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SYNTHESIS OF THREE-TERMINAL ±R,C TWO-PORTS WITH ONE CONTROLLED SOURCE

A THESIS

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SYNTHESIS OF THREE-TERMINAL ±R,C TWO-PORTS

WITH ONE CONTROLLED SOURCE

Approved:

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DEDICATION

This thesis is dedicated to my parents, Pauline Herman and Lyman Winslow Chick, on the occasion of their 29th wedding anniversary.

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I wish also to express appreciation to my wife, Jane, for her assistance and understanding during this time.

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SUMMARY

In this investigation, the necessary and sufficient conditions are established for a 2 X 2 matrix of real rational functions to be realizable as a short-circuit admittance matrix of a transformerless active three-terminal two-port network utilizing either ±R,C or RC networks and one controlled source or one negative impedance converter. A synthesis procedure is developed to realize the admittance matrix that satisfies the sufficient conditions as a two-port network.

The necessary and sufficient conditions for a 2 X 2 matrix to be realizable as a short-circuit admittance matrix of a ±R,C or RC two-port network with one controlled source or one negative impedance converter embedded in it are

1. The given matrix must be expressible as the sum of a ±R,C or RC admittance matrix and a matrix of rank one.

2. The number of non-compact poles that are common to both driving-point admittances of the ±R,C or RC admittance matrix must be equal to or greater than the order of the elements of the matrix of rank one when the latter is expressed with a common denominator.

The method of synthesis requires that the given matrix, which satisfies the realizability conditions, be decomposed into a ±R,C (RC) admittance matrix and a matrix of rank one. [Here ±R,C (RC) indicates two simultaneous developments of the ±R,C and RC class of networks.] The excess admittance that is not needed to realize the matrix of rank one is removed from the ±R,C (RC) admittance matrix and realized separately

as a ±R,C (RC) two-port network. The remaining admittance is realized as a ±R,C (RC) two-port network with a controlled source or negative impedance converter embedded in it. By connecting these networks in parallel the given admittance matrix is realized by an active RC two-port network.

In the situation when the given matrix contains functions with only simple real poles, a decomposition procedure has been developed for determining whether the necessary and sufficient conditions are satisfied. These realizability conditions are expressed in the form of two inequality equations; one is concerned with the residues of the poles located on the negative real axis including the pole at infinity and the other is concerned with the residues of the poles located on the positive real axis including the origin of the complex-frequency plane. The realizability conditions require that a single real constant exist such that both equations are satisfied simultaneously for all poles including the poles at the origin and infinity. The sufficiency of these conditions is established by the synthesis procedure. However, in the special situation when one or both of the given driving-point admittances are ±R,C (RC); then decomposition is always possible and, therefore, a realization can readily be accomplished.

As a collateral interest, the synthesis technique is adapted to realize any two general rational functions as short-circuit admittances of a three-terminal RC two-port network with one controlled source. Transformers are not required. No control can be placed on the other two admittance functions.

As an application of the synthesis procedure, a method is presented whereby equivalent circuits of active devices can be obtained. By way of illustration, equivalent circuits of a transistor are determined. The equivalent circuits are developed from a set of short-circuit admittance functions which are obtained by approximating a set of experimentally determined magnitude and phase characteristics using the half-line approximation technique. This application provides a method to obtain general equivalent circuits for a transistor valid over any desired frequency range with any degrees of predetermined accuracy by synthesis.

Three equivalent circuits of a transistor are presented, each with varying degrees of accuracy and network complication. The first equivalent circuit is developed with the form of the approximating network predetermined. The form of this network is specifically chosen to be a simple three-terminal RC network containing one voltage-controlled current source. The controlling voltage appears across the input terminals of the network. The controlled current source is embedded in the RC network, shunting the output terminals. In order to keep this first model simple, the approximating admittances are allowed only one common pole and two zeros. The equivalent circuit obtained by this procedure is compared with the conventional equivalent circuit for the transistor -- the hybrid-π model -- in terms of network elements, accuracy of approximation, and frequency range. The comparison shows considerable similarity in both circuits for the frequency range from dc to approximately 100 mc.

The accuracy of the approximations is improved in the second and third equivalent circuits. For these networks the approximating admittances contain two common poles and three zeros. This requires increased network complexity. The second model is developed with the form of the network predetermined as in the first network. The accuracy of the approximation in this case is substantially improved. In thd third model the network is not restricted in form and a slight further improvement is obtained over the second model.

CHAPTER I

INTRODUCTION

Active network synthesis has made marked advances in the last decade. This has been partly due to advancement in fields such as semiconductors, low-temperature systems, and feedback amplifiers. With the advent of these devices, the passivity and bilateralness or reciprocity restrictions can be removed from the traditional passive network synthesis requirements that the elements be finite, lumped, linear, passive, and bilateral. At present, numerous active synthesis procedures have been developed, employing such devices as negative impedance converters, controlled sources, gyrators, and negative resistors (1,2). Most of the present active synthesis procedures employing these active devices use positive resistances and positive capacitance as the passive network elements.

This investigation is concerned with determining the necessary and sufficient conditions for realizing a short-circuit admittance matrix, by a two-port network utilizing $\pm R, C$ or RC and one controlled source or one negative impedance converter. A synthesis procedure is developed to realize the admittance matrix that satisfies the sufficient conditions by a two-port network. As an application of the synthesis procedure, equivalent circuits of a transistor are developed from a set of short-circuit admittance functions which have been obtained to approximate a set of experimentally determined magnitude and phase characteristics.

Considerable progress has been accomplished in the active synthesis field using negative resistance, negative impedance converters, and controlled sources. When it is desired that a 2 X 2 matrix of real rational functions be realized by an active RC network, there are only three possible techniques available. None of these techniques is completely suitable for all practical applications.

One of these synthesis techniques was developed by Phillips and Su (3, 4). Their synthesis procedure employs the negative resistance as the active network element. The procedure synthesizes a two-port network using $\pm R,C$ when all three admittance functions y_{11}, $-y_{12}$, and y_{22} are prescribed and satisfy the necessary and sufficient conditions for physical realizability. (Here $\pm R,C$ refer to a class of networks that contains negative resistances, positive resistances, and positive capacitances.)

Sandberg (5,6) has presented two methods for an arbitrary N X N matrix of real rational functions to be realized as a short-circuit admittance matrix of a transformerless active RC N-port network containing either N negative impedance converters whose chain matrices have the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp k \end{bmatrix}$$

or N controlled sources whose admittance matrices have the form

$$[y] = \begin{bmatrix} 0 & 0 \\ g_n & 0 \end{bmatrix}$$

Therefore, an arbitrary 2 X 2 matrix can be realized by an RC two-port network with two negative impedance converters or two controlled sources.

Application of Phillips and Su's synthesis procedure for ±R,C two-ports is very limited because of the restrictive character of the realizable short-circuit admittances. A realization by Sandberg's general synthesis procedure normally requires two controlled sources or two negative impedance converters and a large number of elements. In the light of this situation, there is an obvious void between the synthesis technique of Phillips and Su for realizing a special type of symmetric admittance matrix and Sandberg's realization of a general 2 X 2 rational matrix. The following question arises; "What are the possibilities of a synthesis procedure utilizing ±R,C or RC and one controlled source or one negative impedance converter?"

This problem has been partially considered. The possibility of realizing admittances by RC and one controlled source or one negative impedance converter has been considered by Sipress and Sandberg. Sipress (7) utilized the properties of a negative impedance converter in conjunction with four RC two-port networks to realize any two rational functions as short-circuit admittances of an active RC two-port network. This procedure is so specialized that no control can be placed on the other admittances. Sandberg (6) observed that a necessary condition for a short-circuit admittance matrix to be realizable by an active RC two-port network with one controlled source imbedded in it is that the matrix be expressible as the sum of an RC short-circuit admittance matrix and a matrix of rank one. The exact nature of these two matrices and the sufficiency of the condition have not been established.

Thus, it is highly desirable to develop a method to realize an admittance matrix with ±R,C (RC) and only one controlled source or one negative impedance converter. [Here ±R,C (RC) indicates two simultaneous developments of the ±R,C and RC class of networks]. Additional motivation is obtained when existing synthesis procedures are not suitable for realizing approximated short-circuit admittances of a transistor when only one controlled source is allowed.

The first part of this investigation is concerned with determining the necessary and sufficient conditions for physical realizability of a 2 X 2 matrix by this class of network, and establishing a synthesis procedure to realize the matrix that satisfies these realizability conditions.

The second part of this investigation is concerned with applying the synthesis procedure developed to obtain equivalent circuits of some active devices. The use of equivalent circuits to characterize active devices such as transistors, vacuum tubes, and electric motors is not new. For years physicists have been determining equivalent circuits for active devices mainly on the basis of physical processes that occur within the device. Often circuits obtained this way have varying degrees of accuracy and are only applicable over a limited range of frequency. Thus, the purpose of this part of the investigation is to demonstrate a procedure whereby more general equivalent circuits for these active devices might be obtained.

The procedure for obtaining the equivalent circuits is to first approximate a set of experimentally measured characteristics, specified in both magnitude and phase, in order to obtain a set of short-circuit admittance functions. These functions are then realized by the synthesis procedure. In particular, this procedure is illustrated by obtaining equivalent circuits for a common emitter transistor. This application provides a method to obtain equivalent circuits for a transistor valid over any desired frequency range with any degrees of predetermined accuracy by synthesis.

CHAPTER II

THE NECESSARY AND SUFFICIENT CONDITIONS

The necessary and sufficient conditions for a 2 X 2 matrix to be realizable as a short-circuit admittance matrix of a ±R,C (RC) two-port network with one controlled source embedded in it are

1. The given matrix must be expressible as the sum of a ±R,C (RC) admittance matrix and a matrix of rank one.

2. The number of non-compact poles that are common to both driving-point admittances of the ±R,C (RC) admittance matrix must be equal to or greater than the order of the elements of the matrix of rank one when the latter is expressed with a common denominator.

That these conditions are both necessary and sufficient for this class of networks will be shown presently.

A given 2 X 2 matrix that satisfies the necessary and sufficient conditions for realization by a short-circuit admittance matrix of a two-port ±R,C (RC) network with one controlled source embedded in it is realizable as a parallel combination of two two-ports -- one of which is a ±R,C (RC) network and the other is a two-port that consists of ±R,C (RC) and one controlled source as shown in Figure 1.

Since the realization of ±R,C (RC) two-port network will be used in this development it is convenient to state the necessary and sufficient*

*Sufficient conditions for realizing a set of short-circuit admittance functions as a three-terminal RC two-port network have not been completely established. Thus, even though all necessary conditions have been fulfilled, a realization may not be possible by RC.

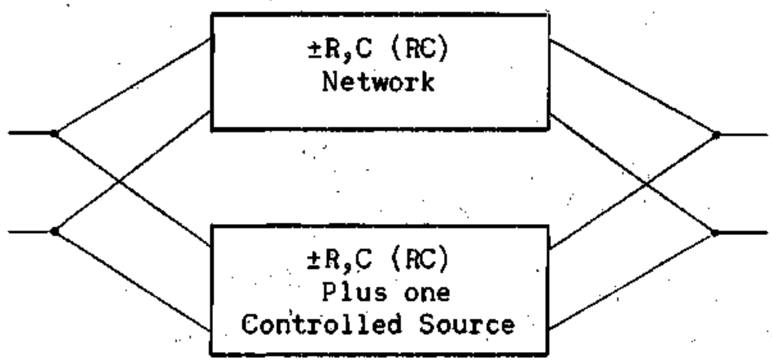


Figure 1. The Network Configuration that Realizes the 2 X 2 Matrix as a Short-Circuited Admittance Matrix.

conditions for realizing this class of networks (8, 9, 10). These conditions shall be expressed in terms of the short-circuit admittance functions,

$$y_{11} = k_{11} + k_{11}^{(\infty)}s + k_{11}^{(o)}/s + \sum_{i=1}^{n} \frac{k_{11}^{(i)}s}{s+\sigma_i}$$

$$-y_{12} = k_{12} + k_{12}^{(\infty)}s + k_{12}^{(o)}/s + \sum_{i=1}^{n} \frac{k_{12}^{(i)}s}{s+\sigma_i} \qquad (1)$$

$$y_{22} = k_{22} + k_{22}^{(\infty)}s + k_{22}^{(o)}/s + \sum_{i=1}^{n} \frac{k_{22}^{(i)}s}{s+\sigma_i}$$

For a ±R,C two-port

1. All coefficients are real,
2. when $\sigma_i > 0$; $k_{11}^{(i)} \geq 0$ and $k_{22}^{(i)} \geq 0$

   when $\sigma_i < 0$; $k_{11}^{(i)} \leq 0$ and $k_{22}^{(i)} \leq 0$

3. $k_{11}^{(\infty)} \geq 0$ and $k_{22}^{(\infty)} \geq 0$; $k_{11}^{(o)} \leq 0$ and $k_{22}^{(o)} \leq 0$

4. in all poles $i = 0,1,2,3,\ldots n,\infty$ $k_{11}^{(i)} k_{22}^{(i)} - (k_{12}^{(i)})^2 \geq 0$

and for an RC two-port network the conditions are

1. all coefficients are real

2. $k_{11}^{(o)} = k_{22}^{(o)} = k_{12}^{(o)} = 0$

3. $k_{11} \geq 0$ and $k_{22} \geq 0$

4. $\sigma_i > 0$, $k_{11}^{(i)} \geq 0$, and $k_{22}^{(i)} \geq 0$

5. in all poles $i = 0,1,2,3,\ldots n,\infty$ $k_{11}^{(i)} k_{22}^{(i)} - (k_{12}^{(i)})^2 \geq 0$

and $k_{11}k_{22} - (k_{12})^2 \geq 0$

Phillips and Su have developed a synthesis procedure which will synthesize a two-port network when all three short-circuit admittance functions, y_{11}, $-y_{12}$, and y_{22} are prescribed and satisfy the necessary and sufficient conditions for physical realizability. Their synthesis procedure utilizes a parallel ladder development in which a network is developed that will realize each type of pole. An attempt has been made to keep the number of negative resistors to a minimum. In Appendix II the networks that realize these poles are presented.

In order to establish the necessary and sufficient conditions for physical realizability of a rational matrix as a ±R,C two-port network with one controlled source embedded in it, the ±R,C (RC) four-port network in Figure 2 is first considered. The short-circuit admittance matrix of this active four-port is symmetric $[y_{ij} = y_{ji}]$ and can be written as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} \qquad (2)$$

A sufficient condition for the transformerless realization of this 4 X 4 matrix is that the matrix be dominant. A matrix is dominant if each of its main-diagonal residues is not less than the sum of the absolute values of all the other residues in that pole in the same row, including the constant term (11).

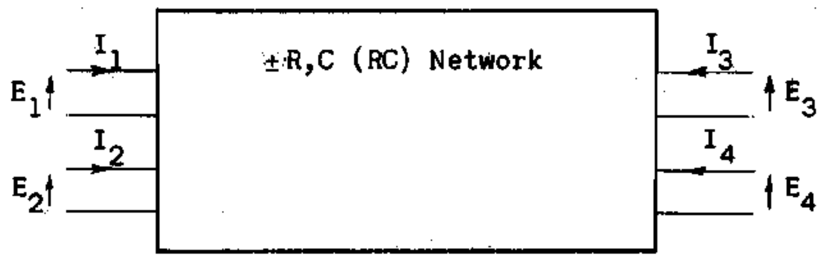


Figure 2. The Arrangement for an Active ±R,C (RC) Four-port Network.

There is no limitation on the type of controlled source to be considered in this research. However, this particular derivation is only applicable to three types of controlled-sources. The fourth type requires a different treatment. The details of this different treatment are discussed in Appendix I. The controlling qualities are placed at port 3 and the controlled quantities at port 4. The controlling quantities [either E_3 or I_3] are related to the controlled quantities [either E_4 or I_4] by matrix relationship

$$\begin{bmatrix} E_4 \\ \\ I_4 \end{bmatrix} = [\beta] \times \begin{bmatrix} E_3 \\ \\ I_3 \end{bmatrix} \tag{3}$$

Here $[\beta]$ is a matrix containing only one real non-zero element. It can be anyone of the following three matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \quad \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

The first matrix is applicable for a current-controlled current source, the second for a current-controlled voltage source, and the third for a voltage-controlled voltage source.

As an example, if the controlled source is a current-controlled voltage source the network configuration is that indicated in Figure 3.

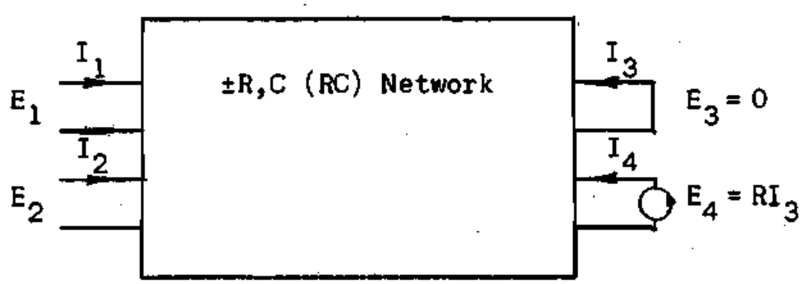


Figure 3. The Arrangement of an Active ±R,C (RC) Two-port Network with One Current-Controlled Voltage Source Embedded in it.

As the controlled source in this development is arbitrary, the network of Figure 3 will be used for the purposes of illustrating the development without loss of generality. Solving for the admittance matrix $[Y]$, which is defined by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [Y] \times \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{5}$$

of the active two-port network of Figure 3 by substituting Equation (3) into Equation (2) and simplifying, yields

$$[Y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} + \frac{R}{1 - Ry_{34}} \begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} [y_{13} \quad y_{23}] \tag{6}$$

It can be deduced that the admittance matrix, is represented by the sum of matrices

$$[y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \tag{7}$$

which is a ±R,C (RC) short-circuit admittance matrix, and

$$[y^{(o)}] = \begin{bmatrix} y_{11}^{(o)} & y_{12}^{(o)} \\ y_{21}^{(o)} & y_{22}^{(o)} \end{bmatrix} = \frac{1}{1 - Ry_{34}} \begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} [y_{13} \quad y_{23}] \tag{8}$$

which is a matrix with rank one.

Therefore, one necessary condition for a matrix to be realizable as a short-circuit admittance matrix of a ±R,C (RC) two-port network with one controlled source embedded in it, is that the matrix be expressible as the sum of two matrices, one having the character of a ±R,C (RC) short-circuit admittance matrix and the other matrix having a rank of one.

The ±R,C (RC) short-circuit admittance matrix in Equation (7) can always be written into the following form,

$$[y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} y'_{11} & y'_{12} \\ y'_{21} & y'_{22} \end{bmatrix} + \begin{bmatrix} y''_{11} & 0 \\ 0 & y''_{22} \end{bmatrix} + \begin{bmatrix} y'''_{11} & 0 \\ 0 & y'''_{22} \end{bmatrix} \qquad (9)$$

The three matrices in Equation (9) are designated as

$$[y'] = \begin{bmatrix} y'_{11} & y'_{12} \\ y'_{21} & y'_{22} \end{bmatrix} \quad [y''] = \begin{bmatrix} y''_{11} & 0 \\ 0 & y''_{22} \end{bmatrix} \quad [y'''] = \begin{bmatrix} y'''_{11} & 0 \\ 0 & y'''_{22} \end{bmatrix}$$

The admittances y'''_{11} and y'''_{22} represent the private* admittances of y_{11} and y_{22}, respectively. The residues in each of the poles in the admittances y'_{11}, y'_{12}, and y'_{22} all satisfy the compact condition. Thus, the admittances y''_{11} and y''_{22} represent the admittance removed from the non-compact poles of the original matrix in the process of making the residues in the poles of y'_{11}, y'_{12}, and y'_{22} compact. This is accomplished by removing the surplus in these non-compact poles in such a way that a non-zero part is assigned to each of the admittances y''_{11} and y''_{22}.

The most general representation of a rational 2 X 2 matrix of rank one is

$$\frac{1}{Q} \begin{bmatrix} P_a P_b & P_b P_c \\ P_a P_d & P_d P_c \end{bmatrix} \qquad (10)$$

---

*The private admittances y'''_{11} and y'''_{22} do not contain any common poles.

The polynomials P_a, P_b, P_c, P_d, and Q are completely general. If a given matrix of rank one does not have the same denominator in its elements, Q polynomial represents the lowest common denominator. This matrix can always be written as

$$\frac{1}{Q}\begin{bmatrix} P_aP_b & P_bP_c \\ P_aP_d & P_dP_c \end{bmatrix} = \frac{1}{Q}\begin{bmatrix} P_b \\ P_d \end{bmatrix}\begin{bmatrix} P_a & P_c \end{bmatrix} \tag{11}$$

Clearly the right hand side of Equation (11) has a form that is similar to that of the admittance matrix of rank one in (8). In order to show how the quantities in Equation (11) can be identified with the admittances in (8), a few definitions are needed.

First let the maximum degree of the polynomials P_aP_b, P_bP_c, P_aP_d, P_dP_c, and Q be equal to or less than M, where M is an arbitrary number. In addition let the degree of P_aP_b be equal to M. Thus, if the degree of P_a is n_1 and that of P_b is n_2, then $n_1n_2 = M$.

Now choose a polynomial $g = g_1g_2$ which contains only simple real zeros (for RC case, g must contain only simple real zeros located on the negative real axis of the complex frequency plane and $g(0) \neq 0$) and is of degree equal to or greater than M. Let the degree of the polynomials g_1 and g_2 be n_1 and n_2', respectively. Then, clearly $n_2' \geq n_1$. Now let

$$1 - Ry_{34} = Q/g$$

or

$$-y_{34} = -\frac{1}{R}(1 - Q/g) \tag{12}$$

In addition, let

$$-y_{14} = \frac{1}{\gamma} \times \frac{P_b}{g_2} \qquad -y_{13} = \frac{\gamma}{R} \times \frac{P_a}{g_1}$$

$$-y_{24} = \frac{1}{\gamma} \times \frac{P_d}{g_2} \qquad -y_{23} = \frac{\gamma}{R} \times \frac{P_c}{g_1} \tag{13}$$

where γ is a constant. Each of these admittances satisfies the requirements of ±R,C (RC) admittance functions. This is clear as the degree of P_a is n_1 and P_b is n_2. Since P_bP_a is of degree equal to or less than M, the maximum degree that P_c can be is n_1. Similarly, the maximum degree that P_d can be is n_2. Hence, each of these admittances has a denominator that has only simple real poles and of order equal to or greater than the order of its numerator.

However, in order to realize the four-port network of which these admittances are elements, obviously the admittances y''_{11} and y''_{22} must contain sufficient ±R,C (RC) admittances such that the transfer admittances in Equation (13) can be realized.

Thus, a second necessary condition for a matrix to be realizable as a short-circuit admittance matrix of a ±R,C (RC) two-port network with one controlled source is that the number of the non-compact poles common to both driving-point admittances of the ±R,C (RC) admittance matrix [i.e. y''_{11} and y''_{22}] must be equal to or greater than the degrees of both numerator and denominator polynomials of the matrix or rank one when the latter is expressed with a common denominator. That is, it is required that y''_{11} and y''_{22} contain at least the admittances P_{11}/g and P_{22}/g, respectively, where these functions are ±R,C (RC) driving-point admittances and are of order equal or greater than M.

We shall now show that these two necessary conditions for a given 2 X 2 matrix to be realizable as a short-circuit admittance matrix of a ±R,C (RC) two-port network with a controlled source are also sufficient by indicating a synthesis procedure.

When given a 2 X 2 matrix that is to be realized by this class of network the first step in the realization procedure is to write this matrix as the sum of a ±R,C (RC) admittance matrix and a matrix of rank one. The ±R,C (RC) admittance matrix is then expressed in the form of Equation (9). Now, consider the ±R,C (RC) admittance matrix y'_{11}, y'_{12} and y'_{22} from Equation (9). This matrix which satisfies the necessary and sufficient condition for realizability as a ±R,C (RC) two-port network, can be realized by a ±R,C (RC) two-port and connected in parallel with the other two-port which realizes the remainder of the given matrix. In addition, the private ±R,C (RC) admittances y'''_{11} and y'''_{22} from Equation (9) satisfy the realizability requirements for ±R,C (RC) driving-point admittances and can, therefore, be realized as ±R,C (RC) admittances and connected in shunt across the terminals of ports 1 and 2, respectively.

In order to realize the remaining ±R,C (RC) admittance and the general matrix of rank one, recall that the remainder of the ±R,C (RC) admittance matrix is now y''_{11} and y''_{22} and that these admittances can be written as P_{11}/g and P_{22}/g, respectively. These ±R,C (RC) admittances satisfy the realizability conditions for ±R,C (RC) driving-point admittances and are of order equal to or greater than M. The transfer admittances $-y_{13}$, $-y_{14}$, $-y_{23}$, $-y_{24}$, and $-y_{34}$ are specified in Equations (12) and (13) and represent ±R,C (RC) transfer admittances. In

(8), these transfer admittances form a matrix of rank one. Thus, it has been shown that the remainder of the given 2 X 2 matrix can be expressed by these admittances.

Prior to showing that these admittances can always be realized as ±R,C (RC) transfer admittances of ±R,C (RC) two-port networks, it can be observed that in Equation (13) all the transfer admittances $-y_{13}$, $-y_{14}$, $-y_{23}$, and $-y_{24}$ contain a constant multiplier, either $1/\gamma$ or γ/R. In addition, the admittances y_{33} and y_{44} from the original four-port network, Equation (2), are not specified in Equation (6). Therefore, it is apparent that these ±R,C (RC) driving-point admittances are arbitrary and can be expressed as $y_{33} = P_{33}/g$ and $y_{44} = P_{44}/g$.

Upon substituting these specified admittances into the 4 X 4 matrix for the four-port ±R,C (RC) network there results

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} \frac{P_{11}}{g} & 0 & -\frac{\gamma}{R}\frac{P_a}{g_1} & -\frac{1}{\gamma}\frac{P_b}{g_2} \\ 0 & \frac{P_{22}}{g} & -\frac{\gamma}{R}\frac{P_c}{g_1} & -\frac{1}{\gamma}\frac{P_d}{g_2} \\ -\frac{\gamma}{R}\frac{P_a}{g_1} & -\frac{\gamma}{R}\frac{P_c}{g_1} & \frac{P_{33}}{g} & \frac{1}{R}(1-\frac{Q}{g}) \\ -\frac{1}{\gamma}\frac{P_b}{g_2} & -\frac{1}{\gamma}\frac{P_d}{g_2} & \frac{1}{R}(1-\frac{Q}{g}) & \frac{P_{44}}{g} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} \qquad (14)$$

Now it is apparent that the constants $1/\gamma$ and γ/R can always be selected such that the dominance condition is satisfied in the first two rows. The functions P_{33}/g and P_{44}/g can always be selected to be large. Since all elements are ±R,C (RC) admittances and the diagonal elements can always be made as large as desired, the 4 X 4 matrix of Equation (14) can always be made to be dominant. To complete the realization these admittances are realized as ±R,C (RC) four-port networks and connected together to realize the ±R,C (RC) network with a controlled source. This network is connected in parallel with the previously realized ±R,C (RC) networks. Thus, the given 2 X 2 matrix is realized and the necessary conditions are also sufficient.

Although the synthesis procedure has been presented from the viewpoint of a particular controlled source, any controlled source could have been used with similar results. In an alternate development it would be necessary to identify the admittances of the matrix of rank one with the polynomials of the matrix of rank one in a similar manner.

Thus it has been shown that if the given matrix $[Y]$, can be expressed as

$$[Y] = [y'] + [y''] + [y'''] + [y^{(o)}] \tag{15}$$

the sum of a ±R,C (RC) admittance matrix and a matrix of rank one where the number of the non-compact poles in the driving-point admittances of the ±R,C (RC) admittances matrix are sufficient, then it is always possible for the given 2 X 2 matrix to be realized as a short-circuit admittance matrix of a ±R,C (RC) two-port network with one controlled source embedded in it.

CHAPTER III

SYNTHESIS OF A MATRIX WITH ONLY SIMPLE REAL POLES

Although the necessary and sufficient conditions for a 2 X 2 matrix to be realizable as a short-circuit admittance matrix of an ±R,C (RC) two-port network with one controlled source have been established and a synthesis procedure indicated in Chapter II, some additional considerations are required. Normally a problem is given in the form of four admittances Y_{11}, Y_{12}, Y_{21}, and Y_{22}, which are to be realized. In order to determine whether these admittances satisfy the necessary and sufficient conditions for a realization, it is required to express these functions as the sum of the matrices in Equation (15). This step is accomplished by first expanding each of the given admittances into partial fraction form. Since complex and multiple poles are readily determined these elements are grouped and identified as part of matrix $[y^{(o)}]$. The remaining functions have only simple real poles. If the residues in these poles satisfy the ±R,C (RC) realizability conditions, then these functions are identified with matrices $[y']$, $[y'']$, and $[y''']$. The remaining functions have simple real poles but do not satisfy the ±R,C (RC) realizability conditions. For these functions to be realized, the residues in each set of poles must be decomposed so that these functions can be appropriately allotted to matrices $[y]$ and $[y^{(o)}]$.

Since the denominator Q of $[y^{(o)}]$ can contain both simple real poles (g_3) and complex and multiple poles (Q'), it is helpful to let

$$
\begin{aligned}
y_{11}^{(o)} &= \frac{P_1P_2}{g_3} + \frac{P_1P_5}{Q'} \\
-y_{12}^{(o)} &= \frac{P_1P_3}{g_3} + \frac{P_1P_6}{Q'} \\
-y_{21}^{(o)} &= \frac{P_4P_2}{g_3} + \frac{P_4P_5}{Q'} \\
y_{22}^{(o)} &= \frac{P_4P_3}{g_3} + \frac{P_4P_6}{Q'}
\end{aligned} \tag{16}
$$

Here polynomials P_1, P_2, P_3, P_4, P_5, P_6, and Q' are completely general. An alternate way of expressing this matrix is obtained by interchanging the quantities of $-y_{12}^{(o)}$ with $-y_{21}^{(o)}$. This is a completely general representation of a matrix of rank one.

Obviously when a given problem requires that the matrix $[y^{(o)}]$ contains only complex and multiple poles (Q'), then $P_2 = P_3 = 0$ and the Equation (16) reduces to Equation (11) with $Q' = Q$. Polynomials P_1, P_4, P_5, P_6, and Q' are readily identified as previously mentioned.

When the given problem requires matrix $[y^{(o)}]$ to contain both types of poles [indicated by g_3 and Q'] methods for decomposition of the functions with simple real poles that are not ±R,C (RC) is less apparent. This is because polynomials P_1 and P_4 appear in the numerators of both the complex and multiple poles (Q') and the simple real poles (g_3). A special case occurs when the given problem specifies that $P_1 = h_1$ and $P_4 = h$, where h_1 and h are constants, then decomposition of the functions that are not ±R,C (RC) can be readily accomplished.

When the matrix $[y^{(o)}]$ contains only simple real poles (g_3), then $P_5 = P_6 = 0$. The problem of determining polynomials P_1, P_2, P_3, P_4, and g_3 to facilitate the decomposition of functions that are not ±R,C (RC) will now be considered.

The Determination of the Realizability of a Matrix

The necessary and sufficient conditions for a 2 X 2 matrix that contains only simple real-axis poles to be realizable as a short-circuit admittance matrix, where this matrix has the form

$$
\begin{aligned}
Y_{11} &= g_{11} + \alpha_{11}^{(\infty)} s + \alpha_{11}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{11}^{(i)} s}{s + \sigma_i} \\
-Y_{12} &= g_{12} + \alpha_{12}^{(\infty)} s + \alpha_{12}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{12}^{(i)} s}{s + \sigma_i} \\
-Y_{21} &= g_{21} + \alpha_{21}^{(\infty)} s + \alpha_{21}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{21}^{(i)} s}{s + \sigma_i} \\
Y_{22} &= g_{22} + \alpha_{22}^{(\infty)} s + \alpha_{22}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{22}^{(i)} s}{s + \sigma_i}
\end{aligned}
\tag{17}
$$

as a ±R,C (RC) two-port network with one controlled source embedded in it are

1. all the coefficients must be real,

2. at least one value of h exists such that the following equations are satisfied for all sets of residues that do not satisfy the realizability conditions of a ±R,C (RC) network:

a. For the pole at infinity and for all poles on the negative real axis,

$$h(a_{12}^{(i)} + a_{21}^{(i)}) - h^2 a_{11}^{(i)} < a_{22}^{(i)} \tag{18}$$

(For an RC realization $h(g_{12} + g_{21}) - h^2 g_{11} < g_{22}$ must also be satisfied)

for $i = 1,2,3,\ldots,n,\infty$.

b. For the pole at zero and for all poles on the positive real axis, the following equation must be satisfied

$$h(a_{12}^{(i)} + a_{21}^{(i)}) - h^2 a_{11}^{(i)} > a_{22}^{(i)} \tag{19}$$

(not applicable for RC case)

for $i = 0,1,2,\ldots, n$

That these conditions represent the necessary and sufficient conditions for this class of network will now be demonstrated. This is accomplished by expressing the given matrix in partial-fraction form and expressing it as the sum of the partial-fraction forms of a ±R,C (RC) short-circuit admittance and a matrix of rank one, and thereby satisfying the first necessary and sufficient condition for a 2 X 2 matrix to be realized by this class of network.

The most general form of $[y^{(o)}]$ when it contains only simple real poles is in Equation (16) with $P_1 = h_1$ and $P_4 = h$, where h_1 and h are constants. In this situation, it is possible to specify both polynomials P_2 and P_3 independently, especially when they are expanded in partial-fraction form. Otherwise, little control can be exercised over the residues. With no loss in generality it is assumed that $h_1 = 1$ and

$$\frac{P_2}{g_3} = d + d^{(o)}/s + d^{(\infty)}s + \frac{d^{(i)}s}{s+\sigma_i}$$

$$\frac{P_3}{g_3} = d_1 + d_1^{(o)}/s + d_1^{(\infty)} + \frac{d^{(i)}s}{s+\sigma_i} \tag{20}$$

$$\frac{hP_2}{g_3} = hd + hd^{(o)}/s + hd^{(\infty)}s + \frac{hd^{(i)}s}{s+\sigma_i}$$

$$\frac{hP_3}{g_2} = hd_1 + hd_1^{(o)}/s + hd_1^{(\infty)}s + \frac{hd_1^{(i)}s}{s+\sigma_i}$$

where all coefficients are real and $i = 1,2,3,\ldots,n$. Equation (17) may now be written as the sum of Equations (1) and (20) as

$$Y_{11} = g_{11} + \alpha_{11}^{(\infty)}s + \alpha_{11}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{11}^{(i)}s}{s+\sigma_i}$$

$$-Y_{12} = g_{12} + \alpha_{12}^{(\infty)}s + \alpha_{12}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{12}^{(i)}s}{s+\sigma_i}$$

$$-Y_{21} = g_{21} + \alpha_{21}^{(\infty)}s + \alpha_{21}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{21}^{(i)}s}{s+\sigma_i}$$

$$-Y_{22} = g_{22} + \alpha_{22}^{(\infty)}s + \alpha_{22}^{(o)}/s + \sum_{i=1}^{n} \frac{\alpha_{22}^{(i)}s}{s+\sigma_i} \tag{21}$$

$$= k_{11} + k_{11}^{(o)}/s + k_{11}^{(\infty)}s + \sum_{i=1}^{n} \frac{k_{11}^{(o)}s}{s+\sigma_i} + d + d^{(o)}/s + d^{(\infty)}s + \sum_{i=1}^{n} \frac{d^{(i)}s}{s+\sigma_i}$$

$$= k_{12} + k_{12}^{(o)}/s + k_{12}^{(\infty)}s + \sum_{i=1}^{n} \frac{k_{12}^{(i)}s}{s+\sigma_i} + d_1 + d_1^{(o)}/s + d_1^{(\infty)}s + \sum_{i=1}^{n} \frac{d_1^{(i)}s}{s+\sigma_i}$$

$$= k_{21} + k_{21}^{(o)}/s + k_{21}^{(\infty)}s + \sum_{i=1}^{n} \frac{k_{21}^{(i)}s}{s+\sigma_i} + hd + hd^{(o)}/s + hd^{(\infty)}s + \sum_{i=1}^{n} \frac{hd^{(i)}s}{s+\sigma_i}$$

$$= k_{22} + k_{22}^{(o)}/s + k_{22}^{(\infty)}s \sum_{i=1}^{n} \frac{k_{22}^{(i)}s}{s+\sigma_i} + hd_1 + hd_1^{(o)}/s + hd_1^{(\infty)}s \sum_{i=1}^{n} \frac{hd_1^{(i)}s}{s+\sigma_i}$$

or

$$\begin{aligned}
\alpha_{11}^{(i)} &= k_{11}^{(i)} + d^{(i)} \\
\alpha_{12}^{(i)} &= k_{12}^{(i)} + d_1^{(i)} \\
\alpha_{21}^{(i)} &= k_{21}^{(i)} + hd^{(i)} \\
\alpha_{22}^{(i)} &= k_{22}^{(i)} + hd_1^{(i)}
\end{aligned} \tag{22}$$

for $i = 0,1,2,3,\ldots,n,\infty$.

To establish the necessary conditions for a 2 X 2 admittance matrix to be realized by this class of network, it is sufficient to establish the necessary restrictions on α's. This can be accomplished by imposing the previously outlined requirements on residues $k_{11}^{(i)}$, $k_{12}^{(i)}$, and $k_{22}^{(i)}$. These conditions are:

A. (i) For $\sigma_i > 0$ as well as $i = \infty$, it is necessary that $k_{11}^{(i)} \geq 0$ and $k_{22}^{(i)} \geq 0$ or

$$d^{(i)} \leq \alpha_{11}^{(i)} \quad \text{and} \quad hd_1^{(i)} \leq \alpha_{22}^{(i)} \tag{23}$$

(For RC case, $k_{11} \geq 0$ and $k_{22} \geq 0$ must be included. Therefore, $d \leq g_{11}$ and $hd_1 \leq g_{22}$)

(ii) For $\sigma_i < 0$ as well as $i = 0$, it is necessary that $k_{11}^{(i)} \leq 0$ and $k_{22}^{(i)} \leq 0$ or

$$d^{(i)} \geq \alpha_{11}^{(i)} \quad \text{and} \quad hd_1^{(i)} \geq \alpha_{22}^{(i)} \tag{24}$$

(Not applicable to RC case.)

B. It is necessary that $k_{12}^{(i)} = k_{21}^{(i)}$ or

$$k_{12}^{(i)} = k_{21}^{(i)} = \alpha_{12}^{(i)} - d_1^{(i)} = \alpha_{21}^{(i)} - hd^{(i)} \tag{25}$$

for $i = 0,1,2,3,\ldots,n,\infty$.

C. It is necessary that $k_{11}^{(i)} \times k_{22}^{(i)} - (k_{12}^{(i)})^2 \geq 0$ or

$$(\alpha_{11}^{(i)} - d^{(i)})(\alpha_{22}^{(i)} - hd_1^{(i)}) - (\alpha_{21}^{(i)} - hd^{(i)})^2 \geq 0 \tag{26}$$

(For RC case, $k_{11} \times k_{22} - (k_{12})^2 \geq 0$ must also be included)

for $i = 0,1,2,\ldots,n,\infty$.

Solving for $d_1^{(i)}$ from Equation (25) yields

$$d_1^{(i)} = (\alpha_{12}^{(i)} - \alpha_{21}^{(i)}) + hd^{(i)} \tag{27}$$

Substituting Equation (27) into Equation (26) and solving for $d^{(i)}$, it is possible, after some manipulation, to write

(i) For $\sigma_i > 0$ as well as $i = \infty$,

$$d^{(i)} \leq \alpha_{11}^{(i)} + \frac{(\alpha_{21}^{(i)} - h\alpha_{11}^{(i)})^2}{h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (h^2\alpha_{11}^{(i)} + \alpha_{22}^{(i)})} \tag{28}$$

assuming $h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (h^2\alpha_{11}^{(i)} + \alpha_{22}^{(i)}) < 0$ or

$$h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - h^2\alpha_{11}^{(i)} < \alpha_{22}^{(i)} \tag{29}$$

Since in Equation (23), $d^{(i)} \leq \alpha_{11}^{(i)}$, Equation (29) must hold. Equation (28) is stronger than Equation (23). Hence Equation (28) will control the value of $d^{(i)}$ in this case.

(For RC case it is also required that

$$d \leq g_{11} + \frac{(g_{21} - hg_{11})^2}{h(g_{12} + g_{21}) - (h^2 g_{11} + g_{22})}$$

and $h(g_{12} + g_{21}) - h^2 g_{11} < g_{22}$)

(ii) For $\sigma_i < 0$, as well as $i = 0$,

$$d^{(i)} \geq \alpha_{11}^{(i)} + \frac{(\alpha_{21}^{(i)} - h\alpha_{11}^{(i)})^2}{h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (h^2\alpha_{11}^{(i)} + \alpha_{22}^{(i)})} \tag{30}$$

assuming $h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (\alpha_{11}^{(i)}h^2 + \alpha_{22}^{(i)}) > 0$ or

$$h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - h^2\alpha_{11}^{(i)} > \alpha_{22}^{(i)} \tag{31}$$

In Equation (24), $d^{(i)} \geq \alpha_{11}^{(i)}$, thus Equation (30) must hold. Equation (30) is stronger than Equation (24). Therefore, Equation (30) will control the value of $d^{(i)}$.

In other words, it is always possible for a single set of real axis poles, including the poles at zero and infinity, to be partitioned into the form of Equation (22) provided a real value of h can be determined such that the appropriate Equation, either (29) or (31), be satisfied.

It can be noted at this point that if the alternate form of the expression for a matrix of rank one is used identical results are obtained. In a similar development, $hd^{(i)}$ and $d_1^{(i)}$ are interchanged in rows 2 and 3 of Equation (16). But, slightly different forms for these equations are obtained if $h = 1$ in Equation (16). In this case the constant is relocated in Equation (22) as

$$\begin{aligned}
\alpha_{11}^{(i)} &= k_{11}^{(i)} + h_1 d^{(i)} \\
\alpha_{12}^{(i)} &= k_{12}^{(i)} + h_1 d_1^{(i)} \\
\alpha_{21}^{(i)} &= k_{21}^{(i)} + d^{(i)} \\
\alpha_{22}^{(i)} &= k_{22}^{(i)} + d_1^{(i)}
\end{aligned} \tag{32}$$

Equations (28) and (29) would now have the form

$$d_1^{(i)} \leq \alpha_{11}^{(i)} + \frac{(\alpha_{12}^{(i)} - h_1\alpha_{22}^{(i)})^2}{h_1(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (h_1^2\,\alpha_{22}^{(i)} + \alpha_{11}^{(i)})} \tag{33}$$

and

$$h_1(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - h_1^2\,\alpha_{22}^{(i)} < \alpha_{11}^{(i)} \tag{34}$$

Similar changes would be reflected into the other equations. Therefore, it can be concluded that the form of Equations (29) and (31) are independent of the original form of the matrix with a rank of one.

Obviously, by choosing an appropriate value of h, it is always possible to satisfy Equation (29) when $\alpha_{11}^{(i)} > 0$ and/or $\alpha_{22}^{(i)} > 0$ and Equation (31) when $\alpha_{11}^{(i)} < 0$ and/or $\alpha_{22}^{(i)} < 0$. However, it is not always possible to select a value of h such that both these equations will be satisfied for all poles. This condition is particularly apparent in Equation (29) when both $\alpha_{11}^{(i)} < 0$ and $\alpha_{22}^{(i)} < 0$ and in Equation (31) when $\alpha_{11}^{(i)} > 0$ and $\alpha_{22}^{(i)} > 0$. Under this latter condition only a very restricted range of values for h will satisfy these equations. Since a solution is not always possible under these conditions, it is desirable to determine when restrictions need to be placed on the α's so that a value of h always exists and thereby guaranteeing that the inequality equations will be satisfied.

In Equation (29) when both $\alpha_{11}^{(i)} < 0$ and $\alpha_{22}^{(i)} < 0$, it is desirable to determine what value of h will make the quantity

$$h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - h^2\alpha_{11}^{(i)}$$

have the most negative value as compared to the negative $\alpha_{22}^{(i)}$. This corresponds to finding the value of h which will make this quantity a maximum negative number. This value of h is determined by differentiating this quantity with respect to h, setting it equal to zero, and solving for h. Thus,

$$\frac{d}{dh}\left[h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - h^2\alpha_{11}^{(i)}\right] = 0$$

or

$$h_{max} = \frac{\alpha_{12}^{(i)} + \alpha_{21}^{(i)}}{2\alpha_{11}^{(i)}}$$

Upon substituting this value of h_{max} into (29) (keeping in mind that $\alpha_{11}^{(i)}$ and $\alpha_{22}^{(i)}$ are negative numbers) there results

$$(\alpha_{12}^{(i)} + \alpha_{21}^{(i)})^2 > 4\,\alpha_{11}^{(i)}\alpha_{22}^{(i)} \tag{35}$$

for $\sigma_i > 0$, including $i = \infty$.

From this equation it is clear that when both $\alpha_{11}^{(i)} < 0$ and $\alpha_{22}^{(i)} < 0$ for poles located on the negative real axis and for the pole at infinity it is necessary for Equation (35) to be satisfied in order for a solution to exist with this class of network.

A similar treatment of Equation (31) when both $\alpha_{11}^{(i)} > 0$ and $\alpha_{22}^{(i)} > 0$ would correspond to establishing the value of h which would make the quantity a minimum. Substituting this value of h into Equation (31), it can be established that a necessary condition for a solution to exist, when both $\alpha_{11}^{(i)} > 0$ and $\alpha_{22}^{(i)} > 0$ for poles located along the positive real axis and for the pole at zero, is that the equation

$$(\alpha_{12}^{(i)} + \alpha_{21}^{(i)})^2 > 4\alpha_{11}^{(i)}\, \alpha_{22}^{(i)} \tag{36}$$

be satisfied for $\sigma_i < 0$, including $i = 0$.

In summary, it has been established that it is always possible to partition any single set of poles that are simple and restricted to the real axis in the complex frequency plane, into a ±R,C (RC) short-circuit admittance matrix and a matrix of rank one provided that

A. For the pole at infinity and for a pole located along the negative real axis, $(\sigma_i > 0)$ if both residues $\alpha_{11}^{(i)} < 0$ and $\alpha_{22}^{(i)} < 0$

(For RC case $g_{11} < 0$ and $g_{22} < 0$ must be included),

or B. For the pole at zero and for a pole located along the positive real axis, $(\sigma_i < 0)$ if both the residues $\alpha_{11}^{(i)} > 0$ and $\alpha_{22}^{(i)} > 0$, the α residues satisfy the equation

$$(\alpha_{12}^{(i)} + \alpha_{21}^{(i)})^2 > 4\, \alpha_{11}^{(i)}\alpha_{22}^{(i)} \qquad (35,36)$$

(For RC case $(g_{12} + g_{21})^2 > 4g_{11}g_{22}$ must be included.)

in order for a solution to exist.

Therefore, it has been established that the only necessary condition, for partitioning a single set of poles into the required form, is that a value of h exists such that Equation (29) or (31) be satisfied. In order to extend the range of poles to be considered to $i = 0,1,2,\ldots,n,\infty$, it is sufficient to require as a necessary condition that at least a single value of h exist such that Equation (29) and (31) are satisfied simultaneously for $i = 0,1,2,\ldots,n,\infty$.

That this condition is sufficient as well as necessary will be now shown by a synthesis procedure.

The Synthesis Procedure

Since the necessary condition is established for an admittance matrix with simple real poles, it is now possible to present a method of synthesis for this admittance matrix to show the sufficiency of this condition. No transformers will be needed. The previous development utilized a current-controlled voltage source. This same controlled source will be used here with no loss in generality.

To facilitate the development of the realization procedure, a partial-fraction expansion is first obtained for each of the individual admittances, Y_{11}, $-Y_{12}$, $-Y_{21}$, and Y_{22}, which comprise the given matrix $[Y]$. This expansion will have the form of Equation (17) which is repeated here.

$$\begin{aligned} Y_{11} &= g_{11} + a_{11}^{(\infty)}s + a_{11}^{(o)}/s + \sum_{i=1}^{n} \frac{a_{11}^{(i)}s}{s+\sigma_i} \\ -Y_{12} &= g_{12} + a_{12}^{(\infty)}s + a_{12}^{(o)}/s + \sum_{i=1}^{n} \frac{a_{12}^{(i)}s}{s+\sigma_i} \\ -Y_{21} &= g_{21} + a_{21}^{(\infty)}s + a_{21}^{(o)}/s + \sum_{i=1}^{n} \frac{a_{21}^{(i)}s}{s+\sigma_i} \end{aligned} \qquad (17)$$

$$Y_{22} = g_{22} + a_{22}^{(\infty)} s + a_{22}^{(o)}/s + \sum_{i=1}^{n} \frac{a_{22}^{(i)} s}{s + \sigma_i}$$

All the residues in each set of poles are examined. Those individual set of admittances which satisfy the necessary and sufficient conditions for a realization by a ±R,C (RC) network are removed from Y_{11}, $-Y_{12}$, $-Y_{21}$, and Y_{22}. These admittances are realized utilizing the synthesis procedure of Phillips and Su [see Appendix II] for a ±R,C realization or any convenient method for an RC realization. This network will be connected in parallel with the ±R,C (RC) network containing one controlled source when the latter network is realized, as indicated in Figure 1.

The remaining sets of residues in each of the admittances Y_{11}, $-Y_{12}$, $-Y_{21}$, and Y_{22} are analyzed individually to determine a single value of h that will simultaneously satisfy the equations

$$h(a_{12}^{(i)} + a_{21}^{(i)}) - h^2 a_{11}^{(i)} < a_{22}^{(i)} \tag{29}$$

(For RC case, also $h(g_{12} + g_{21}) - h^2 g_{11} < g_{22}$) when $\sigma_i > 0$, including $i = \infty$.

And

$$h(a_{12}^{(i)} + a_{21}^{(i)}) - h^2 a_{11}^{(i)} > a_{22}^{(i)} \tag{31}$$

when $\sigma_i < 0$, including $i = 0$.

This single value of h is most readily obtained by determining the permissible range of h in each of the equations. Then, comparing all the ranges of h, it is possible to select a single value which will satisfy all the equations simultaneously.

The range of value that h can assume in any one equation is most readily established by replacing the inequality sign in this equation by the equality sign. The resulting biquadratic equation can be solved for h. Generally two values of h will be obtained. These represent limiting values in the original inequality equation which h should not assume. Generally, it is a simple matter to determine whether the permissible range lies inside or outside these two values of h. Three possible exceptions to this procedure can occur. One of these occurs when the original equation contains only the linear term $(\alpha_{11}^{(i)} = 0)$. In this case the range of h is obvious. The second and third instance occurs when h has complex or repeated roots. These situations result when $\alpha_{11}^{(i)}$ and $\alpha_{22}^{(i)}$ have the same sign and, in addition, $\alpha_{11}^{(i)}\alpha_{22}^{(i)} \geq (\alpha_{12}^{(i)} + \alpha_{21}^{(i)})^2/4$. In these situations, any real value of h [except the repeated value] can be selected, and the appropriate equation will be satisfied when $\sigma_i > 0$ and both $\alpha_{11}^{(i)}$ and $\alpha_{22}^{(i)}$ are positive and when $\sigma_i < 0$ and both $\alpha_{11}^{(i)}$ and $\alpha_{22}^{(i)}$ are negative. Should the signs be reversed on the α's, then no solution would exist as Equations (35) and (36) would be violated.

Once the value of h has been selected, it is possible to rewrite the individual admittances Y_{11}, $-Y_{12}$, $-Y_{21}$, and Y_{22} into the decomposed form of Equation (21).

The value of $d^{(i)}$ has been previously specified by Equation (28) when $\sigma_i > 0$, including $i = \infty$.

$$d^{(i)} \leq \alpha_{11}^{(i)} + \frac{(\alpha_{21}^{(i)} - h\alpha_{11}^{(i)})^2}{h(\alpha_{12}^{(i)} + \alpha_{21}^{(i)}) - (h^2\alpha_{11}^{(i)} + \alpha_{22}^{(i)})} \tag{28}$$

However, if the dominance condition is to be assured for the first two rows of the admittance matrix in Equation (2), (the order of y''_{11} and y''_{22} will be equal to or greater than the order of $[y^{(o)}]$) it is necessary

$$d^{(i)} < a_{11}^{(i)} + \frac{(a_{21}^{(i)} - ha_{11}^{(i)})^2}{h(a_{12}^{(i)} + a_{21}^{(i)}) - (h^2 a_{11}^{(i)} + a_{22}^{(i)})} \tag{37}$$

This condition assures that the residues in each of the poles in the admittance matrix $[y]$ will not be compact.

From Equation (30), $d^{(i)}$ must be specified as

$$d^{(i)} > a_{11}^{(i)} + \frac{(a_{21}^{(i)} - ha_{11}^{(i)})^2}{h(a_{12}^{(i)} + a_{21}^{(i)}) - (h^2 a_{11}^{(i)} + d_{22}^{(i)})} \tag{38}$$

when $a_i < 0$, including $i = 0$. Similarly, this condition assures that the residues in these poles of $[y]$ will be non-compact.

Since $d^{(i)}$ is specified in Equations (37) and (38), $d_1^{(i)}$ is now determined from Equation (27) as

$$d_1^{(i)} = (a_{12}^{(i)} - a_{21}^{(i)}) + hd^{(i)} \tag{27}$$

Since $d_1^{(i)}$ and $d^{(i)}$ are now known, it is possible to solve for the remaining coefficients $k_{11}^{(i)}$, $k_{12}^{(i)}$, and $k_{22}^{(i)}$ from Equation (22). There results

$$k_{11}^{(i)} = a_{11}^{(i)} - d^{(i)} \tag{39}$$

$$k_{12}^{(i)} = k_{21}^{(i)} = a_{12}^{(i)} - d_1^{(i)} \tag{40}$$

and

$$k_{22}^{(i)} = \alpha_{22}^{(i)} - hd_1^{(i)} \tag{41}$$

for $i = 0,1,2,\ldots,n,\infty$.

The only remaining terms to be specified are the conductances d, d_1, k_{11}, k_{12}, and k_{22}. They can be expressed as

$$\begin{aligned} g_{11} &= k_{11} + d \\ g_{12} &= k_{12} + d_1 \\ g_{21} &= k_{21} + hd \\ g_{22} &= k_{22} + hd_1 \end{aligned} \tag{42}$$

Since k_{11} and k_{22} can be either positive or negative in a $\pm R,C$ realization, it is always possible to make this decomposition regardless of the value of h. The only consideration in determining the values of these coefficients is to try and keep the number of negative resistors to a minimum. Unfortunately, there are no general guide lines to ensure that a specific realization uses the minimum number of $-R$'s. This point will be emphasized in the following example.

Having solved for all of the coefficients in Equation (21), this equation can be written into the more compact form

$$\begin{aligned} Y_{11} &= P_{11}/g + P_2/g \\ -Y_{12} &= P_{12}/g + P_3/g \\ -Y_{21} &= P_{12}/g + hP_2/g \\ Y_{22} &= P_{22}/g + hP_3/g \end{aligned} \tag{43}$$

From Equation (43) it is clear that the admittance matrix $[Y]$, can always be realized along the lines previously indicated in Chapter II. This is because each of the constituent matrices of Equation (43) have been determined in such a manner that they will always satisfy the necessary and sufficient conditions for a realization by this class of network. Thus, the necessary conditions for the given 2 X 2 matrix containing only simple real poles are also sufficient.

Special Cases

A special case of considerable interest occurs when Y_{11} and/or Y_{22} of matrix $[Y]$ are ±R,C (RC) driving-point admittances. In this situation it is always possible to realize the given matrix by the synthesis procedure. This results from the fact that both Equations (29) and (31) can always be satisfied for $i = 0,1,2,\ldots,n,\infty$. In particular, consider the case when Y_{22} is a ±R,C admittance. This corresponds to the residues in the poles of Y_{22} having the following properties: when $\sigma_i > 0$, then $\alpha_{22}^{(i)} > 0$ and when $\sigma_i < 0$, then $\alpha_{22}^{(i)} < 0$. Clearly for the case in point, Equations (29) and (31) can always be satisfied with $h = 0$. This is because with $h = 0$, these equations reduce to the necessary requirements that $\alpha_{22}^{(i)} > 0$ when $\sigma_i > 0$ and $\alpha_{22}^{(i)} < 0$ when $\sigma_i < 0$, which must be fulfilled by the fact that Y_{22} is a ±R,C (RC) driving-point admittance. Hence, a solution always exists. If Y_{11} is a ±R,C (RC) driving-point admittance, then, h can be selected large to satisfy both Equations (29) and (31) simultaneously. Possibly a more preferable solution would be obtained in this situation by using the alternate development. Now the constant h_1 in Equation (32), can be set equal to zero as in the case when Y_{22} is a

±R,C admittance. In either situation, the previously developed equations are always valid and a solution readily obtained.

An Example of the Synthesis Procedure

As an example of this synthesis procedure, consider the realization of the following matrix by a network composed of ±R,C and one current-controlled voltage source.

$$
\begin{aligned}
Y_{11} &= \frac{6s}{s+2} - \frac{2s}{s+6} + 2 \\
-Y_{12} &= \frac{3s}{s+2} - \frac{3s}{s+6} \\
-Y_{21} &= \frac{5s}{s+2} - \frac{10s}{s+6} - 1 \\
Y_{22} &= \frac{-7s}{s+2} - \frac{s}{s+6} + 3
\end{aligned}
\qquad (44)
$$

First, it is necessary to establish a value of h to satisfy Equation (29).

For $\sigma_1 = 2$, Equation (29) gives

$$-2h - 6h^2 < -7$$

which requires

$$h < -0.925 \quad \text{or} \quad 1.258 < h$$

For $\sigma_2 = 6$

$$-13h + 2h^2 < -1$$

which gives

$$0.08 < h < 6.42$$

Clearly both equations would be satisfied if $h = 2$. Letting $h = 2$ and solving for $d^{(i)}$ form Equation (37) yield $d^{(i)} < 2.22$ and $d^{(2)} < -3.90$. Thus, the following values for d's are chosen:

$$d^{(1)} = 3/2 \quad d^{(2)} = -\,11/2 \quad d = -1$$

The rest of the coefficients in the expansion of Equation (21) can be solved for. After this has been accomplished, there results

$$\begin{aligned}
Y_{11} &= \left[\frac{9/2\,s}{s+2} + \frac{7/2\,s}{s+6} + 3\right. && + \left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+6} - 1\right. \\
-Y_{12} &= \frac{2s}{s+2} + \frac{s}{s+6} + 1 && + \frac{-5s}{s+2} - \frac{4s}{s+2} - 1 \\
-Y_{21} &= \frac{2s}{s+2} + \frac{s}{s+6} + 1 && + \frac{3s}{s+2} - \frac{11s}{s+6} - 2 \\
Y_{22} &= \left.\frac{3s}{s+2} + \frac{7s}{s+6} + 5\right] && + \left.\frac{-10s}{s+2} - \frac{8s}{s+6} - 2\right]
\end{aligned} \tag{45}$$

The first matrix is a ±R,C admittance matrix in which all residues satisfy the non-compact condition. Rewriting this matrix in the form of Equation (9) yields

$$\begin{aligned}
Y_{11} &= \left[\frac{2s}{s+2} + \frac{s}{s+6} + 1\right. && + \frac{5/2\,s}{s+2} + \frac{5/2\,s}{s+2} + 2 + \left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+2} - 1\right] \\
-Y_{12} &= \frac{2s}{s+2} + \frac{s}{s+6} + 1 && + \left[\frac{-5s}{s+2} - \frac{4s}{s+6} - 1\right] \\
-Y_{21} &= \frac{2s}{s+2} + \frac{s}{s+6} + 1 && + 2\left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+2} - 1\right] \\
Y_{22} &= \left.\frac{2s}{s+2} + \frac{s}{s+6} + 1\right] && + \frac{s}{s+2} + \frac{6s}{s+6} + 4 + 2\left[\frac{-5s}{s+2} + \frac{-4s}{s+6} - 1\right]
\end{aligned} \tag{46}$$

The first group of terms (in brackets) is realized by a ±R,C two-port network. The remaining functions are identified with the form of the active ±R,C network when a current controlled voltage source is used. That is

$$
\begin{aligned}
y''_{11} + \frac{R}{1-Ry_{34}}(y_{14})(y_{13}) &= \frac{5/2\,s}{s+2} + \frac{5/2\,s}{s+2} + 2 + \left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+6} - 1\right] \\
+ \frac{R}{1-Ry_{34}}(y_{14})(y_{23}) &= \qquad\qquad\qquad\qquad + \left[\frac{-5s}{s+2} - \frac{4s}{s+6} - 1\right] \\
+ \frac{R}{1-Ry_{34}}(y_{24})(y_{13}) &= \qquad\qquad\qquad\qquad + 2\left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+6} - 1\right] \\
y''_{22} + \frac{R}{1-Ry_{34}}(y_{24})(y_{23}) &= \frac{s}{s+2} + \frac{6s}{s+6} + 4 + 2\left[\frac{-5s}{s+2} - \frac{4s}{s+6} - 1\right]
\end{aligned} \tag{47}
$$

The individual admittances are readily identified as

$$
\begin{aligned}
&y''_{11} = \frac{5/2\,s}{s+2} + \frac{5/2\,s}{s+6} + 2 \qquad && y''_{22} = \frac{s}{s+2} + \frac{6s}{s+6} + 4 \\
&-y_{13} = \frac{\Upsilon}{R}\left(\frac{3/2\,s}{s+2} - \frac{11/2s}{s+2} - 1\right) \qquad && -y_{23} = \frac{\Upsilon}{R}\left(\frac{-5s}{s+2} - \frac{4s}{s+6} - 1\right) \\
&-y_{14} = \frac{1}{\Upsilon} \qquad -y_{43} = 0 \qquad && -y_{24} = \frac{2}{\Upsilon}
\end{aligned} \tag{48}
$$

Now, the following constants are chosen to simplify the realization:

$$
\frac{\Upsilon}{R} = -\frac{1}{5} \qquad \frac{1}{\Upsilon} = 1.80 \qquad R = -2.78
$$

Therefore

$$
\begin{aligned}
&-y_{14} = 1.80 \qquad && -y_{24} = 3.60 \\
&-y_{13} = \frac{-0.3s}{s+2} + \frac{1.10s}{s+6} + 0.20 \qquad && -y_{23} = \frac{s}{s+2} + \frac{0.8s}{s+6} + 0.20
\end{aligned} \tag{49}
$$

Since y_{33} and y_{44} are arbitrary, they are chosen to be

$$y_{33} = \frac{1.036\,s}{s+2} + \frac{1.9\,s}{s+6} + 0.4$$

$$y_{44} = 5.40 \tag{50}$$

to satisfy the dominance condition. The total network is shown in Figure 4.

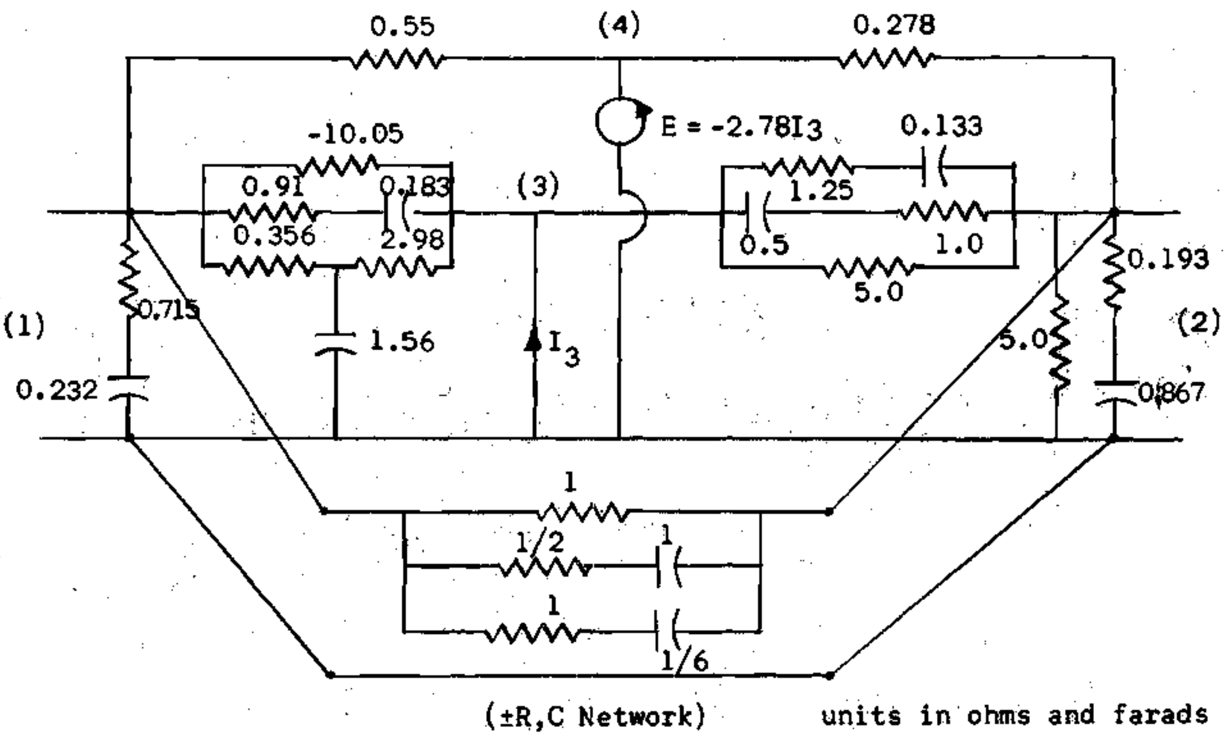


Figure 4. The Active Network that Realizes the Admittance Matrix of Equation (44).

Next if the same admittance is to be realized without using -R's, it would be required that h also satisfy the equation

$$h(g_{12} + g_{21}) - h^2 g_{11} < g_{22}$$

or

$$h(-1) - 2h^2 < 3$$

Clearly $h = 2$ also satisfies this equation. In addition, it is required that d satisfy

$$d < g_{11} + \frac{(g_{21} - hg_{11})^2}{h(g_{12}+g_{21}) - (h^2 g_{11} + g_{22})}$$

which gives $d < -0.77$. If $d = -1$ is chosen, there results

$$\begin{aligned}
Y_{11} &= \left[\frac{2s}{s+2} + \frac{s}{s+6} + 3\right. + \frac{5/2\,s}{s+2} + \frac{5/2\,s}{s+6} + 1 + \left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+6} - 2\right] \\
-Y_{12} &= \frac{2s}{s+2} + \frac{s}{s+6} + 3 \qquad\qquad + \left[\frac{-5s}{s+2} - \frac{4s}{s+6} - 3\right] \\
-Y_{21} &= \frac{2s}{s+2} + \frac{s}{s+6} + 3 \qquad\qquad + 2\left[\frac{3/2\,s}{s+2} - \frac{11/2\,s}{s+6} - 2\right] \\
Y_{22} &= \left.\frac{2s}{s+2} + \frac{s}{s+6} + 3\right] + \frac{s}{s+2} + \frac{6s}{s+6} + 6 + 2\left[\frac{-5s}{s+2} - \frac{4s}{s+6} - 3\right]
\end{aligned} \tag{51}$$

Fortunately, it is possible to realize all the short-circuit admittance functions in this particular example by RC three-terminal networks. In general, there is no guarantee that this can be done. The realized RC network is indicated in Figure 5.

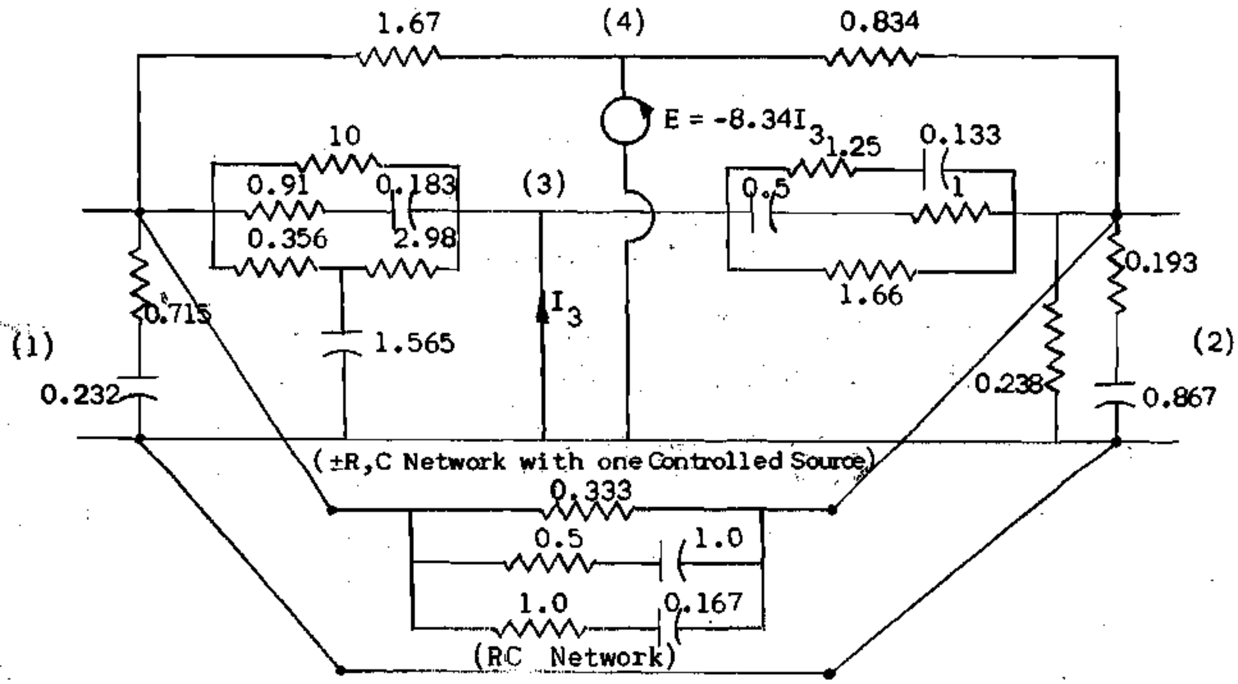


Figure 5. The Active Network that Realized the Admittance Matrix of Equation (44) with RC and One Controlled Source.

The Realization of Any Two General Short-Circuit Admittances

Up to this point, the investigation of ±R,C (RC) networks with one controlled source has been concerned with the realization of an admittance matrix. The admittance matrix realized is not completely general. Limitations are placed on the admittances Y_{11}, Y_{12}, Y_{21}, and Y_{22} in order for a solution to exist. In this section an investigation was conducted to determine whether some admittance functions can be more general or not when fewer admittances are specified. The results of this investigation have established that any two arbitrarily specified rational functions can be realized as the short-circuit admittances of an RC two-port network with one controlled source embedded in it. Since two completely general admittances will be realized; no control can be placed on the other two admittance functions.

In the following three sections the results of this investigation will be presented. The general form of the admittance matrix of the ±R,C (RC) two-port network with a current-controlled voltage source can be written as

$$\begin{aligned} Y_{11} &= y_{11} + \frac{R}{1 - Ry_{34}} y_{14} \times y_{13} \\ -Y_{12} &= -y_{12} + \frac{R}{1-Ry_{34}} y_{14} \times y_{23} \\ -Y_{21} &= -y_{21} + \frac{R}{1-Ry_{34}} y_{24} \times y_{13} \\ Y_{22} &= y_{22} + \frac{R}{1 - Ry_{34}} y_{24} \times y_{23} \end{aligned} \tag{52}$$

This general form for the admittance matrix of a ±R,C (RC) two-port network with this particular controlled source will now be used to realize any two arbitrarily specified rational functions as short-circuit admittance functions. All the individual admittances [represented by lower-case y's] will be restricted to RC functions.

I. Simultaneous Realization of Y_{12} and Y_{21}. To realize the transfer-admittance Y_{12} and Y_{21}, the admittances are expressed as

$$-Y_{12} = \frac{P_{12}(s)}{Q(s)} \quad \text{and} \quad -Y_{21} = \frac{P_{21}(s)}{Q(s)} \tag{53}$$

where these admittances are completely general. $P_{12}(s)$, $P_{21}(s)$, and $Q(s)$ are polynomials. The denominator, $Q(s)$, are assumed to be identical. If this is not the case, the functions may be augmented so that the denominators are identical and $Q(s)$ represents the lowest common denominator.

Now equating Equations (52) and (53) and setting $-y_{12} = -y_{21} = 0$ gives

$$-Y_{12} = \frac{P_{12}(s)}{Q(s)} = \frac{R}{1 - Ry_{34}} y_{14} \times y_{23}$$
$$-Y_{21} = \frac{P_{21}(s)}{Q(s)} = \frac{R}{1 - Ry_{34}} y_{24} \times y_{13} \tag{54}$$

A solution is readily obtained by the following identification:

$$1 - Ry_{34} = \frac{Q(s)}{g(s)} \qquad -y_{14} = \frac{\Upsilon}{R} \times \frac{P_{12}(s)}{g(s)}$$
$$-y_{23} = \frac{1}{\Upsilon} \qquad -y_{24} = \frac{\Upsilon}{R} \times \frac{P_{21}(s)}{g(s)} \qquad -y_{13} = \frac{1}{\Upsilon} \tag{55}$$

Clearly if these transfer admittances are to be RC, it is necessary to select the arbitrary polynomial $g(s)$ such that it contains only simple zeros on the negative real axis with $g(0) \neq 0$ and of degree equal to or greater than the highest degree of the polynomials $P_{12}(s)$, $P_{21}(s)$, and $Q(s)$. These individual admittance functions $-y_{13}$, $-y_{23}$, $-y_{14}$, $-y_{24}$, and $-y_{34}$ are the transfer admittances of a four-port RC network. The driving-point functions of this network y_{11}, y_{22}, y_{33}, and y_{44} have been left unspecified. For a realization to exist these unspecified driving-point admittances should be chosen such that the RC network can be realized readily. Hence, a solution is always possible.

II. Simultaneous Realization of Y_{11} and Y_{22}. These general driving-point admittances can be realized readily by this class of network. First let

$$Y_{11} = \frac{P_{11}(s)}{Q(s)} \quad \text{and} \quad Y_{22} = \frac{P_{22}(s)}{Q(s)} \tag{56}$$

Here again $Q(s)$ represents the augmented common denominator. Choose two arbitrary RC admittances, $y_{11} = K\ p_{11}(s)/g(s)$ and $y_{22} = K_1 p_{22}(s)/g(s)$ where K and K_1 are constants. Let $g(s)$ be a polynomial of degree m which is equal to or greater than the highest degree of the polynomials $P_{11}(s)$, $P_{22}(s)$, and $Q(s)$. Also it is required that $g(0) \neq 0$, $p_{11}(0) \neq 0$ and $p_{22}(0) \neq 0$. Y_{11} and Y_{22} of Equation (56) can be written in terms of Equation (52) as

$$\begin{aligned} \frac{P_{11}(s)g(s) - Kp_{11}(s)Q(s)}{Q(s)\ g(s)} &= \frac{R}{1 - Ry_{34}}\ y_{14} \times y_{13} \\ \frac{P_{22}(s)g(s) - K_1 p_{22}(s)Q(s)}{Q(s)\ g(s)} &= \frac{R}{1 - Ry_{34}}\ y_{24} \times y_{23} \end{aligned} \tag{57}$$

now K and K_1 are chosen such that

$$\begin{aligned} P_{11}(s)g(s) - K\ p_{11}(s)Q(s) &= R_1(s)R_2(s) \\ P_{22}(s)g(s) - K_1\ p_{22}(s)Q(s) &= P_1(s)P_2(s) \end{aligned} \tag{58}$$

where $R_2(s)$ and $P_2(s)$ are polynomials of degree m. Then define

$$1 - Ry_{34} = \frac{Q(s)}{g(s)} \qquad -y_{14} = \frac{1}{\gamma} \times \frac{R_1(s)}{g(s)} \qquad -y_{13} = \frac{\gamma}{R} \times \frac{R_2(s)}{g(s)}$$

$$-y_{24} = \frac{1}{\gamma}\frac{P_1(s)}{g(s)} \qquad -y_{23} = \frac{\gamma}{R} \times \frac{P_2(s)}{g(s)} \tag{59}$$

where γ is a constant. Now the constants γ and R can always be selected so that the dominance condition is satisfied for y_{11} and y_{22}. The driving-point admittances y_{33} and y_{44} have been left unspecified. These admittances can be chosen such that the RC four-port network is easily realized. Therefore, the synthesis procedure is always workable.

III. Simultaneous Realization of Y_{11} and Y_{12}. The simultaneous realization of one driving-point admittance and one transfer admittance is accomplished in a straightforward manner. Although the procedure to be presented is concerned with the realization of Y_{11} and Y_{12}, any other desired combination could be realized in a similar manner by suitably altering this procedure. Therefore, define

$$Y_{11} = \frac{P_{11}(s)}{Q(s)} \quad \text{and} \quad -Y_{12} = \frac{P_{12}(s)}{Q(s)} \tag{60}$$

As before $Q(s)$ represents the augmented denominator. Also as before, select an RC admittance $y_{11} = k\ P_{11}(s)/g(s)$ where k is a constant and $g(s)$ is of degree m which is equal to or greater than the highest degree of the polynomials $P_{11}(s)$, $P_{12}(s)$, and $Q(s)$. In addition, it is required that $g(0) \neq 0$ and $p_{11}(0) \neq 0$. In a similar manner it is possible to write Y_{11} in terms of Equation (52) as

$$\frac{P_{11}(s)g(s) - K\, p_{11}(s)Q(s)}{g(s)\ Q(s)} = \frac{R}{1-Ry_{34}}\, y_{14} \times y_{13} \tag{61}$$

K is selected such that

$$P_{11}(s)g(s) - K\, p_{11}(s)Q(s) = R_1(s)R_2(s) \tag{62}$$

where $R_2(s)$ has m distinct negative real zeros with all positive coefficients. Obviously it is possible to write

$$\begin{aligned} 1 - Ry_{34} &= \frac{Q(s)}{R_2(s)} \\ -y_{14} &= \frac{1}{\Upsilon} \\ \text{and } -y_{13} &= \frac{\Upsilon}{R} \times \frac{R_1(s)}{g(s)} \end{aligned} \tag{63}$$

where each of these functions are RC transfer admittances.

For $-Y_{12}$, when from Equation (52) with $-y_{12} = 0$

$$-Y_{12} = \frac{P_{12}(s)}{Q(s)} = \frac{R}{1-Ry_{34}}\, y_{14} \times y_{23} \tag{64}$$

or

$$\frac{P_{12}(s)}{Q(s)} = \frac{R}{\Upsilon} \times \frac{R_2(s)}{Q(s)}\,(-y_{23}) \tag{65}$$

therefore,

$$-y_{23} = \frac{\Upsilon}{R}\,\frac{P_{12}(s)}{R_2(s)} \tag{66}$$

Thus, the synthesis procedure is complete by selecting the arbitrary RC admittances y_{22}, y_{33}, and y_{44} so that they accommodate the specified transfer admittances and have sufficient magnitude such that the dominance condition is satisfied. Again, the synthesis procedure is always workable.

CHAPTER IV

AN ALTERNATE SYNTHESIS PROCEDURE

The controlled source has had considerable application in the field of active network synthesis. However, it is not the only active device to achieve such distinction. One other active device that also has an enviable reputation in the active network synthesis field is the negative impedance converter. The negative impedance converter, which is often abbreviated as NIC, is an active two-port device that is characterized by the following chain matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \pm k_2 & 0 \\ 0 & \mp 1 \end{bmatrix} \tag{67}$$

Here (k_2) is a constant positive weighting factor. The $\pm$ signs denote the type of NIC. The upper sign is associated with a current-inversion NIC, and the lower sign is associated with a voltage-inversion NIC.

It is of interest to determine if this active device could be embedded in a $\pm$R,C (RC) two-port network such that an admittance matrix which satisfies the necessary and sufficient conditions for a realization with $\pm$R,C and one controlled source could be realized. If this is possible, then an alternate method of realizing this class of matrix can be developed. The result of the investigation of this problem will now be presented.

The circuit of Figure 6 is used as the starting point for this development. The short-circuit admittance matrix for the ±R,C two-port network with one NIC embedded in it can be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [Y] \times \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{68}$$

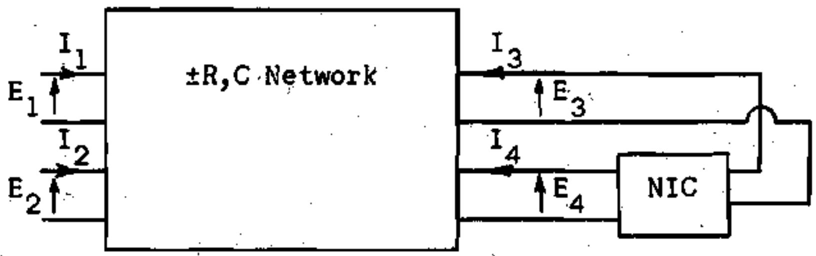


Figure 6. The Arrangement of an Active Two-port Network Containing ±R,C and One NIC.

To establish the character of this admittance matrix, [Y], for the two-port network in Figure 6, it is convenient to consider the ±R,C four-port. The short-circuit admittance matrix for this ±R,C four-port network can be written as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} +y_{11} & +y_{12} & +y_{13} & +y_{14} \\ +y_{21} & +y_{22} & +y_{23} & +y_{24} \\ +y_{31} & +y_{32} & +y_{33} & +y_{34} \\ +y_{41} & +y_{42} & +y_{43} & +y_{44} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} \tag{69}$$

where $y_{ij} = y_{ji}$. The NIC in Figure 6 is connected between ports 3 and 4. From Equation (67) with the lower sign, it is possible to write

$$\begin{bmatrix} E_4 \\ I_4 \end{bmatrix} = \begin{bmatrix} -k_2 & 0 \\ 0 & +1 \end{bmatrix} \times \begin{bmatrix} E_3 \\ -I_3 \end{bmatrix} \tag{70}$$

Now if Equations (70) and (69) are used to solve for the short-circuit admittance matrix, $[Y]$, of the active two-port, there results

$$[Y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} + \frac{1}{(k_2y_{44}-y_{33}-y_{43}+k_2y_{34})} \begin{bmatrix} (y_{13}-k_2y_{14}) \\ (y_{23}-k_2y_{24}) \end{bmatrix} \begin{bmatrix} (y_{31}+y_{41}) & (y_{32}+y_{42}) \end{bmatrix} \tag{71}$$

Clearly in Equation (71), the matrix

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \tag{72}$$

represents a $\pm R,C$ short-circuit admittance matrix. The matrix product

$$\frac{1}{(k_2y_{44}-y_{33}-y_{43}+k_2y_{34})} \begin{bmatrix} (y_{13}-k_2y_{14}) \\ (y_{23}-k_2y_{24}) \end{bmatrix} \begin{bmatrix} (y_{13}+y_{14}) & (y_{32}+y_{42}) \end{bmatrix} \tag{73}$$

has a rank of one. Since Equation (71) has exactly the same form as Equation (6), it can be concluded, that if the general matrix of rank one can be realized by a $\pm R,C$ two-port network with one NIC, the alternate method for realizing the admittance matrix will have been established. Prior to showing that the matrix product in (73) is capable of realizing the general matrix

$$\frac{1}{Q}\begin{bmatrix} P_bP_a & P_bP_c \\ P_dP_a & P_dP_c \end{bmatrix} = \frac{1}{Q}\begin{bmatrix} P_b \\ P_d \end{bmatrix}\begin{bmatrix} P_a & P_c \end{bmatrix} \tag{12}$$

which has a rank one, the following definitions are required.

Let the maximum degree of the polynomials P_bP_a, P_bP_c, P_dP_a, P_dP_c, and Q be M. In addition, let P_bP_a be of maximum degree M, where P_a is of degree n_2 and P_b is of degree n_1. Thus, $n_1n_2 = M$. Now select a polynomial $g = g_1g_2$ which is of degree equal to or greater than M and contains only simple real zeros. (For an RC realization g must have only simple real zeros on the negative axis and $g(0) \neq 0$). Let g_1 and g_2 be of order n_1 and n_2', respectively. Obviously, $n_2' \geq n_2$.

Now it is possible to equate Equation (73) to Equation (12). Then, identify each of the groups of individual admittances such that they realize the general matrix of rank one and at the same time ensuring the individual admittances have ±R,C character. When this is done, there results

$$\begin{gathered} (k_2y_{44}-y_{33}-y_{43}+k_2y_{34}) = \frac{H^2Q}{g} \\ (-y_{13}+k_2y_{14}) = \frac{P_b}{Hg_1} \qquad (-y_{13}-y_{41}) = \frac{P_a}{Hg_2} \\ (-y_{23}+k_2y_{24}) = \frac{P_d}{Hg_1} \qquad (-y_{23}-y_{42}) = \frac{P_c}{Hg_2} \end{gathered} \tag{74}$$

Solving for the individual transfer admittances yields

$$-y_{13} = \frac{1}{H(1+k_2)}\left[\frac{k_2P_b}{g_1} + \frac{P_a}{g_2}\right] \quad -y_{14} = \frac{1}{H(1+k_2)}\left[\frac{P_a}{g_2} - \frac{P_b}{g_1}\right]$$

$$-y_{23} = \frac{1}{H(1+k_2)}\left[\frac{k_2P_d}{g_1} + \frac{P_c}{g_2}\right] \quad -y_{24} = \frac{1}{H(1+k_2)}\left[\frac{P_c}{g_2} - \frac{P_d}{g_1}\right] \tag{75}$$

$$-y_{34} = 0$$

Clearly P_a/g_2, P_c/g_2, P_d/g_1, P_b/g_1 and Q/g satisfy the requirements to be ±R,C (RC) transfer admittances. Thus, $-y_{13}$, $-y_{23}$, $-y_{14}$, $-y_{24}$ and $-y_{34}$ all are ±R,C (RC) transfer admittances. It is clear that it is always possible to realize the general matrix of rank one by ±R,C (RC) transfer admittances.

In order to be able to realize these transfer admittances, as in the case of the controlled source, the number of non-compact poles, that are common to both driving-point admittances [y_{11} and y_{22}] of the ±R,C admittance matrix, must be equal to or greater than M. Thus, if this admittance matrix contains sufficient admittance of the form P_{11}/g and P_{22}/g, clearly it is always possible to realize the four-port network. This is because the constant (H) can always be chosen to ensure that the dominance conditions is satisfied for the first two rows of the four-port admittance matrix. In addition y_{33} and y_{44} can always be selected to facilitate a ±R,C (RC) realization and, at the same time, ensuring that their difference satisfies Equation (74).

Thus, it has been shown that is is possible for a given 2 X 2 matrix that satisfies the necessary and sufficient conditions for a realization by a ±R,C two-port network containing a controlled source to be realized also by a ±R,C two-port network containing an NIC. The synthesis procedure can be carried out along similar lines as presented for synthesizing an active two-port using one controlled source.

CHAPTER V

TRANSISTOR EQUIVALENT CIRCUITS

As an example of the practical application of the synthesis procedure developed, the procedure will be applied to obtain transistor equivalent circuits that are more accurate and valid over a wider range of frequency than the simple conventional equivalent circuits.

For many years, simple equivalent circuits have been used to approximate the performance characteristics of many three-terminal active nonreciprocal devices for small-signal applications. The transistor and the vacuum tube, are but two of many such devices. The equivalent circuit representation for these active devices were primarily determined by physicists who were concerned with physical processes that took place inside the device. The transistors equivalent circuit, for example, was largely determined on the basis of the migration of holes and electrons. Element values for these equivalent circuits are determined by making electrical measurements at a particular frequency. The resulting equivalent circuit is referred to as a low, mid, or high-frequency equivalent circuit depending upon the relative frequency used to determine the element values and the complexity of the equivalent circuit. Usually, the performance characteristics of these equivalent circuits differ in varying degrees from the measured characteristics. An example of these two groups of characteristics are shown in terms of the admittances y_{ie}, $-y_{re}$, $-y_{fe}$ and y_{oe} [which correspond to the conventional two-port designation of Y_{11}, $-Y_{12}$, $-Y_{21}$, and Y_{22} respectively] in Figure 10 as solid and dashed curves for the case of a common-emitter transistor. The solid curves represent actual measured characteristics of the transistor. The dashed curves represent the characteristics that are obtained when the high-frequency hybrid-π equivalent circuit model is used. This hybrid-π equivalent circuit is presented in Figure 7. [This equivalent circuit was calculated specifically for this example. It represents the measured characteristics of the actual transistor in Figure 10 to within ±10% in magnitude and ±10° in phase.]

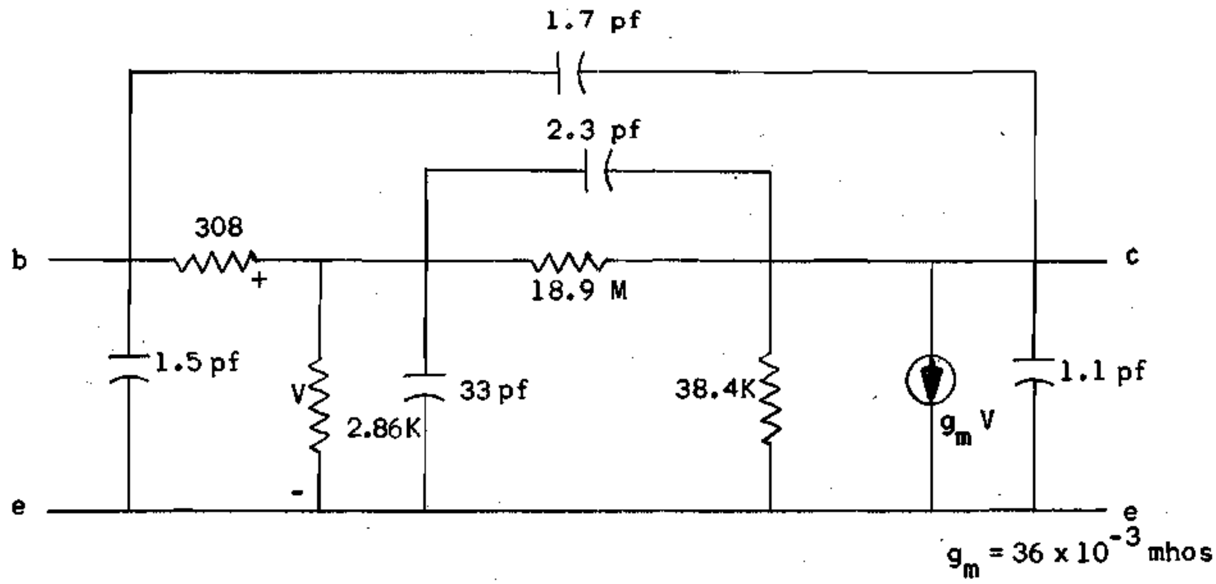


Figure 7. The High-frequency Hybrid-π Model of a Transistor.

It is seen that the conventional type of equivalent circuit, the hybrid-π model, represents the actual transistor rather accurately up to the neighborhood of 10 mc. However, most equivalent-circuit representations of this type do not approximate the transistor performance even this closely. With the demand for greater accuracy that is required in computer simulated studies, it is of interest to determine if a better method can be developed to obtain equivalent circuits for active devices -- a method that would not be as limited in approach as the present technique. The results of this investigation have shown that it is possible to obtain very accurate equivalent circuits for the transistor by using a rigorous synthesis approach. This approach involves first the approximation of a set of measured characteristics that describes the transistor by a set of short-circuit admittance functions. These admittance functions are then realized by the previously developed synthesis procedure to obtain an equivalent-circuit representation. An equivalent circuit developed by this technique has many advantages. Probably the most important of these are that the resulting equivalent circuit is always obtained from the actual characteristics of the active device and that circuit model obtained is representative of the device over a wide range of frequencies.

A transistor is a three-terminal, active, nonreciprocal device that can be completely characterized by a third order indefinite admittance matrix. This matrix has the property that the sum of the elements of every row and of every column equals zero. To be more specific, consider the general three-terminal device of Figure 8. This device is completely

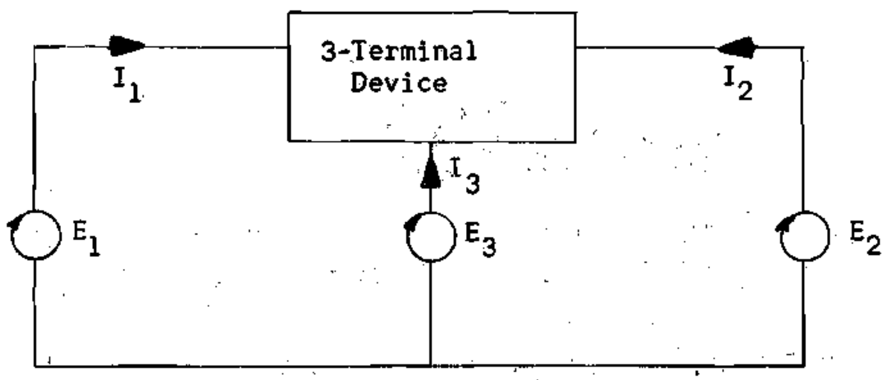


Figure 8. Representation of a Three-terminal, Active, Nonreciprocal Device.

described by the following third-order admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{77}$$

Of all the short-circuit admittances that appear in this matrix only four of these are independent. In order to determine a set of independent admittances, Kirchhoff's current law is applied at the external node of Figure 8. This yields

$$I_1 + I_2 + I_3 = 0 \tag{78}$$

Substituting the admittances of Equation (77) into Equation (78) gives

$$(y_{11}+y_{21}+y_{31})E_1 + (y_{12}+y_{22}+y_{32})E_2 + (y_{13}+y_{23}+y_{33})E_3 = 0 \tag{79}$$

The applied voltages E_1, E_2, and E_3 are arbitrary. To ensure that this condition is preserved in Equation (79), it is required that the coefficient of each of the voltages be zero; that is

$$y_{11} + y_{21} + y_{31} = 0 \qquad y_{12} + y_{22} + y_{32} = 0 \qquad y_{13} + y_{23} + y_{33} = 0 \tag{80}$$

Thus, from this development it is concluded that only four of the short-circuit admittances in Equation (80) can be specified independently.

If any one voltage is set equal to zero in Figure 8, that terminal to which the source is connected becomes the common terminal of the other two sources. For example by setting $E_3 = 0$, terminal 3 becomes a common terminal to sources E_1 and E_2 and the network reduces to the standard two-port designation. In this situation, Equation (77) reduces to

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{81}$$

Hence, these four short-circuited admittances completely specify the active device.

Summing up, it is clear from the foregoing discussion that it is possible to completely describe an active device in two ways -- either by considering it as a three-terminal device and, therefore, describing it by the indefinite admittance matrix or by considering it as a grounded two-port and describe it in terms of a 2 X 2 short-circuit admittance matrix. Either method is completely general and it is easy to go from one designation to the other. Thus for the case in point, these equations offer a very simple analysis of all the possible transistor configurations. In the following presentation the transistor will be considered as a grounded two-port with the emitter as the common terminal.

The Approximation Problem

In order to obtain an equivalent circuit by the synthesis technique developed in this research, a set of four admittance functions must be obtained that not only approximate the measured characteristics [both in magnitude and phase] of the active device but also are realizable by this special class of networks. The approximation method to be presented is developed for the transistor. However, the approach is adaptable to any type of active three-terminal device.

An approximation problem is by its very nature tedious. The task is more difficult when four short-circuit admittance functions $[y_{ie}, -y_{re}, -y_{fe},$ and $y_{oe}]$ are to be simultaneously approximated. When these four functions are specified in magnitude as well as in phase, the task quickly develops into a major undertaking. This is especially so when these four functions are required to be related and integrated into a realizable active RC network. When the active elements are restricted to one controlled source and possibly one or two negative resistors, then the difficulty of the approximation increases considerably.

There are numerous approximation techniques available, such as Taylor, Padé, least mean square, Chebyshev, potential analog, and the half-line approximation technique, to mention but a few. Of all these techniques, only the half-line approximation technique lends itself to simultaneously approximating several functions specified in both magnitude and phase. Fortunately, the half-line approximation technique is directly applicable to the given admittance characteristics. However, a direct application of this method without due regard to a possible network configuration or the number of elements would yield functions of very high order. Each of these functions would have different poles and the realization would be impractical. In an effort to obtain a simple network configuration, the poles of the four admittance functions should coincide.

In each of the three equivalent networks to be presented, the approximation of the magnitude and phase characteristics for each of the admittances is accomplished approximately as follows. A few deviations are discussed with each specific network.

A. The given data of the transistor is plotted (12).

B. Stencils are constructed to represent the response of a single pole (or zero) both in magnitude and phase to the scales of the plotted characteristics.

C. The first step in the actual approximating technique involves locating the first necessary zeros for the four admittance functions. This is accomplished by referring to the magnitude and phase curves for each admittance function simultaneously. The best location of the zero for y_{ie}, $-y_{re}$, $-y_{fe}$, or y_{oe} is determined readily with the aid of the stencils describing the phase and magnitude response for a zero. The zero is located where the closest fit is obtained at the low end of the frequency range of interest for both the magnitude and phase plots. In each case a compromise is made for the best approximation of both the magnitude and phase. The function $-y_{fe}$ requires no initial zero.

D. The second step in the approximating procedure is to determine the best frequency to locate the first common pole to all admittances. This process is usually the most difficult one. The most expedient method is by a slight modification of a trial-and-error procedure. This procedure consists of selecting three preliminary frequencies for the location of the single pole. The response of each pole is plotted individually in both magnitude and phase with the aid of the stencils and a pair of dividers. A comparison of the response of each pole determines the pole which exhibits the closest overall fit on the set of eight curves. With this pole as a reference, two additional poles are plotted in a similar manner, one higher and the other lower, in frequency than the reference. A comparison of the response due to each of these poles indicates whether the reference was a best fit or not. The process is repeated until a best fit is obtained.

Again another compromise is necessary. The best location for the pole to obtain minimum error for each of the functions is not obvious. This is especially apparent when only one common pole is allowed. In this situation the location of the pole is critical. However, when more poles are allowed some readjustment is possible.

E. The remaining steps are merely a repeat of the first and second until the four admittances are approximated to the desired accuracy.

Network I

In the field of network synthesis a unique solution is very rare. More often than not, there is an infinity of solutions for any given problem. So it is with the current problem. The equivalent circuits to be presented in each of the three cases considered are obtained under various network restrictions. These restrictions required slight modifications of the outlined approximation procedure. However, in each case the primary aim is to obtain the best approximate equivalent circuit with the least number of elements. As might be expected, the greater number of network elements allowed the better the possibility of improving the overall approximation. Whenever possible, the error in each of the approximated admittance functions is distributed equally between the magnitude and phase functions.

The first equivalent circuit, Network I, is required to have the form of the network indicated in Figure 9. This network is a simple RC network which contains only one active device -- a voltage-controlled current source. The form of this active network is chosen specifically to enable a comparison of network elements between the hybrid-π model of Figure 7 and the synthesized equivalent circuit, which will be abbreviated as

SEC. Since the first SEC model is to be very simple [comparable to the high-frequency hybrid-π model], the number of allowable poles and zeros the approximating functions y_{ie}, $-y_{re}$, $-y_{fe}$ and y_{oe} can have are also specified. In this simple case the approximating admittances are allowed only one pole and two zeros.

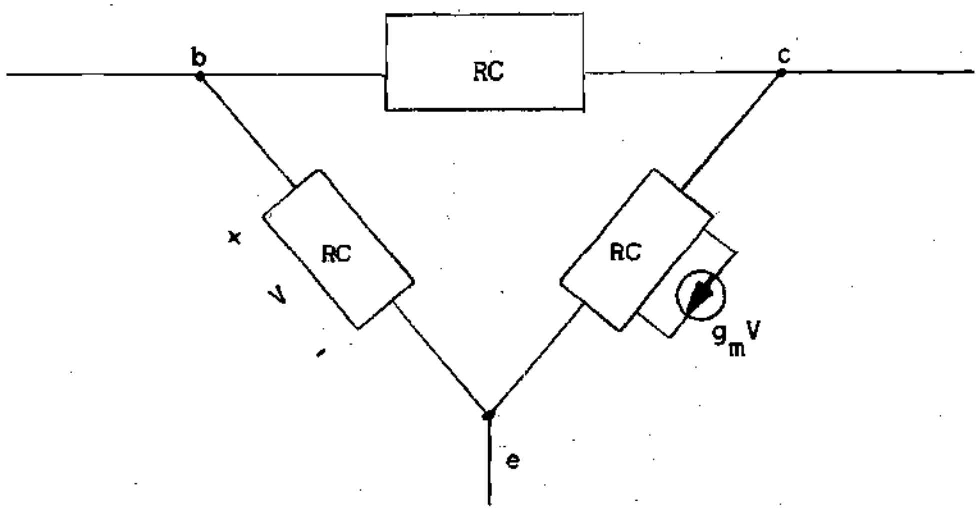


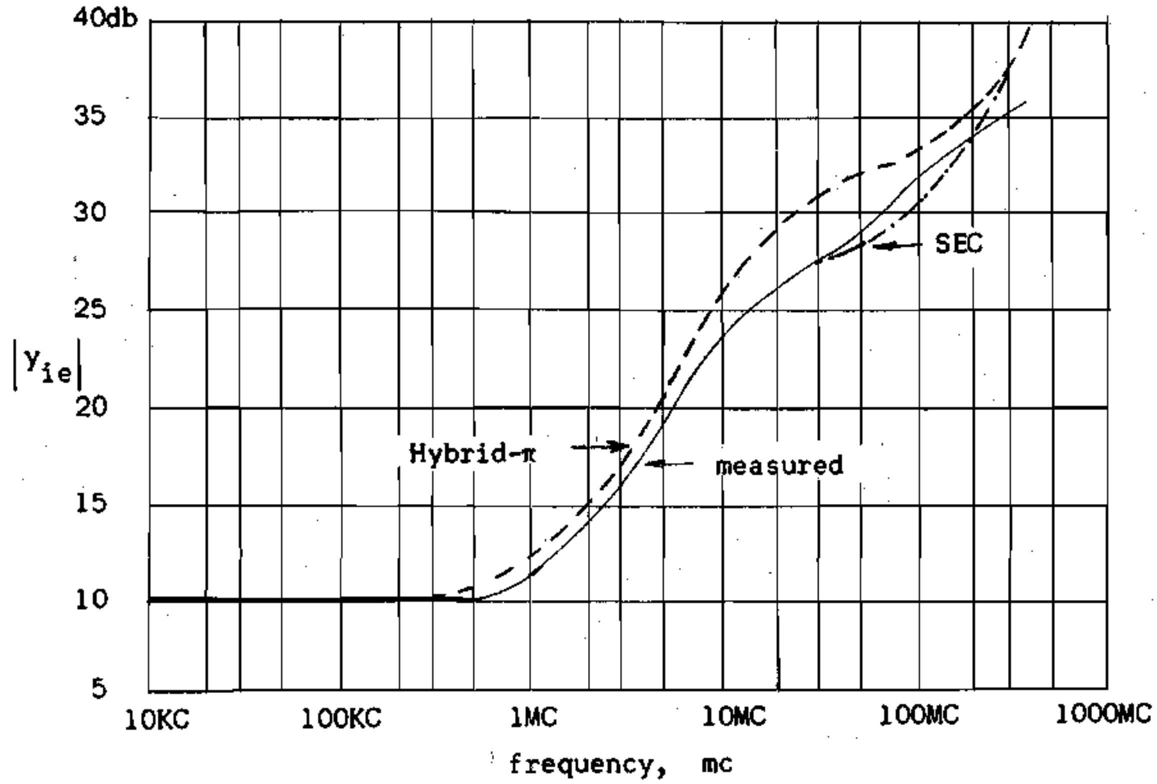
Figure 9. The Form of the Active Network that is to Approximate an Equivalent Circuit for a Transistor.

The character of the approximating admittances y_{ie}, $-y_{re}$, $-y_{fe}$, and y_{oe} is predetermined by the form of the active network. That is, the admittances y_{ie}, $-y_{re}$, and y_{oe} must be RC admittances. The forward transfer admittances $-y_{fe}$ can have $\pm$ R,C character except for the residue in the pole at infinity. This residue must be positive and equal to or less than the corresponding residue in the reverse transfer admittance, $-y_{re}$. In addition, it is required that the residue in the pole at infinity of $-y_{re}$ be equal to or less than the residues in the corresponding poles of the input and output admittances, y_{ie} and y_{oe}. Finally, the poles of all the admittances must be common. If these conditions are not satisfied then a three-terminal RC realization by the simple RC network of Figure 9 would not be possible.

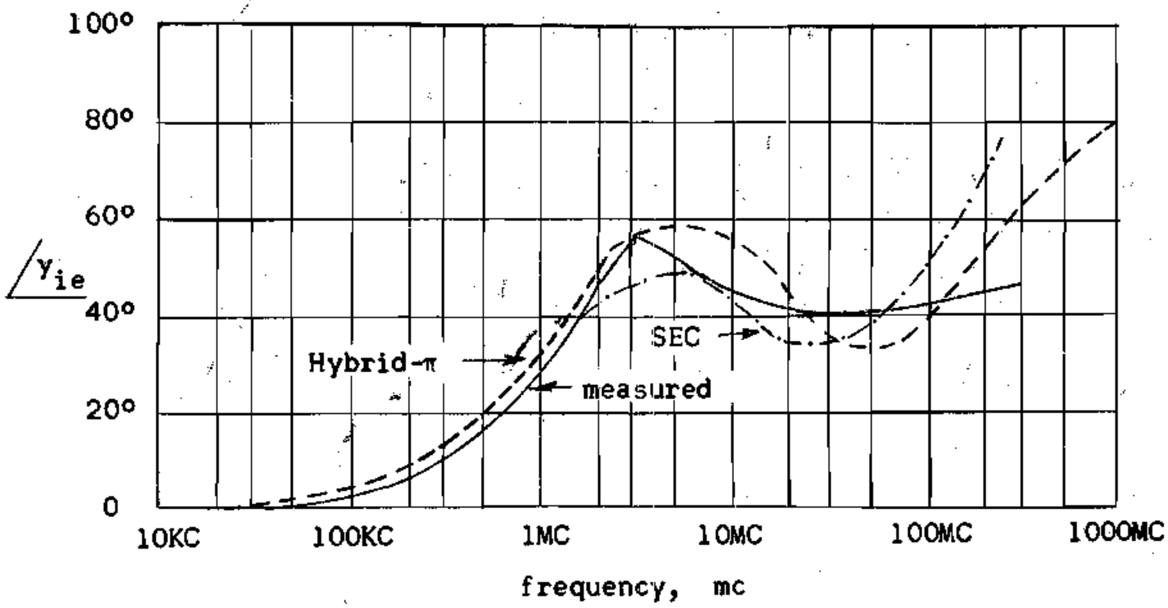
These conditions, placed on the admittances y_{ie}, $-y_{re}$, $-y_{fe}$, and y_{oe}, correspond to restricting the possible locations of the approximating zeros as well as the poles. This further complicates the approximating procedure. This is particularly apparent when the functions $-y_{re}$ and $-y_{fe}$ are being approximated. For example, to improve the approximation of $-y_{re}$ in Figure 10, it is desirable to locate the last zero (the one located at the higher frequency) at a lower frequency than it is located for this example. Also, the approximation of $-y_{fe}$ could be improved by relocating the last zero (a negative zero) at a higher frequency. However, neither can be readily relocated because the present location is the best of a compromise situation. The restriction in this case stems from the requirements placed on the residues in the poles of $-y_{re}$ and $-y_{fe}$ at infinity, to be at most equal.

After applying the approximation procedure and incorporating all the necessary restrictions to ensure that the developed admittances would satisfy the necessary and sufficient conditions for an RC realization the following admittance functions are obtained

$$
\begin{aligned}
y_{ie} &= \frac{K_{11}(s + 1.5 \times 2\pi \times 10^6)(s + 100 \times 2\pi \times 10^6)}{(s + 2\pi \times 11 \times 10^6)} \\
-y_{re} &= \frac{K_{12}(s + 1 \times 2\pi \times 10^3)(s + 22 \times 2\pi \times 10^6)}{(s + 2\pi \times 11 \times 10^6)} \\
-y_{fe} &= \frac{K_{21}(s + 140 \times 2\pi \times 10^6)(s - 210 \times 2\pi \times 10^6)}{(s + 2\pi \times 11 \times 10^6)}
\end{aligned}
\qquad (82)
$$

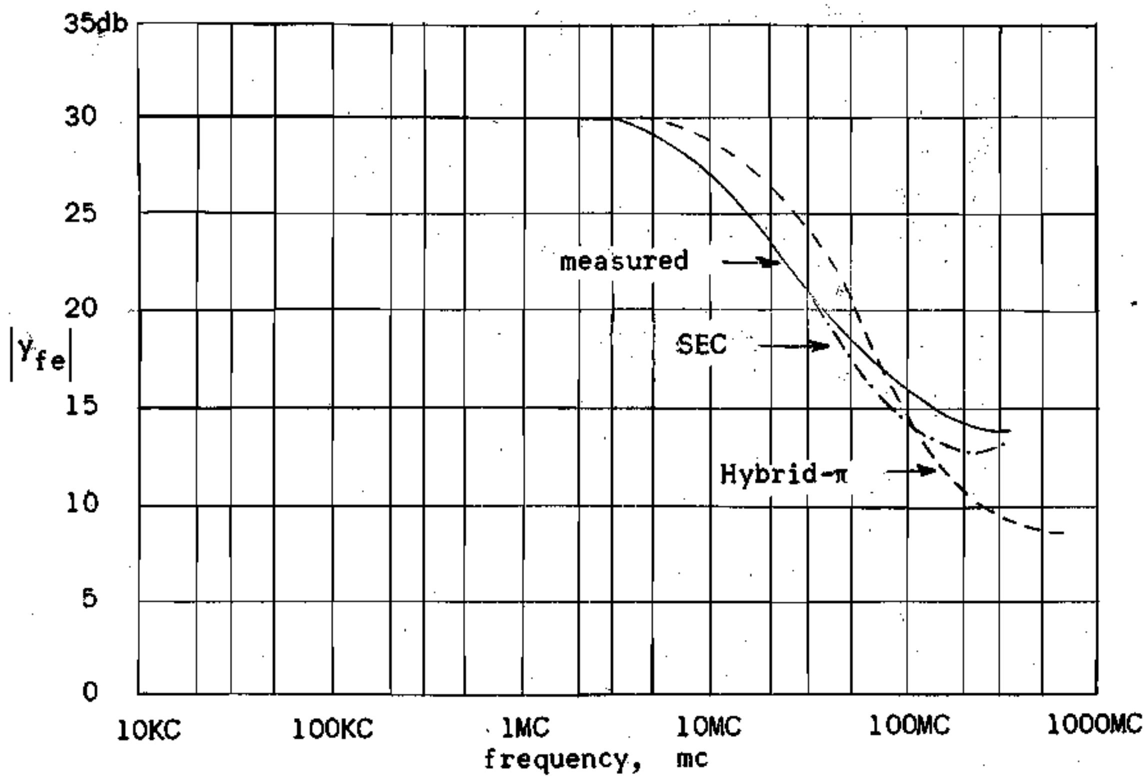


(a)

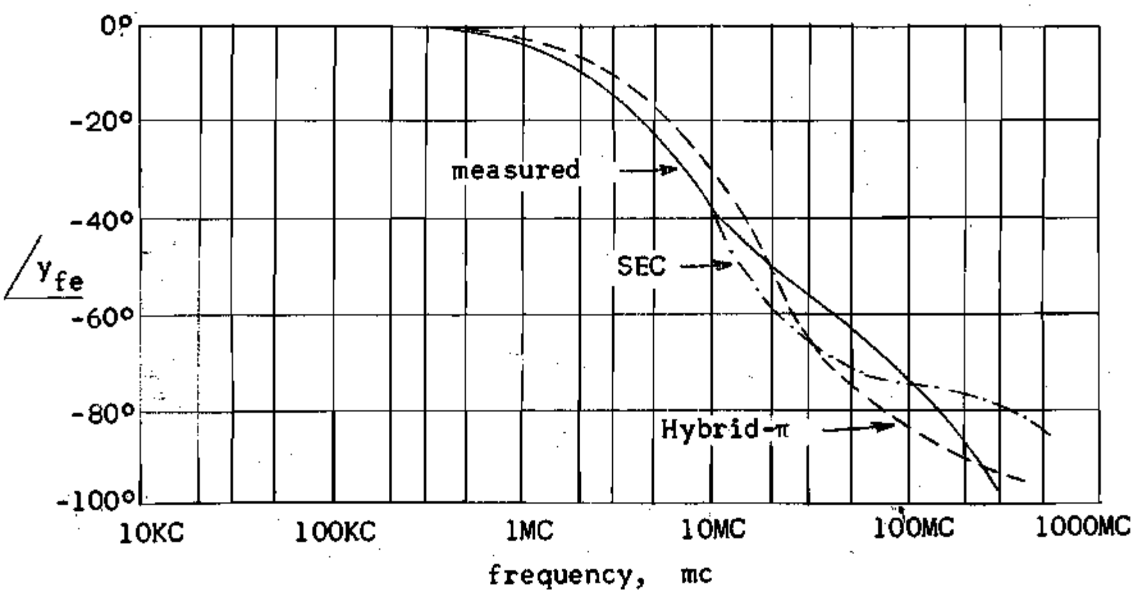


(b)

Figure 10. Experimental Data of y-parameters Versus Frequency (solid line) with Hybrid-π Model (dash line) and SEC (Synthesized equivalent circuit) Model (dot-dashed line) Shown for Comparison.

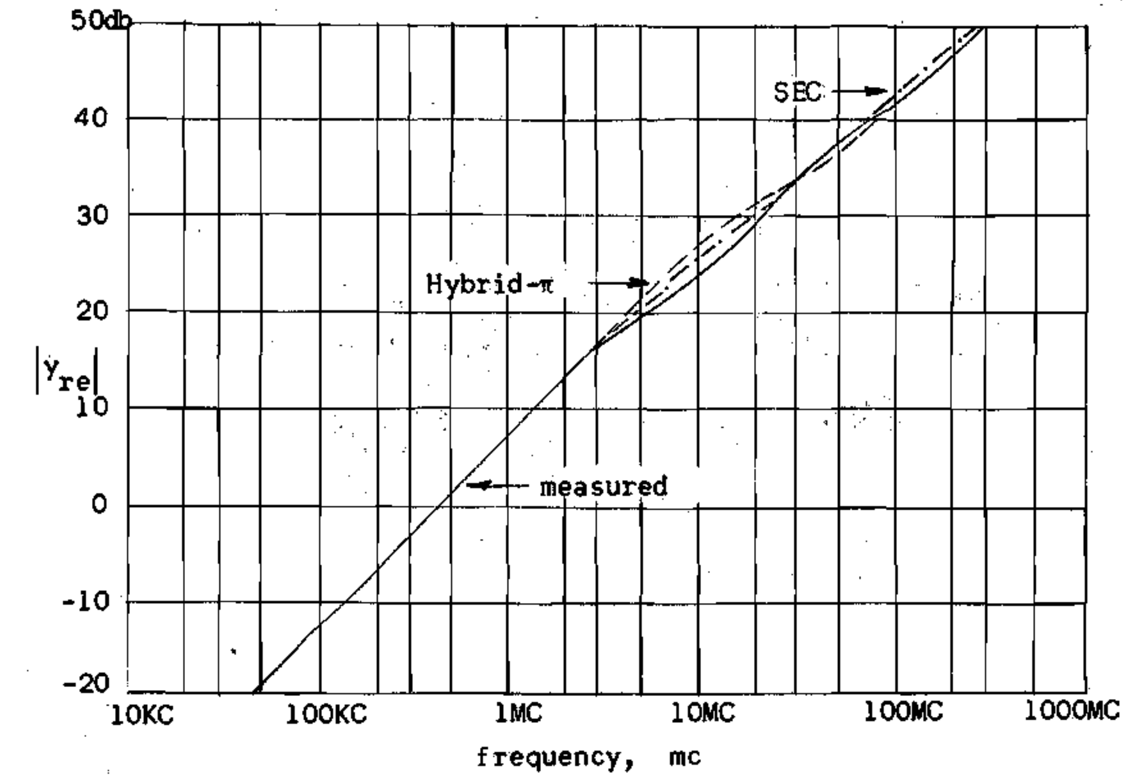


(c)

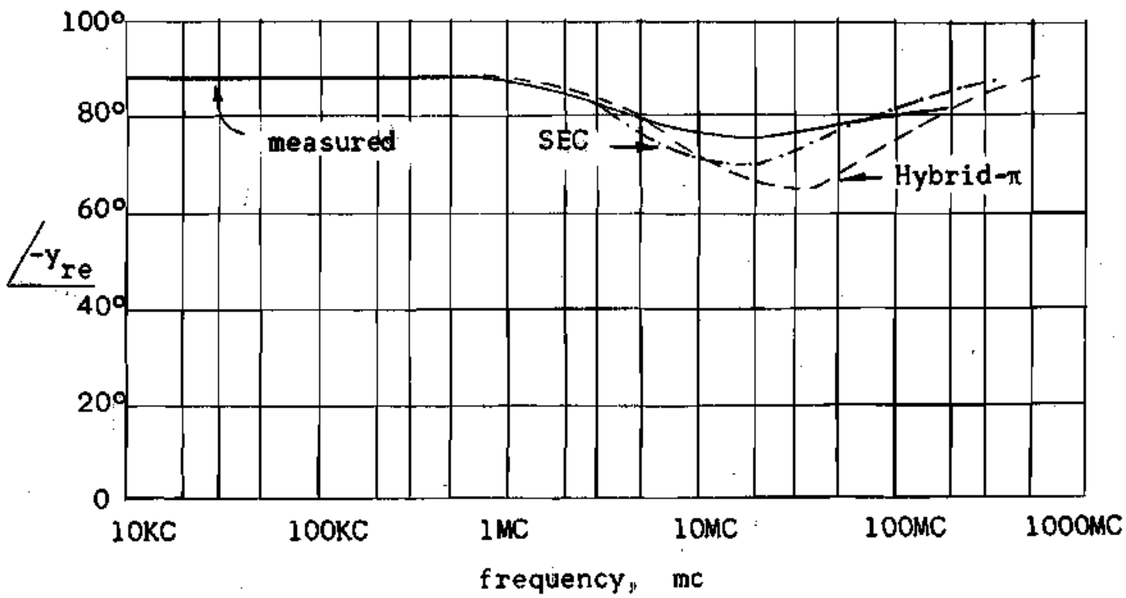


(d)

Figure 10. (Continued)

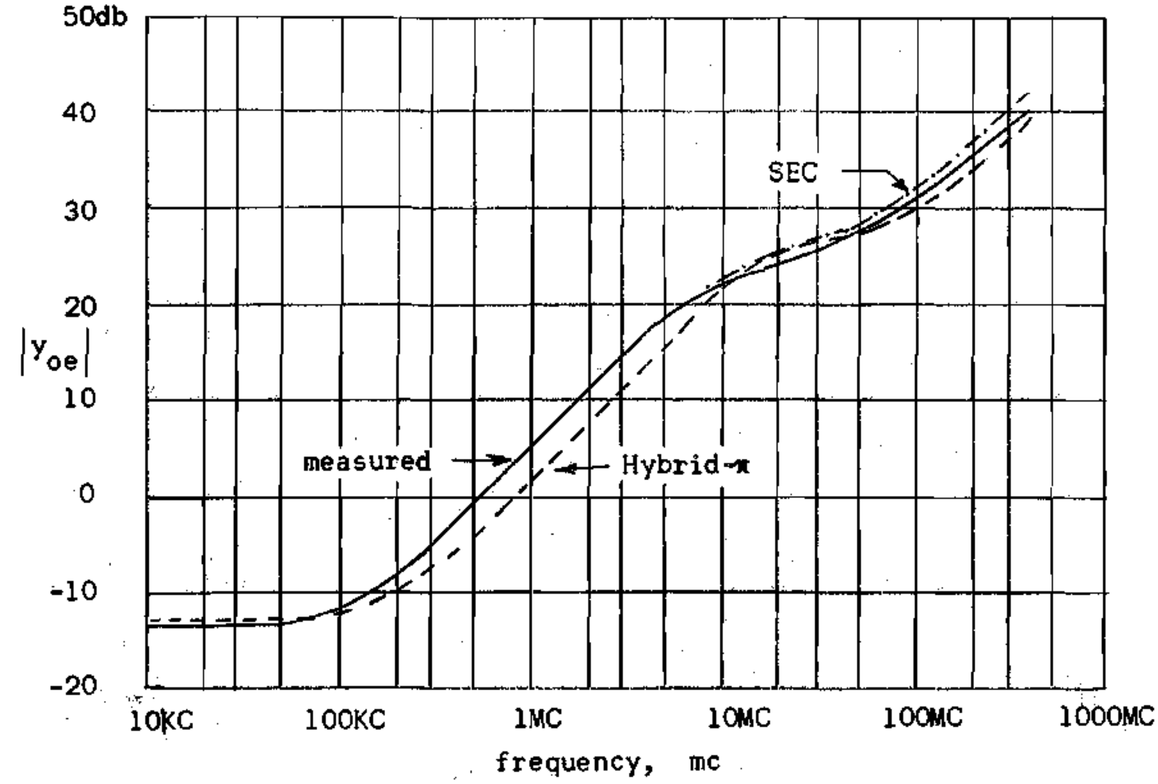


(e)

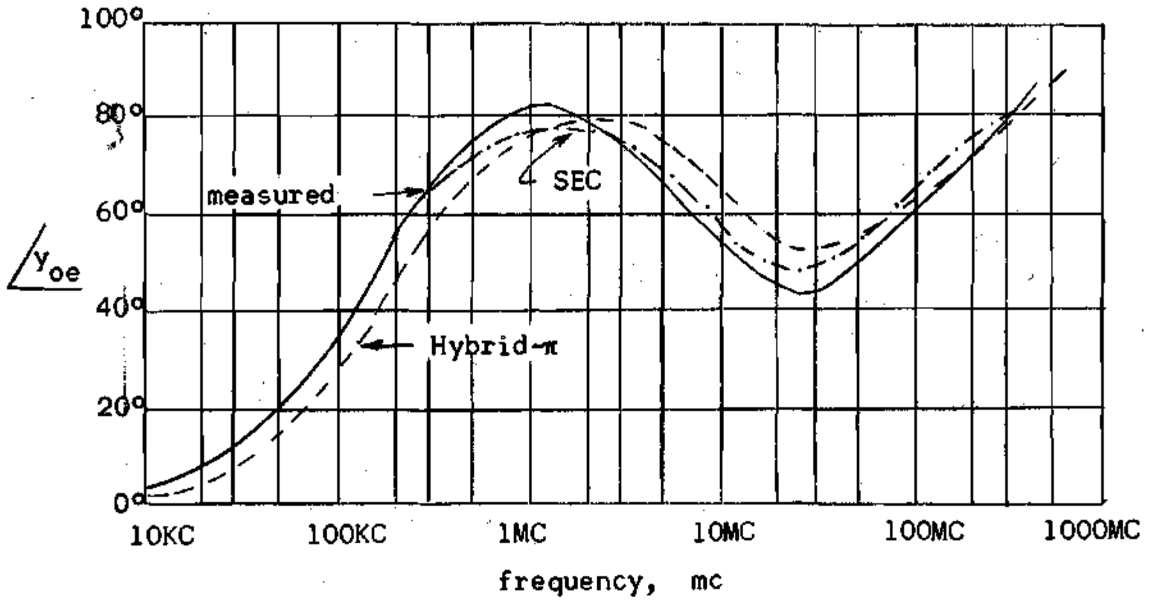


(f)

Figure 10. (Continued)



(g)



(h)

Figure 10. (Continued)

$$y_{oe} = \frac{K_{22}(s + 0.125 \times 2\pi \times 10^6)(s + 55 \times 2\pi \times 10^6)}{(s + 2\pi \times 11 \times 10^6)}$$

These admittances approximate the measured characteristics of a transistor rather closely as can be observed by comparing the solid curves, which represents the measured characteristics, with the dot-dashed curves which represent these approximating functions, in Figure 10.

An equivalent circuit is obtained by applying the synthesis procedure developed previously. First the admittances are expanded into partial-fraction form as

$$\begin{aligned}
y_{ie} &= \left[\frac{1.77s}{s + 69 \times 10^6} + 3.67 \times 10^{-9}s + 0.314\right] \times 10^{-3} \\
-y_{re} &= \left[\frac{0.276s}{s + 69 \times 10^6} + 1.995 \times 10^{-9}s + 0.025 \times 10^{-3}\right] \times 10^{-3} \\
-y_{fe} &= \left[\frac{29.4s}{s + 69 \times 10^6} + 1.995 \times 10^{-9}s - 33.5\right] \times 10^{-3} \\
y_{oe} &= \left[\frac{1.469s}{s + 69 \times 10^6} + 5.37 \times 10^{-9}s + 0.0211\right] \times 10^{-3}
\end{aligned} \tag{83}$$

It can be observed that y_{ie} and y_{oe} are both RC admittances. A realization can be obtained by allowing h to be zero as this is one of the special cases previously mentioned in Chapter III. Matrix (83) can readily be written into the necessary decomposed form of an RC matrix of rank one and realized by the standard synthesis procedure. In this particular case, when a voltage-controlled current source is the active device, the equivalent circuit is shown in Figure 11.

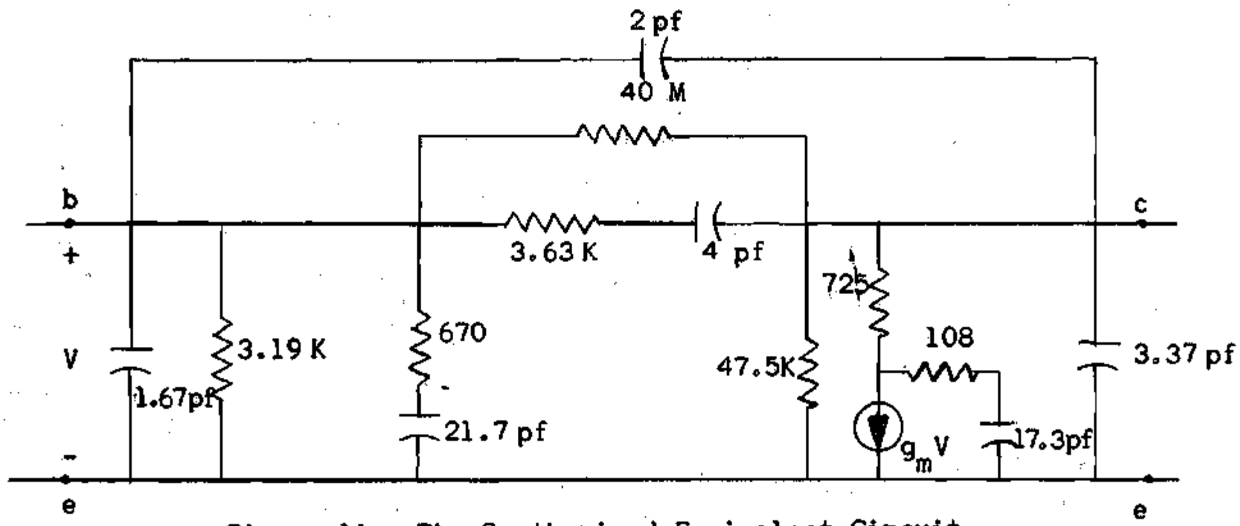


Figure 11. The Synthesized Equivalent Circuit.

Because the SEC model in Figure 11 approximates the measured characteristics of the transistor as accurately as the high-frequency hybrid-π model form dc to 200 mc, a reasonable amount of similarity between the two approximating circuits is expected. For example, the transconductance of the SEC model is 33.5×10^{-3} mhos which is comparable to that of the hybrid-π model, which is 36×10^{-3} mhos.

There are other similarities. For example, simplified high-frequency equivalent circuits are obtained for both the hybrid-π model and the SEC model. In the case of the hybrid-π, a simplified model is obtained by removing the extrinsic header and the overlap-diode capacitances. In the SEC model, the shunt capacitances that exist among all terminal pairs are omitted. The resulting equivalent circuits are shown in Figure 12. Both equivalent circuits are reasonably similar and sufficiently accurate from dc to approximately 10 megacycles.

Similar comparisons can be made. For instance, it is possible to obtain three more simple equivalent-circuit models that are of considerable interest because of the ease these circuits lend themselves to calculations.

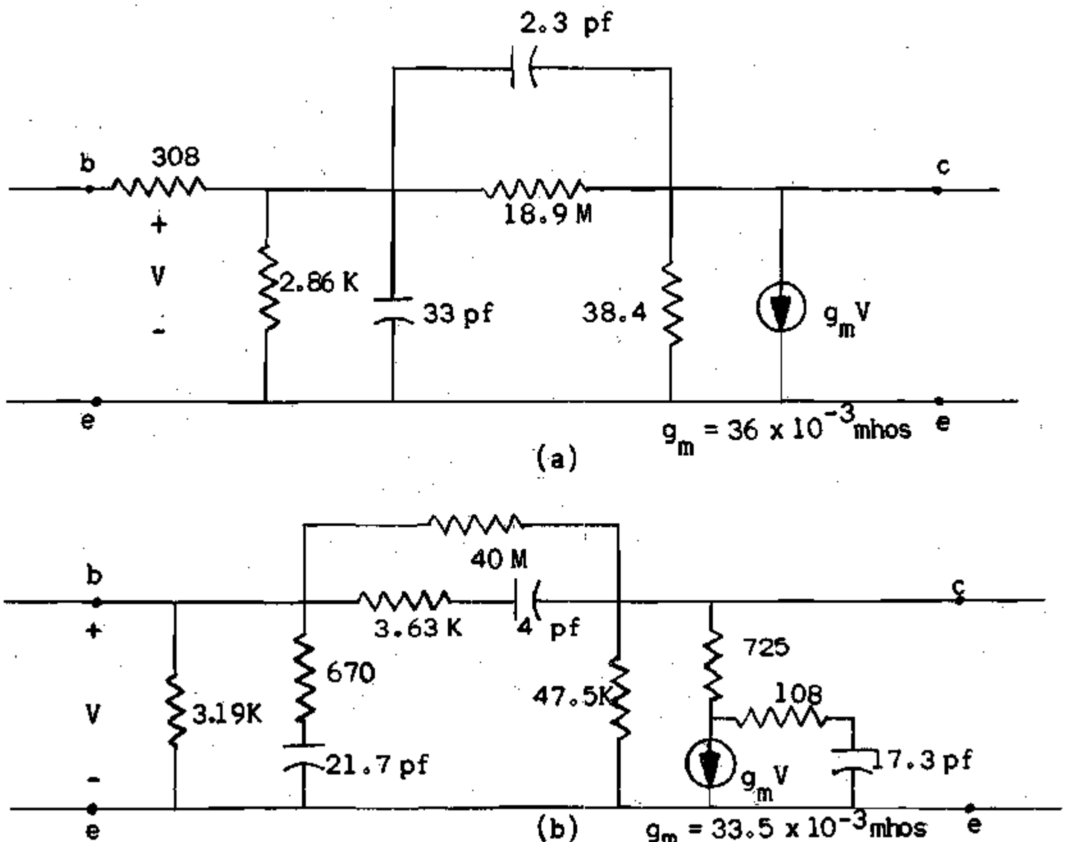


Figure 12. Simplified High-frequency Models. (a) The Hybrid-π Model. (b) The SEC Model. Both Models Valid from dc to Approximately 10 Megacycles.

Prior to presenting these simplified models it is necessary to redevelop two of the admittances of the SEC model in a first Foster form rather than the second Foster form used in Figures 11 and 12. The simplified high-frequency SEC model will now have the form illustrated in Figure 13.

As a first step in obtaining these simplified models a few rough calculations are required to determine the relative importance of the resistors and capacitors that are in shunt in this model. Since the shunting resistors

$$\frac{1}{2\pi R_1 C_1} = \frac{1}{2\pi \times 4.0 \times 10^{-2} \times 40 \times 10^{6}} \cong 1\text{KC} \tag{84}$$

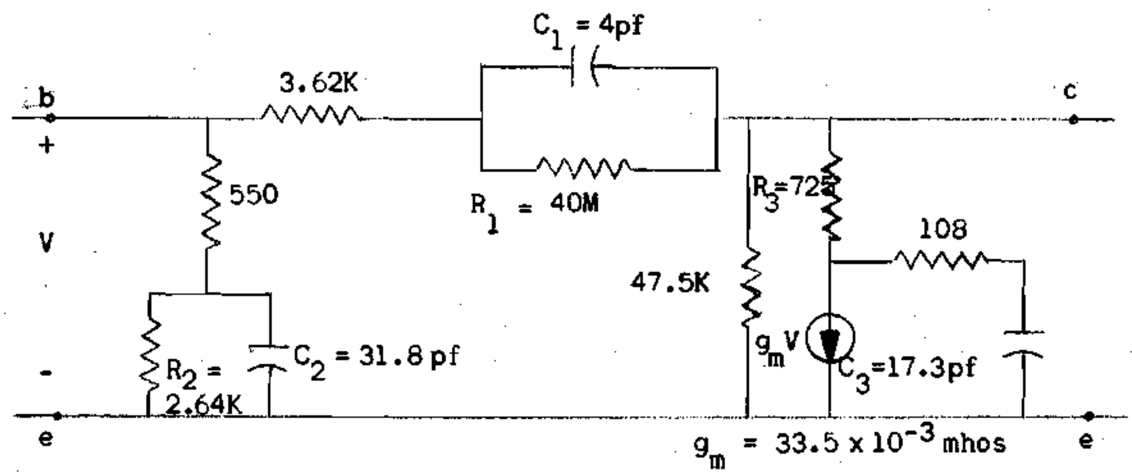


Figure 13. Alternate Form of the Simplified High-frequency SEC Model.

$$\frac{1}{2\pi R_2 C_2} = \frac{1}{2\pi \times 31.8 \times 10^{-12} \times 2.64 \times 10^3} \cong 1.9 \text{ mc} \tag{85}$$

are equal to the reactance of the capacitors at these frequencies. Therefore, at frequencies much below these values the shunting resistors predominate because the reactance of the capacitors are large. Conversely, at higher frequencies the capacitive terms will dominate. Thus, by restricting the frequency range of interest, it is possible to make an engineering approximation and obtain simpler equivalent circuits by neglecting these capacitors and resistors that have a negligible effect in that frequency range. In a like manner

$$\frac{1}{2\pi R_3 C_3} = \frac{1}{2\pi \times 173 \times 10^{-12} \times 0.725\text{K}} \cong 13 \text{ mc} \tag{86}$$

Since the reactance of C_3 is ten times larger than R_3 at 1.3 mc, at frequencies below 1.3 mc it is possible to neglect the effect of the 103-ohm resistor and C_3. In this case, R_3 can be removed as it is in series with the current source.

In an identical manner, when similar calculations are made for the simplified hybrid-π model of Figure 12, the following two frequencies are obtained

$$\frac{1}{2\pi \times 2.3 \times 10^{-12} \times 18.9 \times 10^{6}} \cong 3.7 \text{ KC} \tag{87}$$

$$\frac{1}{2\pi \times 33 \times 10^{-12} \times 2.86 \times 10^{3}} \cong 1.7 \text{ mc} \tag{88}$$

These frequencies are comparable to the previously calculated values for the SEC model.

In the simplification of the hybrid-π model, these two frequencies are used to divide the electrical behavior of the hybrid-π model into three distinct frequency ranges. Since, these frequencies are very close to those calculated for the SEC model, they will also be used to divide the electrical behavior of the SEC model. The three frequency ranges to be considered are low, mid, and high-frequency ranges. For the low-frequency case it is of interest to obtain a simplified equivalent circuit for a frequency range from dc to a few hundred cycles. In this case the capacitors in the SEC model and the hybrid model can be neglected. The resulting equivalent circuits are illustrated in Figure 14. A comparison indicates that if the 18.9M and 40M resistors are neglected, then the input impedance into these circuits are approximately 3.17K for the hybrid-π model and 3.19K for the SEC model.

Extending this comparison further, mid-frequency [approximately 10kc

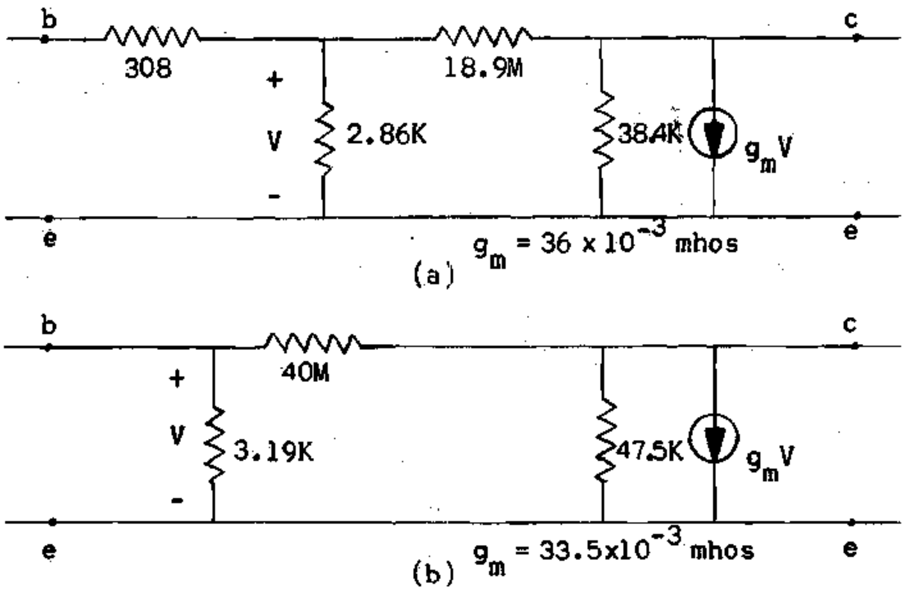


Figure 14. Simplified Low-frequency Equivalent Circuits for (a) the Hybrid-π Model and (b) the SEC Model. These Circuits are Valid from dc to Approximately Several Hundred Cycles.

to 1 mc] equivalent circuits are obtained by neglecting the resistors and capacitors that have insignificant effect. In this particular case, the equivalent circuits are shown in Figure 15, it is clear that both the hybrid-π and the SEC model contain the same number of elements. However, the SEC model is more adaptable to simple calculations because the input voltage is the controlling voltage. Again the range of element values in each circuit are quite similar.

Finally, it is possible to obtain simplified high-frequency [approximately 1mc to 10mc] models of both the SEC and the hybrid-π models. Just as in the previous case, elements are neglected when they have a negligible effect in the circuits. The resulting equivalent circuit models for this case are illustrated in Figure 16.

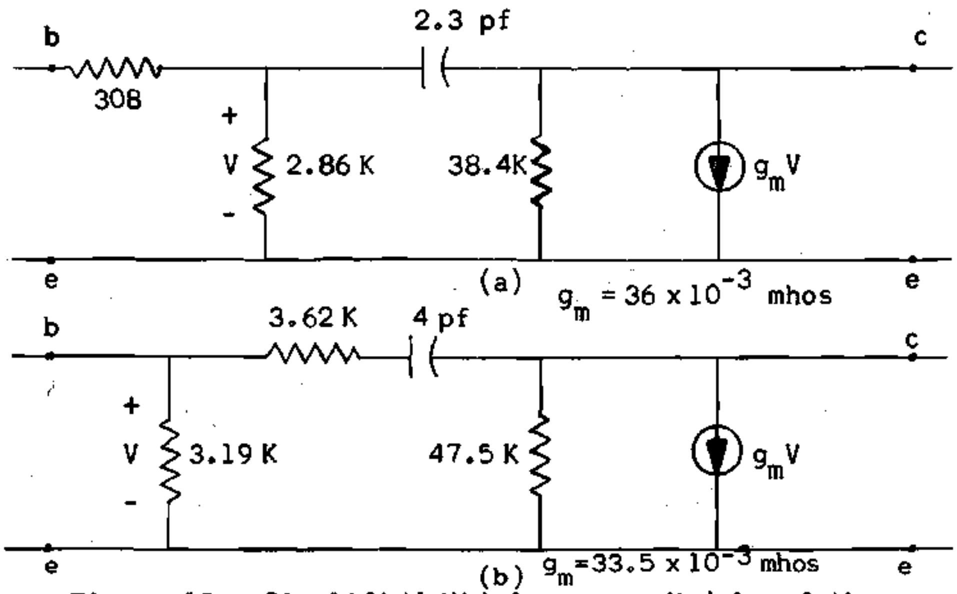


Figure 15. Simplified Mid-frequency Models of the (a) Hybrid-π Model and (b) SEC Model. These Circuits are Valid from about 10kc to Approximately 1mc.

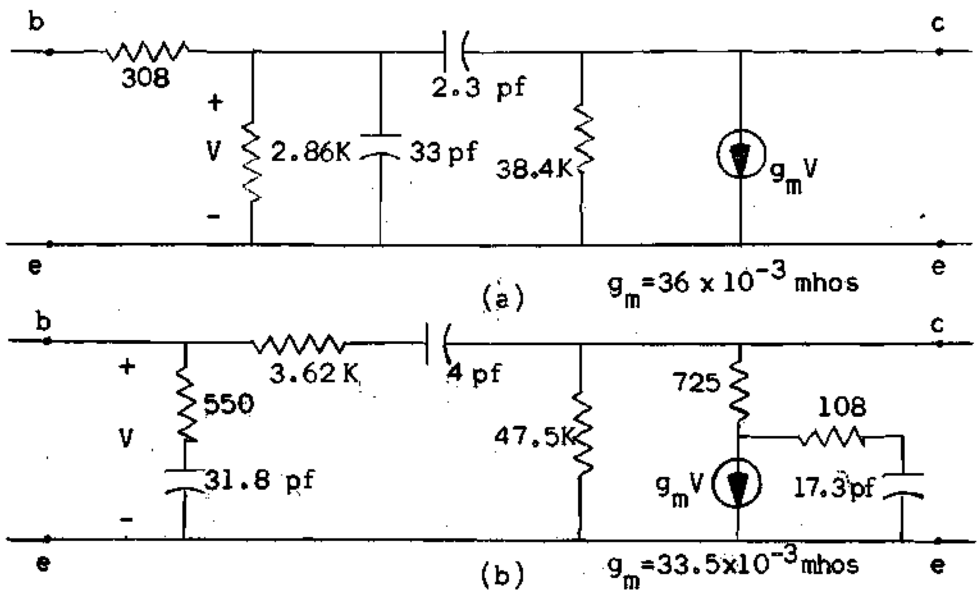


Figure 16. Simplified High-frequency Models of the (a) Hybrid-π Model and (b) SEC Model. These Circuits are Valid from about 1mc to Approximately 10mc.

Network II

The previous development clearly indicates the possibilities of obtaining equivalent circuits to approximate the characteristics of active devices by employing the synthesis technique. The development of Network I is primarily concerned with obtaining a very simple model to approximate the transistor characteristics. As a result, a great deal of accuracy is not attainable, especially at higher frequencies. Clearly, in Figure 10 the approximation of the input admittance y_{ie}, the forward transfer admittance $-y_{fe}$, and the output admittance y_{oe}, could be improved.

It is the purpose of this section to develop and synthesize an equivalent circuit that will be more accurate at high-frequencies than the first SEC model. This circuit is also to have the simple form of the RC network in Figure 9. However, in this network one negative resistor is allowed to give the approximating functions a slightly greater freedom. Furthermore, the approximating functions are allowed to have three zeros and two poles.

The approximation procedure in this case is substantially the same as that indicated for Network I. However, the addition of the negative resistor permits the removal of the previous requirements placed on the pole at infinity for the function $-y_{re}$. Because of this condition, the location of the approximating zeros of $-y_{re}$ are no longer restricted and can be located arbitrarily. The approximation of the input admittance, y_{ie} requires a compromise in locating the last zero. This zero has to be located at a frequency slightly lower than that of the best location in order to be assured of an RC realization. It is also worthwhile to mention that the location of the second pole is truly a compromise. The input admittance, y_{ie} would best be approximated if this pole is located at 160 mc rather than 250 mc. The location of 250 mc is ideal for the reverse transfer admittance, $-y_{re}$. The forward transfer admittance however, would be approximated better if the pole is located at a slightly higher frequency -- approximately 350 mc. Finally, the output admittance, y_{oe} is best approximated if the pole is omitted completely.

As a result of applying the approximation procedure, the following approximating admittance functions are obtained:

$$
\begin{aligned}
y_{ie} &= \frac{K_{ie}(s+2\pi \times 1.5 \times 10^6)(s+2\pi \times 74 \times 10^6)(s+2\pi \times 668 \times 10^6)}{(s+2\pi \times 11 \times 10^6)(s+2\pi \times 250 \times 10^6)} \\
-y_{re} &= \frac{K_{re}(s+2\pi \times 0.01 \times 10^6)(s+2\pi \times 19 \times 10^6)(s+2\pi \times 300 \times 10^6)}{(s+2\pi \times 11 \times 10^6)(s+2\pi \times 250 \times 10^6)} \\
-y_{fe} &= \frac{K_{fe}(s+2\pi \times 70 \times 10^6)(s-2\pi \times 250 \times 10^6)}{(s+2\pi \times 11 \times 10^6)(s+2\pi \times 250 \times 10^6)} \\
y_{oe} &= \frac{K_{oe}(s+2\pi \times 0.125 \times 10^6)(s+2\pi \times 60 \times 10^6)(s+2\pi \times 300 \times 10^6)}{(s+2\pi \times 11 \times 10^6)(s+2\pi \times 250 \times 10^6)}
\end{aligned}
\tag{89}
$$

These admittances approximate the measured characteristics with greater accuracy as illustrated in Figure 17. The solid curve indicates the measured characteristics. The dashed-dotted curve represents the newly approximated admittance functions and the dash curve represents the high-frequency hybrid-π model. Clearly these approximating admittances represent the measured characteristics very closely and with a substantial improvement over the hybrid-π representation. However, this improvement requires

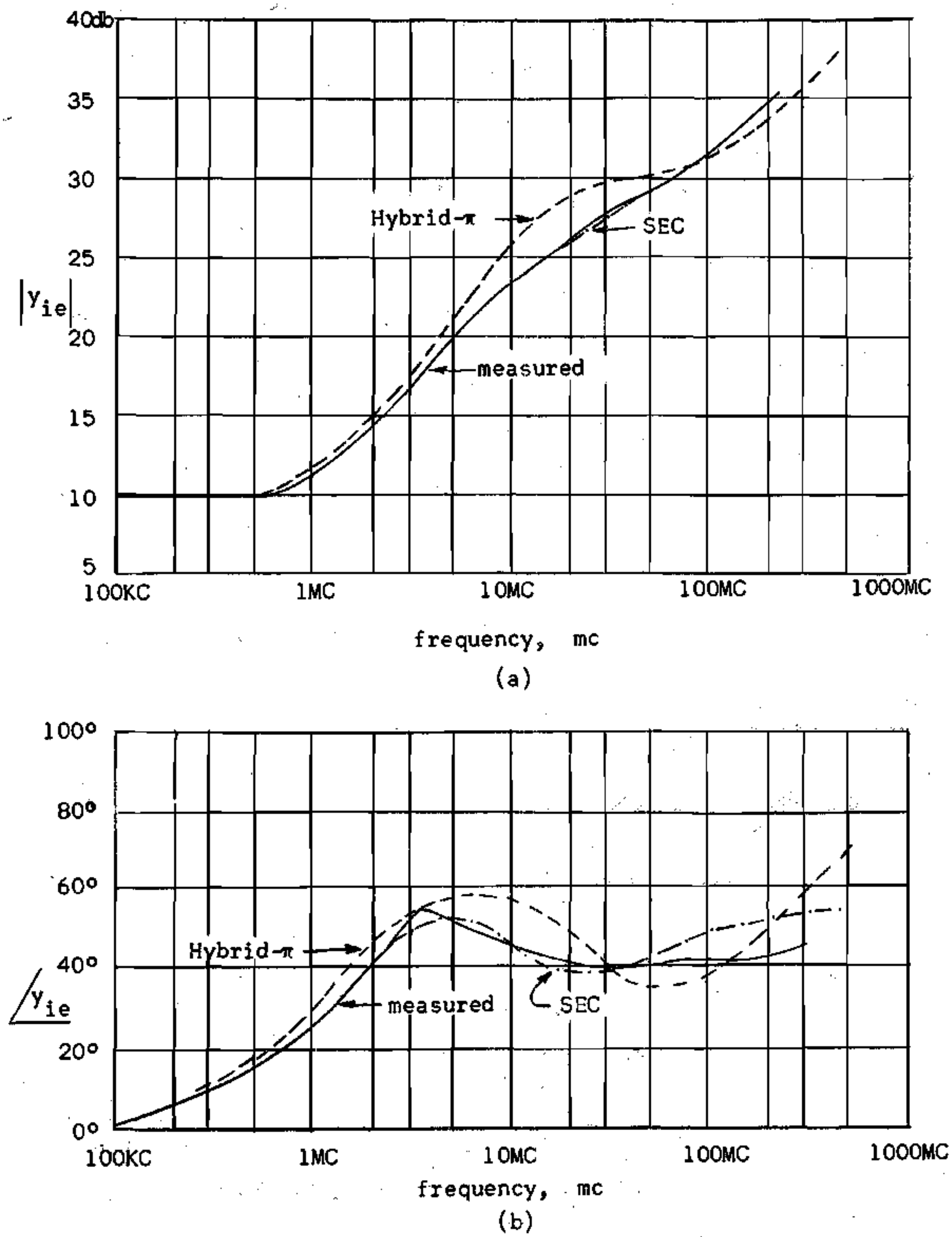
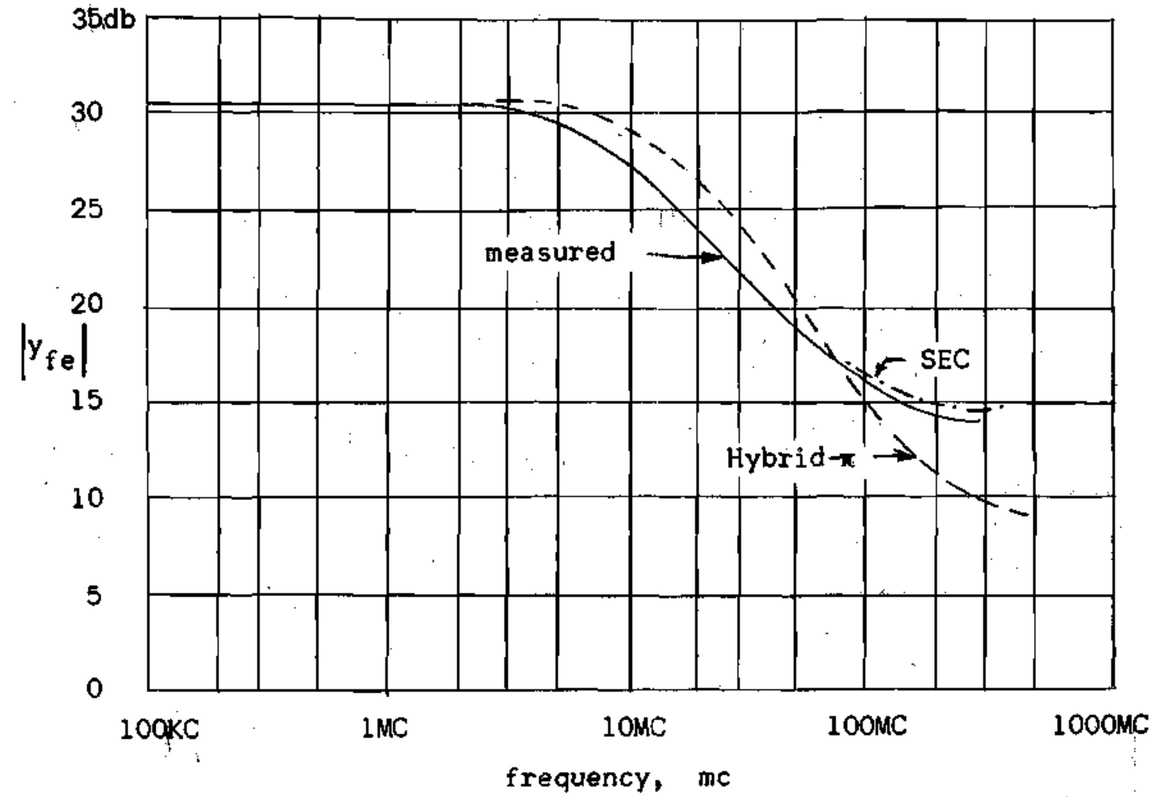
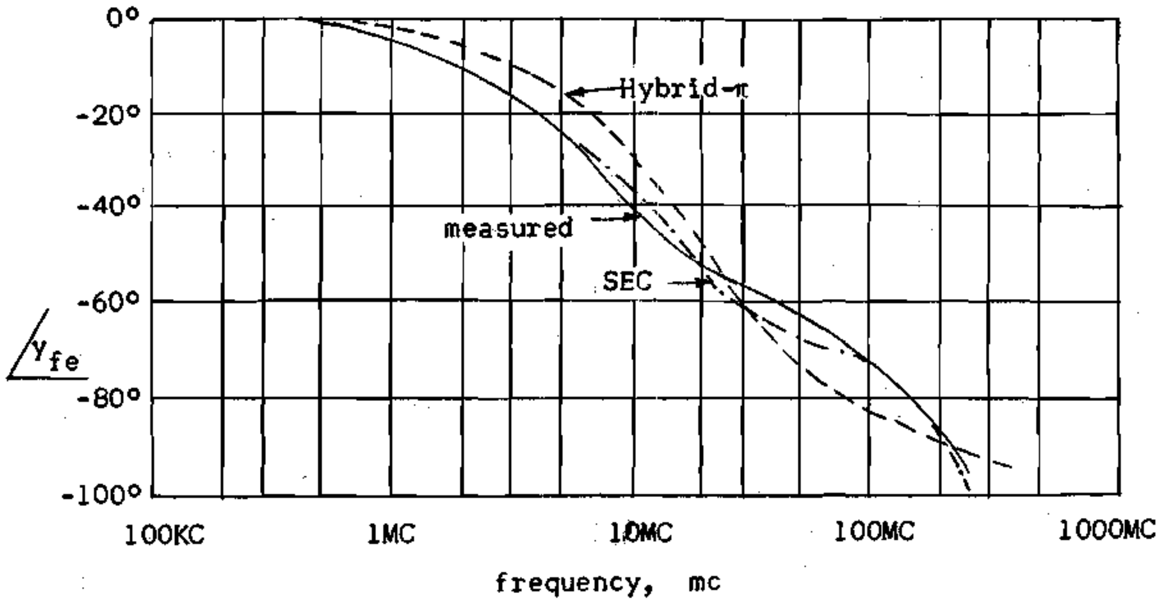


Figure 17. Experimental Data of y-parameters Versus Frequency (solid line) with Hybrid-π Model (dash line) and SEC (synthesized equivalent circuit) Model (dot-dashed line) shown for Comparison.

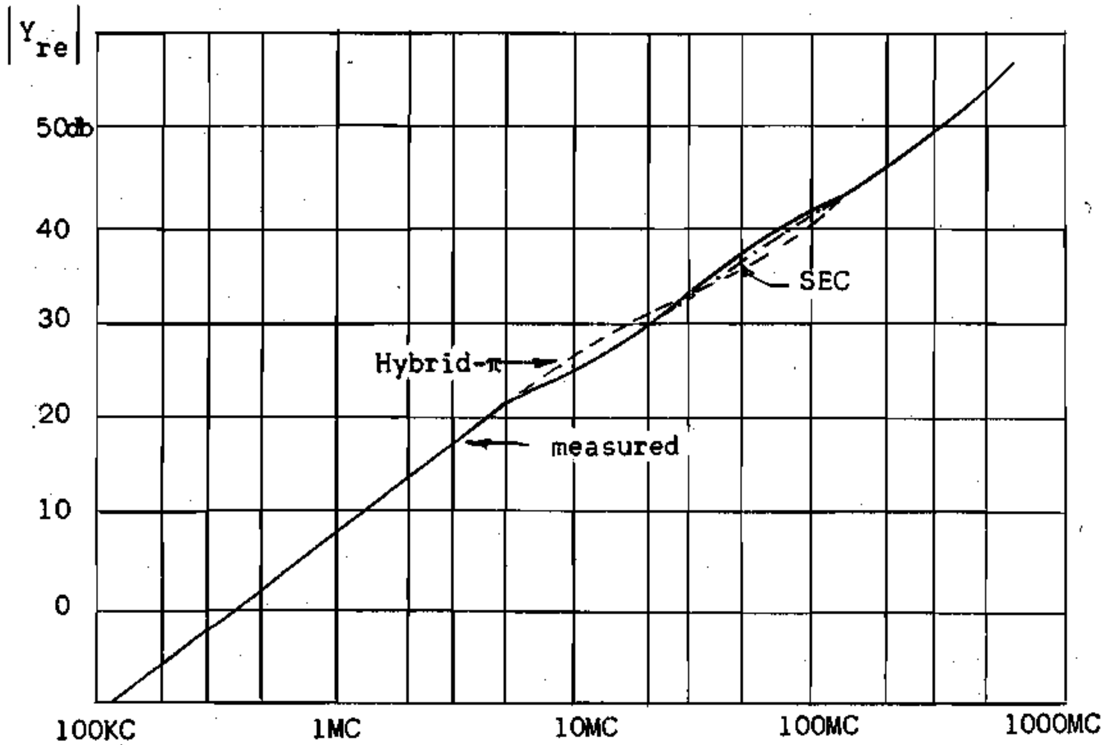


(c)

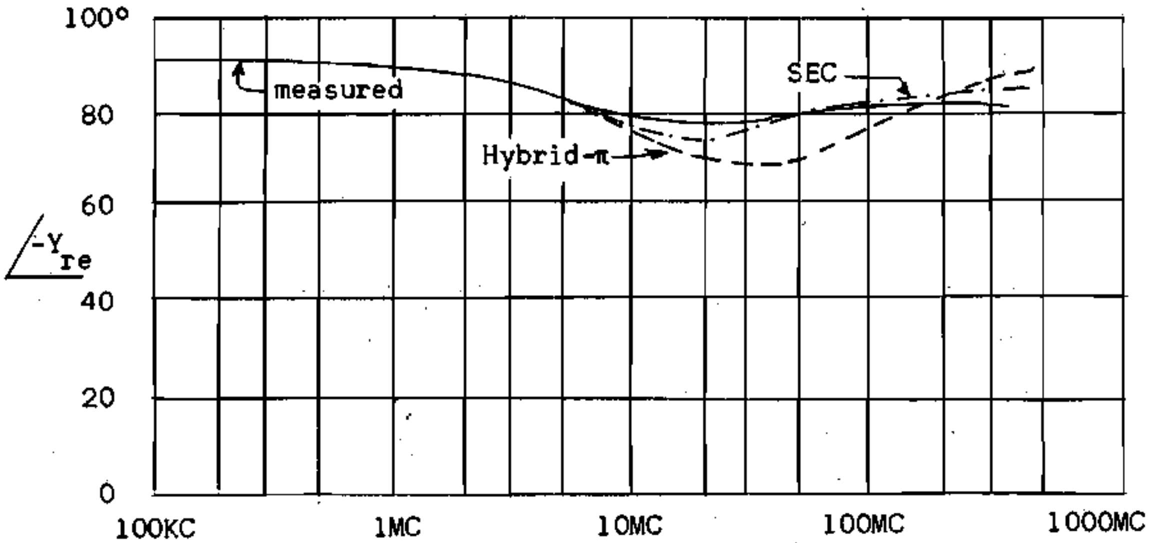


(d)

Figure 17. (Continued)

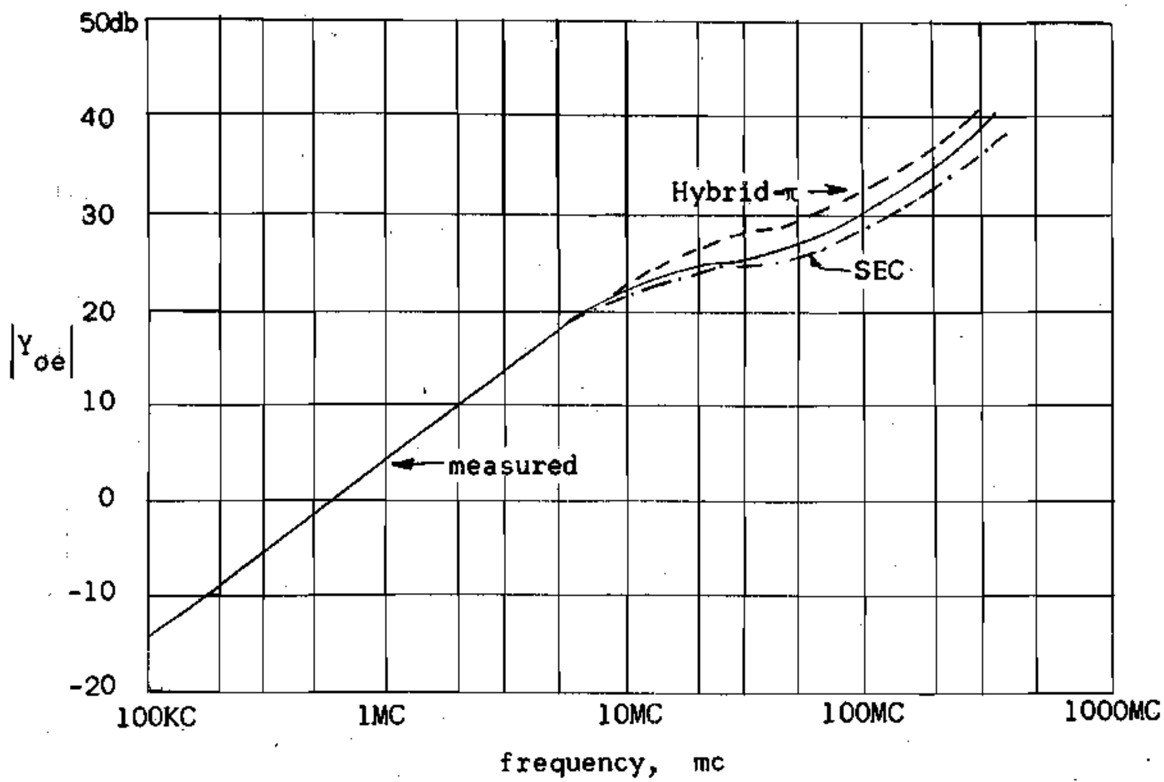


(e)

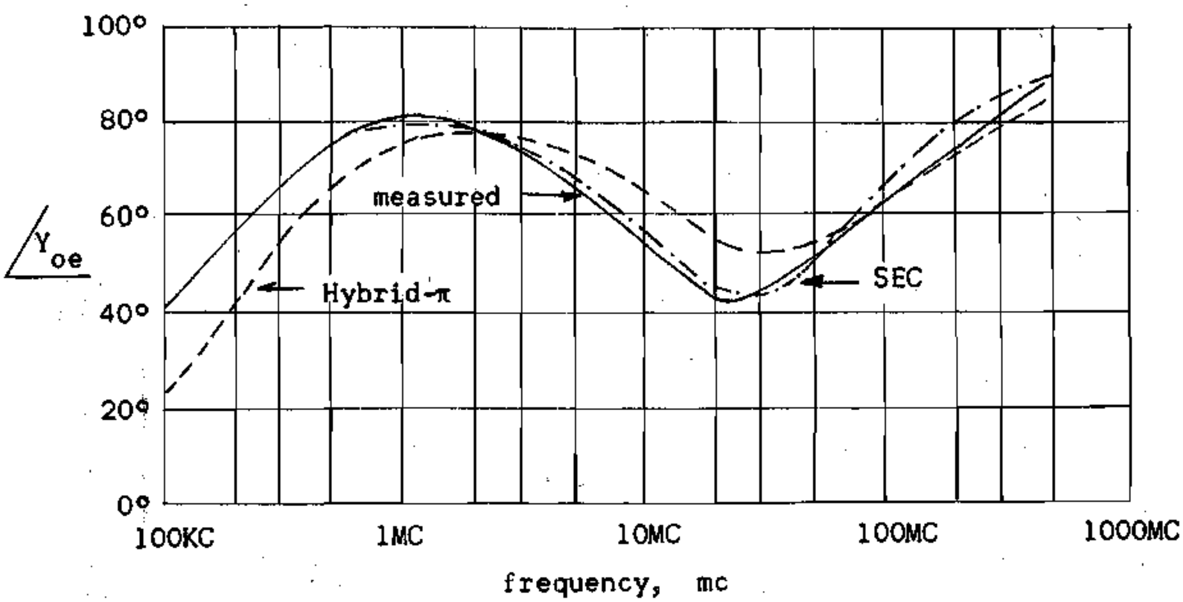


(f)

Figure 17. (Continued)



(g)



(h)

Figure 17. (Continued)

increased network complexity. Specifically a realization of these admittances shows that almost twice as many network elements are required in addition to one negative resistor.

To obtain an equivalent circuit representation from the admittances in Equation (89) it is necessary to write these admittances in the partial-fraction form

$$\begin{aligned}
y_{ie} &= \left[\frac{1.7\,s}{s+2\pi\times 11\times 10^{6}} + \frac{3.49\,s}{s+2\pi\times 250\times 10^{6}} + 1.81\times 10^{-9}s + 0.314\right]\times 10^{-3} \\
-y_{re} &= \left[\frac{0.11s}{s+2\pi\times 11\times 10^{6}} + \frac{0.55s}{s+2\pi\times 250\times 10^{6}} + 1.81\times 10^{-9}s + 0.236\times 10^{-3}\right]\times 10^{-3} \\
-y_{fe} &= \left[\frac{-30.8s}{s+2\pi\times 11\times 10^{6}} + \frac{-7.88s}{s+2\pi\times 250\times 10^{6}} + 33.5\right]\times 10^{-3} \\
y_{oe} &= \left[\frac{1.51s}{s+2\pi\times 11\times 10^{6}} + \frac{1.07s}{s+2\pi\times 250\times 10^{6}} + 4.1\times 10^{-9}s + 0.0211\right]\times 10^{-3}
\end{aligned} \tag{90}$$

Here, as in the previous case, the input admittance and output admittance are obviously RC functions and h can be set equal to zero. By following the synthesis procedure outlined in Chapter III, an equivalent circuit is obtained and presented in Figure 18.

The accuracy of the SEC of Figure 18 in approximating the transistor behavior is quite adequate for almost all practical purposes. This network has been synthesized in such a way that it still has the general appearance of Figure 9, so that the comparison with the hybrid-π model can be made. As the next section will show, still better accuracy can be achieved if the network is not restricted to have the arrangement of Figure 9.

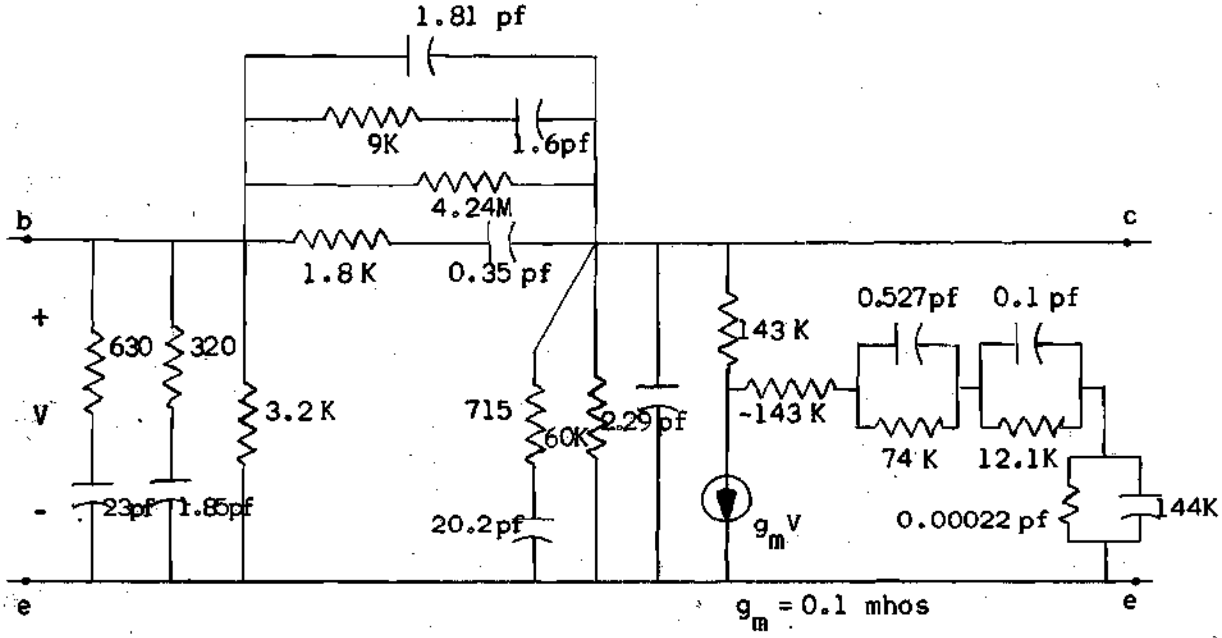


Figure 18. A Synthesized High-Frequency Equivalent Circuit.

Network III

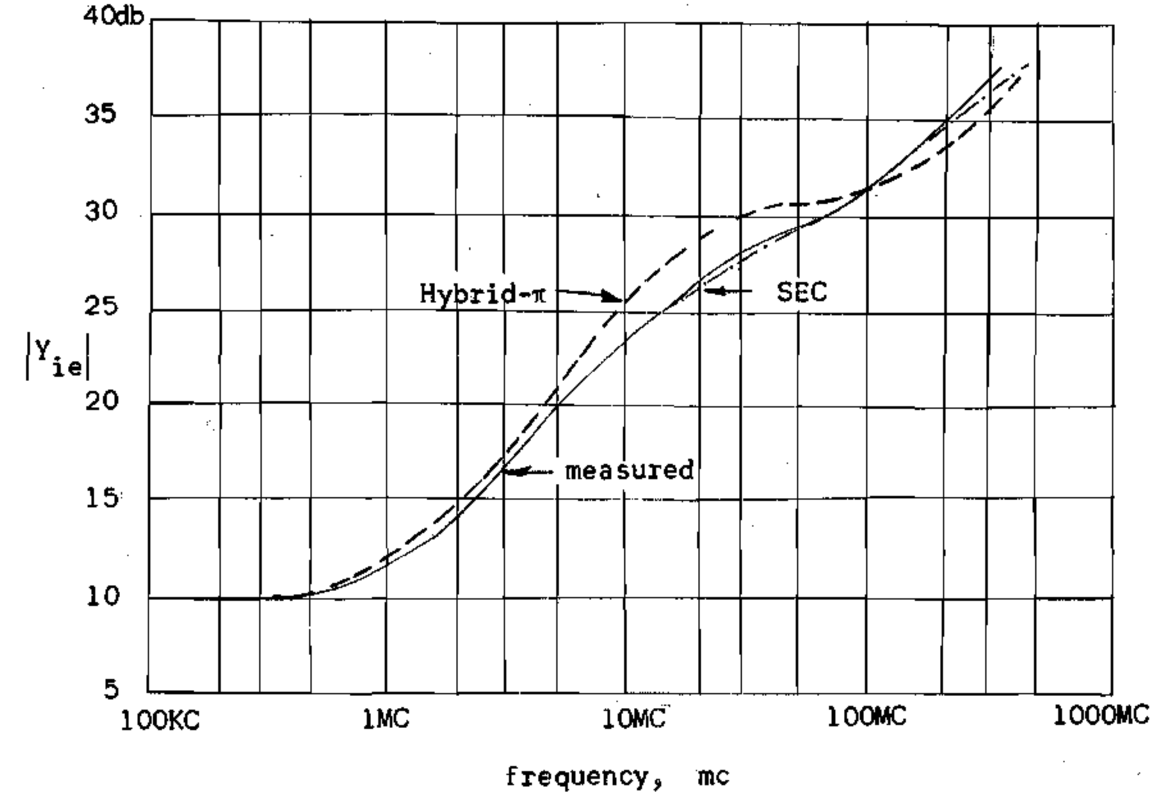
As another alternative development, the restrictions on the network will now be removed. Without this restriction, the approximation problem is less tedious since all limitations on the approximating zeros have been removed. To illustrate the approximation procedure and the improvement obtained when the zeros of the admittance are not restricted, the approximation of the previous example will be improved. This is accomplished by adapting the admittances in Equation (89) to these less stringent conditions. For the input admittance, y_{ie}, this allows the last zero to be relocated to the more desirable location of 876 mc as compared to the previous value of 668 mc. The approximation of the reverse transfer admittance $-y_{re}$, is improved by locating the last zero at a higher frequency of 350 mc as compared to the previous 300 mc location. A slight improvement in the forward transfer admittance $-y_{fe}$, is obtained by locating a zero at 2,200 mc. Lastly, the approximation of the output admittance, y_{oe}, is improved substantially by completely removing the pole located at 250 mc and the zeros at 300 mc.

That these slight adjustments in the approximating functions actually improves the approximation of the measured characteristics is clearly indicated in Figure 19. Here the solid curves represent the measured characteristics and the dot-dash curves represent the newly approximated admittance functions. For comparison purposes the hybrid-π characteristics are plotted as dashed curves.

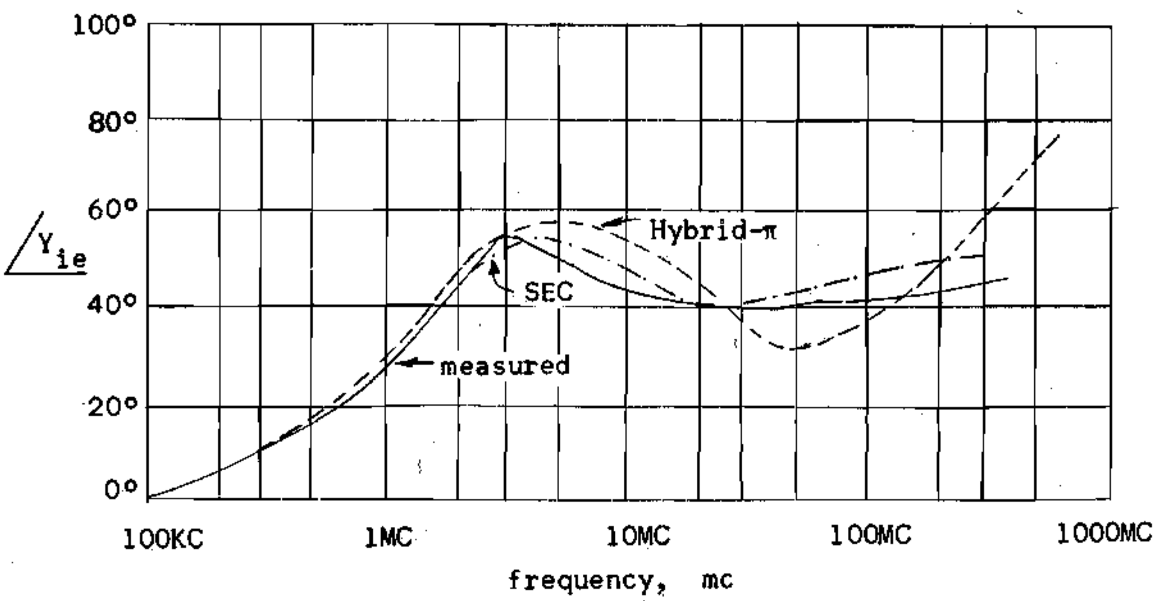
The approximating admittances are

$$\begin{aligned}
y_{ie} &= \frac{K_{ie}(s + 2\pi \times 1.5 \times 10^6)(s + 2\pi \times 74 \times 10^6)(s + 2\pi \times 872 \times 10^6)}{(s + 2\pi \times 11 \times 10^6)(s + 2\pi \times 250 \times 10^6)} \\
-y_{re} &= \frac{K_{re}(s + 2\pi \times 0.01 \times 10^6)(s + 2\pi \times 19 \times 10^6)(s + 2\pi \times 350 \times 10^6)}{(s + 2\pi \times 11 \times 10^6)(s + 2\pi \times 250 \times 10^6)} \\
-y_{fe} &= \frac{K_{fe}(s + 2\pi \times 70 \times 10^6)(s - 2\pi \times 250 \times 10^6)(s + 2\pi \times 2200 \times 10^6)}{(s + 2\pi \times 11 \times 10^6)(s + 2\pi \times 250 \times 10^6)} \\
y_{oe} &= \frac{K_{oe}(s + 2\pi \times 0.125 \times 10^6)(s + 2\pi \times 60 \times 10^6)}{(s + 2\pi \times 11 \times 10^6)}
\end{aligned} \tag{91}$$

To realize these admittances, they are expanded into the following partial-fraction form

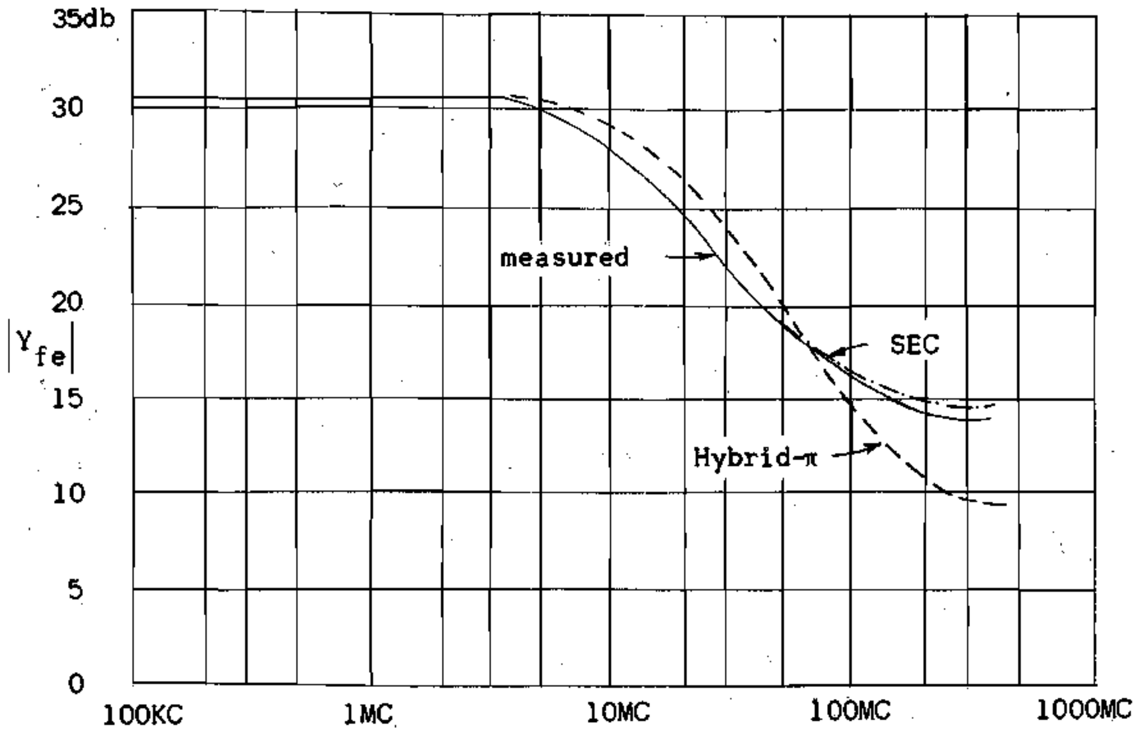


(a)

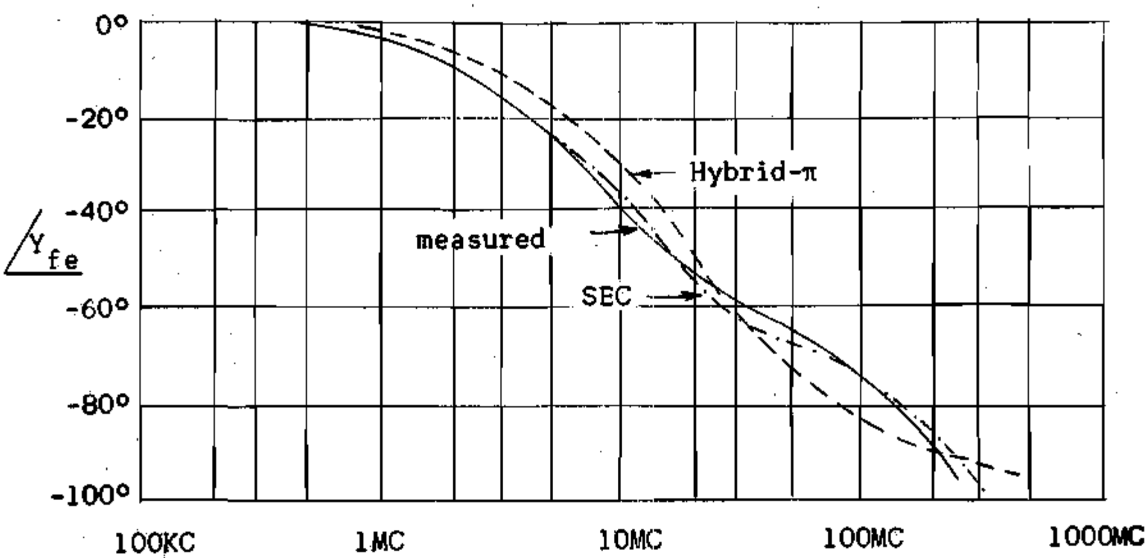


(b)

Figure 19. Experimental Data of y-parameters Versus Frequency (solid line) with Hybrid-π Model (dash line) and SEC (synthesized equivalent circuit) Model (dot-dashed line) shown for Comparison.

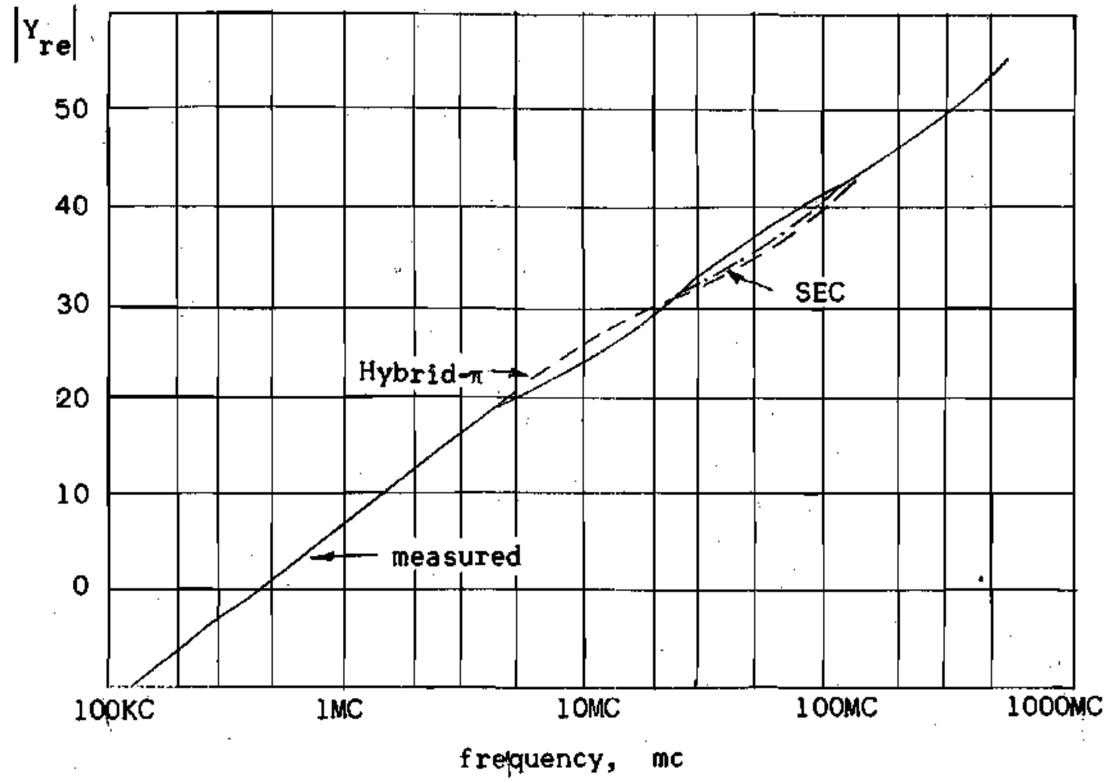


(c)

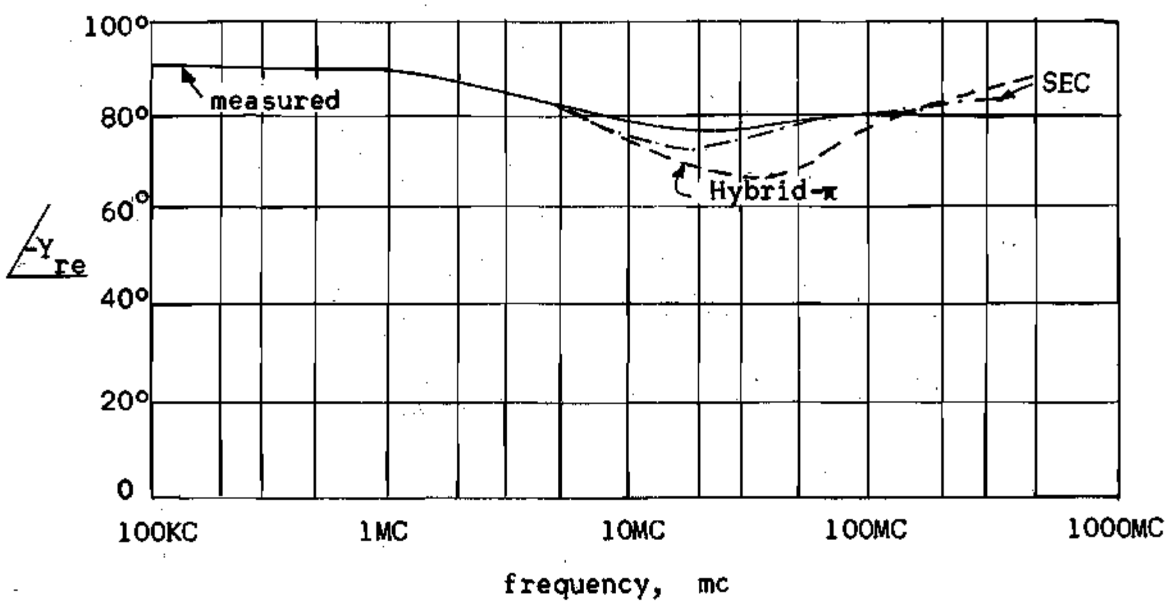


(d)

Figure 19. (Continued)

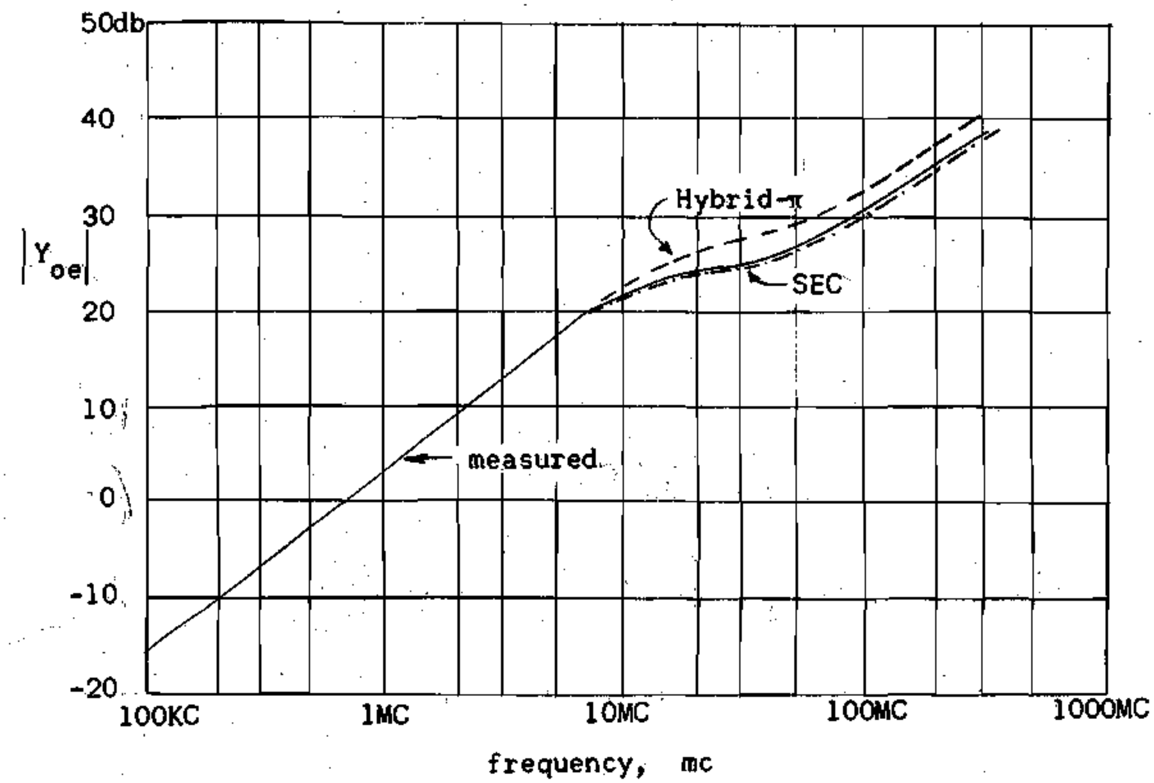


(e)

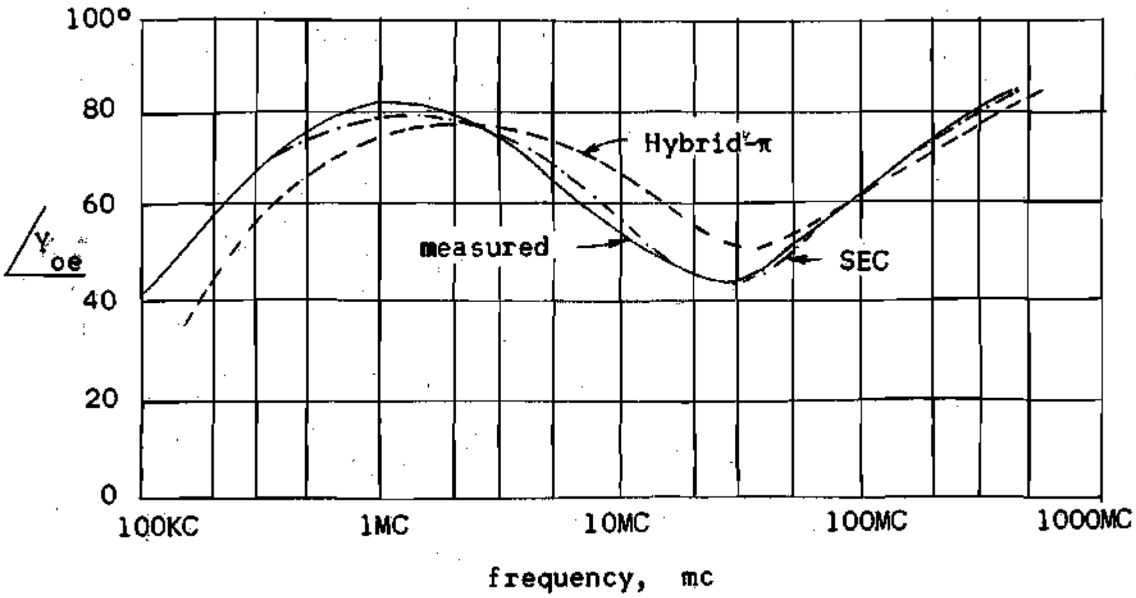


(f)

Figure 19. (Continued)



(g)



(h)

Figure 19. (Continued)

$$y_{ie} = \left[\frac{1.75s}{s + 2\pi \times 11 \times 10^6} + \frac{3.70s}{s + 2\pi \times 250 \times 10^6} + 1.42 \times 10^{-9}s + 0.314\right] \times 10^{-3}$$

$$-y_{re} = \left[\frac{0.101s}{s + 2\pi \times 11 \times 10^6} + \frac{0.95s}{s + 2\pi \times 250 \times 10^6} + 1.565 \times 10^{-9}s + 0.236 \times 10^{-3}\right] \times 10^{-3}$$

$$-y_{fe} = \left[\frac{-32.1s}{s + 2\pi \times 11 \times 10^6} + \frac{-7.35s}{s + 1570 \times 10^6} - 0.4 \times 10^{-9}s + 33.5\right] \times 10^{-3} \qquad (92)$$

$$y_{oe} = \left[\frac{1.54s}{s + 2\pi \times 11 \times 10^6} + 4.93 \times 10^{-9}s + 0.0211\right] \times 10^{-3}$$

These admittances are realized by the standard synthesis procedure of Chapter III. However, here a current-controlled source is used as the active device. The SEC network for this particular case required twenty-two resistors and capacitors and two negative resistors. The SEC is shown in Figure 20.

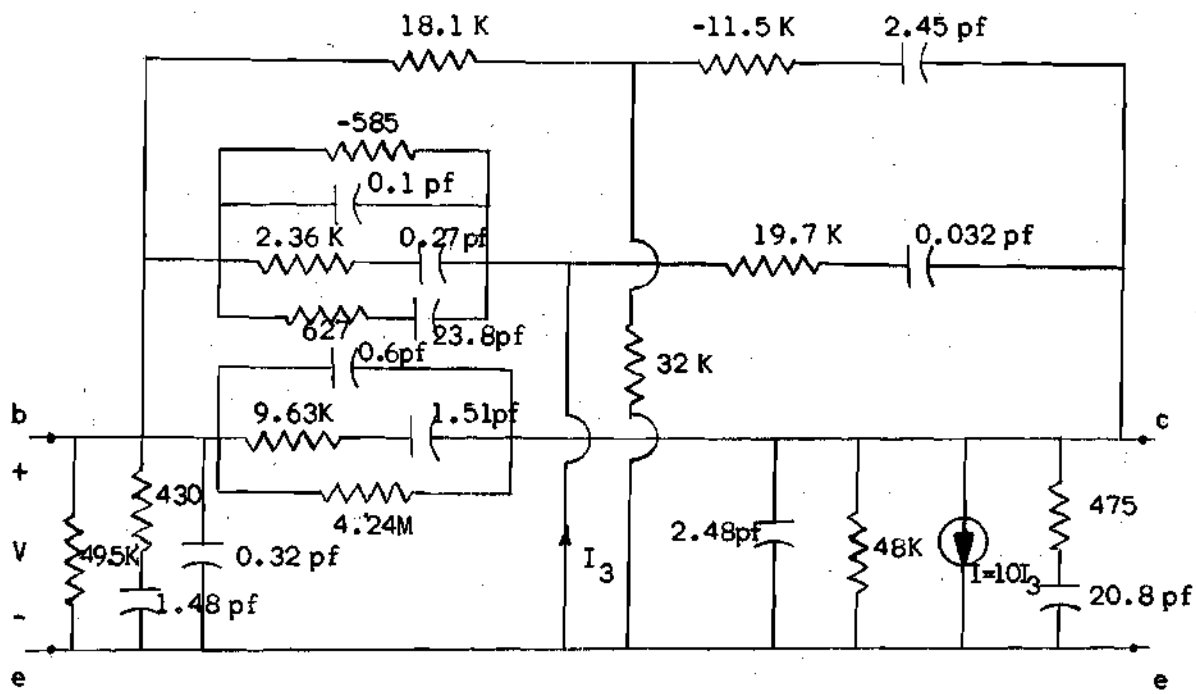


Figure 20. A Synthesized High-frequency Equivalent Circuit.

APPENDICES

APPENDIX I

±R,C (RC) TWO-PORT WITH A VOLTAGE-CONTROLLED CURRENT SOURCE

It will now be shown that a 2 X 2 matrix of real rational functions, that satisfies the realizability conditions for this class of network, can be realized as a 2 X 2 impedance matrix of a ±R,C (RC) two-port network with one voltage-controlled current source. These realizability conditions are

1. The given matrix must be expressible as the sum of a ±R,C (RC) impedance matrix and a matrix of rank one.

2. The number of the non-compact poles that are common to both driving-point impedances of the ±R,C (RC) impedance matrix must be equal to or greater than the order of the elements of the matrix of rank one when the latter is expressed with a common denominator.

To do this it is sufficient to show first that a ±R,C (RC) two-port network with one voltage-controlled current source can be expressed as the sum of a ±R,C (RC) impedance matrix and a matrix of rank one.

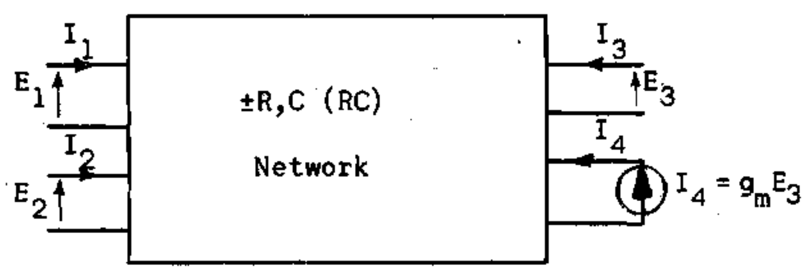


Figure 21. Arrangement of an Active Two-port Network Containing ±R,C (RC) and One Voltage-Controlled Current Source.

The circuit arrangement used in this discussion is shown in Figure 21. The four-port network is a ±R,C (RC) network where port 3 contains the controlling voltage E_3 and the controlled current source I_4 is connected to port 4. The impedance matrix of the active four-port network is defined by the equation

$$[E] = [Z][I]$$

or

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \qquad (93)$$

The open-circuit impedance matrix for the ±R,C (RC) two-port network with the controlled source can be written as

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = [Z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad (94)$$

To establish the character of this impedance matrix, $[Z]$, it is necessary to impose the requirements that $I_3 = 0$ and $I_4 = g_m E_3$ in Equation (93). Now solving for $[Z]$, it is possible to write

$$[Z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} + \frac{g_m}{1 - g_m z_{34}} \begin{bmatrix} z_{14} \\ z_{24} \end{bmatrix} \begin{bmatrix} z_{13} & z_{23} \end{bmatrix} \qquad (95)$$

Clearly in Equation (95), the matrix

$$[z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \tag{96}$$

represents a ±R,C (RC) open-circuit impedance matrix. The matrix product

$$[z^{(o)}] = \frac{g_m}{1 - g_m z_{34}} \begin{bmatrix} z_{14} \\ z_{24} \end{bmatrix} [z_{13} \quad z_{23}] \tag{97}$$

has a rank of one.

The form of Equation (95) is identical to that of Equations (6) and (71) for the development of the admittance cases. Matrix $[Z]$ is the sum of a ±R,C (RC) impedance matrix and a matrix of rank one. It is an easy matter to identify the individual impedances in matrices $[z]$ and $[z^o]$, in a similar manner as presented in Chapters II and IV for the admittance case, as

$$z_{14} = \frac{\Upsilon}{g_m} \times \frac{P_b}{g_2} \qquad z_{13} = \frac{1}{\Upsilon} \times \frac{P_a}{g_1}$$

$$z_{24} = \frac{\Upsilon}{g_m} \times \frac{P_d}{g_2} \qquad z_{23} = \frac{1}{\Upsilon} \times \frac{P_c}{g_1}$$

$$z_{34} = \frac{1}{g_m} (1 - Q/g)$$

In order to ensure that the 4 X 4 matrix $[Z]$ be realizable, it is necessary to show that not only $[Z]$ can always be made dominant, but also that z_{33} and z_{44} can be made as large as desired and, the transfer impedances z_{13}, z_{23}, z_{14}, z_{24}, and z_{34} can be made sufficiently small by selecting γ and g_m large.

Since the dominance condition alone is not always sufficient for the realizability of an open-circuit impedance matrix, it is necessary to prove that the 4 X 4 impedance matrix [Z], can always be made sufficiently dominant so as its inverse is realizable as a short-circuit admittance matrix. This is accomplished by first identifying [Z] as

$$[Z] = \begin{bmatrix} z_{11} & z_{12} & az_{13} & bz_{14} \\ z_{21} & z_{22} & cz_{23} & dz_{24} \\ az_{13} & cz_{23} & fz_{33} & ez_{34} \\ bz_{14} & dz_{24} & ez_{34} & hz_{44} \end{bmatrix} \tag{98}$$

where a, b, c, d, and e are real constants and h and f are real positive constants.

Assume that [Z] is nonsingular and solving for $[Z]^{-1} = [Y]$ where

$$[Y] = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix} \tag{99}$$

It is possible to write

$$y_{11} = \frac{1}{\Delta}\Big[hf\ z_{22}z_{33}z_{44} - z_{22}(ez_{34})^2 - hz_{44}(cz_{23})^2 + 2z_{34}ez_{24}dz_{23}c - (dz_{24})^2 z_{23}f\Big] \tag{100}$$

$$-y_{12} = \frac{-1}{\Delta}\Big[-az_{13}ez_{34}dz_{24} - hf\ z_{12}z_{33}z_{44} + z_{12}(ez_{34})^2 + az_{13}cz_{23}hz_{44} - dz_{14}cz_{23}ez_{34} - bz_{14}fz_{33}dz_{24}\Big] \tag{101}$$

$$-y_{13} = \frac{-1}{\Delta}\Big[z_{12}cz_{23}hz_{44} - z_{12}dz_{24}ez_{34} - az_{13}z_{22}hz_{44} + az_{13}(dz_{24})^2 + bz_{14}z_{22}ez_{34} - bz_{14}cz_{23}dz_{24}\Big] \tag{102}$$

$$-y_{14} = \frac{-1}{\Delta}\Big[-z_{12}cz_{23}ez_{34} + z_{12}fz_{33}dz_{24} + az_{13}z_{22}ez_{34} - az_{13}dz_{24}cz_{23} - bz_{14}z_{22}fz_{33} - bz_{14}(cz_{23})^2\Big] \tag{103}$$

$$y_{22} = \frac{1}{\Delta}\Big[z_{11}fz_{33}hz_{44} - z_{12}(ez_{34})^2 - (ez_{34})^2 hz_{44} -(bz_{14})^2 fz_{33} + 2ez_{34}bz_{14}az_{13}\Big] \tag{104}$$

$$-y_{23} = \frac{-1}{\Delta}\Big[-z_{11}cz_{23}hz_{44} + z_{11}dz_{24}ez_{34} + az_{13}z_{12}hz_{44} - az_{13}dz_{24}bz_{14} - bz_{14}ez_{34}z_{12} + (bz_{14})^2 cz_{23}\Big] \tag{105}$$

$$-y_{24} = \frac{-1}{\Delta}\Big[z_{11}cz_{23}ez_{34} - z_{11}dz_{24}fz_{33} - az_{13}z_{12}ez_{34} \quad (az_{13})^2 dz_{24} + bz_{14}z_{12}fz_{33} - bz_{14}cz_{23}az_{13}\Big] \tag{106}$$

$$y_{33} = \frac{1}{\Delta}\Big[(z_{11}z_{22} - z_{12}^2)hz_{44} - z_{11}(dz_{24})^2 + 2z_{12}dz_{24}bz_{14} -(bz_{14})^2\ z_{22}\Big] \tag{107}$$

$$-y_{34} = \frac{-1}{\Delta}\Big[(z_{11}z_{22} - z_{12}^2)ez_{34} - z_{11}dz_{24}cz_{23} + z_{11}dz_{24}az_{13} \qquad (108)$$

$$bz_{14}z_{12}cz_{23} - bz_{14}z_{22}az_{13}\Big]$$

$$y_{44} = \frac{1}{\Delta}\Big[(z_{11}z_{22} - z_{12}^2)fz_{33} - z_{11}(cz_{23})^2 - (az_{13})^2 z_{22} \qquad (109)$$

$$+ 2z_{12}cz_{23}az_{13}\Big]$$

$$\Delta = \Big[(z_{11}z_{22} - z_{12})^2 hf\ z_{33}z_{44} - (z_{11}z_{22} - z_{12}^2)(ez_{34})^2 \qquad (110)$$

$$+ (cz_{23})^2[-z_{11}hz_{44} + (bz_{14})^2] + (dz_{24})^2[-z_{11}fz_{33} + (az_{13})^2]$$

$$+ (bz_{14})^2[-z_{22}fz_{33}] + 2bz_{14}[z_{12}dz_{24}fz_{33} - z_{12}cz_{23}ez_{34}$$

$$- az_{13}cz_{23}dz_{24} + az_{13}z_{22}ez_{34}] - (az_{13})^2 z_{22}hz_{44}$$

$$+ 2bz_{14}az_{13}(cz_{23}hz_{44} - ez_{34}dz_{24}) + 2z_{11}ez_{34}dz_{24}dz_{23}\Big]$$

Now is a, b, c, d, and e are selected to be small in comparison with h and f, which are selected large, and if each z_{jk} is replaced by its residue $k_{jk}^{(i)}$, where $i = 0, 1, 2, \ldots, n, \infty$ in Equations (100) through (110), it is possible to show that the dominance of the residues in the diagonal of the matrix [Y] can always be satisfied. This is accomplished by first expressing

$$\text{res}\,[y_{11}] \geq |\text{res}\,[y_{12}]| + |\text{res}[y_{13}]| + |\text{res}[y_{14}]| \qquad (111)$$

$$\text{res}\,[y_{22}] \geq |\text{res}\,[y_{12}]| + |\text{res}[y_{23}]| + |\text{res}[y_{24}]| \qquad (112)$$

$$\text{res}\,[y_{23}] \geq |\text{res}\,[y_{13}]| + |\text{res}[y_{23}]| + |\text{res}[y_{34}]| \qquad (113)$$

$$\text{res}\,[y_{44}] \geq |\text{res}\,[y_{14}]| + |\text{res}\,[y_{24}]| + |\text{res}\,[y_{34}]| \qquad (114)$$

[The situation presented here is for the RC case with poles located on the negative real axis including the pole at infinity of the complex frequency plane. A similar development is possible for the ±R,C case.] Neglecting the insignificant small terms, there results

$$\text{res}[y_{11}] \cong hf\ k_{33}^{(i)}\ k_{44}^{(i)}\ k_{22}^{(i)} \geq |hf\ k_{33}^{(i)}\ k_{44}^{(i)}\ k_{12}^{(i)}| \qquad (115)$$

$$\text{res}[y_{22}] \cong hf\ k_{33}^{(i)}\ k_{44}^{(i)}\ k_{11}^{(i)} \geq |hf\ k_{33}^{(i)}\ k_{44}^{(i)}\ k_{12}^{(i)} \qquad (116)$$

$$\text{res}[y_{33}] \cong [k_{11}^{(i)}\ k_{22}^{(i)} - (k_{12}^{(i)})^2]hk_{44}^{(i)} \geq \qquad (117)$$

$$|[k_{11}^{(i)}k_{22}^{(i)} - (k_{12}^{(i)})^2]ek_{34}^{(i)}|$$

$$\text{res}[y_{44}] \cong [k_{11}^{(i)}\ k_{22}^{(i)} - (k_{12}^{(i)})^2]fk_{33}^{(i)} \geq \qquad (118)$$

$$|[k_{11}^{(i)}k_{22}^{(i)} - (k_{12}^{(i)})^2]ek_{34}^{(i)}|$$

These equations reduce to the following requirements

$$k_{22}^{(i)} \geq |k_{12}^{(i)}| \qquad (119)$$

$$k_{11}^{(i)} \geq |k_{12}^{(i)}| \qquad (120)$$

$$hk_{44}^{(i)} \geq |ek_{34}^{(i)}| \qquad (121)$$

$$fk_{33}^{(i)} \geq |ek_{34}^{(i)}| \qquad (122)$$

For matrix [z] to be a ±R,C (RC) open-circuit impedance matrix, one realizability condition is

$$k_{11}^{(i)}\, k_{22}^{(i)} - (k_{12}^{(i)})^2 \geq 0$$

for $i = 0,1,2,\ldots,n,\infty$. (For RC case $k_{11}k_{22} - (k_{22})^2 \geq 0$ must also be satisfied). Recall that in the development for realizing a short-circuit admittance matrix [Y], the short-circuit ±R,C (RC) admittance matrix [y] in Equation (9) is expressed as

$$[y] = [y'] + [y''] + [y'''] \qquad (9)$$

where

$$[y''] = \begin{bmatrix} y''_{11} & 0 \\ 0 & y''_{22} \end{bmatrix}$$

Since the off-diagonal terms of this matrix are zero, $k_{12}^{(i)} = 0$ for $i = 0,1,2,\ldots,n,\infty$ including $k_{12} = 0$. Thus, Equations (119) and (120) now require only that

$$k_{11}^{(i)} \geq 0 \quad k_{22}^{(i)} \geq 0$$

This condition is always satisfied because [z] is a ±R,C (RC) open-circuit impedance matrix. Clearly Equations (121) and (122) will always be satisfied also as $h \gg |e|$ and $f \gg |e|$.

Therefore it can be concluded that it is always possible to realize the 4 X 4 matrix [Z] by realizing it's inverse as a short-circuit admittance matrix [Y].

APPENDIX II

THE SYNTHESIS OF ±R,C NETWORKS

The general synthesis technique developed by Phillips and Su for synthesizing a two-port network when all three short-circuit admittance functions, y_{11}, $-y_{12}$, and y_{22}, are prescribed and satisfy the realizability conditions for a realization by ±R,C two-port network involves developing a separate component network for each type of pole that is realizable by a ±R,C network. The short-circuit admittance functions are then realized by the parallel connection of the required component networks. The component networks will be presented in Figures 22, 23, 24, and 25 in the order that the functions appear in Equation (1). All of these networks are developed for the case of compact poles. If the pole is not compact, sufficient admittance should be subtracted from y_{11} or y_{22} to make the pole compact. This admittance can be realized in shunt across the input or output of the network. The number of negative resistors are reduced by combining parallel resistors.

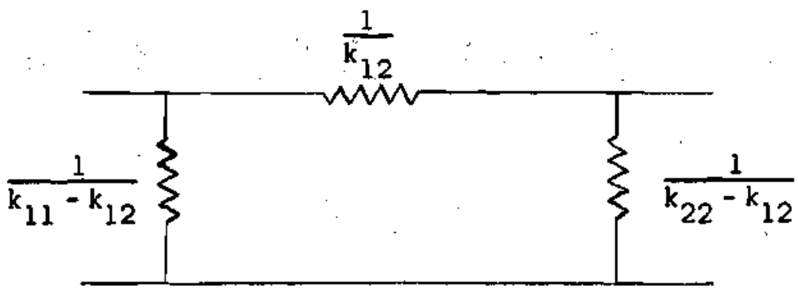


Figure 22. Network for Realizing Conductance Terms.

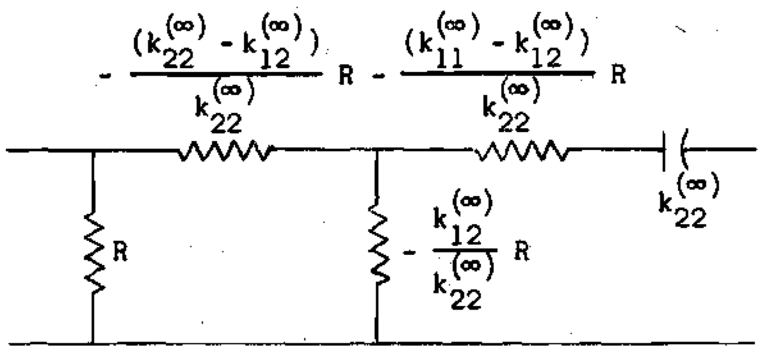


Figure 23. Network for Realizing the Pole at Infinity. Choose R Negative.

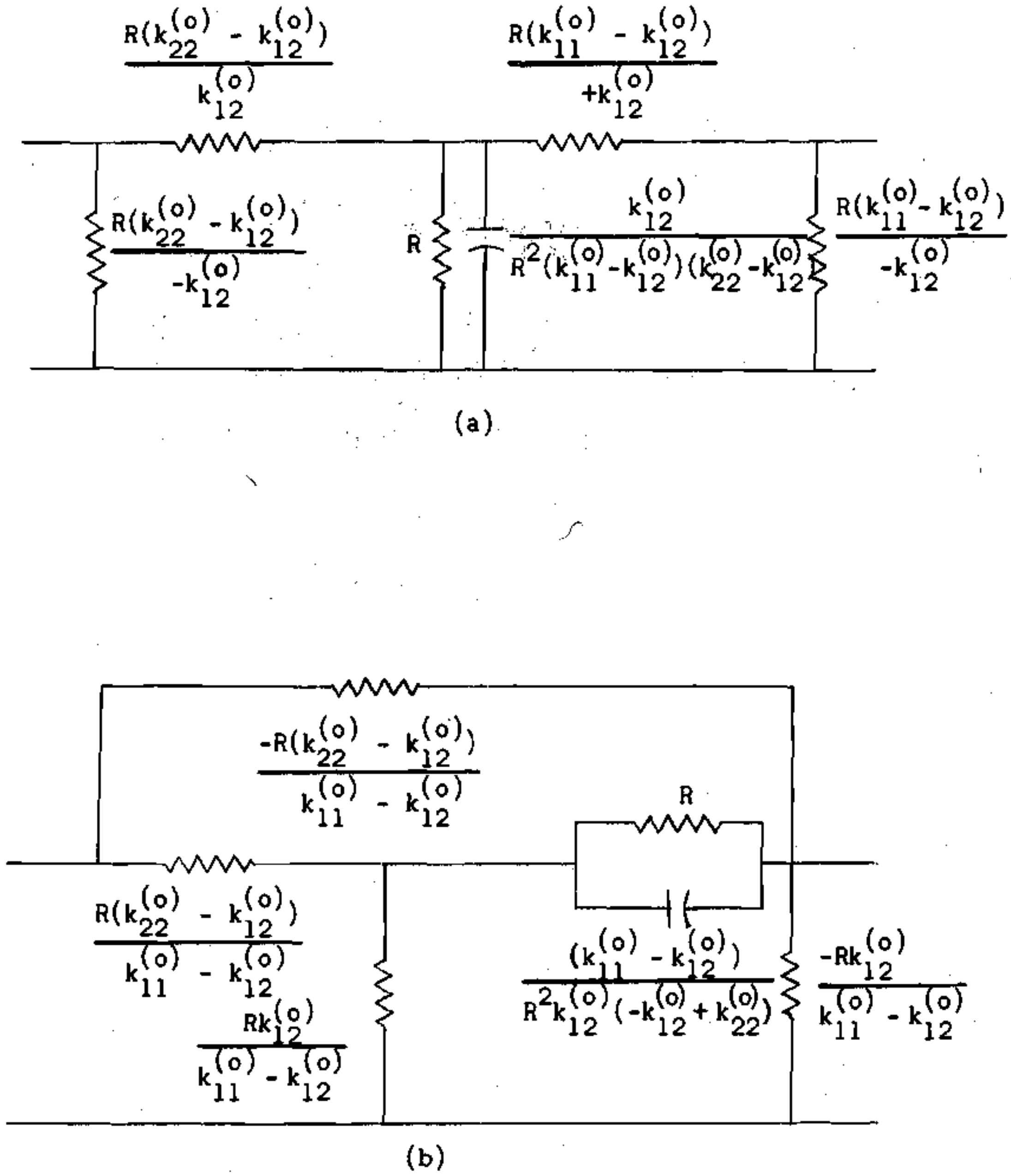


Figure 24. Networks for Realizing a Pole at the Origin. Network (a) is used when $k_{12}^{(o)}$ is Positive. Network (b) is used when $k_{12}^{(o)}$ is Negative and $k_{11}^{(o)} > |k_{12}^{(o)}|$. When $k_{11}^{(o)} < |k_{12}^{(o)}|$, Network (b) is used with the Input and Output Ports Reversed and $k_{11}^{(o)}$ and $k_{22}^{(o)}$ Interchanged. Choose R Negative.

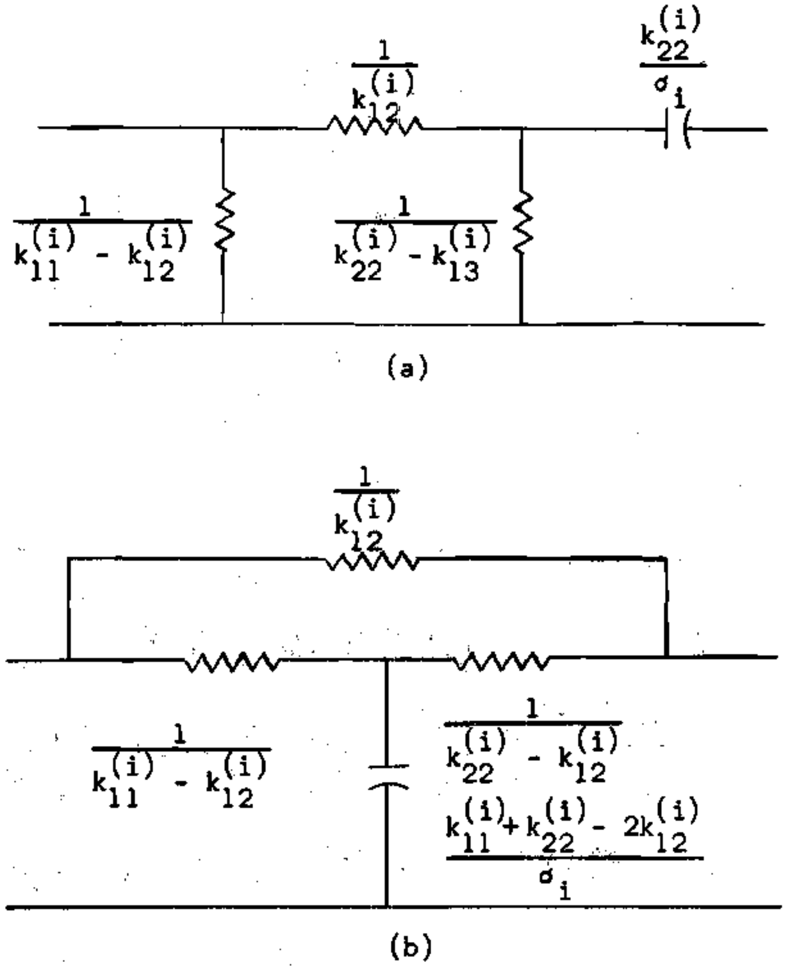


Figure 25. Networks for Realizing a Pole on the Real Axis. Network (a) is used for $k_{12}^{(i)}$ Positive and $k_{11}^{(i)} < |k_{12}^{(i)}|$, if $k_{11}^{(i)} > |k_{12}^{(i)}|$ then Input and Output Ports are Reversed and $k_{11}^{(i)}$ and $k_{22}^{(i)}$ are Interchanged. Network (b) is used for $k_{12}^{(i)}$ Negative.

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