# On Matchings and Covers 

Penny Haxell<br>University of Waterloo

## Matchings and Covers

Let $\mathcal{H}$ be a hypergraph.
A matching in $\mathcal{H}$ is a set of disjoint edges.
A cover of $\mathcal{H}$ is a set of vertices that meets every edge.

General Problem: To bound the minimum size of a cover of $\mathcal{H}$ in terms of the maximum size of a matching in $\mathcal{H}$.

## Triangle matching

Let $G$ be a graph. A triangle matching or triangle packing of $G$ is a set of pairwise edge-disjoint triangles in $G$.


The parameter $\nu(G)$ is defined to be the maximum size of a triangle matching in $G$.

## Covering triangles by edges

A cover of the triangles in the graph $G$ is a set of edges $C$ of $G$ such that every triangle of $G$ contains an edge of $C$.


The parameter $\tau(G)$ is defined to be the minimum size of a cover of $G$.

## Comparing $\nu(G)$ and $\tau(G)$

For every graph $G$ we have $\nu(G) \leq \tau(G)$.


For every graph $G$ we have $\tau(G) \leq 3 \nu(G)$.


## Tuza's Conjecture

Conjecture (Tuza 1984): For every graph $G$,

$$
\tau(G) \leq 2 \nu(G) .
$$

## Results on Tuza's Conjecture

- known for certain special classes of graphs, including $K_{5}$-free chordal graphs (Tuza 1990), odd-wheel-free and four-colourable graphs (Aparna Lakshmanan, Bujtás and Tuza 2011)
- known for planar graphs (Tuza 1990), and more generally graphs without subdivisions of $K_{3,3}$ (Krivelevich 1995)
- weighted versions of the problem have been studied (Chapuy, DeVos, McDonald, Mohar and Scheide 2011)
- for every graph $G$ we have $\tau(G) \leq\left(3-\frac{3}{19}\right) \nu(G)$.
- If true, Tuza's Conjecture is best possible.

$$
A \underset{\nabla}{\infty}
$$

## Fractional matching

Let $G$ be a graph. A fractional triangle matching of $G$ is a function $p$ that assigns to each triangle $t$ of $G$ a non-negative real number, such that for each edge $e$ of $G$ we have

$$
\sum_{t \ni e} p(t) \leq 1
$$

Thus a triangle matching $\mathcal{S}$ corresponds to a fractional triangle matching in which each triangle in $\mathcal{S}$ gets value 1 and all others get 0.

## Fractional triangle cover

A fractional cover of the triangles in $G$ is a function $c$ that assigns to each edge of $G$ a non-negative real number, such that for each triangle $t$ of $G$ we have

$$
\sum_{e \in t} c(e) \geq 1 .
$$

Thus a cover $C$ of the triangles in $G$ corresponds to a fractional cover in which each vertex in $C$ gets value 1 and all other vertices get 0 .

## Fractional parameters

The fractional parameter $\nu^{*}(G)$ is defined to be the maximum of $\sum_{t \in G} p(t)$ over all fractional triangle matchings $p$ of $G$.

The parameter $\tau^{*}(G)$ is the minimum of $\sum_{e \in G} c(e)$ over all fractional covers $c$ of $G$.

Then we know that $\nu(G) \leq \nu^{*}(G)$ and $\tau(G) \geq \tau^{*}(G)$.

The Duality Theorem of linear programming tells us that

$$
\tau^{*}(G)=\nu^{*}(G)
$$

## Fractional results on Tuza's Conjecture

Theorem (Krivelevich 1995): Let $G$ be a graph. Then

- $\tau^{*}(G) \leq 2 \nu(G)$.
- $\tau(G) \leq 2 \nu^{*}(G)$.


## An asymptotic result

Theorem: Let $G$ be a graph with $n$ vertices. Then

$$
\tau(G) \leq 2 \nu(G)+o\left(n^{2}\right) .
$$

This follows from the fractional version $\tau(G) \leq 2 \nu^{*}(G)$ together with the following general packing theorem.

Theorem (PH, Rödl): Let $G$ be a graph with $n$ vertices, and let $H$ be any fixed graph (for example, a triangle). Then

$$
\nu_{H}^{*}(G) \leq \nu_{H}(G)+o\left(n^{2}\right) .
$$

## A closer look - the role of $K_{4}$

(A) Tuza's Conjecture is true for planar graphs, and best possible because of $K_{4}$. What can we say about planar graphs for which $\tau(G)$ is close to $2 \nu(G)$ ? Are they close to being disjoint unions of $K_{4}$ 's?
(B) The fractional result $\tau^{*}(G) \leq 2 \nu(G)$ of Krivelevich is best possible because of $K_{4}$. What can we say about graphs for which $\tau^{*}(G)$ is close to $2 \nu(G)$ ? Are they close to being disjoint unions of $K_{4}$ 's?

## On Question (A)

Theorem (Cui, PH, Ma 2009) Let $G$ be a planar graph, and suppose

$$
\tau(G)=2 \nu(G)
$$

Then $G$ is an edge-disjoint union of $K_{4}$ 's and edges, such that every triangle is contained in exactly one of the $K_{4}$ 's.


## On Question (A)

Theorem (PH, Kostochka, Thomassé 2011) Let $G$ be a planar graph that does not contain $K_{4}$. Then

$$
\tau(G) \leq 1.5 \nu(G)
$$

Moreover equality holds if and only if $G$ is the edge-disjoint union of 5 -wheels (plus possibly some edges not in any triangle).


## (B): A stability theorem

Theorem (PH, Kostochka, Thomassé 2011) Let $G$ be a graph such that $\tau^{*}(G) \geq 2 \nu(G)-x$. Then $G$ contains $\nu(G)-\lfloor 10 x\rfloor$ edge-disjoint $K_{4}$-subgraphs plus an additional $\lfloor 10 x\rfloor$ edge-disjoint triangles.

Note that just these $K_{4}$ 's and triangles witness the fact that

$$
\tau^{*}(G) \geq 2 \nu(G)-\lfloor 10 x\rfloor .
$$

The proof also shows that if $G$ is $K_{4}$-free then

$$
\tau^{*}(G) \leq 1.8 \nu(G)
$$

## Stability for Tuza's conjecture

Could there be a similar stability theorem for Tuza's Conjecture?

The only known graphs for which equality holds for Tuza's Conjecture are (disjoint unions of) $K_{4}$ and $K_{5}$. Could it be true that every graph for which $\tau(G)$ is close to $2 \nu(G)$ contains many $K_{4}$ 's?

## NO.

For each $\epsilon>0$, there exists a $K_{4}$-free graph $G_{\epsilon}$ such that $\tau\left(G_{\epsilon}\right)>$ $(2-\epsilon) \nu\left(G_{\epsilon}\right)$.

For large $n$, let $H$ be an $n$-vertex triangle-free graph with independence number $\alpha(H)<n^{2 / 3}$. $\left(R(3, t)\right.$ is of order $t^{2} / \log t$.)


Form a graph $G$ by adding a new vertex $v_{0}$ and joining it to all vertices in $H$.

Then a triangle matching in $G$ corresponds to a matching in $H$, so

$$
\nu(G) \leq n / 2 .
$$

A cover in $G$ corresponds to the complement of an independent vertex set in $H$. Thus

$$
\tau(G) \geq n-n^{2 / 3} .
$$

## Some proof ideas

Theorem (PH, Kostochka, Thomassé 2011) Let $G$ be a planar graph that does not contain $K_{4}$. Then $\tau(G) \leq 1.5 \nu(G)$. Moreover equality holds if and only if $G$ is the edge-disjoint union of 5 -wheels (plus possibly some edges not in any triangle).

To prove this we first show
Theorem Let $G$ be a FLAT graph that does not contain $K_{4}$. Then $\tau(G) \leq$ $1.5 \nu(G)$. Moreover equality holds if and only if $G$ is the edge-disjoint union of 5 -wheels (plus possibly some edges not in any triangle).

Here flat means that every edge is in at most two triangles.

## The triangle graph

Let $G$ be a graph. The triangle graph $T(G)$ of $G$ has vertex set the set of triangles of $G$. Two vertices of $T(G)$ are joined by an edge if and only if the corresponding triangles of $G$ share an edge of $G$.

If $G$ is flat then $T(G)$ is a triangle-free subcubic graph.
A triangle matching in $G$ corresponds to an independent set in $T(G)$, and a cover of the triangles in $G$ corresponds to an edge cover of the vertices in $T(G)$.

By Gallai's Theorem this is determined by the maximum size of a matching in $T(G)$.

## Flat graphs and planar graphs

To study triangle matching and covering in flat graphs we need

- bounds on the independence number of triangle-free subcubic graphs,
- Tutte's 1 -factor theorem.

If $G$ is planar and has no separating triangle then it is flat.
To study triangle matching and covering in flat graphs we need in addition to analyse the effect of separating triangles.

## Some proof ideas

Theorem (PH, Kostochka, Thomassé 2011) Let $G$ be a graph such that $\tau^{*}(G) \geq 2 \nu(G)-x$. Then $G$ contains $\nu(G)-\lfloor 10 x\rfloor$ edge-disjoint $K_{4}$-subgraphs plus an additional $\lfloor 10 x\rfloor$ edge-disjoint triangles.

Proof idea: choose a maximum triangle matching and a lexicographically largest set of edge-disjoint special substructures in $G$. Define a fractional cover based on this set.


## The role of $K_{4}$

We are still far from understanding the role of $K_{4}$ even in these special cases of the conjecture. For example:

Q: What is the smallest possible value of $c$ such that

$$
\tau^{*}(G) \leq c \nu(G)
$$

for every $K_{4}$-free graph $G$ ?
We know that $c \leq 1.8$, and the 5 -wheel shows that $c \geq 1.25$.

