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PROJECT INITIATION

Date February 28, 1964

Project Title: A Study of Differential Equations Related to the Response of Shells of Revolution to Blast Loading

Project No.: A-765

Project Director: J. T. Wang

Sponsor: National Aeronautics and Space Administration

Effective: 2-1-64 Estimated to run until: 1-31-65

Type agreement: Research Grant No. NsG 571

Amount: $17,840
Reports: Status Reports - required semiannually
Final Report - required upon completion

Contact Person: Dr. T.L.K. Smull, Director
Grants and Research Contracts Division
Office of Space Science and Applications
National Aeronautics and Space Administration
Washington, D. C. 20546

Assigned to Mechanical Sciences Division

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GEORGIA INSTITUTE OF TECHNOLOGY
Engineering Experiment Station

PROJECT TERMINATION

Date December 2, 1965

PROJECT TITLE: A Study of Differential Equations Related to the Response of Shells of Revolution to Blast Loading

PROJECT NO: A-765

PROJECT DIRECTOR: J. T. Wang

SPONSOR: National Aeronautics and Space Administration

TERMINATION EFFECTIVE: 11-30-65

CHARGES SHOULD CLEAR ACCOUNTING BY: All acceptable charges have cleared and overrun must be transferred to continuation project (B-910) being conducted through the School of Engineering Mechanics.

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GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA 30332

A-765

August 3, 1964

Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington 25, D. C.

Attention: Code SC

Subject: Research Grant No. NsG-571
Semiannual Status Report for the Period
February 1, 1964 to August 1, 1964
(Georgia Tech Project No. A-765)

Gentlemen:

The following represents a summary of the project status during the subject report period.

1. The Status

The official approval of the research grant was received April 8, 1964. Since the spring quarter started March 31, 1964, faculty work-loads had already been established and would have been difficult to reallocate. Also, it was not possible to obtain a graduate student to assist on the project so close to the end of the academic year. It was for these reasons that work on the project was not started until June 15, 1964. Due to this delay, it is probable that an extension of time may be needed. No money was drawn from the project funds during the period February 1 to June 15, 1964.

2. Personnel

Mr. Wolfram Stadler, graduate student in the School of Engineering Mechanics, was employed as a graduate assistant (1/2 time) beginning June 15, 1964. He received the degree Master of Science of Aerospace Engineering from Georgia Tech in June 1964, and is continuing his Ph.D. program in Engineering Mechanics. He will be assisting in all phases of investigation pertaining to the project. The work accomplished by him will be considered as contributory material for his Ph.D. thesis. One additional graduate assistant may be employed, starting September 1964.

3. Progress During the Reporting Period

(a) Mr. Stadler worked for approximately two weeks on a bibliography of references pertinent to the project. He will also be responsible for keeping this bibliography up to date.

(b) The governing differential equations related to the axisymmetric response of a conical shell subject to blast load were formulated, based on the membrane theory of shells. The technique of separation of variables was employed in obtaining the general solution. An analysis is being made, applicable to the response of a conical shell. It is intended that a manuscript for this portion of information will be prepared.

(c) A further method being applied to obtain the solution of the system, is that of integral transforms. The problem was not suitable to the application of Hankel transforms, and it seems, at the moment, that the use of Laplace transforms may result in a satisfactory solution.

4. Plans for Next Reporting Period

(a) It is hoped to extend the applicability of integral transforms to the solution of dynamic response problems involving blast loads.

(b) The study will be extended to other shells of revolution. For the most part, elastic membrane theory will be used in formulating the problem.

(c) Progress on the project has been quite satisfactory and no major difficulties are foreseen at the present time.

The expenditures on the subject grant to date are as follows:

Direct Salary and Wages

Project Director	$ 1,697.21
Graduate Assistant	600.00
	$ 2,297.21
Materials and Supplies	$ 1.15
Overhead (20% of Direct Cost)	$ 459.67
Grand Total	$ 2,758.03

Respectfully submitted.

James Ting-Shun Wang
Project Director

JTSW/c

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA 30332

February 3, 1965

Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington 25, D. C.

Attention: Code SC

Subject: Research Grant No. NsG-571
Semiannual Status Report for the Period
August 1, 1964 to January 31, 1965
(Georgia Tech Project No. A-765)

Gentlemen:

The following represents a summary of the project status during the subject report period.

1. The Status

As was pointed out in the previous semiannual report, dated August 3, 1964, work on the project did not start until June 15, 1964. Professor Robert E. Stiemke, Administrator of Research and Secretary of the Georgia Tech Research Institute has submitted a request dated December 21, 1964, to Dr. T. L. K. Smull for an extension of time from January 31, 1965 to June 15, 1965.

2. Personnel

Mr. Chi-Wen Lin was employed as a graduate assistant (1/3 time) beginning September 1, 1964. He received a Master of Science degree in Engineering from the University of Florida in August of 1964 and is continuing his Ph.D. program in Engineering Mechanics here at the Georgia Institute of Technology. He will be assisting in all phases of investigation pertaining to the project. The work accomplished by him will be considered as contributory material for his Ph.D. thesis.

3. Progress During the Reporting Period

(a) On September 1, 1964, a manuscript entitled "Axisymmetric Response of a Conical Shell to Blast Load," was submitted to the Acoustic Society of America for review. No word has been heard from the Society concerning the acceptability of the manuscript.

(b) A simply supported cylindrical panel under a time-dependent load has been studied. The general solution to the governing differential equation has been obtained by use of an integral transform technique.

A specific example is currently being worked out. Preliminary computer results based on the general solution seem to compare favorably with experimental data presented in the paper "Blast Loading of Small Buried Arches, Journal of Structural Division, ASCE, October, 1964," by Jay R. Allgood. More computational work concerning wider aspects of the specific example will be carried on.

(c) The general problem concerning the axisymmetric response of paraboloidal shells of revolution has been formulated according to membrane theory. The non-constant coefficients involved in the system of governing differential equations are lengthy and complex. To obtain a general solution to the system of differential equations is difficult. The current attempt has been focused on a truncated paraboloidal shell of revolution. Since the metric tensor varies slowly from point to point on the middle surface of the shell at the region away from the tip, an approximation is made that the derivatives of the metric tensor are negligible. The governing differential equations have been simplified to some extent. It is believed that a solution to the resulting equations can be obtained.

(d) The axisymmetric response of a spherical shell to a time-dependent load based on bending theory has been studied. It is found that a series solution in terms of Legendre polynomial is capable of describing the response of a complete spherical shell, a simply supported hemi-spherical shell, and a clamped hemi-spherical shell with modified boundary conditions.

4. <u>Plans for Next Reporting Period</u>

(a) Additional computer work will be made on the specific example of the cylindrical panel in order to have a better understanding concerning the response of a shell subject to blast loading.

(b) The solutions for a truncated paraboloidal shell of revolution based on the approximation mentioned in 3 (c) will be sought. It is hoped that the investigation of the truncated one will give better insight into the general problem so that solutions may be obtained for general paraboloidal shells of revolution.

(c) Numerical examples will be made for a complete spherical and a hemi-spherical shell. An attempt will be made to find solutions for a segment of spherical shell. The task is seemingly more difficult than that of a complete or a hemi-spherical shell.

(d) The work on the project is progressing satisfactorily.

The expenditures on the subject grant to date are estimated as follows:

Direct Salary and Wages	
Project Director	$ 4,577.79
Graduate Assistant (1/2 time)	2,475.00
Graduate Assistant (1/3 time)	1,000.00
	$ 8,052.79
Materials and Supplies	$ 20.00
Overhead (20% of Direct Cost)	$ 1,614.56
	$ 9,667.35
Computer	$ 100.00
Grand Total	$ 9,767.35

The Grand Total includes estimated expenditures during January of 1965. The exact figures for the January month will be shown in the transaction in February.

Respectfully submitted,

James Ting-Shun Wang
Project Director

JTSW/c

GEORGIA INSTITUTE OF TECHNOLOGY
ENGINEERING EXPERIMENT STATION
ATLANTA, GEORGIA 30332
August 12, 1965

Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington 25, D. C.

Attention: Code SC

Subject: Research Grant No. NsG-571
Semiannual Status Report for the Period
February 1, 1965 to July 31, 1965
(Georgia Tech Project No. A-765)

Gentlemen:

The following represents a summary of the project status during the subject report period.

1. The Status

The official expiration date of the research grant was February 1, 1965. Approval from the Office of Grants and Research Contracts dated March 31, 1965 for continuing research under subject grant through June 15, 1965, was received.

An official proposal for the second year (beginning June 16, 1965 and ending June 15, 1966) was submitted to the Office of Grants and Research Contracts on February 18, 1965. The approval for the extension for one additional year was verbally confirmed by Dr. J. G. Etgen, Office of the Basic Research and Applied Mathematics.

2. Personnel

Messrs. Wolfram Stadler and Chi-Wen Lin, graduate research assistants of Engineering Mechanics, will continue to assist in all phases of investigation pertaining to the project. Mr. Lin has successfully passed the qualifying examination which is required for all possible Ph.D. candidates in Engineering Mechanics at the Georgia Institute of Technology. His work load for the project has been changed from one-third time to five-twelfth time since June 16, 1965.

3. Progress During the Reporting Period

(a) No final word has been received from the Acoustic Society of America concerning the acceptability of the manuscript "Axisymmetric Response of a Conical Shell to Blast Load."

(b) A manuscript entitled "Dynamic Response of a Cylindrical Shell Segment Subjected to an Arbitrary Loading," was submitted to and was accepted by the Ninth Midwestern Mechanics Conference. The paper will be presented at the conference, August 18, 1965, at the University of Wisconsin, Madison, Wisconsin and will be published in the Proceedings.

(c) The axisymmetric response of a complete cylindrical shell fixed at both ends has been studied. When the longitudinal inertial term is omitted, the preliminary study seems to indicate that a general solution can be obtained by use of an integral transform technique.

(d) The study of the axisymmetric response of paraboloidal shells of revolution has been continued. The current effort has been concentrated in analyzing the free vibration case. The two coupled governing differential equations formulated according to the membrane theory may be decoupled. The resulting equations become extremely lengthy. For a truncated shell where the meridional lines are slightly curved, additional approximation may be made by neglecting certain small quantities. The governing differential equations may be reduced to canonical form. A numerical scheme by use of finite difference technique has been established for finding the natural frequencies. The recent computer results for a relatively shallow paraboloidal shell of revolution seem to compare favorably with the lowest natural frequency obtained by C. N. DeSilva and G. E. Tersteeg in the paper "Axisymmetric Vibrations of Thin Elastic Shells," Jour. Acoustical Society of America, April 1964, for a spherical shell of comparable dimensions.

The general problem concerning the axisymmetric vibration of a paraboloidal shells of revolution based on bending theory has been formulated. The equations of motion obtained are very lengthy and complex. Additional effort in checking the final expressions is needed. Only numerical methods will be attempted in finding the solution of the problem.

(e) The study of the axisymmetric response of a spherical shell has been continued. The recent effort has been concentrated on hemi-spherical shell.

4. Plans for Next Reporting Period

(a) A specific example concerning the axisymmetric response of a complete cylindrical shell with fixed edges will be worked out. Further investigation by including all inertial terms for axisymmetric as well as asymmetric cases will be attempted.

(b) Additional computer work will be made on the paraboloidal shells of revolution based on membrane as well as bending theories.

(c) The study concerning a hemi-spherical shell will be continued. The results may be extended to analyze the free vibration of a composite shell (cylinder with a spherical bottom). An attempt will also be made to find solutions for a spherical shell segment.

(d) The work on the project is progressing satisfactorily.

The expenditures on the subject grant through July 31, 1965 are estimated as follows:

Direct Salary and Wages	
Project Director	$ 7,148
Graduate Assistant	4,050
Graduate Assistant	2,175
Other Personal for Computer Programming and Report Reproduction	400
	$ 13,773
Materials and Supplies	$ 100
Overhead (20% of Direct Cost)	$ 2,775
	$ 16,648
Computer	$ 500
Grand Total	$ 17,148

Respectfully submitted,

James Ting-Shun Wang
Project Director

JTSW/c

AXISYMMETRIC RESPONSE OF CONICAL SHELLS TO BLAST LOAD

By James Ting-Shun Wang
Assistant Professor
Engineering Mechanics

Project A-765

Research Grant NsG-571

August 1964

Performed for
Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington, D. C.

ABSTRACT

The dynamic response of thin elastic conical shells subject to blast load is formulated according to membrane theory. The solutions are obtained by the technique of separation of variables with Poisson's ratio neglected.

NOMENCLATURE

All symbols are defined in the text where they first appear, and some of the major symbols are listed below:

a_n, b_n	Generalized Fourier coefficients
E	Modulus of elasticity
e_{xx}, $e_{\theta\theta}$	Strain components
h	Thickness of shell
ℓ	Length of shell
$\mathcal{L}$, $\mathcal{D}$	Linear differential operators
m, n	Indices
N_{xx}, $N_{\theta\theta}$	Stress resultants
p	Transient pressure
p_o	Peak overpressure
$r(x)$	Weight function
t	Time
t_d	Duration of transient pressure
v, w	Displacement components
x, θ	Coordinates
α_n, β_n	Separation constants
α	Half apical angle
ρ	Mass density of material
ν	Poisson's ratio
ω_n, $\bar{\omega}_n$	Circular frequencies

INTRODUCTION

A theoretical study of the axisymmetric response of conical shells to blast load is presented in this investigation. The shell is elastic and homogeneous. Two partial differential equations which govern the displacement components in normal and meridianal directions are derived. The solutions are obtained by use of the technique of separation of variables. The governing differential equations are essentially unseparable. To overcome this difficulty, an approximation is made so that the equation of motion for free vibration is satisfied to its mean value along a generator. The results are expressed in series form.

Some progress has been made in the analytic treatment of the response of cylindrical and spherical shells to blast load; it appears that very limited work on the conical shell has been presented. The investigations made by Bluhm [4], Herrmann and Mirsky [8] are good for conical shells with small apical angles.

FORMULATION OF THE PROBLEM

The axisymmetric response of a thin conical shell under blast load will be formulated based upon elastic membrane shell theory. The meridianal lines and parallel circles will be used as the coordinate system (x, θ) as shown in Figure (1). The usual assumptions for thin shells, such as given by Timoshenko [14], are used. Due to symmetry of loading and the geometry of the structure, the displacement or motion is independent of coordinate θ. Furthermore, no shears exist on the meridianal lines.

By summing forces in x and normal directions (Figure (1)), the equations of motion are found to be

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{x} N_{xx} - \frac{1}{x} N_{\theta\theta} = \rho h \frac{\partial^2 v}{\partial t^2} - p_x \tag{1}$$

$$\frac{N_{\theta\theta}}{x \tan \alpha} = \rho h \frac{\partial^2 w}{\partial t^2} - p_n \tag{2}$$

where v and w are respectively the components of displacements in the x and normal directions. p_x and p_n are components of loading in the x and n directions. N_{xx} and $N_{\theta\theta}$ are stress resultants in the x and circumferential directions. ρ is the mass density of the material.

The stress, strain, and displacement relationships are

$$e_{xx} = \frac{\partial v}{\partial x} \tag{3a}$$

$$e_{\theta\theta} = \frac{1}{x}(v - w \cot \alpha) \tag{3b}$$

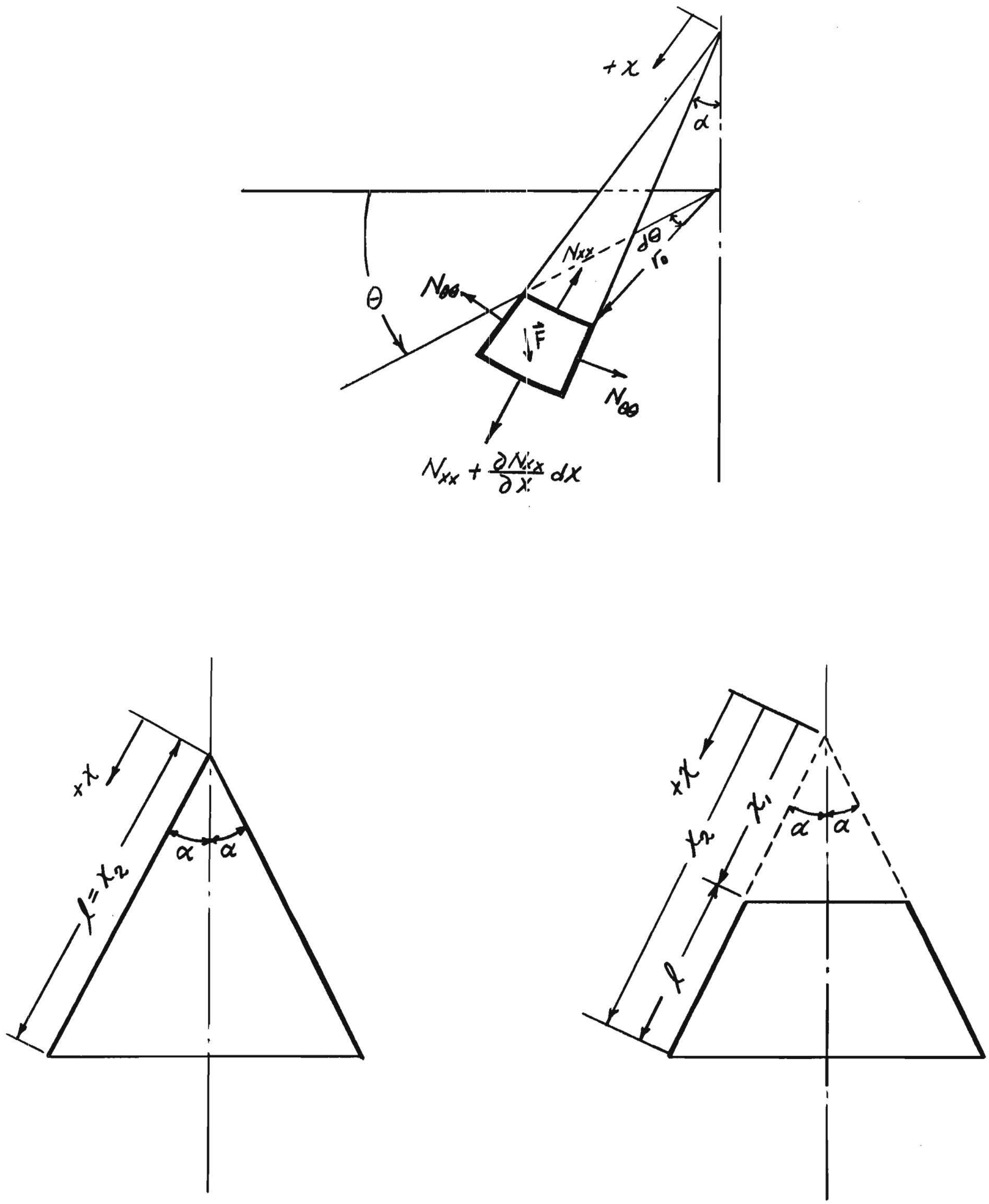


Figure 1. Coordinate System and Symbols.

$$N_{xx} = \frac{Eh}{1 - \nu^2} (e_{xx} + \nu e_{\theta\theta}) = \frac{Eh}{1 - \nu^2}\left[\frac{\partial v}{\partial x} + \frac{\nu}{x} (v - w \cot \alpha)\right] \quad (4a)$$

$$N_{\theta\theta} = \frac{Eh}{1 - \nu^2} (e_{\theta\theta} + \nu e_{xx}) = \frac{Eh}{1 - \nu^2}\left[\frac{1}{x} (v - w \cot \alpha) + \nu \frac{\partial v}{\partial x}\right] \quad (4b)$$

where e_{xx} and $e_{\theta\theta}$ are components of strain in x and θ directions, respectively. E is the modulus of elasticity and ν is the Poisson's ratio.

Substitution of Equations (4a) and (4b) into Equations (1) and (2) yields the following governing differential equations:

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} - \frac{\nu}{x} \cot \alpha \frac{\partial w}{\partial x} - \frac{v}{x^2} + \frac{w}{x^2} \cot \alpha = \frac{\rho(1 - \nu^2)}{E}\frac{\partial^2 v}{\partial t^2} - \frac{p_x(1 - \nu^2)}{Eh} \quad (5a)$$

$$\frac{\nu}{x}\frac{\partial v}{\partial x} + \frac{v}{x^2} - \frac{w}{x^2} \cot \alpha = \frac{\rho(1 - \nu^2)}{E} \tan \alpha \frac{\partial^2 w}{\partial t^2} - \frac{p_n(1 - \nu^2)}{Eh} \tan \alpha \quad (5b)$$

The possible boundary conditions are as follows:

1. Closed Cone:

$v(o, t)$ is finite

$v(x_2, t) = 0$ if supported at $x = x_2$

$$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x} (v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t}} = 0 \text{ if free at } x = x_2.$$

$$v(x, o) = \frac{\partial v}{\partial t} (x, o) = w(x, o) = \frac{\partial w}{\partial t} (x, o) = 0$$

2. Truncated Cone:

Case I. Supported in x-direction at both ends.

(a) With zero initial displacements and velocity

$$v(x_1, t) = v(x_2, t) = v(x, o) = \frac{\partial v}{\partial t}(x, o) = 0$$

$$w(x, o) = \frac{\partial w}{\partial t}(x, o) = 0 \tag{6a}$$

(b) With initial displacements and zero initial velocity

$$v(x_1, t) = v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = \frac{\partial w}{\partial t}(x, o) = 0$$

$$v(x, o) = F(x), \; w(x, o) = G(x) \tag{6b}$$

Case II. Supported in x-direction at $x = x_1$ and free at $x = x_2$

(a) With zero initial displacements and velocity

$$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x}(v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t}} = 0$$

$$v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = v(x, o) = 0 \tag{6c}$$

$$w(x, o) = \frac{\partial w}{\partial t}(x, o) = 0$$

(b) With initial displacements and zero initial velocity

$$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x}(v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t_1}} = 0$$

$$v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = \frac{\partial w}{\partial t}(x, o) = 0 \tag{6d}$$

$$v(x, o) = F(x)$$

$$w(x, o) = G(x)$$

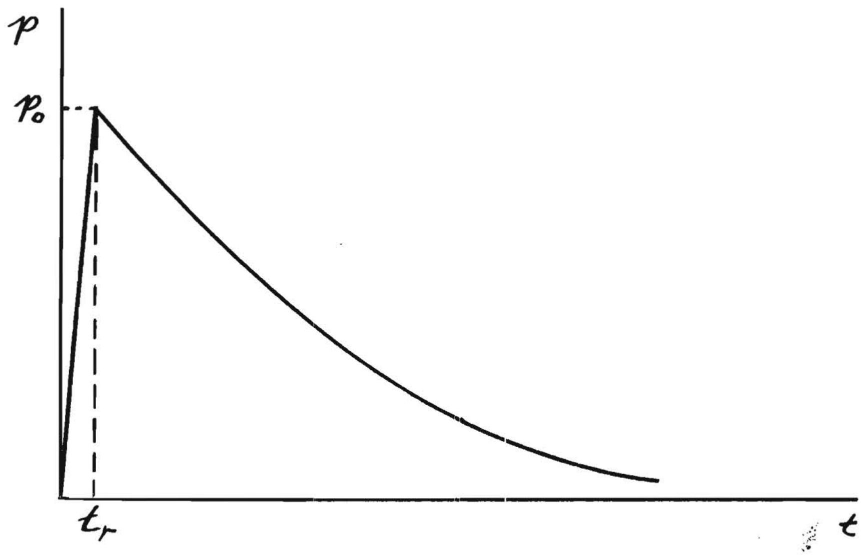
Since the shock wave front, in general, travels with very high speed, it is reasonable to assume that the blast loading function varies with respect to time only. The actual loading function is shown in Figure (2a). Since the rise time t_r is usually short, the relation between force and time may be approximated by the following continuous functions:

$$p = p_o(1 - \frac{t}{t_d})\ e^{-t/t_d}$$

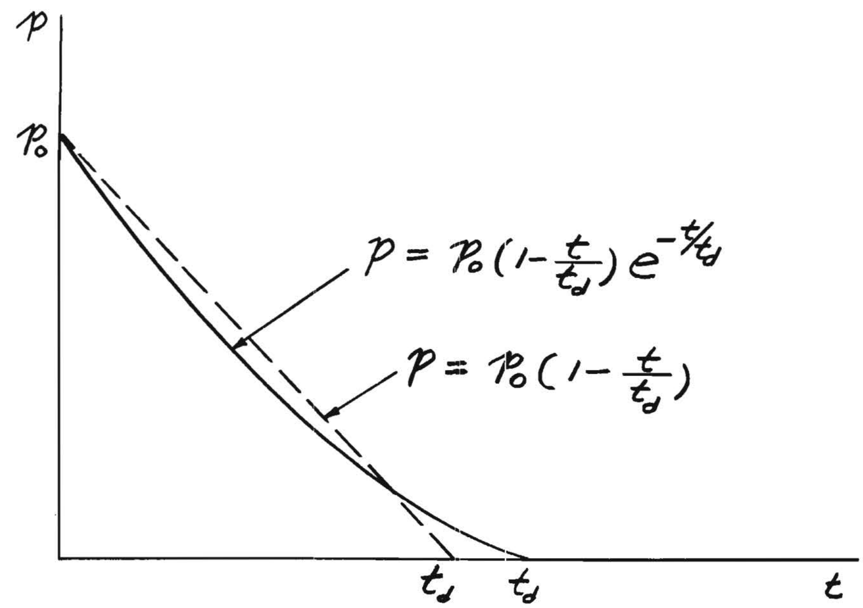
or

$$p = p_o(1 - t/t_d)$$

which are shown in Figure (2b). t_d is the duration of the load and p_o is the peak overpressure.



(a)



(b)

Figure 2. Load-Time Relation.

ANALYSIS

For a simpler case, consider Poisson's ratio to be small and thus negligible and the loading as normal to the surface. The governing differential Equations (5a) and (5b) then reduce to the following form:

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} - \frac{v}{x^2} + \frac{w}{x^2}\cot\alpha = \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} \tag{7a}$$

$$\frac{v}{x^2} - \frac{w}{x^2}\cot\alpha = -\frac{p}{Eh}\tan\alpha + \frac{\rho}{E}\tan\alpha\frac{\partial^2 w}{\partial t^2} \tag{7b}$$

The homogeneous solution, neglecting the forcing function, will be sought first. By adding Equations (7a) and (7b), we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} = \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} + \frac{\rho}{E}\tan\alpha\frac{\partial^2 w}{\partial t^2} \tag{8a}$$

Equations (8a) and the following equation will be used for seeking homogeneous solutions:

$$\frac{v}{x^2} - \frac{w}{x^2}\cot\alpha = \frac{\rho}{E}\tan\alpha\frac{\partial^2 w}{\partial t^2} \tag{8b}$$

For possible separation of variables, the homogeneous solutions are assumed in the following form:

$$w_c = \sum_{n=0}^{\infty} W_n(x)\, f_n(t) \tag{9}$$

$$v_c = \sum_{n=0}^{\infty} \bar{V}_n(x)\, \bar{g}_n(t) \tag{10}$$

Substitution of Equations (9) and (10) into Equation (8a) yields

$$\left(\frac{d^2\bar{V}_n}{dx^2} + \frac{1}{x}\frac{d\bar{V}_n}{dx}\right)\bar{g}(t) = \frac{\rho}{E}\bar{V}_n \frac{d^2 g}{dt^2} + \frac{\rho}{E}\tan\alpha\, W_n \frac{d^2 f}{dt^2} \tag{11}$$

For Equation (15) to be separable, we take

$$\frac{d^2V_n}{dx^2} + \frac{1}{x}\frac{dV_n}{dx} + \beta_n^2 \frac{\rho}{E} V_n = 0 \tag{12}$$

$$\tan\alpha\, W_n = -\alpha_n^2 \bar{V}_n = -V_n \tag{13}$$

$$\frac{d^2\bar{g}}{dt^2} + \beta_n^2 \bar{g}_n = \alpha_n^2 \frac{d^2 f}{dt^2} \tag{14}$$

where β_n and α_n are separation constants. α_n is perfectly arbitrary since we may take

$$g_n = \frac{\bar{g}_n}{\alpha_n^2}$$

and

$$V_n = \alpha_n^2 \bar{V}_n$$

Equation (10) is thus reduced to

$$v_c = \sum_{n=0}^{\infty} \bar{V}_n(x)\, \bar{g}_n(t) = \sum_{n=0}^{\infty} V_n(x)\, g_n(t) \tag{10a}$$

It is seen that the condition shown in Equation (13) will not satisfy Equation (8b). This indicates that the governing differential Equations (8a) and (8b) are essentially unseparable. To overcome this difficulty, Equation (8b) may be replaced by an equivalent condition based on a physical point of view. It is known that Equation (8b) represents the equation of motion of an infinitesimal element of the shell in the normal direction. We write the equation of motion for an element along the total length of a generator as the condition equivalent to Equation (8b), i.e.

$$\int_{x_1}^{x_2} \frac{1}{x}\,(v - w \cot\alpha)\, dx = \int_{x_1}^{x_2} \frac{\rho}{E} \tan\alpha \frac{\partial^2 w}{\partial t^2}\, dx \tag{15}$$

Substitution of Equations (9), (10a) and (13) into Equation (15) yield

$$(g_n + \cot^2\alpha\, f_n) \int_{x_1}^{x_2} \frac{V_n(x)}{x}\, dx = - \frac{d^2 f_n}{dt^2} \int_{x_1}^{x_2} \frac{\rho}{E}\, x\, V_n(x)\, dx \tag{16}$$

We shall use the absolute values of the integrands to evaluate the integrals shown in Equation (16). The integrals when divided by the length of the shell, ℓ, will be interpreted as the mean values of the functions under the integrals. Equation (16) thus reduces to

$$g_n(t) + \cot^2\alpha\, f_n(t) = - \eta_n \frac{d^2 f_n}{dt^2} \tag{17}$$

where

$$\eta_n = \frac{\int_{x_1}^{x_2} \frac{\rho}{E} \, x \, V_n \, \text{sgn} \, V_n \, dx}{\int_{x_1}^{x_2} \frac{V_n}{x} \, \text{sgn} \, V_n \, dx} \tag{18}$$

This approximation, Equations (15) through (18), will mean that Equation (8b) is satisfied to its mean value along a generator.

Elimination of $g_n(t)$ between Equations (14) and (17) yields

$$- \cot^2 \alpha \frac{d^2 f_n}{dt^2} - \eta_n \frac{d^4 f_n}{dt^4} - \beta_n^2 \cot^2 \alpha \, f_n - \beta_n^2 \eta_n \frac{d^2 f_n}{dt^2} = \frac{d^2 f_n}{dt^2} \tag{19}$$

or

$$\frac{d^4 f_n}{dt^4} + K_n(\eta_n, \beta_n) \frac{d^2 f_n}{dt^2} + H_n(\eta_n, \beta_n) \, f_n = 0 \tag{20}$$

where

$$K_n = \frac{1}{\eta_n}(\text{cosec}^2 \alpha + \beta_n^2 \eta_n) \tag{21a}$$

$$H_n = \frac{\beta_n^2 \cot^2 \alpha}{\eta_n} \tag{21b}$$

The general solution of Equation (19) becomes

$$f_n = \bar{A}_n \cos \omega_n t + \bar{B}_n \sin \omega_n t + \bar{C}_n \cos \bar{\omega}_n t + \bar{D}_n \sin \bar{\omega}_n t \tag{22}$$

where

$$\omega_n^2 = \frac{+K_n + \sqrt{(K_n)^2 - 4 H_n}}{2} \tag{22a}$$

and

$$\bar{\omega}_n^2 = \frac{+K_n - \sqrt{(K_n)^2 - 4 H_n}}{2} \tag{22b}$$

are two natural frequencies associated with each n. This fact has also appeared in the discussions given by Baker [2] and Lamb [10] for spherical shells.

$g_n(t)$ may be obtained by substituting Equation (22) into (17); we have

$$\begin{aligned} g_n(t) &= (\eta_n \omega_n^2 - \cot^2 \alpha)(\bar{A}_n \cos \omega_n t + \bar{B}_n \sin \omega_n t) \\ &+ (\eta_n \bar{\omega}_n^2 - \cot^2 \alpha)(\bar{C}_n \cos \bar{\omega}_n t + \bar{D}_n \sin \bar{\omega}_n t) \end{aligned} \tag{23}$$

It is seen that Equation (12) is the standard form for Bessel's equation of zero order with solution

$$V_n = \hat{A}_n J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x) + \hat{B}_n Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x) \tag{24}$$

where J_n and Y_n are Bessel's function of first and second kind, respectively.

If the shell forms a closed cone, $\hat{B}_n$ must be zero since v should be finite at x = 0. The eigenvalues β_n may be generated by applying the geometric boundary condition at $x = x_2$.

Case I. Supported condition, $v(x_2, t) = 0$ from Equation (22)

$$J_o(\beta_n \sqrt{\frac{\rho}{E}} x_2) = 0 \tag{25a}$$

and the roots are

$$\beta_n \sqrt{\frac{\rho}{E}} x_2 = 2.405, 5.520 \ldots.$$

Case II. Free end condition $\frac{\partial v}{\partial x}(x_2, t) = 0$ which leads to

$$J_o'(\beta_n \sqrt{\frac{\rho}{E}} x_2) = -J_1(\beta_n \sqrt{\frac{\rho}{E}} x_2) = 0 \tag{25b}$$

and the roots are

$$\beta_n \sqrt{\frac{\rho}{E}} x_2 = 0, 3.832, 7.016 \ldots.$$

For a truncated conical shell, geometric boundary conditions listed in Equation (6) will be applied to Equation (24) in order to determine β_n and the ratio of $\hat{A}_n$ to $\hat{B}_n$.

Case I. The conditions $v(x_1,t) = v(x_2,t) = 0$ lead to the following homogeneous algebraic equations:

$$\begin{bmatrix} J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1) & Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1) \\ J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2) & Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2 \end{bmatrix} \begin{bmatrix} \hat{A}_n \\ \hat{B}_n \end{bmatrix} = 0 \qquad (26)$$

For non-trivial solutions, the determinant of the coefficient matrix must vanish. Hence,

$$\frac{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = \frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} \qquad (27)$$

and

$$\frac{\hat{A}_n}{\hat{B}_n} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}$$

Case II. The conditions $v(x_2,t) = \frac{\partial v}{\partial x}(x_1,t) = 0$ result in

$$\frac{\hat{A}_n}{\hat{B}_n} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = -\frac{Y_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)} \qquad (28a)$$

and

$$\frac{J_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = \frac{Y_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} \qquad (28b)$$

The solutions of Equations (27) and (28) will give the eigenvalues β_n. The first several roots of Equation (27) and (28) can be found in [1] or [9]. It is a generally known fact that for large x, the Bessel functions involved in the Equation (24) behave very much like trigonometric functions. This phenomenon can be revealed from Equation (12).

$$\frac{d^2V_n}{dx^2} + \frac{1}{x}\frac{dV_n}{dx} + \beta_n^2 \frac{\rho}{E} V_n = 0 \tag{29}$$

By a change of variable of the form

$$U_n = V_n \sqrt{x}$$

Equation (16a) is transformed into

$$\frac{d^2U_n}{dx^2} + (\beta_n^2 \frac{\rho}{E} - \frac{1}{x^2}) U_n = 0 \tag{29a}$$

If the values of x are large we may assume

$$\frac{1}{x^2} \ll \beta_n^2 \frac{\rho}{E}$$

and neglect it in Equation (29a). The solution of Equation (29a) is seen to be

$$V_n = \frac{U_n}{\sqrt{x}} = \frac{1}{\sqrt{x}} [\hat{A}_n \cos(\beta_n \sqrt{\frac{\rho}{E}}\, x) + \hat{B}_n \sin(\beta_n \sqrt{\frac{\rho}{E}}\, x)] \tag{30}$$

The eigenvalues β_n for the two cases shown in Equation (6) are

Case I.

$$\beta_n = \frac{n\pi}{\ell}\sqrt{\frac{E}{\rho}} \tag{31a}$$

and

$$\hat{A}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_1\right)\hat{B}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_2\right)\hat{B}_n \tag{31b}$$

Case II.

$$\beta_n = \frac{(2n+1)\pi}{2\ell}\sqrt{\frac{E}{\rho}} \tag{31c}$$

and

$$\hat{A}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_2\right)\hat{B}_n = \cot\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_1\right)\hat{B}_n \tag{31d}$$

where $\ell = x_2 - x_1$. Equation (30) will be valid for truncated conical shells.

Substitution of Equations (22) and (24) or (22) and (30) into Equations (10a) and (9), the following general homogeneous solutions are obtained:

$$V_c = \sum_{n=0}^{\infty} X_n(x)\left[(\eta_n\omega_n^2 - \cot^2\alpha)(A_n \cos\omega_n t + B_n \sin\omega_n t)\right.$$

$$\left. + (\eta_n\bar{\omega}_n^2 - \cot^2\alpha)(C_n \cos\omega_n t + D_n \sin\omega_n t)\right] \tag{32a}$$

and

$$W_c = \sum_{n=0}^{\infty} - X_n(x)(A_n \cos \omega_n t + B_n \sin \omega_n t + C_n \cos \bar{\omega}_n t + D_n \sin \bar{\omega}_n t) \quad (32b)$$

where

$$X_n(x) = \lambda_n J_o(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) + Y_o(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) \;\ldots\; \text{exact} \quad (32c)$$

or

$$X_n(x) = \frac{1}{\sqrt{x}} [\lambda_n \cos(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) + \sin(\beta_n \sqrt{\tfrac{\rho}{E}}\, x)] \;\ldots\; \text{approx} \quad (32d)$$

in which

$$\lambda_n = \hat{A}_n/\hat{B}_n \quad (32e)$$

The particular solutions will be sought next. Let $\mathcal{L} = \frac{\partial}{\partial t}$ and $\mathcal{D} = \frac{\partial}{\partial x}$ be the linear operators. By eliminating v between Equations (8a) and (8b), the particular solution for w is taken as

$$w_p = \frac{1}{wt\ \alpha \mathcal{D}(x\mathcal{D})} (x)\{1 - F(\mathcal{L},\mathcal{D}) + F^2(\mathcal{L},\mathcal{D})\}$$

$$\{\frac{3 \tan \alpha}{Eh} p - \frac{p \tan \alpha}{E^2 h} x^2 \frac{\partial^2 p}{\partial t^2}\} + C_1 \log x + C_2 \quad (33)$$

where

$$F(\mathcal{L},\mathcal{D}) = \frac{p}{E}(3 \tan \alpha - \cot \alpha) \frac{\mathcal{L}^2}{\mathcal{D}(x\mathcal{D})} - (\frac{p}{E})^2 \tan \alpha x^2 \frac{\mathcal{L}^4}{\mathcal{D}(x\mathcal{D})} \quad (34)$$

Substitution of Equation (33) into Equation (8b), we obtain the particular solution for v

$$v_p = w_p \cot \alpha - \frac{x^2}{Eh} \tan \alpha p + \frac{\rho x^2}{E} \tan \alpha \frac{\partial^2 w_p}{\partial t^2} \qquad (35)$$

C_1 and C_2 will be determined by satisfying the geometric boundary conditions for v at both ends of the shell.

The particular solutions are then expanded into the following series:

$$v_p = \sum_{n=0}^{\infty} a_n X_n(x) \qquad (35a)$$

and

$$w_p = \sum_{n=0}^{\infty} b_n \cot \alpha \, X_n(x) \qquad (35b)$$

where

$$a_n = \frac{\int_{x_1}^{x_2} r(x) V_p X_n(x) dx}{\int_{x_1}^{x_2} r(x) X_n^2(x) dx} \qquad (36a)$$

$$b_n = \frac{\int_{x_1}^{x_2} r(x) W_p X_n(x) dx}{\int_{x_1}^{x_2} r(x) X_n^2(x) dx} \qquad (36b)$$

where $r(x)$ is the weight function. It should be noted that a_n and b_n, the generalized Fourier coefficients, are functions of t.

The final solutions of v and w are obtained by combining Equations (32a) to (35a) and (32b) to (35b), respectively. Or,

$$w = \sum_{n=0}^{\infty} -\cot\alpha \, X_n(x)(A_n \cos\omega_n t + B_n \sin\omega_n t$$

$$+ C_n \cos\bar{\omega}_n t + D_n \sin\bar{\omega}_n t - b_n) \tag{37a}$$

and

$$v = \sum_{n=0}^{\infty} X_n(x)[(\eta_n\omega_n^2 - \cot^2\alpha)(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

$$+ (\eta_n\bar{\omega}_n^2 - \cot^2\alpha)(C_n \cos\bar{\omega}_n t + D_n \sin\bar{\omega}_n t) + a_n] \tag{37b}$$

The following initial conditions will be used to determine the unknown coefficients A_n, B_n, C_n, and D_n:

$$v(x,o) = \frac{\partial v}{\partial t}(x,o) = w(x,o) = \frac{\partial w}{\partial t}(x,o) = 0 \tag{38}$$

By applying conditions listed in Equation (38) to Equations (37a) and (37b), we obtain

$$A_n = \frac{-b_n(0)(\eta_n\bar{\omega}_n^2 - \cot^2\alpha) - a_n(0)}{\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \tag{39a}$$

$$C_n = \frac{b_n(0)(\eta_n\omega_n^2 - \cot^2\alpha) + a_n(0)}{\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \tag{39b}$$

$$B_n = \frac{-\frac{\partial b_n}{\partial t}(0)(\eta\bar{\omega}_n^2 - \omega t^2\alpha) - \frac{\partial a_n}{\partial t}(0)}{\omega_n\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \tag{39c}$$

$$D_n = \frac{\frac{\partial b_n}{\partial t}(0)(\eta\omega_n^2 - \cot^2\alpha) + \frac{\partial a_n}{\partial t}(0)}{\bar{\omega}_n\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \tag{39d}$$

From Equation (22a) and (22b), it is seen that

$$\omega_n^2 - \bar{\omega}_n^2 = \sqrt{K_n^2 - 4\ H_n} \tag{40}$$

The response, after the shock wave has passed the structure, will be

$$w = \sum_{n=0}^{\infty} -\cot\alpha\ X_n(x)(A_n' \cos\omega_n t + B_n' \sin\omega_n t$$

$$+ C_n' \cos\bar{\omega}_n t + D_n' \sin\bar{\omega}_n t) \tag{41a}$$

and

$$v = \sum_{n=0}^{\infty} X_n(x)[\eta_n\omega_n^2 - \cot^2\alpha)(A_n' \cos\omega_n t + B_n' \sin\omega_n t)$$

$$+ (\eta\bar{\omega}_n^2 - \cot^2\alpha)(C_n' \cos\bar{\omega}_n t + D_n' \sin\omega_n t)] \tag{41b}$$

The unknown coefficients will be determined by using the known conditions

$$v(x,t_d),\ \frac{\partial v}{\partial t}(x,t_d),\ w(x,t_d) \text{ and } \frac{\partial w}{\partial t}(x,t_d)$$

which are evaluated from Equations (37a) and (37b).

EXAMPLE

For illustrative purposes, an example for a truncated conical shell with both ends supported is presented here. The boundary conditions shown in Equation (6a) are thus used. The following data are considered:

$$x_2 = 1.5\, x_1, \quad \alpha = \pi/4$$

$$p = p_o(1 - t/t_d) \tag{42}$$

The solutions of Equation (27) tabulated in [9] indicate that

$$\beta_n^2 \frac{\rho}{E} \geq \frac{1/2(6.27)^2}{\ell^2} \gg \frac{1}{\ell^2}$$

where $\ell = x_2 - x_1$. Therefore, the homogeneous solution shown in Equation (32d) will be used.

The particular solutions for this case will be taken as

$$w_p = \frac{3}{Eh}\left(\frac{x^2}{4} + \bar{\bar{C}}_1 \log x + \bar{C}_2\right) p_o(1 - t/t_d) \tag{43a}$$

and

$$v_p = \frac{3}{Eh}\left(- \frac{x^2}{12} + \bar{C}_1 \log x + \bar{C}_2\right) p_o(1 - t/t_d) \tag{43b}$$

where

$$\bar{C}_1 = \frac{x_1^2 - x_2^2}{12 \log \dfrac{x_1}{x_2}} \tag{44a}$$

and

$$\bar{C}_2 = \frac{x_2^2 \log x_1 - x_1^2 \log x_2}{12 \log \frac{x_1}{x_2}} \tag{44b}$$

It is seen from Equations (43a) and (43b) that

$$\frac{\partial a_n(0)}{\partial t} = - \frac{a_n(0)}{t_d} \tag{45a}$$

and

$$\frac{\partial b_n(0)}{\partial t} = - \frac{b_n(0)}{t_d} \tag{45b}$$

From Equation (18), we have

$$\eta_n = \frac{\frac{\rho}{E} \int_{x_1}^{x_2} x \sin \frac{n\pi x}{\ell} \operatorname{sgn}(\sin \frac{n\pi x}{\ell})\, dx}{\int_{x_1}^{x_2} \frac{1}{x} \sin \frac{n\pi x}{\ell} \operatorname{sgn}(\sin \frac{n\pi x}{\ell})\, dx} \tag{46a}$$

in which the integral involved in the denominator will be taken in the following form:

$$\int_{\xi_1}^{\xi_2} \frac{1}{\xi} \sin \frac{n\pi\xi}{\ell}\, d\xi = \sum_{m=1}^{\infty} m! \left(\frac{\ell}{n\pi}\right)^{2m-1} \left[\frac{\cos \frac{n\pi\xi_1}{\ell}}{(\xi_1)^{2m-1}} - \frac{\cos \frac{n\pi\xi_2}{\ell}}{(\xi_2)^{2m-1}} \right] \tag{46b}$$

The series shown in Equation (46b) converges rapidly and if only the first term is considered, Equation (46a) is approximately equal to

$$\eta_n = \frac{5n\ell \frac{\rho}{E}}{\frac{5}{6} + \sum_{j=1}^{n-1} \frac{2n}{(2n + j)}} \qquad (46c)$$

From Equations (21) and (22), we have

$$K_n = \frac{1}{\eta_n}\left[2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\,\eta_n\right] \qquad (46d)$$

$$H_n = \frac{1}{\eta_n}\left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho} \qquad (46e)$$

and

$$\left\{\begin{matrix} \omega_n^2 \\ \\ \bar{\omega}_n^2 \end{matrix}\right\} = \frac{2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\,\eta_n}{2\eta_n} \left\{\begin{matrix} + \\ - \end{matrix}\right\} \frac{1}{2}\sqrt{\left[\frac{2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\,\eta_n}{\eta_n}\right]^2 - \frac{4}{\eta_n}\left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}} \qquad (46f)$$

By use of Equations (32d), (36) and (43), the Fourier coefficients a_n and b_n become

$$a_n = \frac{6P_o}{Eh\ell}\left(1 - \frac{t}{t_d}\right)\int_{x_1}^{x_2}\left(-\frac{x^2}{12} + \bar{C}_1 \log x + \bar{C}_2\right)\sqrt{x}\,\sin\frac{n\pi x}{\ell}\,dx \qquad (47a)$$

$$b_n = \frac{6p_o}{Eh\ell}\left(1 - \frac{t}{t_d}\right)\int_{x_1}^{x_2}\left(\frac{x^2}{4} + \bar{C}_1 \log x + \bar{C}_2\right)\sqrt{x}\,\sin\frac{n\pi x}{\ell}\,dx \tag{47b}$$

or

$$a_n = \frac{6p_o}{Eh}\ell^{5/2}\left(1 - \frac{t}{t_d}\right)\left\{-\frac{1}{12}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2}x^{1/2}\log x \sin\frac{n\pi x}{\ell}dx\right\} \tag{48a}$$

$$b_n = \frac{6p_o}{Eh}\ell^{5/2}\left(1 - \frac{t}{t_d}\right)\left\{\frac{1}{4}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2}x^{1/2}\log x \sin\frac{n\pi x}{\ell}\,dx\right\} \tag{48b}$$

where

$$h_n = \left(\frac{1}{\ell}\right)^{7/2}\int_{x_1}^{x_2}x^{5/2}\sin\frac{n\pi x}{\ell}\,dx$$

$$= \left(\frac{1}{n\pi}\right)^{7/2}\Big[-(3n\pi)^{5/2}\cos 3n\pi + (2n\pi)^{5/2}\cos 2n\pi$$

$$-\frac{15}{8}\Big\{-\sqrt{3n\pi}\cos 3n\pi + \sqrt{2n\pi}\cos 2n\pi$$

$$+\sqrt{\frac{\pi}{2}}\,C(\sqrt{6n}) - \sqrt{\frac{\pi}{2}}\,C(\sqrt{4n})\Big\}\Big]$$

$$k_n = \left(\frac{1}{\ell}\right)^{3/2}\int_{x_1}^{x_2}x^{1/2}\sin\frac{n\pi x}{\ell}\,dx$$

$$= \left(\frac{1}{n\pi}\right)^{3/2}\Big[-\sqrt{3n\pi}\cos 3n\pi + \sqrt{2n\pi}\cos 2n\pi$$

$$+ \quad \frac{\pi}{2}\left\{C(\sqrt{6n}) - C(\sqrt{4n})\right\}\Big]$$

in which C is the Fresnel's integral and is tabulated in [9]. Substituting $t = 0$ into Equation (48) we have

$$a_n(0) = 6\ell^{5/2}\frac{p_o}{Eh}\left[-\frac{1}{12}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2} x^{1/2}\log x \sin\frac{n\pi x}{\ell}\,dx\right] \tag{49a}$$

$$b_n(0) = 6\,\frac{p_o\ell^{5/2}}{Eh}\left[\frac{1}{4}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2} x^{1/2}\log x \sin\frac{n\pi x}{\ell}\,dx\right] \tag{49b}$$

The transient response is then determined by substituting Equations (44) through (49) into Equation (39) and then (37). The response for $t > t_d$ can readily be determined by use of Equation (41).

CONCLUDING REMARKS

The governing differential equations were made separable with approximation. The natural frequencies ω_n and $\bar{\omega}_n$ corresponding to specific modes are therefore not exact. It is of interest to note that there exists two natural frequencies corresponding to each mode. This fact is true for the extensional vibration for spherical shells as discussed in [2] and [10]. No exact solution or experimental data seemed to be available at the present time in order to evaluate the error, due to the approximation made in the analysis, involved in the present solution. The method of solution presented in this study is straightforward. However, the solutions expressed in series form appear to be lengthy when numerical results are needed. High speed electronic computers may be efficiently employed for this purpose. Since the response of one case may look entirely different from the other case even for two identical shells, if the durations of the load are different; hence, no numerical example is presented in this paper.

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DYNAMIC RESPONSE OF A CYLINDRICAL SHELL SEGMENT SUBJECTED TO AN ARBITRARY LOADING

By

Wolfram Stadler[1]

James T. S. Wang[2]

ABSTRACT

Closed form expressions are obtained for the displacement, in the radial, circumferential and axial directions respectively, for a cylindrical shell segment subtended by an arbitrary angle θ_0 and subjected to an arbitrary load-distribution. Laplace and finite Fourier sine and cosine transforms are employed to accomplish the solution. A numerical example, utilizing an idealized equivalent triangular blast load, is included to provide a comparison with available experimental data as found in the literature. Furthermore, the effect of the magnitude of the thickness to radius of curvature ratio $(\frac{h}{R})$, and of the negligence of the inertial terms in the axial and circumferential directions, on the frequencies, are investigated.

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[1] Graduate Student, School of Engineering Mechanics.

[2] Associate Professor of Engineering Mechanics. Georgia Institute of Technology. Atlanta 13, Georgia 30332

INTRODUCTION

Experimental data, concerning the response of buried arches or tubes to blast wave loading is available to some extent; however, adequate theories and theoretical solutions upon which the design of such structures may be based are lacking in the literature. The present solution is to provide a step in this direction.

The investigation is restricted to the dynamic response of a simply supported cylindrical shell segment subjected to an arbitrary loading. The basic system of equations is taken directly from [5] and modified only by the inclusion of the inertia terms and forcing functions. The equations then are non-dimensionalized with respect to the length of the arch, L, the subtended angle θ_o and the decay time of the applied blast load, in order to simplify the necessary calculations. This system of linear partial differential equations is reduced to a linear algebraic system in the transformed displacement functions, by eliminating the time dependence by means of the Laplace-transformation and the spatial dependence by means of double, finite Fourier sine and cosine transforms. A simple application of Cramer's rule, and the inversion of the transformed displacement functions results in the final, closed-form solutions.

A numerical example, the dynamic response of a semicircular cylindrical shell, is included to provide a qualitative comparison to experimentally obtained data [1]. The comparison here was possible only so far as the order of magnitude is concerned, since the arch tested in [1] was stiffened considerably by the surrounding soil so that it was to be expected that the measured deflections resulting from the dynamic load would be less than those obtained by truncating the theoretical series solutions obtained in this paper. The times at which the peak deflections occurred also compare favorably.

The effect on the natural frequencies of the system, of the omission of the inertial terms in the axial and circumferential directions for varying values of the ratio $\frac{h}{R}$ is found to be negligible (of the order of 1%) for the higher frequencies, up to a certain limiting value, beyond which the inertial terms must be included.

Notation:

-	Bar above a letter denotes the Laplace transform of a function with respect to the non-dimensional time variable τ
(c)(s)	Superscripts (s) and (c) denote finite Fourier sine and cosine transforms respectively.
(cs)(cc)(ss)	Superscripts (cs), (cc) and (ss) denote successive finite Fourier transforms.
[]	Denote references in the bibliography.

Symbols:

d	Equivalent decay time.
E	Modulus of elasticity
h	Thickness of shell.
L	Length of shell.
M	Moment.
N	Normal or shear force.
p_u, p_v, p_w	Surface loading components in the directions indicated by the subscripts.
R	Radius of the cylindrical shell segment.
s	Laplacian parameter.
t	Time variable.
u,v,w	Displacements in the x,y,z-directions (Figure 1).
ν	Poisson's ratio.
ρ	Density.

Equations of Motion

The static equations of general cylindrical shell theory may be found in [5]. These equations are modified by the inclusion of the inertial forces and arbitrary forcing functions. They thence have the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2R}\frac{\partial^2 v}{\partial x \partial \theta} - \frac{\nu}{R}\frac{\partial w}{\partial x} + \frac{1-\nu}{2R^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{(1-\nu^2)}{Eh}\left(p_u - \rho h \frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{1+\nu}{2R}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{R^2 \partial \theta^2} - \frac{\partial w}{R^2 \partial \theta} = -\frac{(1-\nu^2)}{Eh}\left(p_v - \rho h \frac{\partial^2 v}{\partial t^2}\right) \qquad (1)$$

$$\nu \frac{\partial u}{\partial x} + \frac{1}{R}\frac{\partial v}{\partial \theta} - \frac{w}{R} - \frac{h^2 R}{12}\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}\frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{1}{R^4}\frac{\partial^4 w}{\partial \theta^4}\right) = \frac{-R(1-\nu^2)}{Eh}\left(p_w - \rho h \frac{\partial^2 w}{\partial t^2}\right)$$

where the loading components p_u, p_v, and p_w are functions of x, θ, and t.

The problem is completely formulated, when the following boundary and initial conditions are included:

a) initial conditions:

$$w(x,\theta,0) = u(x,\theta,0) = v(x,\theta,0) = 0;$$

$$w_t(x,\theta,0) = u_t(x,\theta,0) = v_t(x,\theta,0) = 0;$$

b) boundary conditions:

i) imposed on θ

$$u(x,0,t) = u(x,\theta_o,t) = 0;$$

$$v_\theta(x,0,t) = v_\theta(x,\theta_o,t) = 0; \qquad (2)$$

$$w(x,0,t) = w(x,\theta_o,t) = 0;$$

$$w_{\theta\theta}(x,0,t) = w_{\theta\theta}(x,\theta_o,t) = 0;$$

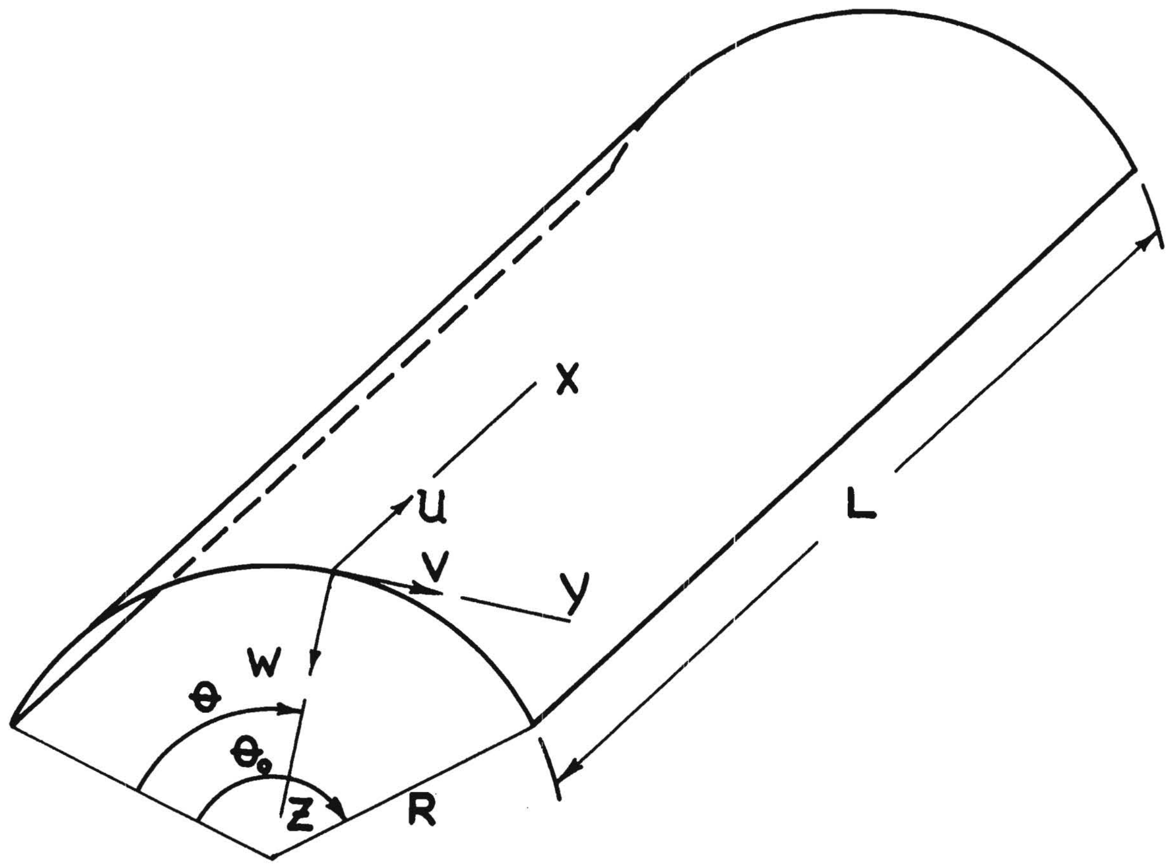

FIG. 1
CYLINDRICAL SHELL SEGMENT

ii) imposed on x

$$u_x(0,\theta,t) = u_x(L,\theta,t) = 0;$$
$$v(0,\theta,t) = v(L,\theta,t) = 0;$$
$$w(0,\theta,t) = w(L,\theta,t) = 0;$$
$$w_{xx}(0,\theta,t) = w_{xx}(L,\theta,t) = 0,$$

where the subscript notation has been used to denote partial differentiation.

It is convenient to non-dimensionalize equations (1), and (2), by introducing the following non-dimensional ratios:

$$\xi = \frac{x}{L}; \; \varphi = \frac{\theta}{\theta_o}; \; \tau = \frac{t}{d}; \; U = \frac{u}{L}; \; V = \frac{v}{L}; \; W = \frac{w}{L}; \; p_u^* = \frac{p_u}{E}; \; p_v^* = \frac{p_v}{E}; \; p_w^* = \frac{p_w}{E},$$

where d is the equivalent decay time of the shock-load. This characteristic constant is used since the prime concern here is blast and impulsive loading. The substitution of these non-dimensional variables in equation (1) and the subsequent Laplace transformation of these equations, in conjunction with the initial conditions results in

$$\frac{\partial^2\bar{U}}{\partial\xi^2} + a_1\frac{\partial^2\bar{V}}{\partial\xi\partial\varphi} - a_2\frac{\partial\bar{W}}{\partial\xi} + a_3\frac{\partial^2\bar{U}}{\partial\varphi^2} = -a_4\bar{p}_u^* + a_5 s^2\bar{U} \qquad \text{(a)}$$

$$\frac{\partial^2\bar{V}}{\partial\xi^2} + b_1\frac{\partial^2\bar{U}}{\partial\xi\partial\varphi} - b_2\frac{\partial\bar{W}}{\partial\varphi} + b_3\frac{\partial^2\bar{V}}{\partial\varphi^2} = -b_4\bar{p}_v^* + b_5 s^2\bar{V} \qquad \text{(b)} \qquad (3)$$

$$\frac{\partial\bar{U}}{\partial\xi} + c_1\frac{\partial\bar{V}}{\partial\varphi} - c_2\bar{W} - c_3\frac{\partial^4\bar{W}}{\partial\xi^4} - c_4\frac{\partial^4\bar{W}}{\partial\xi^2\partial\varphi^2} - c_5\frac{\partial^4\bar{W}}{\partial\varphi^4} \qquad \text{(c)}$$

$$= -c_6\bar{p}_w^* + c_7 s^2\bar{W}$$

where

$$a_1 = \frac{L}{\theta_o}\,\frac{1+\nu}{2R} \qquad b_1 = \frac{L(1+\nu)}{R\theta_o(1-\nu)} \qquad c_1 = \frac{L}{\nu R\theta_o} \qquad c_6 = \frac{R(1-\nu^2)}{h\nu}$$

$$a_2 = \frac{\nu L}{R} \qquad b_2 = \frac{2L^2}{R^2\theta_o(1-\nu)} \qquad c_2 = \frac{L}{\nu R} \qquad c_7 = \frac{R(1-\nu^2)\rho L}{E\nu d^2}$$

$$a_3 = \frac{L^2(1-\nu)}{2R^2\theta_o^2} \qquad b_3 = \frac{2L^2}{R^2\theta_o^2(1-\nu)} \qquad c_3 = \frac{h^2R}{12L^3\nu}$$

$$a_4 = \frac{L(1-\nu^2)}{h} \qquad b_4 = \frac{2L(1+\nu)}{h} \qquad c_4 = \frac{h^2}{6RL\nu\theta_o^2}$$

$$a_5 = \frac{(1-\nu^2)\rho L^2}{Ed^2} \qquad b_5 = \frac{2(1+\nu)L^2\rho}{Ed^2} \qquad c_5 = \frac{h^2L}{12\nu R^3\theta_o^4}$$

and where it is assumed that the p's are Laplace-transformable.

The linear system (3) of partial differential equations is now transformed to a linear system of algebraic equations by the application of finite Fourier sine and cosine transforms [4](see also appendix), with the boundary conditions (2), i.e.

$$\begin{aligned}
&(A_{11} + s^2)\,\bar{U}^{(cs)}(m,n,s) + A_{12}\bar{V}^{(sc)}(m,n,s) + A_{13}\bar{W}^{(ss)}(m,n,s) = \bar{Q}_1(m,n,s)\\
&A_{21}\bar{U}^{(cs)}(m,n,s) + (A_{22} + s^2)\bar{V}^{(sc)}(m,n,s) + A_{23}\bar{W}^{(ss)}(m,n,s) = \bar{Q}_2(m,n,s)\\
&A_{31}\bar{U}^{(cs)}(m,n,s) + A_{32}\bar{V}^{(sc)}(m,n,s) + (A_{33} + s^2)\bar{W}^{(ss)}(m,n,s) = \bar{Q}_3(m,n,s).
\end{aligned} \tag{4}$$

Here, the superscripts indicate the type of transformation and the order in which they were carried out. Furthermore, the arguments (m,n,s) of the transformed functions are order-preserving with respect to the arguments (ξ,φ,τ) of the original functions. The coefficients A_{ij} in equation (4) are given by

$$
\begin{aligned}
A_{11} &= \frac{\alpha_m^2 + a_3\beta_n^2}{a_5}; \quad A_{12} = \frac{a_1\alpha_m\beta_n}{a_5}; \quad A_{13} = \frac{a_2\alpha_m}{a_5}; \quad \bar{Q}_1 = \frac{a_4}{a_5}\,\bar{p}_u^{*(cs)}; \\
A_{21} &= \frac{\alpha_m\beta_n b_1}{b_5}; \quad A_{22} = \frac{\alpha_m^2 + b_3\beta_n^2}{b_5}; \quad A_{23} = \frac{b_2\beta_n}{b_5}; \quad \bar{Q}_2 = \frac{b_4}{b_5}\,\bar{p}_v^{*(sc)}; \qquad (5) \\
A_{31} &= \frac{\alpha_m}{c_7}; \quad A_{32} = \frac{c_1\beta_n}{c_7}; \quad A_{33} = \frac{c_2 + c_3\alpha_m^4 + c_4\alpha_m^2\beta_n^2 + c_5\beta_n^4}{c_7}; \quad \bar{Q}_3 = \frac{c_6}{c_7}\,\bar{p}_w^{*(ss)},
\end{aligned}
$$

where $\alpha_m = \pi m$ and $\beta_n = \pi n$ are the transform parameters for ξ and φ respectively.

An investigation of these coefficients A_{ij} reveals that they are symmetric, which was to be expected from the symmetry of the original equations. This symmetry becomes quite useful if one observes that the s^2's may be interpreted as the eigenvalues of a real, symmetric matrix, obtained from equation (4), and as such must be real [2]. With this in mind, Cramer's rule may be applied to equation (4) and the solutions written in the form:

$$
\begin{aligned}
\bar{U}^{(cs)}(m,n,s) &= \sum_{j=1}^{3} \bar{Q}_j\,\bar{K}_{1j}(m,n,s) \\
\bar{V}^{(sc)}(m,n,s) &= \sum_{j=1}^{3} \bar{Q}_j\,\bar{K}_{2j}(m,n,s) \qquad (6) \\
\bar{W}^{(ss)}(m,n,s) &= \sum_{j=1}^{3} \bar{Q}_j\,\bar{K}_{3j}(m,n,s)
\end{aligned}
$$

where the $\bar{Q}_j$'s are the loading functions as defined in (5) and where

$$\bar{K}_{ij}(m,n,s) = \frac{C_{ij}}{s^2+\omega_1^2} + \frac{D_{ij}}{s^2+\omega_2^2} + \frac{E_{ij}}{s^2+\omega_3^2} \tag{7}$$

with

$$C_{11} = \frac{(\omega_2^2-\omega_3^2)[\omega_1^4-(A_{22}+A_{33})\omega_1^2+A_{22}A_{33}-A_{23}^2]}{\Delta(\omega^2)}$$

$$C_{12} = C_{21} = \frac{(\omega_2^2-\omega_3^2)[A_{12}\omega_1^2+A_{13}A_{23}-A_{12}A_{33}]}{\Delta(\omega^2)}$$

$$C_{13} = C_{31} = \frac{(\omega_2^2-\omega_3^2)[A_{13}\omega_1^2+A_{12}A_{23}-A_{13}A_{22}]}{\Delta(\omega^2)}$$

$$C_{22} = \frac{(\omega_2^2-\omega_3^2)[\omega_1^4-(A_{11}+A_{33})\omega_1^2+A_{11}A_{33}-A_{13}^2]}{\Delta(\omega^2)}$$

$$C_{23} = \frac{(\omega_2^2-\omega_3^2)[A_{23}\omega_1^2+A_{12}A_{13}-A_{11}A_{23}]}{\Delta(\omega^2)}$$

$$C_{33} = \frac{(\omega_2^2-\omega_3^2)[\omega_1^4-(A_{11}+A_{22})\omega_1^2+A_{11}A_{22}-A_{12}^2]}{\Delta(\omega^2)}$$

$$D_{11} = \frac{(\omega_3^2-\omega_1^2)[\omega_2^4-(A_{22}+A_{33})\omega_2^2+A_{22}A_{33}-A_{23}^2]}{\Delta(\omega^2)}$$

$$D_{12} = D_{21} = \frac{(\omega_3^2-\omega_1^2)(A_{12}\omega_2^2+A_{13}A_{23}-A_{12}A_{33})}{\Delta(\omega^2)}$$

$$D_{13} = D_{31} = \frac{(\omega_3^2-\omega_1^2)(A_{13}\omega_2^2+A_{12}A_{23}-A_{13}A_{22})}{\Delta(\omega^2)}$$

$$D_{22} = \frac{(\omega_3^2-\omega_1^2)[\omega_2^4-(A_{11}+A_{33})\omega_2^2+A_{11}A_{33}-A_{13}^2]}{\Delta(\omega^2)}$$

$$D_{23} = D_{32} = \frac{(\omega_3^2-\omega_1^2)\ [A_{23}\omega_2^2+A_{12}A_{13}-A_{11}A_{23}]}{\Delta(\omega^2)}$$

$$D_{33} = \frac{(\omega_3^2-\omega_1^2)[\omega_2^4-(A_{11}+A_{22})\omega_2^2+A_{11}A_{22}-A_{12}^2]}{\Delta(\omega^2)}$$

$$E_{11} = \frac{(\omega_1^2-\omega_2^2)[\omega_3^4-(A_{22}+A_{33})\omega_3^2+A_{22}A_{33}-A_{23}^2]}{\Delta(\omega^2)}$$

$$E_{12} = E_{21} = \frac{(\omega_1^2-\omega_2^2)[\ A_{12}\omega_3^2+A_{13}A_{23}-A_{12}A_{33}]}{\Delta(\omega^2)}$$

$$E_{13} = E_{31} = \frac{(\omega_1^2-\omega_2^2)[\ A_{13}\omega_3^2+A_{12}A_{23}-A_{13}A_{22}]}{\Delta(\omega^2)}$$

$$E_{22} = \frac{(\omega_1^2-\omega_2^2)[\omega_3^4-(A_{11}+A_{33})\omega_3^2+A_{11}A_{33}-A_{13}^2]}{\Delta(\omega^2)}$$

$$E_{23} = E_{32} = \frac{(\omega_1^2-\omega_2^2)[\ A_{23}\omega_3^2+A_{12}A_{13}-A_{11}A_{23}]}{\Delta(\omega^2)}$$

$$E_{33} = \frac{(\omega_1^2-\omega_2^2)[\omega_3^4-(A_{11}+A_{22})\omega_3^2+A_{11}A_{22}-A_{12}^2]}{\Delta(\omega^2)}$$

and

$$\Delta(\omega^2) = (\omega_1^2 - \omega_2^2)(\omega_1^2 - \omega_3^2)(\omega_2^2 - \omega_3^2)$$

as obtained by some algebraic manipulation and by means of a partial fractions separation.

The ω_i's are obtained by setting $s^2 = -\lambda_i$ in the determinant of the coefficient matrix of equations (4), and then solving the cubic equation

$$\lambda^3 + \theta_1\lambda^2 + \theta_2\lambda + \theta_3 = 0 \tag{9}$$

obtained by expansion of the determinant. Here, θ_1, θ_2 and θ_3 are given by

$$\theta_1 = (A_{11}+A_{22}+A_{33})$$

$$\theta_2 = (A_{11}A_{22}+A_{11}A_{33}+A_{22}A_{33}-A_{12}^2-A_{13}^2-A_{23}^2)$$

$$\theta_3 = (A_{11}A_{22}A_{33}+2A_{12}A_{13}A_{23}-A_{12}^2A_{33}-A_{13}^2A_{22}-A_{23}^2A_{11})$$

From Descartes' rule for the roots of a polynominal it is known that (9) can have only negative roots as long as $\theta_1, \theta_2, \theta_3 > 0$. Since this must be the case, if the solution is to remain stable, it is justifiable to write the solution in the form (6).

The inversion of equations (6) with respect to the Laplace transformation is now accomplished through the application of the convolution integral defined by

$$L^{-1}\left\{f(s)g(s)\right\} = F(\tau)*G(\tau) = \int_0^\tau F(\tau-t)G(t)dt$$

The result is

$$
\begin{aligned}
U^{(cs)}(m,n,\tau) &= \sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{1j}(m,n,\tau-\zeta)d\zeta \\
V^{(sc)}(m,n,\tau) &= \sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{2j}(m,n,\tau-\zeta)d\zeta \qquad (10) \\
W^{(ss)}(m,n,\tau) &= \sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{3j}(m,n,\tau-\zeta)d\zeta
\end{aligned}
$$

where

$$Q_1(m,n,\zeta) = kp_u^{*(cs)}(m,n,\zeta)$$

$$Q_2(m,n,\zeta) = kp_v^{*(cs)}(m,n,\zeta)$$

$$Q_3(m,n,\zeta) = kp_w^{*(ss)}(m,n,\zeta)$$

since

$$\frac{a_4}{a_5} = \frac{b_4}{b_5} = \frac{c_6}{c_7} = \frac{Ed^2}{\rho hL} = k.$$

Also,

$$
\begin{aligned}
L^{-1}\left\{\bar{K}_{ij}(m,n,s)\right\} &= K_{ij}(m,n,\tau) \\
&= \frac{C_{ij}}{\omega_1}\sin\omega_1\tau + \frac{D_{ij}}{\omega_2}\sin\omega_2\tau + \frac{E_{ij}}{\omega_3}\sin\omega_3\tau,
\end{aligned}
$$

$L^{-1}\left\{\ \right\}$ denoting the inverse Laplace transform.

The final step in obtaining the solution of the system of equations (1) is the inversion of the Fourier sine and cosine transforms. These operations are carried out with the use of the inverse transforms as defined in the appendix. The inversion may be separated into two parts:

a) inversion w.r.t. φ

$$U^{(c)}(m,\varphi,\tau) = 2\sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{1j}(m,n,\tau-\zeta)d\zeta\right\}\sin \beta_n\varphi$$

$$V^{(s)}(m,\varphi,\tau) = V^{(sc)}(m,0,\tau) + 2\sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{2j}(m,n,\tau-\zeta)d\zeta\right\}\cos\beta_n\varphi$$

$$W^{(s)}(m,\varphi,\tau) = 2\sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{3j}(m,n,\tau-\zeta)d\zeta\right\}\sin \beta_n\varphi \tag{11}$$

b) inversion w.r.t. ξ

$$U(\xi,\varphi,\tau) = U^{(c)}(0,\varphi,\tau) + 4\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{1j}(m,n,\tau-\zeta)d\zeta\right\}\cdot$$

$$\sin \beta_n\varphi \cos \alpha_m\xi$$

$$V(\xi,\varphi,\tau) = 2\sum_{m=1}^{\infty} \left\{V^{(sc)}(m,0,\tau) + 2\sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{2j}(m,n,\tau-\zeta)d\zeta\right\}\cos\beta_n\varphi\right\}$$

$$\sin \alpha_m\xi \tag{12}$$

$$W(\xi,\varphi,\tau) = 4\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{\sum_{j=1}^{3} \int_{o}^{\tau} Q_j(m,n,\zeta)K_{3j}(m,n,\tau-\zeta)d\zeta\right\}\sin \beta_n\varphi \sin \alpha_m\xi$$

Equations (12) represent the solution of the dynamic equations of a cylindrical shell segment subjected to an arbitrary loading, arbitrary to the extent, naturally, that the loading function be Fourier transformable.

The expressions for the stresses and moments in terms of the displacements are given by [5]

$$
\begin{aligned}
N_{xx} &= N[\frac{\partial u}{\partial x} + \frac{\nu}{R}(\frac{\partial v}{\partial \theta} - w)]; \\
N_{\theta\theta} &= N[\frac{1}{R}(\frac{\partial v}{\partial \theta} - w) + \nu\frac{\partial u}{\partial x}]; \\
N_{x\theta} &= \frac{1}{2}N(1-\nu)[\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}]; \\
M_{xx} &= -D[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{R^2}(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2})]; \\
M_{\theta\theta} &= -D[\frac{1}{R^2}(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}) + \nu\frac{\partial^2 w}{\partial x^2}]; \\
M_{x\theta} &= D(1-\nu)\frac{1}{R}[\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial \theta}].
\end{aligned}
\tag{13}
$$

where $N = \dfrac{Eh}{1-\nu^2}$ and $D = \dfrac{Eh^3}{12(1-\nu^2)}$. Note that equations (13) are in terms of the dimensional variables.

Comparison with Experimental Data

Unfortunately, no experimental data are available which correspond exactly to simple support conditions. Of the considered experimental data, that which most closely approximates the case of simply supported edges is obtained from [1], where a buried arch subjected to short and long duration blast-loading is considered. In order to keep the numerical calculations as simple as possible, this type of loading is simulated by an equivalent (equivalent in the sense of the same total impulse) triangular pulse load of the form

$$p_w = p_o \left(1 - \frac{t}{d}\right),$$

where p_o represents the peak overpressure, and d is the equivalent decay time. Transformation and substitution in equation (12) results in

$$w(\xi,\varphi,\tau) = \frac{4p_o d^2}{h\rho L\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [1-(-1)^m-(-1)^n+(-1)^{m+n}].$$

$$\left\{\frac{C_{33}}{\omega_1^2}\left[1-\tau-\cos\omega_1\tau+\frac{1}{\omega_1}\sin\omega_1\tau\right] + \frac{D_{33}}{\omega_2^2}\left[1-\tau-\cos\omega_2\tau+\frac{1}{\omega_2}\sin\omega_2\tau\right] + \frac{E_{33}}{\omega_3^2}\left[1-\tau-\cos\omega_3\tau+\frac{1}{\omega_3}\sin\omega_3\tau\right]\right\}$$

$$\sin \alpha_m\xi \sin \beta_n\varphi. \tag{14}$$

This expression, with the particular values:

$$h = .0478 \text{ in}, \ L = 57.6 \text{ in}, \ \rho = 1.3564 \times 10^{-3} \ \frac{\text{lb sec}^2}{\text{in}^4},$$

$$p_o = 7.5 \text{ psi}, \ d = 76 \text{ msec}, \ E = 30 \times 10^6 \text{ psi}, \ \theta_o = \pi$$

is used to obtain the deflections in non-dimensional form.

The experimental and theoretical data can be compared only in order of magnitude due to the discrepancies mentioned above. In the numerical evaluation of equation (12), care must be exercised in truncating the resultant series, since convergence is slow. The difficulty arises for fixed m, with n increasing, since the radial frequency, ω_1, initially decreases, and thereafter monotonically increases for some n depending on m. The

TABLE 1

Numerical Comparison with Experimental Data

	Experimental	Theoretical
$\bar{\omega}_1 \binom{m=1}{n=3} \frac{\text{cyd}}{\text{msec}}$	.105	.132
Peak Crown Deflection in inches	.172	.535
Max. Response at Crown in msec	15	36

interval, in which ω_1 decreases, becomes smaller with a decrease in θ_o, i.e. approaching the shallow shell range, and with an increase in $\frac{h}{R}$. From Figure 2 it is apparent that, in view of the large difference in frequencies the omission of the inertial terms for u and v has little effect for the lower modes. Figure 3 indicates that there is a peak deflection at 13 msec. This, however, is a relative maximum; the maximum occurs at 36 msec as indicated in Table 1. The difference in the numerical values of the deflections is explained in [1] as the result of a difference in soil density, i.e. small reductions in soil density resulted in a large percentage increase in the deflection. The frequency in Table 1, corresponding to the first inextensional symmetrical mode, was measured with no endwalls in the arch. Since simple supports were assumed in the numerical example, an increase in the frequency is to be expected. The absence, in the theoretical response curve (Figure 3), of the damping exhibited for the experimental deflection curve is to be attributed to the rough approximation given by the triangular load.

Effect of the Omission of the Inertial Terms in the Axial and Circumferential Directions.

The investigation of the omission of the inertial terms is restricted to the effect on the radial frequency. The effect on the deflection is not considered. The frequencies, as obtained here, are compared to those calculated from the expression [4]

$$\omega_1^{*} = \frac{1}{h\rho R^2}\left[\frac{N}{R^2}(\lambda_m^2 + \mu_n^2)^2 + \frac{Eh\lambda_m^4}{(\lambda_m^2+\mu_n^2)^2}\right] \tag{15}$$

where

$$N = \frac{Eh^3}{12(1-\nu^2)};\ \lambda_m = \frac{m\pi R}{L};\ \mu_n = \frac{n\pi}{\theta_o}.$$

Equation (15) is based on shallow shell theory, omitting the inertial terms in the u and v -directions. For the higher frequencies there is virtually no error introduced by using (15), even out of shallow shell range. However, the error in the fundamental frequency (m=1, n=1) increases as θ_o increases, and becomes 41% for $\theta_o=\pi$. The frequencies increase with increasing $\frac{h}{R}$ up to $\frac{h}{R} = \frac{1}{20}$, which was taken to be the upper bound for thin shell theory [3]. Again, the error introduced by using (15) instead of (9) to obtain the frequencies is negligible.

It must be emphasized that the range of applicability of equation (15) is limited to a certain frequency range. As was discovered by E. Reissner [6] for spherical shells, there also exist limiting values of the frequencies of a cylindrical shell segment, beyond which, the inertial terms must be included. It was not yet possible to determine an indicative parameter. The range of applicability of (15) decreases as $\frac{h}{R}$ increases and as θ_o decreases. As can be seen from Table 2, equation (15) is useless for $\frac{h}{R} = \frac{1}{20}$ when $n \geq 15$, $m = 1$, or for $\frac{h}{R} = \frac{1}{45}$ when $n \geq 31$, $m = 1$. (The latter values do not appear in the table).

TABLE 2

NON-DIMENSIONAL FREQUENCIES
(m = 1)

	$\theta_o = \pi$		$\theta_o = \frac{\pi}{3}$									
$\frac{h}{R}$	$\frac{1}{300}$		$\frac{1}{300}$		$\frac{1}{200}$		$\frac{1}{100}$		$\frac{1}{45}$		$\frac{1}{20}$	
h	ω_1	ω_1*	ω_1	ω_1*	ω_1	ω_1*	ω_1	ω_1*	ω_1	ω_1*	ω_1	ω_1*
1	214.5	302.1	47.1	49.7	47.9	50.5	51.6	54.4	67.8	71.5	120.37	126.97
3	49.7	52.6	63.2	63.5	98.8	99.5	197.5	198.6	438.3	440.9	987.0	992.9
5	26.5	27.1	174.5	174.9	273.9	274.4	547.7	548.8	1215.9	1218.4	2738.2	2744.3
7	37.1	37.5	342.0	342.3	536.6	537.2	1073.1	1074.3	2382.3	2384.9	5365.2	5371.5
9	59.3	59.7	565.2	565.6	886.9	887.5	1773.7	1774.9	3937.6	3940.2	8867.7	8874.5
11	88.1	88.5	844.3	844.6	1324.7	1325.3	2649.4	2650.6	5881.7	5884.4	13245	13253
13	122.9	123.3	1179.1	1179.5	1850.1	1850.1	3700.3	3701.5	8214.6	8217.4	18498	18507
15	163.6	164.0	1570.0	1570.2	2463.2	2463.7	4926.4	4927.5	10936.0	10939.0	21723	24637
17	210.1	210.5	2016.4	2016.7	3163.8	3164.3	6327.5	6328.7	14047.0	14049.0	24619	31643
19	262.5	262.8	2518.7	2519.0	3952.0	3952.1	7904.0	7905.1	17546.0	17549.0	27514	39525

Conclusions

The response of a cylindrical shell segment, subtended by an arbitrary angle θ_o and subjected to an arbitrary loading, is obtained, based on thin shell theory.

In summary:

1. Inertial terms affect mainly the frequency corresponding to the first inextensional mode in the radial direction. ($n = 1$, m increasing).
2. The introduced error increases with decreasing $\frac{h}{R}$.
3. To obtain the frequencies corresponding to the higher longitudinal modes ($n \geq 15$ for $\frac{h}{R} = \frac{1}{20}$ and $m = 1$) the inertial terms must be included.
4. Comparisons with experimental data are favorable as far as order of magnitude is concerned.

APPENDIX

The Fourier sine and cosine transforms are given by [4]:

a) Single transforms:

Sine transform: $$f^{(s)}(n) = \int_0^a f(\varphi)\sin\beta_n\varphi d\varphi.$$

Cosine transform: $$f^{(c)}(n) = \int_0^a f(\varphi)\cos\beta_n\varphi d\varphi.$$

Single inverse transforms:

Inverse sine transform: $$f(\varphi) = \frac{2}{a}\sum_{n=1}^{\infty} f^{(s)}(n)\sin\beta_n\varphi.$$

Inverse cosine transform: $$f(\varphi) = \frac{1}{a}f^{(c)}(0) + \frac{2}{a}\sum_{n=1}^{\infty} f^{(c)}(n)\cos\beta_n\varphi.$$

b) Simple double transforms:

Double sine transform: $$f^{(ss)}(m,n) = \int_0^b \int_0^a f(\xi,\varphi)\sin\alpha_m\xi\sin\beta_n\varphi d\varphi d\xi.$$

Double cosine transform: $$f^{(cc)}(m,n) = \int_0^b \int_0^a f(\xi,\varphi)\cos\alpha_m\xi\cos\beta_n\varphi d\varphi d\xi.$$

Inverse simple double transforms:

Inverse double sine transform: $$f(\xi,\varphi) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^{(ss)}(m,n)\sin\alpha_m\xi\sin\beta_n\varphi.$$

Inverse double cosine transform:

$$f(\xi,\varphi) = \frac{1}{a^2} f^{(cc)}(0,0) + \frac{2}{a^2}\sum_{n=1}^{\infty} f^{(cc)}(0,n)\cos\beta_n\varphi + \frac{2}{a^2}\sum_{m=1}^{\infty}\left\{f^{(cc)}(m,0)+2\sum_{n=1}^{\infty} f^{(cc)}(m,n)\right.$$

$$\left.\cos\alpha_m\xi\right\}\cos\beta_n\varphi.$$

c) Mixed double transforms:

Sine-cosine transform: $$f^{(sc)}(m,n) = \int_0^b \int_0^a f(\xi,\varphi)\sin\alpha_m\xi\cos\beta_n\varphi d\varphi d\xi$$

Cosine-sine transform: $$f^{(cs)}(m,n) = \int_0^b \int_0^a f(\xi,\varphi)\cos\alpha_m\xi\sin\beta_n\varphi d\varphi d\xi$$

Inverse mixed double transforms:

Inverse sine-cosine transform: $$f(\xi,\varphi) = \frac{2}{a^2}\sum_{m=1}^{\infty} \left\{f^{(sc)}(m,0) + 2\sum_{n=1}^{\infty} f^{(sc)}(m,n) \cos\beta_n\varphi\right\}\sin\alpha_m\xi$$

Inverse cosine-sine transform: $$f(\xi,\varphi) = \frac{2}{a^2}\sum_{n=1}^{\infty} f^{(cs)}(0,n)\sin\beta_n\varphi + \frac{4}{a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f^{(cs)}(m,n)\sin\beta_n\varphi\cos\alpha_m\xi$$

where $\alpha_m = \frac{m\pi}{b}$ and $\beta_n = \frac{n\pi}{a}$.

Note that, in general

$$f^{(sc)}(m,n) \neq f^{(cs)}(m,n)$$

It is clear that when these transform methods are applied the relevant assumptions concerning the functions to be transformed are made, namely that they satisfy Dirichlet's conditions in their respective intervals, and that the iterated integrals may be taken successively, i.e.

$$f^{(s)}(m,\varphi) = \int_0^b f(\xi,\varphi)\sin\alpha_m\xi\, d\xi$$

and

$$f^{(sc)}(m,n) = \int_0^a f^{(s)}(m,\varphi)\cos\beta_n\varphi d\varphi.$$

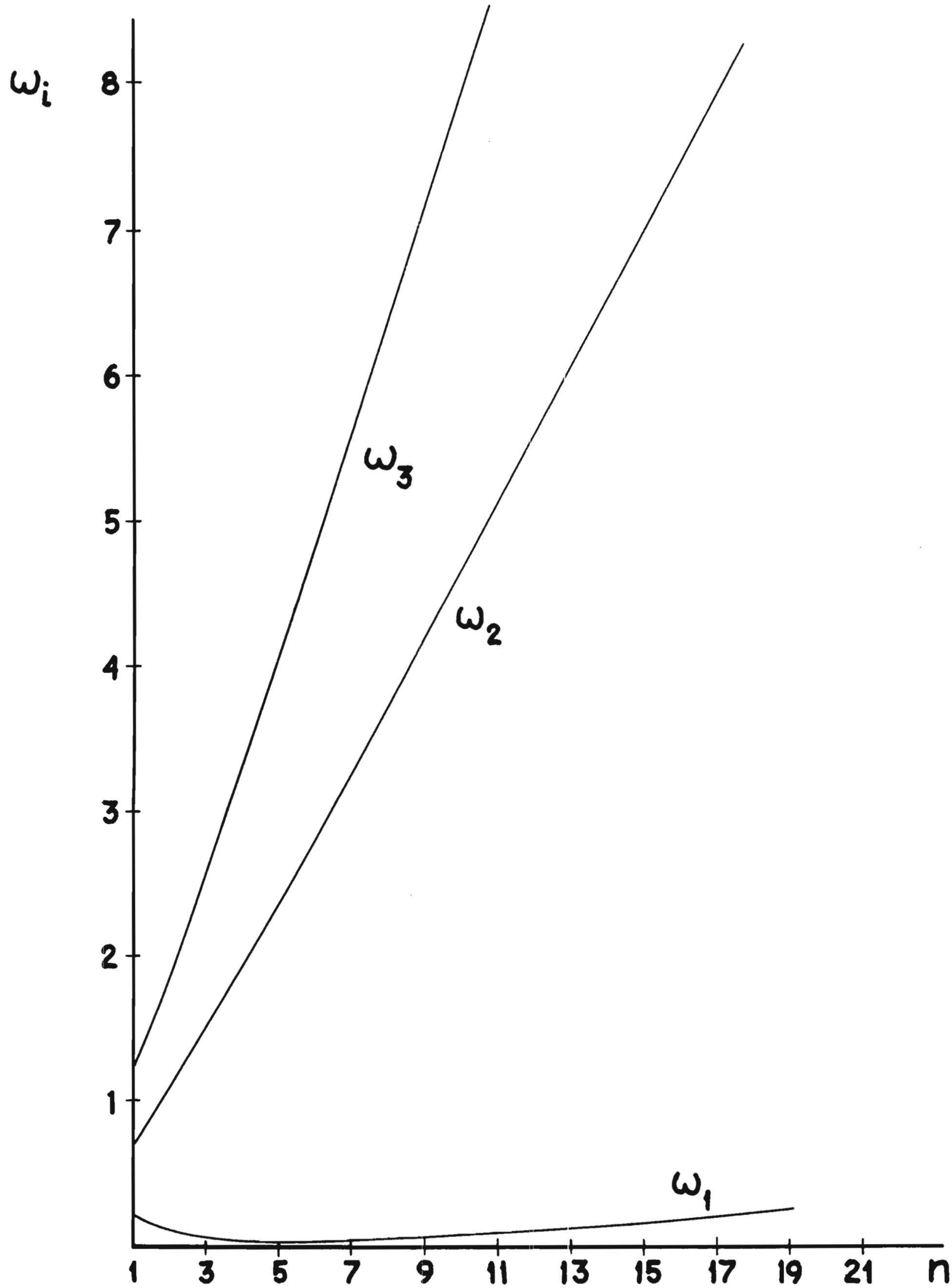

FIG. 2
NONDIM. FUNDAMENTAL FREQUENCIES

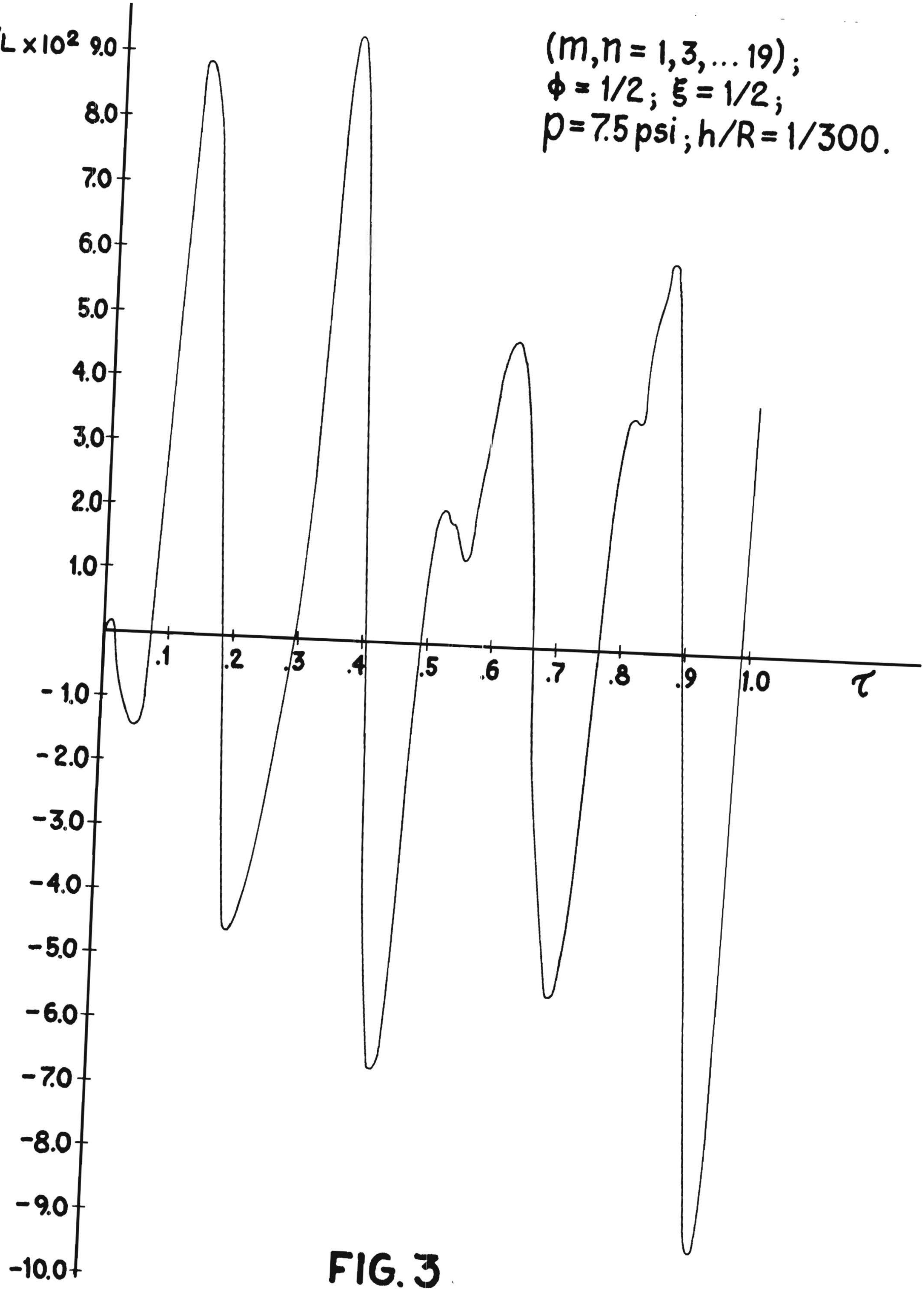

FIG. 3
W/L VS τ

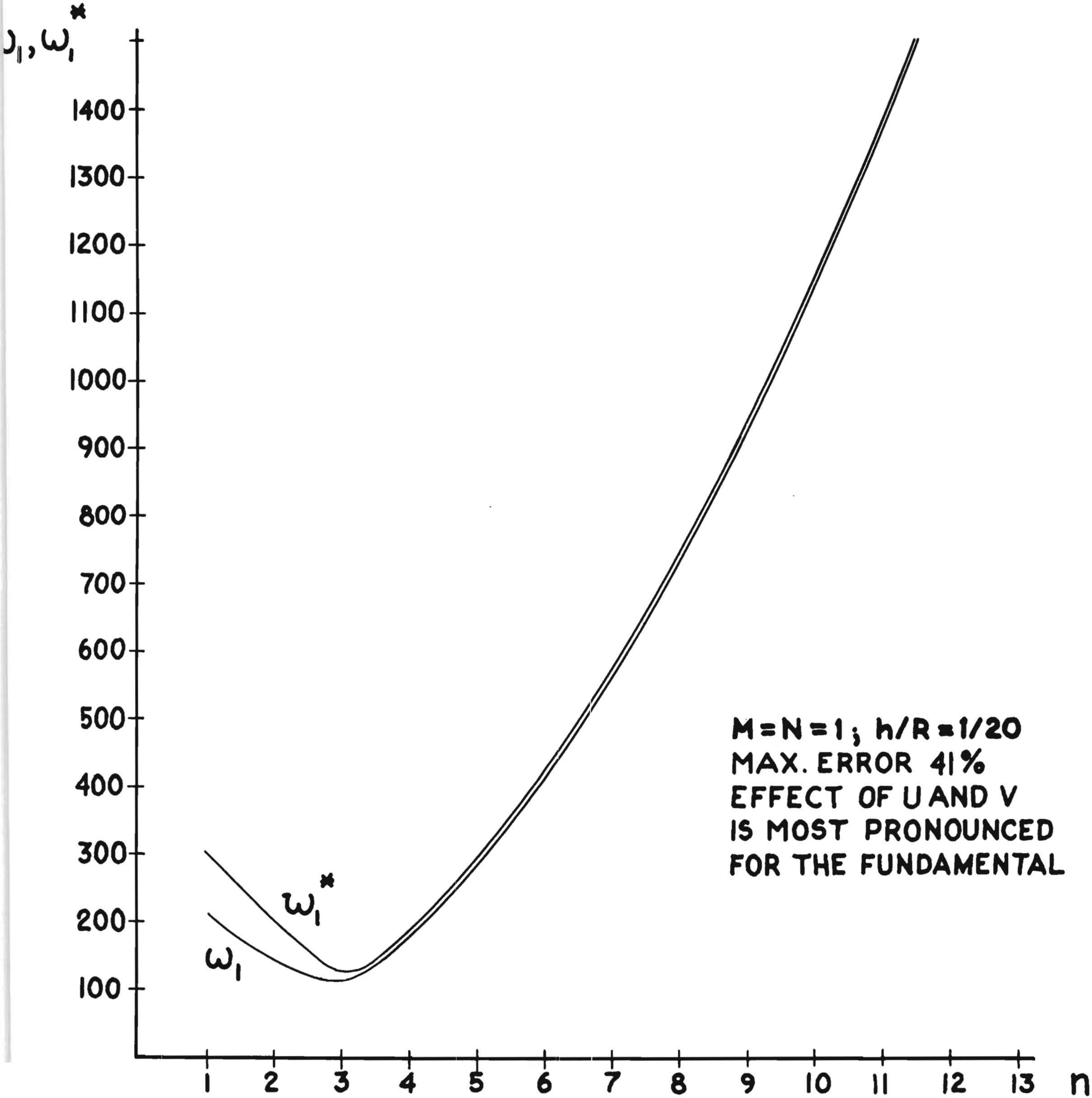

FIG. 4

ω_1, ω_1^* VS n

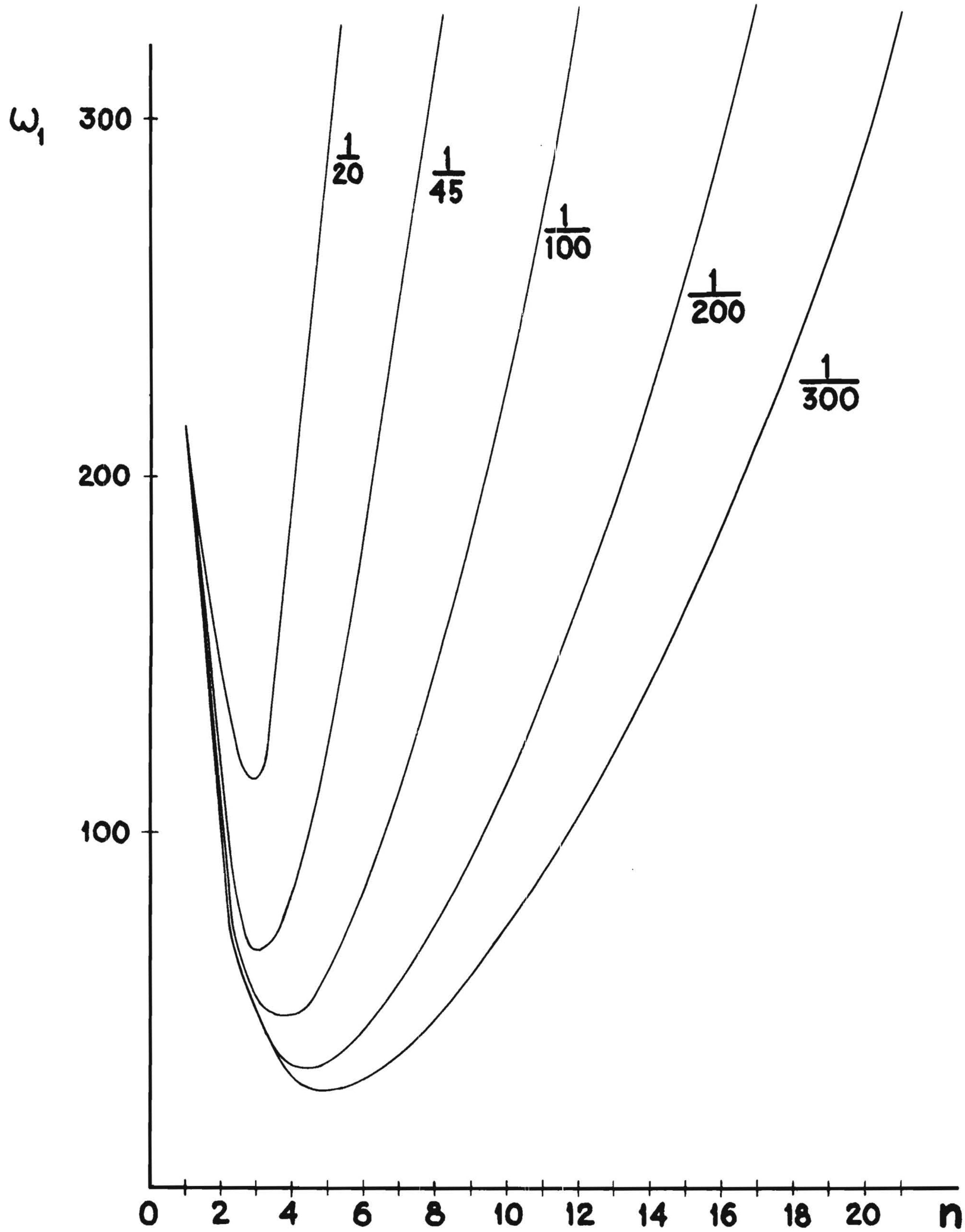

FIG. 5

ω_1 VS n FOR VARYING h/R

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