Operator algebras and data hiding in topologically ordered systems

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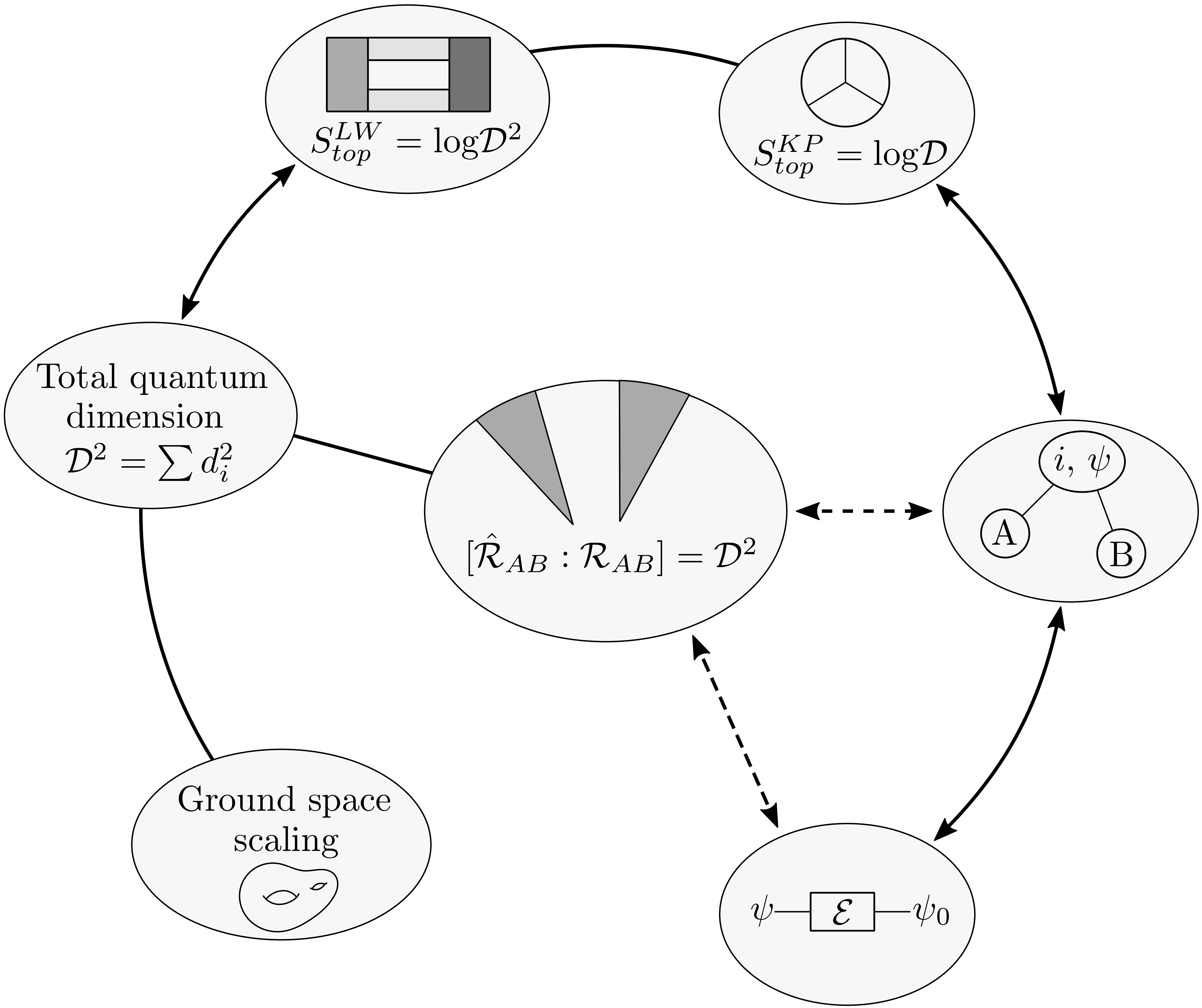
UC Davis & RWTH Aachen

arXiv:1608.02618

9 October 2016
QMath 13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$S_{top}^{KP} = \log \mathcal{D}$
Total quantum dimension
$\mathcal{D}^2 = \sum d_i^2$
$[\hat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] = \mathcal{D}^2$
$i,\, \psi$
A
B
Ground space scaling
ψ
$\mathcal{E}$
ψ_0

Topological order

Topological phases

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Quantum phase outside of Landau theory

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- ground space degeneracy

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fusion, **braiding**, **charge conjugation**, …

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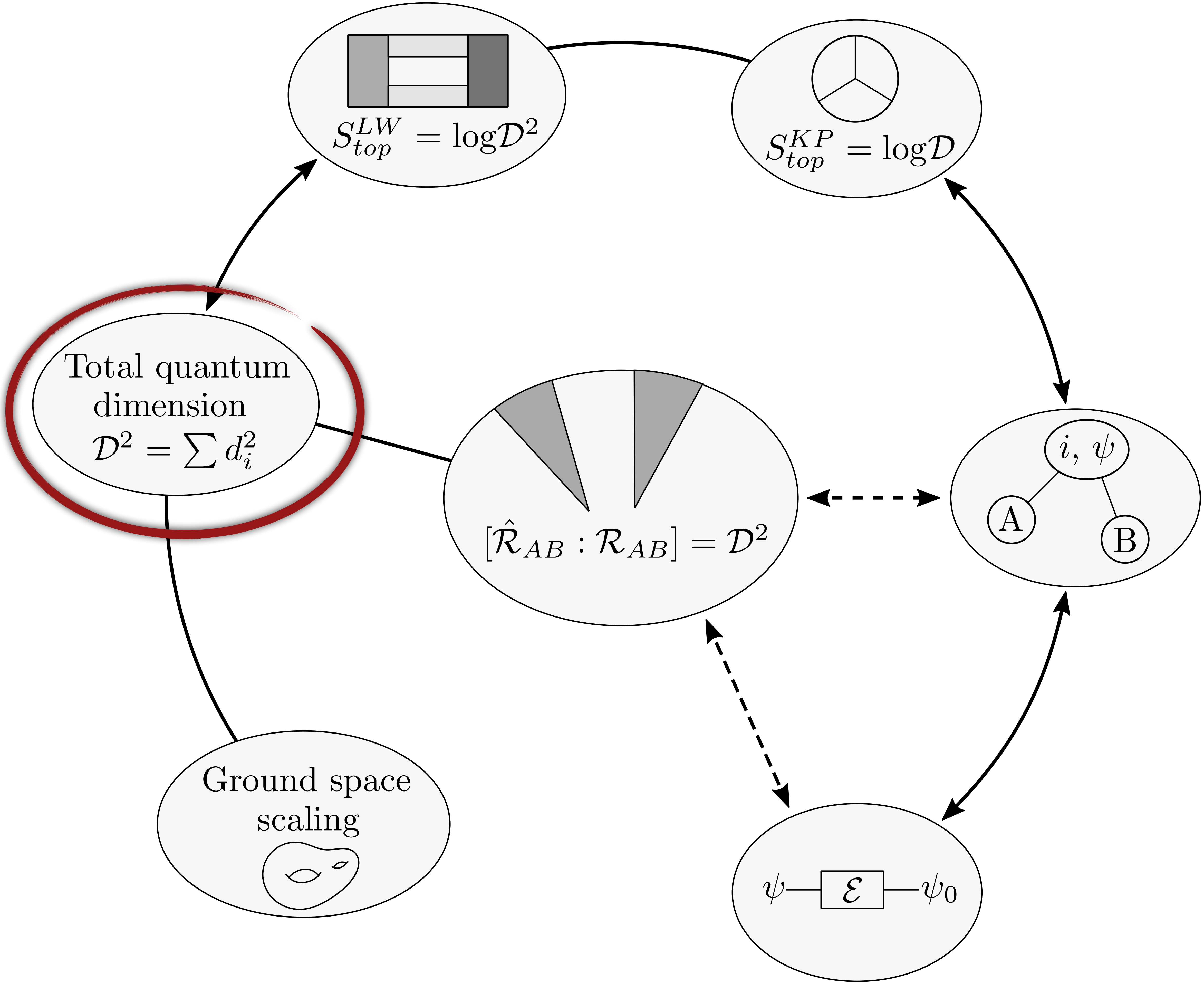
Quantum dimension $\mathcal{D}^2 = \sum_i d(\rho_i)^2$



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Topological entanglement entropy

Area law for top. ordered states:

$$S_\Lambda = \alpha|\partial\Lambda| - \gamma + \cdots$$

Kitaev & Preskill (06), Levin & Wen (06)

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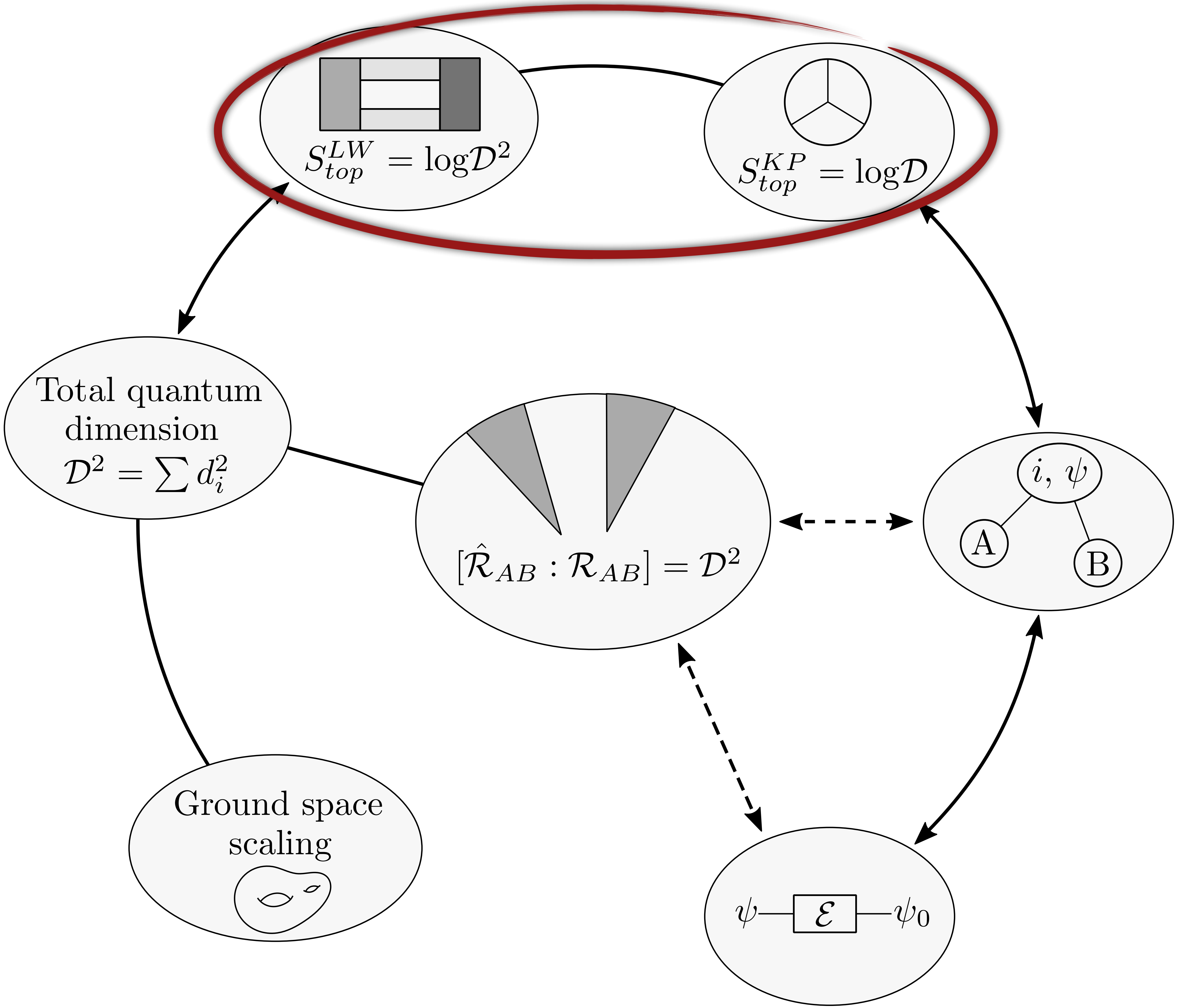
Universal constant: $\gamma = \log \mathcal{D}$

Kitaev & Preskill (06), Levin & Wen (06)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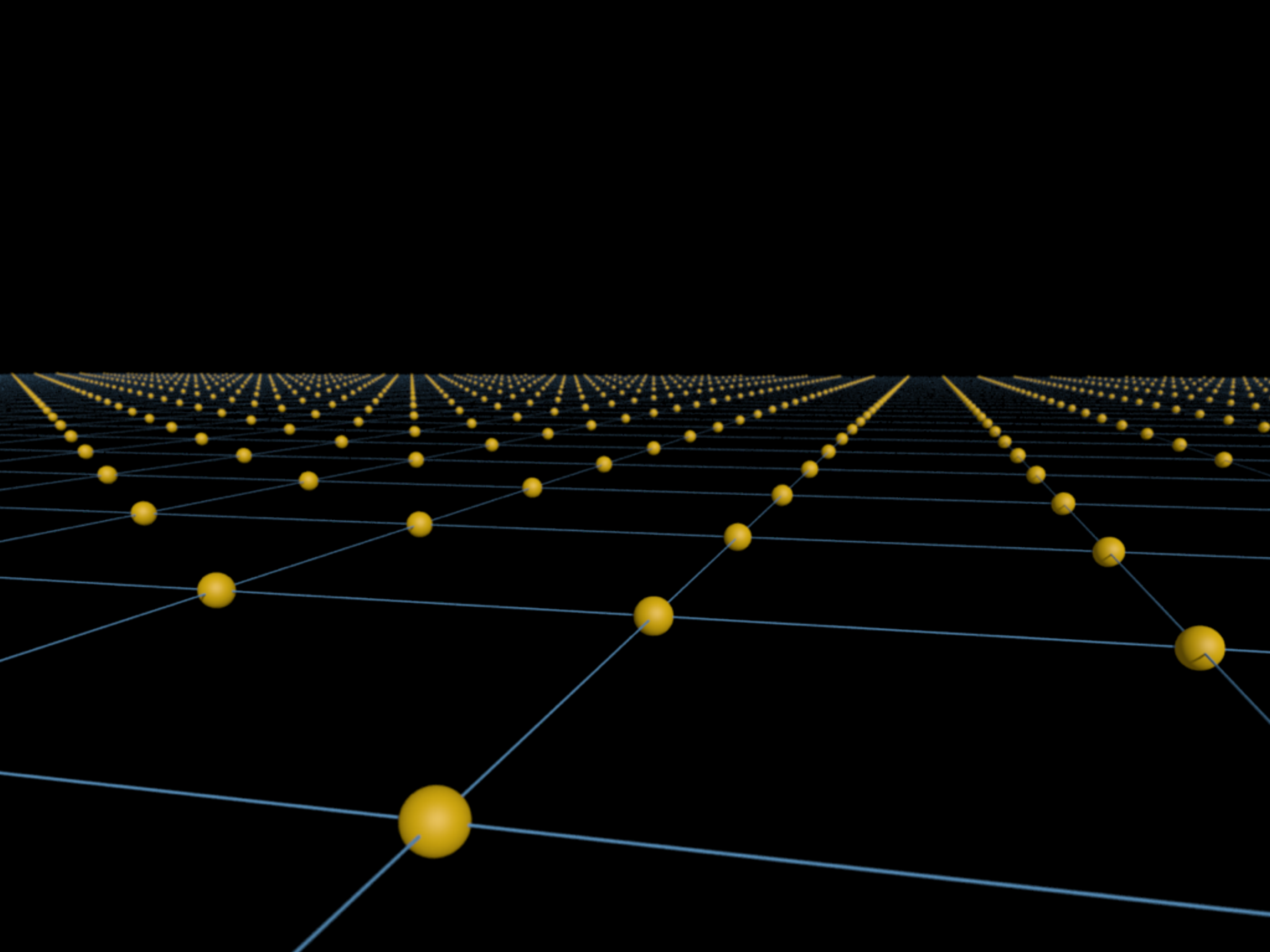
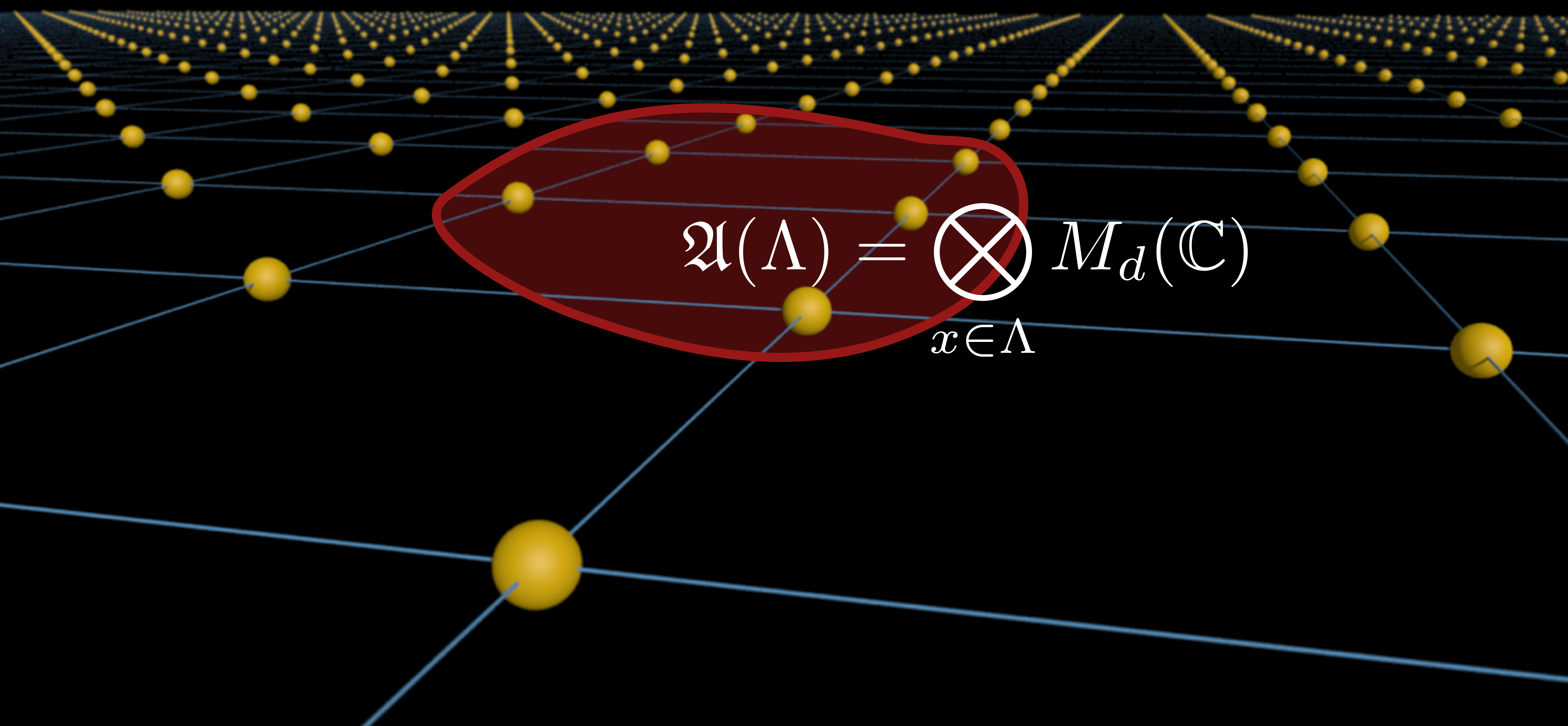
Technical framework

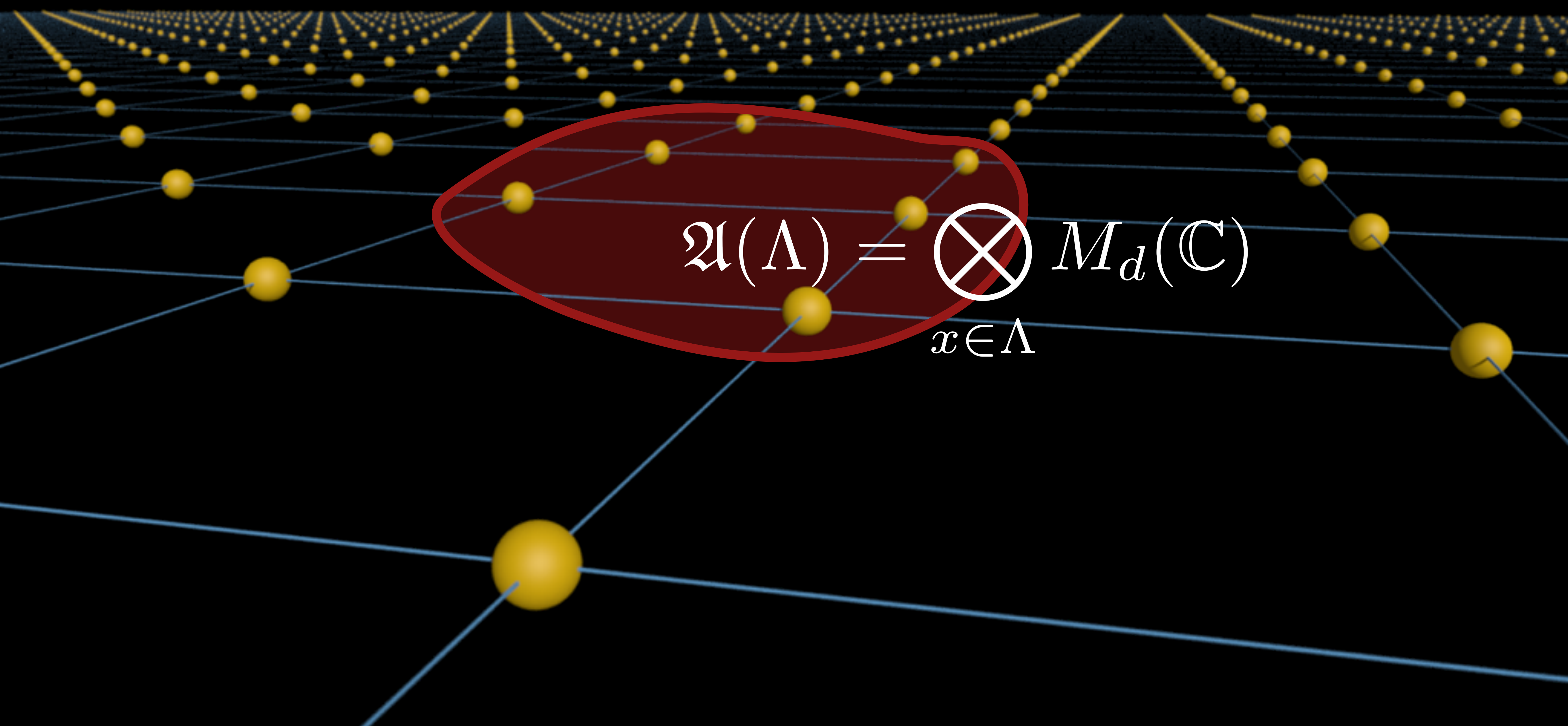


Quasi-local algebra $\mathfrak{A} = \overline{\bigcup_{\Lambda} \mathfrak{A}(\Lambda)}^{\|\cdot\|}$



$$\mathfrak{A}(\Lambda) = \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$$

and **local Hamiltonians** $H_\Lambda \in \mathfrak{A}(\Lambda)$



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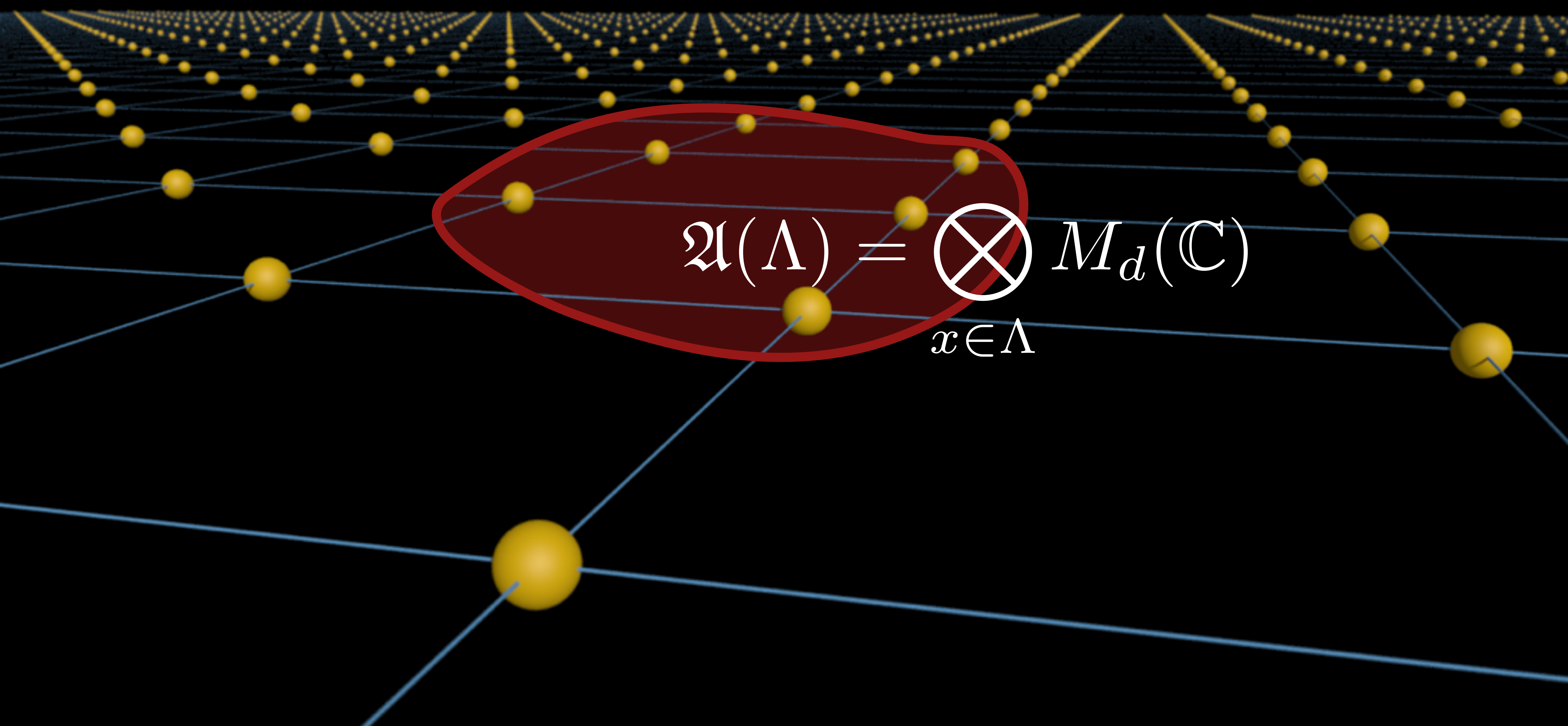
ground state representation π_0



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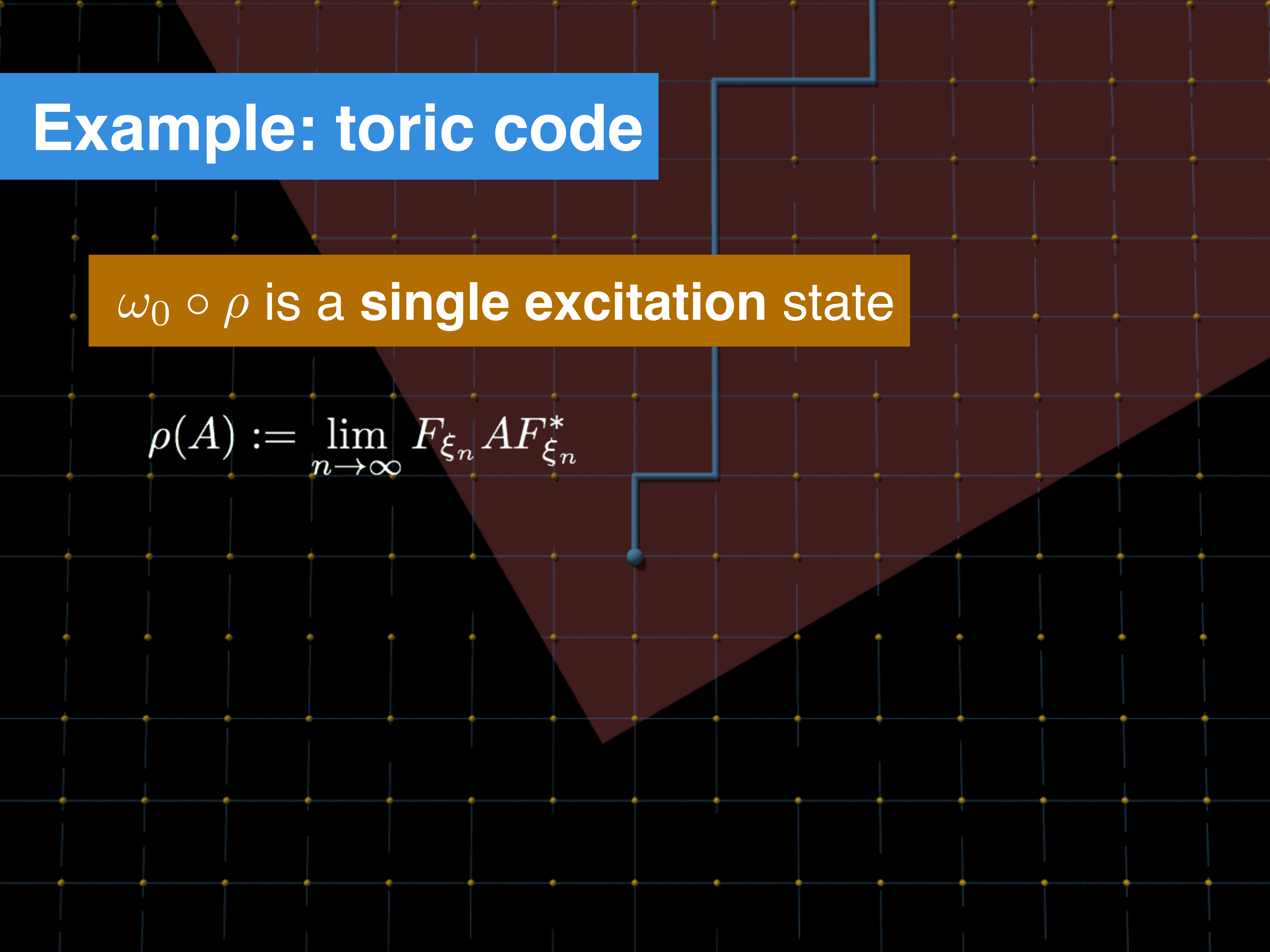

Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*$$

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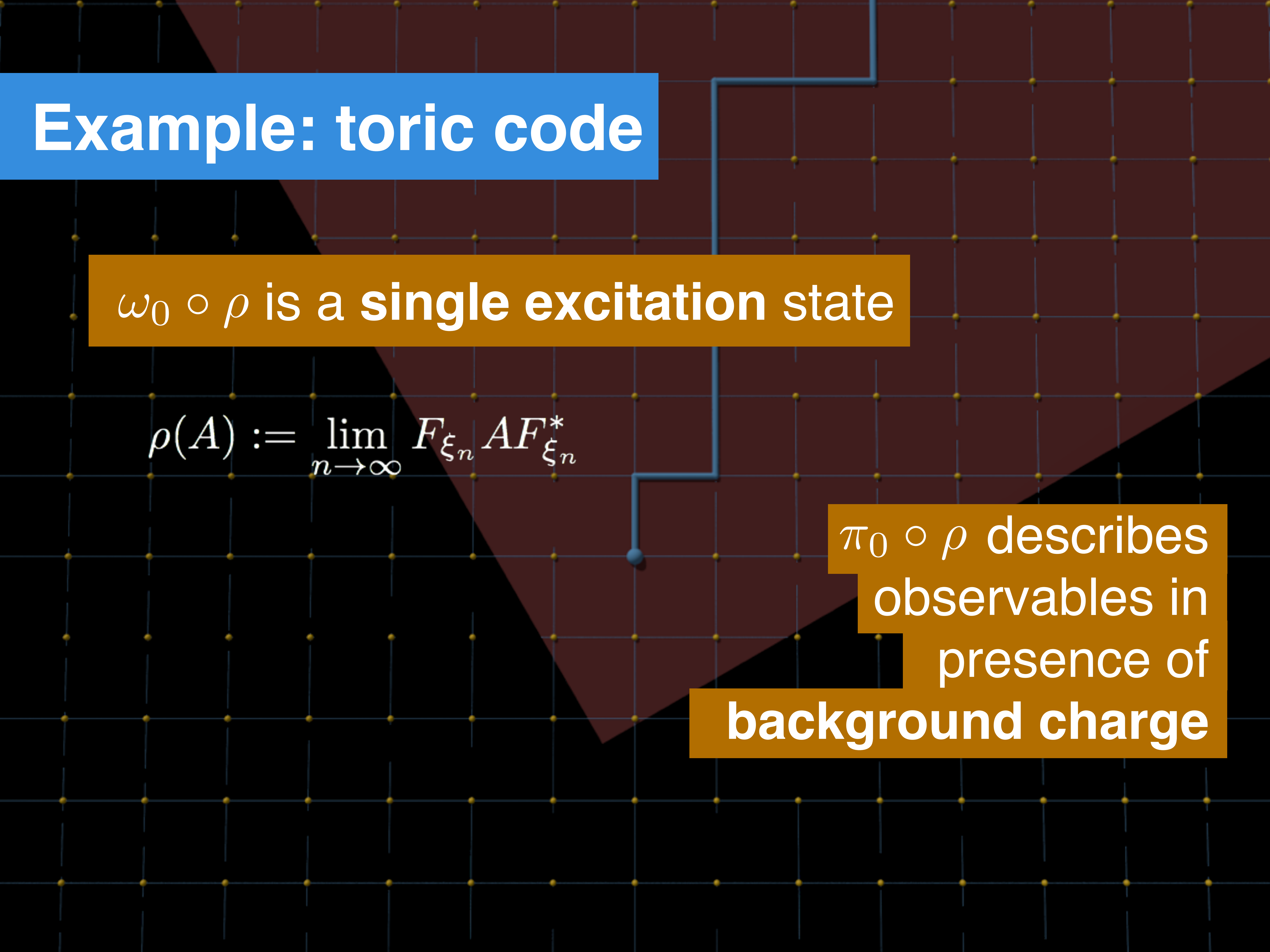


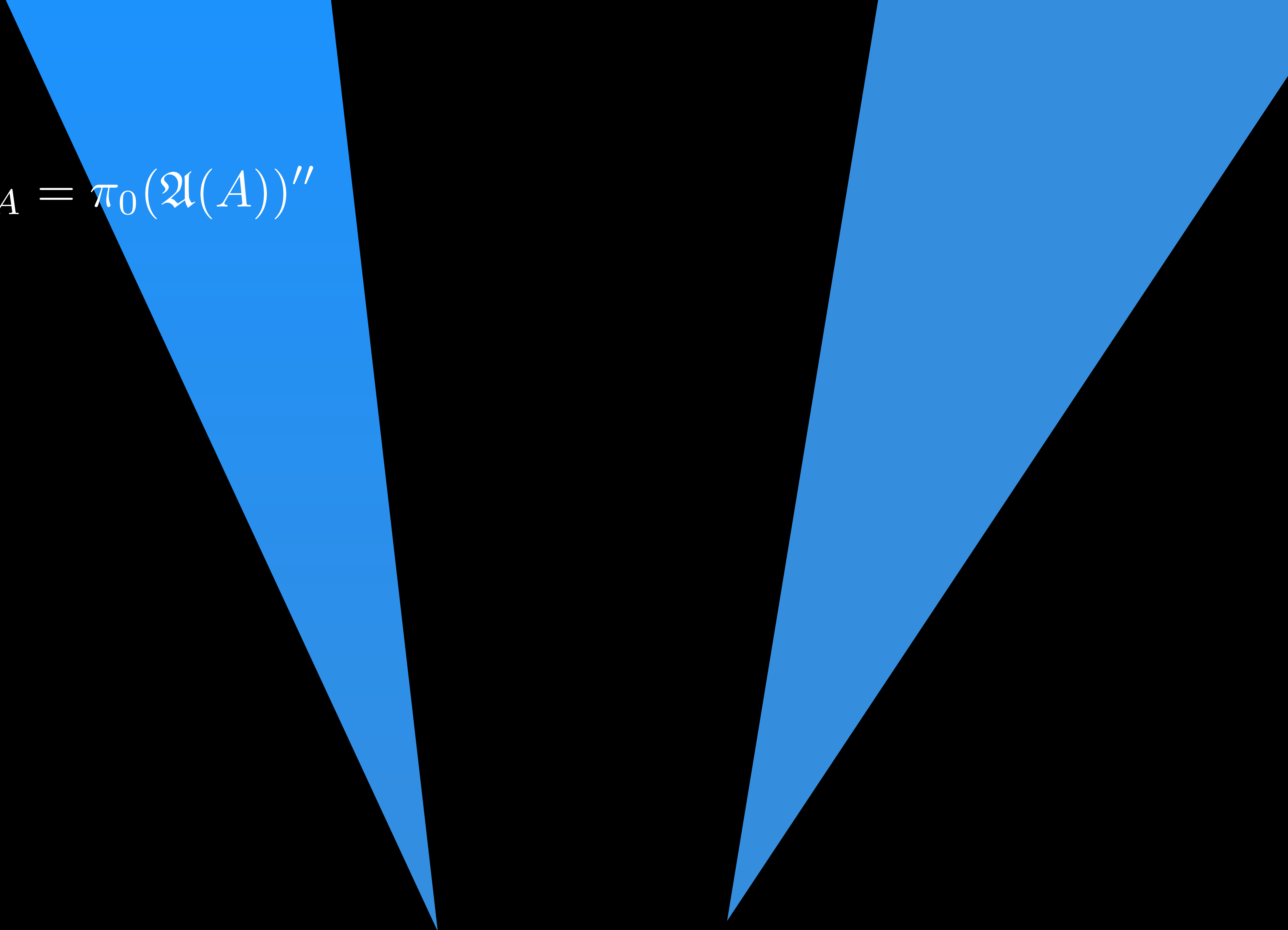
$\omega_0 \circ \rho$ is a **single excitation** state

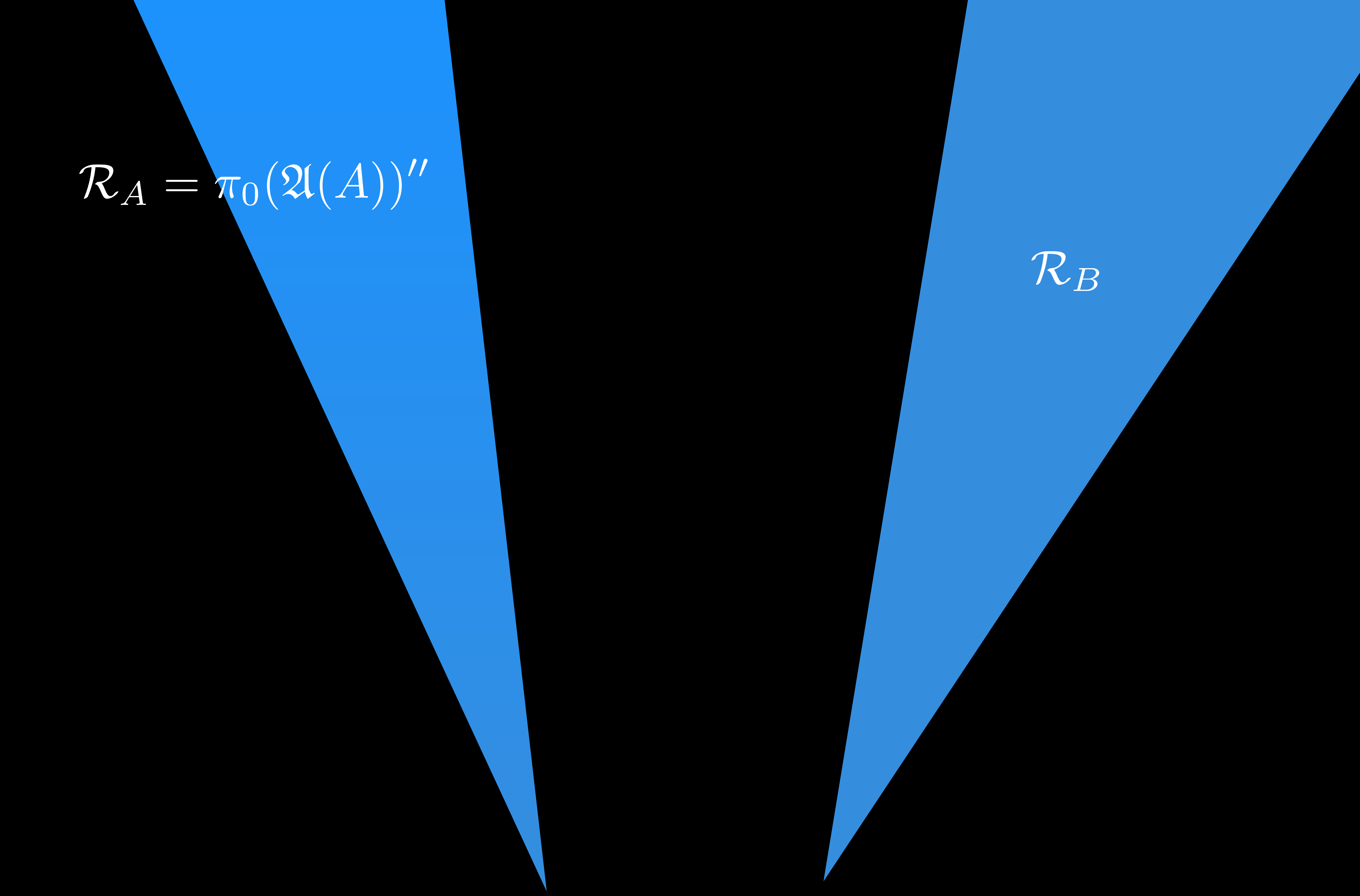
$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of **background charge**

Quantum dimension

$$\mathcal{R}_A = \pi_0(\mathfrak{A}(A))''$$

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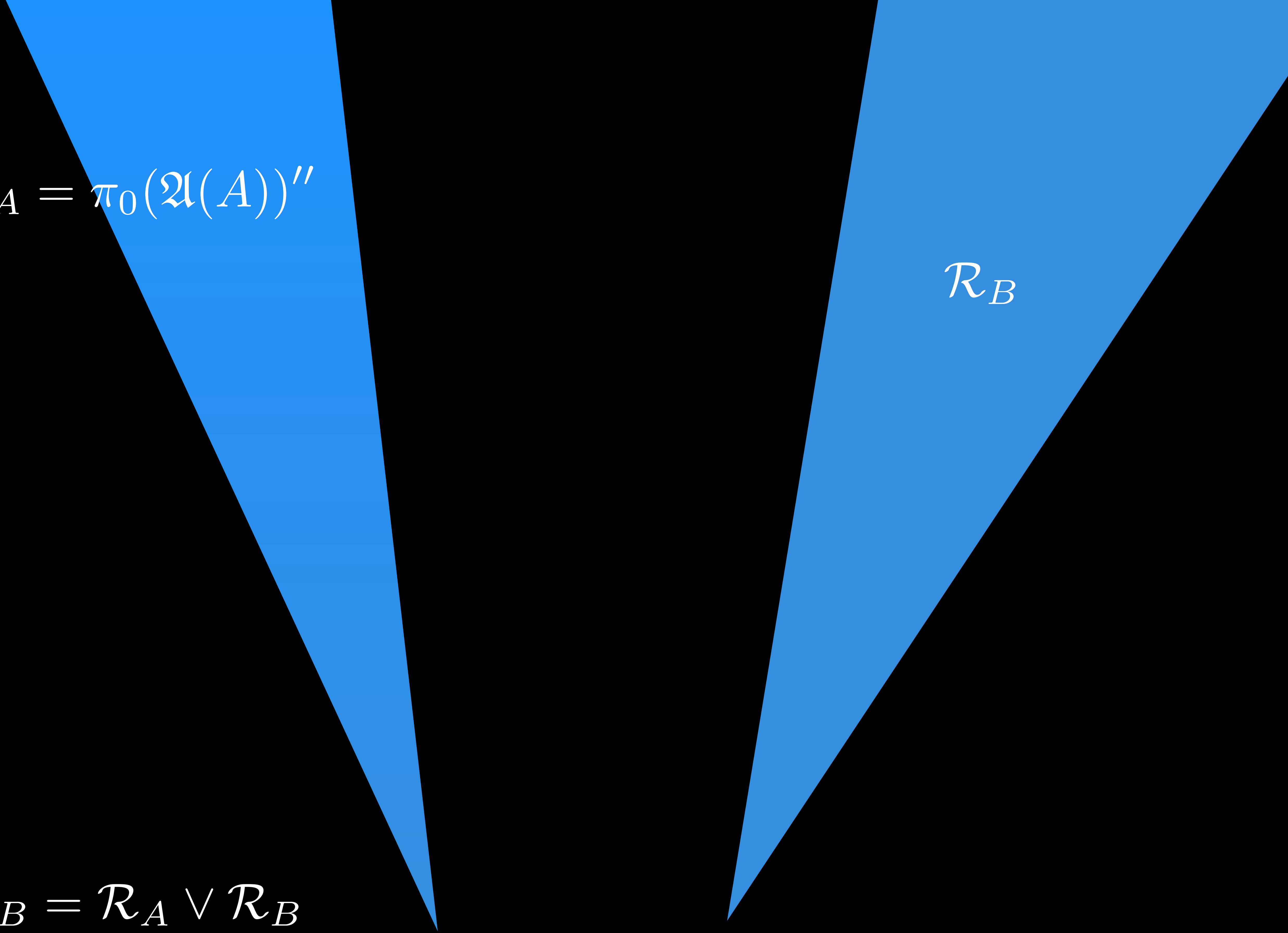

$$\mathcal{R}_{AB} = \mathcal{R}_A \vee \mathcal{R}_B$$

$$\widehat{\mathcal{R}}_{AB} = \pi_0(\mathfrak{A}((A \cup B)^c))'$$

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

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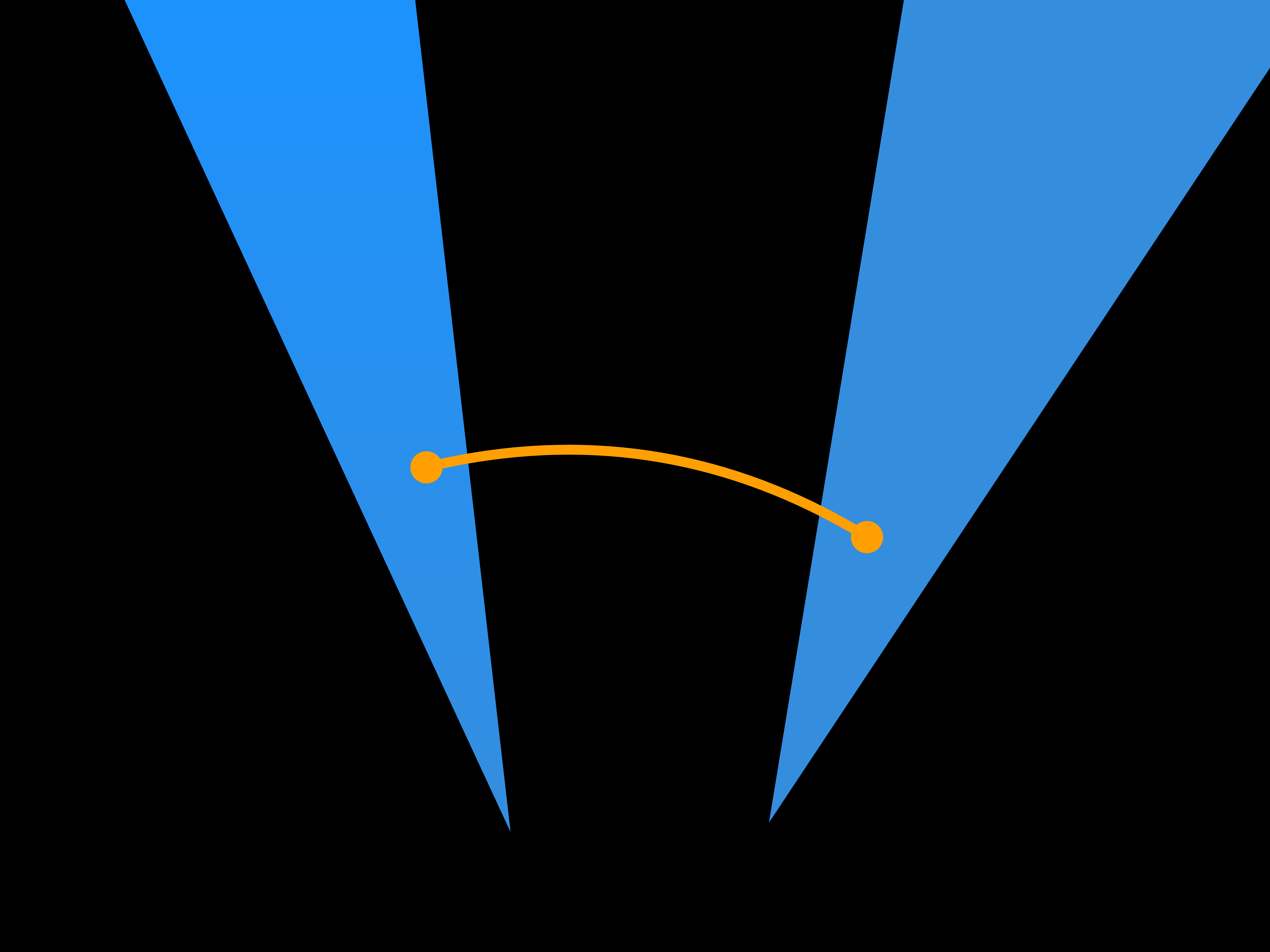
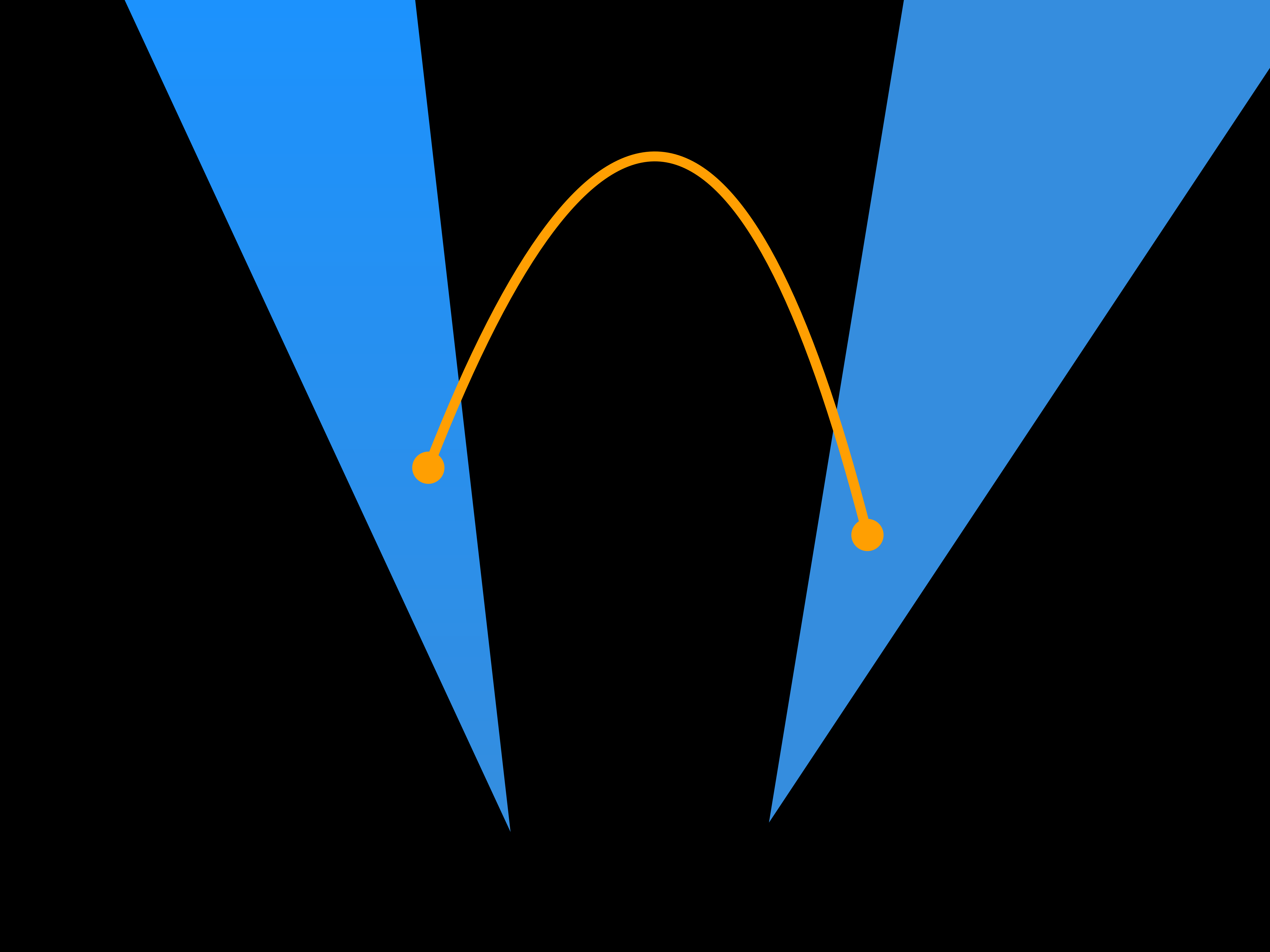
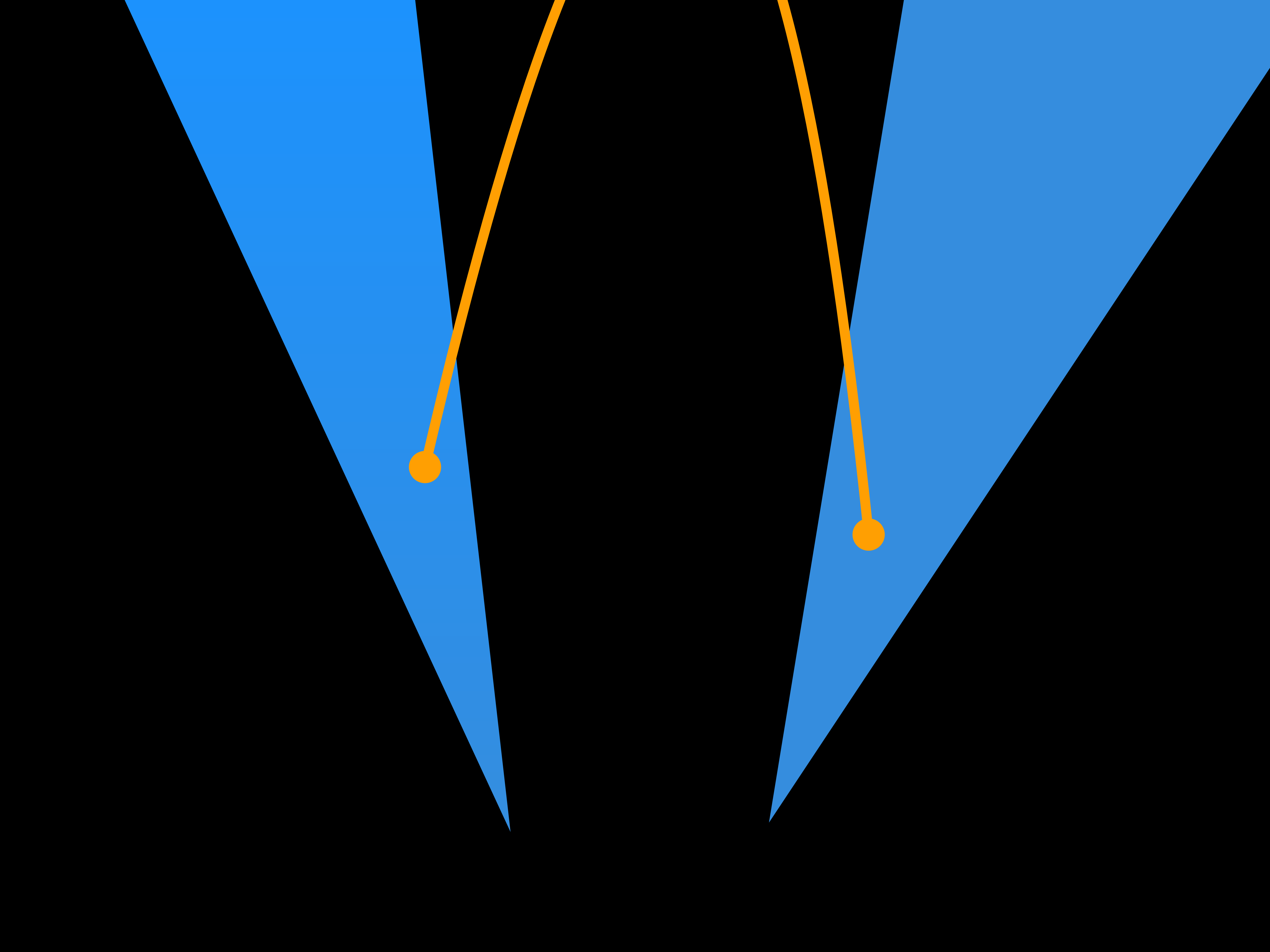
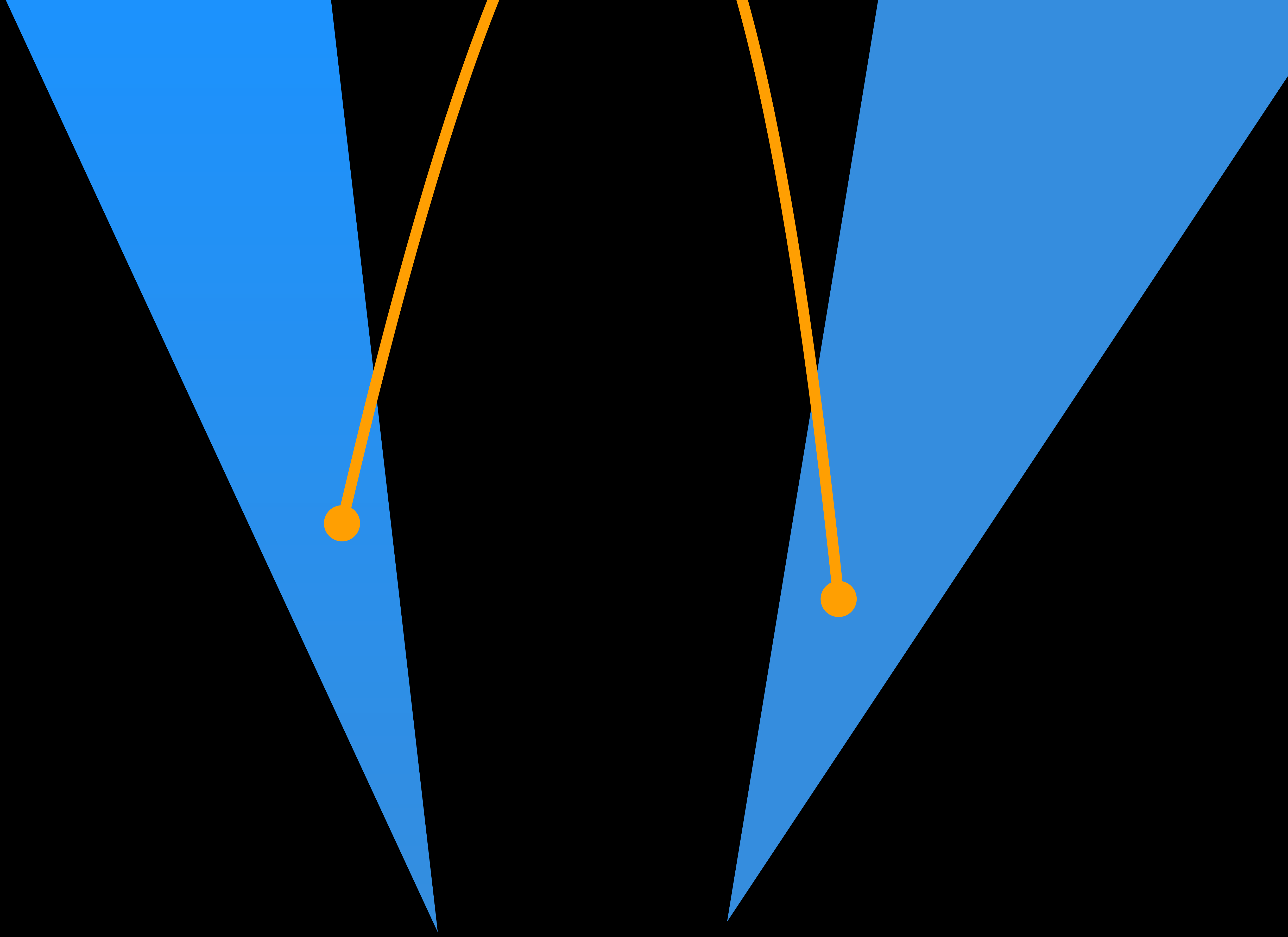
but:

$$\mathcal{R}_{AB} \subsetneqq \widehat{\mathcal{R}}_{AB}$$

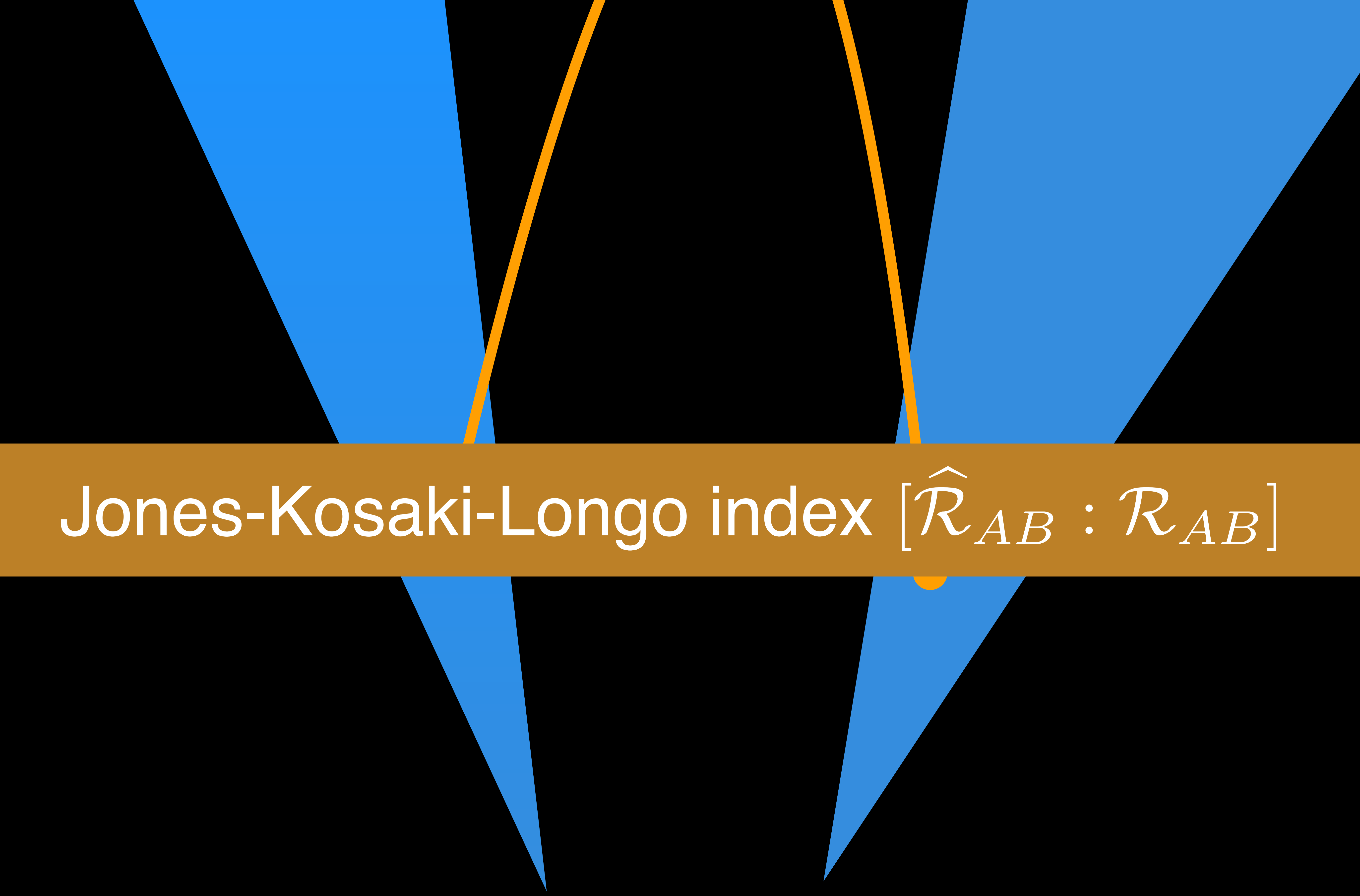








Weak-operator limit is in $\widehat{\mathcal{R}}_{AB}$



Weak-operator limit is in $\widehat{\mathcal{R}}_{AB}$

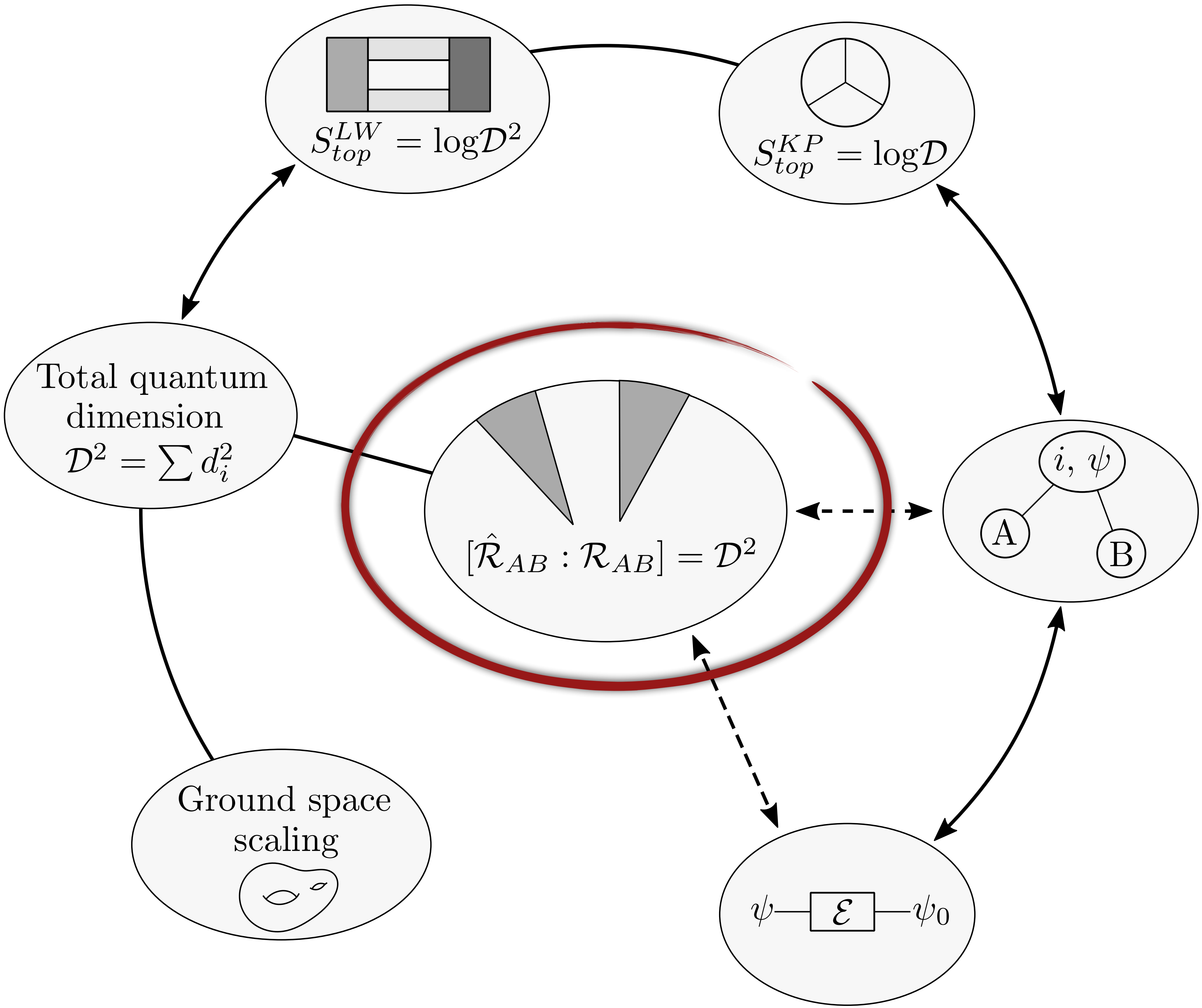
Theorem

The number of excitation types is bounded by

$$\mu_{\pi_0} = \sup_{A \cup B} [\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}]$$

If all excitations have conjugates, μ_{π_0} is equal to the **total quantum dimension**.

PN, J. Math. Phys. ’13
Kawahigashi, Longo & Müger, Commun. Math. Phys. ’01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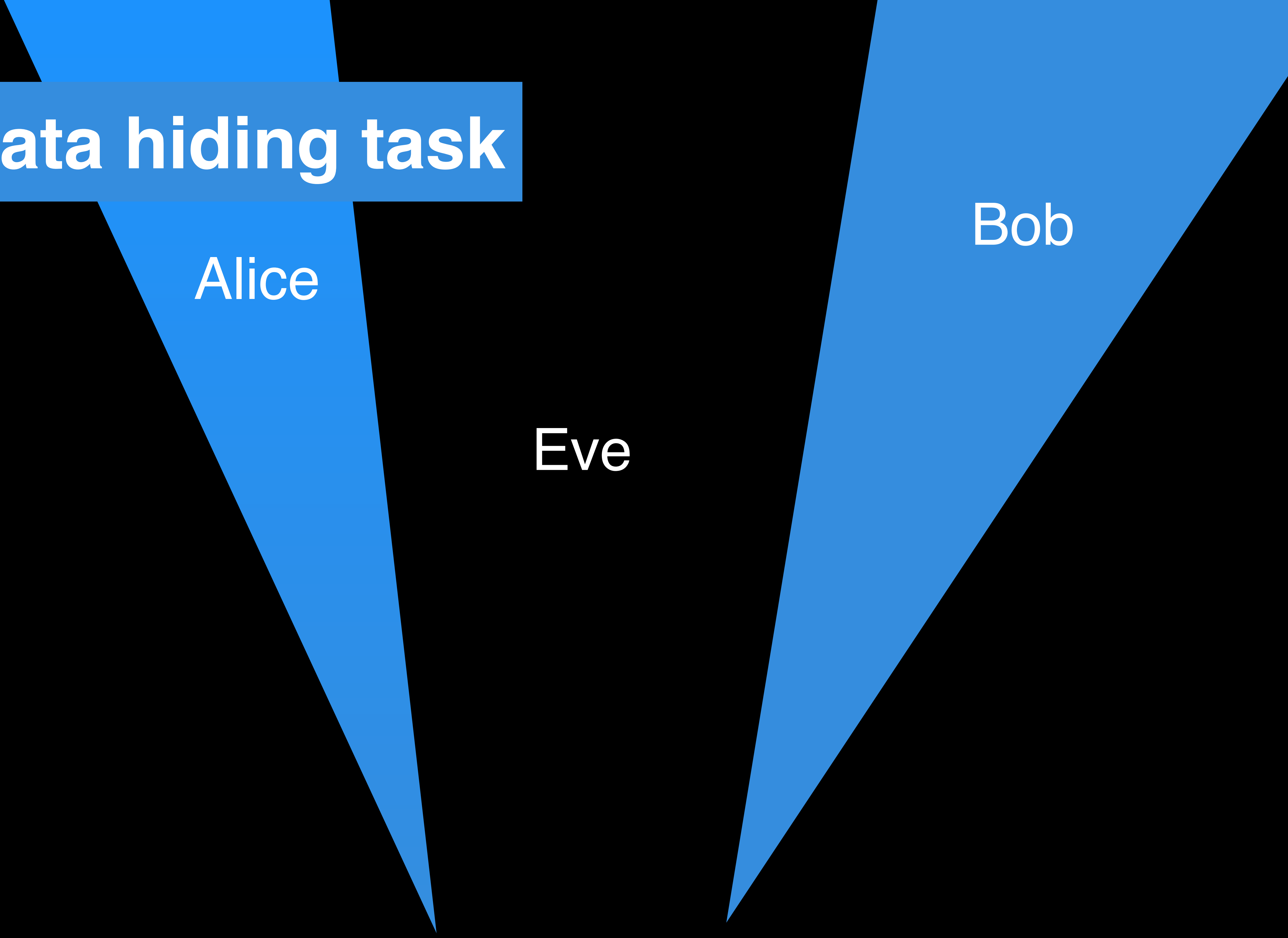
Data hiding

A data hiding task



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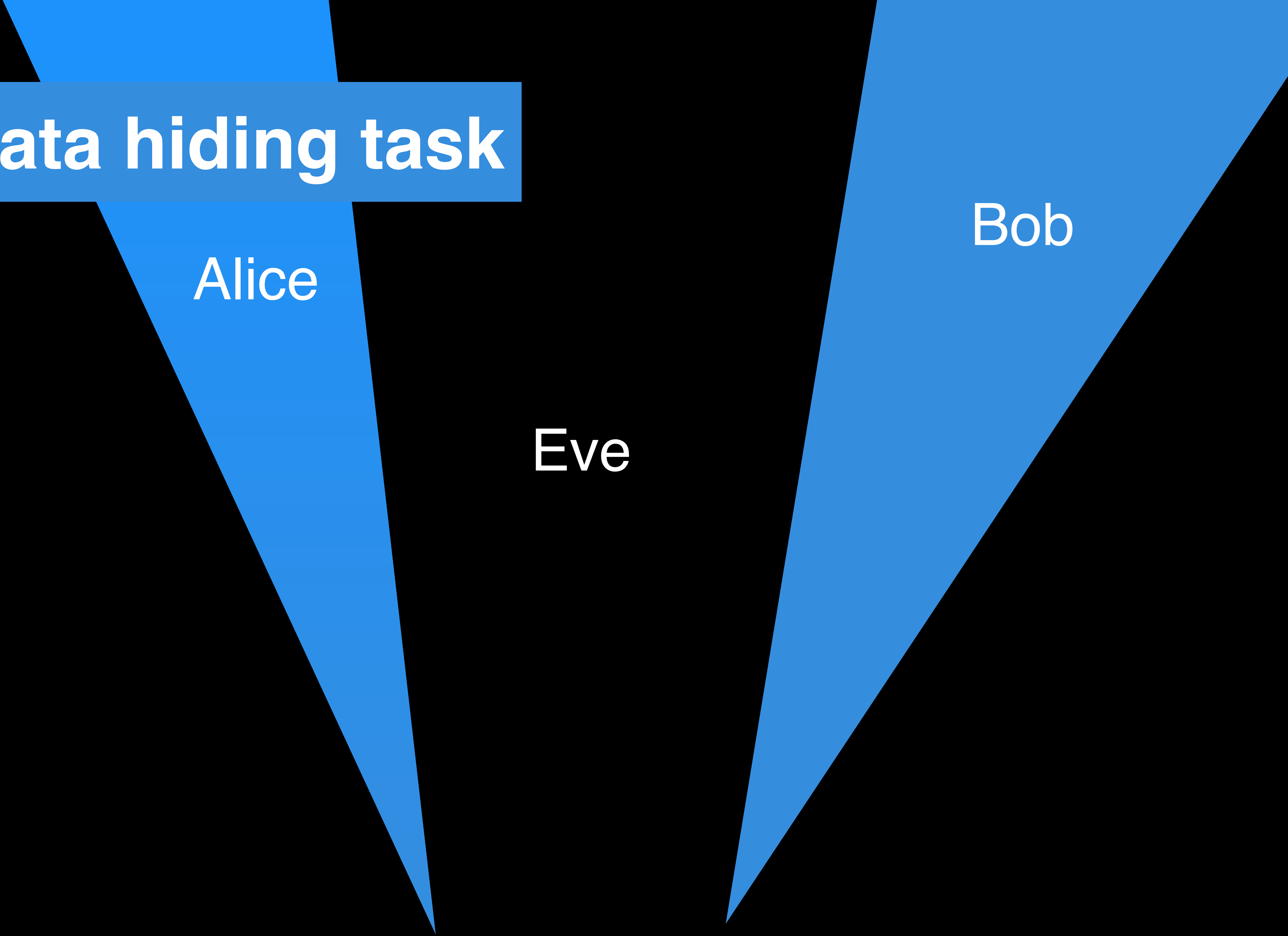
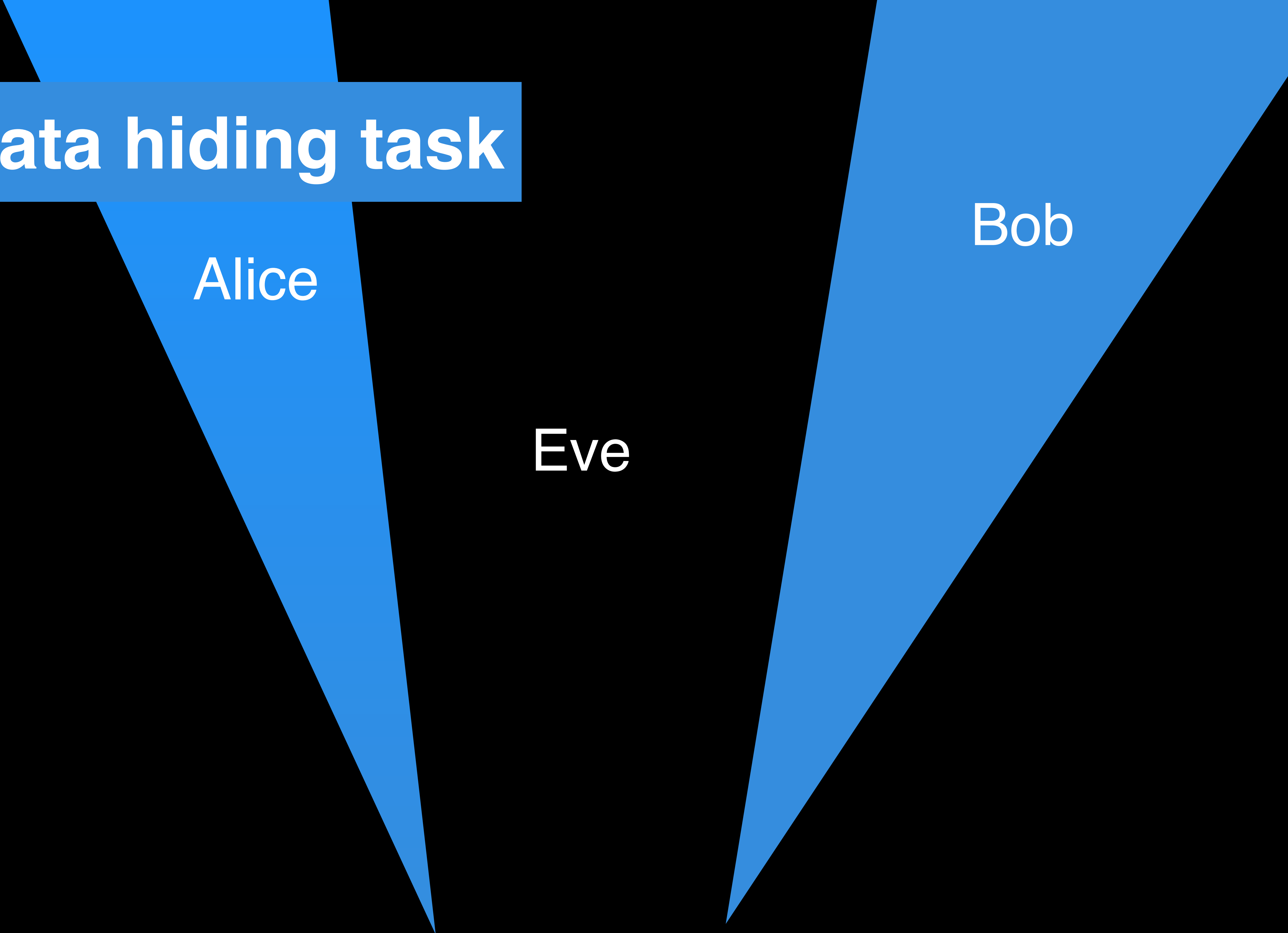


Operations in $\widehat{\mathcal{R}}_{AB}$ are invisible to Eve

A data hiding task

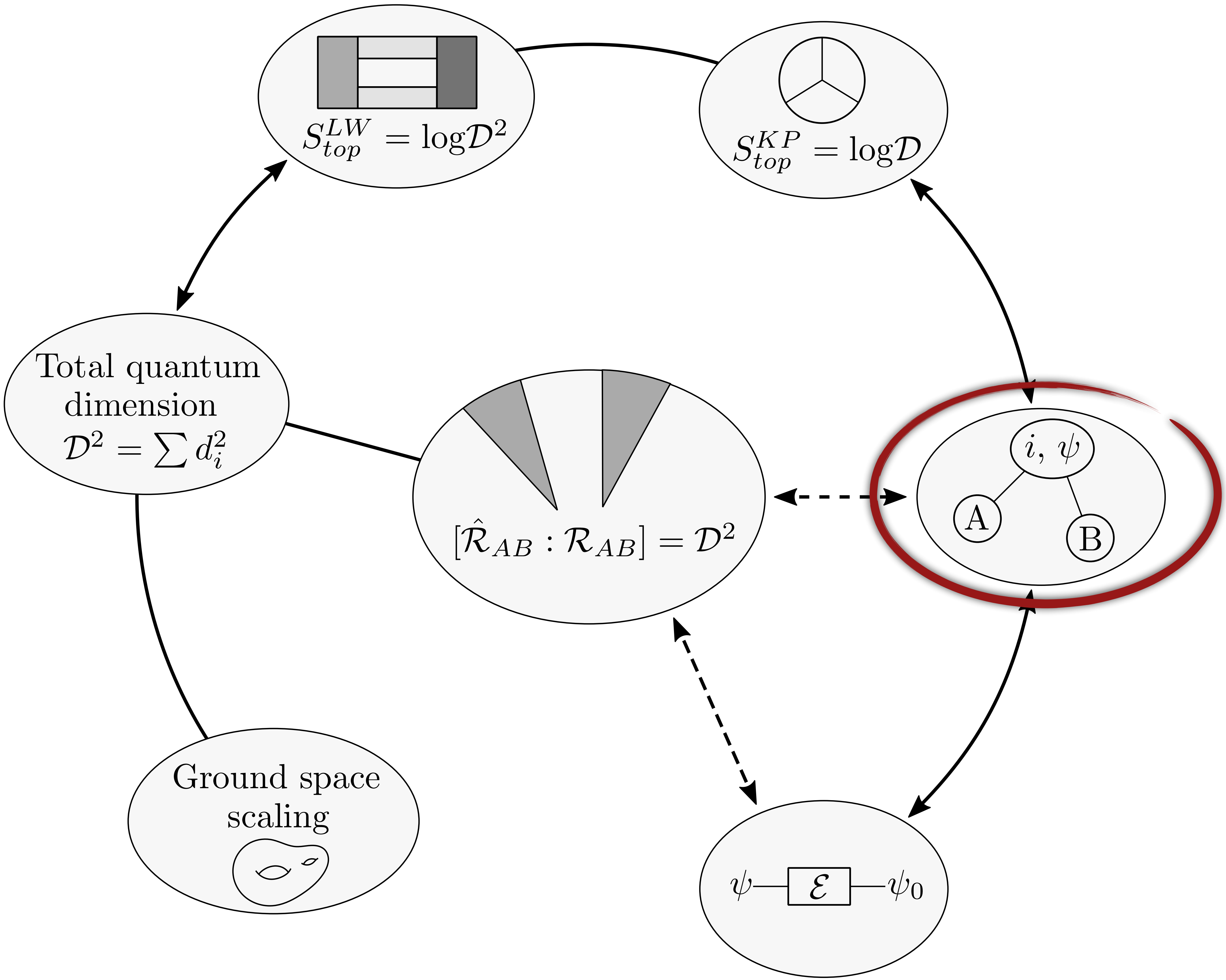


and can be used to create charge pairs

A data hiding task

Similar conclusion: TEE
as a **secret sharing capacity**

Kato, Furrer & Murao, Phys. Rev. A., ’16



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Distinguishing states

Alice prepares a mixed state ρ:

$$\rho = \sum_{i=1}^{n} p_i \rho_i$$

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Can Bob recover $\{p_i\}$?

Holevo χ quantity

In general not exactly:

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Generalisation of **Shannon information**

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In general not exactly:

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$$= \sum_i p_i S(\rho_i, \rho)$$

Generalisation of **Shannon information**

Optimal strategy

Want to compare $\widehat{\mathcal{R}}$ and $\mathcal{R}$:

$$H_\phi(\widehat{\mathcal{R}}|\mathcal{R}) = \sup_{(\phi_i)} \left(\sum_i [S(p_i\phi_i, \phi) - S(p_i\phi_i \restriction \mathcal{R}, \phi \restriction \mathcal{R})] \right)$$

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$$= \sup_{(\phi_i)} (\chi(\{p_i\}, \{\phi_i\}) - \chi(\{p_i\}, \{\phi_i \restriction \mathcal{R}\}))$$

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\begin{aligned}
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&= \sup_{(\phi_i)} \left(\chi(\{p_i\}, \{\phi_i\}) - \chi(\{p_i\}, \{\phi_i \upharpoonright \mathcal{R}\})\right)
\end{aligned}
$$

Δ_χ

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\end{aligned}
$$

Shirokov & Holevo, arXiv:1608.02203

A quantum channel

For finite index inclusion $\mathcal{R} \subset \widehat{\mathcal{R}}$

$$\mathcal{E} : \widehat{\mathcal{R}} \to \mathcal{R}, \qquad \mathcal{E}(X^* X) \geq \frac{1}{[\widehat{\mathcal{R}} : \mathcal{R}]} X^* X$$

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quantum channel, describes the **restriction** of operations

Quantum dimension and entropy

$$\log[\widehat{\mathcal{R}} : \mathcal{R}] = \sup_{\phi:\phi\circ\mathcal{E}=\phi} H_\phi(\widehat{\mathcal{R}}|\mathcal{R})$$

Hiai, J. Operator Theory, ’90; J. Math. Soc. Japan, ‘91

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gives an **information-theoretic** interpretation to quantum dimension

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Completely different methods from Kato/Furrer/Murao, PRA **93**, 022317 (2016)

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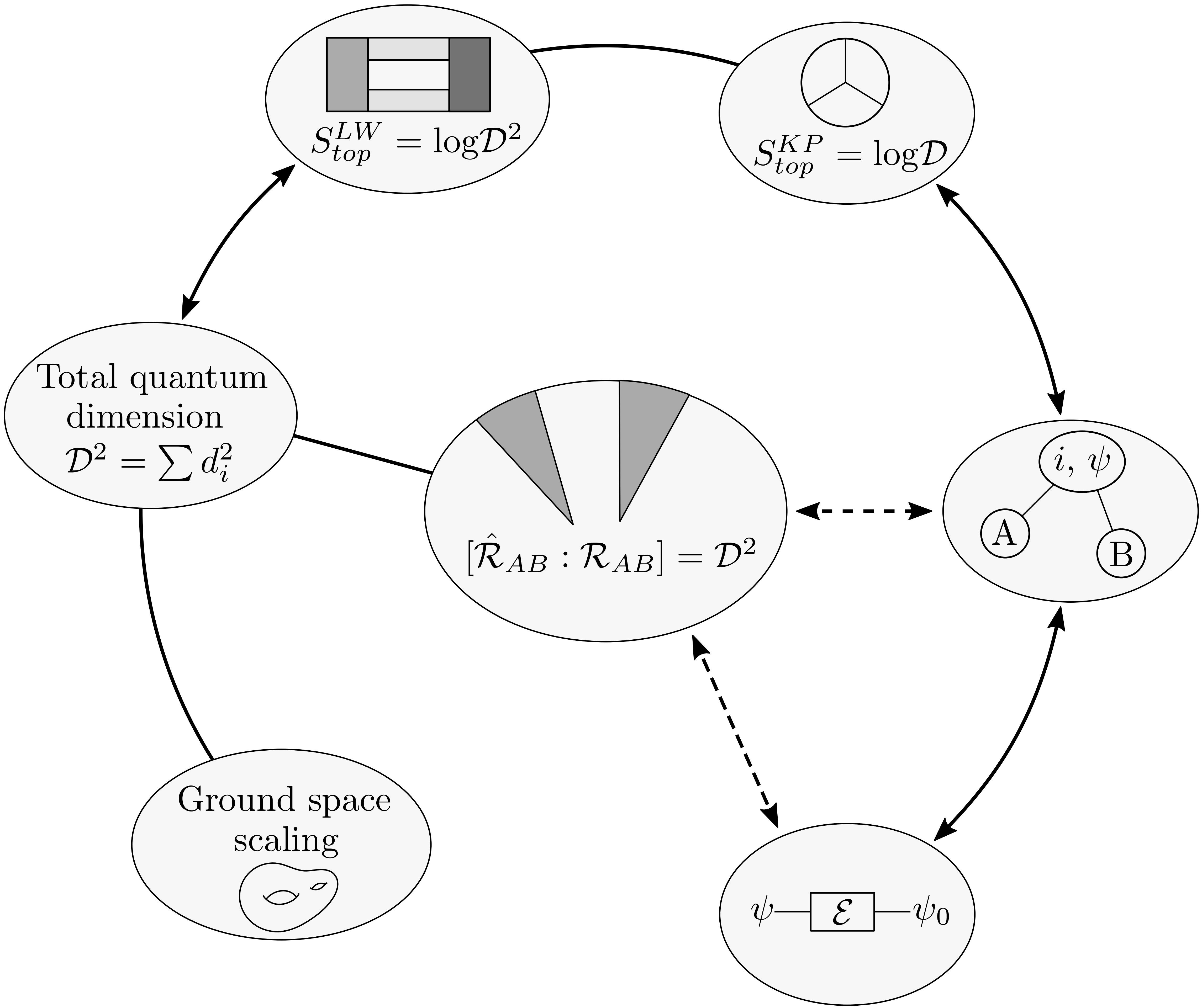
- No finite dimensional analogue to index
- Can use powerful methods from mathematics
- Right framework to study stability?



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