COUNTING HAMILTONIAN CYCLES IN PLANAR TRIANGULATIONS

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Dedicated to my family

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SUMMARY

Whitney showed that every planar triangulation without separating 3-cycles is Hamiltonian. This result was extended to all 4-connected planar graphs by Tutte. Hakimi, Schmeichel, and Thomassen showed the first lower bound $n/\log_2 n$ for the number of Hamiltonian cycles in every n-vertex 4-connected planar triangulation and, in the same paper, they conjectured that this number is at least $2(n-2)(n-4)$, with equality if and only if G is a double wheel. We show that every 4-connected planar triangulation on n vertices has $\Omega(n^2)$ Hamiltonian cycles. Moreover, we show that if G is a 4-connected planar triangulation on n vertices and the distance between any two vertices of degree 4 in G is at least 3, then G has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Terminology and Notations

We list some definitions and notations that will be used throughout the dissertation. Readers are referred to Graph Theory textbook by Bondy and Murty [21] and Diestel [30] for any terminology or notations that we may have missed in this section.

(1) **Basic Graph Terminology:** A graph $G = (V, E)$ is a pair (V, E) such that V is the vertex set and $E \in 2^{\binom{V}{2}}$ is the edge set. We use $V(G), E(G)$ to denote the vertex set and the edge set of G respectively.

A *path* on n $(n \geq 2)$ vertices in a graph G is a sequence of distinct vertices $v_1v_2 \cdots v_n$ such that $v_iv_{i+1} \in E(G)$ for each $i \in [n-1]$. A *cycle* on n $(n \geq 3)$ vertices in a graph G is a sequence of vertices $v_1v_2 \cdots v_nv_1$ such that $v_1v_2 \cdots v_n$ is a path in G and $v_1v_n \in E(G)$. A graph G is called *connected* if for any two vertices in G, there exists a path between them. A graph G is k*-connected* if it has more than k vertices and if it remains connected when fewer than k vertices are removed. The *distance* between two vertices u, v in a graph G, denoted by $d_G(u, v)$, is the number of edges in a shortest path in G connecting them. A graph G is called *Hamiltonian* if it contains a *Hamiltonian cycle*, i.e., a cycle that uses every vertex in G. For a positive integer k, a k*-cycle* is a cycle (of length k) using k vertices in G. A k-cycle C in a connected graph G is said to be *separating* if the graph obtained from G by deleting C is not connected. A separating 3-cycle is also called a *separating triangle*.

Let G be a graph. For $v \in V(G)$, we use $N_G(v)$ (respectively, $N_G[v]$) to denote the neighborhood (respectively, closed neighborhood) of v, and use $d_G(v)$ to denote $|N_G(v)|$, which is the *degree* of v in G. For a positive integer r, a graph G is called

r-*regular* if every vertex has degree r in G; moreover if $r = 3$, G is also called a *cubic* graph. Given a path P and distinct vertices $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y. If H is a subgraph of G, we write $H \subseteq G$. For any set R consisting of vertices of G and 2-element subsets of $V(G)$, we use $H + R$ (respectively, $H - R$) to denote the graph with vertex set $V(H) \cup (R \cap V(G))$ (respectively, $V(H) \backslash (R \cap V(G))$) and edge set $E(H) \cup (R \cap \binom{V(H) \cup (R \cap V(H))}{2})$ (respectively, $E(H) \backslash (R \cap \binom{V(H) \cup (R \cap V(H))}{2})$). If $R = \{\{x, y\}\}$ (respectively, $R = \{v\}$), we write $H + xy$ (respectively, $H + v$) instead of $H + R$, and write $H - xy$ (respectively, $H - v$) instead of $H - R$.

Let G and H be graphs. We use $G \cup H$ and $G \cap H$ to denote the *union* and *intersection* of G and H, respectively. Thus, $V(G \cup H) = V(G) \cup V(H), E(G \cup H) = E(G) \cup E(H)$ and $V(G \cap H) = V(G) \cap V(H), E(G \cap H) = E(G) \cap E(H)$. The *join* of G and H, denoted $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, i.e., $V(G[S]) = S$ and $E(G[S]) = E(G) \cap \binom{S}{2}$, and let $G - S = G[V(G) \backslash S]$. A set $S \subseteq V(G)$ is a *cut* in G if $G - S$ has more components than G, and if $|S| = k$ then S is a cut of *size* k or k-*cut* for short. For a subgraph T of G, we often write $G - T$ for $G - V(T)$ and write $G[T]$ for $G[V(T)]$.

(2) **Basic Planar Graph Terminology:** A graph is called *planar* if it can be drawn in the plane with no crossing edges. A graph drawn in the plane without crossing edges is called a *plane graph*.

For any plane graph G, the connected regions of $\mathbb{R}^2 \backslash G$ are called *faces* of G, and the set of gaces of G is denoted $F(G)$. Two elements of $V(G) \cup E(G)$ are *cofacial* if they are incident with a common face of G. For any positive integer k, G is said to be k-*face-colorable* if all faces of G can be colored by k colors such that no two faces incident with a common edge use the same color.

A *closed walk* in a graph G is defined as a sequence of vertices, starting and ending at the same vertex, such that any two consecutive vertices in this sequence are adjacent in G. Every face of a plane graph G is bounded by a closed walk in G called the *boundary* of the face. We call a face f of G whose boundary consists of k vertices (including repetitions) a *face of size* k. A cycle is called a *facial cycle* if it is the boundary of a face. Any face bounded by a triangle is known as a *triangular face*. It is well-known that every face of a 2-connected plane graph is bounded by a cycle. If G is a finite plane graph, then exactly one face of G is not bounded and we call it the *infinite face* or *outer face* of G. The *outer walk* of G is the boundary of the infinite face of G. If the outer walk is a cycle in G, we call it *outer cycle* instead. If all vertices of G are incident with its infinite face, then we say that G is an *outer planar* graph.

Let G be a plane graph. The *dual graph* of G is the graph G^* such that every face f in G corresponds to a vertex f^* in G^* and every edge e in G correspondings an edge e^* in G^*, and two vertices f_1^*, f_2^* in G^* are joined by the edge e^* in G^* if and only if their corresponding faces f_1 and f_2 in G are both incident with the edge e. It is easy to see that $(G^*)^* \cong G$.

A *planar triangulation* is an edge-maximal planar graph on at least three vertices, i.e., every face in any drawing of a planar triangulation in the plane is bounded by a triangle. Every planar triangulation corresponds to a unique plane triangulation up to homeomorphism. By Euler's theorem ($|V(G)|-|E(G)|+|F(G|=2$ for a connected plane graph G), an n-vertex plane triangulation has exactly $3n-6$ edges. For a cycle C in G, we use $\overline{C}$ to denote the subgraph of G consisting of all vertices and edges of G contained in the closed disc in the plane bounded by C. The *interior* of C is then defined as the subgraph $\overline{C} - C$. For any distinct vertices $u, v \in V(C)$, we use uCv to denote the subpath of C from u to v in clockwise order.

(3) **Interval Notation:** For integers n, m with $m \geq n \geq 1$, we use the notation $[n] = \{1, 2, \cdots, n\}$, and $[n, m] = \{n, n+1, \cdots, m-1, m\}$.

(4) **Asymptotic Notation:** Give two functions $f, g : \mathbb{Z}^+ \to \mathbb{R}$, we say $f = O(g)$ if there exist some constant C and some integer n_0 such that for all $n \geq n_0$, $|f(n)| \leq Cg(n)$. We say $f = \Omega(g)$ if $g = O(f)$. We say $f = o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and we say $f = \omega(g)$ if $g = o(f)$.

1.2 Background and history

1.2.1 The Four Color Theorem

We begin with the Four Color Theorem stating that

Theorem 1.2.1 (Appel and Haken [10, 11, 12]). *Every plane graph is* 4*-face-colorable.*

This statement was conjectured by Guthrie in 1852, and remained open until a proof was found by Appel and Haken [10, 11, 12] in 1976. But the proof by Appel and Haken is not completely satisfactory as it uses a computer and cannot be verified by hand. This motivated Robertson, Sanders, Seymour, and Thomas to give a simper proof in [66] but their proof is also computer assisted.

It can be shown easily that each plane graph containing a Hamiltonian cycle is 4-face-colorable. To see this, let G be a plane graph and C be a Hamiltonian cycle in G. We consider the faces in the interior of C, which induce a tree in G^*, and hence we can use two colors to color those faces. Similarly, the faces in the exterior of C can be colored with two colors. Therefore, every plane graph containing a Hamiltonian cycle is 4-face-colorable.

Tait [70] in 1880 gave a false proof of the Four Color Theorem by assuming that every 3-connected cubic planar graph is Hamiltonian. It was not until 1946 that Tutte [76] found the first counterexample: There exists a 3-connected cubic planar graph with no Hamiltonian cycle. See Figure 1.1. More examples can be found in [39]. However, all known examples

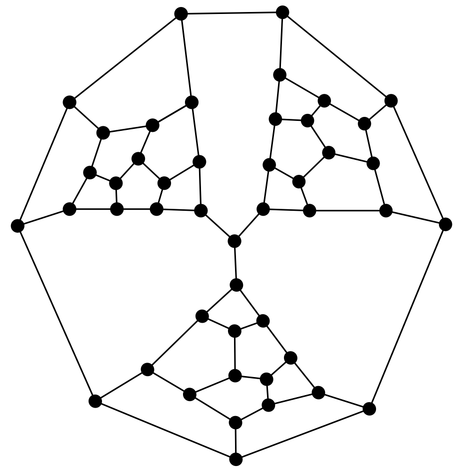

Figure 1.1: Tutte's example.

contain odd cycles. Indeed, Barnette [13] in 1969 proposed the following conjecture, which still remains open.

Conjecture 1.2.2 (Barnette [13]). *Every* 3*-connected cubic planar bipartite graph is Hamiltonian.*

Each known non-Hamiltonian 3-connected cubic plane graph also has a face of size 7 or more (see [7], [79]). It was conjectured by Barnette and independently by Goodey [35] that every 3-connected cubic plane graph with all faces of size at most 6 is Hamiltonian. Kardoš [44] proved this conjecture and showed that

Theorem 1.2.3 (Kardoš [44]). *Every* 3*-connected cubic plane graph with faces of size at most* 6 *is Hamiltonian.*

It is interesting to know which planar graphs contain Hamiltonian cycles.

1.2.2 Circumference of 3-connected planar graphs

Note that there exist 3-connected planar triangulations that do not have a Hamiltonian cycle. For example, let G be a plane triangulation on n ($n \geq 5$) vertices. We obtain a new graph G' by inserting a vertex to each face of G and connecting this vertex to the three vertices on the boundary of that face. Note that the vertices in $V(G')\backslash V(G)$ form an independent set in G'. Since G has exactly $2n-4$ faces,

$$|V(G')| = n + (2n-4) = 3n-4.$$

G' is a planar triangulation and it is 3-connected but G' contains no Hamiltonian cycle. Otherwise there exist two vertices in $V(G')\backslash V(G)$ that are adjacent in G'. Moreover, there are many 3-connected planar graphs containing no Hamiltonian cycles (see [39]). Hence one can ask: What is the length of a longest cycle in a 3-connected planar graph? Is there any lower bound for it?

Definition 1.2.4. The *circumference* $circ(G)$ of a graph G is the length of a longest cycle in G.

Moon and Moser [63] in 1963 constructed 3-connected planar graphs G with $circ(G) \leq 9|V(G)|^{\log_3 2}$ and implicitly conjectured that every n-vertex 3-connected planar graph G has circumference $\Omega(n^{\log_3 2})$. In 1966, Barnette [14] showed

Theorem 1.2.5 (Barnette [14])**.** *Every n-vertex 3-connected planar graph G has a cycle of length at least $c\sqrt{\lg n}$, i.e., $circ(G) \geq c\sqrt{\lg n}$ for some constant c.*

This bound was improved by Clark [27] in 1985 who showed the following result

Theorem 1.2.6 (Clark [27])**.** *Every n-vertex 3-connected planar graph G has a cycle of length at least $e^{\sqrt{\frac{1}{6}\lg n}}$, i.e., $circ(G) \geq e^{\sqrt{\frac{1}{6}\lg n}}$.*

In 1992, Jackson and Wormald [41] gave a polynomial lower bound.

Theorem 1.2.7 (Jackson and Wormald [41])**.** *Every n-vertex 3-connected planar graph G has a cycle of length at least βn^{α}, i.e., $circ(G) \geq \beta n^{\alpha}$, where β is some constant and $\alpha \approx 0.207$.*

Gao and Yu [33] improved the value of α and generalized this result to graphs on the projective plane, or the torus, or the Kelin bottle.

Theorem 1.2.8 (Gao and Yu [33]). *Let G be a 3-connected graph on n vertices. If G can be embedded in the plane or or the projective plane, or the torus, or the Kelin bottle, $circ(G) \geq \Omega(n^{0.4})$.*

In 2002, Chen and Yu [25] solved the conjecture implicitly proposed by Moon and Moser [63] and gave a better construction of 3-connected planar graphs with circumference $O(|V(G)|^{\log_3 2})$.

Theorem 1.2.9 (Chen and Yu [25]). *Let G be a 3-connected graph on n vertices. If G can be embedded in the plane or or the projective plane, or the torus, or the Kelin bottle, $circ(G) \geq \Omega(n^{\log_3 2})$.*

1.2.3 Hamiltonian cycles in 4-connected planar graphs

The situation is different when 3-connected is replaced by 4-connected. Whitney [78] in 1931 showed

Theorem 1.2.10 (Whitney [78]). *Every planar triangulation without separating triangles is Hamiltonian.*

Observe that planar triangulations without separating triangles are 4-connected except it is K_3 or K_4. In 1956, Tutte [75] extended Whitney's result by showing the following.

Theorem 1.2.11 (Tutte [75]). *Every 4-connected planar graph is Hamiltonian.*

Definition 1.2.12. A graph is *Hamiltonian connected* if for any two distinct vertices u and v, there exists a Hamiltonian path between u and v.

Observe that if a graph G is Hamiltonian connected, then G is Hamiltonian (as G has a Hamiltonian path between u and v for any edge $uv \in E(G)$). In 1983, Thomassen [74] further strengthened Tutte's result by showing the following.

Theorem 1.2.13 (Thomassen [74]). *Every 4-connected planar graph is Hamiltonian connected.*

These results have been extended to graphs on other surfaces. Thomas and Yu [71] in 1994 extended Tutte's result to the projective plane.

Theorem 1.2.14 (Thomas and Yu [71])**.** *Every* 4*-connected projective-planar graph is Hamiltonian.*

Kawarabayashi and Ozeki [45] in 2015 further strengthened the result of Thomas and Yu to the following.

Theorem 1.2.15 (Kawarabayashi and Ozeki [45])**.** *Every* 4*-connected projective-planar graph is Hamiltonian connected.*

Note that this result cannot be generalized to 4-connected graphs on the torus or on the Klein bottle (There are 4-connected graphs on the torus or on the Klein bottle which are not Hamiltonian-connected).

Altshuler [9] considered the graphs on torus and showed in 1972 that

Theorem 1.2.16 (Altshuler [9])**.** *The followings are true:*

(1) Every 6*-connected graph on the torus is Hamiltonian.*

(2) Every cubic graph on the torus with each face of size 6 *is Hamiltonian.*

(3) Every 4*-regular graph on the torus with each face of size* 4 *is Hamiltonian.*

Thomas and Yu [72] in 1997 improved the first result of Altshuler.

Theorem 1.2.17 (Thomas and Yu [72])**.** *Every* 5*-connected toroidal graph is Hamiltonian.*

Kawarabayashi and Ozeki [46] generalized this result.

Theorem 1.2.18 (Kawarabayashi and Ozeki [46])**.** *Every* 5*-connected toroidal graph is Hamiltonian connected.*

It is possible that the condition on the connectivity above can be weakened if instead one asks for a Hamiltonian path instead of a Hamiltonian cycle. Indeed, Thomas, Yu, and Zang [73] in 2005 considered 4-connected toroidal graphs and proved that

Theorem 1.2.19 (Thomas, Yu, and Zang [73]). *Every 4-connected toroidal graph contains a Hamiltonian path.*

There also have been extensive investigations on the existence of Hamiltonian cycles in random graphs. For results along this line, we refer the readers to [47, 65, 48, 17, 1, 8].

1.2.4 Cycle spectrum of 4-connected planar graphs

In this section, we discuss the pancyclicity in planar graphs.

Definition 1.2.20. The *cycle spectrum* $\mathcal{C}(G)$ of a simple graph G is the set of all possible lengths of cycles in G. A graph G on at least three vertices is said to be *pancyclic* if G contains cycles of all possible lengths, i.e., $\mathcal{C}(G) = \{3, 4, \cdots, |V(G)|\} = [3, |V(G)|]$.

Bondy suggested that it is usually hardest to guarantee the existence of a Hamiltonian cycle for the cylce spectrum of a graph. In 1973, Bondy [20] proposed his famous meta-conjecture.

Conjecture 1.2.21 (Bondy [20]). *Any non-trivial condition which implies that a graph is Hamiltonian, also implies that this graph is pancyclic (up to a small class of exceptional graphs).*

For example, Dirac proved that every n-vertex ($n \geq 3$) graph with minimum degree at least $n/2$ is Hamiltonian, and Bondy [19] proved the following.

Theorem 1.2.22 (Bondy [19]). *Let G be an n-vertex graph with minimum degree $n/2$. Then either G is pancyclic or G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.*

For more results concerning Hamiltonian degree conditions that imply pancyclicity, we refer the readers to [15, 69, 18, 26, 31].

By Tutte's result, Bondy's conjecture suggests that 4-connected planar graphs are pancyclic except for some simple families of graphs. Malkevitch [59] found a simple family of exceptions, of which every member is Hamiltonian but contains no 4-cycle. There are two interesting conjectures concerning cycle spectrum of 4-connected planar graphs, proposed by Bondy in 1973 and by Malkevitch in 1988 respectively.

Conjecture 1.2.23 (Bondy [20])**.** *Every 4-connected planar graph contains cycles of all lengths from 3 to $|V(G)|$, with the possible exception of one even length.*

Conjecture 1.2.24 (Malkevitch [60])**.** *Every 4-connected planar graph is pancyclic if it contains a cycle of length 4.*

A 4-connected planar graph need not contain a cycle of length 4 by a construction in [59]. Plummer [64] in 1975 proposed a conjecture stating that any graph obtained from a 4-connected planar graph by deleting one vertex has a Hamiltonian cycle. This conjecture follows from a theorem of Tutte in [75]. Plummer [64] also conjectured that any graph obtained from a 4-connected planar graph by deleting two vertices has a Hamiltonian cycle. This conjecture was proved by Thomas and Yu [71]. Note that deleting three vertices from a 4-connected planar graph may result in a graph which is not 2-connected and hence, has no Hamiltonian cycle. However, Sanders [67] in 1996 showed that in any 4-connected planar graph with at least six vertices, there are three vertices whose deletion results in a Hamiltonian graph. In 2004, Chen, Fan, and Yu [24] showed that any n-vertex 4-connected graph also has a cycle of length k for each $k \in \{n-4, n-5, n-6\}$ with $k \geq 3$. It was shown by Cui, Hu, and Wang [29] in 2009 that any n-vertex 4-connected graph with $n \geq 10$ has a cycle of length $n-7$. We list those results below.

Theorem 1.2.25 (Tutte)**.** *If G is an n-vertex 4-connected planar graph, then G contains a cycle of length $n-1$, i.e., $n-1 \in \mathcal{C}(G)$.*

Theorem 1.2.26 (Thomas and Yu [71])**.** *If G is an n-vertex 4-connected planar graph, then $n-2 \in \mathcal{C}(G)$.*

Theorem 1.2.27 (Sanders [67])**.** *If G is an n-vertex 4-connected planar graph with $n \geq 6$, then $n-3 \in \mathcal{C}(G)$.*

Theorem 1.2.28 (Chen, Fan, and Yu [24]). *If G is an n-vertex 4-connected planar graph with $n \geq 9$, then $\{n-4, n-5, n-6\} \subseteq \mathcal{C}(G)$.*

Theorem 1.2.29 (Cui, Hu, and Wang [29]). *If G is an n-vertex 4-connected planar graph with $n \geq 10$, then $n-7 \in \mathcal{C}(G)$.*

What about small cycle lengths? Wang and Lih [77] in 2002 showed

Theorem 1.2.30 (Wang and Lih [77]). *If G is a 4-connected planar graph, $3, 5 \in \mathcal{C}(G)$.*

This result was improved by Fihavž, Juvan, Mohar, and Škrekovski [32] to

Theorem 1.2.31 (Fihavž, Juvan, Mohar, and Škrekovski [32]). *If G is a 4-connected planar graph, then $3, 5, 6 \in \mathcal{C}(G)$.*

Recently, Lo [54] showed

Theorem 1.2.32 (Lo [54]). *If G is an 4-connected planar graph, for any $k \in \{\lfloor \frac{|V(G)|}{2} \rfloor, \lfloor \frac{|V(G)|}{2} \rfloor + 1, \cdots, \lceil \frac{|V(G)|}{2} \rceil + 3\}$ with $3 \leq k \leq |V(G)|$, $k \in \mathcal{C}(G)$.*

In addition, Lo [53] gave a lower bound for $|\mathcal{C}(G)|$.

Theorem 1.2.33 (Lo [53]). *If G is an n-vertex 4-connected planar graph with $n \geq 3$, then $|\mathcal{C}(G)| \geq \lceil \frac{n}{2} \rceil + 2$.*

Using a similar idea as in [53] and the Hamiltonicity of 4-connected planar graphs, Mohar and Shantanam [62] proved the following.

Theorem 1.2.34 (Mohar and Shantanam [62]). *If G is an n-vertex 4-connected planar graph and e is an edge in G, then e is contained in at least $\lceil \frac{n}{2} \rceil + 1$ cycles of pairwise distinct lengths.*

It is an interesting problem to study the cycle spectrum of 4-connected planar graphs without 4 cycles. Horňák and Kocková [40] in 2008 proved

Theorem 1.2.35 (Horňák and Kocková [40]). *If G is a 4-connected planar graph containing no cycle of length 4, then $7 \in \mathcal{C}(G)$.*

Madaras and Tamášová [58] showed

Theorem 1.2.36 (Madaras and Tamášová [58]). *If G is a 4-connected planar graph containing no cycle of length 4, then $8, 9 \in \mathcal{C}(G)$.*

Very recently, Lo [52] proved

Theorem 1.2.37 (Lo [52]). *If G is a 4-connected planar graph containing no cycle of length 4, then for any $k \in \{\lfloor |V(G)|/2 \rfloor, \lfloor |V(G)|/2 \rfloor + 1, \cdots, |V(G)|\}$, $k \in \mathcal{C}(G)$.*

For the size of the cycle spectrum of 4-connected planar graphs without 4-cycles, Mohar and Shantanam [62] gave the following lower bound.

Theorem 1.2.38 (Mohar and Shantanam [62]). *If G is an n-vertex 4-connected planar graph containing no cycle of length 4, then $|\mathcal{C}(G)| \geq \lceil \frac{5n}{6} \rceil + 2$.*

1.2.5 Counting Hamiltonian cycles

It is a natural problem to consider the number of Hamiltonian cycles in a graph, which also has applications in coding theory according to [3, 4, 5]. For the results concerning the number of Hamiltonian cycles in random graphs, see [43, 28, 34, 61]. In this subsection, we focus on counting Hamiltonian cycles in planar triangulations, which is the main theme of this dissertation.

The problem of determining the number of Hamiltonian cycles in 4-connected planar triangulations was initated by Hakimi, Schmeichel, and Thomassen who showed in 1979 that

Theorem 1.2.39 (Hakimi, Schmeichel, and Thomassen [38]). *Every 4-connected planar triangulation on n vertices has at least $n/\log_2 n$ Hamiltonian cycles.*

In the same paper, they conjectured a lower bound which is quadratic in the number of vertices and realized by the double wheel.

Definition 1.2.40. A *double wheel* is a planar triangulation obtained from a cycle by adding two vertices and all edges from these two vertices to the vertices of the cycle. See the double wheel on 10 vertices in Figure 1.2.

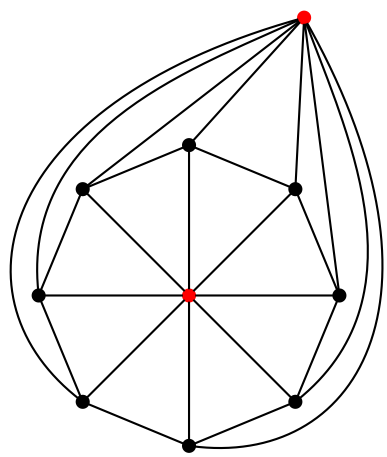

Figure 1.2: Double wheel on 10 vertices.

Now we count the number of the Hamiltonian cycles in a double wheel graph G on n vertices. For each Hamiltonian cycle H in G, consider the choices of the two edges incident with each of the vertices v_1, v_2 of degree $n-2$ in H. If the two edges incident with v_1 in H are cofacial, then the two edges incident with v_2 in H are also cofacial. There are $(n-2)(n-3)$ such Hamiltonian cycles. Now suppose the two edges incident with v_1 in H are not cofacial. We have two ways to extend this path to a Hamiltonian cycle. Hence there are $2(\binom{n-2}{2} - (n-2))$ such Hamiltonian cycles. Therefore, the number of Hamiltonian cycles in G is exactly

$$(n-2)(n-3) + 2(\binom{n-2}{2} - (n-2)) = 2(n-2)(n-4).$$

Conjecture 1.2.41 (Hakimi, Schmeichel, and Thomassen [38]). *If G is a 4-connected planar triangulation on n vertices, then G has at least $2(n-2)(n-4)$ Hamiltonian cycles, with equality if and only if G is a double wheel.*

It was not until recently that Brinkmann, Souffriau, and Van Cleemput [23] gave the first linear lower bound by showing

Theorem 1.2.42 (Brinkmann, Souffriau, and Van Cleemput [23]). *Every n-vertex 4-connected planar triangulation contains $\frac{12}{5}(n-2)$ Hamiltonian cycles.*

Subsequently, Brinkmann and Van Cleemput [22] proved linear lower bounds for 4-connected plane graphs and plane graphs with at most one 3-cut and sufficiently many edges. Since then, there has been more progress on this problem. Lo [55] showed that

Theorem 1.2.43 (Lo [55]). *Every n-vertex 4-connected planar triangulation with $O(\log n)$ separating 4-cycles has $\Omega((n/\log n)^2)$ Hamiltonian cycles.*

We in [51] improved this result by showing that

Theorem 1.2.44 (Liu and Yu [51]). *Every n-vertex 4-connected planar triangulation with $O(n/\log n)$ separating 4-cycles has $\Omega(n^2)$ Hamiltonian cycles.*

Very recently, Lo and Qian [56] further showed that

Theorem 1.2.45 (Lo and Qian [56]). *Every n-vertex 4-connected planar or projective planar triangulation with $O(n)$ separating 4-cycles has $2^{\Omega(n)}$ Hamiltonian cycles.*

Thus Conjecture 1.2.41 holds for large graphs with $O(n)$ separating 4-cycles. Note that the condition on $O(n)$ separating 4-cycles restricts the planar triangulation to have $O(n)$ degree 4 vertices and an n-vertex 4-connected planar triangulation can have $\Omega(n)$ degree 4-vertices.

The number of Hamiltonian cycles in a planar triangulation G can be significantly larger if one increases the connectivity or the minimum degree of G. Böhme, Harant, and Tkáč [16] in 1999 showed that

Theorem 1.2.46 (Böhme, Harant, and Tkáč [16]). *Every n-vertex 5-connected planar triangulation has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.*

In 2020, Alahmadi, Aldred, and Thomassen [2] improved this bound by showing that

Theorem 1.2.47 (Alahmadi, Aldred, and Thomassen [2]). *Every n-vertex 5-connected planar or projective planar triangulation has $2^{\Omega(n)}$ Hamiltonian cycles.*

Note that the more recent result of Lo and Qian [56] is stronger, but the technique used in [2] played an important role in [56]. We [51] weakened the hypothesis of 5-connectivity in the Böhme-Harant-Tkáč result and showed that

Theorem 1.2.48 (Liu and Yu [51]). *Every n-vertex 4-connected planar triangulation with minimum degree 5 has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.*

1.3 Main results

In this section, we state the main results of this dissertation. Recall the Hakimi-Schmeichel-Thomassen Conjecture on the number of Hamiltonian cycles in a 4-connected planar triangulation on n vertices, which states that if G is a 4-connected planar triangulation on n vertices then G has at least $2(n-2)(n-4)$ Hamiltonian cycles, with equality if and only if G is a double wheel.

Building upon techniques from the previous results, and using some new ideas, we settled the conjecture above asymptotically by showing the following.

Theorem 1.3.1 (Liu, Wang, and Yu [50]). *If G is a 4-connected planar triangulation on n vertices, then G has at least cn^2 Hamiltonian cycles, where $c = (12 \times 90 \times 541 \times 301)^{-2}/2$.*

We also observed that the relative locations of degree 4 vertices play an essential role for 4-connected planar triangulations to have exponentially many Hamiltonian cycles. If any two degree 4 vertices are far from each other in a 4-connected planar triangulation G, we can find exponentially many Hamiltonian cycles in G.

Theorem 1.3.2 (Liu, Wang, and Yu [50])**.** *There exists a constant $c > 0$ such that for any 4-connected planar triangulation G on n vertices in which the distance between any two vertices of degree 4 is at least three, G has at least $2^{cn^{1/4}}$ Hamiltonian cycles.*

1.4 Proof sketch

From the results in section 1.2, we know that if we want to solve Conjecture 1.2.41 we need to deal with the separating 4-cycles in an n-vertex 4-connected planar triangulation G. Hence we try to analyse the structure of the interior of a separating 4-cycle in G (Lemmas 2.2.2 and 2.2.3).

Note that the interior of a separating 4-cycle can be a degree 4 vertex. What happens if G has a lot of degree 4-vertices? We can imagine that G may contain two vertices u, v of degree 4 that are adjacent to each other. It is natural to consider contracting the edge uv and applying induction to this smaller graph G' as G' is still a 4-connected planar triangulation and $|V(G')| = n - 1$.

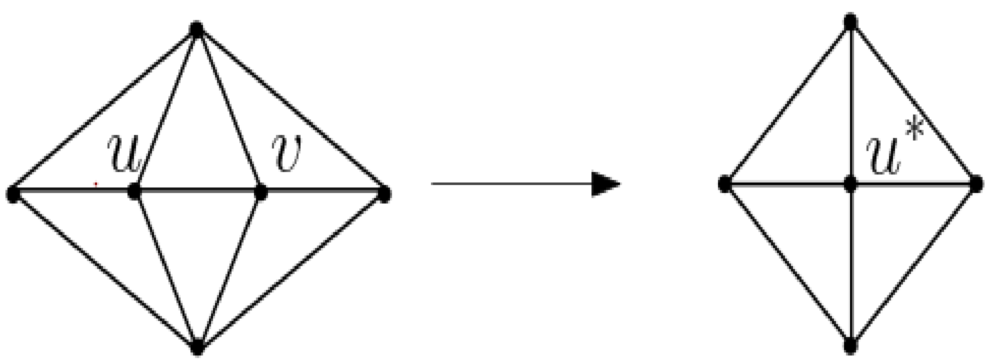

Figure 1.3: Contracting the edge uv.

Observe that each Hamiltonian cycle in G' can be modified to a Hamiltonian cycle in G containing the edge uv. By induction, G' has at least $2(n-3)(n-5)$ Hamiltonian cycles. Therefore, we only need to find the extra

$$2(n-2)(n-4) - 2(n-3)(n-5) = 2(n-7)$$

Hamiltonian cycles (not from induction) to show Conjecture 1.2.41.

Since all Hamiltonian cycles from induction contain the edge uv, the extra Hamiltonian cycles could be the cycles not containing the edge uv in G. As G is a planar triangulation, the edge uv is contained in the boundary of exactly two triangular faces. Moreover, uv is contained in exactly two triangles T_1, T_2 in G due to the connectivity of G. For each triangle T of T_1, T_2, the Hamiltonian cycles through the two edges in $E(T)\backslash\{uv\}$ do not contain the edge uv. This motivates us to show that the number of Hamiltonian cycles through any two edges in a triangle of G is linear in n (Lemma 3.1.2).

Now we can assume that any two degree 4 vertices in G are not adjacent. We want to show such G contains $\Omega(n^2)$ Hamiltonian cycles. How can we find this many distinct Hamiltonian cycles in G? Tutte's theorem states that 4-connected planar graphs are Hamiltonian. Can we still get a 4-connected graph by removing some edges from a 4-connected planar triangulation G? Following an idea from Alahmadi, Aldred, and Thomassen [2], we can find an independent set S such that for any edge set F consisting of exactly one edge incident with each vertex in S, the graph G' obtained from G by removing the edges in F is still 4-connected, and hence G' has a Hamiltonian cycle and this cycle contains no edges in F. Now we see what will happen if $G - F$ is not 4-connected.

Suppose $G' = G - F$ is not 4-connected. Then let K be a minimum cut set of $G' = G - F$. Therefore, $|K| \leq 3$ and $G' - K$ is not connected. Since each edge in G is contained in exactly two triangles (each is the boundary of a triangular face), Alahmadi, Aldred, and Thomassen in [2] observed that every edge $e = uv$ in G between two components of $G' - K$ belongs to F and the exactly two vertices contained in $N(u) \cap N(v)$ are contained in the cut set K.

Thus, if $G - F$ is not 4-connected, we have that two vertices in S are contained in a separating 4-cycle (the first case in Figure 1.4), or two vertices in S are contained in a separating 5-cycle (the second case in Figure 1.4), or three vertices in S occur in some 6-cycle (the third case in Figure 1.4), or some vertex in S is contained in a separating 4-cycle

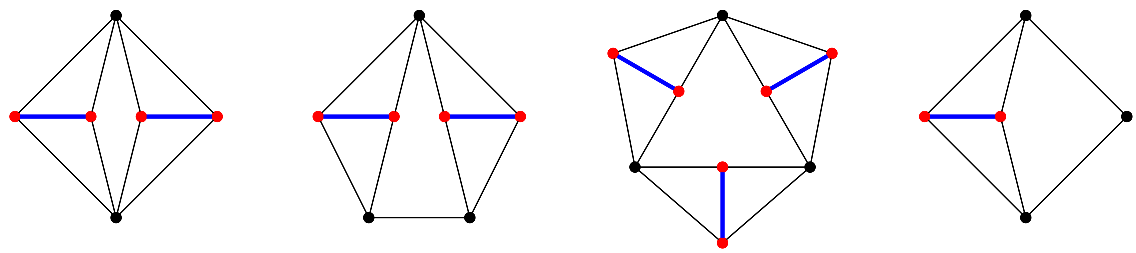

Figure 1.4: Blue edges belong to F; black vertices belong to K.

or some vertex in S is adjacent to three vertices in a separating 4-cycle (the last case in Figure 1.4). Alahmadi, Aldred, and Thomassen in [2] found such good independent set of vertices of degree 5 or 6 in G with size $\Omega(n)$ in n-vertex 5-connected planar triangulations. Given such an independent set S of vertices of degree 5 or 6 in G, the number of the choices of F is $5^{a_5}6^{a_6}$, where a_i is the number degree i vertices in S for $i \in \{5, 6\}$. For each such F, note that $G - F$ is 4-connected and has a Hamiltonian cycle. For each Hamiltonian cycle we obtain in this way, it can be picked by at most $(5-2)^{a_5}(6-2)^{a_6} = 3^{a_5}4^{a_5}$ times. Then G has at least

$$\frac{5^{a_5}6^{a_6}}{3^{a_5}4^{a_5}} = (5/3)^{a_5}(3/2)^{a_6} \geq (3/2)^{a_5+a_6} = (3/2)^{|S|}$$

Hamiltonian cycles. This implies that every n-vertex 5-connected planar triangulation has $2^{\Omega(n)}$ Hamiltonian cycles as $|S| = \Omega(n)$.

However, in a 4-connected planar triangulation G, we cannot find such a good independent set S of vertices of degree 5 or 6 in G with size $\Omega(n)$ since G can have many degree 4 vertices (the number of degree 5 or 6 vertices can be very small) and many separating 4-cycles (we cannot avoid the first case and the last case in Figure 1.4). It is natural to ask when G contains such a good independent S. Lo [55] showed either there are two vertices in G with at least t common neighbors or G has an independent set S of size $\Omega(n/t)$ such that the first three cases in Figure 1.4 cannot occur by using the results in [2].

Suppose there exist $v, x \in V(G)$ such that they have many (a large constant is enough and t can be a constant) common neighbors. Then we can find a separating 4-cycle of G such that the interior of G has size at least two but not too large size. Therefore we can apply induction. Now we can assume that G has an independent set S of size $\Omega(n)$ such that the first three cases in Figure 1.4 cannot occur. To deal with the lase case in Figure 1.4, we need to know how to deal with degree 4 vertices. For each degree 4 vertex in S, if we remove an edge incident with it then the remaining graph is not 4-connected as the other three neighbors form a 3-cut. We observed that we can remove one edge in the cycle induced by the neighbors of this degree 4 vertex. Since G does not have two adjacent

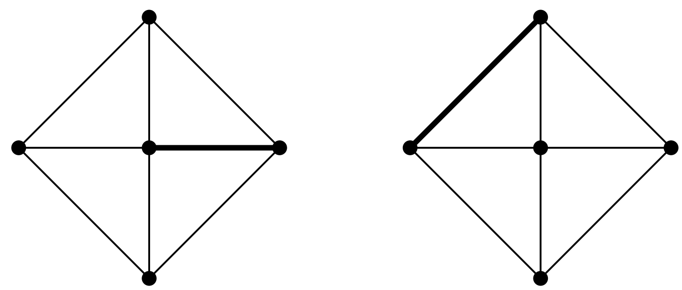

Figure 1.5: Edges related to a degree 4 vertex.

degree 4 vertices, we can show that the graph obtained by removing an edge from the cycle induced by the neighbors of a degree 4 vertex is still 4-connected (see Lemma 2.3.4). Finally we can find an edge set F of size $\Omega(n)$ (by the independent set S) in G such that $G - F$ is 4-connected. Here we introduce a result of Sanders [68].

Theorem 1.4.1 (Sanders [68])**.** *If G is a 4-connected planar graph and $e_1, e_2 \in E(G)$, then there exists a Hamiltonian cycle in G containing e_1 and e_2.*

Now there exists an edge set F in G with size $\Omega(n)$ such that $G - F$ is still 4-connected. It follows from Sanders's theorem that there exists a Hamiltonian cycle in G through $e, f \in F$ but no edge in $F \backslash \{e, f\}$. Therefore, there are at least $\binom{|F|}{2}$ Hamiltonian cycles in G, and hence G has $\Omega(n^2)$ Hamiltonian cycles as $|F| = \Omega(n)$. Note that we did not use Sanders's theorem to prove Theorem 1.3.1 and instead, we used a lemma (Lemma 3.1.1) we showed by using the tools about 'Tutte path' and 'Tutte cycle', which is also used in the proof of $\Omega(n)$ Hamiltonian cycles through two cofacial edges in G.

We give a proof sketch of Theorem 1.3.1 here. To prove Theorem 1.3.1, we apply induction on the number of vertices in the planar triangulation G.

- Base case: The assertion holds when $n \leq 1/\sqrt{c}$ as every 4-connected planar graph is Hamiltonian by Tutte's theorem and $cn^2 \leq 1$ if $n \leq 1/\sqrt{c}$.
- Induction step: We may assume $n > 1/\sqrt{c}$ and the statement holds for 4-connected planar triangulations on fewer than n vertices. We consider two cases.

Case 1. G contains two adjacent vertices of degree 4, say u and v.

We contract the edge uv to a vertex and apply induction to the new graph G^* on $n-1$ vertices. This gives $c(n-1)^2$ Hamiltonian cycles in G, of which all contain the edge uv. In G, the edge uv is contained in exactly two triangles, say T_1 and T_2. It follows from Lemma 3.1.2 that there are at least $2c_1 n$ Hamiltonian cycles in G avoiding the edge uv (as we let the remaining two edges in $E(T_i)\backslash\{uv\}$ be e_1 and e_2 in Lemma 3.1.2 for each $i = 1, 2$). Hence in total there exist at least

$$c(n-1)^2 + 2c_1 n \geq cn^2$$

Hamiltonian cycles in G.

Case 2. No two vertices of degree 4 in G are adjacent.

It can be shown that G contains an independent set I of vertices of degree at most 6 with $|I| \geq n/12$. By Lemma 2.3.8 (with $t = 10$), either (1) there exist distinct vertices $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 10$, or (2) G contains $S \subseteq I$ such that $|S| \geq c_2 n$ ($c_2^2/2 = c$) and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

(1) For the former case, G contains a separating 4-cycle, say D, whose interior has at least two but at most $n/4$ vertices. Let $\overline{D}$ denote the subgraph of G consisting of all vertices and edges of G contained in the closed disc in the plane bounded by D. We contract the interior of D, i.e., $\overline{D} - D$, to a vertex u_0 and obtain a smaller graph G_0 with $3n/4 \leq |V(G_0)| \leq n-1$. By induction, G_0 has at least $c(3n/4)^2$ Hamiltonian cycles. Note that for each Hamiltonian cycle H_0 in G_0, $H_0 - u_0$ union a Hamiltonian path between a and b in $\overline{D} - (V(D)\backslash\{a,b\})$, where $a, b \in N_{H_0}(u_0)$ is a Hamiltonian cycle in G. It follows from Lemma 2.2.4 that $\overline{D} - (V(D)\backslash\{a,b\})$ has at least two Hamiltonian paths between a and b for any distinct $a, b \in V(D)$ as no two vertices of degree 4 in G are adjacent. Therefore, G has at least

$$2 \cdot c(3n/4)^2 \geq cn^2$$

Hamiltonian cycles.

(2) For the latter case, under the assumptions of G, we use the large independent set S to show that G has

$$|S|(|S|+1)/2 \geq c_2^2 n^2/2 = cn^2$$

Hamiltonian cycles (by Lemma 2.3.4, the counting idea of Alahmadi et al, Lemma 2.1.4 and Lemma 2.1.7).

From time to time, we observe that the distance of degree 4 vertices in G can play an important role to find many Hamiltonian cycles. If the distance of any two degree 4 vertices is large, we can use the tools of showing Theorem 1.3.1 to find a lot of pairwise distinct Hamiltonian cycles in G directly, and we showed Theorem 1.3.2. We also give a proof sketch of Theorem 1.3.2 here.

(i) We may assume that G has an independent set S of vertices of degree 5 or 6 such that $|S| = \Omega(n^{3/4})$ and S has some good properties. Otherwise, G has two vertices v, x such that $|N(x) \cap N(v) \cap I| \geq cn^{1/4}$ for some constant $c > 0$ by Lemma 2.3.8. This implies that G contains $\Omega(n^{1/4})$ separating 4-cycles such that all of them have disjoint interior. We contract the interior of each such separating 4-cycles and use Lemma 2.2.4 to show G has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.

(ii) We may assume that there exists $S_1 \subseteq S$ such that, $|S_1| \geq |S|/2$ and each vertex in S_1 is contained in a separating 4-cycle D in G whose interior has at least two vertices. Suppose not. We can use Lemma 2.3.4 and the counting idea in [2] to find $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.

(iii) We may assume that there exists a sequence of $\Omega(n^{1/2})$ separating 4-cycles with "nested" interiors and each 4-cycle is a maximal separating 4-cycle containing a vertex in S_1. Otherwise, there exist at least $\Omega(n^{1/4})$ separating 4-cycles in G such that all of them have disjoint interior (We still contract the interior of each separating 4-cycle and find the desired number of Hamiltonian cycles). This sequence of "nested" separating 4-cycles has good properties which allows us to find $2^{\Omega(n^{1/4})}$ Hamiltonian cycles in G.

1.5 Organization

This dissertation is organized as follows.

In chapter 2, we first cite some known results on "Tutte paths" and "Tutte cycles" in planar graphs. Such results will be used to find a Hamiltonian cycle through specific edges in a planar graph. Then we show several results on the number of Hamiltonian paths between two given vertices in planar graphs (analyzing the interior of a separating 4-cycle in a planar triangulation). Finally, we discuss an idea similar to the idea in [2] for finding an edge set F in a 4-connected planar triangulation G such that removing F from G still gives a 4-connected graph.

In chapter 3, we prove Theorem 1.3.1. We first show that every n-vertex 4-connected planar triangulation G has $\Omega(n)$ Hamiltonian cycles through two specified edges in any given triangle. Moreover, if G does not contain two adjacent vertices of degree 4, then G has $\Omega(n^2)$ Hamiltonian cycles. We then use these results and apply induction on n to complete the proof of Theorem 1.3.1.

In chapter 4, we consider 4-connected planar triangulations G in which any two vertices of degree 4 have distance at least three. We show that either G has a large independent set with nice properties, or G has many separating 4-cycles with pairwise disjoint interiors, or G has many "nested" separating 4-cycles. In all cases, we can find the desired number of Hamiltonian cycles in G.

CHAPTER 2

PRELIMINARIES

2.1 Tutte paths and Tutte cylces

We first give some definitions. Let G be a graph and $H \subseteq G$.

Definition 2.1.1. An H-*bridge* of G is a subgraph of G induced by either an edge in $E(G) \backslash E(H)$ with both incident vertices in $V(H)$, or all edges in $G - H$ with at least one incident vertex in a single component of $G - H$.

Definition 2.1.2. For an H-bridge B of G, the vertices in $V(B \cap H)$ are the *attachments* of B on H.

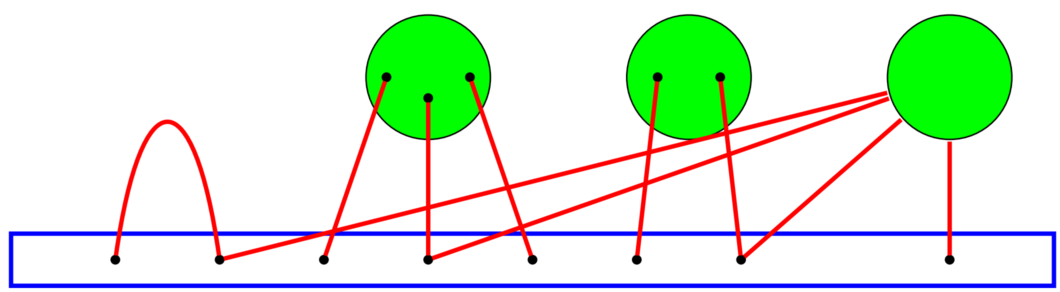

Figure 2.1: H-bridges.

In Figure 2.1, H is the blue part and the green parts represent the components of $G-H$. We denote the four H-bridges from the left to the right by B_1, B_2, B_3, B_4 respectively. B_1 consists exactly one edge contained in $E(G) \backslash E(H)$ and it has two attachments on H. Note that B_2 has three attachments on H, B_3 has two attachments on H and B_4 has four attachments on H. Observe that for each $i \in \{2, 3, 4\}$, the attachments of B_i form a cut set in G.

Definition 2.1.3. A path (or cycle) P in a graph G is called a *Tutte path* (or *Tutte cycle*) if every P-bridge of G has at most three attachments on P. If, in addition, every P-bridge of

G containing an edge of some subgraph C of G has at most two attachments on P, then P is called a *C-Tutte path* (or *C-Tutte cycle*) in G.

Thomassen [74] proved the following result on Tutte paths in 2-connected planar graphs.

Lemma 2.1.4 (Thomassen [74]). *Let G be a 2-connected plane graph and C be its outer cycle, and let $x \in V(C)$, $y \in V(G)\backslash\{x\}$, and $e \in E(C)$. Then G has a C-Tutte path P between x and y such that $e \in E(P)$.*

Note that Lemma 2.1.4 implies that every 4-connected planar graph is Hamiltonian connected and has a Hamiltonian cycle through two given edges that are cofacial.

We also need the following result of Thomas and Yu [71], which was used to extend Tutte's theorem on Hamiltonian cycles in planar graphs to projective planar graphs.

Lemma 2.1.5 (Thomas and Yu [71]). *Let G be a 2-connected plane graph with outer cycle C, and let $u, v \in V(C)$ and $e, f \in E(C)$ such that u, e, f, v occur on C in clockwise order. Then G has a uCv-Tutte path P between u and v such that $e, f \in E(P)$.*

Definition 2.1.6. A *circuit graph* is an ordered pair (G, C) consisting of a 2-connected plane graph G and a facial cycle C of G such that, for any 2-cut U of G, each component of $G - U$ contains a vertex of C.

Jackson and Yu [42] showed that every circuit graph (G, C) has a C-Tutte cycle through a given edge of C and two other vertices.

Lemma 2.1.7 (Jackson and Yu [42]). *Let (G, C) be a circuit graph, and let r, z be vertices of G and $e \in E(C)$. Then G contains a C-Tutte cycle H such that $e \in E(H)$ and $r, z \in V(H)$.*

2.2 Separating 4-cycles

Definition 2.2.1. A *near triangulation* is a plane graph in which all faces except possibly its infinite face are bounded by triangles.

We considered the number of Hamiltonian paths between two given vertices in the outer cycle of a near triangulation. We use Lemmas 2.1.4 and 2.1.5 to prove the following two lemmas.

Lemma 2.2.2 (Liu and Yu [51]). *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that $G \neq C + vx$ and G has no separating triangles. Then one of the following holds:*

(i) *$G - \{v, x\}$ has at least two Hamiltonian paths between u and w.*

(ii) *$G - \{v, x\}$ is a path between u and w and, hence, $G - \{v, x\}$ is outer planar.*

Proof. If $vx \in E(G)$, then $G = C + vx$ or G has a separating triangle, contradicting our assumption. So $vx \notin E(G)$. Then $G - \{v, x\}$ has a path from u to w, say Q. Since G has no separating triangles, each block of $G - \{v, x\}$ contains an edge of Q. Hence, the blocks of $G - \{v, x\}$ can be labeled as $B_1, \ldots, B_t$ and the cut vertices of $G - \{v, x\}$ can be labeled as $b_1, \ldots, b_{t-1}$ such that $V(B_i \cap B_{i+1}) = \{b_i\}$ for $i = 1, \ldots, t-1$, and $V(B_i \cap B_j) = \emptyset$ when $|i - j| \geq 2$. Let $b_0 = u$ and $b_t = w$. Moreover, let C_i denote the outer walk of B_i for $1 \leq i \leq t$. See Figure 2.2.

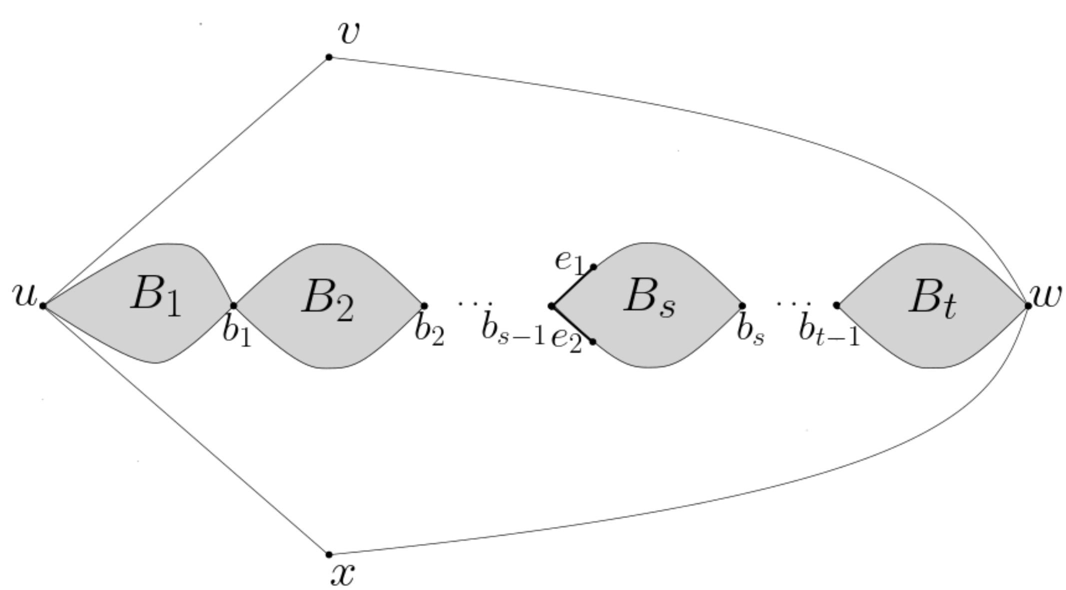

Figure 2.2: The blocks $B_1, \ldots, B_t$.

If $|V(B_i)| = 2$ for $1 \leq i \leq t$, then (ii) holds. Hence, we may assume that $|V(B_s)| \geq 3$ for some s, where $1 \leq s \leq t$. Then $b_{s-1}b_s \notin E(B_s)$, as otherwise, $vb_{s-1}b_sv$ or $xb_{s-1}b_sx$ would be a separating triangle in G. Let e_1, e_2 be the edges of C_s incident with b_{s-1}. By Lemma 2.1.4, B_s has a C_s-Tutte path P_s^j between b_{s-1} and b_s such that $e_j \in E(P_s^j)$, for $j = 1, 2$. Since G has no separating triangles, P_s^1 and P_s^2 are Hamiltonian paths in B_s.

For each $1 \leq i \leq t$ with $i \neq s$, if $|V(B_i)| \geq 3$, we apply Lemma 2.1.4 to B_i and find a Hamiltonian path P_i between b_{i-1} and b_i in B_i; if $|V(B_i)| = 2$, let $P_i = b_{i-1}b_i$. Then $(\bigcup_{i \neq s} P_i) \cup P_s^1$ and $(\bigcup_{i \neq s} P_i) \cup P_s^2$ are distinct Hamiltonian paths in $G - \{v, x\}$ between u and w. So (i) holds. □

Lemma 2.2.3 (Liu and Yu [51]). *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that G has no separating triangles. Then one of the following holds:*

(i) *$G - \{w, x\}$ has at least two Hamiltonian paths between u and v.*

(ii) *$G - \{w, x\}$ is an outer planar near triangulation.*

Proof. We apply induction on $|V(G)|$. If $|V(G)| = 4$ then we see that (i) holds trivially. So assume $|V(G)| \geq 5$. Then $uw, vx \notin E(G)$, as G has no separating triangles.

We may assume that u, v each have at least two neighbors in $V(G) \backslash V(C)$. For, otherwise, by symmetry assume that u has a unique neighbor in $V(G) \backslash V(C)$, say u'. Now $G' := G - u$ is a near triangulation with outer cycle $C' := u'vwxu'$ and G' has no separating triangles. Hence, by induction, $G' - \{w, x\}$ is an outerplanar near triangulation, or $G' - \{w, x\}$ has at least two Hamiltonian paths between u' and v. In the former case, (i) holds; in the latter case, (ii) holds by extending the Hamiltonian paths in G' from u' to u along the edge $u'u$.

Next, we claim that $(G - \{w, x\}) - u$ or $(G - \{w, x\}) - v$ is 2-connected. For, suppose $(G - \{w, x\}) - u$ is not 2-connected. Then $(G - \{w, x\}) - u$ can be written as the union of two subgraphs B_1 and B_2 such that $|V(B_1 \cap B_2)| \leq 1$, $B_1 - B_2 \neq \emptyset$, and $B_2 - B_1 \neq \emptyset$. Without loss of generality, assume that $v \in V(B_2)$. (Indeed, $v \in V(B_2) \backslash V(B_1)$.) We further choose B_1, B_2 to minimize B_1. Then B_1 is connected and B_1 has no cut vertex. By planarity, there exists a unique vertex $y \in N_G(w) \cap N_G(x)$. If $y \in V(B_2)$ then $V(B_1 \cap B_2) \cup \{u, x\}$ is a 2-cut in G or induces a separating triangle in G, a contradiction. So $y \in V(B_1)\backslash V(B_2)$. Now u has a neighbor in $V(B_1)\backslash V(B_2)$; as otherwise, $V(B_1 \cap B_2) \cup \{w, x\}$ is a 2-cut in G or induces a separating triangle in G, a contradiction. This implies that $G[B_1 + u]$ is 2-connected. Now, we repeat this argument for $(G - \{w, x\}) - v$. Suppose $(G - \{w, x\}) - v$ is not 2-connected. Then $(G - \{w, x\}) - v$ can be written as the union of two subgraphs B_1' and B_2' such that $|V(B_1' \cap B_2')| \leq 1$, $B_1' - B_2' \neq \emptyset$, $B_2' - B_1' \neq \emptyset$, and $u \notin V(B_1')\backslash V(B_2')$. Then, since $G[B_1 + u]$ is 2-connected and $y \in V(B_1)\backslash V(B_2)$, we have $y \in V(B_2')$. Now, $V(B_1' \cap B_2') \cup \{v, w\}$ is a 2-cut in G or induces a separating triangle in G, a contradiction.

By symmetry, we may assume that $H := (G - \{w, x\}) - u$ is 2-connected. Let D denote the outer cycle of H and let $u_1, u_2 \in N_G(u) \cap V(D)$ such that $u_1 \in N_G(x)$ and $u_2 \in N_G(v)$. Since u has at least two neighbors in $V(G)\backslash V(C)$, $u_1 \neq u_2$. Let $y \in N_G(w) \cap N_G(x)$. Choose an edge $e \in E(D)$ incident with y, and an edge $f \in E(D)$ incident with u_1. By Lemma 2.1.4, H has a D-Tutte path P between u_1 and v such that $e \in E(P)$. By Lemma 2.1.5, H has a vDu_2-Tutte path Q between u_2 and v such that $e, f \in E(Q)$. Since G has no separating triangles, we see that both P, Q are Hamiltonian paths in H. Now $P \cup u_1u$ and $Q \cup u_2u$ are distinct Hamiltonian paths in $G - \{w, x\}$ between u and v, and (ii) holds. $\square$

By the above two lemmas, we have the following observation about degree 4 vertices and the number of Hamiltonian paths in a near triangulation.

Lemma 2.2.4 (Liu, Wang, and Yu [50]). *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that $|V(G)| \geq 6$ and G has no separating triangles. Suppose there exist distinct $a, b \in V(C)$ such that $G - (V(C)\backslash\{a, b\})$ has at most one Hamiltonian path between a and b. Then G has two adjacent vertices of degree 4 that are contained in $V(G)\backslash V(C)$.*

Proof. By symmetry, we only need to consider two cases: $\{a, b\} = \{u, w\}$ or $\{a, b\} = \{u, v\}$. If $\{a, b\} = \{u, w\}$ and $G - (V(C)\backslash\{a, b\}) = G - \{v, x\}$ has at most one Hamiltonian path between u and w, then by Lemma 2.2.2, $G - \{v, x\}$ is a path. Hence, by planarity, all vertices in $V(G)\backslash V(C)$ have degree 4 in G; so the assertion holds as $|V(G)\backslash V(C)| \geq 2$.

Now suppose $\{a, b\} = \{u, v\}$ and there exists at most one Hamiltonian path between u and v in $G - (V(C)\backslash\{a, b\}) = G - \{w, x\}$. Then by Lemma 2.2.3, $G - \{w, x\}$ is an outer planar near triangulation. Let $D = u_1u_2 \dots u_tu_1$ denote the outer cycle of $G - \{w, x\}$ such that $u_1 = u$ and $u_t = v$. Note that $t \geq 4$ and that u_i is adjacent to w or x for every $i \in [t]$. Let u_s, where $s \in [t]$, be the common neighbor of w and x in $V(D)$. (The existence of u_s is guaranteed by the fact that G is a near triangulation with outer cycle $uvwxu$.) Since G has no separating triangles, $2 \leq s \leq t - 1$ and every edge of $(G - \{w, x\}) - E(D)$ is incident with both paths $u_1 \dots u_{s-1}$ and $u_{s+1} \dots u_t$. It follows that $d_G(u_s) = 4$. Moreover, $s \geq 3$ and $d_G(u_{s-1}) = 4$, or $s \leq t - 2$ and $d_G(u_{s+1}) = 4$, as $|V(G)\backslash V(C)| \geq 2$ and G is a near triangulation. This completes the proof of the lemma. $\square$

2.3 A technique of Alahmadi, Aldred, and Thomassen

Let S be an independent set in a 4-connected planar triangulation G and $F \subseteq E(G)$ consist of $|S|$ edges incident with S. Alahmadi *et al.* [2] observed that $G - F$ is not 4-connected only if some vertex in S is contained in a separating 4-cycle, or some vertex in S is adjacent to three vertices in a separating 4-cycle, or two vertices in S are contained in a separating 5-cycle, or three vertices in S occur in some diamond-6-cycle.

Definition 2.3.1. A *diamond*-6-*cycle* is a graph isomorphic to the graph shown on the left in Figure Figure 2.3, in which the vertices of degree 3 are called *crucial* vertices. (A *diamond*-4-*cycle* is a graph isomorphic to the graph shown on the right in Figure Figure 2.3, where the two degree 3 vertices not adjacent to the degree 2 vertex are its *crucial* vertices.)

Definition 2.3.2. We say that S *saturates* a 4-cycle or 5-cycle C in G if $|S \cap V(C)| = 2$, and S *saturates* a diamond-6-cycle D in G if S contains three crucial vertices of D.

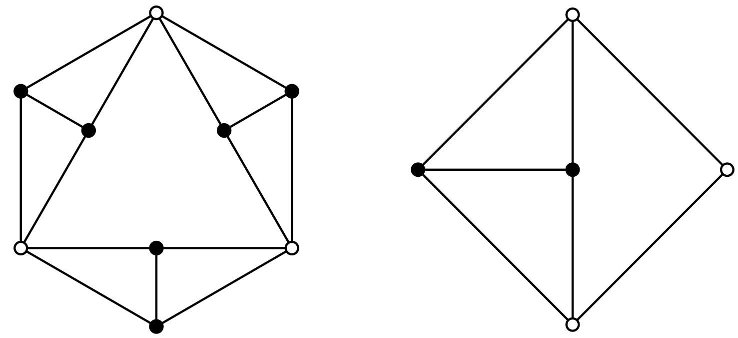

Figure 2.3: Diamond-6-cycle (left); diamond-4-cycle (right); solid vertices represent the crucial vertices.

For an n-vertex 5-connected planar triangulation G, Alahmadi *et al.*[2] showed that there exists an independent set S consisting of $\Omega(n)$ vertices of degree at most 6 in G, such that $G-F$ is 4-connected for each set F consisting of $|S|$ edges of G that are incident with S. Then it follows from a simple calculation that G has $2^{\Omega(n)}$ Hamiltonian cycles. Such large independent sets need not exist in 4-connected planar triangulations because of the existence of vertices of degree 4 or separating 4-cycles.

Next, we prove two lemmas that will help us deal with vertices of degree 4 and separating 4-cycles. Let G be a plane graph. Suppose u is a vertex of degree at most 6 in G. Define the *link* of u in G, denoted by A_u, as

$$A_u = \begin{cases} E(G[N(u)]), & \text{if } d(u) = 4, \\ \{e \in E(G) : e \text{ is incident with } u \text{ and } G-e \text{ is 4-connected}\}, & \text{if } d(u) \in \{5,6\}. \end{cases}$$

Note that in Figure 2.4, the link of the vertex of degree 4 in the left consists of the four red edges and that blue edge is not contained in the link of the vertex of degree 5 in the right.

Lemma 2.3.3 (Liu, Wang, and Yu [50])**.** *Let G be a 4-connected planar triangulation. Suppose S is an independent set of vertices of degree at most 6 in G such that, for any $u \in S$ with $d(u) \in \{5,6\}$, no degree 4 neighbors of u are adjacent in G. Then the*

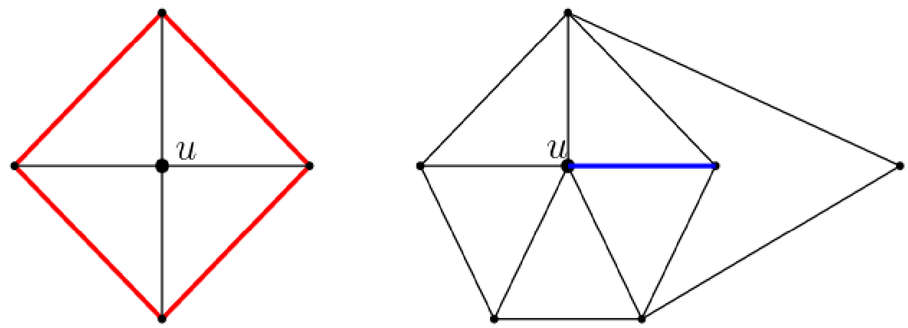

Figure 2.4: The link sets.

following statements hold:

(i) *For $u \in S$ with $d(u) \in \{5,6\}$, $\{v \in N(u) : uv \notin A_u\}$ is independent in G and, hence, $|A_u| \geq \lceil d(u)/2 \rceil$.*

(ii) *If S saturates no 4-cycle in G, then, for any distinct $u_1, u_2 \in S$, $E(G[N[u_1]]) \cap E(G[N[u_2]]) = \emptyset$.*

Proof. Suppose $u \in S$ and $d(u) \in \{5,6\}$, and suppose there exist two edges $e_1 = uv_1, e_2 = uv_2 \in E(G) \backslash A_u$ with $v_1v_2 \in E(G)$. Let $v_0, v_3 \in N(u) \backslash \{v_1, v_2\}$ be the neighbors of v_1, v_2 in $G[N(u)]$, respectively. Since $G - e_1$ is not 4-connected, there exists a vertex $z \in V(G)$ such that $\{z, v_0, v_2\}$ is a 3-cut in $G - e_1$. Since G is a planar triangulation, we have $zv_0, zv_2 \in E(G)$. Since $G - e_2$ is not 4-connected, we see from planarity that $\{z, v_1, v_3\}$ is a 3-cut in $G - e_2$. Thus, $zv_1, zv_3 \in E(G)$ as G is a planar triangulation. Since G has no separating triangles, we have $d(v_1) = d(v_2) = 4$, a contradiction. Thus, (i) holds.

For (ii), suppose S saturates no 4-cycle in G, and let $u_1, u_2 \in S$ be distinct. Suppose there exists $e \in E(G[N[u_1]]) \cap E(G[N[u_2]])$. Since S is an independent set, it follows that u_1, u_2, and the two vertices incident with e form a 4-cycle in G, contradicting the assumption that S saturates no 4-cycle in G. Hence $E(G[N[u_1]]) \cap E(G[N[u_2]]) = \emptyset$. □

The following lemma is derived by using an idea similar to one in [2].

Lemma 2.3.4 (Liu, Wang, and Yu [50]). *Let G be a 4-connected planar triangulation and S be an independent set of vertices of degree at most 6 in G. Suppose that $A_u \neq \emptyset$ for all $u \in S$, that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G, and that no degree 4 vertex of G in S has a neighbor of degree 4 in G. Let $F \subseteq \bigcup_{u\in S} A_u$ with $|F \cap A_u| \leq 1$ for all $u \in S$. Then $G - F$ is 4-connected.*

Proof. Suppose there exists some $F \subseteq \bigcup_{u\in S} A_u$ such that $|F \cap A_u| \leq 1$ for all $u \in S$, and $G - F$ is not 4-connected. Let K be a minimum cut of $G - F$; so $|K| \leq 3$. Let G_1, G_2 be subgraphs of $G - F$ such that $G - F = G_1 \cup G_2$, $V(G_1 \cap G_2) = K$, $E(G_1 \cap G_2) = \emptyset$, and $V(G_i) \neq K$ for $i = 1, 2$. Let F' be the set of the edges of G between $G_1 - K$ and $G_2 - K$. Then $F' \subseteq F$ and $F' \neq \emptyset$ (as $G - K$ is connected).

Observation 1. *Since G is a 4-connected planar triangulation, for each $e \in F'$, the two vertices incident with e have exactly two common neighbors, which must be contained in K.*

Observation 2. *For any two edges $e_1, e_2 \in F'$, there do not exist distinct vertices $u, v \in K$ such that all vertices incident with e_1 or e_2 are contained in $N_G(u) \cap N_G(v)$. For, otherwise, $G[N_G[u] \cup N_G[v]]$ contains a 4-cycle with two vertices in S, contradicting the assumption that S saturates no 4-cycle in G.*

By Observation 1, $|K| \geq 2$. By Observations 1 and 2, $|F'| \leq \binom{|K|}{2}$. Hence, $1 \leq |F'| \leq 3$. Moreover, $|K| = 3$ as, otherwise, $|K| = 2$ and $|F'| \leq \binom{2}{2} = 1$, contradicting the assumption that G is 4-connected. By the definition of A_u, if $e \in A_u \cap F'$ and $d_G(u) = 4$, then $u \in K$; if $e \in A_u \cap F'$ and $d_G(u) \in \{5, 6\}$, then e is incident with u and $u \notin K$.

Suppose $|F'| = 1$ and let $e \in F'$ with $e \in A_u$ for some $u \in S$. If $d_G(u) = 5$ or 6, then u is incident with e and $G - e$ is 4-connected by the definition of A_u, contradicting the fact that K is a 3-cut of $G - F' = G - e$. Thus $d_G(u) = 4$ and $u \in K$. Let $e = w_1w_2$ and $K = \{u, v, w\}$ such that $w_1 \in V(G_1)\backslash V(G_2)$, $w_2 \in V(G_2)\backslash V(G_1)$, and $N_G(w_1) \cap N_G(w_2) = \{u, v\}$. Again since G is a planar triangulation and K is a 3-cut in $G - e$, we have $wu, wv \in E(G)$. Hence $C_1 = uw_1vwu$ and $C_2 = uw_2vwu$ are 4-cycles in G. Let $x \in N_G(u)\backslash\{w, w_1, w_2\}$. Then $G[N_G(u)] = xw_2w_1wx$ or $G[N_G(u)] = xw_1w_2wx$. In the former case, $V(G_1)\backslash K = \{w_1\}$ as, otherwise, $\{w_1, w, v\}$ would be a 3-cut in G; so w_1 and u are two adjacent vertices of degree 4 in G, a contradiction. In the latter case, $V(G_2)\backslash K = \{w_2\}$ as, otherwise, $\{w_2, w, v\}$ would be a 3-cut in G; so w_2 and u are two adjacent vertices of degree 4 in G, a contradiction.

If $|F'| = 2$ and let $F' = \{e_1, e_2\}$, then by Observations 1 and 2, each vertex in K is adjacent to both vertices incident with some edge in F', and exactly one vertex of K is adjacent to all vertices incident with e_1 or e_2. Hence, some 5-cycle in the subgraph of G induced by K and the vertices incident with F' contains two vertices from S, contradicting the assumption that S saturates no 5-cycle in G.

Hence, $|F'| = 3$, and let $e_1, e_2, e_3 \in F'$ where $e_i \in A_{u_i}$ and $u_i \in S$ for $i = 1, 2, 3$. Since S is independent and saturates no 4-cycle or 5-cycle, F' is a matching in G. If two vertices in $\{u_1, u_2, u_3\}$ have degree 4 in G, then these two vertices are contained in K and in a 4-cycle in G, a contradiction. If exactly one vertex in $\{u_1, u_2, u_3\}$, say u_1, has degree 4 in G, then $u_1 \in K$ and u_1 must be adjacent to a vertex in $\{u_2, u_3\}$, contradicting the assumption that S is independent. So u_1, u_2, and u_3 all have degree 5 or 6 in G. But then by Observations 1 and 2, we see that the subgraph of G induced by K and the vertices of G incident with F' contains a diamond-6-cycle in which u_1, u_2, u_3 are three crucial vertices, contradicting the assumption that S saturates no diamond-6-cycle. $\square$

We also need the following two lemmas from Lo [55] and Alahmadi *et al.* [2], that will help us to find an independent set saturating no 4-cycle, or 5-cycle, or diamond-6-cycle.

Lemma 2.3.5 (Lo [55]). *Let G be a 4-connected planar triangulation and let S be an independent set of vertices of degree at most 6 in G, such that S saturates no 4-cycle in G. Then there exists a subset $S' \subseteq S$ of size at least $|S|/541$ such that S' saturates no 5-cycle in G.*

Lemma 2.3.6 (Alahmadi, Aldred, and Thomassen [2]; Lo [55]). *Let G be a 4-connected planar triangulation and let S be an independent set of vertices of degree at most 6 in G, such that S saturates no 4-cycle in G. Then there exists a subset $S' \subseteq S$ of size at least $|S|/301$ such that S' saturates no diamond-6-cycle in G.*

We need another result from Lo [55], which shows that any 4-connected planar triangulation either has a large independent set saturating no 4-cycle, or contains two vertices with many common neighbors.

Lemma 2.3.7 (Lo [55]). *Let G be a 4-connected planar triangulation. Let S be an independent set of vertices of degree at most 6 in G, and S' be a maximal subset of S such that S' saturates no 4-cycle in G. Then there exist distinct vertices $v, x \in V(G)$ such that $|(N(v) \cap N(x)) \cap S| \geq |S|/(9|S'|)$.*

The following result can be easily deduced from the previous three lemmas.

Lemma 2.3.8 (Liu, Wang, and Yu [50]). *Let G be a 4-connected planar triangulation on n vertices. Let I be an independent set of vertices of degree at most 6 in G. For any positive integer t, one of the following statements holds:*

(i) *There exist distinct vertices $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq t$.*

(ii) *There is a subset $S \subseteq I$, such that $|S| > |I|/(t \times 9 \times 541 \times 301)$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.*

Proof. Let S_1 be a maximal subset of I such that S_1 saturates no 4-cycle in G. If $|S_1| \leq |I|/(t \times 9)$, then by Lemma 2.3.7 there are distinct vertices v, x in G such that

$$|N(v) \cap N(x) \cap I| \geq |I|/(9|S_1|) \geq |I|/(9|I|/(t \times 9)) = t;$$

so (i) holds.

Now suppose $|S_1| > |I|/(t \times 9)$. By Lemmas 2.3.5 and 2.3.6, there exists $S \subseteq S_1$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G, and

$$|S| \geq |S_1|/(541 \times 301) > |I|/(t \times 9 \times 541 \times 301);$$

thus (ii) holds. $\square$

CHAPTER 3

QUDRATIC BOUND

3.1 Hamiltonian cycles through two cofacial edges

We start with a technical lemma for finding distinct Hamiltonian cycles. Recall the link A_u for a vertex u of degree at most 6 in a plane graph G.

$$A_u = \begin{cases} E(G[N(u)]), & \text{if } d(u) = 4, \\ \{e \in E(G) : e \text{ is incident with } u \text{ and } G - e \text{ is } 4\text{-connected}\}, & \text{if } d(u) \in \{5, 6\}. \end{cases}$$

Lemma 3.1.1 (Liu, Wang, and Yu [50])**.** *Let G be a 4-connected plane graph and $e \in E(G)$. Suppose u is a vertex of degree at most 6 in G such that $G[N(u)]$ is a cycle and $e \notin E(G[N[u]])$. Moreover, assume that if $d(u) \in \{5, 6\}$ then $\{v \in N(u) : uv \notin A_u\}$ is an independent set in G, and that if $d(u) = 4$ then there exist two nonadjacent neighbors of u each having degree at least 5 in G. Then the following statements hold.*

(i) *G has a Hamiltonian cycle through e as well as an edge in A_u.*

(ii) *For any $y \in V(G) \backslash \{u\}$ cofacial with e but not incident with e, $G - y$ has a Hamiltonian cycle through e and an edge in A_u not incident with y.*

Proof. Let $y \in V(G) \backslash \{u\}$ be cofacial with e but not incident with e. Consider a drawing of G in which y is contained in the infinite face of $G - y$. Let C denote the facial cycle of G containing e and y, and C' denote the outer cycle of $G - y$. Then $e \in E(C')$ as y is cofacial with e and not incident with e. Since G is 4-connected, both (G, C) and $(G - y, C')$ are circuit graphs.

Case 1. $d_G(u) \in \{5, 6\}$ and $d_G(u) - |A_u| \leq 1$.

By Lemma 2.1.7, G has a C-Tutte cycle D through e, and $G-y$ has a C'-Tutte cycle D_1 through e. Since G is 4-connected, D is a Hamiltonian cycle in G, and D_1 is a Hamiltonian cycle in $G-y$. Since $d_G(u) - |A_u| \leq 1$, both D and D_1 contain some edge in A_u. Thus, (i) and (ii) hold.

Case 2. $d_G(u) \in \{5, 6\}$ and $d_G(u) - |A_u| = 2$.

Then let $r \in N_G(u)$ such that $ur \notin A_u$, and let $G' := G - ur$.

Let C_1 be a facial cycle of G' containing e. Since G is 4-connected, (G', C_1) is a circuit graph. By Lemma 2.1.7, G' has a C_1-Tutte cycle D_1 through e, r, and u, which is a Hamiltonian cycle in G containing e. Since $d_G(u) - |A_u| = 2$ and $ur \notin A_u$, D_1 must contain an edge in A_u. Hence, (i) holds.

Let C_2 be the outer cycle of $G' - y$. Since G is 4-connected, $(G' - y, C_2)$ is a circuit graph. By Lemma 2.1.7, $G' - y$ has a C_2-Tutte cycle D_2 through e and every vertex in $\{r, u\}\backslash\{y\}$. Now D_2 is a Hamiltonian cycle in $G - y$ containing e as G is 4-connected. Moreover, D_2 contains an edge in A_u as $d_G(u) - |A_u| = 2$ and $ur \notin A_u$. Hence, (ii) holds.

Case 3. $d_G(u) \in \{5, 6\}$ and $d_G(u) - |A_u| \geq 3$.

Since $\{v \in N_G(u) : uv \notin A_u\}$ is an independent set in G, $|A_u| \geq \lceil d_G(u)/2 \rceil = 3$ (as $d_G(u) \in \{5, 6\}$). Hence, $d_G(u) = 6$ and $|A_u| = 3$ (as $d_G(u) - |A_u| \geq 3$). Let $r_1, r_2 \in N_G(u)$ such that $ur_1, ur_2 \notin A_u$, and let $G' = G - \{ur_1, ur_2\}$. Note that $r_1r_2 \notin E(G)$.

Let C_1 be a facial cycle of G' containing e. Then (G', C_1) is a circuit graph as G is 4-connected and $d_G(u) = 6$. It follows from Lemma 2.1.7 that G' has a C_1-Tutte cycle D_1 through e, r_1, and r_2. If D_1 is a Hamiltonian cycle in G, then (i) holds, since D_1 contains an edge of A_u (because $d_G(u) - |A_u| = 3$ and $ur_1, ur_2 \notin A_u$). So suppose $V(G)\backslash V(D_1) \neq \emptyset$.

Then there exists a D_1-bridge of G', say B, such that $V(B)\backslash V(D_1) \neq \emptyset$ and $V(B \cap D_1) \leq 3$. Observe that $u \in V(B)\backslash V(D_1)$ otherwise, $V(B \cap D_1)$ is a 3-cut in G, a contradiction. Since $G[N_G(u)]$ is a cycle and $r_1r_2 \notin E(G)$, $V(B) \cap V(r_1D_1r_2 - \{r_1, r_2\}) \neq \emptyset$ and $V(B) \cap V(r_2D_1r_1 - \{r_1, r_2\}) \neq \emptyset$. Thus, since $|V(B) \cap V(D_1)| \leq 3$, we may assume $V(B) \cap V(r_2D_1r_1 - \{r_1, r_2\}) = \{z\}$. Now since $d_G(u) = 6$, $\{u, z\}$ is a 2-cut in G, or

$\{u\} \cup (V(B \cap D_1) \backslash \{z\})$ is a 3-cut in G, a contradiction.

Let C_2 be the outer cycle of $G' - y$. Since G is 4-connected and $d_G(u) = 6$, $(G' - y, C_2)$ is a circuit graph. By Lemma 2.1.7, $G' - y$ has a C_2-Tutte cycle D_2 through e and every vertex in $\{r_1, r_2\} \backslash \{y\}$. Similarly, we can show that D_2 is a Hamiltonian cycle in $G - y$ containing e as G is 4-connected and D_2 is a C_2-Tutte cycle. (In particular, note that if B is a D_2-bridge and $V(B) \backslash V(D_2)$ contains a neighbor of y, then $|V(B) \cap V(D_2)| \leq 2$.) Moreover, D_2 contains an edge in A_u as $|A_u| = 3$ and $ur_1, ur_2 \notin A_u$. Hence, (ii) holds.

Case 4. $d_G(u) = 4$.

Let $G[N_G(u)] = x_1x_2x_3x_4x_1$. By our assumption on u, two nonadjacent neighbors of u must each have degree at least 5 in G. Without loss of generality, assume that $d_G(x_2) \geq 5$ and $d_G(x_4) \geq 5$.

We claim that $(G - u) + x_1x_3$ or $(G - u) + x_2x_4$ is 4-connected. For, suppose $(G - u) + x_1x_3$ is not 4-connected, and let S be a 3-cut in $(G - u) + x_1x_3$. Then $\{x_1, x_3\} \subseteq S$, and $S \cup \{u\}$ is a 4-cut in G separating x_2 and x_4. Suppose G_1, G_2 are the components of $G - (S \cup \{u\})$ containing x_2, x_4, respectively. Since $d_G(x_{2i}) \geq 5$ for $i = 1, 2$, $|V(G_i)| \geq 2$ for $i = 1, 2$. Let $w_i \in V(G_i) \backslash \{x_{2i}\}$ for $i = 1, 2$. Since G is 4-connected, there exist a path Q_i' from w_i to x_1 in $(G - (S \backslash \{x_1\})) - x_{2i}$ and a path Q_i'' from w_i to x_3 in $(G - (S \backslash \{x_3\})) - x_{2i}$. Observe that $V(Q_i' \cup Q_i'') \subseteq (V(G_i) \backslash \{x_{2i}\}) \cup \{x_1, x_3\}$. Hence, $G - \{u, x_2, x_4\}$ has two internally disjoint paths between x_1 and x_3. This implies that $(G - u) + x_2x_4$ is 4-connected.

So without loss of generality assume that $G^* = (G - u) + x_1x_3$ is 4-connected and that the edge x_1x_3 is inside the face of $G - u$ bounded by $x_1x_2x_3x_4x_1$. Let G' be the plane graph obtained from G^* by inserting two vertices r and z into the faces of G^* bounded by $x_1x_2x_3x_1$ and $x_1x_3x_4x_1$, respectively, and then adding edges rx_i for $i = 1, 2, 3$ and zx_i for $i = 1, 3, 4$.

Since G^* is 4-connected, (G', C) is a circuit graph. By Lemma 2.1.7, G' has a C-Tutte cycle D' containing e, r, and z, which is a Hamiltonian cycle in G' as G^* is 4-connected. It is easy to check that D' can be modified at r and z to give a Hamiltonian cycle in G

containing e and an edge in A_u; so (i) holds.

To prove (ii), we apply Lemma 2.1.7 to the circuit graph $(G' - y, C_1)$, where C_1 is the outer cycle of $G' - y$ containing e. Then $G' - y$ has a C_1-Tutte cycle D_1 through e, r, and z. (Note that if $uy \in E(G)$, then $\{r, z\} \cap V(C_1) \neq \emptyset$.) Since G^* is 4-connected, D_1 is a Hamiltonian cycle in $G' - y$. It is straightforward to check that D_1 can be modified to give a Hamiltonian cycle in $G - y$ containing e and an edge in A_u not incident with y. $\square$

We now prove that in a 4-connected planar triangulation on n vertices, any two cofacial edges are contained in $\Omega(n)$ Hamiltonian cycles.

Lemma 3.1.2 (Liu, Wang, and Yu [50]). *Let n be an integer with $n \geq 4$, G be a 4-connected planar triangulation on n vertices, T be a facial triangle in G, and $e_1, e_2 \in E(T)$. Then G contains at least $c_1 n$ Hamiltonian cycles through e_1 and e_2, where $c_1 = (12 \times 63 \times 541 \times 301)^{-1}$.*

Proof. We apply induction on n. Since G is a 4-connected plane graph and e_1, e_2 are cofacial in G, it follows from Lemma 2.1.4 that G has a Hamiltonian cycle through e_1 and e_2. So the assertion holds when $n \leq 1/c_1$. Now assume $n > 1/c_1$ and the assertion holds for 4-connected planar triangulations on fewer than n vertices.

Consider a drawing of G in which T is its outer cycle. Let $y \in V(T)$ be incident with both e_1 and e_2, and let e_3 be the edge in $E(T)\backslash\{e_1, e_2\}$.

We may assume that if there exist two adjacent vertices u_1, u_2 in G with $d_G(u_1) = d_G(u_2) = 4$, then $u_1u_2 = e_3$ or $y \in \{u_1, u_2\}$. For, suppose there exist $u_1, u_2 \in V(G)\backslash\{y\}$ such that $d_G(u_1) = d_G(u_2) = 4$ and $u_1u_2 \neq e_3$. We contract the edge u_1u_2 to obtain a planar triangulation G^* on $n-1$ vertices. (We retain the edges e_1 and e_2.) Note that G^* is 4-connected (as $n > 1/c_1 > 6$) and T is a triangle in G^*. So by induction, G^* has $c_1(n-1)$ Hamiltonian cycles through e_1 and e_2. Observe that all such cycles in G^* can be modified to give $c_1(n-1)$ distinct Hamiltonian cycles in G through the edges e_1, e_2, and u_1u_2. Therefore, it suffices to show that G has a Hamiltonian cycle through e_1 and e_2 but not u_1u_2, as $c_1(n-1)+1 \geq c_1 n$. So let C_1 denote the outer cycle of $G_1 := (G-y)-u_1u_2$. Observe that $e_3 \in E(C_1)$ and (G_1, C_1) is a circuit graph as G is 4-connected and planar. By Lemma 2.1.7, G_1 contains a C_1-Tutte cycle H_1 through e_3, u_1, and u_2. Moreover, H_1 is a Hamiltonian cycle in G_1 (since G is 4-connected). Therefore, $(H_1 - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G through e_1, e_2 and avoiding u_1u_2.

Since G has minimum degree at least 4 and $|E(G)| = 3n-6$ by Euler's formula, we have

$$\begin{aligned}
2(3n-6) &= 2|E(G)| \\
&= \sum_{\{v \in V(G): 4 \leq d(v) \leq 6\}} d(v) + \sum_{\{v \in V(G): d(v) \geq 7\}} d(v) \\
&\geq 4|\{v \in V(G) : d(v) \leq 6\}| + 7(n - |\{v \in V(G) : d(v) \leq 6\}|) \\
&= 7n - 3|\{v \in V(G) : d(v) \leq 6\}|.
\end{aligned}$$

It follows that $|\{v \in V(G) : d(v) \leq 6\}| \geq n/3 + 4$. By the Four Color Theorem, there exists an independent set I of vertices of degree at most 6 in G with $I \cap V(T) = \emptyset$ and $|I| \geq (n/3+4-3)/4 \geq n/12$. By Lemma 2.3.8 (with $t = 7$), either there exist distinct $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 7$, or there is a subset $S \subseteq I$ such that $|S| > c_1 n$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G. Moreover, $S \cap V(T) = \emptyset$ as $I \cap V(T) = \emptyset$.

Case 1. There exist distinct $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 7$.

Recall that any two adjacent degree 4 vertices of G cannot be contained in $V(G) \backslash V(T)$. Since $|N(v) \cap N(x) \cap I| \geq 7$, G has at least two separating 4-cycles D_1 and D_2, such that $|V(\overline{D_i})| \geq 6$ for $i = 1, 2$, and $\overline{D_1} - D_1$ and $\overline{D_2} - D_2$ are disjoint. Without loss of generality, we may assume $|V(\overline{D_1} - D_1)| \leq n/2$. By our assumptions on G and applying Lemma 2.2.4, we see that $\overline{D_1} - (V(D_1) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b for any distinct $a, b \in V(D_1)$.

Let G_1^* be obtained from G by contracting $\overline{D_1} - D_1$ to a new vertex v_1. Observe that G_1^* is a 4-connected planar triangulation with outer cycle T. It follows by induction that G_1^* has at least $c_1(n - |V(\overline{D_1} - D_1)| + 1) \geq c_1 n/2$ Hamiltonian cycles through e_1 and e_2. For each such Hamiltonian cycle in G_1^*, say H^*, let $a_1, b_1 \in N_{G_1^*}(v_1)$ such that $a_1 v_1 b_1 \subseteq H^*$. We can then form a Hamiltonian cycle in G through e_1 and e_2 by taking the union of $H^* - v_1$ and a Hamiltonian path between a_1 and b_1 in $\overline{D_1} - (V(D_1)\backslash\{a_1, b_1\})$. Thus G has at least $2(c_1 n/2) = c_1 n$ Hamiltonian cycles through e_1 and e_2.

Case 2. There is an independent set S of vertices of degree at most 6 in G such that $|S| > c_1 n$, $S \cap V(T) = \emptyset$, and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

If there exist distinct $u_1, u_2 \in S$ such that $|N_G(u_i) \cap V(T)| \geq 2$ for $i \in [2]$, then u_1, u_2 are contained in a 4-cycle or a 5-cycle in G, a contradiction. Hence, at most one vertex in S, say x, is adjacent to two vertices in $V(T)$. Let $S' = S$ if x does not exist, and $S' = S\backslash\{x\}$ if x exists. Hence, $|S'| \geq |S| - 1$ and for all $u \in S'$, $|N_G(u) \cap V(T)| \leq 1$.

Next we show that S' satisfies the conditions of Lemma 2.3.3 and Lemma 2.3.4. First suppose $d_G(y) > 4$. If two degree 4 vertices are adjacent in G, they must be the two vertices in $V(T)\backslash\{y\}$. Hence, for any $u \in S'$, since $|N_G(u) \cap V(T)| \leq 1$, if $d_G(u) = 4$ then u is not adjacent to a degree 4 vertex in G, and if $d_G(u) \in \{5, 6\}$ then no degree 4 neighbors of u are adjacent in G. Now assume that $d_G(y) = 4$. Notice that for any $v \in N_G(y)$, $|N_G(v) \cap V(T)| = 2$, and thus, $N_G(y) \cap S' = \emptyset$. Hence, for any $u \in S'$, if $d_G(u) = 4$ then u is adjacent to no degree 4 vertex in G, and if $d_G(u) \in \{5, 6\}$ then no degree 4 neighbors of u are adjacent in G. Therefore, S' satisfies the conditions of Lemma 2.3.3 and Lemma 2.3.4.

Let $k := |S'|$ and $S' = \{u_1, u_2, \ldots, u_k\}$. Recall the definition of A_{u_i} for $i \in [k]$, and let $A_i := A_{u_i}\backslash\{e \in E(G) : e \text{ is incident with } y\}$. By Lemma 2.3.3, $E(G[N_G[u_i]]) \cap E(G[N_G[u_j]]) = \emptyset$ for $i \neq j$, and if $d_G(u_i) \in \{5, 6\}$ then $\{v \in N_G(u_i) : vu_i \notin A_{u_i}\}$ is independent in G; so $|A_{u_i}| \geq 3$ for all $u_i \in S'$. Hence, $|A_i| \geq 2$ for all $i \in [k]$. Note that $e_3 \notin E(G[N_G[u_i]])$, as $S' \cap V(T) = \emptyset$ and $|N_G(u_i) \cap V(T)| \leq 1$. We now find $k + 1 > c_1 n$

Hamiltonian cycles $H_1, \ldots, H_{k+1}$ in G, as follows.

Let $F_0 = \emptyset$ and $X_1 := G - F_0 = G$. Note that $e_3 \notin E(G[N_G[u_1]])$ and by Lemma 2.3.3, u_1 satisfies the conditions of Lemma 3.1.1 (with u_1, e_3, X_1 as u, e, G in Lemma 3.1.1, respectively). Since y is cofacial with e_3 but not incident with e_3, it follows from (ii) of Lemma 3.1.1 that $X_1 - y = G - y$ has a Hamiltonian cycle D_1 through e_3 and an edge $f_1 \in A_1$. Hence, $H_1 = (D_1 - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G through e_1, e_2, and $f_1 \in A_1$. Set $F_1 = \{f_1\}$.

Suppose for some $j \in [k+1]$ ($j \geq 2$) we have found an edge set $F_{j-1} = \{f_1, \ldots, f_{j-1}\}$ where $f_i \in A_i$ for each $i \in [j-1]$, and a Hamiltonian cycle H_l in $X_l := G - F_{l-1}$ for each $l \in [j-1]$, such that $\{e_1, e_2, f_l\} \subseteq E(H_l)$ and $F_{l-1} \cap E(H_l) = \emptyset$. Consider the graph $X_j := G - F_{j-1}$. By Lemma 2.3.4, X_j is 4-connected. When $j = k+1$, $X_{k+1} := G - F_k$ is 4-connected; so by Lemma 2.1.4, X_{k+1} has a Hamiltonian cycle H_{k+1} through e_1 and e_2. We stop this process and output the desired $H_1, \ldots, H_{k+1}$. Now suppose $j \leq k$. Note that $G[N_G[u_j]]$ is a subgraph of X_j (as $E(G[N_G[u_j]]) \cap A_{u_l} = \emptyset$ for any $l \in [j-1]$), and that $e_3 \in E(X_j) \backslash E(G[N_G[u_j]])$. We now show that u_j satisfies the conditions of Lemma 3.1.1 (with u_j, e_3, X_j as u, e, G in Lemma 3.1.1, respectively). Since $u_j \in S'$, $X_j - f$ is 4-connected for any $f \in A_{u_j}$ (by Lemma 2.3.4), and the link of u_j in X_j is A_{u_j} (as $G[N_G[u_j]] \subseteq X_i \subseteq G$). Hence, if $d_{X_j}(u_j) = d_G(u_j) \in \{5, 6\}$ then by Lemma 2.3.3, $\{v \in N_{x_j}(u_j) : vu_j \notin A_{u_j}\} = \{v \in N_G(u_j) : vu_j \notin A_{u_j}\}$ is independent in G (hence, in X_j); if $d_{X_j}(u_j) = d_G(u_j) = 4$ then all neighbors of u_j each have degree at least 5 in X_j, as $A_{u_j} = E(G[N_G(u_j)])$ and $X_j - f$ is 4-connected for any $f \in A_{u_j}$. Therefore, by (ii) of Lemma 3.1.1, $X_j - y$ has a Hamiltonian cycle D_j through e_3 and some edge $f_j \in A_j$. Now $H_j = (D_j - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G such that $\{e_1, e_2, f_j\} \subseteq E(H_j)$. Note that $F_{j-1} \cap E(H_j) = \emptyset$ as $D_j \subseteq X_j$. Set $F_j = F_{j-1} \cup \{f_j\}$.

Therefore, G has at least $k+1 = |S'| + 1 > c_1 n$ Hamiltonian cycles through e_1, e_2. $\square$

3.2 Proof of quadratic bound

Proof of Theorem 1.3.1. Let $c_2 := (12 \times 90 \times 541 \times 301)^{-1}$ and $c = c_2^2/2$. We show that every 4-connected planar triangulation on n vertices has at least cn^2 Hamiltonian cycles. It is easy to check that the assertion holds when $n \le 1/\sqrt{c} = \sqrt{2}/c_2$ as every 4-connected planar graph is Hamiltonian by Tutte's theorem (or by Lemma 2.1.4). Hence we may assume that $n > \sqrt{2}/c_2$ and that the assertion holds for 4-connected planar triangulations on fewer than n vertices.

Case 1. G contains two adjacent vertices of degree 4.

Let $u_1, u_2 \in V(G)$ such that $u_1u_2 \in E(G)$ and $d_G(u_1) = d_G(u_2) = 4$. Let G^* be the graph obtained from G by contracting the edge u_1u_2 to a new vertex u^*. By induction, G^* has at least $c(n-1)^2$ Hamiltonian cycles from which we obtain at least $c(n-1)^2$ Hamiltonian cycles in G through the edge u_1u_2.

Let x_1, x_2, x_3, x_4 be the vertices that occur on $G[N_{G^*}(u^*)]$ in the clockwise order such that $N_G(u_1) \cap N_G(u_2) = \{x_2, x_4\}$. Note that $u_1u_2x_2u_1, u_1u_2x_4u_1$ are two triangles in G. By Lemma 3.1.2, G has at least c_1n Hamiltonian cycles through u_1x_{2i} and u_2x_{2i} for each $i \in [2]$. Observe that if H is a Hamiltonian cycle in G through u_1x_{2i} and u_2x_{2i}, then H is a Hamiltonian cycle in $G - u_1u_2$. Therefore, G contains at least $2c_1n$ Hamiltonian cycles all avoiding the edge u_1u_2. Hence, there exist at least $c(n-1)^2 + 2c_1n \ge cn^2$ Hamiltonian cycles in G.

Case 2. No two vertices of degree 4 in G are adjacent.

Recall that G contains an independent set I of vertices of degree at most 6 with $|I| \ge n/12$. By Lemma 2.3.8 (with $t = 10$), either there exist distinct vertices $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \ge 10$, or G contains $S \subseteq I$ such that $|S| \ge c_2n$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

Suppose the former case holds. Since no two vertices of degree 4 in G are adjacent, we can find a separating 4-cycle D such that $1 < |V(\overline{D} - D)| \le n/4$. We contract $\overline{D} - D$ to a vertex and denote this new graph by G_0. Note that G_0 is a 4-connected planar triangulation with $3n/4 \leq |V(G_0)| = n - |V(\overline{D} - D)| + 1 \leq n - 1$; so G_0 has at least $c(3n/4)^2$ Hamiltonian cycles by induction. Therefore, G has at least $2c(3n/4)^2 \geq cn^2$ Hamiltonian cycles, as by Lemma 2.2.4, $\overline{D} - (V(D) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b for any distinct $a, b \in V(D)$.

Now assume that there exists an independent set S of vertices of degree at most 6 in G with $|S| \geq c_2 n$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G. By our assumptions on G, S satisfies the conditions in Lemma 2.3.3 and Lemma 2.3.4. Let $S = \{u_1, u_2, \ldots, u_k\}$. Recall the definition of A_{u_i}, the link of u_i for each $i \in [k]$.

Let $F = \{f_1, f_2, \ldots, f_k\}$, where $f_i \in A_{u_i}$ for $i \in [k]$. Let $F_j = \{f_1, \ldots, f_j\}$ for each $j \in [k]$ and let $F_0 = \emptyset$.

We claim that for each integer $j \in [k]$, there exists a collection of Hamiltonian cycles in G, say $\mathcal{C}_j$, such that $|\mathcal{C}_j| = k - j + 1$ and every cycle in $\mathcal{C}_j$ contains f_j but no edge from F_{j-1}. For each $j \in [k]$, let $X_j := G - F_{j-1}$. By Lemma 2.3.4, X_j is 4-connected for each j. If $j = k$, it follows from Lemma 2.1.4 that X_k has a Hamiltonian cycle $H_{k+1}^{(k)}$ through the edge f_k. Moreover, $F_{k-1} \cap E(H_{k+1}^{(k)}) = \emptyset$ as $H_{k+1}^{(k)} \subseteq X_k$. Let $\mathcal{C}_k = \{H_{k+1}^{(k)}\}$.

Now assume $j < k$. Let $F_j^{(j)} = \emptyset$ and $Y_{j+1}^{(j)} := X_j - F_j^{(j)} = X_j$. Note that $f_j \in X_j$ as $X_j = G - F_{j-1}$. Note that u_{j+1} satisfies the conditions in Lemma 3.1.1 (with u_{j+1}, f_j, X_j as u, e, G in Lemma 3.1.1, respectively), since $u_{j+1} \in S$, S satisfies the conditions in Lemmas 2.3.3 and 2.3.4, and $F_{j-1} \subseteq \cup_{i=1}^{j-1} A_{u_i}$ if $j \geq 1$. Then by (i) of Lemma 3.1.1, X_j has a Hamiltonian cycle $H_{j+1}^{(j)}$ containing f_j and some edge $f_{j+1}^{(j)} \in A_{u_{j+1}}$. Set $F_{j+1}^{(j)} = \{f_{j+1}^{(j)}\}$. For $j + 2 \leq l \leq k + 1$, suppose we have found an edge set $F_{l-1}^{(j)} = \{f_{j+1}^{(j)}, \ldots, f_{l-1}^{(j)}\}$, where $f_t^{(j)} \in A_{u_t}$ for $j + 1 \leq t \leq l - 1$, such that, for each $j + 1 \leq t \leq l - 1$, $Y_t^{(j)} := X_j - F_{t-1}^{(j)}$ is 4-connected and has a Hamiltonian cycle $H_t^{(j)}$ through f_j and $f_t^{(j)}$. Consider $Y_l^{(j)} := X_j - F_{l-1}^{(j)}$. Then $Y_l^{(j)}$ is 4-connected by Lemma 2.3.4. If $l = k + 1$, then, by Lemma 2.1.4, $Y_{k+1}^{(j)} := X_j - F_k^{(j)}$ has a Hamiltonian cycle $H_{k+1}^{(j)}$ through f_j and $F_k^{(j)} \cap H_{k+1}^{(j)} = \emptyset$. We stop the process and output the desired $\mathcal{C}_j = \{H_{j+1}^{(j)}, H_{j+2}^{(j)}, \ldots, H_{k+1}^{(j)}\}$. Now assume that

$l < k+1$. Then $G[N_G[u_l]]$ is a subgraph of $Y_l^{(j)}$, and u_l in $Y_l^{(j)}$ satisfies the conditions in Lemma 3.1.1. Since $f_j \in E(Y_l^{(j)}) \backslash E(G[N_G[u_l]])$, we apply (i) of Lemma 3.1.1 to find a Hamiltonian cycle $H_l^{(j)}$ in $Y_l^{(j)}$ through f_j and an edge $f_l^{(j)}$ in A_{u_l}. Set $F_l^{(j)} = F_{l-1}^{(j)} \cup \{f_l^{(j)}\}$.

Hence, by the above claim, G has at least

$$\sum_{i=1}^{k} |\mathcal{C}_j| \geq \sum_{j=1}^{k} (k+1-j) = k(k+1)/2 > c_2^2 n^2/2 = cn^2$$

Hamiltonian cycles. $\square$

CHAPTER 4

RESTRICTING DEGREE 4 VERTICES

4.1 Nested separating 4-cycles

In this section, we show several lemmas about nested separating 4-cycles in a planar triangulation.

Lemma 4.1.1. *Let G be a 4-connected planar triangulation. Let S be an independent set in G saturating no 4-cycle or 5-cycle in G. Let $u, u' \in S$ be distinct and let D_u and $D_{u'}$ be 4-cycles containing u and u', respectively. Then $|V(D_u) \cap V(D_{u'})| \leq 2$, $V(D_u) \cap V(D_{u'}) \cap S = \emptyset$, and if $V(D_u) \cap V(D_{u'})$ consists of two vertices, say a and b, then $ab \in E(D_u) \cap E(D_{u'})$.*

Proof. Note that $u' \notin V(D_u)$ since $\{u, u'\}$ saturates no 4-cycle in G. Similarly, $u \notin V(D_{u'})$. So $V(D_u) \cap V(D_{u'}) \cap S = \emptyset$. Moreover, $|V(D_u) \cap V(D_{u'})| \leq 2$ since otherwise u, u' are contained in a 4-cycle in $G[D_u + u']$, contradicting the assumption that S saturates no 4-cycle in G.

Now suppose $V(D_u) \cap V(D_{u'}) = \{a, b\}$ with $a \neq b$. If $ab \in E(D_u) \backslash E(D_{u'})$ then $G[D_u + u']$ has a 5-cycle containing u and u', contradicting the assumption that S saturates no 5-cycle in G. So, $ab \notin E(D_u) \backslash E(D_{u'})$. Similarly, $ab \notin E(D_{u'}) \backslash E(D_u)$. If $ab \notin E(D_u) \cup E(D_{u'})$, then u, u' are contained in a 4-cycle in G, a contradiction. Thus, $ab \in E(D_u) \cap E(D_{u'})$. □

We now prove a technical lemma, which will be used in the proof of Lemma 4.1.3 to produce two Hamiltonian paths in a near triangulation.

Lemma 4.1.2. *Let G be a near triangulation with outer cycle C, and let $x_1, w_1, w_2, x_2 \in V(C)$ be distinct and occur on C in clockwise order such that $x_1x_2, w_1w_2 \in E(C)$ and each* *edge of* $G - E(C)$ *is incident with both* $x_1 C w_1$ *and* $w_2 C x_2$*. Let* $N_G(x_1) \cap N_G(x_2) = \{r\}$ *and* $N_G(w_1) \cap N_G(w_2) = \{y\}$*, and assume* $r \notin \{y, w_1, w_2\}$ *and* $y \notin \{r, x_1, x_2\}$*. Suppose any two degree* 3 *vertices of* G *contained in* $V(G)\backslash\{x_1, x_2, w_1, w_2\}$ *have distance at least three in* G*. Then* $G - \{x_1, x_2, w_1, w_2\}$ *has a Hamiltonian path between* r *and* y*.*

Proof. Note that $|V(G)| \geq 6$ as $r \notin \{y, w_1, w_2\}$ and $y \notin \{r, x_1, x_2\}$. We apply induction on $|V(G)|$. Without loss of generality, we may assume $r \in V(x_1 C w_1)$. Then $d_G(x_1) = 2$.

Suppose $|V(G)| = 6$. If $ry \in E(G)$ then we are done. So assume $ry \notin E(G)$. Then $y \in V(w_2 C x_2)$, $x_2 w_1 \in E(G)$, and $d_G(r) = d_G(y) = 3$. This gives a contradiction since $d_G(r, y) = 2$.

Now assume $|V(G)| > 6$. We have two cases: $y \in V(x_1 C w_1)$ or $y \in V(w_2 C x_2)$.

Case 1. $y \in V(x_1 C w_1)$. Then $d_G(w_1) = 2$.

Consider $G_1 = G - w_1$. Let y' denote the unique vertex in $N_{G_1}(y) \cap N_{G_1}(w_2)$. If $y' \notin \{r, x_2\}$ then, by induction, $G_1 - \{x_1, x_2, y, w_2\}$ has a Hamiltonian path H_1 between r and y'; so $H_1 + y'y$ gives a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$. Hence, we may assume $y' \in \{r, x_2\}$.

If $y' = r$ then $ry, rw_2 \in E(G)$ and $d_G(y) = 3$. Since $|V(G)| \geq 7$, $|V(w_2 C x_2)| \geq 3$. Now, y and a degree 3 vertex of G contained in $V(w_2 C x_2)\backslash\{x_2, w_2\}$ are distance 2 apart in G, a contradiction. So $y' = x_2$. Then $x_2 y, x_2 w_2 \in E(G)$. Hence, since each edge of $G - E(C)$ is incident with both $x_1 C w_1$ and $x_2 C w_2$, $V(rCy) \subseteq N_G(x_2)$, and all vertices in $V(rCy - y)$ have degree 3 in G. Since $|V(G)| \geq 7$ and $x_2 w_2 \in E(G)$, $rCy - y$ contains two adjacent vertices of degree 3 in G, a contradiction.

Case 2. $y \in V(w_2 C x_2)$. Then $d_G(w_2) = 2$.

Consider $G_2 = G - w_2$. Let y' denote the unique vertex in $N_{G_2}(y) \cap N_{G_2}(w_1)$. Similar to Case 1, if $y' \notin \{r, x_2\}$ then, by induction, $G_2 - \{x_1, x_2, y, w_1\}$ has a Hamiltonian path H_2 between r and y'. Hence, $H_2 + y'y$ is a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$.

If $y' = r$, then let x_2' be the neighbor of x_2 on w_2Cx_2; now $rx_2' \cup yCx_2'$ is a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$. If $y' = x_2$, then $x_2w_1, x_2y \in E(G)$. It follows that $d_G(r) = d_G(y) = 3$, which gives a contradiction since $d_G(r, y) = 2$. $\square$

Recall that for a cycle D in G, $\overline{D}$ is the subgraph of G consisting of all vertices and edges of G contained in the closed disc bounded by D. In the proof of Theorem 1.3.2, we will need to consider the subgraphs of a planar triangulation that lie between two separating 4-cycles and use the following result on Hamiltonian paths in those subgraphs.

Lemma 4.1.3. *Let G be a 4-connected planar triangulation in which the distance between any two vertices of degree 4 is at least three. Let S be an independent set in G such that S saturates no 4-cycle or 5-cycle in G. Let $u, u' \in S$ be distinct, and $D_u, D_{u'}$ be separating 4-cycles in G containing u and u', respectively. Suppose $\overline{D_{u'}} \subseteq \overline{D_u}$, and $D_{u'}$ is a maximal separating 4-cycle containing u' in G, i.e., $\overline{D_{u'}}$ is not contained in $\overline{D}$ for any other separating 4-cycle $D \neq D_{u'}$ with $u' \in V(D)$. Let H denote the graph obtained from $\overline{D_u}$ by contracting $\overline{D_{u'}} - D_{u'}$ to a new vertex z so that H is a near triangulation with outer cycle D_u. Then one of the following holds:*

(i) *For any distinct $a, b \in V(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b.*

(ii) *There exist distinct $a, b \in V(D_u)$ such that $H - (V(D_u) \backslash \{a, b\})$ has a unique Hamiltonian path, say P, between a and b; but for any distinct $c, d \in V(D_u)$ with $\{c, d\} \neq \{a, b\}$, $H - (V(D_u) \backslash \{c, d\})$ has at least two Hamiltonian paths between c and d and avoiding an edge of P incident with z.*

Proof. Let $D_u = uvwxu$ and $D_{u'} = u'v'w'x'u'$. Without loss of generality, assume that u, v, w, x occur on D_u in clockwise order, and u', v', w', x' occur on $D_{u'}$ in clockwise order.

By Lemma 4.1.1, we have $|V(D_u) \cap V(D_{u'})| \leq 1$ or $|E(D_u) \cap E(D_{u'})| = 1$. Thus, $|V(H)| \geq 7$, and for any distinct $a, b \in V(D_u)$ with $ab \notin E(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ is not a path. So by Lemma 2.2.2, we have

Claim 1. *For any distinct $a, b \in V(D_u)$ with $ab \notin E(D_u)$, $H - (V(D_u)\backslash\{a, b\})$ has at least two Hamiltonian paths between a and b.*

Claim 2. *We may assume $V(D_u) \cap V(D_{u'}) \neq \emptyset$.*

Proof. For, suppose $V(D_u) \cap V(D_{u'}) = \emptyset$. Then z is not incident with the infinite face of $H - D_u$. So for any distinct $a, b \in V(D_u)$ with $ab \in E(D_u)$, $H - (V(D_u)\backslash\{a, b\})$ cannot be an outer planar graph. Thus, by Lemma 2.2.3, $H - (V(D_u)\backslash\{a, b\})$ has at least two Hamiltonian paths between a and b. So (i) holds by Claim 1. □

Claim 3. *We may further assume that $|V(D_u) \cap V(D_{u'})| = 2$.*

Proof. For, suppose $V(D_u) \cap V(D_{u'})$ consists of exactly one vertex, say y. Then $y \in \{v, w, x\} \cap \{v', w', x'\}$. We show that (i) holds. By Claim 1, it suffices to consider distinct $a, b \in V(D_u)$ with $ab \in E(D_u)$. By Lemma 2.2.3, it suffices to show that $H - (V(D_u)\backslash\{a, b\})$ is not outer planar.

Let $a, b \in V(D_u)$ with $ab \in E(D_u)$. If $y \in \{a, b\}$, then z is not incident with the infinite face of $H - (V(D_u)\backslash\{a, b\})$; so $H - (V(D_u)\backslash\{a, b\})$ is not outer planar. Hence we may assume that $y \notin \{a, b\}$. Let $D_u = yy_1y_2y_3y$ and assume that y, y_1, y_2, y_3 occur on D_u in clockwise order. Then $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$.

First, assume that $y \in \{v', x'\}$. We consider $y = v'$, as the other case $y = x'$ is symmetric. If $H - \{y, y_3\}$ is outer planar, then u' is adjacent to the vertices y, y_3 in $V(D_u)$. Since $yy_3 \in E(D_u)$, $G[D_u + u']$ has a 5-cycle containing u and u', contradicting the assumption that S saturates no 5-cycle. Hence, $H - (V(D_u)\backslash\{y_1, y_2\}) = H - \{y, y_3\}$ is not outer planar. It remains to consider $H - (V(D_u)\backslash\{y_2, y_3\}) = H - \{y, y_1\}$. Suppose $H - \{y, y_1\}$ is outer planar. Then w' and x' are incident with the infinite face of $H - \{y, y_1\}$ and $w'y_1 \in E(G)$. We claim that $x'y_1 \in E(G)$; otherwise $x'y \in E(G)$, implying that $x'w'yx'$ or $x'u'yx'$ is a separating triangle in G, a contradiction. But then $D = u'yy_1x'u'$ is a separating 4-cycle in G containing u', and $\overline{D}$ properly contains $\overline{D_{u'}}$, contradicting the maximality of $D_{u'}$.

Suppose $y = w'$. For $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$, if $H - (V(D_u)\backslash\{a, b\})$ is not outer planar, then $u'y_3 \in E(G)$ or $u'y_1 \in E(G)$. So $D = u'y_3w'x'u'$ or $D = u'y_1w'v'u'$ is a separating 4-cycle in G such that $\overline{D}$ properly contains $\overline{D_{u'}}$. Thus, $H - (V(D_u)\backslash\{a, b\})$ cannot be outer planar for $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$. $\square$

By Claim 3, $|E(D_u) \cap E(D_{u'})| = 1$; so $V(D_u) \cap V(D_{u'}) = \{v, w\}$ or $V(D_u) \cap V(D_{u'}) = \{w, x\}$. By the symmetry among the edges in D_u and between the two orientations of D_u, we may further assume $V(D_u) \cap V(D_{u'}) = \{v, w\}$.

Claim 4. *For $\{a, b\} \subseteq V(D_u)$ with $ab \in E(D_u)$, if $\{a, b\} \neq \{u, x\}$, then $H - (V(D_u)\backslash\{a, b\})$ has at least two Hamiltonian paths between a and b.*

Proof. For $\{a, b\} = \{v, w\}$, since z is not incident with the infinite face of $H - \{x, u\}$, $H - (V(D_u)\backslash\{a, b\}) = H - \{x, u\}$ is not outer planar and has at least two Hamiltonian paths between a and b by Lemma 2.2.3.

For $\{a, b\} = \{u, v\}$ or $\{a, b\} = \{w, x\}$, $H - (V(D_u)\backslash\{a, b\})$ cannot be outer planar. Otherwise, one can check that $\{u, u'\}$ is contained in a 4-cycle or 5-cycle in G. Hence, by Lemma 2.2.3, there exist at least two Hamiltonian paths between a and b in $H - (V(D_u)\backslash\{a, b\})$ when $\{a, b\} = \{u, v\}$ or $\{a, b\} = \{w, x\}$. $\square$

By Claim 4, we may assume that $H - \{v, w\}$ has a unique Hamiltonian path P between u and x, as otherwise (i) holds. It follows from Lemma 2.2.3 that $H - \{v, w\}$ is an outer planar near triangulation. Let y' denote the vertex in $V(D_{u'})\backslash\{u', v, w\}$; so $y' = x'$ or $y' = v'$. Note that $V(uPz) \subseteq N_H(v)$, that $V(xPz) \subseteq N_H(w)$, and that P contains $u'zy'$. Let r denote the unique vertex in $N_H(u) \cap N_H(x)$. Observe that $\{v, w\} = \{v', w'\}$ or $\{v, w\} = \{w', x'\}$. Recall the definition of diamond-4-cycle in Figure Figure 2.3.

Claim 5. *There exists a vertex $y \in V(P)\backslash\{u, x, z\}$ such that $yu' \in E(P)$ and $D' := G[D_{u'} + y]$ is a diamond-4-cycle with u' and y as crucial vertices. Moreover, $r \notin \{y, y'\}$.*

Proof. First, suppose $\{v, w\} = \{v', w'\}$, i.e. $v = v'$ and $w = w'$. Then $y' = x'$. Hence, there exists a vertex y in $V(uPu')\backslash\{u, u'\}$ such that $yu' \in E(P)$ and $yv \in E(G)$. If $yx' \notin E(G)$, then u' has a neighbor z' in $V(xPx')$ since $H - \{v, w\}$ is an outer planar near triangulation; now $u'v'w'z'u'$ is a separating 4-cycle in G containing u' (as $z'w = z'w' \in E(G)$), contradicting the maximality of $D_{u'}$. Therefore, $yx' \in E(G)$ and $G[D_{u'} + y]$ is a diamond-4-cycle with crucial vertices u' and y. Moreover, $r \notin \{y, x'\} = \{y, y'\}$; otherwise, $uvu'yu$ (when $r = y$) or $uvu'x'u$ (when $r = y'$) is a 4-cycle saturated by S, a contradiction.

Now assume that $\{v, w\} = \{w', x'\}$, i.e., $v = w'$ and $w = x'$. Then $y' = v'$. Observe that $u'x \notin E(G)$, otherwise $uvwu'xu$ is a 5-cycle in G saturated by S, a contradiction. Hence, there exists $y \in V(u'Px)\backslash\{u', x\}$ such that $yu' \in E(P)$ and $yw \in E(G)$. Now $yv' \in E(G)$ by the maximality of $D_{u'}$. Therefore, $G[D_{u'} + y]$ is a diamond-4-cycle in G in which u', y are crucial vertices. If $r = y' = v'$ then $uvwu'y'u$ is a 5-cycle in G containing $\{u, u'\}$, and if $r = y$ then $uyu'wxu$ is a 5-cycle in G containing $\{u, u'\}$. This contradicts the assumption that S saturates no 5-cycle in G, completing the proof of Claim 5. □

We need another claim, in order to show that for any $\{c, d\} \neq \{u, x\}$, $H - (V(D_u)\backslash\{c, d\})$ has at least two Hamiltonian paths between c and d and not containing $u'zy'$. Let $H' := H - (V(D_u) \cap V(D_{u'}) \cup \{z\}) = H - \{v, w, z\}$. Then H' is an outer planar near triangulation and $H' \subseteq G$.

Claim 6. *$r \in V(H')\backslash\{y, u', y'\}$ and $H' - \{u, x, u', y'\}$ has a Hamiltonian path P_1 between r and y.*

Proof. Since $r \in N_G(u)$ and S is independent, $r \neq u'$. By Claim 5, $r \notin \{y, y'\}$ and $y \notin \{u, x, y'\}$. Thus, $r \notin \{y, y', u'\}$.

Let C denote the outer cycle of H'. Then $ux, u'y' \in E(C)$. We may assume $y' = x'$; the other case is similar.

Since G contains no separating triangle, each edge in $H' - E(C)$ is incident with both

uPu' and xPy'. Since $V(uPu') \subseteq N_G(v)$ and $V(xPy') \subseteq N_G(w)$, every degree 4 vertex of G in $V(H')\backslash\{u, x, u', y'\}$ has degree 3 in H'. Hence, by assumption of the lemma, the distance between any two degree 3 vertices of H', contained in $V(H')\backslash\{u, x, u', y'\}$, is at least three in H'. Applying Lemma 4.1.2 to H', we see that $H' - \{u, x, u', y'\}$ has a Hamiltonian path P_1 between r and y. $\square$

Let $Q_1 := P_1 \cup yu'y'z$ and $Q_2 := P_1 \cup yy'u'z$. Then Q_1 and Q_2 are two distinct Hamiltonian paths between r and z in $H - V(D_u)$, and neither contains $u'zy'$. We now show that (ii) holds with $\{a, b\} = \{u, x\}$. Let $c, d \in V(D_u)$ be distinct such that $\{c, d\} \neq \{u, x\}$. Observe that one vertex in $\{c, d\}$ is a neighbor of r and the other is a neighbor of z. We may assume $c \in N_H(r)$ and $d \in N_H(z)$. Then $cr \cup Q_1 \cup zd$, $cr \cup Q_2 \cup zd$ are two distinct Hamiltonian paths in $H - (V(D_u)\backslash\{c, d\})$ between c and d and not containing $u'zy'$. $\square$

We also need the following result, which is given implicitly in the proof of Theorem 1.3 in [51].

Lemma 4.1.4 (Liu and Yu [51]). *Let G be a 4-connected planar triangulation. Assume that G contains a collection of separating 4-cycles, say $\mathcal{D} = \{D_1, D_2, \dots, D_{t+1}\}$, such that $\overline{D_1} \supseteq \overline{D_2} \supseteq \cdots \supseteq \overline{D_{t+1}}$. For $j \in [t]$, let G_j be the graph obtained from $\overline{D_j}$ by contracting $\overline{D_{j+1}} - D_{j+1}$ to a new vertex, denoted by z_{j+1}. Suppose the conclusion of Lemma 4.1.3 holds for G_j and z_{j+1} (as H and z, respectively, in Lemma 4.1.3). Then G has at least $2^{\sqrt{t}}$ Hamiltonian cycles.*

Proof. Let G_0 be the graph obtained from G by contracting $\overline{D_1} - D_1$ to the vertex z_1. We see that G_0 is 4-connected. Hence G_0 has a Hamiltonian cycle, say F_0, and let $a_1, b_1 \in V(D_1)$ such that $a_1 z_1 b_1 \subseteq F_0$. We now define a rooted tree T whose root r represents F_0, and whose leaves are Hamiltonian cycles in G. Note that G_1 is a near triangulation with outer cycle D_1 and no separating triangles. For each Hamiltonian path P_1 in $G_1 - (V(D_1)\backslash\{a_1, b_1\})$ between a_1 and b_1, $F_1 := (F_0 - z_1) \cup P_1$ is a Hamiltonian cycle in $G_1 \cup (G - (\overline{D_1} - D_1))$; we add a neighbor to r in T to represent F_1. This defines all vertices of T at distance 1 from the root r. Now, suppose we have defined all vertices of T at distance s from r, for some $s \in [t-1]$, each of which represents a Hamiltonian cycle in $G_s \cup (G - (\overline{D_s} - D_s))$. To define the vertices of T at distance $s+1$ from r, we let v be an arbitrary vertex in T that is at distance s from r. Then v represents a Hamiltonian cycle F_s in $G_s \cup (G - (\overline{D_s} - D_s))$. Let $a_{s+1}, b_{s+1} \in V(D_{s+1})$ such that $a_{s+1}z_{s+1}b_{s+1} \subseteq F_{s+1}$. For each Hamiltonian path P_{s+1} in $G_{s+1} - (V(D_{s+1}) \backslash \{a_{s+1}, b_{s+1}\})$ between a_{s+1} and b_{s+1}, $F_{s+1} := (F_s - z_{s+1}) \cup P_{s+1}$ is a Hamiltonian cycle in $G_{s+1} \cup (G - (\overline{D_{s+1}} - D_{s+1}))$; we add a neighbor to v in T to represent F_{s+1}. Suppose we defined all vertices of T at distance at most t from r. For each vertex v of distance t from r in T, v represents a Hamiltonian cycle F_t in $G_t \cup (G - (\overline{D_t} - D_t))$. Let $a_{t+1}, b_{t+1} \in V(D_{t+1})$ such that $a_{t+1}z_{t+1}b_{t+1} \subseteq F_t$. For each Hamiltonian path P_{t+1} between a_{t+1} and b_{t+1} in $\overline{D_{t+1}} - (V(D_{t+1}) \backslash \{a_{t+1}, b_{t+1}\})$, $F_{t+1} = (F_t - z_{t+1}) \cup P_{t+1}$ is a Hamiltonian cycle in G. Then we add a neighbor to v to represent F_{t+1} and we finished constructing T. Hence the leaves of T correspond to distinct Hamiltonian cycles in G. Note that, by construction, the distance in T between the root and any leaf is $t+1$.

If T has no path of length $\sqrt{t}$ whose internal vertices are of degree 2 in T, We obtain the tree T^* from T by contracting all edges of T incident with degree 2 vertices in T. Then all vertices in T^*, except the leaves and possibly the root, have degree at least 3. Since each leaf of T has distance t from the root r, the distance between the root and any leaf in T^* is at least $\sqrt{t}$. Hence, T^* and, thus, T both have at least $2^{\sqrt{t}}$ leaves. Therefore, G has at least $2^{\sqrt{t}}$ Hamiltonian cycles.

Now suppose T has a path of length $\sqrt{t}$ whose internal vertices are of degree 2 in T. This implies that for some $k \in \{0, 1, \ldots, t - \sqrt{t}\}$, all $j \in \{k+1, \ldots, k+\sqrt{t}\}$, there exist $a_j, b_j \in V(D_j)$ such that $G_j - (V(D_j) \backslash \{a_j, b_j\})$ has a unique Hamiltonian path P_j between a_j and b_j such that $a_{j+1}z_{j+1}b_{j+1} \subseteq P_j$. It follows from Lemma 4.1.3 that for any distinct $c_j, d_j \in V(D_j)$ with $\{c_j, d_j\} \neq \{a_j, b_j\}$, $G_j - (V(D_j) \backslash \{c_j, d_j\})$ has at least two Hamiltonian paths between c_j and d_j and avoiding the edge $a_{j+1}z_{j+1}$ or the edge $b_{j+1}z_{j+1}$.

Without loss of generality, we may assume $k = 0$.

Recall G_0. By Sanders's Theorem, G_0 has a Hamiltonian cycle F_0 through the edges c_1z_1, d_1z_1 such that $\{c_1, d_1\} \neq \{a_1, b_1\}$. We now define a new rooted tree T' whose root r' represent F_0 and whose leaves are Hamiltonian cycles in G. Note that $G_1 - (V(D_1)\backslash\{c_1, d_1\})$ has at least two Hamiltonian paths between c_1 and d_1 and not containing $a_2z_2b_2$. For each such Hamiltonian path P_1, we see that $(F_0-z_1)\cup P_1$ is a Hamiltonian cycle in $G_1\cup(G-(\overline{D_1}-D_1))$, and we add a vertex to T' representing $(F_0-z_1)\cup P_1$ and make it adjacent to r'. This defines all vertices of T' within distance 1 from r'. Note that $d_{T'}(r') \geq 2$. Now suppose we have defined the vertices of T' at distance s from r for some $s \in [\sqrt{t}-1]$, each representing a Hamiltonian cycle in $G_s \cup (G - (\overline{D_s} - D_s))$ not containing $a_{s+1}z_{s+1}b_{s+1}$. To define the vertices of T' that are at distance $s+1$ from r', let v be an arbitrary vertex of T' at distance s from r'. Then v corresponds to a Hamiltonian cycle F_s in $G_s \cup (G - (\overline{D_s} - D_s))$ not containing $a_{s+1}z_{s+1}b_{s+1}$. Let $c_{s+1}, d_{s+1} \in V(D_{s+1})$ be distinct such that $c_{s+1}z_{s+1}d_{s+1} \subseteq F_s$. Then $\{c_{s+1}, d_{s+1}\} \neq \{a_{s+1}, b_{s+1}\}$. Hence, by Lemma 4.1.3, $G_{s+1} - (V(D_{s+1})\backslash\{c_{s+1}, d_{s+1}\})$ has at least two Hamiltonian paths between c_{s+1} and d_{s+1} and not containing $a_{s+2}z_{s+2}b_{s+2}$. For each such path P_{s+1}, $(F_s-z_{s+1})\cup P_{s+1}$ is a Hamiltonian cycle in $G_{s+1} \cup (G - (\overline{D_{s+1}} - D_{s+1}))$ not containing $a_{s+2}z_{s+2}b_{s+2}$, and we add a neighbor to v in T' to represent $(F_s - z_{s+1}) \cup P_{s+1}$. Thus, $d_T(v) \geq 3$. We repeat this process for $s = 1, \ldots, \sqrt{t} - 1$. Note that the number of vertices of T' with distance $i \in [\sqrt{t}]$ is at least 2^i.

For an arbitrary vertex u of T' that has distance $q = \sqrt{t}$ from r' in T', it represents a Hamiltonian cycle F_q in $G_q \cup (G - (\overline{D_q} - D_q))$. Assume $c_{q+1}z_{q+1}d_{q+1} \subseteq F_q$. By Lemma 2.2.2 or Lemma 2.2.3, there exist at least one Hamiltonian path P_{q+1} between c_{q+1} and d_{q+1} in $\overline{D_{q+1}}-(V(D_{q+1})\backslash\{c_{q+1}, d_{q+1}\})$. Then $(F_q-z_{q+1})\cup P_{q+1}$ is Hamiltonian cycle in G. So T' has at least $2^q = 2^{\sqrt{t}}$ leaves. Hence, G has at least $2^{\sqrt{t}}$ Hamiltonian cycles. $\square$

4.2 Proof of exponential bound

Proof of Theorem 1.3.2. Note that, for any two distinct vertices x, y of degree 4 in G, we have $N_G(x) \cap N_G(y) = \emptyset$ as $d_G(x, y) \geq 3$. Hence, the number of vertices of degree 4 in G is at most $n/5$. Thus, since $|E(G)| = 3n - 6$ and $\delta(G) \geq 4$, there exist at least $n/5$ vertices of degree 5 or 6 in G. Then by the Four Color Theorem, there is an independent set I such that every vertex in I has degree 5 or 6 in G and $|I| \geq (n/5)/4 = n/20$. We may assume that

(1) G has an independent set $S \subseteq I$ of size $\Omega(n^{3/4})$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

For, otherwise, by Lemma 2.3.8, there exist distinct $v, x \in V(G)$ such that $|N_G(v) \cap N_G(x) \cap I| \geq c_0 n^{1/4}$ for some constant $c_0 > 0$. Since any two vertices of degree 4 in G have distance at least three, $G[N_G[v] \cup N_G[x]]$ contains separating 4-cycles $C_1, \ldots, C_k$ in G, where $k \geq c_0 n^{1/4} - 1$, such that $|V(\overline{C_i})| \geq 6$ for each $i \in [k]$, and $\overline{C_i} - C_i$, $\overline{C_j} - C_j$ are disjoint whenever $1 \leq i \neq j \leq k$. Let G^* be the graph obtained from G by contracting $\overline{C_i} - C_i$ to a new vertex v_i, for $i \in [k]$. Then G^* is a 4-connected planar triangulation and, hence, has a Hamiltonian cycle, say H.

Let $a_i, b_i \in N_{G^*}(v_i)$ such that $a_i v_i b_i \subseteq H$ for $i \in [k]$. Since $|V(\overline{C_i})| \geq 6$ and no vertices of degree 4 in G are adjacent, it follows from Lemma 2.2.4 that $\overline{C_i} - (V(C_i) \backslash \{a_i, b_i\})$ has at least two Hamiltonian paths between a_i and b_i. We can form a Hamiltonian cycle in G by taking the union of $H - \{v_i : i \in [k]\}$ and one Hamiltonian path between a_i and b_i in $\overline{C_i} - (V(C_i) \backslash \{a_i, b_i\})$ for each $i \in [k]$. Thus, G has at least $2^k \geq 2^{c_0 n^{1/4} - 1}$ Hamiltonian cycles and we are done. This completes the proof of (1).

For each $u \in S$, recall the link A_u defined in Section 2. We may assume that

(2) there exists $S_1 \subseteq S$ such that $|S_1| \geq |S|/2$ and, for each $u \in S_1$, $d_G(u) - |A_u| \geq 2$ and u is contained in a separating 4-cycle D in G with $|V(\overline{D})| \geq 6$.

Suppose we have $S_2 \subseteq S$ with $|S_2| \geq |S|/2$ such that $d_G(u) - |A_u| \leq 1$ for all $u \in S_2$. Hence, for any $u \in S_2$, $|A_u| \geq 4$ if $d_G(u) = 5$; and $|A_u| \geq 5$ if $d_G(u) = 6$. Let F be any subset of $E(G)$ with $|F| = |S_2|$ and $|F \cap A_u| = 1$ for each $u \in S_2$. By Lemma 2.3.4, $G - F$ is 4-connected; so $G - F$ has a Hamiltonian cycle by Tutte's theorem (or by Lemma 2.1.4). Let $\mathcal{C}$ be a collection of Hamiltonian cycles in G by taking precisely one Hamiltonian cycle in $G - F$ for each choice of F. Let a_5 and a_6 denote the number of vertices in S_2 of degree 5 and 6 in G, respectively. There are at least $4^{a_5}5^{a_6}$ choices of the edge set $F \subseteq E(G)$. Each Hamiltonian cycle of G in $\mathcal{C}$ is chosen at most $(5-2)^{a_5}(6-2)^{a_6} = 3^{a_5}4^{a_6}$ times. Thus

$$|\mathcal{C}| \geq (4/3)^{a_5}(5/4)^{a_6} \geq (5/4)^{a_5+a_6} = (5/4)^{|S_2|} \geq (5/4)^{\Omega(n^{3/4})}.$$

Hence, we may assume that there exists $S_1 \subseteq S$ such that $|S_1| \geq |S|/2$ and $d_G(u) - |A_u| \geq 2$ for all $u \in S_1$. For each $u \in S_1$, since $d_G(u) - |A_u| \geq 2$, there exist at least two edges e_1 and e_2 incident with u such that $G - e_i$ is not 4-connected for $i \in [2]$. Since u has at most one neighbor of degree 4 in G (by assumption), there exists $i \in [2]$ such that a 3-cut of $G - e_i$ and u induce a separating 4-cycle D_u in G with $|V(\overline{D_u})| \geq 6$. This completes the proof of (2).

For each $u \in S_1$, we choose a maximal separating 4-cycle D_u containing u. Note that $|V(\overline{D_u})| \geq 6$. Let $\mathcal{D} = \{D_u : u \in S_1\}$. Since S_1 saturates no 4-cycle, $D_u \neq D_{u'}$ for any distinct $u, u' \in S_1$ and $|\mathcal{D}| = |S_1| \geq |S|/2$. By Lemma 4.1.1, for any distinct $D_1, D_2 \in \mathcal{D}$, either $\overline{D_1} - D_1$ and $\overline{D_2} - D_2$ are disjoint, or $\overline{D_1}$ contains $\overline{D_2}$ or vice versa. We may assume that

(3) there exist $D_1, D_2, \ldots, D_{t+1} \in \mathcal{D}$, where $t = \Omega(n^{1/2})$, such that $\overline{D_1} \supseteq \overline{D_2} \supseteq \cdots \supseteq \overline{D_{t+1}}$.

For, otherwise, since $|\mathcal{D}| = |S_1| \geq |S|/2 = \Omega(n^{3/4})$, there exist separating 4-cycles $D_1', \ldots, D_k' \in \mathcal{D}$, where $k = \Omega(n^{1/4})$, such that $|V(\overline{D_i'})| \geq 6$ for $i \in [k]$, and $\overline{D_i'} - D_i'$, $\overline{D_j'} - D_j'$ are disjoint for $1 \leq i \neq j \leq k$. Hence, G has at least 2^k Hamiltonian cycles, as shown in the first paragraph in the proof of (1). This completes the proof of (3).

For each $j \in [t]$, let G_j denote the graph obtained from $\overline{D_j}$ by contracting $\overline{D_{j+1}} - D_{j+1}$ to a new vertex z_{j+1}. Note that G_j is a near triangulation with outer cycle D_j and that G_j contains the 4-cycle D_{j+1}.

By Lemma 4.1.1 and the definition of $\mathcal{D}$, we see that D_{j+1}, D_j, G_j, and G (as $D_{u'}, D_u, H, G$, respectively, in Lemma 4.1.3) for $j \in [t]$, satisfy the conditions in Lemma 4.1.3. Hence, by Lemma 4.1.3 and Lemma 4.1.4, G has at least $2^{\sqrt{t}} = 2^{\Omega(n^{1/4})}$ Hamiltonian cycles. $\square$

CHAPTER 5

FUTURE WORK

In this chapter, we list some results and open problems related to counting cycles in planar graphs.

5.1 Hakimi-Schmeichel-Thomassen Conjecture

The main result of this dissertation states that every 4-connected planar triangulation on n vertices has $\Omega(n^2)$ Hamiltonian cycles, and provides evidence that the extremal graph contains a large double wheel structure. However, the Hakimi-Schmeichel-Thomassen Conjecture remains open in its exact form.

Conjecture 5.1.1 (Hakimi, Schmeichel, and Thomassen [38]). *If G is a 4-connected planar triangulation on n vertices then G has at least $2(n-2)(n-4)$ Hamiltonian cycles, with equality if and only if G is a double wheel.*

To show the exact bound $2(n-2)(n-4)$, it is natural to first consider the planar triangulations that are 'close' to a double wheel graph. We calculated the number of Hamiltonian cycles in a 4-connected n-vertex planar triangulation G such that the graph H obtained from G by contracting an edge e is a double wheel. We know that the number of Hamiltonian cycles in G is at least the sum of the number of Hamiltonian cycles in H and $2(2n-7)$. Let $C(G)$ denote the set of all Hamiltonian cycles in G. We wish to construct a sequence $G_0, G_1, \ldots, G_k$ for some integer k such that $G_0 = G$, $|V(G_i)| \leq n$ for $i \in [k]$, G_k is a double wheel, and $|C(G_{i-1})| \geq |C(G_i)| + 2(|V(G_{i-1})| - |V(G_i)|)(|V(G_{i-1})| + |V(G_i)| - 6)$ for $i \in [k]$. We hope to figure out graph operatins that we could use to obtain this sequence.

5.2 Counting Hamiltonian cycles in graphs embeddable in other surfaces

The problem of counting Hamiltonian cycles in graphs embeddable on other surfaces has also attracted a lot of attention (see [2, 16, 56]). In particular, the existence of Hamiltonian cycles in graphs embeddable on a projective plane or torus has been investigated in, e.g., [71, 45, 9, 72, 46, 73].

It is possible that these previous results on the existence of Hamiltonian cycles in graphs embeddable on other surfaces of higher genus can be combined with our methods to obtain similar counting results on Hamiltonian cycles in certain triangulations on other surfaces.

Problem 5.2.1. *Generalize the results in counting Hamiltonian cycles in planar triangulations to triangulations on the surfaces other than the sphere, in particular, the projective plane.*

5.3 Counting k-cycles in planar triangulations

Motivated by Bondy's meta conjecture [20], it is natural to consider the problem of counting cycles of other lengths in Hamiltonian graphs.

Definition 5.3.1. The *girth* of a graph G is the length of a shortest cycle in G. A graph G is said to be *weakly pancyclic* if G contains cycles of all lengths between its girth and its circumference.

Note that any planar triangulation G is weakly pancyclic. Let C be a longest cycle in G. Consider the graph $G_1 = \overline{C}$ (the subgraph of G consisting all vertices and edges of G contained in the closed disc bounded by C), which is a near triangulation with outer cycle C. We can obtain a sequence of 2-connected graphs $G_1, G_2, \cdots, G_t$ such that G_{i+1} is obtained from G_i by removing a degree 2-vertex in the outer cycle of G_i for each $i \in [t-1]$, and G_t is a triangle. Let C_i denote the outer cycle of G_i for each $i \in [t]$. Observe that either $|E(C_{i+1})| = |E(C_i)| - 1$ for every $i \in [t-1]$. Therefore, it follows that G is weakly pancyclic. Hence any 4-connected planar triangulation is pancyclic.

By Euler's formula, an n-vertex planar triangulation G has exactly $2n-4$ triangular faces and hence G has $\Omega(n)$ triangles. Let $C_k(G)$ denote the number of k-cycles in G. Hakimi and Schmeichel [37] in 1979 gave tight upper and lower bounds on the number of triangles (3-cycles) and 4-cycles in planar triangulations, and characterized the extremal graphs. They also gave upper and lower bounds on the number of 5-cycles.

Theorem 5.3.2 (Hakimi and Schmeichel [37])**.** *Let G be an n-vertex planar triangulation. Then*

(1) $2n-4 \leq C_3(G) \leq 3n-8$ *for* $n \geq 6$*, where the lower bound is attained if and only if G is 4-connected, and the upper bound is attained if and only if G is obtained from K_3 by recursively placing a vertex of degree 3 inside a face, and joining this new vertex to the three vertices incident to that face.*

(2) $3n-6 \leq C_4(G) \leq (n^2+3n-22)/2$ *for* $n \geq 5$*, where the lower bound is attained if and only if $n = 5$ or G is 5-connected, and the upper bound is attained if and only if G is the join of P_{n-2} and K_2.*

(3) $6n \leq C_5(G) \leq 5n^2 - 26n$ *for* $n \geq 8$*, and there are infinitely many triangulations attaining this lower bound $6n$.*

For the second result of this theorem, Alameddine [6] pointed out that for $n = 7, 8$, there is another planar triangulation on n vertices attaining the upper bound $(n^2+3n-22)/2$ for the number of 4-cycles.

For the number of 5-cycles in an n-vertex planar triangulation, Hakimi and Schmeichel [37] conjectured an upper bound $2n^2-10n+12$; moreover, this bound is achieved by the double wheel on n-vertices. Recently, Győri, Paulos, Salia, Tompkins, and Zamora [36] confirmed this conjecture by showing the following.

Theorem 5.3.3 (Győri, Paulos, Salia, Tompkins, and Zamora [36])**.** *Let G be an n-vertex planar triangulation. Then*

(1) $C_5(G) \leq 2n^2 - 10n + 12$ *for* $n = 6$ *or* $n \geq 8$,

(2) $C_5(G) \leq 6$ *for* $n = 5$,

(3) $C_5(G) \leq 41$ *for* $n = 7$.

Moreover, they characterized the n-vertex planar graphs that attain the upper bound for each $n \geq 5$.

It is natural to consider the following general problem.

Problem 5.3.4. *Determine the minimum number of k-cycles in an n-vertex planar triangulation for each $k \in [6, n]$.*

While it is tempting to conjecture every planar triangulation G has $\Omega(n)$ many k-cycles for every $3 \leq k \leq circ(G)$, it is not true in general. In particular, Hakimi, Schmeichel, and Thomassen [38] constructed an infinite family of planar triangulations with exactly four Hamiltonian cycles. One cannot hope for constructions with fewer Hamiltonian cycles, as Kratochvil and Zeps [49] showed that if a triangulation different from K_4 contains a Hamiltonian cycle, then it contains at least four of them.

Very recently, Lo and Zamfirescu [57] prove the following bound.

Theorem 5.3.5 (Lo and Zamfirescu [57])**.** *Every n-vertex planar triangulation G has $\Omega(n)$ many k-cycles for each $k \in \{3, 4, \cdots, 3 + \max\{rad(G^*), \lceil (\frac{n-3}{2})^{\log_3 2} \rceil\}\}$, where $rad(G^*)$ is the radius of the dual graph of G.*

Recall that Moon and Moser [63] showed that there are infinitely many n-vertex triangulations with no k-cycle for all $k > 9n^{log_3 2}$.

5.4 k-cycles in 4-connected planar triangulations

Now we focus on 4-connected planar triangulations. The results of Tutte in [75] imply that every n-vertex 4-connected planar triangulation G has $\Omega(n)$ cycles of length $n - 1$. Our result (in particular, Lemma 3.1.2) implies that G has $\Omega(n^2)$ many $(n-1)$-cycles. It follows from a theorem of Thomas and Yu [71] that G also has $\Omega(n^2)$ cycles of length $n-2$. A result of Sanders in [67] implies that G has $\Omega(n)$ many $(n-3)$-cycles. This motivates the following problem.

Problem 5.4.1. *Determine the best (asymptotic) lower bound on the number of k-cycles in a 4-connected planar triangulation on n vertices.*

Lo and Zamfirescu [57] show that every n-vertex planar triangulation with at most one separating triangle contains $\Omega(n)$ k-cycles for every $k \in \{3, \cdots, n\}$. In particular, this implies that every 4-connected n-vertex planar triangulation has $\Omega(n)$ k-cycles for every $k \in \{3, \cdots, n\}$, i.e., 4-connected planar triangulations are *linearly pancyclic*. In the same paper, Lo and Zamfirescu [57] also show that under certain circumstances, 4-connected triangulations contain a quadratic number of k-cycles for many values of k linear in n. They mentioned the following open problem.

Problem 5.4.2. *Determine whether there exists an integer k such that every n-vertex ($n \geq k$) 4-connected planar triangulation contains $\Omega(n^2)$ k-cycles.*

5.5 k-cycles in 5-connected planar triangulations

Recall that Alahmadi, Aldred, and Thomassen [2] showed that every n-vertex 5-connected planar or projective planar triangulation has $2^{\Omega(n)}$ Hamiltonian cycles. We [51] recently showed every n-vertex 4-connected planar triangulation with minimum degree 5 has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles. It is natural to consider the problem of counting cycles of other lengths in 5-connected planar triangulations. In particular, in a paper of Lo and Zamfirescu [57], they mentioned the following open question.

Problem 5.5.1. *Is it true that for every 5-connected planar triangulation G there exists a subset S of the cycle spectrum of G such that $|S| = \omega(1)$ and G contains exponentially many k-cycles for every $k \in S$?*

5.6 Pancyclicity in 4-connected planar graphs

We end with the following conjecture by Malkevitch [60] concerning the pancyclicity of 4-connected planar graphs.

Conjecture 5.6.1 (Malkevitch [60]). *Every 4-connected planar graph containing a cycle of length 4 is pancyclic.*

It is also interesting to investigate the cycle spectrum for 4-connected planar graph without 4-cycles.

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