$$

$$\text{(iii)}\quad \lambda_n = n(n-1)(\frac{\lambda_2}{2}) - n(n-2)\lambda_1, \quad n = 0,1,2,\ldots,N\ .$$

In the sequel, whenever reference is made to the differential equation (2), it will be understood that α, β, γ, δ, ε, and λ_n have the values prescribed by Theorem 2.2.

<u>Definition 2.1</u>. Let $b_n = \frac{B_n}{A_n}$, $n = 0,1,2,\ldots, N-1$, and $c_n = \frac{C_n}{A_nA_{n-1}}$, $n = 1,2,\ldots,N-1$. Define $g_1(n)$ and $g_2(n)$ as follows for $n = 1,2,\ldots,N-2$.

$$g_1(n) = [(n+1)b_{n+1} + (-n+1)b_n - b_1 - b_o]\lambda_2$$

$$+ [(-2n-1)b_{n+1} + (2n-3)b_n + b_1 + 3b_o]\lambda_1$$

$$g_2(n) = [(n+1)b_nb_{n+1} - nb_n^2 - b_ob_1 + c_1 - (2n+1)c_{n+1}$$

$$+ (2n-3)c_n]\lambda_2 + [(-2n-1)b_nb_{n+1} + (2n-1)b_n^2$$

$$+ b_ob_1 + b_o^2 + 4n\,c_{n+1} + (-4n+8)c_n]\,\lambda_1$$

<u>Theorem 2.3</u>. The polynomial $\varphi_n(x)$ is a solution of (2) for

$n = 0,1,2,\ldots,N$ if and only if $g_1(n) = g_2(n) = 0$ for $n = 1,2,\ldots,N-2$.

To insure that equation (2) is nontrivial, at least one of λ_1 or λ_2 must be nonzero. By use of the argument advanced in [2, pp. 9-10], the choices $\lambda_1 = 1$, $\lambda_2 = [(b_1 - b_o)^2 + 4(c_1 + c_2)]/3c_2$, and

$$\Delta \equiv (2b_2 - b_1 - b_o)[(b_1 - b_o)^2 + 4(c_1 + c_2)] + 9c_2(b_o - b_2) = 0$$

are justified. With these values of λ_1 and λ_2 inserted in the expressions for $g_1(n)$ and $g_2(n)$, Theorem 2.3 provides a necessary and sufficient condition that $\varphi_n(x)$ be a solution of the <u>nontrivial</u> differential equation (2) for $n = 0,1,2,\ldots,N$. In all that follows, the values above for λ_1 and λ_2 and the condition $\Delta = 0$ will be assumed.

Another fact will prove to be of importance in future analysis. With $\lambda_o = 0$ and $\lambda_1 = 1$ it follows from (iii) of Theorem 2.3 that $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,j=0,1,2,\ldots,N)$ if and only if

$$\lambda_2 \neq \frac{2(i+j-2)}{i+j-1} \quad (i \neq j,\ i+j \geq 2;\ i,j = 0,1,2,\ldots,N).$$

Consequently, Theorem 2.3 and the results following it can be summarized as a single theorem.

<u>Theorem 2.4</u>. $\varphi_n(x)$, $n = 0,1,2,\ldots,N$, is a solution of the nontrivial differential equation (2), in which $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,j=0,1,2,\ldots,N)$, if and only if

(i) $\Delta = 0$;

(ii) $[(b_o - b_1)^2 + 4(c_1 + c_2)]/3c_2$ is different from $\frac{2(m-2)}{m-1}$ for $m = 2,3,\ldots,2N-1$;

(iii) $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, where $g_1(n)$ and $g_2(n)$ are evaluated with $\lambda_1 = 1$ and $\lambda_2 = [(b_o - b_1)^2 + 4(c_1 + c_2)]/3c_2$.

As an important closing remark to this chapter, it should be noted that the equations $g_1(n) = 0 = g_2(n)$, $n = 1,2,\ldots,N-2$, can be regarded as a system of linear first-order nonhomogeneous difference equations:

$$[(n+1)\lambda_2 - (2n+1)]b_{n+1} - [(n-1)\lambda_2 - (2n-3)]b_n \qquad (3)$$

$$= (b_o + b_1)\lambda_2 - b_1 - 3b_o \;;$$

$$[(2n+1)\lambda_2 - 4n]c_{n+1} - [(2n-3)\lambda_2 - (4n-8)]\,c_n$$

$$= [(n+1)b_n b_{n+1} - nb_n^2 - b_o b_1 + c_1]\lambda_2$$

$$+ (-2n-1)b_n b_{n+1} + (2n-1)b_n^2 + b_o b_1 + b_o^2 ,$$

$$n = 1,2,\ldots, N-2 .$$

Again, it is understood that $\Delta = 0$ and that λ_1 and λ_2 have the values prescribed in part (iii) of Theorem 2.4. The requirement that λ_2 be different from $\frac{2(m-2)}{m-1}$ for $m = 2,3,\ldots,2N-1$ implies that the difference equations (3) have no singular points; hence, the unique solution to $g_1(n) = 0$ (for given b_o, b_1, and λ_2) is

$$b_n = \frac{[(b_o+b_1)(\lambda_2-2) + (b_1-b_o)][(n-1)(\lambda_2-2) + 2]n - 2b_o(\lambda_2-2) + 2b_o}{2[n(\lambda_2-2)+1][(n-1)(\lambda_2-2)+1]} , \qquad (4)$$

$$n = 2,3,\ldots,N-1 ,$$

as deduced in [2, p. 12]. The equation $g_2(n) = 0$ for the unknown c_n is considerably less tractable because of the more complicated nonhomogeneous term. However, this equation has been solved in the special cases $\lambda_2 = 2$ and $b_o = b_1$ [2, pp. 12-13]. The solution in the general case will be given in Chapter IV.

CHAPTER III

PROPER THREE-TERM RECURSIVE FINITE STURM-LIOUVILLE POLYNOMIAL SEQUENCES

The objects of Chapter III are as follows:

(1) to give necessary and sufficient conditions that a proper three-term recursive finite Sturm-Liouville polynomial sequence be of Hermite, extended generalized Laguerre, extended generalized Bessel, or generalized Jacobi type; and

(2) to give a complete <u>mathematical</u> classification of all proper three-term recursive finite Sturm-Liouville polynomial sequences.

Characterization of Finite Hermite, Laguerre, Bessel and Jacobi Polynomial Sequences

Definition 3.1. Let n be a nonnegative integer. The Hermite polynomial of degree n, denoted by H_n, is defined by

$$H_n(t) = \sum_{k=o}^{[\frac{n}{2}]} \frac{(-1)^k \, n!}{k!(n-2k)!} (2t)^{n-2k} ,$$

where $[\frac{n}{2}]$ means the greatest integer less than or equal to $\frac{n}{2}$.

Lemma 3.1. Let n be a nonnegative integer. Then $H_n(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2ny = 0 ; \tag{5}$$

and if $\psi_n(t)$ is a polynomial of degree n in t which also satisfies (5), $\psi_n(t) = D_n H_n(t)$, where D_n is a term depending possibly on n but not on t.

Both of the results above are well-known [4]; so they are not re-proved here.

The following two theorems provide a necessary and sufficient condition that a finite sequence $\{\varphi_n(x)\}_{n=0}^{N}$ generated as in (1) be -- apart from a linear change of variable and multiplicative factors independent of x -- a finite sequence of Hermite polynomials in the variable x. Here, as in all that follows, N is a fixed but arbitrary integer greater than or equal to three, and D_n denotes a nonzero term independent of x.

Theorem 3.1. If for $n = 0,1,2,\ldots,N$ the coefficients of $\varphi_n(x)$ satisfy ① $b_1 = b_0$, ② $\lambda_2 = 2$, and ③ $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, then $\varphi_n(x) = D_n H_n\left(\frac{1}{\sqrt{2c_1}}\,x + \frac{b_0}{\sqrt{2c_1}}\right)$, $n = 0,1,2,\ldots,N$, where the principal square root is taken.

Proof. If conditions ① - ③ hold, Theorem 2.3 implies that $\varphi_n(x)$ satisfies a differential equation of the form (2) in which $\lambda_1 = 1$, $\lambda_2 = 2$, and $b_1 = b_0$. Consequently, $\varphi_n(x)$ is a solution of

$$c_1 \frac{d^2y}{dx^2} - (x + b_0)\frac{dy}{dx} + ny = 0\,, \qquad (6)$$

$n = 0,1,2,\ldots,N$. Under the change of variable $t = \frac{1}{\sqrt{2c_1}}(x + b_0)$, where again the principal square root is taken, (6) becomes (5); hence, $\psi_n(t) \equiv \varphi_n(\sqrt{2c_1}\,t - b_0)$ is a polynomial of degree n in t which satisfies (5), $n = 0,1,2,\ldots,N$. By Lemma 3.1, $\psi_n(t) = D_n H_n(t)$; that is, $\varphi_n(x) = D_n H_n\left(\frac{x}{\sqrt{2c_1}} + \frac{b_o}{\sqrt{2c_1}}\right)$, $n = 0,1,2,\ldots,N$.

Theorem 3.2. If there exist a nonzero constant μ and a constant ν such that $\varphi_n(x) = D_n H_n(\mu x + \nu)$, $n = 0,1,2,\ldots,N$, then the coefficients of $\varphi_n(x)$ satisfy conditions ① - ③ of Theorem 3.1.

Proof. If such constants μ and ν exist, it follows from Lemma 3.1 that $\varphi_n(x)$ is a solution of

$$\frac{1}{2\mu^2}\frac{d^2y}{dx^2} + \left(-x - \frac{\nu}{\mu}\right)\frac{dy}{dx} + ny = 0 , \tag{7}$$

$n = 0,1,2,\ldots,N$. Equation (7) is a nontrivial differential equation of the form (2) in which $\lambda_1 = 1$ and $\lambda_2 = 2$. Hence, by Theorem 2.3, $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$; and, from part (i) of Theorem 2.2,

$$0 = \beta = [(A_oB_1 + 3A_1B_o) - 2(A_oB_1 + A_1B_o)]/2A_oA_1 = \tfrac{1}{2}(b_o - b_1).$$

Clearly Theorems 3.1 and 3.2 can be combined into an "if and only if theorem" as follows:

> For $n = 0,1,2,\ldots,N$ the coefficients of $\varphi_n(x)$ satisfy conditions ① - ③ if and only if there exist a nonzero constant μ and a constant ν such that $\varphi_n(x) = D_n H_n(\mu x + \nu)$, $n = 0,1,2,\ldots,N$.

However, for clarity and ease in applying these theorems, it seems better to state the results separately. For if ①, ②, and ③ hold, Theorem 3.1 specifies $\varphi_n(x)$, $n = 0,1,2,\ldots,N$ to within a multiplicative factor independent of x; and Theorem 3.2 -- in contrapositive form -- states that if any of ①, ②, or ③ fail to hold, it is impossible for $\left\{\varphi_n(x)\right\}_{n=0}^{N}$ to be -- within a linear change of variable -- $\left\{D_n H_n(x)\right\}_{n=0}^{N}$. It should be noted that if conditions ① and ② are true and if ③ holds only for $n = 1,2,\ldots,k$, $1 \leq k \leq N-3$, then the conclusion of Theorem 3.1 is valid for $n = 0,1,2,\ldots,k+2$.

The remaining part of this section is devoted to theorems which concern the existence of finite Laguerre, Bessel, and Jacobi polynomial sequences. These theorems are much like Theorems 3.1 and 3.2 both in formulation and method of proof. For each of the three types of polynomial sequences mentioned, there are two applicable theorems which can be combined, if desired, into an "if and only if theorem." However, in order to facilitate their application and improve the clarity of the exposition, they are stated separately. This format is consistent with that followed for the Hermite case.

Definition 3.2. Let n be a nonnegative integer and $\underline{a}$ any complex number. The extended generalized Laguerre polynomial of degree n, denoted by L_n^a, is defined by

$$L_n^a(t) = \sum_{k=0}^{n} \binom{a+n}{n-k} \frac{(-1)^k t^k}{k!},$$

where $\binom{a+n}{n-k}$ is a generalized binomial coefficient defined by

$$\binom{a+n}{n-k} = \frac{\prod_{j=1}^{n-k} (a+k+j)}{(n-k)!} \quad \text{if } 0 \leq k \leq n-1$$

and $\binom{a+n}{0} = 1$.

The definition of L_n^a given here agrees with that of the classical generalized Laguerre polynomial of degree n whenever $a > -1$. Because no such requirement is made of a in this development, the adjective "extended" has been added in the definition.

Lemma 3.2. Let n be a nonnegative integer and a any complex number. Then $L_n^a(t)$ satisfies the differential equation

$$t\,\frac{d^2y}{dt^2} + (a + 1 - t)\,\frac{dy}{dt} + ny = 0\;; \tag{8}$$

and if $\psi_n(t)$ is a polynomial of degree n in t which also satisfies (8), $\psi_n(t) = D_n L_n^a(t)$.

The conclusions of Lemma 3.2 are well-known if $a > -1$ [4]. For the general case, direct substitution shows that $L_n^a(t)$ satisfies (8). If $\psi_n(t) = \sum_{k=o}^{\infty} a_k t^k$ is substituted in (8) (with $a_k = 0$ for $k \geq n + 1$), if a_k is expressed in terms of a_o, and if the resulting expression for a_k is compared with the coefficient of t^k in the definition of $L_n^a(t)$, the second conclusion of Lemma 3.1 follows.

Theorem 3.3. If for $n = 0,1,2,\ldots,N$ the coefficients of $\varphi_n(x)$ satisfy ① $b_1 \neq b_o$, ② $\lambda_2 = 2$, and ③ $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, then $\varphi_n(x) = D_n L_n^a(\mu x + \nu)$, where $a = \dfrac{4c_1}{(b_o - b_1)^2} - 1$, $\mu = \dfrac{2}{b_o - b_1}$, and $\nu = \dfrac{2}{b_o - b_1}\left(b_o + \dfrac{2c_1}{b_o - b_1}\right)$, for $n = 0,1,2,\ldots,N$.

Proof. If conditions ① - ③ are satisfied, it follows from Theorem 2.3 that $\varphi_n(x)$ is a solution of (2) in which $\lambda_1 = 1$, $\lambda_2 = 2$, and $b_1 \neq b_o$. Hence, for $n = 0,1,2,\ldots,N$ $\varphi_n(x)$ is a solution of

$$\left[\frac{(b_o - b_1)x}{2} + \frac{(b_o^2 - b_o b_1 + 2c_1)}{2}\right]\frac{d^2y}{dx^2} - (x + b_o)\frac{dy}{dx} + ny = 0\,, \tag{9}$$

which transforms to (8) -- with $a = \dfrac{4c_1}{(b_o - b_1)^2} - 1$ -- under the change of variable $t = \dfrac{2}{b_o - b_1}\left(x + b_o + \dfrac{2c_1}{b_o - b_1}\right)$. By Lemma 3.2

$$\varphi_n\left(\frac{b_o - b_1}{2}t - b_o - \frac{2c_1}{b_o - b_1}\right) \equiv \psi_n(t) = D_n L_n^a(t)\ ;$$

or, in terms of x, $\varphi_n(x) = D_n L_n^a(\mu x + \nu)$, $n = 0,1,2,\ldots,N$, where μ, ν, and a are as specified in the theorem.

Theorem 3.4. If there exist a nonzero constant μ and constants ν and a such that $\varphi_n(x) = D_n L_n^a(\mu x + \nu)$, $n = 0,1,2,\ldots,N$, then the coefficients of $\varphi_n(x)$ satisfy conditions ① - ③ of Theorem 3.3.

Proof. If such constants μ, ν, and a exist, it follows from Lemma 3.2 that $\varphi_n(x)$ is a solution of

$$\frac{(\mu x + \nu)}{\mu^2}\frac{d^2y}{dx^2} + \frac{(a + 1 - \mu x - \nu)}{\mu}\frac{dy}{dx} + ny = 0\ , \tag{10}$$

$n = 0,1,2,\ldots,N$. Equation (10) is of the form (2) in which $\lambda_1 = 1$ and $\lambda_2 = 2$. Consequently, by Theorem 2.3, $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$; and from part (i) of Theorem 2.2, $0 \neq \frac{1}{\mu} = \beta = \frac{b_o - b_1}{2}$.

Definition 3.3. Let n be a positive integer, b a nonzero complex number, and a a complex number not an integer in $[-2(n-1), -(n-1)]$. The extended generalized Bessel polynomial of degree n, denoted by $B_n^{(a,b)}$, is defined by

$$B_n^{(a,b)}(t) = \sum_{k=o}^{n} \binom{n}{k} (n + k + a - 2)^{(k)} (\tfrac{t}{b})^k ,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $(n + k + a - 2)^{(k)} = \prod_{j=1}^{k} (n + k + a - j - 1)$ if $1 \leq k \leq n$, and $(n + k + a - 2)^{(0)} = 1$. For nonzero complex b, the generalized Bessel polynomial of degree zero is defined as $B_o^{(a,b)}(t) \equiv 1$.

The definition above agrees with that of the generalized Bessel polynomial of degree n provided $\underline{a}$ is other than a nonpositive integer [3]. Because a less stringent requirement is placed on a in the present work, the adjective "extended" has been added. The condition imposed on a for $n \geq 1$ is the minimal one which guarantees that the expression for $B_n^{(a,b)}$ is a polynomial of degree exactly n.

Lemma 3.3. Let n be a nonnegative integer. Then $B_n^{(a,b)}(t)$ is a solution of the differential equation

$$t^2 \frac{d^2y}{dt^2} + (at + b) \frac{dy}{dt} - n(n - 1 + a)y = 0 ; \tag{11}$$

and if $\psi_n(t)$ is a polynomial of degree n in t that also satisfies (11), in which a and b fulfill the requirements of Definition 3.3, then $\psi_n(t) = D_n B_n^{(a,b)}(t)$.

The results of Lemma 3.3 are indicated in [3] for the case in which a is not a nonpositive integer and $b \neq 0$. For the general case in which it is assumed only that a is not an integer in $[-2(n-1),-(n-1)]$ $(n \geq 1)$ and $b \neq 0$, use of the techniques outlined in the remarks following Lemma 3.2 suffices to prove the lemma. When

$\psi_n(t) = \sum_{k=o}^{\infty} a_k t^k$ is substituted into (11), the recurrence formula for the coefficients a_k is

$$ba_{k+1} = \frac{(n - k)(n + k + a - 1)}{k + 1} a_k , \tag{12}$$

$k = 0,1,2,\ldots,n-1$; and $a_k = 0$ for $k \geq n+1$. Equation (12) is of importance in the next section of this chapter.

Theorem 3.5. If for $n = 0,1,2,\ldots,N$ the coefficients of $\varphi_n(x)$ satisfy ① $\lambda_2 \neq 2$, ② $(\lambda_2 - 1)^2(b_o - b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1 = 0$, ③ $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, and ④ $\lambda_2 \neq \frac{2(m-2)}{m-1}$ for $m = 2,3,\ldots,2N-1$, then $\varphi_n(x) = D_n B_n^{(a,2)}(\mu x + \nu)$ for $n = 0,1,2,\ldots,N$, where $a = \frac{2}{\lambda_2 - 2}$, $\mu = \frac{2(\lambda_2 - 2)^2}{(b_o - b_1)(\lambda_2 - 1)}$, and

$$\nu = \frac{(\lambda_2 - 2)[(b_o + b_1)\lambda_2 - (3b_o + b_1)]}{(\lambda_2 - 1)(b_o - b_1)} .$$

Proof. Whenever conditions ① - ③ hold, Theorem 2.3 implies that $\varphi_n(x)$ satisfies (2) with $\lambda_1 = 1$, $\lambda_2 \neq 2$, and

$$0 = D \equiv \beta^2 - 4\alpha\gamma = \frac{(\lambda_2 - 1)^2(b_o - b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1}{4} . \tag{13}$$

Therefore, for $n = 0,1,2,\ldots,N$ $\varphi_n(x)$ is a solution of

$$\frac{(2 - \lambda_2)}{2}(x - x^*)^2 \frac{d^2y}{dx^2} - (x + b_o)\frac{dy}{dx} + [\frac{n(n-1)}{2}\lambda_2 - n(n-2)]y = 0, \tag{14}$$

where $x^* = \dfrac{\lambda_2(b_o + b_1) - (3b_o + b_1)}{2(2 - \lambda_2)}$. If the change of variable $t = x - x^*$ is made, equation (14) transforms to an equation of the form (11) in which $a = \dfrac{2}{\lambda_2 - 2}$ and $b = \dfrac{(b_o - b_1)(\lambda_2 - 1)}{(\lambda_2 - 2)^2}$. Clearly $b \neq 0$; for if it were, ② would become $\lambda_2(\lambda_2 - 2)c_1 = 0$, which is a contradiction because $\lambda_2 \neq 0$, $\lambda_2 \neq 2$, and $c_1 \neq 0$ (note that this is the first place in the proof that ④ has been utilized, an observation which will prove useful in the next section of the chapter). Also, a is not an integer in $[-2(N-1), 0]$; for if $\dfrac{2}{\lambda_2 - 2} = a = -j$, where j is an integer satisfying $1 \leq j \leq 2N - 1$, then $\lambda_2 = \dfrac{2(j-1)}{j}$, which contradicts ④. By Lemma 3.3 $\varphi_n(t + x^*) \equiv \psi_n(t) = D_nB_n^{(a,b)}(t) = D_nB_n^{(a,2)}(\frac{2t}{b})$. Hence, for $n = 0,1,2,\ldots,N$ $\varphi_n(x) = D_nB_n^{(a,2)}[\frac{2(x-x^*)}{b}]$. The conclusion of the theorem follows immediately.

Theorem 3.6. If there exist a nonzero constant μ and constants v, a, and b such that $\varphi_n(x) = D_nB_n^{(a,b)}(\mu x + v)$, $n = 0,1,2,\ldots,N$, then the coefficients of $\varphi_n(x)$ satisfy conditions ① - ④ of Theorem 3.5.

Proof. The existence of such constants, Lemma 3.3, and the fact that a cannot be zero imply that $\varphi_n(x)$ satisfies

$$-\frac{1}{a}\left(x + \frac{v}{\mu}\right)^2 \frac{d^2y}{dx^2} + \left(-x - \frac{v}{\mu} - \frac{b}{a\mu}\right)\frac{dy}{dx} + n\left(\frac{n-1}{a} + 1\right)y = 0, \quad (15)$$

$n = 0,1,2,\ldots,N$. Equation (15) is a nontrivial differential equation of the form (2) in which $\lambda_1 = 1$, $\lambda_2 = 2(\frac{1}{a} + 1) \neq 2$, and the discriminant

$D = 0 = \dfrac{(\lambda_2 - 1)^2(b_o - b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1}{4}$. By Theorem 2.3 $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$; and because $\dfrac{2}{\lambda_2 - 2} = a$ is (from Definition 3.3 applied to $n = 1,2,\ldots,N$) not an integer in $[-2(N-1), 0]$, it follows that $\lambda_2 \neq \dfrac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$.

Definition 3.4. Let n be a positive integer and let a and b be any two complex constants such that $n + a + b + k \neq 0$ for each integer k between one and n inclusive -- so that $a + b$ is not an integer in $[-2n, -(n+1)]$. The generalized Jacobi polynomial of degree n, denoted by $P_n^{(a,b)}$, is defined by

$$P_n^{(a,b)}(t) = \frac{\prod_{j=1}^{n}(a+j)}{n!} + \frac{1}{n!}\sum_{k=1}^{n-1}\left[\binom{n}{k}\prod_{j=1}^{k}(n+a+b+j)\prod_{m=1}^{n-k}(a+k+m)\left(\frac{t-1}{2}\right)^k\right]$$

$$+ \frac{\prod_{j=1}^{n}(n+a+b+j)}{n!}\left(\frac{t-1}{2}\right)^n,$$

where it is understood that a summand is omitted when $n = 1$. For any two complex numbers a and b, the generalized Jacobi polynomial of degree zero is defined as $P_o^{(a,b)}(t) \equiv 1$.

Definition 3.4 agrees with that of the classical Jacobi polynomial of degree n provided $a > -1$ and $b > -1$. The adjective "generalized" has been added to indicate a relaxation of these restrictions. The condition imposed on $a + b$ for $n \geq 1$ is the minimal one which guarantees that $P_n^{(a,b)}$ is of degree n.

Lemma 3.4. Let n be a nonnegative integer. Then $P_n^{(a,b)}(t)$ is a solution of the differential equation

$$(1-t^2)\frac{d^2y}{dt^2}+[b-a-(a+b+2)t]\frac{dy}{dt}+n(n+a+b+1)y=0\,; \qquad (16)$$

and if $\psi_n(t)$ is a polynomial of degree n in t that also satisfies (16), in which a and b fulfill the requirements of Definition 3.4, then $\psi_n(t) = D_n P_n^{(a,b)}(t)$.

The assertions of Lemma 3.4 are verified in [4] for the case in which $a > -1$ and $b > -1$. For the case in which it is assumed only that $a + b$ is not an integer in $[-2n, -(n+1)]$ $(n \geq 1)$, the remarks made after the statement of Lemma 3.2 again apply. In this connection, use of the solution $\psi_n(t) = \sum_{k=o}^{\infty} a_k(t-1)^k$, where $a_k = 0$ for $k \geq n+1$, rather than $\psi_n(t) = \sum_{k=o}^{\infty} a_k t^k$ simplifies the verification.

<u>Theorem 3.7</u>. If for $n = 0,1,2,\ldots,N$ the coefficients of $\varphi_n(x)$ satisfy ① $\lambda_2 \neq 2$, ② $(\lambda_2-1)^2(b_o-b_1)^2 + 4\lambda_2(\lambda_2-2)c_1 \neq 0$, ③ $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, and ④ $\lambda_2 \neq \frac{2(m-2)}{m-1}$ for $m = 1,2,\ldots,2N-1$, then for $n = 0,1,2,\ldots,N$ $\varphi_n(x) = D_n P_n^{(a,b)}[1-2(\mu x+\nu)]$, where $a = \frac{2(x_1+b_o)}{(\lambda_2-2)(x_1-x_2)} - 1$, $b = \frac{2(x_2+b_o)}{(\lambda_2-2)(x_2-x_1)} - 1$, $\mu = \frac{1}{x_2-x_1}$, $\nu = -\frac{x_1}{x_2-x_1}$, and

$$x_2 = \frac{\lambda_2(b_o+b_1) - (b_1+3b_o)}{2(2-\lambda_2)} + \frac{1}{2}\sqrt{\frac{(\lambda_2-1)^2(b_o-b_1)^2+4\lambda_2(\lambda_2-2)c_1}{(2-\lambda_2)^2}}$$

and $x_1 = \frac{\lambda_2(b_o+b_1) - (b_1+3b_o)}{2-\lambda_2} - x_2$ are the two distinct zeros of the coefficient of $\frac{d^2y}{dx^2}$ in (2) (the square root used being the principal one). Moreover, neither a nor b is an integer in $[-(N-1), -1]$.

Proof. If conditions ① - ③ hold, Theorem 2.3 implies that $\varphi_n(x)$ is a solution of (2) in which $\lambda_1 = 1$, $\lambda_2 \neq 2$, and $D \neq 0$. Consequently, $\varphi_n(x)$ satisfies

$$(1 - \frac{\lambda_2}{2})(x - x_1)(x - x_2)\frac{d^2y}{dx^2} - (x+b_o)\frac{dy}{dx} + [\frac{n(n-1)\lambda_2}{2} - n(n-2)]y = 0, \tag{17}$$

$n = 0,1,2,\ldots,N$, where the unequal quantities x_1 and x_2 are as specified in the statement of the theorem. If the change of variable $t = 1 - \frac{2}{x_2 - x_1}(x - x_1)$ is made, (17) transforms to an equation of the form (16) in which $a = \frac{2(x_1 + b_o)}{(\lambda_2 - 2)(x_1 - x_2)} - 1$ and $b = \frac{2(x_2 + b_o)}{(\lambda_2 - 2)(x_2 - x_1)} - 1$. Condition ④, used in this proof for the first time, forces $a + b = \frac{2}{\lambda_2 - 2} - 2$ to be other than an integer in $[-2N, -2]$; therefore, by Lemma 3.4,

$$\varphi_n\left[\frac{(x_2 - x_1)(1 - t)}{2} + x_1\right] \equiv \psi_n(t) = D_n P_n^{(a,b)}(t)$$

so that

$$\varphi_n(x) = D_n P_n^{(a,b)}[1 - 2\left(\frac{x}{x_2 - x_1} - \frac{x_1}{x_2 - x_1}\right)], \quad n=0,1,2,\ldots,N.$$

The condition concerning a and b is now easy to prove. From the general recursion formula for $P_{n+1}^{(a,b)}$ in terms of $P_n^{(a,b)}$ and $P_{n-1}^{(a,b)}$ [4], which is valid for all values of a and b permitted in

Definition 3.4, one need only note that the coefficient of $P_{n-1}^{(a,b)}$ includes multiplicative factors $n+a$ and $n+b$. The assertion is now evident since $\{\varphi_n(x)\}_{n=o}^{N} \equiv \{D_n P_n^{(a,b)}[1 - 2(\mu x + \nu)]\}_{n=o}^{N}$ is to be a proper three-term recursive finite sequence. The last sentence in Theorem 3.7 is of consequence in the next section of this chapter.

Theorem 3.8. If there exist a nonzero constant μ and constants ν, a, and b such that $\varphi_n(x) = D_n P_n^{(a,b)}[1-2(\mu x+\nu)]$, $n = 0,1,2,\ldots,N$, then the coefficients of $\varphi_n(x)$ satisfy conditions ① - ④ of Theorem 3.7.

Proof. The existence of such constants, Lemma 3.4, and the fact that $a + b + 2$ cannot be zero imply that $\varphi_n(x)$ is a solution of

$$\frac{(\mu x + \nu)(1 - \mu x - \nu)}{\mu^2(a+b+2)} \frac{d^2y}{dx^2} + \left[\frac{a+1}{\mu(a+b+2)} - \frac{(\mu x + \nu)}{\mu}\right]\frac{dy}{dx} + \frac{n(n+a+b+1)}{a+b+2}\, y = 0, \tag{18}$$

$n = 0,1,2,\ldots,N$. Equation (18) is a nontrivial differential equation of the form (2) in which $\lambda_1 = 1$, $\lambda_2 = \frac{2(a+b+3)}{a+b+2} \neq 2$, and $D \neq 0$. Therefore, $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, by Theorem 2.3; and, since $\frac{2}{\lambda_2 - 2} - 2 = a + b$ is not an integer in $[-2N, -2]$, $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$.

Classification of the Polynomials Generated

In this section all types of proper three-term recursive finite sequences of polynomials whose coefficients satisfy the hypothesis of Theorem 2.3 are identified. Diagrams summarize the results at the end of the discussion.

For the case in which $\lambda_2 = 2$, the classification has already been completed. Specifically, if $b_1 = b_o$ Theorem 3.1 settles the issue; and if $b_1 \neq b_o$ Theorem 3.3 applies. Suppose that $\lambda_2 \neq 2$ and $\lambda_2 \neq \frac{2(m-2)}{m-1}$ for $m = 2,3,\ldots,2N-1$. If the discriminant D of the coefficient of $\frac{d^2y}{dx^2}$ in (2) is zero, Theorem 3.5 provides the answer; and if $D \neq 0$, Theorem 3.7 suffices. Consequently, the following two cases remain to be considered: (a) $\lambda_2 \neq 2$, $\lambda_2 = \frac{2(m_o - 2)}{m_o - 1}$ for some integer m_o in $[2, 2N-1]$, $D = 0$; (b) $\lambda_2 \neq 2$, $\lambda_2 = \frac{2(m_o - 2)}{m_o - 1}$, $D \neq 0$.

From the proof of Theorem 3.5, whenever the first case occurs $\varphi_n(t + x^*) \equiv \psi_n(t)$ is a polynomial of degree n in t which satisfies (11) in which $a = \frac{2}{\lambda_2 - 2}$ and $b = \frac{(b_o - b_1)(\lambda_2 - 1)}{(\lambda_2 - 2)^2}$, $n = 0,1,2,\ldots,N$. Suppose first that $b \neq 0$. Then substitution of $\psi_n = \sum_{k=o}^{\infty} a_k t^k$, where $a_k = 0$ for $k \geq n + 1$, into (11) leads to recursion formula (12) in which $b \neq 0$. It is clear from (12) that $\psi_n(t)$ is of degree $n \geq 1$ if and only if $\frac{2}{\lambda_2 - 2} = a \neq -2(n-1)$; and, because this inequality must be true for $n = 1,2,\ldots,N$, $\frac{2}{\lambda_2 - 2}$ cannot be an integer in $[-2(N-1), 0]$. This result implies that $\lambda_2 \neq \frac{2(m-2)}{m-1}$ for $m = 2,3,\ldots,2N-1$ -- a contradiction.

If $0 = b = \frac{(b_o - b_1)(\lambda_2 - 1)}{(\lambda_2 - 2)^2}$, the condition $D = 0$ reduces to $\lambda_2(\lambda_2 - 2)\, c_1 = 0$ so that $\lambda_2 = 0$; hence, $b = 0$ implies $\lambda_2 = 0$ and $b_o = b_1$. Furthermore, $a = \frac{2}{\lambda_2 - 2} = -1$ so that (12) reduces to

$$a_k(k - n)[k - (2 - n)] = 0 \tag{19}$$

for $0 \leq k \leq n$. Since $\psi_n(t)$ is to be of degree n for $n = 0,1,2,\ldots,N$, (19) implies that $\psi_n(t) = d_n t^n$, $n = 0,1,3,\ldots,N$, and $\psi_2(t) = d_2 t^2 + e_2$, where each $d_j \neq 0$ and e_2 is arbitrary. Suppose that $N \geq 4$. Then, since $\varphi_n(x)$ must be generated as in (1) and $\{\varphi_n(x)\}_{n=o}^{N} \equiv \{\psi_n(x-x^*)\}_{n=o}^{N}$,

$$\begin{aligned} d_4(x - x^*)^4 &= (A_3 x + B_3) d_3 (x - x^*)^3 - C_3[d_2(x - x^*)^2 + e_2] \\ &= [A_3(x - x^*) + (B_3 + A_3 x^*)] d_3 (x - x^*)^3 \\ &\qquad - C_3 d_2 (x - x^*)^2 - C_3 e_2 \,. \end{aligned} \tag{20}$$

But (20) implies that $C_3 = 0$ -- a contradiction; so N must be equal to three. If $N = 3$ and $e_2 = 0$, a contradiction arises because $\varphi_2(x)$ is not generated from $\varphi_o(x)$ and $\varphi_1(x)$ as prescribed in (1) with $A_1 C_1 \neq 0$. If $e_2 \neq 0$, the finite sequence $\{\varphi_n(x)\}_{n=o}^{3}$ is easily seen to be a proper three-term recursive finite polynomial sequence. Equations (3) with $\lambda_1 = 1$, $\lambda_2 = 0$, $b_1 = b_o$, and $N = 3$ are equivalent to $b_2 = b_o$ and $c_2 = -c_1$. It follows from (1) that $\varphi_n(x) = D_n(x + b_o)^n$, $n = 0,1,3$, and $\varphi_2(x) = D_2[(x + b_o)^2 - c_1]$, where $D_o = 1$. Thus, the discussion of the first case is complete.

Before undertaking the analysis of the remaining case, it is helpful to verify the following lemma.

<u>Lemma 3.5</u>. Let n be a positive integer, and let a and b be any two complex constants such that $n + a + b + k = 0$ for some integer k satisfying $1 \leq k \leq n$. If $\psi_n(t)$ is a polynomial of degree n in t which satisfies (16), then a must be an integer lying in $[-n, -1]$ such

that $1 \leq k \leq -a \leq n$. Moreover, when all of the above conditions on a and b are fulfilled, $r(t) \equiv (1-t)^{-a} P_{n+a}^{(-a,b)}(t)$ is a polynomial of degree n in t and $s(t) \equiv P_{k-1}^{(a,b)}(t)$ is a polynomial of degree $k-1$ in t.

Proof. The recursion formula occurring when $\psi_n(t) = \sum_{m=o}^{\infty} a_m (t-1)^m$ is substituted into (16) is

$$(m+a)a_m = \frac{[n-(m-1)](n+a+b+m)a_{m-1}}{2m} \qquad (21)$$

for $1 \leq m \leq n$, and $a_m = 0$ for $m \geq n+1$. If a is not an integer in $[-n, -1]$, it follows from (21) that $a_k = a_{k+1} = \ldots = a_n = 0$ so that $\psi_n(t)$ is not of degree n. If a is an integer in $[-n, -1]$, say $a = -i$ where $i+1 \leq k \leq n$, (21) shows that $a_o = a_1 = \ldots = a_{i-1} = 0$, a_i is arbitrary, $a_{i+1}, a_{i+2}, \ldots, a_{k-1}$ are expressible in terms of a_i (provided $i+2 \leq k \leq n$), and $a_k = a_{k+1} = \ldots = a_n = 0$. Again, $\psi_n(t)$ is not of degree n. Finally, if $a = -i$ is an integer in $[-n, -1]$ and $1 \leq k \leq i \leq n$, (21) implies that $a_1, a_2, \ldots, a_{k-1}$ are expressible in terms of a_o (provided $k \geq 2$), $a_k = a_{k+1} = \ldots = a_{i-1} = 0$ (provided $i \geq k+1$), a_i is arbitrary, and $a_{i+1}, a_{i+2}, \ldots, a_n$ are expressible in terms of a_i (provided $i \leq n-1$). In this case

$$\psi_n(t) = \sum_{m=o}^{k-1} a_m (t-1)^m + \sum_{m=i=-a}^{n} a_m (t-1)^m , \qquad (22)$$

where a_o is arbitrary, a_i is nonzero, and the remaining a_m -- if any -- are expressed in terms of a_o or a_i by means of recursion formula (21).

If $a = -n$ the assertion for r is trivial. If $1 \leq -a \leq n-1$, $n + a - a + b + j = n - n - a - k + j = (-a - k) + j \geq j \neq 0$ for each integer j such that $1 \leq j \leq n + a$. By Definition 3.4 and the remark concerning $a + b$ in the paragraph following that definition, $P_{n+a}^{(-a,b)}(t)$ is a polynomial of degree $n + a$ in t. The assertion concerning r is then immediate. Similarly, if $k = 1$ the assertion for s is trivial; and if $2 \leq k \leq -a \leq n$, $k - 1 + a + b + j = k - 1 + a - a - n - k + j = -(n + 1) + j \neq 0$ for each integer j satisfying $1 \leq j \leq k - 1$. Therefore, $s(t)$ is a polynomial of degree $k - 1$ in t.

It is interesting to note in the special circumstances of Lemma 3.5 that the expression for $P_n^{(a,b)}(t)$ reduces to the zero function. Thus, in some instances there is a polynomial solution to (16) of degree n in t which is not $P_n^{(a,b)}(t)$. This peculiar polynomial will later be identified as a linear combination of two functions -- one a Jacobi polynomial and the other a product of a Jacobi polynomial and a power of $1-t$.

The necessary mathematical machinery is now available for study of the second case. As in the proof of Theorem 3.7, the conditions $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, $\lambda_2 \neq 2$, $D \neq 0$, and $\lambda_2 = \frac{2(m_o - 2)}{m_o - 1}$ for some integer m_o in $[2, 2N - 1]$ imply that

$$\varphi_n\left[\frac{(x_2 - x_1)(1 - t)}{2} + x_1\right] \equiv \psi_n(t)$$

is a polynomial of degree n in t which satisfies (16) in which $a = \frac{2(x_1 + b_o)}{(\lambda_2 - 2)(x_1 - x_2)} - 1$ and $b = \frac{2(x_2 + b_o)}{(\lambda_2 - 2)(x_2 - x_1)} - 1$. From this starting point all but two sets of circumstances can be shown not to exist.

First, it is noted from the discussion in [2, pp. 18 - 23] that $\lambda_2 = \frac{2(m_o - 2)}{m_o - 1}$ implies $c_{m_o+1} = 0$. Hence, in order to guarantee $c_n \neq 0$, $n = 1,2,3,\ldots,N-1$, m_o must satisfy $N - 1 \leq m_o \leq 2N - 1$. Next, $\lambda_2 = \frac{2(m_o - 2)}{m_o - 1}$ implies the existence of a first integer n_o, $1 \leq n_o \leq N$, and a corresponding integer k_o, $1 \leq k_o \leq n_o \leq N$, such that $n_o + a + b + k_o = 0$. For if not, it follows as in the proof of Theorem 3.7 that $\varphi_n(x) = D_n P_n^{(a,b)}[1 - 2(\frac{x}{x_2 - x_1} - \frac{x_1}{x_2 - x_1})]$, $n = 0,1,2,\ldots,N$; consequently, by Theorem 3.8 $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$ -- a contradiction. In fact, the integer n_o must be strictly greater than one; for if $n_o = 1$, $k_o = 1$ so that $0 = a + b + 2 = \frac{2}{\lambda_2 - 2}$.

By Lemma 3.5 a must be an integer in $[-n_o, -1]$ and must satisfy $1 \leq k_o \leq -a \leq n_o$. Moreover, since $\psi_n(t)$ is a polynomial of degree n which satisfies (16) for $n = 0,1,2,\ldots,n_o - 1$, Lemma 3.4 implies $\psi_n(t) = D_n P_n^{(a,b)}(t)$, $n = 0,1,2,\ldots,n_o-1$; that is, $\varphi_n(x) = D_n P_n^{(a,b)}\left[1 - 2\left(\frac{x}{x_2 - x_1} - \frac{x_1}{x_2 - x_1}\right)\right]$, $n = 0,1,2,\ldots,n_o - 1$. By replacing N by $n_o - 1$ and assuming that $n_o \geq 4$, it follows from the last statement of Theorem 3.7 that a cannot be an integer in $[-(n_o - 2), -1]$. The subcases $n_o = 2$ and $n_o = 3$ are taken up in subsequent paragraphs.

The discussion of the preceding paragraph proves that either $n_o = -a$ or $n_o = 1 - a$ when $n_o \geq 4$. Thus, by the definition of n_o, $n_o = -a$; and consequently, from $n_o + a + b + k_o = 0$, $k_o = -b$ so that b is an integer in $[-n_o, -1]$. In fact, if $a + b = \frac{2}{\lambda_2 - 2} - 2 = -(m_o + 1)$ is even, $n_o + a + b + k_o = 0$ implies $k_o = n_o = -(\frac{a+b}{2})$; and if

$a + b$ is odd, $n_o + a + b + k_o = 0$ implies $k_o = n_o - 1 = \frac{-(a+b)+1}{2} - 1$. Since a is an integer in $[-n_o, -1]$ and $1 \leq k_o \leq -a = n_o \leq n_o$, (22) with $n = n_o$, $a = -n_o$, and $k = k_o$ yields

$$\psi_{n_o}(t) = \sum_{m=o}^{k_o-1} a_m(t-1)^m + a_{n_o}(t-1)^{n_o}, \tag{23}$$

where $a_o = \psi_{n_o}(1) = \varphi_{n_o}(x_1)$, $a_{n_o} \neq 0$, and the remaining a_m -- if any -- are determined via (21) in which n is replaced by n_o, a by $-n_o$, and b by $-k_o$.

Clearly $a_{n_o}(t-1)^{n_o} = E_{n_o}(1-t)^{n_o} P_o^{(n_o,-k_o)}(t)$, where E_{n_o} is a nonzero term independent of t. Also, a routine computation utilizing recursion formula (21) with $n = n_o$, $a = -n_o$, $b = -k_o$ and a comparison of like terms yield $\sum_{m=o}^{k_o-1} a_m(t-1)^m = F_{n_o} P_{k_o-1}^{(-n_o,-k_o)}(t)$, where F_{n_o} is a term independent of t which is nonzero if and only if $a_o \neq 0$. Therefore, (23) becomes $\psi_{n_o}(t) = E_{n_o}(1-t)^{n_o} P_o^{(n_o,-k_o)}(t) + F_{n_o} P_{k_o-1}^{(-n_o,-k_o)}(t)$.

If $n_o = N$, the nature of the sequence $\{\varphi_n(x)\}_{n=o}^{N}$ is determined; if not, the case $n = n_o + 1$ must be considered. In this situation $(n_o + 1) + a + b + (k_o - 1) = 0$. If $k_o = 1$, $\psi_n(t) = D_n P_n^{(-n_o,-k_o)}(t)$, $n = n_o + 1, n_o + 2, \ldots, N$, as guaranteed by Lemma 3.4. If $k_o \geq 2$, $(n_o + 1) + a + b + (k_o - 1) = 0$, $a = -n_o$ is an integer in $[-(n_o+1), -1]$, and $1 \leq k_o - 1 \leq -a = n_o \leq n_o + 1$. Hence, Lemma 3.5 with $n = n_o + 1$ and $k = k_o - 1$ applies so that (22) becomes

$$\psi_{n_o+1}(t) = \sum_{m=o}^{k_o-2} a_m(t-1)^m + \sum_{m=n_o}^{n_o+1} a_m(t-1)^m , \qquad (24)$$

where $a_o = \psi_{n_o+1}(1) = \varphi_{n_o+1}(x_1)$, $a_{n_o} \neq 0$, and the remaining a_m are determined from (21) in which n is replaced by $n_o + 1$, a by $-n_o$, and b by $-k_o$. With the aid of recursion formula (21), one can show -- as in the preceding case when n equaled n_o and k equaled k_o -- that (24) yields $\psi_{n_o+1}(t) = E_{n_o+1}(1-t)^{n_o} P_1^{(n_o,-k_o)}(t) + F_{n_o+1}P_{k_o-2}^{(-n_o,-k_o)}(t)$, where E_{n_o+1} is a nonzero term independent of t and F_{n_o+1} is a term independent of t which is nonzero if and only if $a_o \neq 0$.

The process of reasoning indicated above for values $n = n_o$, $k = k_o$ and $n = n_o+1$, $k = k_o+1$ can be repeated verbatim for values of n and k increasing and decreasing respectively in steps of unity until either $n_o + j > N$ or $k_o - j \leq 0$ for some positive integer j. After this point is reached, the remaining polynomials generated in the finite sequence $\{\psi_n(t)\}_{n=o}^{N}$, if any, are guaranteed by Lemma 3.4 to be $D_nP_n^{(-n_o,-k_o)}(t)$. For the general case in which $\psi_n(t)$ is different from $D_nP_n^{(-n_o,-k_o)}(t)$,

$$\psi_{n_o+m}(t) = E_{n_o+m}(1-t)^{n_o} P_m^{(n_o,-k_o)}(t) + F_{n_o+m}P_{k_o-(m+1)}^{(-n_o,-k_o)}(t), \quad (25)$$

where E_{n_o+m} is a nonzero term independent of t, F_{n_o+m} is a term independent of t which is nonzero if and only if $a_o = \psi_{n_o+m}(1) = \varphi_{n_o+m}(x_1) \neq 0$, and integer m satisfies $0 \leq m \leq j = \min(N - n_o, k_o-1) \leq N - 4$. As before, (25) is verified in a routine fashion by using (21) in which $n = n_o + m$, $a = -n_o$, $b = -k_o$ to determine the coefficients of ψ_{n_o+m} in (22) with $k = k_o - m$ and $a = -n_o$ and then comparing like terms in (25). That the Jacobi polynomials displayed in (25) are of degree indicated by their subscripts is a consequence of Lemma 3.5. The nature of $\{\psi_n(t)\}_{n=o}^{N}$ has now been specified; therefore

$$\{\varphi_n(x)\}_{n=o}^{N} \equiv \left\{\psi_n\left[1 - 2\left(\frac{x}{x_2 - x_1} - \frac{x_1}{x_2 - x_1}\right)\right]\right\}_{n=o}^{N}$$

is also determined. This completes the analysis of the second case <u>except</u> when $n_o = 2$ or $n_o = 3$.

If $n_o = 2$, then $2 + a + b + k_o = 0$ for $1 \leq k_o \leq 2 \leq N$ so that either $k_o = 1$ or $k_o = 2$. Moreover, by Lemma 3.5, a must be an integer in $[-2, -1]$ and satisfy $1 \leq k_o \leq -a \leq n_o = 2 \leq N$. If $k_o = 1$, either $a = -1$ or $a = -2$. Suppose $a = -1$. Then, by (21) and (22) with $n = n_o = 2$, $k = k_o = 1$, and $a = -1$,

$$\psi_2(t) = a_o + a_1(t-1)[1 + \frac{(t-1)}{4}], \tag{26}$$

where $a_o = \psi_2(1) = \varphi_2(x_1)$ and $a_1 \neq 0$. A simple comparison shows that $\psi_2(t) = E_2(1-t)P_1^{(1,-2)}(t) + F_2P_o^{(-1,-2)}(t)$, where E_2 is a nonzero term independent of t and F_2 is a term independent of t which is nonzero if and only if $a_o \neq 0$. Also, since $-3 = -(n_o + k_o) = a + b = \frac{2}{\lambda_2 - 2} - 2 = -(m_o + 1)$, $m_o = 2$ so that $\lambda_2 = 0$; and, since $N - 1 \leq m_o \leq 2N - 1$ is to be satisfied, $N = 3$. A similar analysis can be performed for the case $n_o = 2$, $k_o = 1$, and $(a,b) = (-2, -1)$.

In addition to the two sets of circumstances listed above, the following can arise: $n_o = 2$, $k_o = 2$, $(a, b) = (-2, -2)$; $n_o = 3$, $k_o = 1$, $(a, b) = (-1, -3)$ or $(-2, -2)$ or $(-3, -1)$; $n_o = 3$, $k_o = 2$, $(a,b) = (-2, -3)$ or $(-3, -2)$; and $n_o = 3$, $k_o = 3$, $(a,b) = (-3, -3)$. For each possibility a set of permissible values of N can be deduced by means of the relations $-(n_o + k_o) = a + b = \frac{2}{\lambda_2 - 2} - 2 = -(m_o + 1)$ and $N - 1 \leq m_o \leq 2N - 1$. In general the following representations are found:

$$\psi_{n_o}(t) = E_{n_o}(1 - t)^{-a} P_{n_o+a}^{(-a,b)}(t) + F_{n_o} P_{k_o-1}^{(a,b)}(t) ,$$

where E_{n_o} is a nonzero term independent of t and F_{n_o} is a term independent of t which is nonzero if and only if $a_o = \psi_{n_o}(1) = \varphi_{n_o}(x_1) \neq 0$;

$$\psi_{n_o+1}(t) = E_{n_o+1}(1 - t)^{-a} P_{n_o+a+1}^{(-a,b)}(t) + F_{n_o+1} P_{k_o-2}^{(a,b)}(t)$$

provided $k_o \geq 2$ and $N \geq n_o + 1$; and

$$\psi_{n_o+2}(t) = E_{n_o+2}(1-t)^{-a} P_{n_o+a+2}^{(-a,b)}(t) + F_{n_o+2} P_{k_o-3}^{(a,b)}(t)$$

provided $N \geq n_o + 2$ and $k_o = 3$. In the preceding expressions, the Jacobi polynomials displayed are of degree indicated by their subscripts -- by Lemma 3.5. The remaining polynomials in the finite sequence $\{\psi_n(t)\}_{n=o}^{N}$ satisfy $\psi_n(t) = D_n P_n^{(a,b)}(t)$. The structure of $\{\varphi_n(x)\}_{n=o}^{N}$ for the exceptional values $n_o = 2$ and $n_o = 3$ is thus known.

The results of the classification above are represented schematically in the following figures. The constants μ, ν, a, b, n_o, and k_o appearing

$\lambda_2 = 2$

- $b_1 = b_o$
  - $\varphi_n(x) = D_n H_n(\mu x + \nu)$
- $b_1 \neq b_o$
  - $\varphi_n(x) = D_n L_n^a(\mu x + \nu)$

$\lambda_2 \neq 2$ and $\lambda_2 \neq \dfrac{2(m_o - 2)}{m_o - 1}$

- $D = 0$
  - $\varphi_n(x) = D_n B_n^{(a,b)}(\mu x + \nu)$
- $D \neq 0$
  - $\varphi_n(x) = D_n P_n^{(a,b)}[1 - 2(\mu x + \nu)]$

$\lambda_2 \neq 2, \quad \lambda_2 = \dfrac{2(m_o - 2)}{m_o - 1}, \quad D = 0$

- $\lambda_2 = 0$ and $N = 3$

  $$\varphi_o(x) = 1$$
  $$\varphi_1(x) = D_1(x + b_o)$$
  $$\varphi_2(x) = D_2[(x + b_o)^2 - c_1]$$
  $$\varphi_3(x) = D_3(x + b_o)^3$$

- Either $\lambda_2 \neq 0$ or $N \geq 4$
  - No Such Case

Figure 1. Classification of the Polynomials Generated by the Recursive Relation (1).

$$\lambda_2 \neq 2, \quad \lambda_2 = \frac{2(m_o - 2)}{m_o - 1}, \quad D \neq 0$$

- $2 \leq m_o \leq N - 2$
  - No Such Case
- $N - 1 \leq m_o \leq 2N - 1$
  - a not an integer in $[-n_o, -1]$
    - No Such Case
  - a an integer in $[-n_o, -1]$
    - $n_o \geq 4$
    - $n_o = 2$ or $n_o = 3$
      - As indicated on page 32

$$\varphi_n(x) = D_n P_n^{(-n_o,-k_o)}[1 - 2(\mu x+\nu)] \quad (n=0,1,2,\ldots,n_o-1)$$

$$\varphi_{n_o}(x) = E_{n_o}[2(\mu x+\nu)]^{n_o} P_o^{(n_o,-k_o)}[1-2(\mu x+\nu)] + F_{n_o} P_{k_o-1}^{(-n_o,-k_o)}[1-2(\mu x+\nu)]$$

$$\vdots$$

$$\varphi_{n_o+m}(x) = E_{n_o+m}[2(\mu x+\nu)]^{n_o} P_m^{(n_o,-k_o)}[1-2(\mu x+\nu)] + F_{n_o+m} P_{k_o-(m+1)}^{(-n_o,-k_o)}[1-2(\mu x+\nu)]$$

$$\vdots$$

$$\varphi_{n_o+j}(x) = E_{n_o+j}[2(\mu x+\nu)]^{n_o} P_j^{(n_o,-k_o)}[1-2(\mu x+\nu)] + F_{n_o+j} P_{k_o-(j+1)}^{(-n_o,-k_o)}[1-2(\mu x+\nu)],$$

where $j = \min(N - n_o, \; k_o - 1)$,

$$\varphi_n(x) = D_n P_n^{(-n_o,-k_o)}[1-2(\mu x+\nu)] \quad (n = n_o+j+1,\; n_o+j+2, \ldots, N \text{ -- provided } n_o + j < N)$$

Figure 2. Classification of the Polynomials Generated (cont.).

are as previously defined in the text; the index n runs from zero through N, inclusive; and the integer m_o lies in $[2, 2N-1]$. It is assumed that $\{\varphi_n(x)\}_{n=o}^{N}$ is generated as in (1) and that its coefficients satisfy $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$.

As a point in passing, it should be noted that the set of all finite sequences $\{\varphi_n(x)\}_{n=o}^{N}$ satisfying the conditions indicated in Figure 2 is nonempty; that is, a finite sequence $\{\varphi_n(x)\}_{n=o}^{N}$ in which some of its members are a linear combination of two types of Jacobi polynomials (rather than one only) does exist. An example follows.

Let $m_o = 2k+1$, where k is an integer such that $N-1 \leq 2k+1 \leq 2N-1$. Then choose $b_n = b_o$, $n = 0,1,2,\ldots,N-1$, where $b_o \neq 0$, and $c_n = \dfrac{c_1(2k-1)(2k+2-n)n}{(2k+1-2n)(2k+3-2n)}$, $n = 1,2,\ldots,N-1$, where $c_1 \neq 0$. The coefficients b_n and c_n so chosen satisfy $g_1(n) = 0$ and $g_2(n) = 0$ $(n = 1,2,\ldots,N-2)$, respectively [2, pp. 18 - 21]; $c_n \neq 0$ for $n = 1,2,\ldots,N-1$; and $D \equiv \dfrac{(\lambda_2-1)^2(b_o-b_1)^2+4\lambda_2(\lambda_2-2)c_1}{4} = \lambda_2(\lambda_2-2)c_1 \neq 0$. Moreover, the equation $0 = n_o + a + b + k_o = n_o + \dfrac{2}{\lambda_2-2} - 2 + k_o = n_o - (m_o+1) + k_o = n_o - (2k+2) + k_o$ for $1 \leq k_o \leq n_o \leq N$ implies $n_o = k+1 = k_o$; and $a = \dfrac{2(b_o+x_1)}{(\lambda_2-2)(x_1-x_2)} - 1 = -(k+1) = -n_o$ is an integer in $[-n_o, -1]$. Consequently, if $\{\varphi_n(x)\}_{n=o}^{N}$ is generated as in (1) with the above set of c_n and b_n, $n = 1,2,\ldots,N-1$, the finite sequence will be of the desired type.

CHAPTER IV

SIMPLIFICATION OF THE THEORY AND DISCUSSION OF SOME APPLICATIONS

Solution of the Equation $g_2(n) = 0$

The solution of equations (3) is obtained in this section under the conditions $\lambda_2 \neq 2$ and $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$. The solution when $\lambda_2 = 2$ is already known [2, equations (11)], and the remaining stipulation -- equivalent to $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,\ j = 0,1,2,\ldots,N)$ -- is the minimal one which guarantees that these equations have no singular points. The solution of the system clearly facilitates usage of the theorems developed in Chapter III; for instead of testing the coefficients c_n and b_n one by one in the relatively complicated difference equations $g_1(n) = g_2(n) = 0$, $n = 1,2,\ldots,N-2$, one can merely check whether c_n and b_n are of the proper form.

As was indicated in Chapter II, the first of equations (3) has previously been solved under the stipulation $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$. Its solution appears as equation (4), Chapter II.* It remains, therefore, to consider only the coupling equation of the system -- $g_2(n) = 0$.

The difference equation $g_2(n) = 0$, $n = 1,2,\ldots,N-2$, can be rewritten as

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*Even if the difference equation $g_1(n) = 0$ has a singular point, its solution is still known (see [2], pp. 17 - 20).

$$[(2n+1)(\lambda_2-2)+2]c_{n+1} - [(2n-3)(\lambda_2-2)+2]c_n = b_n\Big\{[(n+1)(\lambda_2-2)+1]b_{n+1} \tag{27}$$

$$- [n(\lambda_2-2)+1]b_n\Big\} - b_o[(\lambda_2-1)b_1 + b_o] + \lambda_2 c_1 ,$$

$n = 1,2,\ldots,N-2$. Define $\lambda \equiv \lambda_2 - 2$, $g_n \equiv (2n-1)\lambda + 2$, and $f_n \equiv n\lambda + 1$. Then, after the shift in index $m = n + 1$, (27) can be shortened to

$$g_m c_m - g_{m-2} c_{m-1} = b_{m-1}(f_m b_m - f_{m-1} b_{m-1}) \tag{28}$$

$$- b_o(f_1 b_1 - f_o b_o) + g_1 c_1 ,$$

$m = 2,3,\ldots,N-1$. It will be convenient in what follows to allow m to take on the value 1. If c_o is defined to be zero, (28) reduces to the identity $\lambda_2 c_1 = \lambda_2 c_1$ when $m = 1$. Therefore, nothing is lost by this expansion of index range.

If (28) is multiplied by g_{m-1}, the left-hand side can be written as $\nabla(g_m g_{m-1} c_m)$, where ∇ is the first-order difference operator defined by $\nabla h_m = h_m - h_{m-1}$. Since

$$\sum_{m=1}^{n} \nabla(g_m g_{m-1} c_m) = g_n g_{n-1} c_n ,$$

$n = 2,3,\ldots,N-1$, the problem of solving (28) for c_n in a tractable form reduces to summing the right-hand side of (28) multiplied by g_{m-1}.

To begin this task, it is noted from (4) that for $m = 1,2,\ldots,N-1$

$$b_m = b_o + \frac{Jm}{f_m}\left(1 + \frac{1}{f_{m-1}}\right) , \tag{29}$$

where $J = \frac{1}{2}(b_1 - b_o)f_1$. Also, since $\frac{m}{f_m}\left(1 + \frac{1}{f_{m-1}}\right) = \frac{1}{\lambda}\left(1 + \frac{\lambda - 1}{f_m f_{m-1}}\right)$,

$$b_m = \frac{1}{\lambda}\left[b_o\lambda + J + \frac{J(\lambda - 1)}{f_m f_{m-1}}\right] , \tag{30}$$

$m = 1,2,\ldots,N-1$. From (29), $f_m b_m = b_o f_m + J\ (m + \frac{m}{f_{m-1}})$; consequently,

$$\nabla(f_m b_m) = b_o\lambda + J - \frac{J(\lambda - 1)}{f_{m-1} f_{m-2}} . \tag{31}$$

Equations (30) and (31) yield an expression for the first term on the right-hand side of (28):

$$b_{m-1}\ \nabla\ (f_m b_m) = \frac{J^2}{\lambda}\left[1 - \frac{(\lambda - 1)^2}{f_{m-1}^2 f_{m-2}^2}\right] + b_o^2\lambda + 2b_o J , \tag{32}$$

$m = 1,2,\ldots,N-1$. But, by definition of J, $b_o^2\lambda + 2b_oJ = b_o(f_1b_1 - f_ob_o)$; hence, the last two terms of the right-hand side of (32) will eliminate the second term on the right-hand side of (28). Therefore, after multiplying (28) by g_{m-1}, one obtains

$$\nabla(g_m g_{m-1} c_m) = \frac{J^2}{\lambda}\left[g_{m-1} - \frac{(\lambda-1)^2 g_{m-1}}{f_{m-1}^2 f_{m-2}^2}\right] + g_1 c_1 g_{m-1} , \tag{33}$$

$m = 1,2,\ldots,N-1$. With the aid of the difference formula

$$\nabla\left[\frac{m(f_{m-2}+1)}{f_{m-1}^2}\right] = \frac{(\lambda-1)^2 g_{m-1}}{f_{m-1}^2 f_{m-2}^2} ,$$

(33) can be rewritten as

$$\nabla(g_m g_{m-1} c_m) = \frac{J^2}{\lambda}\left\{g_{m-1} - \nabla\left[\frac{m(f_{m-2}+1)}{f_{m-1}^2}\right]\right\} + g_1 c_1 g_{m-1}\ , \tag{34}$$

$m = 1,2,\ldots,N-1$; and since $\sum_{m=1}^{n} g_{m-1} = n(f_{n-2} + 1)$, one can now sum both sides of (34) from 1 to n to obtain

$$g_n g_{n-1} c_n = n(f_{n-2} + 1)\left[\frac{J^2}{\lambda}\left(1 - \frac{1}{f_{n-1}^2}\right) + g_1 c_1\right]\ , \tag{35}$$

$n = 2,3,\ldots,N-1$. The condition $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, implies that $g_n g_{n-1} \neq 0$, $n = 2,3,\ldots,N-1$. Consequently, in terms of the original notation,

$$c_n = \frac{n[(n-2)(\lambda_2-2) + 2]}{[(2n-1)(\lambda_2-2)+2][(2n-3)(\lambda_2-2)+2]}\left\{\lambda_2 c_1 + \frac{1}{4}(b_1-b_o)^2(\lambda_2-1)^2(n-1)\frac{(n-1)(\lambda_2-2)+2}{[(n-1)(\lambda_2-2)+1]^2}\right\}\ , \tag{36}$$

$n = 2,3,\ldots,N-1$. It should be noted here that if $\lambda_2 = 2$, (36) reduces to the known solution of Jayne. Hence, equations (36) and (4) can be taken as the unique solution (for given b_o, b_1, c_1, and λ_2) of (3), the only restriction being that $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$.

Applications of the Theory

A discussion of some uses of the previous theory is now given. These applications will be concerned with a class of coupled linear systems of which the coupled harmonic oscillators shown in Figure 3 may be regarded as a prototype.

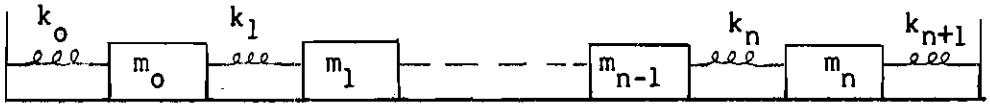

Figure 3. A System of Coupled Harmonic Oscillators.

The system indicated is assumed to be dissipationless, and the springs are linear and massless. The term k_o may be zero; all other k_n and all m_n are to be positive.

Starting with a mass m_o and two springs having spring constants k_o and k_1, one can, by successively adding a mass m_j and a spring with spring constant k_{j+1}, $j = 1,2,\ldots,N-1$, construct a finite sequence $\{S_n\}_{n=1}^{N}$ of systems. The characteristic polynomials for such systems -- obtained by assuming solutions to the differential equations of motion of the form $x_n = T_n e^{i\omega t}$ where T_n is real, ω is positive, and x_n is the displacement of mass m_n from equilibrium -- are polynomials in ω^2 which satisfy a three-term recursion formula of the type studied in previous chapters with

$$b_n = \frac{-(k_n + k_{n+1})}{m_n}, \quad n = 0,1,2,\ldots,N-1, \tag{37}$$

and

$$c_n = \frac{k_n^2}{m_n m_{n-1}}, \quad n = 1,2,\ldots,N-1 . \tag{38}$$

These results have, over an infinite index range, been noted earlier by

Jayne [2, pp. 26-28].

In the sequel, the finite sequence $\left\{S_n\right\}_{n=1}^{N}$ of systems will be referred to as a physical Hermite sequence of order N if, for $n = 1,2,\ldots,N$, the characteristic polynomial of system S_n is a polynomial $\varphi_n(\omega^2)$ which is -- apart from a linear change of variable and a multiplicative constant -- $H_n(\omega^2)$. Similar definitions are to hold for the terms "physical generalized Laguerre system of order N," "physical extended generalized Bessel system of order N," and "physical generalized Jacobi system of order N."

There are at least two types of situations which can be encountered while analyzing the configuration of Figure 3 with $n = N-1$. First, one can be given a particular system and asked if it is of Hermite, Laguerre, Bessel, or Jacobi type of order N. In this instance the question of physical realizability does not arise; the system is already built. So the spring constants and masses are, a priori, positive except possibly for k_o. Hence, Theorems 3.1 - 3.8 can be utilized to give a complete answer. If a certain set of conditions is satisfied, the answer will be "yes"; and the type, the linear change of variable, and appropriate parameters, if any, will be uniquely specified. If any one of the conditions is not satisfied, the answer will be "no." If the coefficients b_n and c_n of $\varphi_n(\omega^2)$ satisfy all conditions except (4) and (36), and if these equations hold for $n = 2,3,\ldots,j$, $2 \leq j \leq N-2$, then the system obtained by detaching masses $m_{N-1}, m_{N-2},\ldots,m_{j+1}$, and springs with spring constants $k_N, k_{N-1},\ldots,k_{j+2}$ will be one of the four types of order $j+1$. On the other hand, one may wish to construct his own physical system of order N which is one of the four types. In this case physical realizability is an added factor. Theorems 3.1 - 3.8 state necessary and sufficient conditions on $b_n \equiv \frac{-(k_n + k_{n+1})}{m_n}$ and $c_n \equiv \frac{k_n^2}{m_n m_{n-1}}$ which must be met in the choice of k_n and m_n; but the added physical requirements that $k_n > 0$, $n = 1,2,\ldots,N$, $k_o \geq 0$, and $m_n > 0$, $n = 0,1,2,\ldots,N-1$, must also be taken into account.

Jayne [2, Chapter IV] dealt with the latter sort of problem when the system is to be of infinite order and deduced existence theorems for each of the four types. Specifically, he proved that only the Jacobi- and Laguerre-type systems of infinite order can ever be generated, and the latter case can occur only if $\lim_{n \to \infty} \frac{m_{n+1}}{m_n} = 1$. He then posed the same sort of existence question for the general finite case. Constructions of physical systems of arbitrary preassigned order N which exemplify each of the four types will be performed in the next chapter. Thus, Jayne's existence question can be answered affirmatively in all four cases for the general finite system.

One advantage which occurs when the system is one of the four types is the ease with which its natural frequencies can be calculated. For example, if the configuration is a physical Hermite system of order N, and if $\{t_m\}_{m=1}^{N}$ represents the finite sequence of zeros of $H_N(t)$, then the equations $\frac{1}{\sqrt{2c_1}}\omega^2 + \frac{b_o}{\sqrt{2c_1}} = t_m$, $m = 1,2,\ldots,N$ -- obtained by setting the argument of characteristic polynomial φ_N equal to t_m -- will yield the desired frequencies. Since the zeros of the Hermite polynomials are tabulated to a high degree of accuracy in many places, $\{t_m\}_{m=1}^{N}$ may be considered as known for all practical purposes. Similar remarks hold for the Laguerre, Bessel, and Jacobi cases. In this connection it should be pointed out that much is known about the zeros of these classes of polynomials [4]; and, therefore, if a tabulation of the zeros of the particular polynomial involved is not available, such a compilation could easily be performed with the aid of an electronic computer and standard numerical procedures. Of course, if one wishes to compute the natural frequencies of any subsystem S_j $(1 \leq j \leq N - 1)$, he need only think of detaching masses $m_{N-1}, m_{N-2}, \ldots, m_j$ and springs having spring constants k_N, $k_{N-1}, \ldots, k_{j+1}$ and then analyzing the truncated system as before with N replaced by j.

For the sake of completeness in this chapter, two final facts are pointed out. First, no spring-mass combination of order three can ever be constructed in which the characteristic polynomials $\varphi_n(\omega^2)$ are as listed on the bottom of Figure 1 (see p. 33). This conclusion follows from the fact that $0 < \lambda_2 = \dfrac{(b_o - b_1)^2 + 4(c_1 + c_2)}{3c_2}$ in any spring-mass configuration while $\lambda_2 = 0$ for the polynomials in question. Secondly, the existence question for a spring-mass combination in which the conditions of Figure 2 hold (see p. 34) is not considered in this paper; because even if such a system were constructable, a knowledge of the zeros of the component terms of φ_n would not aid in the determination of the zeros of φ_n unless $F_{n_o+m} = 0$ for $m = 0,1,2,\ldots,j$, where $j = \min(N - n_o, k_o - 1)$.

CHAPTER V

PHYSICAL HERMITE, LAGUERRE, BESSEL, AND JACOBI SYSTEMS OF FINITE ORDER

Necessary Conditions for the Construction of Such Systems

As was pointed out in the second section of Chapter IV, the construction of a physical Hermite, Laguerre, Bessel, or Jacobi system of order N can be performed only if the added restrictions $b_n < 0$, $n = 0,1,2,\ldots,N-1$, and $c_n > 0$, $n = 1,2,\ldots,N-1$ are taken into account. These restrictions arise because of the physical interpretation of k_n and m_n as positive quantities (except possibly for k_o) and the relationships between k_n, m_n, b_n, and c_n expressed by (37) and (38). In the next four theorems the constraints on b_n and c_n will be transformed into equivalent conditions involving only b_o, b_1, c_1, λ_2, and N.

Notice first that if $\lambda_2 = 2$, equations (4) and (36) reduce to

$$b_n = b_o + (b_1 - b_o)n, \quad n = 2,3,\ldots,N-1, \tag{39}$$

and

$$c_n = \frac{(n^2 - n)}{4}(b_1 - b_o)^2 + nc_1, \quad n = 2,3,\ldots,N-1, \tag{40}$$

respectively. From these relations the first of the four theorems follows as a trivial consequence; it is stated simply for completeness.

<u>Theorem 5.1</u>. If $\lambda_2 = 2$ and $b_1 = b_o$, then $c_n > 0$, $n = 1,2,\ldots,N-1$, and $b_n < 0$, $n = 0,1,2,\ldots,N-1$, if and only if $c_1 > 0$

and $b_o < 0$. Thus, if a physical Hermite system of order N is constructed, $b_o < 0$ and $c_1 > 0$ must occur in addition to the hypothesis of Theorem 3.1.

When $b_1 \neq b_o$, the right-hand side of (39) is negative for $n = 2,3,\ldots,N-1$ if and only if $b_1 < b_o(1 - \frac{1}{n})$, and this is true if and only if $b_1 < b_o(1 - \frac{1}{N-1})$ since b_1 and b_o are negative. The following result is then evident.

<u>Theorem 5.2</u>. If $\lambda_2 = 2$ and $b_1 \neq b_o$, then $c_n > 0$, $n = 1,2,\ldots,N-1$, and $b_n < 0$, $n = 0,1,2,\ldots,N-1$, if and only if $b_o < 0$, $b_1 < 0$, $b_1 < b_o(1 - \frac{1}{N-1})$, and $c_1 > 0$. Thus, if a physical extended generalized Laguerre system of order N is constructed, $b_o < 0$, $b_1 < 0$, $b_1 < b_o(1 - \frac{1}{N-1})$, and $c_1 > 0$ must occur in addition to the hypothesis of Theorem 3.3.

It is worth noting that any Laguerre-type system of order N which is constructed will fall into the classical generalized Laguerre category and cannot be of the extended generalized Laguerre type. For suppose that such a system is built. Then, by Theorems 3.3 and 3.4, its characteristic polynomials $\varphi_n(\omega^2)$ are the polynomials $D_n L_n^a(\mu\omega^2 + \nu)$, where $a = \dfrac{4c_1}{(b_o - b_1)^2} - 1$. The conditions $b_o < 0$, $b_1 < 0$, and $c_1 > 0$ imply $a > -1$. Hence, the polynomials are of the classical type.

<u>Theorem 5.3</u>. If $\lambda_2 \neq 2$, if $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, and if $D = 0$, then b_n expressed in (4) is negative for $n = 0,1,2,\ldots,N-1$ and c_n expressed in (36) is positive for $n = 1,2,\ldots,N-1$ if and only if $b_o < 0$, $b_1 < 0$, $c_1 > 0$, $2 - \frac{2}{2N-3} < \lambda_2 < 2$, and either

$$b_1 < b_o\left(\frac{3 - \lambda_2}{\lambda_2 - 1}\right)\left\{1 - \frac{2}{[2 + (N-2)(\lambda_2-2)](N-1)}\right\} \tag{41}$$

or

$$b_o\left(\frac{3 - \lambda_2}{\lambda_2 - 1}\right)\left\{1 - \frac{2}{[2 + (N-2)(\lambda_2-2)](N-1)}\right\} < b_1 < b_o\left(\frac{3 - \lambda_2}{\lambda_2 - 1}\right) \cdot \left\{1 - \frac{2}{[2 + (N-3)(\lambda_2-2)](N-2)}\right\} . \tag{42}$$

Thus, if a physical extended generalized Bessel system of order N is constructed, then $b_o < 0$, $b_1 < 0$, $c_1 > 0$, $2 - \frac{2}{2N - 3} < \lambda_2 < 2$, and either (41) or (42) must all occur in addition to the hypothesis of Theorem 3.5.

Proof. Suppose first that $c_n > 0$, $n = 1,2,\ldots,N-1$, and $b_n < 0$, $n = 0,1,2,\ldots,N-1$. Then

$$0 = D \equiv \frac{(\lambda_2-1)^2(b_o-b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1}{4}$$

implies $0 < \lambda_2 < 2$; and the solution for c_n displayed in (36) reduces to

$$c_n = \frac{c_1\lambda_2 n[(n-2)(\lambda_2-2) + 2]}{[(2n-1)(\lambda_2-2)+2][(2n-3)(\lambda_2-2)+2][(n-1)(\lambda_2-2)+1]^2} , \tag{43}$$

$n = 2,3,\ldots,N-1$. Denote the bracketed expression in the numerator of (43) by $(C)_n$, the left bracketed expression in the denominator of (43) by $(A)_n$ and the middle bracketed expression in the denominator of (43) by $(B)_n$. If $n = 2$, $(B)_2$ and $(C)_2$ are positive; and, since $c_2 > 0$, (43) implies $(A)_2 > 0$. Thus, $\frac{4}{3} < \lambda_2$. If $N = 3$ the assertion concerning λ_2 is verified. Suppose that $N \geq 4$ and that $(A)_j$, $(B)_j$, and $(C)_j$ are positive for some integer j satisfying $2 \leq j \leq N-2$. Then $(B)_{j+1} > 0$ because $(B)_{j+1} = (A)_j$. Since $\lambda_2 - 2 < 0$, $(B)_n \leq (C)_n$ for $n \geq 1$; consequently $(C)_{j+1} > 0$. Since $c_{j+1} > 0$ by hypothesis, (43) implies $(A)_{j+1} > 0$. Hence, by finite induction, $(A)_n$, $(B)_n$, and $(C)_n$ are positive for $n = 2,3,\ldots,N-1$; that is, $\lambda_2 > 2 - \frac{2}{2n-1}$, $\lambda_2 > 2 - \frac{2}{2n-3}$, and $(n-2)\lambda_2 > 2(n-2) - 2$, $n = 2,3,\ldots,N-1$. The last three inequalities are equivalent to $\lambda_2 > 2 - \frac{2}{2N-3}$. To deduce (41) or (42), first set $\delta \equiv 2 - \lambda_2$. Then $0 < \delta < \frac{2}{2N-3}$, $0 < (n-1)\delta < 1$ for $n = 2,3,\ldots,N-1$, and $0 < n\delta < 1$ for $n = 2,3,\ldots,N-2$. In terms of δ, (4) can be written as

$$b_n = \frac{[b_1(1-\delta) - b_o(1+\delta)][2-(n-1)\delta]n + 2b_o(1+\delta)}{2(1-n\delta)[1-(n-1)\delta]}, \tag{44}$$

$n = 2,3,\ldots,N-1$; and the denominator will be positive for $n = 2,3,\ldots,N-2$. According to whether $(N-1)\delta < 1$ or $(N-1)\delta > 1$, (41) or (42) will occur. For suppose $(N-1)\delta < 1$. Then $b_n < 0$ for $n = 2,3,\ldots,N-1$ if and only if the numerator of (44) is negative for $n = 2,3,\ldots,N-1$. This is true if and only if

$$b_1 < b_o\left(\frac{1+\delta}{1-\delta}\right)\left\{1 - \frac{2}{[2-(n-1)\delta]n}\right\} \tag{45}$$

which is equivalent to

$$b_1 < b_o\left(\frac{1+\delta}{1-\delta}\right)\left\{1 - \frac{2}{[2-(N-2)\delta](N-1)}\right\}, \tag{46}$$

since the expression in braces in (45) is a positive strictly increasing function of n and b_o and b_1 are negative. If $1 < (N-1)\delta$, $b_n < 0$ for $n = 2,3,\ldots,N-1$ if and only if (46) holds with N replaced by $N-1$ and

$$[b_1(1-\delta) - b_o(1+\delta)][2-(N-2)\delta](N-1) + 2b_o(1+\delta) > 0 \, . \tag{47}$$

Inequality (47) is true if and only if

$$b_1 > b_o \left(\frac{1+\delta}{1-\delta}\right) \left\{1 - \frac{2}{[2-(N-2)\delta](N-1)}\right\}. \tag{48}$$

Inequality (46) (with N replaced by $N-1$) and (48) yield (42). Note that the case $(N-1)\delta = 1$ cannot occur since $(N-1)\delta = 1$ implies $\lambda_2 = \frac{2[(2N-1)-2]}{(2N-1)-1}$ -- a forbidden value of λ_2.

Conversely, suppose that $b_o < 0$, $b_1 < 0$, $c_1 > 0$, $2 - \frac{2}{2N-3} < \lambda_2 < 2$, and that either (41) or (42) holds. All steps taken in the necessity part of the proof for the derivation of (41) and (42) are reversible. Hence, $b_n < 0$, $n = 0,1,2,\ldots,N-1$. When $2 - \frac{2}{2N-3} < \lambda_2 < 2$, terms $(A)_n$, $(B)_n$, and $(C)_n$ are positive for $n = 2,3,\ldots,N-1$. Consequently, by (43), $c_n > 0$, $n = 1,2,\ldots,N-1$. This completes the proof.

Theorem 5.4. If $\lambda_2 \neq 2$, if $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, and if $D \neq 0$, then b_n expressed in (4) is negative for $n = 0,1,2,\ldots,N-1$ and c_n expressed in (36) is positive for $n = 1,2,\ldots,N-1$ if and only if $b_o < 0$, $b_1 < 0$, $c_1 > 0$, $2 - \frac{2}{2N-3} < \lambda_2$, and either (41) or (42) holds. Thus, if a physical generalized Jacobi system of order N is constructed, then $b_o < 0$, $b_1 < 0$, $c_1 > 0$, $2 - \frac{2}{2N-3} < \lambda_2$, and either (41) or (42) must all occur in addition to the hypothesis of Theorem 3.7.

Proof. The proof is similar to that of Theorem 5.3 and therefore will be somewhat brief. In (36) let $[(2n-1)(\lambda_2-2)+2] \equiv (A)_n$, $[(2n-3)(\lambda_2-2)+2] \equiv (B)_n$, $[(n-2)(\lambda_2-2)+2] \equiv (C)_n$, and $[(n-1)(\lambda_2-2)+2] \equiv (D)_n$, and suppose first that $c_n > 0$, $n = 1,2,\ldots,N-1$ and $b_n < 0$, $n = 0,1,2,\ldots,N-1$. Then $\lambda_2 \equiv \dfrac{(b_o-b_1)^2 + 4(c_1+c_2)}{3c_2} > 0$ so that $(B)_2$, $(C)_2$, and $(D)_2$ are positive. Since $c_2 > 0$, $(A)_2 > 0$; so the case $N = 3$ is settled. If $N \geq 4$ and $(A)_j$, $(B)_j$, $(C)_j$, and $(D)_j$ are positive for some integer j such that $2 \leq j \leq N-2$, $(B)_{j+1} = (A)_j > 0$. If $\lambda_2 > 2$, $(A)_{j+1}$, $(C)_{j+1}$, and $(D)_{j+1}$ are all positive; and if $0 < \lambda_2 < 2$, $(B)_n \leq (D)_n \leq (C)_n$ for $n \geq 2$. Consequently, both $(C)_{j+1}$ and $(D)_{j+1}$ are positive; and since $c_{j+1} > 0$, $(A)_{j+1} > 0$. Hence, whenever $\lambda_2 > 0$ each of the four terms is positive for $n = 2,3,\ldots,N-1$. It follows that $\lambda_2 > 2 - \dfrac{2}{2N-3}$. The verification of (41) and (42) can be repeated verbatim from the proof of Theorem 5.3 as can the sufficiency part of this theorem.

Now suppose that a physical Jacobi-type system of order N is built. It will next be shown that the Jacobi characteristic polynomials generated are of the classical type if and only if $\lambda_2 > 2$. Thus, if $2 - \dfrac{2}{2N-3} < \lambda_2 < 2$ and $\lambda_2 \neq \dfrac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, the characteristic polynomials will fall into the generalized Jacobi category. The last two conditions are clearly equivalent to $2 - \dfrac{2}{2N-3} < \lambda_2 < 2$ and $\lambda_2 \neq \dfrac{2N-3}{N-1}$.

Whenever a Jacobi-type system of order N is constructed, Theorems 3.7 and 3.8 imply that for $n = 0,1,2,\ldots,N-1$ $\varphi_n(\omega^2) = D_n P_n^{(a,b)}[1-2(\mu\omega^2+\nu)]$, where $a = \dfrac{2(x_1+b_o)}{(\lambda_2-2)(x_1-x_2)} - 1$ and $b = \dfrac{2(x_2+b_o)}{(\lambda_2-2)(x_2-x_1)} - 1$. If $b_1 = b_o$,

$a+1 = b+1 = \frac{1}{\lambda_2 - 2}$ so that $a > -1$ and $b > -1$ if and only if $\lambda_2 > 2$. If $b_1 \neq b_o$,

$$a + 1 = \frac{1}{\lambda_2 - 2}\left[1 + \frac{(b_1-b_o)(\lambda_2-2)}{|b_1 - b_o||\lambda_2 - 2|\sqrt{1 + \frac{4\lambda_2(\lambda_2-2)c_1}{(b_o-b_1)^2(\lambda_2-1)^2}}}\right] \quad (49)$$

and

$$b + 1 = \frac{1}{\lambda_2 - 2}\left[1 - \frac{(b_1-b_o)(\lambda_2-2)}{|b_1 - b_o||\lambda_2 - 2|\sqrt{1 + \frac{4\lambda_2(\lambda_2-2)c_1}{(b_o-b_1)^2(\lambda_2-1)^2}}}\right], \quad (50)$$

where the square root used in both cases is the principal one. If $\lambda_2 > 2$, the right-hand side of (49) and (50) is positive; and, consequently, the Jacobi polynomials in question are of the classical type. If $2 - \frac{2}{2N-3} < \lambda_2 < 2$ and $\lambda_2 \neq \frac{2N-3}{N-1}$, the bracketed expressions in (49) and (50) cannot simultaneously be negative; so $a > -1$ and $b > -1$ cannot occur. The Jacobi polynomials in this case must be of the generalized type.

As a final remark in this section, the significance of the parameter λ_2 is mentioned. Mathematically, the division in the classification scheme when $\lambda_2 = 2$ and $\lambda_2 \neq 2$ is to be expected. For with $\lambda_1 = 1$, 2 is the only value of λ_2 for which the coefficient of $\frac{d^2y}{dx^2}$ in (2) does not involve x^2; and therefore the character of the polynomial solutions of (2) will be decidedly different in the cases $\lambda_2 = 2$ and $\lambda_2 \neq 2$. Moreover, the omitted values $\frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, in the Bessel and Jacobi cases are a direct reflection of the requirement that φ_n be of degree n, $n = 0,1,2,\ldots,N$. On the other hand, when b_n and c_n are identified with spring constants k_n and masses m_n as prescribed in (37) and (38), the question of a meaningful physical interpretation of the important cases $\lambda_2 = 2$ and $\lambda_2 \neq \frac{2(m-2)}{m-1}$, $m = 2,3,\ldots,2N-1$, arises. Such an interpretation, if it exists, has not yet been found; it remains an open question.

Some Constructions of the Four Types of Systems

The last section contained a listing of all conditions on b_n and c_n which must be satisfied in order to construct finite physical systems of the desired types. Unfortunately, the satisfaction of these conditions does not guarantee the positiveness of k_n and m_n in (37) and (38). For suppose all necessary conditions on b_n and c_n are met. With b_o, b_1, c_1, and λ_2 specified, the quantities b_n and c_n are completely determined by (4) and (36) for $n = 2,3,\ldots,N-1$. Equations (37) and (38) then form a finite system of coupled nonlinear difference equations in the variables k_n and m_n. Failure to satisfy the restrictions on k_n and m_n in the term-by-term computation of these quantities can only occur if in (37) $k_j \leq 0$ for some integer j in $[0, N]$. In this case the system will be one of the four types of order $j-1$, provided $j \geq 4$; but it will not be of order N.

The possibility of constructing one of the four types of systems hinges on the answer to the following question: Is it possible to choose b_o, b_1, c_1, and λ_2 such that all necessary conditions on b_n and c_n for the particular type of system hold and such that, with the possible exception $k_o = 0$, the solution pair $\{k_n, m_n\}$ to the finite difference system (37) and (38) is positive? This existence query was first posed by Jayne [2, p. 28]. It will be answered here for all four types of systems. In each case the affirmative answer is proved by a certain class of examples. Thus, not only is the question of physical realizability answered; a method of selecting b_o, b_1, c_1, and λ_2 is also given. To begin the discussion an important equation is derived.

Suppose that the existence question can be answered affirmatively. Then $k_o + k_1 = -m_o b_o$ and $\frac{k_n^2}{c_n m_n} = m_{n-1} = \frac{-(k_{n-1} + k_n)}{b_{n-1}}$ for $1 \le n \le N-1$. It follows that $\frac{k_n^2 b_n}{c_n(k_n + k_{n+1})} = \frac{k_{n-1} + k_n}{b_{n-1}}$ for $1 \le n \le N-1$. If it is further assumed that k_o is chosen as positive, the last equation can be rewritten as

$$1 + \frac{k_{n+1}}{k_n} = \frac{b_{n-1} b_n}{c_n \left(1 + \frac{k_n}{k_{n-1}}\right)\left(\frac{1}{1 + k_n/k_{n-1} - 1}\right)} , \tag{51}$$

$1 \le n \le N-1$. For $0 \le n \le N-1$, let $v_{n+1} \equiv 1 + \frac{k_{n+1}}{k_n}$. Then (51) becomes

$$v_{n+1} = \frac{b_{n-1} b_n}{c_n}\left(1 - \frac{1}{v_n}\right), \tag{52}$$

$1 \le n \le N-1$ with $v_1 \equiv 1 + \frac{k_1}{k_o}$. Equation (52) provides the key for the forthcoming development of examples.

For the Hermite-type system, Theorem 5.1 specifies the conditions on b_n and c_n which must be satisfied for any successful construction.

In this case b_n and c_n are determined by (39) and (40) in which $b_1 = b_o$; the difference system of (37) and (38) reduces to

$$k_n + k_{n+1} = -m_n b_o, \quad n = 0,1,2,\ldots,N-1 , \tag{53}$$

and

$$k_n^2 = nm_{n-1}m_n c_1, \quad n = 1,2,\ldots,N-1 ; \tag{54}$$

and (52) becomes

$$v_{n+1} = \frac{b_o^2}{nc_1}\left(1 - \frac{1}{v_n}\right) \tag{55}$$

for $1 \le n \le N-1$ with $v_1 \equiv 1 + \frac{k_1}{k_o}$. Let $k_o > 0$, $k_1 \ge k_o$, and $m_o > 0$ be chosen and set $b_o = \frac{-(k_o + k_1)}{m_o}$. Then set $c_1 = \frac{kb_o^2}{4(N-1)}$, where k is a number in $(0, 1]$. With these choices $v_1 \ge 2$; and, by finite induction,

$$v_{n+1} = \frac{b_o^2}{nc_1}\left(1 - \frac{1}{v_n}\right) \ge \frac{b_o^2}{nc_1}\left(\frac{1}{2}\right) = \frac{2(N-1)}{kn} \ge \frac{2}{k} \ge 2$$

for $1 \le n \le N-1$. Hence, $\frac{k_{n+1}}{k_n} = v_{n+1} - 1 \ge 1$; and, since $k_1 \ge k_o > 0$, $k_n > 0$ for $0 \le n \le N$. Consequently, if k_n is computed from (55), it is positive and satisfies

$$k_n^2 = \frac{nc_1}{b_o^2}(k_{n-1} + k_n)(k_n + k_{n+1}) \tag{56}$$

for $1 \leq n \leq N-1$. Use these quantities in (53) to compute m_n for $1 \leq n \leq N-1$. The m_n so calculated will all be positive; and (54) will be satisfied by virtue of (56). It follows that the indicated choices for b_o, b_1, c_1, and λ_2 guarantee that the quantities k_n and m_n computed by means of (53) and (54) are all positive. By Theorem 3.1, the members of the resultant class of Hermite systems of order N have characteristic polynomials

$$\varphi_n(\omega^2) = D_n H_n\left(\frac{\omega^2}{\sqrt{2c_1}} + \frac{b_o}{\sqrt{2c_1}}\right) = D_n H_n\left(\frac{-\sqrt{2(N-1)}}{b_o\sqrt{k}}\,\omega^2 - \frac{\sqrt{2(N-1)}}{\sqrt{k}}\right),$$

$n = 0,1,2,\ldots,N$.

The existence question in the Laguerre case can be handled readily by means of the system of infinite order given by Jayne. In that example $\{k_n, m_n\}_{n=o}^{\infty} = \{n, 1\}_{n=o}^{\infty}$. Thus, in order to produce a finite Laguerre system of order N, one need only select $k_n = n$ for $0 \leq n \leq N$ and $m_n = 1$ for $0 \leq n \leq N-1$. The resultant characteristic polynomials are $\varphi_n(\omega^2) = D_n L_n^{\,o}(\omega^2)$, $n = 0,1,2,\ldots,N$. However, in order to afford some variety of choice in the constructions, other examples are now given.

The necessary conditions on b_n and c_n for the Laguerre-type system are provided by Theorem 5.2. Under these stipulations, b_n and c_n are prescribed by (39) and (40) in which $b_1 \neq b_o$; the difference system of (37) and (38) reduces to

$$k_n + k_{n+1} = m_n[(n-1)b_o - nb_1], \quad n = 0,1,2,\ldots,N-1, \tag{57}$$

and

$$k_n^{\,2} = m_{n-1}m_n\left[\frac{(n^2-n)(b_o-b_1)^2}{4} + nc_1\right],\ n = 1,2,\ldots,N-1; \tag{58}$$

and (52) becomes

$$v_{n+1} = \frac{4[(n-1)b_o-nb_1][(n-2)b_o - (n-1)b_1]}{(n^2-n)(b_o-b_1)^2 + 4nc_1}\left(1 - \frac{1}{v_n}\right) \tag{59}$$

for $1 \leq n \leq N-1$. In order to help make clear an appropriate choice of b_o, b_1, and c_1, a lemma is first proved.

Lemma 5.1. Let b_o, b_1, and c_1 satisfy only the requirements $b_1 < b_o < 0$ and $c_1 > 0$. Then a choice of such quantities can be made so that $[\,\cdot\,]_n \geq [\,\cdot\,]_{N-1} = 4$ for $1 \leq n \leq N-1$, where $[\,\cdot\,]_n$ is the coefficient of $(1 - \frac{1}{v_n})$ in (59).

Proof. The conclusion of the lemma is valid if and only if

$$\frac{(n^2-n)(b_o-b_1)^2+(2n-1)b_o(b_1-b_o)+b_o^2}{[(N-1)^2-(N-1)](b_o-b_1)^2+(2N-3)b_o(b_1-b_o)+b_o^2} \geq \frac{(n^2-n)(b_o-b_1)^2+4nc_1}{[(N-1)^2-(N-1)](b_o-b_1)^2+4(N-1)c_1}, \tag{60}$$

$n = 1,2,\ldots,N-1$, and

$$\frac{[(N-1)^2-(N-1)](b_o-b_1)^2 + (2N-3)b_o(b_1-b_o) + b_o^2}{[(N-1)^2-(N-1)](b_o-b_1)^2 + 4(N-1)c_1} = 1\,. \tag{61}$$

To satisfy (60) and (61), it is sufficient to require that

$$b_o(b_1-b_o)(2n-1) + b_o^2 \geq 4nc_1\ , \tag{62}$$

$n = 1,2,\ldots,N-1$, and

$$b_o(b_1-b_o)(2N-3) + b_o^2 = 4(N-1)c_1 \quad . \tag{63}$$

If (62) and (63) are satisfied, $b_o(b_1 - b_o)(N-1-n) \leq 2c_1(N-1-n)$; and, consequently, $c_1 \geq \frac{b_o(b_1 - b_o)}{2}$. Write

$$c_1 = \frac{kb_o(b_1 - b_o)}{2} \quad , \tag{64}$$

where $k \geq 1$. Then (63) is satisfied if and only if

$$c_1 = \frac{kb_o^2}{4(N-1)(k-1) + 2} \, . \tag{65}$$

Substitution of (65) into (64) yields

$$b_1 = b_o \left[1 + \frac{1}{2(N-1)(k-1) + 1} \right] . \tag{66}$$

The above choices of b_1 and c_1 in terms of k and the negative quantity b_o satisfy $c_1 > 0$, $b_1 < b_o < 0$, and (63); and substitution of these values into the left and right-hand sides of (62) verifies the correctness of that inequality. The proof of Lemma 5.1 is now complete.

To return to the question of constructing systems of Laguerre type, the technique demonstrated in the Hermite case for the choice of b_o, b_1, and c_1 can again be used, Lemma 5.1 providing a guide. Choose $k_o > 0$, $k_1 \geq k_o$, $m_o > 0$ and set $b_o = \frac{-(k_o + k_1)}{m_o}$. Then choose $k \geq 1$ and set $c_1 = \frac{kb_o^2}{4(N-1)(k-1) + 2}$. Finally, set $b_1 = b_o \left[1 + \frac{1}{2(N-1)(k-1) + 1}\right]$. From the proof of Lemma 5.1, $v_{n+1} \geq 4(1 - \frac{1}{v_n})$ for $1 \leq n \leq N-1$. Therefore, since $v_1 \geq 2$, $v_n \geq 2$ for $1 \leq n \leq N$.

It follows that if k_n is computed from (59), it will be positive and satisfy

$$k_n^2 = \frac{(k_{n-1} + k_n)(k_n + k_{n+1})}{[\ \cdot\ ]_n} \qquad (67)$$

for $1 \leq n \leq N-1$. Use these quantities in (57) to compute m_n for $1 \leq n \leq N-1$. The m_n so calculated will all be positive; and (58) will be satisfied by virtue of (67). Thus, the selection process given for b_o, b_1, c_1, and λ_2 yields a class of finite Laguerre systems. By Theorem 3.3, the characteristic polynomials of these systems are $\varphi_n(\omega^2) = D_n L_n^a(\mu\omega^2 + \nu)$ for $0 \leq n \leq N$, where $a = 2k[2(N-1)(k-1)+1]-1$, $\mu = -\frac{2}{b_o}[2(N-1)(k-1) + 1]$, and $\nu = 2(k-1)[2(N-1)(k-1) + 1]$.

Two other observations of interest in the Laguerre case may be noted. First, another example of an infinite-order Laguerre system can be given. To see this, set $k = 1$ in the foregoing discussion. Then $b_1 = 2b_o$ and $c_1 = \frac{b_o^2}{2}$ are independent of N, and (59) reduces to $v_{n+1} = 4(1 - \frac{1}{v_n})$ for $1 \leq n \leq N-1$. If one continues to compute v_{n+1} for $n \geq N$ in the last equation, $v_n \geq 2$ for $n \geq 1$ results. Therefore, if k_n and m_n are computed for $n \geq 0$ from (57) and (58), the solution pair $\left\{k_n, m_n\right\}_{n=o}^{\infty}$ to the <u>extended</u> finite system of difference equations remains positive. In particular, if $k_1 = k_o > 0$, then $v_n = 2$ for $n \geq 1$ so that $\left\{k_n, m_n\right\}_{n=o}^{\infty} = \left\{k_o, \frac{m_o}{n+1}\right\}_{n=o}^{\infty}$. The characteristic polynomials for the case $k = 1$ are $\varphi_n(\omega^2) = D_n L_n^1(\frac{-2\omega^2}{b_o})$ for $n \geq 0$. Secondly, there are many examples of Laguerre systems of order N in the class previously developed which cannot be extended to an infinite-order Laguerre system. A proof of this assertion by contradiction follows.

Let b_o, b_1, c_1, and λ_2 be determined as in the foregoing class of examples but restrict k only by

$$k \geq \frac{8-5\alpha}{8-2\alpha} - \frac{1}{4(N-1)} + \sqrt{\frac{1}{16(N-1)^2} + \frac{3\alpha^2(\alpha-1)}{4(\alpha-4)^2}}\ , \tag{68}$$

where α is some chosen integer greater than or equal to 5. That k is greater than 1 is evident. It will be shown that no member of this subcollection of finite Laguerre systems can be extended to an infinite-order Laguerre system. For suppose the contrary. Then, for each nonnegative integer n, k_n and m_n of the hypothesized system satisfy (37) and (38) and are positive. Consequently,

$$k_{n+1} = -m_n b_n - k_n = \frac{-k_n^2 b_n}{m_{n-1} c_n} - k_n = k_n\left[\frac{k_n}{m_{n-1}}\left(\frac{-b_n}{c_n}\right) - 1\right] \tag{69}$$

must hold for $n \geq 1$. Now in any infinite-order Laguerre system in which $k_o > 0$,

$$\frac{k_n}{m_{n-1}} = -\frac{k_n b_{n-1}}{k_{n-1} + k_n} = -\frac{b_{n-1}}{1 + \frac{k_{n-1}}{k_n}} < -b_{n-1} \tag{70}$$

for $n \geq 1$. Hence, from (39), (40), (69), and (70),

$$k_{n+1} < k_n\left\{\frac{4[n+2(N-1)(k-1)][n+1+2(N-1)(k-1)]}{n^2 - n + 2nk[2(N-1)(k-1)+1]} - 1\right\} \tag{71}$$

must hold for $n \geq 1$. The expression in braces in (71) must be positive for $n \geq 1$. In particular, this must be true if $n = \alpha(N-1)$ so that

$$4[\alpha(N-1) + 2(N-1)(k-1)][\alpha(N-1) + 1 + 2(N-1)(k-1)] > \qquad (72)$$
$$\alpha(N-1)\left\{\alpha(N-1) - 1 + 2k[2(N-1)(k-1) + 1]\right\} .$$

Inequality (72) is equivalent to

$$4(N-1)^2(4-\alpha)k^2 + [4(N-1)^2(5\alpha-8) + 2(N-1)(4-\alpha)]k \qquad (73)$$
$$+ [(N-1)^2(3\alpha^2 - 16\alpha + 16) + (N-1)(5\alpha-8)] > 0 ;$$

and, for fixed integers $N \geq 3$ and $\alpha \geq 5$, the left-hand side of (73) is a parabola in the variable k that opens downward. The larger zero of this parabola is given by the right-hand side of (68). Equation (73) therefore furnishes the desired contradiction.

The analysis in the Bessel case is much more difficult to perform than in either the Hermite or Laguerre case. The added difficulty stems from the relatively complicated expressions for b_n and c_n in (4) and (43) and the greater number of conditions on b_o, b_1, c_1, and λ_2 that must be satisfied. However, a technique similar to that illustrated in the prior two developments of examples again proves successful in producing a class of examples of Bessel-type systems of order N.

To satisfy the conditions $D = 0$, $\lambda_2 \neq 0$, and $\lambda_2 \neq 2$, b_o cannot equal b_1 and b_o, b_1, c_1, and λ_2 must be related by

$$\lambda_2 = 1 \pm \sqrt{\frac{4c_1}{(b_o-b_1)^2 + 4c_1}} . \qquad (74)$$

The condition $2 - \frac{2}{2N-3} < \lambda_2 < 2$ implies $\lambda_2 > \frac{4}{3}$. Hence, only the expression containing the positive square root term is acceptable in (74).

Set $\lambda_2 = 2 - \frac{4}{4N+1}$. Then all necessary conditions on λ_2 are met; and, in fact, $\lambda_2 \neq \frac{2(m-2)}{m-1}$ for $m = 2,3,4,\ldots$. With this choice of λ_2, the relationship between b_o, b_1, and c_1 given by (74) is

$$c_1 = \frac{(b_o - b_1)^2 (4N-3)^2}{32(4N-1)} ; \tag{75}$$

and, since $(N-1)(2-\lambda_2) < 1$, the proof of Theorem 5.3 shows that b_o and b_1 must be related by (41). Thus,

$$b_1 < b_o \frac{(N-2)(2N+3)(4N+5)}{(N-1)(2N+5)(4N-3)} \tag{76}$$

must be true. As in the Laguerre case, a lemma is advantageous at this point.

<u>Lemma 5.2</u>. Let b_o, b_1, and c_1 be related by (75) and (76) and subject also to $b_o < 0$, $b_1 < 0$. Then a choice of such quantities can be made so that $\frac{b_{n-1} b_n}{c_n}$, the coefficient of $(1 - \frac{1}{v_n})$ in (52), is greater than or equal to 4 for $1 \leq n \leq N-1$, where b_n is determined by (4), c_n is determined by (43), and $\lambda_2 = 2 - \frac{4}{4N+1}$.

<u>Proof</u>. Let

$$b_1 = \frac{2N^3 b_o}{2N^3 + 1} \tag{77}$$

and

$$c_1 = \frac{(4N-3)^2 \, b_o^2}{32(2N^3+1)^2 (4N-1)} . \tag{78}$$

Then (75) and (76) hold; and a straightforward calculation shows that the proposed inequality is equivalent to

$$\begin{aligned}
&8[4(N-n)+7][4(N-n)+3]\Big\{32N^3(N-n)^2+48N^3(N-n)+10N^3-16N^2n+8Nn^2 \\
&+16N^2-32Nn+10n^2+24N-15n+5\Big\}\Big\{32N^3(N-n)^2+112N^3(N-n) \\
&+90N^3-16N^2n+8Nn^2+32N^2-48Nn+10n^2+64N-35n+30\Big\} \geq \\
&\qquad [4(N-n)+1][4(N-n)+9](4N-3)^2(4N+1)^2(4Nn-2n^2+5n)
\end{aligned} \tag{79}$$

for $1 \leq n \leq N-1$. The minimum value for each of the two terms in braces in (79) occurs at $n = N-1$, and these minimum values are positive. Thus, for $1 \leq n \leq N-1$, each term in parentheses, brackets, or braces in (79) is positive. Since $[4(N-n)+7][4(N-n)+3] > [4(N-n)+1][4(N-n)+9]$ for $1 \leq n \leq N-1$, it follows that (79) can be proved by verifying a modification of (79) in which the two bracketed terms on either side of the inequality sign have been deleted. Since both terms in braces are positive, the minimum value of the left-hand side of (79) with the two bracketed terms deleted occurs when $n = N-1$; and the maximum value of the right-hand side of (79) with the two bracketed terms deleted also occurs when $n = N-1$. Consequently, to verify (79) it is sufficient only to verify (79) less the four bracketed terms when $n = N-1$. Another routine computation shows that when this last condition is expanded, it is equivalent to the inequality

$$49{,}248N^6 - 5{,}184N^5 + 32{,}822N^4 + 32{,}400N^3 + 6{,}546N^2 + 11{,}097N + 6{,}021 \geq 0 ,$$

which is obviously valid. The choice of b_1 and c_1 in terms of the negative quantity b_o therefore satisfies the conclusion of the lemma.

With the aid of Lemma 5.2, examples of Bessel systems of order N can easily be given. Choose $k_o > 0$, $k_1 \geq k_o$, $m_o > 0$ and set $b_o = \frac{-(k_o+k_1)}{m_o}$. Let b_1 and c_1 be given by (77) and (78), respectively. Then, by Lemma 5.2, $v_{n+1} \geq 4(1 - \frac{1}{v_n})$ for $1 \leq n \leq N-1$; and therefore, since $v_1 \geq 2$, $v_n \geq 2$ for $1 \leq n \leq N$. Thus, if k_n is computed from (52) in which b_n is given by (4), c_n is as specified in (43), and $\lambda_2 = 2 - \frac{4}{4N+1}$, then k_n will satisfy

$$k_n^2 = \frac{c_n}{b_n b_{n-1}} (k_{n-1} + k_n)(k_n + k_{n+1}) \tag{80}$$

for $1 \leq n \leq N-1$ and will be positive. Use these quantities in (37) to compute m_n for $1 \leq n \leq N-1$. The m_n so calculated will all be positive; and (38) will be satisfied by virtue of (80). By Theorem 3.5, the characteristic polynomials of the resultant generalized Bessel systems are $\varphi_n(\omega^2) = D_n B_n^{(a,2)}(\mu\omega^2+\nu)$ for $0 \leq n \leq N$, where $a = \frac{-(4N+1)}{2}$, $\mu = \frac{32(2N^3+1)}{(4N+1)(4N-3)b_o}$, and $\nu = \frac{4(16N^3+4N+5)}{(4N-3)(4N+1)}$.

The existence question in the Jacobi case is easy to handle because of prior results by Jayne. Let r be an arbitrary positive constant. Then the systems obtained by selecting positive spring constants $k_n = k_o r^n$ and positive masses $m_n = m_o r^n$ for $n \geq 0$ form a class of infinite-order Jacobi systems. Hence, in order to produce finite Jacobi systems of order N, one need only select $k_n = k_o r^n$ for $0 \leq n \leq N$ and $m_n = m_o r^n$ for $0 \leq n \leq N-1$. By Theorem 3.7, these systems have characteristic polynomials $\varphi_n(\omega^2) = D_n P_n^{(a,b)}[1-2(\mu\omega^2+\nu)]$ for $0 \leq n \leq N$, where

$a = b = \frac{1}{2}$, $\mu = \frac{m_o}{4k_o \sqrt{r}}$, and $\nu = -\frac{1}{4\sqrt{r}}(\sqrt{r} - 1)^2$. Thus, $\varphi_n(\omega^2) = D_n U_n[1 - 2(\mu\omega^2 + \nu)]$ for $0 \leq n \leq N$, where U_n is the Tchebycheff polynomial of degree n of the second kind. An appreciable variety of constructions is afforded by the above class of systems. Hence, no further examples of the Jacobi type are given in this paper.

CHAPTER VI

COMMENTS ON THE CONSTRUCTIONS OF THE FOUR TYPES OF SYSTEMS

In Chapter V the question of theoretical realizability for the construction of N^{th} order systems of each of the four types was answered. In this chapter some remarks are made about possible criteria for deciding whether the theoretically realizable systems previously proposed are, in some practical sense, realistic. The discussion applies to linear systems of which the dissipationless spring-mass combination is the prototype and is predicated upon the thought that a system in which a hair spring is attached to a hundred-ton mass should probably not be considered realistic. In addition, comments and numerical examples are given that illustrate certain properties of the classes of examples presented in Chapter V.

The question of whether a given construction is physically realistic ultimately depends on the situation for which the spring-mass combination is purported to be a suitable model. One possible set of criteria for realisticness is

$$\frac{1}{3} \leq \frac{m_{i+1}}{m_i} \leq 3, \quad \frac{1}{5} \leq \frac{k_{i+1}}{k_i} \leq 5 \;, \tag{81}$$

$$\frac{1}{10} \leq \frac{m_i}{m_j} \leq 10 \;, \quad \frac{1}{20} \leq \frac{k_i}{k_j} \leq 20 \;,$$

$$\frac{1}{2} \leq \frac{1}{2\pi}\sqrt{\frac{k_i + k_{i+1}}{m_i}} \leq 20 \;,$$

where integers i and j vary over the full index range in a system of order N, the lower bound being 1 in the requirements involving spring constant ratios when $k_o = 0$. Such conditions imply that masses which are near each other in the system and springs which are near each other in the system do not differ by "too much," that the overall growth (or decay) of the mass and spring constant values remains "reasonable," and that the natural frequencies of subsystems consisting only of a mass and its two adjoining springs are limited to a range of the kind often observed. For certain situations, these five requirements might not be appropriate for physical realism. In the following discussion, however, most of the attention is focused on this set of properties.

For the class of Hermite systems presented in Chapter V, $k_1 \geq k_o > 0$, $m_o > 0$, $b_o = \dfrac{-(k_o + k_1)}{m_o}$, $c_1 = \dfrac{kb_o^2}{4(N-1)}$ for some number k in $(0,1]$, and succeeding k_n and m_n are determined by (53) and (54). Set $k_1 = rk_o$, where $r \geq 1$. Then $b_o = \dfrac{-(1+r)k_o}{m_o}$, $c_1 = \dfrac{k(r+1)^2 k_o^2}{4(N-1)m_o^2}$,

$\dfrac{k_1}{k_o} = r \geq 1$, $\dfrac{m_1}{m_o} = 4\left(\dfrac{r}{1+r}\right)^2 \dfrac{(N-1)}{k} \geq N-1$, $\dfrac{k_2}{k_o} = r\left[\dfrac{4(N-1)r}{k(1+r)} - 1\right] \geq 2N-3$,

$\dfrac{m_2}{m_o} = \dfrac{1}{2}\left[\dfrac{4(N-1)r}{k(1+r)} - 1\right]^2 \geq \dfrac{1}{2}(2N-3)^2$, and $\dfrac{k_3}{k_o} = \left[\dfrac{4(N-1)r}{k(1+r)} - 1\right]\left\{r\left[\dfrac{2(N-1)}{k} - \dfrac{3}{2}\right] - \dfrac{1}{2}\right\} \geq 2(2N-3)(N-2)$. If all five of conditions (81) must hold for the spring-mass combination to be considered realistic, the preceding inequalities imply that these systems do not qualify except possibly when $N = 3$. In this case, if r and k are both taken to be one, $k_1 = k_o$, $k_2 = 3k_o$, $k_3 = 6k_o$, $m_1 = 2m_o$, $m_2 = \frac{9}{2} m_o$, and

$\frac{k_i + k_{i+1}}{m_i} = - b_o = \frac{2k_o}{m_o}$ for $i = 0,1,2$; and such a system is acceptable provided k_o and m_o are positive and satisfy $\frac{\pi^2}{2} \leq \frac{k_o}{m_o} \leq 800\pi^2$. It is interesting to note, however, that even though the rate of growth of the spring constants and masses in this class of examples is perhaps unrealistically rapid, the natural frequencies of many of the systems fall within a physically plausible range. For example, if $r = k = 1$ and $N = 20$, the equations $\varphi_n(\omega^2) = D_n H_n\left(\frac{\omega^2}{\sqrt{2c_1}} + \frac{b_o}{\sqrt{2c_1}}\right)$, $n = 0,1,2,\ldots,20$, become $\varphi_n(\omega^2) = D_n H_n\left[\sqrt{38}\left(\frac{m_o\omega^2}{2k_o} - 1\right)\right]$, $n = 0,1,2,\ldots,20$. The natural frequencies of this 20^{th} order Hermite system are determined from

$$f_m = \frac{1}{2\pi}\sqrt{\frac{2k_o}{m_o}\left[\frac{t_m}{\sqrt{38}} + 1\right]}\ , \quad m = 1,2,\ldots,20, \quad \text{where } \{t_m\}_{m=1}^{20} \text{ is the finite}$$

sequence of zeros of $H_{20}(x)$; and a brief calculation shows that they satisfy $0.079\sqrt{\frac{k_o}{m_o}} < f_m < 0.309\sqrt{\frac{k_o}{m_o}}$. If $\frac{k_o}{m_o}$ is chosen so that $10 \leq \sqrt{\frac{k_o}{m_o}} \leq 60$, then $\frac{1}{2} < f_m < 20$ for $1 \leq m \leq 20$. If $3 \leq N \leq 19$ and if $r = k = 1$, the same type of behavior for f_m, $m = 1,2,\ldots,N$, can be guaranteed. This is easily verified by inspection of a table of zeros of $H_n(x)$ for $3 \leq n \leq 19$ [1] or by use of known bounds for the largest and smallest zeros of $H_n(x)$ [4].

Although the existence question for Hermite systems of order $N \geq 3$ has been answered affirmatively, the last paragraph points out that it remains an open question as to whether Hermite systems of order N can ever be "realistic" in the sense described earlier, since the systems of the class used to prove realizability do not seem to be realistic. For a low-order system this question can easily be answered after some insight is gained into what relationships between k_o, k_1, m_o, and m_1 are conducive to a successful construction. For example, if $\frac{k_i + k_{i+1}}{m_i} = - b_o = \frac{k_o + k_1}{m_o}$ is large enough to satisfy the natural frequency requirement, if k_1 is chosen as relatively small, and if c_1 is taken large enough so that the value of m_1 used in the calculation of k_2 forces k_2 to remain small and less than or equal to k_1, Hermite systems of the desired type of order 6 or less can be produced. Specifically, set $k_o = \frac{5}{2}$, $k_1 = \frac{1}{2}$, $m_o = \frac{1}{48}$, $-b_o = 144$, and $c_1 = (12)^3$. Then $m_1 = \frac{1}{144}$, $k_2 = \frac{1}{2}$, $m_2 = \frac{1}{96}$, $k_3 = 1$, $m_3 = \frac{1}{54}$, $k_4 = \frac{5}{3}$, $m_4 = \frac{25}{(144)(8)}$, $k_5 = \frac{35}{24}$, $m_5 = \frac{49}{(144)(30)}$, and $k_6 = \frac{7}{40}$. Since $m_6 = \frac{1}{(96)(40)}$ and $k_7 < 0$, this is the maximum size system for such a choice of k_o, k_1, m_o, and c_1. Notice that the system of (53) and (54) is homogeneous in k_n and m_n for fixed b_o and c_1 so that the values of k_n and m_n may be scaled as desired. Hence, for any positive number s, $k_o = \frac{5}{2} s$, $k_1 = \frac{s}{2}$, $m_o = \frac{s}{48}$, $m_1 = \frac{s}{144}$, et cetera, are "realistic" values for the masses and spring constants in a Hermite system of order 6 or less. The characteristic polynomials generated in this example are given by $\varphi_n(\omega^2) = D_n H_n[\sqrt{6}\,(\frac{\omega^2}{144} - 1)]$, $n = 0,1,2,\ldots,6$; and the six natural frequencies are determined from $f_m = \frac{6}{\pi}\sqrt{\frac{t_m}{\sqrt{6}} + 1}$, where $\{t_m\}_{m=1}^{6}$ is the finite sequence of zeros of $H_6(x)$. It follows that $0.381 < f_m < 2.674$ for $1 \leq m \leq 6$. If a large-order Hermite configuration satisfying these restrictions is required, it seems likely that a systematic search procedure for determining appropriate values of k_o, k_1, m_o, and m_1 can be incorporated into a computer program.

The procedure described in Chapter V for the construction of a class of N^{th} order Laguerre systems was to choose $k_o > 0$, $k_1 \geq k_o$, $m_o > 0$, set $b_o = \frac{-(k_o + k_1)}{m_o}$, choose $k \geq 1$, set $c_1 = \frac{kb_o^2}{4(N-1)(k-1)+2}$, set $b_1 = b_o[1 + \frac{1}{2(N-1)(k-1)+1}]$, and then compute subsequent k_n, m_n from equations (57) and (58). For systems of order ten or lower, many combinations of k_o, k_1, m_o, and k lead to spring-mass configurations satisfying each of the five conditions (81). For example, if $k = 1$ and $k_1 = k_o$, the resultant infinite-order Laguerre system has values $k_n = k_o$ and $m_n = \frac{m_o}{n+1}$ for $n \geq 0$. The bound on the overall growth of the masses -- $\frac{1}{10} \leq \frac{m_i}{m_j} \leq 10$ for $i,j = 0,1,2,\ldots,N-1$ -- can be met only if $N \leq 10$. In this case, the natural frequency restriction -- $\pi \leq \sqrt{\frac{k_i + k_{i+1}}{m_i}} \leq 40\pi$ for $i = 0,1,2,\ldots,N-1 = 9$ -- can be satisfied only if $\pi^2 \leq \frac{2k_o(i+1)}{m_o} \leq 1600\,\pi^2$ for $i = 0,1,2,\ldots,9$. The last condition is fulfilled whenever $k_o > 0$ and $m_o > 0$ are chosen so that $\frac{\pi^2}{2} \leq \frac{k_o}{m_o} \leq 80\pi^2$. If $k_1 = k_o$ and if $k > 1$ is chosen sufficiently close to 1 so that $(k-1)(N-1)$ remains "small," other Laguerre systems of the desired type having order less than 10 can be given. As to be expected, these systems have values of k_n and m_n much like those of the preceding example whenever $(k-1)(N-1) << 1$. As another example, the infinite-order Laguerre system in which $\{k_n, m_n\}_{n=o}^{\infty} = \{n, m_o\}_{n=o}^{\infty}$ can be considered. In this case all restrictions are satisfied for $3 \leq N \leq 20$ provided $m_o > 0$ is chosen so that $\frac{39}{1600\pi^2} \leq m_o \leq \frac{1}{\pi^2}$. If Laguerre systems of order larger than 20 that satisfy all five conditions are required, no example given in this paper fulfills the need. In this case, whether such a system can ever be constructed is still an open question. Again, however, because of the still relatively simple expressions for b_n and c_n, it seems likely that a search procedure for appropriate values of k_o, k_1, m_o, m_1, and k_2 can be programmed to a computer in order to resolve this question.

For the Bessel examples given, $\lambda_2 = 2 - \frac{4}{4N+1}$, $k_1 \geq k_o > 0$, $m_o > 0$, $b_o = \frac{-(k_o + k_1)}{m_o}$, $c_1 = \frac{(4N-3)^2 b_o^2}{32(2N^3+1)(4N-1)}$, $b_1 = \frac{2N^3 b_o}{2N^3+1}$, and k_n, m_n are computed from (37) and (38), where b_n is determined by (4) and c_n by (43). Set $k_1 = rk_o$, where $r \geq 1$. Then $b_o = -\frac{(1+r)k_o}{m_o}$, $c_1 = \frac{(4N-3)^2(1+r)^2 k_o^2}{32(2N^3+1)^2(4N-1)m_o^2}$, $\frac{k_1}{k_o} = r \geq 1$, and $\frac{m_1}{m_o} = 32(\frac{r}{1+r})^2 \frac{[2N^3+1]^2(4N-1)}{(4N-3)^2} > 8N^5$. Consequently, no one of the given examples of Bessel spring-mass combinations falls within the bounds prescribed by the five criteria (81). Small-order Bessel systems can be produced that lie within the tolerance limits, but the computation involved is tedious. A computer search program for developing examples of Bessel systems having the required properties might be possible, but it would probably be more sophisticated than one for the Hermite or Laguerre cases because of the greater complexity of the equations involved.

The Jacobi systems presented have $k_n = k_o r^n$ for $0 \leq n \leq N$ and $m_n = m_o r^n$ for $0 \leq n \leq N - 1$, where r is an arbitrary positive constant. The five stipulations to be met imply that k_o, m_o, and r must satisfy $\frac{1}{3} \leq r \leq 3$, $\frac{1}{10} \leq r^{i-j} \leq 10$ for $i,j = 0,1,2,\ldots,N-1$, $\frac{1}{20} \leq r^N \leq 20$, and $\pi^2 \leq \frac{k_o}{m_o}(1+r) \leq 1600\,\pi^2$. Many such values of k_o, m_o, and r can be chosen for a specified value of N, but the range of permissible values of r is clearly dictated by the size of N. For large values of N, r is forced to be very close to 1.

CHAPTER VII

A SUMMARY OF THE MAJOR RESULTS

The purpose of Chapter VII is to provide a concise listing of the major conclusions of this paper. Relevant chapter and page numbers are included within the list to serve as an aid to the reader.

(1) Hermite, extended generalized Laguerre, extended generalized Bessel, and generalized Jacobi polynomial sequences, each having more than three terms, can be characterized in terms of the coefficients occurring in the recursive relation (1) to within a determinable linear change of variable and computable multiplicative factors independent of x (see Chapter III, Theorems 3.1 - 3.8). Note: For infinite sequences, the integer N appearing in these theorems must be thought of as having increased without bound before the theorems are applied.

(2) A proper three-term recursive finite Sturm-Liouville polynomial sequence $\{\varphi_n(x)\}_{n=0}^{N}$ in which $N \geq 3$ must be exactly one of six types: Hermite, extended generalized Laguerre, extended generalized Bessel, generalized Jacobi, or one of two other unnamed kinds (see Chapter III, pp. 23-35).

(3) The system of difference equations $g_1(n) = g_2(n) = 0$ that first arose in Chapter II of [2] can be solved in closed form over either a finite or infinite index range, provided that neither of the equations has a singular point (see Chapter IV, pp. 36-39). This solution simplifies the application of the theory developed in Chapter III and some of the work

done in [2]. <u>Note</u>: Even if the difference equation $g_1(n) = 0$ has a singular point, its solution is still known (see [2], pp. 17-20).

(4) For an arbitrary preassigned positive integer N, it is possible to build linear dissipationless spring-mass systems S_n having secular polynomials of Hermite, Laguerre, Bessel, or Jacobi type for $1 \leq n \leq N$ (see Chapter V, pp. 51-63).

(5) Linear dissipationless spring-mass systems of Laguerre type always lead to classical generalized Laguerre secular polynomials, but linear dissipationless spring-mass systems of the Jacobi type can lead to either classical Jacobi or generalized Jacobi secular polynomials (see Chapter V, pp. 45, 49, and 50). <u>Note</u>: In the infinite-order system, generalized Jacobi polynomials cannot occur; for in this case $\lambda_2 > 2$ (see Theorem 5.4 and let N tend to infinity), and the result on pages 49 and 50 can be applied.

(6) A further investigation into the possibility of devising efficient numerical procedures for the construction of "realistic" finite linear dissipationless spring-mass combinations of each of the four types (Hermite, Laguerre, Bessel, and Jacobi) remains to be performed.

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