Large interaction asymptotics for the Spin-Boson model

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Joint work with Jacob Schach Møller

Let $\mathcal{H}_b = L^2(\mathbb{R}^\nu)$ denote the single particle Hilbert space for the bosons. The total state space then becomes $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F}_s(\mathcal{H}_b)$.

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The Hamiltonian describing the system is

$$H_{e,g} = e\sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + g\sigma_x \otimes \phi(v)$$

where ω is the kinetic energy operator of one boson, $v \in \mathcal{H}_b$, $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices, $2e$ is the gap in the two-level system and g is the strength of the interaction.

First a basic statement

Theorem

Assume ω is injective and non-negative. Assume furthermore that $v \in \mathcal{D}(\omega^{-1/2})$. Then $H_{e,g}$ is self-adjoint for all e, g real. For $e, g \in \mathbb{R}$ we define $E_{e,g} = \inf \sigma(H_{e,g})$. Then $(e,g) \mapsto E_{e,g}$ is concave and in particular continuous.

The Spin Boson model-Introduction

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The following theorem goes back to a result due to Gerard and a paper by Herbst and Hassler.

Theorem

Assume ω *is injective and non-negative. Assume furthermore that* $v \in \mathcal{D}(\omega^{-1})$ *and* $m = \operatorname{essinf} \omega > 0$ *or* ω *goes to infinity at infinity. Then* $E_{e,g}$ *is an eigenvalue for* $H_{e,g}$ *for all* $e, g \in \mathbb{R}$. *Furthermore, it is non degenerate.*

The Spin Boson model- Diagonalization

First we introduce the so called "Fiber-Hamiltonians" corresponding to the spin boson model. They are operators on $\mathcal{F}_s(\mathcal{H}_b)$ given by the formal expression

$$F_{e,g} = e(-1)^N + d\Gamma(\omega) + g\varphi(v)$$

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Define $\Sigma(e,g) = \inf(\sigma(F_{e,g}))$.
There is a unitary map $U : \mathcal{H} \to \mathcal{F}_s(\mathcal{H}_b) \oplus \mathcal{F}_s(\mathcal{H}_b)$ such that

$$UHU^* = F_{-e,g} \oplus F_{e,g}$$

so we may just analyze the fibers!

The fibers

Theorem

Assume that $v/\omega \in \mathcal{H}_b$. Define

$$U_g = exp(ig\phi(iv/\omega))$$
$$\tilde{F}_{e,g} = U_g F_{e,g} U_g^* + g^2 \|v/\omega^{1/2}\|^2$$

Then $\tilde{F}_{e,g}$ converges to $d\Gamma(\omega)$ in strong resolvent sense. If the mass is positive then the convergence is in uniform resolvent sense.

One may then easily prove the following.

Corollary

Assume that $m > 0$ then $\Sigma(e,g) + g^2\|v/\omega^{1/2}\|^2$ converges to 0 for g tending to ∞. Furthermore for g large enough $F_{e,g}$ has a non degenerate ground state and we may pick the ground state vectors ψ_g such that

$$\lim_{g\to\infty} \|\psi_g - U_g^*\Omega\| = 0$$

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One may also show that

$$\lim_{g\to\infty} |\langle \psi_g, N\psi_g\rangle - g^2\|v/\omega\|^2| = 0$$

Conclusion

It follows directly from the equation

$$UH_{e,g}U^* = F_{-e,g} \oplus F_{e,g}$$

that $H_{e,g}$ will have two non degenerate eigenvalues in the mass gap $[E(e,g), E(e,g) + m]$ for sufficiently large g. (The eigenvalues in each fiber are different due to non degeneracy of the total system)

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that $H_{e,g}$ will have two non degenerate eigenvalues in the mass gap $[E(e,g), E(e,g)+m]$ for sufficiently large g. (The eigenvalues in each fiber are different due to non degeneracy of the total system) Furthermore, the difference between the eigenvalues goes to 0 for g tending to infinity. Furthermore any higher order eigenvalues in the mass gap must converge to $E(e,g)+m$ as g tends to infinity.

Thank You For Your Attention