In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

7/25/68

GROUP THEORETIC STRUCTURES IN THE FIXED CHARGE TRANSPORTATION PROBLEM

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

Curtis Johnston Tompkins

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Industrial and Systems Engineering

Georgia Institute of Technology

July, 1971

GROUP THEORETIC STRUCTURES IN THE

FIXED CHARGE TRANSPORTATION PROBLEM

Approved:

Chairman

Date approved by Chairman: Aug. 25, 1971

ACKNOWLEDGMENTS

Dr. V. E. Unger deserves my gratitude for introducing me to group theoretic integer programming and for his many contributions to the work presented in this thesis. His guidance and direction are greatly appreciated.

Dr. J. A. Harrison contributed "above and beyond the call of duty" to my doctoral research. His conscientious and helpful attitude as a member of my reading committee will long be remembered and appreciated.

Dr. D. E. Fyffe and Dr. L. A. Johnson have my sincere thanks for guidance and inspiration throughout my doctoral program at the Georgia Institute of Technology. I also wish to thank Dr. Fyffe for serving on my reading committee, and I thank Dr. Johnson and Dr. W. R. Smythe, Jr. for participating in the final oral examination.

Two individuals have had particularly strong influence on me during my doctoral program. Dr. R. N. Lehrer, Director of the School of Industrial and Systems Engineering, has provided inspirational leadership and support. My special thanks are due to Dr. Lehrer for his help and understanding during my last year at Georgia Tech. Dr. H. E. Smalley, Director of the Health Systems Research Center, introduced me to health systems applications of industrial engineering and was my advisor during the doctoral study. I am grateful to him for providing many exciting opportunities to learn about the health care delivery field and to accept responsibilities for interesting projects at the Research Center. If it had not been for Dr. Smalley, I probably would not have attended Georgia Tech.

Mr. Louis P. Pante, Director of Operations Research at the American Can Company, contributed to the author's perspective of the problem investigated in this research; his interest in this work is appreciated. The support of my research by the American Can Company is gratefully acknowledged.

Many fellow graduate students deserve my thanks for positive contributions and inspirations, directly or indirectly, during my doctoral program. These fine people are too numerous to mention individually, but I express my genuine appreciation to them all.

I also wish to thank Mrs. Betty Sims for the excellent job she has done in typing the final draft of this thesis, and Mrs. Dollie Metts and Mrs. Lourdes Abella for their typing of earlier drafts and for their enduring and friendly attitudes throughout the documentation phase of the research project.

My most special thanks go to my wife Kathy for her encouragement, patience, endurance, and that magic ingredient love. I am greatly indebted to her and my sons Robert and Joseph for their sacrifice and support.

This work is dedicated to my parents who have most generously tolerated the absence of their only grandchildren and who, somehow, engendered in me a thirst for knowledge and a spirit of optimism.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

The purpose of this research is to investigate the underlying algebraic structures of certain fixed-charge transportation problems. Primary objectives are to determine inherent group theoretic properties of fixed-charge transportation problems and to specify how these properties may be exploited advantageously.

The group problem is especially structured with predictable interrelationships among the group variables. It is shown that the order of the subgroups is bounded by $\max\{\min(D_j, S_i)\}\ \forall(i,j)$ where D_j is the demand at destination j and S_i is the supply available at source i. When a fixed-charge transportation problem is solved as a linear program, the number of y_{ij} (0-1) variables represented in the optimal LP basis is also the dimensionality of the group problem constraint. The components of the group elements are exclusively members of $\{0,1,-1\}$, but the group elements corresponding to y_N are always null. The interrelationships among the variables will not permit strict interpretation of the group theoretic (asymptotic) approach in some instances; specifically, additional procedural steps may be required to produce a feasible (nonnegative) solution.

A group theoretic solution procedure is devised which guarantees an optimal, feasible solution to the fixed-charge transportation problem. The procedure begins by solving the associated linear program, thereby establishing an upper bound on the objective of the group problem and

providing information necessary for formulation of the group problem. Upper bounds are established on each group variable, and enumeration proceeds by investigating certain subsets of the variables. Bounding continues as the search progresses through the subgroups, and as a better feasible solution is discovered for the group problem, feasibility is checked in the original problem.

CHAPTER I

INTRODUCTION

Purpose

The purpose of this research is to investigate the underlying algebraic structures of certain fixed-charge transportation problems. It is hoped that detailed knowledge of these structures will enable one to determine inherent characteristics and properties of fixed-charge transportation problems that can be exploited in solution procedures and that may provide a more solid basis for choosing among algorithms. (We will sometimes refer to the fixed-charge transportation problem as the FCTP.)

Objectives

The primary objectives of this research are to determine inherent group theoretic properties of fixed-charge transportation problems and to specify how these properties may be exploited advantageously in respect to algorithmic solution procedures. A secondary objective is to gain experience in the application of group theory to linear decision models. Fortunately, this secondary objective is satisfied as a natural consequence of activities directed toward the primary objectives.

Formulation of the Problems

The problem investigated in this dissertation is a special case of the mixed-integer linear programming problem. Hadley [40] demonstrates how to convert the fixed-charge problem to an integer programming problem. The relationships among the fixed-charge transportation problem and other fixed-charged problems are discussed by Ellwein [25].

The *mixed integer problem* has the following form:

$$\text{(P1)} \qquad \begin{aligned} &\text{Minimize} \quad c_1x + c_2y \\ &\text{subject to} \quad A_1x + A_2y \ge b \\ &x \ge 0,\ y \ge 0 \text{ and integer} \end{aligned}$$

where c_1, c_2, b are given vectors and A_1, A_2 given matrices of appropriate dimension. Efficient algorithms are yet to be developed for handling large mixed integer problems. In order to solve large fixed-charge problems and other special classes of problems, the structure of the specific problem at hand must be exploited. Thus, we are motivated to focus on the following problems.

The *linear fixed-charge problem* [42,65,66] may be stated as follows:

$$\text{(P2)} \qquad \begin{aligned} &\text{Minimize} \quad cx + fy \\ &\text{subject to} \quad Dx \ge b \\ &-Ix + My \ge 0 \\ &Cy \le e \\ &x \ge 0,\ y = 0 \text{ or } 1, \end{aligned}$$

where I is the identity and M a diagonal matrix of upper bounds m_j on the values x_j, the components of $\mathbf{x}$. The constraint $Cy \le e$ represents a limitation on the number of resources (e.g., plants, machines). This problem exists when a fixed charge is incurred for using an activity at all; that is, the cost for using activity x_j is $c_j x_j + f_j$ if $x_j > 0$ and is zero if $x_j = 0$.

The *fixed-charge transportation problem* (FCTP) [4,51,56,64] is a particular type of linear fixed-charge problem. In this case, we have a transportation problem in which a fixed charge arises for every route used. Thus, we have problem (P2) where $D\mathbf{x} \ge b$ are taken to be the ordinary transportation problem constraints and $Cy \le e$ is void:

$$
\text{(P3)} \qquad
\begin{aligned}
\text{Minimize} \quad & \sum_i \sum_j c_{ij} x_{ij} + \sum_i \sum_j f_{ij} y_{ij} \\
\text{subject to} \quad & \sum_i x_{ij} \ge D_j, \quad \forall j \\
& \sum_j x_{ij} \le S_i, \quad \forall i \\
& x_{ij} - m_{ij} y_{ij} \le 0, \quad \forall (i,j) \\
& x_{ij} \ge 0, \quad \forall (i,j) \\
& y_{ij} = 0 \text{ or } 1, \quad \forall (i,j)
\end{aligned}
$$

where D_j is the demand at destination j, S_i is the supply available at source i, m_{ij} is the capacity of route (i,j), j=1,...,n, and i=1,...,m.

Importance of the Problem

Fixed-charge problems occur widely in business and industry; yet, only recently have reasonably effective solution methods been developed [19,20,22,25,37,49] to solve, often suboptimally, these problems. The emphasis to date has been on developing better implicit enumeration approaches for problems with fixed-charges. These approaches are closely tied to computing machinery and can be evaluated only by the efficiency of its application to numerical problems of significance [8]. Unfortunately, research and development along the lines of implicit enumeration has revealed either little or nothing about the *structures* of the problems.

Problems of public facility location (e.g., hospitals, clinics, schools), private facility location (e.g., plants, warehouses, banks, retail outlets), routing with fixed-charges attached to the opening of routes, product-mix, and other activities that incur set-up or expansion costs are plentiful. Much work remains before effective, efficient optimization methods can be developed and applied to these various fixed-charge problems.

As Spielberg [64] has said of the FCTP (P3): "While the statement of the problem is simple, its practical solution is known to offer great difficulties."

Scope and Limitations

The research reported herein considers the group structure of the fixed-charge transportation problem (FCTP).

Linearity is assumed throughout this investigation except for the obvious non-linearities created by the fixed charges. It is also assumed that all parameter and coefficient values are known with certainty.

Recommendations are made for subsequent research based on these findings.

Organization

Chapter II presents a general survey of literature pertinent to the fixed-charge transportation problem and highlights several important works.

Chapter III discusses the basic concepts of group theoretic integer programming used in the research.

Chapter IV presents an analysis of the group structure of the fixed-charge transportation problem, develops several theorems based on the group structure, and applies these theorems in a group theoretic solution procedure.

Chapter V concludes the research with a summary of the results and recommendations for further research.

A summary of the fundamentals of group theory is presented in Appendix A. The remaining appendices report experimental results.

CHAPTER II

THE FIXED-CHARGE TRANSPORTATION PROBLEM

Introduction

The fixed-charge transportation problem (P3) is a frequently occurring special type of fixed-charge problem that may pertain in various settings. The literature dealing directly with the FCTP is relatively limited; however, a much wider range of publications has been produced on the more general fixed-charge problem (P2). To develop a perspective of the problem domain, let us briefly discuss the general fixed-charge problem, and then focus on the fixed-charge transportation problem.

Fixed-Charge Problems

Hirsch and Dantzig presented a fundamental exposition of the fixed-charge problem in a Rand paper in 1954; this definitive work was republished as [42] in 1968. One of the prominent results reported by Hirsch and Dantzig is that the fixed-charge objective function is concave and that the minimization of a concave functional, defined over a convex polyhedron, takes on its minimum at an extreme point. Thus, all of the methods developed to solve (P2) are so-called extreme point methods.

Hadley [40] indicated in 1964 how the fixed-charge problem (P2) can be written as a mixed-integer program (P1). The implication was that any mixed-integer algorithm could be used to solve the problem exactly. Unfortunately, available mixed integer algorithms are computationally feasible only for small problems. Hadley's formulation was followed by a series of attempts by other researchers to obtain exact solutions to the fixed-charge problem. Most notable among these activities are the works of Gray [37], Steinberg [65,66], Murty [55], and Jones and Soland [49].

Gray [37] developed a decomposition algorithm similar to Benders' algorithm. There are two major differences between Gray's method and Benders' method:

1. The Benders algorithm requires the solution of a series of integer programs whereas Gray's algorithm involves only one integer program.

2. The Benders algorithm, being a general algorithm, does not make use of the specific structure of the problem. Gray's algorithm makes use of the relations of the integer and continuous variables.

Gray reports that his method requires an average of 16 minutes to solve a 5×7 fixed-charge transportation problem and as much as 22 minutes to solve a 30-site warehouse location problem.

Steinberg [65,66] presents a branch and bound algorithm and compares the algorithm's computational speed with that of several heuristic algorithms. The heuristics, which almost always provide very good, if not optimal, solutions, are several orders of magnitude faster than Steinberg's method which requires as much as 47 minutes on an IBM 360/50 to solve a 15×30 problem.

Murty [55] uses the result of Hirsch and Dantzig [42]; i.e., that the minimum of the fixed-charge problem will occur at an extreme point of the constraint set. He presents an algorithm for finding the adjacent extreme points and for searching systematically among these extreme points for the minimum total cost. Computational results are reported for Murty's algorithm by Gray [38].

Jones and Soland [49] report a branch and bound algorithm for the fixed-charge problem with piecewise linear costs. Essentially this is the problem of incurring fixed-charges at several stages or levels of production. The additional fixed charges occur for such things as plant expansion. Jones and Soland indicate reasonably favorable computational experience with their method.

A considerable amount of effort has been devoted to developing computational schemes to generate good, if not optimal, solutions to fixed-charge problems. Cooper and Drebes [19], Cooper and Olson [20], Steinberg [65], Walker [69], and Denzler [22] all report good computational results with their approximate solution methods.

Cooper and Drebes' [19] approximate method uses adjacent extreme point methodology. Their computational experience indicates that the method will yield the optimal solution a high percentage of the time and, when not optimal, it provides a good approximation. There are two particularly interesting features of the Cooper-Drebes method:

1. Objective function costs are recalculated at certain stages in their algorithm as

$$\hat{c}_j = c_j + \frac{f_j}{x_j}, \quad x_j > 0.$$

2. At certain times in the calculation, a vector is chosen to enter the basis with the least fixed charge of the non-basic variables. At other times, a vector is chosen to leave with the highest fixed charge in the basic set and a vector then enters according to simplex rules.

Cooper and Olson [20] build on the work of Cooper and Drebes in an attempt to improve the approximate methods using basic perturbation techniques. The Cooper-Olson perturbation approach is significantly more effective than the earlier heuristics of Cooper and Drebes according to the computational experience reported in [20]. The same test problems are used in both [19] and [20].

Steinberg [65] and Walker [69] both use the linear programming criterion for a vector to enter the basis. Walker's computational experience is especially encouraging. Cooper and Drebes [19] randomly generated 290 - (5×10) fixed charge problems to test their method. Steinberg presents these problems and their solutions in his thesis [65]. Walker and Steinberg both use the Cooper-Drebes problems to test their algorithms (as did Cooper and Olson), and it is on the basis of experience with this commonly-used problem set that Walker's methods may be judged superior. In fact, Walker's method determined the optimal solution to all test problems in relatively fewer iterations than the other approximate methods. Walker presents a counter-example for his methods.

The Fixed-Charge Transportation Problem

Since the fixed-charge transportation problem is a particular case of the general fixed-charge problem, the approaches discussed above apply to the FCTP. Additionally, Kuhn and Baumol [51], Balinski [4], Spielberg [64], Dwyer [24], and Robers and Cooper [56] have specifically investigated the FCTP. Kuhn and Baumol [51] deal with the FCTP in terms of the Navy's periodic redistribution of inventories for over 7,000 different stock items. They present a computationally simple "forced degeneracy" method that makes small adjustments in the right-hand side of (P3). Unfortunately, the test problem results reported by Kuhn and Baumol indicate that their method may produce a terminal objective function value that is as much as 29 per cent greater than the true minimum.

Balinski [4] replaces the non-linear fixed-charge objective function with an approximate linear objective function, and solves the resulting problem using the standard transportation algorithm. He also finds bounds on the optimal exact solution. Robers and Cooper [56] refine Balinski's method and produce significantly more accurate solutions on test problems reported in [56]; in fact, the Robers-Cooper method yielded optimal solutions for nearly all problems tested.

Spielberg [64] applies Benders' partitioning procedure to the fixed-charge transportation problem to obtain an exact solution. He reports that his method, which employs the stopped simplex method and a branch and bound scheme, is effective for problems with less than 150 fixed charges; however, for large problems computational results are discouraging.

Dwyer [24] applies his method of completely reduced matrices to the FCTP. He presents exact and approximate methods for finding the most degenerate solution for the case of equal fixed charges which are large compared to variable costs; this most degenerate solution is shown to be optimal. Dwyer also discusses approximate solutions for the case of unequal fixed charges.

We briefly investigate the work of Balinski [4], Robers and Cooper [56], Gray [37], and Murty [55] pertinent to the FCTP in the following sections.

Balinski's Approximation Method

Balinski [4] presents an approximation procedure for the fixed-charge transportation problem based on the following theorems.

Theorem 2.1. Let $(\overline{P3})$ be the program (P3) with integer constraints ignored. $\{\bar{x}_{ij},\bar{y}_{ij}\}$ is a solution to $(\overline{P3})$ only if $\bar{x}_{ij} = m_{ij}\bar{y}_{ij}$.

The proof of Theorem 2.1 as offered by Balinski proceeds as follows: Consider $(\overline{P3})$. If $\bar{y}_{ij} = 0$ then $\bar{x}_{ij} = 0$, and thus $\bar{x}_{ij} = m_{ij}\bar{y}_{ij}$. Otherwise, suppose $\bar{y}_{ij} > 0$ and $\bar{x}_{ij} < m_{ij}\bar{y}_{ij}$. Then $\bar{y}_{ij}$ can be decreased without violating the constraints of (P3) but with a decrease in the value of the objective function. Thus $\bar{x}_{ij} = m_{ij}\bar{y}_{ij}$ for any solution $\bar{x}_{ij}$, $\bar{y}_{ij}$ to $(\overline{P3})$.

Theorem 2.2. There exist solutions $\{x^o_{ij},y^o_{ij}\}$ to (P3) such that the x^o_{ij} are integers.

This theorem follows from the unimodular property of the submatrix of constraint coefficients corresponding to the column vectors associated with the x_{ij}, $\forall(i,j)$.

The approximation procedure is a sequence of three basic steps.

Step 1. Given (P3) derive a problem (P3*) by letting $c'_{ij} = c_{ij} + \dfrac{f_{ij}}{\min(D_j, S_i)}$.

$$\text{(P3*)} \qquad \begin{aligned} &\text{Minimize} \quad \lambda_{(P3^*)} = \sum_i \sum_j c'_{ij} x_{ij} \\ &\text{subject to} \quad \sum_j x_{ij} \le S_i, \quad \forall i \\ &\qquad\qquad\quad\;\; \sum_i x_{ij} \ge D_j, \quad \forall j \\ &\qquad\qquad\qquad\quad x_{ij} \ge 0 \end{aligned}$$

Step 2. Find an integer solution $\{x'_{ij}\}$ to (P3*) and its value λ_{P3^*} by using some transportation problem algorithm.

Step 3. Determine a feasible solution $\{x^*_{ij}, y^*_{ij}\}$ to (P3) by letting

$$x^*_{ij} = y^*_{ij} = 0 \quad \text{if } x'_{ij} = 0,$$

and

$$x^*_{ij} = x'_{ij} \quad \text{and} \quad y^*_{ij} = 1 \quad \text{if } x'_{ij} > 0.$$

Denote the value of the objective function in (P3) in this case by $\lambda(x^*, y^*)$.

Balinski shows that λ_{P3*} is a lower bound on the optimal value for (P3); $\lambda(x^*,y^*)$ is an upper bound on the optimal value for (P3).

Robers and Cooper [56] make two relevant observations about Balinski's method. First, it would seem that the approximation should be more accurate when there are many more destinations than sources. Second, it seems that the approximation should become more accurate as the fixed charges become smaller.

To evaluate Balinski's method, Robers and Cooper used three statistics: the location index, the error percentage, and the interval width percentage. Where the exact optimal solution value for (P3) is denoted by Z^*, the three statistics of Robers and Cooper are defined as follows:

1. The location index (L.I.)

   $$\text{L.I.} = (Z^*-\lambda_{P3*})/\left(\lambda(x^*,y^*)-\lambda_{P3*}\right)$$

   L.I. measures where Z^* falls in the interval $[\lambda_{P3*},\lambda(x^*,y^*)]$.

2. The error percentage (E.P.)

   $$\text{E.P.} = 100\left(\lambda(x^*,y^*)-Z^*\right)/Z^*$$

3. The interval width percentage (I.W.P.)

   $$\text{I.W.P.} = 100\left(\lambda(x^*,y^*)-\lambda_{P3*}\right)/Z^*$$

   This statistic measures the width of the interval $[\lambda(x^*,y^*),\lambda_{P3*}]$.

The results of the evaluation of Balinski's method by Robers and Cooper are summarized in the remaining paragraphs of this section.

The lower bound λ_{P3*} is generally well below the value Z^* of the true solution.

All three statistics tend to increase as fixed charges are made larger.

The error percentage and interval width percentage both tend to increase as the problem size increases; fortunately, the statistics do not increase very rapidly as the problem size increases.

The approximation tends to improve as the ratio of the number of destinations to the number of sources increases. This is true regardless of the magnitude of the fixed charges.

Robers and Cooper Approximation Method

Robers and Cooper [56] extend the method of Balinski by searching adjacent extreme points of the convex set of feasible solutions to (P3) beginning with the solution produced by Balinski's method. This procedure is based on a theorem by Balinski which, in turn, is a special case of a theorem due to Hirsch and Dantzig [42]. The theorem stated by Balinski is now given.

Theorem 2.3. Any solution $\{x_{ij}^{o}, y_{ij}^{o}\}$ to (P3) is a vertex of the polyhedral convex constraint set of $(\overline{P3})$ (the program (P3) with integer constraints ignored).

The method of Robers and Cooper may be summarized as the following step-wise procedure.

Step 1. Find the solution $\{x_{ij}'\}$ by Balinski's approximation method. Call this the current solution.

Step 2. Calculate all of the $z_{ij}-c_{ij}$ for the current solution.

Step 3. Consider individually the variables not presently in the basis and find the change in cost if each is allowed to become positive. This is accomplished as follows:

a. Determine the basic loop involving *each* non-basic variable x'_{st}.

b. Find the smallest element x'_{uv} in the loop which decreases as x'_{st} increases. If more than one vector satisfies this condition, go to e.

c. Compute

$$\Delta_{st} = -(z_{st}-c_{st})x'_{uv} - f_{uv} + f_{st}$$

where Δ_{st} is the increase (decrease if negative) in the value of the objective function which would result if x'_{uv} were replaced by x'_{st}.

d. Determine

$$\Delta_{min} = \min\{\Delta_{st}\}$$

over all Δ_{st} calculated for the non-basic variables. If $\Delta_{min} < 0$, the variable x_{st} which yielded Δ_{min} is allowed to enter the basis to produce a new current solution which we again denote $\{x'_{ij}\}$; return to 2. If $\Delta_{min} \geq 0$, terminate with the current solution $\{x'_{ij}\}$ being the best solution available by means of this algorithm.

e. If a tie exists among k variables (which we can, for convenience, denote by $x'_{pq}, x'_{uv}, \ldots$) we would compute

$$\Delta_{st} = -(z_{st}-c_{st})x'_{uv} - f_{pq} - f_{uv} - \dots + f_{st}$$

where we subtract the k fixed costs associated with the k basic variables which are tied. Go to d.

Computational experience reported by Robers and Cooper is encouraging for two reasons. First, all but two of the 280 experimental problems designed by Robers and Cooper were optimized by their method; second, their experience seems to indicate that fixed-charge transportation problems are well behaved at least in the neighborhood of the optimal solution.

According to Robers and Cooper the average computation time for the 280 experimental problems was one minute per problem on the IBM 7072. These 280 problems were of the following sizes:

Number of Sources	14	8	6	5
Number of Destinations	14	24	30	35
Number of Problems Solved	40	160	40	40
Number of Optimal Solutions	39	160	40	39

It is interesting that, while Robers and Cooper had solved 11 28×28 problems and 5 48×48 problems by Balinski's method, they indicate they did not solve the larger problems by their method "because the computation time would have been excessive."

<u>Gray's Exact Solution Method</u>

Gray [37] presents an algorithm that searches systematically among the extreme points defined by the fixed charges and iteratively decreases the maximum allowable fixed cost. Gray's algorithm may be summarized as follows:

Step 1. Solve problem (P3) by Balinski's approximation method to determine $\lambda(x^*,y^*)$, the upper bound on the solution value as defined earlier. Solve the associated transportation problem $\big($(P3) *ignoring* fixed costs and setting all $y_{ij}=1\big)$ to determine what Gray calls cx_o, the minimum variable cost. An upper bound on the total fixed charge may now be obtained as FMAX = $\lambda(x^*,y^*) - cx_o$, i.e., FMAX = $\sum_i \sum_j f_{ij}y_{ij}$, and this relation is used as a constraint in the problem.

Step 2. Generate y vectors (0,1 elements) that satisfy the condition FMAX $\geq \sum_i \sum_j f_{ij}y_{ij}$ as well as the following conditions:

a. At least n and at most m+n-1 of the fixed-charge variables (y) are equal to 1, and the others are equal to zero (where m = number of sources, n = number of destinations). We might write this as

$$n \leq \sum_i \sum_j y_{ij} \leq m + n - 1,$$

$y_{ij} = 0$ or 1, $\forall i,j$. Further, we may write

$$\sum_{i=1}^{m} y_{ij} \geq 1, \quad \forall j,$$

to assure at least one route being open to each destination.

b. $\sum_{i=1}^{m} S_i y_{ij} \geq D_j$, $\forall j$ where S_i denotes *currently* available supplies at source i; i.e., available supply through *open routes* must be greater than or equal to demand at the respective destinations.

Step 3. Solve the transportation problem with unit costs for closed routes set equal to M (some very large number). If total cost is better than any found so far, store the result and return to 1 to compute a new upper bound FMAX; otherwise, return to 2.

The algorithm terminates when no new 0-1 vectors can be found in Step 2. The optimal solution is the lowest total cost solution found during the computations.

Gray indicates on page 86 of [37] that his algorithm works well for problems of size up to 6×8.

Murty's Exact Solution Method

Murty [55] describes a method that searches systematically among the extreme points of the transportation subproblems and iteratively decreases the maximum allowable variable cost. Essentially, Murty uses an adjacent extreme point method (that initializes at the solution with value cx_o as defined in conjunction with Gray's Method) to generate extreme points of the convex constraint set for the transportation problem in "rank order"; i.e., in increasing order of $\sum_i \sum_j c_{ij}x_{ij}$.

As the *rth* extreme point is generated, the variable cost Z_r and the fixed cost D_r are calculated and used to calculate the following quantities:

$$\delta_r = \min_{k=1,\ldots,r} \{Z_k - Z_1 + D_k - D_o\}$$

and

$$\Delta_r = Z_r - Z_1'$$

where $D_o = \min_k D_k$.

Optimality is recognized when

$$\delta_r \leq \Delta_r,$$

and the best current solution must be optimal.

Gray [37] indicates that his algorithm seems to be particularly suitable when the fixed costs are larger compared to the variable costs, whereas the Murty algorithm seems to be more suitable for large variable and small fixed costs.

Concluding Remarks

Gray [37,38], Murty [55], and Spielberg [64] have presented exact solution methods for the fixed-charge transportation problem. Gray reports satisfactory experience with his algorithm for problems up to 6×8 in size. Murty's method appears to be similarly limited according to the results reported in [38]. According to [8, page 26] Spielberg's experience with Benders' method indicated that particular approach is not effective for more than, say, 150 fixed charges (routes).

No one has specifically used group theory in his analysis of the fixed-charge transportation problem. The regularity of the problem structure has been exploited, but the underlying group structures have never been thoroughly investigated, as far as is known. Perhaps the reason for this will be made clear in Chapter IV when we see that one of the primary "measures of attractiveness" for a problem in respect to group theory is, in some cases, misleading; that is, the order of the group is usually quite large. It is conjectured that this fact has deterred researchers from using a group theoretic approach to explore the FCTP and other fixed-charge problems.

CHAPTER III

GROUP THEORETIC INTEGER PROGRAMMING

Introduction

The purpose of this chapter is to present some of the definitions, concepts, and theorems from group theoretic integer programming relevant to this research. A summary of the fundamentals of group theory is presented in Appendix A. Most of the material presented in this chapter is based on Gomory's asymptotic theory [33,34,35].

Balinski and Spielberg [8] partition integer programming methods into three main areas; these three areas are identified as algebraic, combinatorial, and implicitly enumerative. We might add approximative or heuristic methods as a fourth area. Group theoretic integer programming methods generally are classified as algebraic and have their origin in Gomory's cutting-plane work during the late 1950's. A thorough review of the evolution of algebraic (and other) approaches to integer programming is given by Balinski [6,7] and Balinski and Spielberg [8]. Paralleling [8], Hu places particular emphasis on group theoretic methods in his textbook [46]. We shall now discuss Gomory's concept of applying group theory to the integer programming problem.

Fundamental Methodology

Consider the linear integer program

$$\text{(P4)} \qquad \begin{aligned} &\text{Maximize} \quad z = cx \\ &\text{subject to} \quad Ax = b \\ &x \geq 0, \text{ integer} \end{aligned}$$

We proceed to solve (P4) by first solving the linear program associated with (P4); that is, we ignore the integrality constraint. The optimal L.P. solution is found to be $x_B = B^{-1}b$, $x_N = 0$, with basis matrix B, non-basis matrix N, and value $z = c_B x_B$. If x_B is not integer, we see that x_N must be increased to some nonnegative integer vector x_N^* so that $x_B^* = B^{-1}b - B^{-1}Nx_N^*$ also is integer and nonnegative.

Suppose rank A = m. Let A = [B,N] where B is m×m of rank m and N is m×n. Let $c = (c_B, c_N)$ and $x = (x_B, x_N)$. Then (P4) becomes

$$\begin{aligned} &\text{Maximize} \quad c_B x_B + c_N x_N \\ &\text{subject to} \quad Bx_B + Nx_N = b \\ &x_B, x_N \geq 0, \text{ integers.} \end{aligned}$$

Let M(I) be the module of all integer m-vectors and M(B) be the module of integer multiples of the columns of B. Let G be a factor group such that G = M(I)/M(B).

Let ϕ be the natural homomorphism of $M(I) \to M(I)/M(B)$. Since Ax = b or [B,N]x = b, it follows that $Bx_B + Nx_N = b$ and $x_B + B^{-1}Nx_N = B^{-1}b$ where x_B is now restricted to be integer-valued. Since ϕ is a homomorphism, it follows that $\phi(Bx_B) + \phi(Nx_N) = \phi(b)$. $Bx_B \in \mathrm{Ker}\ \phi$ so $\phi(Nx_N) = \phi(b)$. Let $N = [\alpha_1,\ldots,\alpha_n]$. Then $\phi([\alpha_1,\ldots,\alpha_n]x_N) = \phi(\sum_j \alpha_j x_j) = \sum_j \phi(\alpha_j)x_j = \phi(b)$. Let $\phi(\alpha_j) = g_j$; $\phi(b) = g_o$. Then the condition $Ax = b$ is equivalent to

$$\sum_{j=1}^{n} g_j x_j = g_o. \tag{1}$$

It is possible that some of the g's are equal; i.e., $\phi(\alpha_i) = \phi(\alpha_j)$ $(i\neq j)$. Let the set of images of all non-basic column vectors be the set η. Let $n' = |\eta| \leq D = |\det B|$. If $g\epsilon\eta$ let $t(g) = \sum_{j\epsilon J} x_j$ where $J = \{j|\phi(\alpha_j)=g\}$. Let t be the vector whose coordinates are the t(g). Then (1) is equivalent to

$$\sum_{g\epsilon\eta} t(g)\cdot g = g_o.$$

Since $Bx_B + Nx_N = b$, it follows that $x_B + B^{-1}Nx_N = B^{-1}b$, and that $c_Bx_B + c_Nx_N = c_BB^{-1}b - (c_BB^{-1}N-c_N)x_N$. Hence, maximizing $c_Bx_B + c_Nx_N$ is equivalent to minimizing $c_N^*x_N$ where $c_N^* = c_BB^{-1}N - c_N$.

Let $c^*(g) = \min_j c_j^*$, $\forall j \ni \phi(\alpha_j) = g$. Then it follows that minimizing $c_N^*x_N$ subject to $Bx_B + Nx_N = b$, $x_N \geq 0$, integer, is equivalent to

$$\text{(P5)}\qquad \begin{aligned} &\text{Minimize} && \sum_{g\epsilon\eta} c^*(g)t(g) \\ &\text{subject to} && \sum_{g\epsilon\eta} t(g)\cdot g = g_o \\ &&& t(g) \geq 0,\ \text{integers} \end{aligned}$$

EXAMPLE: Consider the problem

$$\text{Maximize} \quad x_1 + 5x_2$$

$$\text{subject to} \quad 4x_1 + x_2 \le 2$$

$$5x_1 + 15x_2 \le 9$$

$$x \ge 0, \text{ integers},$$

which we put in the form of problem (P4) as follows:

$$\text{Maximize} \quad x_1 + 5x_2$$

$$\text{(I)} \qquad \text{subject to} \quad 4x_1 + x_2 + s_1 = 2$$

$$5x_1 + 15x_2 + s_2 = 9$$

$$x \ge 0, \text{ integers}.$$

Solving the associated L.P. of (I) we find

$$x_B = \begin{bmatrix} s_1 \\ x_2 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 7/5 \\ 3/5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 15 \end{bmatrix}, \quad N = \begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} 1 & -1/15 \\ 0 & 1/15 \end{bmatrix}, \quad B^{-1}N = \begin{bmatrix} 55/15 & -1/15 \\ 5/15 & 1/15 \end{bmatrix},$$

$$\phi B^{-1}N = \begin{bmatrix} 10/15 & 14/15 \\ 5/15 & 1/15 \end{bmatrix}, \quad \phi B^{-1}b = \begin{bmatrix} 6/15 \\ 9/15 \end{bmatrix},$$

$$c_N^* = c_B B^{-1}N - c_N = [10/15 \quad 5/15],$$

$$D = |\det B| = 15.$$

We now have the group problem

$$\text{Minimize } 10/15\ t_1 + 5/15\ t_2$$

$$\text{(II)} \qquad \text{subject to } \begin{bmatrix} 10/15 \\ 5/15 \end{bmatrix} t_1 + \begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix} t_2 = \begin{bmatrix} 6/15 \\ 9/15 \end{bmatrix}$$

$$t \geq 0, \text{ integers.}$$

Thus, $g_1 = \begin{bmatrix} 10/15 \\ 5/15 \end{bmatrix}$, d_1 = order of $g_1 = 3$,

$g_2 = \begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix}$, d_2 = order of $g_2 = 15$,

$g_o = \begin{bmatrix} 6/15 \\ 9/15 \end{bmatrix}$, d_o = order of $g_o = 5$.

As we will see, there are only two irreducible feasible solutions to (II); they are

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

and

$$t = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

where the latter is optimal.

Thus,

$$x_N^* = \begin{bmatrix} x_1^* \\ s_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

and it follows that

$$x_B^* = B^{-1}b - B^{-1}Nx_N^* = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \ngeq 0.$$

This is a "non-asymptotic" problem as defined by Gomory [33]; that is, $x_B^* \ngeq 0$. In such a case, we must search back through the set of feasible solutions, beginning with the "next best" solution. Obviously in our present example, there is only one other feasible solution; that is, $x_N = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$ which produces $x_B^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ which is feasible and, therefore, the optimal solution to problem (I) is $x_1 = 0$, $x_2 = 0$.

Gomory has given a sufficient condition for the asymptotic case to prevail; that is, for $x_B^* \geq 0$. This condition is stated as Theorem 3.1.

Theorem 3.1. Let $K_B = \{y | B^{-1}y \geq 0\}$, and let $K_B(d)$ be the cone of points in K_B at a euclidean distance of d or more from the frontier of K_B. Further, let $D = |\det B|$ and let ℓ_{max} be the (euclidean) length of the longest nonbasic column of A. If $b \in K_B[\ell_{max}(D-1)]$, then any optimal solution to (P5) is a feasible (hence optimal) solution to (P4).

This condition is not necessary for an asymptotic solution to occur; since it is not a particularly tight condition, we would expect an optimal solution to (P5) to solve (P4) often, even for right-hand sides that do not satisfy the condition. In a recent paper [1], Balas has dealt with the applicability of Theorem 3.1 to the 0-1 integer program. His conclusion is somewhat startling: Not only can the right-hand side vector of a 0-1 problem never belong to the cone $K_B[\ell_{max}(D-1)]$, but it cannot even belong to a cone obtained from the latter by replacing $\ell_{max}(D-1)$, a number larger than the determinant of B, by any number greater than 1. Balas' conclusion is stated as Theorem 3.2.

Theorem 3.2. If (P4) is a 0-1 program, $b \notin K(d)$ whenever $d > 1$.

Balas presents a proof of Theorem 3.2 in [1]. He makes clear that Theorem 3.2 does not mean that group theory is irrelevant for the 0-1 case for, in spite of the theorem, an optimal solution to (P5) may still turn out to be feasible for the initial 0-1 program.

Let us now identify G and η in problem (I). Referring to our notation in Appendix A, we see that $G = gp(g_2) = gp\left(\begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix}\right)$; i.e., G is a cyclic group generated by $g_2 = \begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix}$. We may write G as

$$G = (g_2, 2g_2, 3g_2, \ldots, 15g_2) = \left(\begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix}, \begin{bmatrix} 13/15 \\ 2/15 \end{bmatrix}, \begin{bmatrix} 12/15 \\ 3/15 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

On the other hand, $\eta = \left(\begin{bmatrix} 10/15 \\ 5/15 \end{bmatrix}, \begin{bmatrix} 14/15 \\ 1/15 \end{bmatrix}\right)$ with $n' = |\eta| = 2$.

The convex hull of integer solutions to (P4) is denoted P_x; the convex hull of integer solutions to (P5) is denoted by P_η. P_η is a convex cone in n'-space. There is a point t of P_η associated with any point (x_B, x_N) of P_x such that the following conditions hold:

1. (x_B, x_N) is a vertex of $P_x \Longleftrightarrow$ the corresponding t is a vertex of P_η;

2. $\sum_{j=1}^{n} \bar{\pi}_j x_j \geq \bar{\pi}_o$ is an (n-1)-dimensional face (hyperplane) of $P_x \Longleftrightarrow \sum_{g\in\eta} \pi(g)t(g) \geq \pi_o$ is an (n'-1)-dimensional face of P_η, where $\pi(g) = \bar{\pi}_j$ for $g = \phi\alpha_j$;

3. if t^* is a vertex of P_η minimizing $\sum_{g\in\eta} c^*(g)t(g)$, then the corresponding vertex $x^* = (x^*_B, x^*_N)$ of P_x solves (P4) (except possibly $x^*_B \ngeq 0$) where x^*_N is defined as follows: $x^*_k = t(g)^*$ for exactly one k satisfying $c_k = \min\{c_j | \phi\alpha_j = g\}$, $x^*_j = 0$ otherwise; and $x^*_B = B^{-1}b - B^{-1}Nx^*_N$.

Thus, we realize that we may confine our investigation to P_η and minimization over P_η. The conditions listed above tell us that the extreme points or vertices of P_η constitute the set of possible solutions to (P4) and that the faces of P_η, given by $\sum_{g\in\eta} \pi(g)t(g) \geq \pi_o$, provide valid inequalities or cuts for (P4). We may write the following theorem.

<u>Theorem 3.3</u>. $\sum_{g\in\eta} \pi(g)t(g) \geq \pi_o > 0$ is a face of $P_\eta \Longleftrightarrow \pi = (\pi(g_1), \ldots, \pi(g_{n'}))$ is a basic feasible solution to $\sum_{g\in\eta} \pi(g)t(g) \geq \pi_o$

for all $t \in T$ where $T = \{t \mid \sum_{g \in \eta} g t(g) = g_0,\ t(g) \geq 0$, integer , i.e., T is the set of all nonnegative integer solutions to (P5).

Theorem 3.3 is proved in [35]. Note that T contains an infinite number of points; this may seem to diminish the worth of Theorem 3.3. However, let us proceed to the following definition and theorems.

Definition. An integer point t of P_η is *irreducible* if for any set of integers s(g) and r(g) the conditions $0 \leq s(g) \leq t(g)$, $0 \leq r(g) \leq t(g)$, and $\sum_{g \in \eta} s(g) \cdot g = \sum_{g \in \eta} r(g) \cdot g$ imply $r(g) = s(g)$ for all $g \in \eta$.

Theorem 3.4. Every vertex of P_η is irreducible.

Theorem 3.5. If $t \geq 0$ is irreducible, then $\prod_{g \in \eta} \left(1+t(g)\right) \leq D$ where D is the order of the group G.

Corollary 3.1. If t is an irreducible point of P_η, then $\sum_{g \in \eta} t(g) \leq D - 1$.

Theorems 3.4 and 3.5 and Corollary 3.1 are proved in [35]. These theorems place an upper bound on the number of meaningful components of T defined in Theorem 3.3; that is, there are only a finite number of elements of T that are irreducible, and all other elements of T are superfluous in the sense of Theorem 3.5 and Corollary 3.1. We may note one other fact in this regard. *Each* component t(g) has an upper bound $|g| - 1$, where $|g|$ denotes the order of the group element g.

Theorem 3.6. $t(g) < |g|$ for all $g \in \eta$.

Proof. We know that $|g|\cdot g = 0$, where 0 is the identity of G. Any $t(g) \geq |g|$ is equivalent to some $t(g) < |g|$ in respect to the group relation $\sum_{g\in\eta} t(g)\cdot g = g_o$ and in respect to the inequality $\sum_{g\in\eta} \pi(g)t(g) \geq \pi_o$. That is, $t(g)\cdot g = [t(g)+|g|]\cdot g \bmod |g|$.

The group G has been defined as $G = M(I)/M(B)$ and is isomorphic to the direct sum of cyclic subgroups $G_1,\ldots,G_r$ such that

$$G = M(I)/M(B) = G_1 \oplus \ldots \oplus G_r \cong Z_{q_1} \oplus \ldots \oplus Z_{q_r}.$$

Gomory [35] uses this isomorphic relationship in conjunction with a result produced over 100 years ago (in 1861) by Smith [63]; Smith's result is given as follows:

Theorem 3.7. Given a nonsingular n×n integer matrix B, there exist n×n unimodular matrices R and C such that S = RBC is a diagonal matrix with positive diagonal elements such that $q_{11}|q_{22}|\ldots|q_{nn}$.

Gomory shows that the diagonal elements $q_{11},q_{22},\ldots,q_{nn}$ of the matrix S (called the "Smith Normal Matrix") corresponding to the optimal basis B are the orders of the corresponding cyclic subgroups $G_1,\ldots,G_n$. This result is developed in [35] and [46] and is one of the primary components of the foundation of group theoretic integer programming.

One of the immediate consequences of the above result is that (P5) may be rewritten as follows:

Beginning with the problem in the form

$$\text{Maximize} \quad c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N$$

$$\text{subject to} \quad B x_B + N x_N = b$$

$$x_B, x_N \geq 0 \text{ and integer,}$$

the Smith Normal Matrix, S, is calculated according to Theorem 3.7 such that S = RBC;

$$S = \begin{bmatrix} q_1 & & & \\ & q_2 & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & q_m \end{bmatrix}$$

where $q_i > 0$, $\forall i$, and $q_1 | q_2 | \ldots | q_m$. Dropping the nonnegativity constraint on x_B (as we did earlier) and premultiplying the constraint by the matrix R the problem becomes

$$\text{Maximize} \quad c_B B^{-1} b - c_N^* x_N$$

$$\text{subject to} \quad RBx_B + RNx_N = Rb$$

$$x_B \text{ integer, } x_N \geq 0 \text{ and integer}$$

where $c_N^* = c_B B^{-1} N - c_N$ as before.

Dropping the constant term $c_B B^{-1} b$, we may write

$$\text{Minimize} \quad c_N^* x_N$$

$$\text{(P6)} \qquad \text{subject to} \quad \{RN \bmod S.1\}x_N = \{Rb \bmod S.1\} \bmod S.1$$

$$x_N \geq 0 \text{ and integer,}$$

where

$$S.1 = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}$$

and {mod S.1} implies that the ith row is taken modulo q_i, and where we have recognized that

$$\begin{aligned} (RBx_B) \bmod S.1 &= (RBCy_B) \bmod S.1 \\ &= (Sy_B) \bmod S.1 \\ &= 0 \end{aligned}$$

where $y_B = C^{-1}x_B$.

We write (P6) in terms of $t(g) = \sum_{j \in J} x_j$ where $J = \{j \mid R(\alpha_j) \bmod S.1 = g\}$; the result is problem (P5) (as stated earlier):

(P5)
$$\text{Minimize} \quad \sum_{g \in \eta} c^*(g)t(g)$$
$$\text{subject to} \quad \sum_{g \in \eta} t(g) \cdot g = g_o$$
$$t(g) \geq 0, \text{ integers.}$$

EXAMPLE: In our earlier example we found $B = \begin{bmatrix} 1 & 1 \\ 0 & 15 \end{bmatrix}$. The equivalent Smith Normal Matrix is simply $S = \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix}$ where $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Thus, $G = G_1 \oplus G_{15} \cong Z_1 \oplus Z_{15}$, and we have the group minimization problem,

$$\text{Minimize } 10/15t_1 + 5/15t_2$$
$$\text{subject to } \begin{bmatrix} 4 \\ 5 \end{bmatrix} t_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$
$$t_1, t_2 \geq 0, \text{ integer}$$

or equivalently,

$$\text{Minimize } 10/15t_1 + 5/15t_2$$
$$\text{subject to} \quad 5t_1 + \quad 1t_2 = 9$$
$$t_1, t_2 \geq 0, \text{ integer.}$$

Group Minimization Algorithms

Several algorithms have been presented in the literature for solving the group minimization problem (P5). Gomory developed a dynamic programming algorithm in [35] that is discussed by Balinski and

Spielberg [8], Kortanek and Jeroslow [50], and Hu [46]; Gomory's basic method is extended by White in [70]. Hu presents another algorithm in [45] and indicates that, whereas Gomory's algorithm entails $2D^2$ to $4D^2$ operations, Hu's method never requires more than $2D^2$ operations. Fewer memory locations are needed for Hu's method than for Gomory's method, and it is sometimes possible to truncate computation early in Hu's algorithm. An interesting feature of both algorithms is that (P5) is solved for all possible right-hand sides.

As illustrated earlier in this chapter, the optimal solution to the group minimization problem will not always produce a feasible solution to the original problem; i.e., for some $x_N^*, x_B^* \ngeq 0$. White's algorithm is motivated by this fact. His algorithm not only produces optimal solutions for the group minimization problem, but second, third, and in general "*rth* best" solutions. The first of these solutions that is feasible for the general problem is optimal. This feature proves to be particularly necessary for some fixed-charge transportation problems.

Shapiro has presented several methods for solving the group problem. The algorithms presented in [59] and [60] proceed along the lines of dynamic programming until an optimal solution occurs for the group problem (P5). If this optimal solution yields an infeasible integer solution to (P1), a search procedure is used to find a feasible optimal solution.

Three of Shapiro's advisees at M.I.T. have written theses on group-theory related topics. Baxter [9] attempted to combine group theoretic methods with the branch and bound algorithm of Little, et al. [52] for the traveling salesman problem. Thiriez [67] takes advantage of small-ordered groups in the set-covering problem in an application of group theory to the airline crew scheduling problem. Wolsey [8] develops a method for mixed integer programs.

Shapiro shows in [61] that the generalized linear programming approach of Brooks and Geoffrion [17] for estimating generalized LaGrange multipliers is almost algorithmically equivalent to Gomory's cutting-plane method. These two methods can be combined to produce a single cut which can be shown to be stronger in a cost sense than the combination of all the cuts suggested by Gomory in [35]. A primary feature of Shapiro's algorithmic procedures based on generalized LaGrange multipliers is that the faces of the integer polyhedron P_x are implicitly considered as constraints. As such the multipliers are an improvement over the generalized LaGrange multiplier methods based strictly on linear programming solutions of (P1). Unfortunately, this particular method does not guarantee an optimal solution will be found and identified.

Gorry and Shapiro [36] exploit two main ideas: (1) that a wide variety of existing methods for integer programming can be analyzed and compared from the common viewpoint of group theory, and (2) that an adaptive integer programming algorithm should be controlled by a supervisor which performs four main functions: set-up, directed search, subproblem analysis, and prognosis. The set-up function of the supervisor attempts to structure a given problem during the early stages of computation that the methods to be applied will be more effective. These methods include group optimization, cutting plane, surrogate constraint, LaGrangian, and search methods. If some type of enumeration is required, then the directed search function guides the search and, at each computational stage, selects the most promising subproblem to be analyzed. The subproblem analysis function selects a sequence of analytic methods to be applied to a selected subproblem. Finally, the prognosis function maintains upper and lower bounds on the cost of an optimal solution and recommends termination when the predicted change in the objective function as a result of additional computation is marginal. The supervisor of the adaptive algorithm makes decisions primarily on the basis of structural insights derived from the group theoretic approach. Encouraging computational experience is reported by Gorry and Shapiro.

Gorry and Shapiro recognize in [36] at least part of a fundamental idea that appears to promise significant gains in integer programming methodology: that there is an equivalence among many seemingly different integer programming problems and methods. Shapiro devotes reference [62] to what he calls cost-equivalent group problems; he discusses Gomory's fundamental concept that for a certain class of integer programming problems, the original integer problem has at least one optimal solution in common with a group problem. Gomory refers to these integer problems as asymptotic integer programs; Shapiro prefers to call them *steady-state* integer programs. The original integer programming problem is said to be *cost-equivalent* to the group problem. Shapiro shows that there are cost equivalent group problems for all integer programming problems.

Bradley makes considerable inroads into the idea of equivalence in [11] and [12]. He shows that every integer programming problem is equivalent to infinitely many other integer programming problems. The solution to any one problem in this equivalence class is sufficient to determine the solution to every other problem in the class; every problem in the class may be constructed from the original problem. Given any integer programming problem, Bradley shows that it is always possible to construct an equivalent problem that will be, in general, easier to solve than the original problem.

Glover [29] presents an implicit enumeration algorithm for solving (P5). Computational experience reported by Glover is encouraging. Glover and Litzler [30] develop an extension of Glover's algorithm for the general all-integer programming problem.

Glover and Devine [23] extend the work of Gomory [34] by developing a method for generating a subclass of the faces of the polyhedron P_η; faces in this subclass are called nested faces.

The most recent work of Johnson and Gomory [48] extends the asymptotic theory, advanced in [35] for all-integer problems, to mixed-integer programming problems. This should prove particularly helpful for fixed-charge problems (other than the fixed-charge transportation problem which is essentially all-integer).

Concluding Remarks

We have seen that an all-integer problem (P4) can be solved by Gomory's group theoretic (asymptotic) method using any one of several algorithms currently available or by one of the extensions of Gomory's method. We note that non-prime ordered groups may be decomposed into the direct sum of r cyclic subgroups, and the resulting r subproblems solved to find the solution to the original problem. The construction of the Smith Normal Matrix corresponding to the optimum L.P. basis will give the orders of the subgroups and may be used to establish problem (P5).

No one has specifically applied group theoretic approaches to fixed-charge problems, although we note that Wolsey [72] incidentally indicates the subgroup structure for two fixed-charge problems. The mixed-integer structures of fixed-charge problems in general should make them prime applications for the recently developed theory of Gomory and Johnson [26]. The inherent all-integer structure of the fixed-charge transportation problem makes it amenable to most of the general theory of group theoretic integer programming.

CHAPTER IV

GROUP THEORETIC STRUCTURES IN THE FIXED-CHARGE TRANSPORTATION PROBLEM

Introduction

In this chapter, we investigate the structures of the coefficient matrix of the FCTP and the corresponding group minimization problem. It is shown that the route capacities may be set such that the order of the subgroups is bounded from above by $\max\{\min(D_j, S_i)\}$, $\forall(i,j)$, where D_j is the demand at destination j and S_i is the supply available from source i. Components of the group elements are restricted to certain values (0,1,-1), interrelationships among the FCTP variables are preserved in the group problem, and group equivalence and dominance properties are discovered. It is found that the y_{ij} (0-1) variables are never represented explicitly in the group problem. A group theoretic solution procedure is presented. Applying group theory to the integer subproblem in Benders Partitioning Procedure is investigated; the unpartitioned problem proves to be more susceptible to the group theoretic approach than does the Benders problem.

Matrix Structures

We investigate the structures of the coefficient matrices of the FCTP when solved as an unpartitioned problem and when solved by Bender's partitioning procedure. The fundamental characteristics of the related group problems are reported.

The Unpartitioned Problem

We rearrange (P3) slightly to conform with the formulation used in the experiments that follow.

$$\text{Minimize} \quad \sum_i \sum_j c_{ij} x_{ij} + \sum_i \sum_j f_{ij} y_{ij}$$

$$\text{(P3')} \qquad \begin{aligned} \text{subject to} \quad & \sum_i x_{ij} && \geq D_j, \ \forall j \\ & \sum_j x_{ij} && \leq S_i, \ \forall i \\ & -x_{ij} + m_{ij} y_{ij} && \geq 0, \quad \forall (i,j) \end{aligned}$$

$$x_{ij}, y_{ij} \geq 0, \ y_{ij} = 0 \text{ or } 1, \ \forall (i,j)$$

Subtracting surplus and adding slack variables, we may write the coefficient matrix of (P3') as follows:

$$\left[\begin{array}{cccc|c|cc|c}
I_{n\times n} & I_{n\times n} & \cdots & I_{n\times n} & & -I_{n\times n} & 0_{n\times m} & \\
E_{1\times n} & 0_{1\times n} & \cdots & 0_{1\times n} & & & & \\
0_{1\times n} & E_{1\times n} & \cdots & 0_{1\times n} & 0_{(n+m)\times mn} & 0_{m\times n} & I_{m\times m} & 0_{(m+n)\times mn} \\
\vdots & \vdots & \ddots & \vdots & & & & \\
0_{1\times n} & 0_{1\times n} & \cdots & E_{1\times n} & & & & \\
\hline
\multicolumn{4}{c|}{-I_{mn\times mn}} & M_{mn\times mn} & \multicolumn{2}{c|}{0_{mn\times(n+m)}} & -I_{mn\times mn}
\end{array}\right]$$

where n = the number of destinations, m = the number of sources, E denotes a matrix of 1's, and M is a diagonal matrix whose diagonal elements are the route capacities (m_{ij}).

This is obviously a very sparse, block-angular matrix containing p elements, where p = (mn+m+n)(3mn+m+n), only mn of which are not 0, 1, or -1. These mn elements are the diagonal elements m_{ij} of the submatrix $M_{mn\times mn}$. The number of non-zero elements in the coefficient matrix of (P3') is q = 5mn + m + n. We may calculate the density of the matrix as q/p. Figure 1 illustrates a coefficient matrix for the FCTP where there are two sources and three destinations.

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

Figure 1. Coefficient Matrix for a 2×3 FCTP

Let us partition this matrix (which we denote as A) as $A = [A_1, A_2, A_3]$ where

$$A_1 = \left[\begin{array}{c|c|c|c}
I_{n\times n} & I_{n\times n} & \cdots & I_{n\times n} \\
\hline
E_{1\times n} & 0_{1\times n} & \cdots & 0_{1\times n} \\
\hline
0_{1\times n} & E_{1\times n} & \cdots & 0_{1\times n} \\
\hline
\vdots & \vdots & \cdots & \vdots \\
\hline
0_{1\times n} & 0_{1\times n} & \cdots & E_{1\times n} \\
\hline
\multicolumn{4}{c}{-I_{mn\times mn}}
\end{array}\right]$$

$$A_2 = \left[\begin{array}{c} 0_{(n+m)\times mn} \\ \hdashline M_{mn\times mn} \end{array}\right], \qquad \text{and} \qquad A_3 = \left[\begin{array}{cc:c} -I_{n\times n} & 0_{n\times m} & \\ \hdashline 0_{m\times n} & I_{m\times m} & 0_{(m+n)\times mn} \\ \hdashline \multicolumn{2}{c:}{0_{mn\times(n+m)}} & -I_{mn\times mn} \end{array}\right]$$

Suppose (P3') were solved as a linear programming problem where the integer restrictions on y_{ij} $\forall(i,j)$ has been removed.[1] The basic variables, y_B, are not necessarily integers. We know that at least n of the $y_{ij} \epsilon y_B$ are positive since at least one route must be open to each destination. Further, we may have some slack or surplus variables in the optimal LP basis; we will denote these basic slack and surplus variables as s_B so that the basis is composed of column vectors corresponding to $[x_B, y_B, s_B]$.[2] Figure 2 illustrates a basis for the coefficient matrix that was given in Figure 1.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 \\ \hline
0 & 25 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25
\end{bmatrix}$$

Figure 2. A Basis Matrix for a 2×3 FCTP

---

[1] One can easily show that $y_{ij} \leq 1$, $\forall(i,j)$, in the LP solution. See, for example, Theorem 2.1 on page 11 of this thesis.

[2] Note that vectors corresponding to artificial variables in the basis at the zero level may be removed from the optimal LP basis by interchanging those vectors with the vectors associated with the appropriate slack or surplus variables.

Thus we see that the elements of B will belong exclusively to the set $\{0,1,-1,m_{ij}\}$ where, in fact, the m_{ij} are exclusively elements of the columns corresponding to y_B. The determinant of B, therefore, gains magnitude predominantly due to the $m_{ij}\epsilon B$. When we consider the related group problem in terms of its group G, we recall from Chapter III that the order of the group, $D = |G|$, is equal to the absolute value of the determinant of B; i.e., $D = |\det B|$.

<u>Theorem 4.1</u>. In the FCTP with optimal LP basis B, $|\det B| = \prod_{m_{ij}\epsilon B} m_{ij}$.

Proof. By structure of the coefficient matrix A, the $m_{ij}\epsilon B$ are located independently; i.e., in different columns and rows. $\{m_{pg},m_{rs},\ldots,m_{vw}\} = \{m_{ij}\epsilon B\}$ where $p \neq r \neq v$, $q \neq s \neq w$, etc. Consequently, cofactor expansion of B will not eliminate any $m_{ij}\epsilon B$ from consideration.

Begin cofactor expansion about the columns corresponding to y_B; i.e., about the columns in submatrix $A_2 = \left[\begin{array}{c} 0_{(n+m)\times mn} \\ \hline M_{mn\times mn} \end{array}\right]$. A typical column of A_2 is of the form

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{ij} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow (i+m+n)\textit{th} \text{ row} \\ \\ \\ \\ \end{array}$$

where m_{ij} is the (i+m+n)*th* element, and the remaining elements of the column are zeros.

Consider what happens as we initiate the cofactor expansion about the column corresponding to some $y_{ij} \epsilon y_B$:

$$\det B = 0 + \ldots + 0 + (-1)^{i+m+n+j} m_{ij} B'_{i+m+n,j} + 0 + \ldots + 0$$

$$= (-1)^{i+m+n+j} m_{ij} B'_{i+m+n,j},$$

where $B'_{k\ell}$ denotes the minor found by deleting row k and column ℓ of B.

Let the minor that remains (after expanding about the p columns of A_2 in B) be denoted as B^P. We know that $B^P \neq 0$ since B is, by definition, nonsingular. Further, we see that the columns of the matrix associated with B^P are members of either A_1 or of A_1 and A_3; thus, $b^P_{ij} \epsilon \{0,1,-1\}$ where b^P_{ij} denotes an element of the matrix associated with the minor B^P. We now need to show that the absolute value of B^P is one, and Theorem 4.1 will be proved since we have $|\det B| = \prod_{m_{ij} \epsilon B} m_{ij} B^P$.

There are two fundamental cases that might exist at this point; we enumerate them as follows:

Case I. The matrix associated with B^P contains only rows corresponding to the transportation problem portion of the FCTP; i.e. the remaining matrix contains only rows corresponding to the constraints $\sum_i x_{ij} \geq D_j$, $\forall j$, and $\sum_j x_{ij} \leq S_i$, $\forall i$. It is well known that this matrix is unimodular and nonsingular; therefore, $|B^P| = 1$.

Case II. The matrix associated with B^p contains rows described in Case I *plus* a portion of one or more rows from rows m+n+1 through m+n+mn of the basis; i.e., we have rows pertaining to the "transportation problem portion" of the basis *and* a portion of one or more rows from the submatrix

$$\left[-I_{mn\times mn} \;\vdots\; 0_{mn\times(n+m)} \;\vdots\; -I_{mn\times mn} \right]$$

where the portion of the M submatrix in B was removed by the cofactor expansion. It is likely that some of the -1 elements in the above submatrix were removed by the cofactor expansion about the $m_{ij}\epsilon B$. We now show that cofactor expansion about certain of the remaining -1 elements in the matrix associated with B^p will reduce this case to Case I; i.e., we can reduce the matrix of Case II to the matrix pertaining to the transportation problem portion of the basis.

The two submatrices of interest are both derived from the submatrices of the form $-I_{mn\times mn}$. There are three possibilities at this point; all three or a subset could occur simultaneously:

(i) the only nonzero element is a -1 from the right-hand -I submatrix;

(ii) the only nonzero element is a -1 from the left-hand -I submatrix;

(iii) there are two -1 elements in a particular row, one -1 from each of the -I submatrices. In case (iii), we expand about the column containing the right-hand -1 element since the only nonzero element in this column is the -1 (by construction of the coefficient matrix). This leaves the single -1 element in the row of interest, and we handle this as described earlier in this paragraph. Thus, Case II reduces to Case I, and $|B^P| = 1$; therefore, Theorem 4.1 is true.

The effect of Theorem 4.1 would appear devastating to any attempt to use group theoretic integer programming for the FCTP in most cases. The corresponding group problem (P5), where components of the group elements are of the form n/D, will probably be computationally infeasible since the methods for solving (P5) typically require from $2D^2$ operations (in the case of Hu's algorithm [45]) to $4D^2$ operations (in case of Gomory's algorithm [35]). The decomposed group problem (P5) will often be computationally unattractive because the largest-ordered subgroup will exhibit very large order. This is illustrated by some experimental results for the (very small) FCTP with two sources and three destinations with supplies and demands as shown in Table 1. The orders of the subgroups as well as the order of the complete group would seem to preclude direct application of group theoretic methods. We will return to this point in subsequent sections.

Table 1. Representative Subgroup and Group Orders

Problem Number	Supplies at Source 1	2	Demands at Destination 1	2	3	Subgroup Orders $q_i > 1$	Group Order D
1	60	60	32	27	25	21600	21600
2	95	75	50	33	28	66,23100	1524600
3	150	200	103	122	35	103,439810	45300430
4	150	175	15	122	71	129930	129930
5	200	175	155	122	71	155,1342610	208104550
6	150	125	55	22	71	11,7810	85910

Motivated by the evidence of experimental results shown in Table 1 and by Theorem 4.1, we turn to Bender's Partitioning Procedure with the clear hope that the partitioned problem structure may be more amenable to the group theoretic approach.

Bender's Partitioned Problem

We noted in the previous section that the unpartitioned FCTP has a very simple coefficient matrix structure; in fact, the only elements of the coefficient matrix that are not 0, 1, or -1 are the m_{ij}'s. We have a considerably different situation in the case of the problem partitioned by Bender's method. In Bender's procedure we have the integer subproblem

$$\text{(P7)} \qquad \begin{aligned} &\text{Minimize} \quad z \\ &\text{subject to} \quad z \geq fy + \bar{u}^1(b-A_2y) \\ &\qquad\qquad\quad \vdots \quad \vdots \qquad \vdots \\ &\qquad\qquad\quad z \geq fy + \bar{u}^k(b-A_2y) \\ &\qquad\qquad\quad \sum_{i=1}^{m} y_{ij} \geq 1, \quad \forall j, \\ &\qquad\quad y = 0 \text{ or } 1 \end{aligned}$$

as described in [10], [64] and Appendix C of this thesis, where f is the vector of fixed charges, A_2 is the matrix of coefficients for y in (P3'); i.e.,

$$A_2 = \left[\begin{array}{c} 0_{(n+m)\times mn} \\ \hdashline M_{mn\times mn} \end{array}\right],$$

$\bar{\bar{u}}^{\ell}$ is a vector of dual variables corresponding to the transportation problem that results from opening a certain (*ℓth*) subset of the routes and ignoring all fixed charges, and $\sum y_{ij} \geq 1$, $\forall j$, implies that at least one route must be open to each destination.

We see that $\bar{\bar{u}}(b-A_2y)$ has the following form:

$$\bar{\bar{u}}(b-A_2y) = [u_1, u_2, \ldots, u_{m+n+mn}] \begin{bmatrix} D_1 \\ \vdots \\ D_n \\ -S_1 \\ \vdots \\ -S_m \\ m_{11}y_{11} \\ \vdots \\ m_{1n}y_{1n} \\ m_{21}y_{21} \\ \vdots \\ m_{2n}y_{2n} \\ \vdots \\ m_{m1}y_{m1} \\ \vdots \\ m_{mn}y_{mn} \end{bmatrix}$$

$$= (u_1D_1+u_2D_2+\ldots+u_nD_n-u_{n+1}S_1-u_{n+2}S_2-\ldots-u_{n+m}S_m) +$$

$$u_{n+m+1}m_{11}y_{11} + \ldots + u_{n+m+mn}m_{mn}y_{mn} =$$

$$= K + u_{n+m+1}m_{11}y_{11} + \ldots + u_{n+m+mn}m_{mn}y_{mn},$$

where K is the sum of the constant terms contained in the parentheses. Thus, we can rewrite (P7) as (P7'):

Minimize z

(P7') subject to
$$a_{1z}z - (f_{11}-m_{11}u^1_{m+n+1})y_{11} - \ldots - (f_{mn}-m_{mn}u^1_{m+n+mn})y_{mn} \geq K^1$$

$$a_{2z}z - (f_{11}-m_{11}u^2_{m+n+1})y_{11} - \ldots - (f_{mn}-m_{mn}u^2_{m+n+mn})y_{mn} \geq K^2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{\ell z}z - (f_{11}-m_{11}u^\ell_{m+n+1})y_{11} - \ldots - (f_{mn}-m_{mn}u^\ell_{m+n+mn})y_{mn} \geq K^\ell$$

$$\sum_{i=1}^{m} y_{ij} \geq 1,\ \forall j,\quad 0 \leq y_{ij} \leq 1,\quad \text{integer},\quad \forall i,j,$$

where a_{pz} is a scalar by which row p has been multiplied to remove all fractional coefficients.

The coefficient matrix of (P7') may be written as

$$\left[\begin{array}{c|cccc}
A_{\ell\times 1} & \multicolumn{4}{c}{-F_{\ell\times mn}} \\
\hline
 & I_{n\times n} & I_{n\times n} & \cdots & I_{n\times n} \\
\cline{2-5}
O_{(n+mn)\times 1} & \multicolumn{4}{c}{I_{mn\times mn}}
\end{array}\right]$$

where the submatrix $F_{\ell\times mn}$ contains the elements $(f_{ij}-m_{ij}u^k_{m+n+(i\cdot j)})$. When we add slack variables to (P7') and rearrange the rows of the problem in an obvious way, we have the following coefficient matrix for the integer subproblem in Bender's procedure:

$$\left[\begin{array}{c|cccc|cc|c}
 & I_{n\times n} & I_{n\times n} & \cdots & I_{n\times n} & -I_{n\times n} & O_{n\times mn} & \\
\cline{2-7}
O_{(n+mn)\times 1} & \multicolumn{4}{c|}{I_{mn\times mn}} & O_{mn\times n} & I_{mn\times mn} & O_{(mn+n)\times\ell} \\
\hline
A_{\ell\times 1} & \multicolumn{4}{c|}{-F_{\ell\times mn}} & \multicolumn{2}{c|}{O_{\ell\times(mn+n)}} & -I_{\ell\times\ell}
\end{array}\right]$$

Suppose we solve the associated LP of (P7') ignoring the integer restriction. Let us again refer to the optimal LP basis as B where we see that $B = [z,y_B,s_B]$; i.e., B is composed of the vectors corresponding to z, y_B, and s_B where s_B denotes the basic slack variables. By the constraint $\sum_{i=1}^{m} y_{ij} \geq 1$, $j=1,\ldots,n$, we see that at least n of the y_{ij} are going to be basic; z will always be basic when the cost coefficients and demand are positive by the very nature of the problem. The coefficient matrix for an integer subproblem in Bender's procedure and the corresponding basis are illustrated in Figures 3 and 4, respectively, for the same 2×3 problem (two sources and three destinations) previously shown in Figures 1 and 2.

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -253 & -263 & -273 & -283 & -293 & -303 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
3 & -759 & -789 & -819 & 31151 & 26121 & 9091 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

Figure 3. Coefficient Matrix for an Integer Subproblem in Benders' Procedure

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -253 & -263 & -273 & -283 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -759 & -789 & -819 & 31151 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 4. A Basis Matrix for an Integer Subproblem in Benders' Procedure

The magnitude of the determinant of B is affected predominantly by the elements of the submatrix $-F_{\ell \times mn}$. It is interesting to note that the elements of F are a composite of the fixed charges, the route capacities, and (by virtue of the dual variables) the variable transportation costs.

Experience with the group theoretic aspects of the Benders problem is discussed more fully in Appendix C; for illustrative purposes here, let us list in Table 2 typical group orders encountered in solving some 2×3 example problems.

Table 2. Group Order in Benders' Problem

Problem	Number of Constraints of Type (1) as Below	Order of Group $\|\det B\|$
A	3	12,475
B	3	349,015,500
C	2	32,000
D	4	34,300,800
E	3	381,848,726
F	5	$> 10^9$

(1) $z \geq fy + u^{\ell}(b-A_2)$.

The point of Table 2 is that we are no better off in respect to the group theoretic approach in Bender's problem than we were in the unpartitioned problem.

As we view the structures of the unpartitioned problem and the Benders' problem, the relatively simple structure of the unpartitioned problem appears more amenable to further analysis. Let us proceed by focusing on the properties of the route capacities (m_{ij}) in the unpartitioned problem.

Route Capacities in the FCTP

Balinski [4] defines $m_{ij} = \min(S_i,D_j)$ in (P3). This follows from the constraints

$$\sum_j x_{ij} \le S_i, \quad \forall i,$$

and (2)

$$\sum_i x_{ij} \ge D_j, \quad \forall j.$$

We see that if $\min(S_i,D_j) = S_i$, no more than $x_{ij} = S_i$ *can* be shipped over route (i,j); if $\min(S_i,D_j) = D_j$, no more than $x_{ij} = D_j$ *would* be shipped over route (i,j) since the total variable cost would be increased unnecessarily by the movement of undemanded units. It is important to note that no other mathematical reason exists for restricting $m_{ij} = \min(S_i,D_j)$; however, in the uncapacitated transportation problem we require that $m_{ij} \ge \min(S_i,D_j)$. By the constraints (2),

$$x_{ij} \le \min(D_j,S_i)$$

when $m_{ij} \ge \min(D_j,S_i)$. We see that setting $m_{ij} \ge \min(D_j,S_i)$ does not affect the relationship

$$x_{ij} > 0 \Rightarrow y_{ij} = 1,$$

$$x_{ij} = 0 \Rightarrow y_{ij} = 0.$$

We are motivated to specify the m_{ij} such that the group structure is enhanced. It is evident that the order of the group will be quite large because m_{ij} is restricted by $m_{ij} \geq \min(D_j, S_i)$ and because $D = \prod_{m_{ij} \in B} m_{ij}$ (by Theorem 4.1). Suppose we could promote the decomposability of the group into the direct sum of cyclic subgroups such that the subgroup orders are equal and bounded by $\max\{\min(D_j, S_i)\}$ $\forall(i,j)$. We show in the next theorem that the m_{ij} can be chosen such that decomposability of the group is indeed affected.

Theorem 4.2. Where $G \cong Z_{q_1} \oplus \ldots \oplus Z_{q_r}$ and $q_1 | q_2 | \ldots | q_r$, $m_{ij} = \mu$ $\forall(i,j) \Rightarrow q_r = q_{r-1} = \ldots = q_1 = \mu \Rightarrow k = r$ where k is the number of $m_{ij} \in B$ and r is the number of $q_i > 1$.

Proof. $B \sim S$, where S is the Smith Normal Matrix whose diagonal elements are $1,\ldots,1,q_1,\ldots,q_r$ where $q_1 > 1$ and $q_1 | q_2 | \ldots | q_r$. To construct S from B move the columns corresponding to A_2 to the right-hand side of B by interchanging columns. Place the $m_{ij} \in B$ on the main diagonal of B by interchanging rows. By the proof of Theorem 4.1, the submatrix composed of the rows and columns not containing the m_{ij} elements is unimodular (i.e., has a determinant of 1 or -1). Thus, by elementary row and column operations, a diagonal matrix can be formed from this submatrix with only 1's and -1's on the main diagonal. At this point, any nonzero entries remaining in the rows containing the m_{ij} elements can be transformed to zeros by elementary row operations. To obtain the Smith Normal Matrix, it remains only to convert all -1's to +1. It follows that $q_r = q_{r-1} = \ldots = q_1 = \mu$, and the number of $m_{ij} \in B$ equals the number of diagonal elements greater than 1 in S. Thus, Theorem 4.2 is true.

Now we desire to choose an appropriate value for μ. Since the only restriction presently imposed on the m_{ij}'s is that $m_{ij} \geq \min(D_j, S_i)$, $\forall(i,j)$, we *could* set the m_{ij}'s equal to very large values from the standpoint of this single restriction. Specifically, we can *not* justify setting any $m_{pq} > \max\{\min(D_j,S_i)\}$, $\forall(i,j)$; therefore, we restrict the m_{ij} such that $\min(D_p,S_q) \leq m_{pq} \leq \max\{\min(D_j,S_i)\}$, $\forall(i,j)$. It follows that we may set $m_{ij} = \mu = \max\{\min(D_j,S_i)\}$ $\forall(i,j)$. In this case, we will have a group G of order $D = \prod_{m_{ij} \in B} m_{ij} = \mu^k$ which is decomposable into the direct sum of k subgroups each of order $\mu = \max\{\min(D_j,S_i)\}$ $\forall(i,j)$.

Consider the problem whose coefficient matrix and basis were shown in Figures 1 and 2. The Smith Normal Matrix for this problem is given in Figure 5. We see that D = 466,560,000 and $G = G_1 \oplus G_2 \cong Z_{21600} \oplus Z_{21600}$. Setting $\mu = \max\{\min(32,60), \min(27,60), \min(25,60)\} = 32$, we obtain the problem whose coefficient matrix, basis matrix, and Smith Normal Matrix are given in Figures 6, 7, and 8, respectively. We see from the Smith Normal Matrix that $G = G_1 \oplus G_2 \oplus G_3 \oplus G_4 \cong Z_{32} \oplus Z_{32} \oplus Z_{32} \oplus Z_{32}$.

The Group Problem

Consider the group problem (P5) corresponding to the fixed charge transportation problem where we set $m_{ij} = \mu = \max\{\min(D_j,S_i)\}$ $\forall(i,j)$. We state the following theorem.

Theorem 4.3. The number of components taken modulo μ in each element of the group problem (P5) equals the number of $y_{ij} \epsilon y_B$.

Proof. For each $y_{ij} \epsilon y_B$ there is a corresponding $m_{ij} \epsilon B$; therefore, a basis B containing k y_{ij} vectors will contain the corresponding k m_{ij} elements. In the proof of Theorem 4.2, we saw that k m_{ij} elements equal to a common value μ implies that $\delta_1 = \delta_2 = \ldots = \delta_k = \mu$ and, therefore, that $q_1 = \ldots = q_k = \mu$. Thus, Theorem 4.3 is true.

EXAMPLE: Consider the FCTP where we wish to

$$\text{Minimize } 32x_{11} + 31x_{12} + 30x_{13} + 29x_{21} + 28x_{22} + 27x_{23}$$
$$+ 253y_{11} + 263y_{12} + 273y_{13} + 283y_{21} + 293y_{22} + 303y_{23}$$

subject to demands at destinations 1, 2, and 3 of 32, 27, and 25, respectively, and supply availability at sources 1 and 2 of 60 and 60. The coefficient matrix is given as Figure 6 where all route capacities have been set equal to 32. The optimal LP basis is given in Figure 7 and the Smith Normal Matrix is Figure 8. The cost vector c^*_N = [0 0 39.91 39.22 38.53 2.06 7.91 8.22 8.53 8.84 9.16 9.47]; the non-basis matrix is given in Figure 9; the row transformation matrix R is given in Figure 10. The resulting group problem (P5) is as follows:

$$\text{Minimize } 7.91t_1 + 8.22t_2 + 8.53t_3 + 2.06t_4 + 8.84t_5 + 9.16t_6 + 9.47t_7$$

$$\text{subject to } \begin{bmatrix}31\\0\\0\\0\end{bmatrix}t_1 + \begin{bmatrix}31\\1\\31\\0\end{bmatrix}t_2 + \begin{bmatrix}31\\1\\0\\31\end{bmatrix}t_3 + \begin{bmatrix}31\\1\\0\\0\end{bmatrix}t_4 + \begin{bmatrix}0\\31\\0\\0\end{bmatrix}t_5 + \begin{bmatrix}0\\0\\31\\0\end{bmatrix}t_6 + \begin{bmatrix}0\\0\\0\\31\end{bmatrix}t_7 = \begin{bmatrix}16\\16\\27\\25\end{bmatrix} \pmod{32}$$

$$t_j \geq 0, \text{ integer}, j=1,\ldots,7.$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 021600 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 02160 & 0
\end{bmatrix}
$$

Figure 5. The Smith Normal Matrix Corresponding to the Basis Matrix Shown in Figure 2

$$
\left[\begin{array}{ccccccccccccccccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
$$

Figure 6. Coefficient Matrix for a 2×3 FCTP Where $m_{ij} = 32, \forall(i,j)$

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0
\end{bmatrix}$$

Figure 7. Basis Matrix Corresponding to the Coefficient Matrix Shown in Figure 6

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32
\end{bmatrix}$$

Figure 8. Smith Normal Matrix Corresponding to the Basis Matrix Shown in Figure 7

$$\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
32 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

Figure 9. Non-Basis Matrix Corresponding to the Coefficient Matrix Shown in Figure 6

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

Figure 10. Row Transformation Matrix Corresponding to the Basis Shown in Figure 7

The Smith Normal Matrix

Many of the structural properties of the group problem constraint are shown, in the remainder of this chapter, to be attributable to the structure of the row transformation matrix R. We recall that S = RBC where R is unimodular and corresponds to elementary row operations. The following equivalence holds:[1]

$$\left[\begin{array}{c|c|c|c} B & I & N & b \\ \hline I & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{c|c|c|c} S & R & RN & Rb \\ \hline C & 0 & 0 & 0 \end{array}\right].$$

R, RN, and Rb may be generated simultaneously from I, N, and b, respectively, as S is being generated from B by applying the same elementary row operations to I, N, b, and B. We proceed by investigating the steps in a Smith Normal Algorithm[2] which is outlined as follows:

1. By interchanging columns, move columns from A_2 in B to the right-hand side of B; i.e., move columns corresponding to $y_{ij} \epsilon y_B$ to the right side of B.

2. By interchanging rows, place the $m_{ij} \epsilon B$ on the main diagonal of B.

3. Let $s(D_j)$ denote the surplus variable corresponding to the "destination demand constraint" for destination j. If $s(D_j) \in s_B$, add α_{D_j} to $\alpha_{x_{ij}}$ where $x_{ij} \in x_B$, thereby creating $\alpha'_{x_{ij}}$ with a 1 in row n + i and a -1 in row m + in + j.

4. Let $s(S_i)$ denote the slack variable corresponding to the "source availability constraint" for source i. If $s(S_i) \in s_B$, subtract

---

[1]See, for example, theorems 1-29 and 1-30 in [21].

[2]The reader unfamiliar with the Smith Normal Algorithm is referred to Hu [45]. In this section we take advantage of the specialized nature of the FCTP by a somewhat modified procedure.

α_{s_i} from $\alpha_{x_{ij}}$ where $x_{ij} \in x_B$, thereby creating $\alpha'_{x_{ij}}$ with a 1 in row j and a -1 in row m+in+j.

5. Let s(i,j) denote the slack variable corresponding to the "route capacity constraint" for route (i,j). If $s(i,j) \in s_B$ and $x_{ij} \in x_B$, subtract $\alpha_{s(i,j)}$ from $\alpha_{x_{ij}}$ thereby deleting the -1 from row m+in+j of $\alpha_{x_{ij}}$.

6. By interchanging rows, place the nonzero elements (1 or -1 corresponding to $s(D_j)$, $s(S_i)$, and s(i,j)) in A_3 on the main diagonal. Linear independence of the basis vectors precludes both y_{ij} and s(i,j) being in the basis.

7. If $x_{ij} \in x_B$ but $y_{ij} \notin y_B$ and $s(i,j) \notin s_B$, row m+in+j contains only one nonzero element: the -1 in $\alpha_{x_{ij}}$. Add row m+in+j to row j if $s(D_j) \notin s_B$ and/or to row n+i if $s(S_i) \notin s_B$. This creates a negative unit vector in $\alpha_{x_{ij}}$ (where $x_{ij} \in x_B$ and $y_{ij} \in y_N$).

8. By interchanging columns and rows, place the -1's (in the negative unit vectors created in step 7) on the main diagonal and change these -1's to +1's by multiplying their rows by -1.

At this point, the last mn rows and possibly some of the first m+n rows of B have been diagonalized. The columns created in steps 7 and 8 and the columns associated with A_2 and A_3 in B are on the right side of the matrix. We now turn to the diagonalization of the remaining portion of the first m+n rows. The remaining columns associated with A_1 in B are collected on the left side of the matrix; these columns are of three types:

(i) those vectors with a 1 in row j ($1 \le j \le n$) being the only nonzero element in the first m+n rows;

(ii) those vectors with a 1 in row n+i ($1 \le i \le m$) being the only nonzero element in the first m+n rows;

(iii) those vectors with 1's in rows j and n+i being the only nonzero elements in the first m+n rows.

We are now able to state the next steps in the Smith Normalization process.

9. Use type (i) columns to convert type (iii) columns to type (ii) columns, and use type (ii) columns to convert type (iii) columns to type (i) columns. Continue this conversion process until all type (iii) columns have been converted to type (i) and type (ii) columns.

10. Interchange rows and columns to place the 1's in type (i) and type (ii) columns on the main diagonal. This results in a diagonal matrix except for the nonzero elements in the last r rows. Multiply those rows containing a -1 on the main diagonal by -1 to create 1's on the main diagonal.

11. By adding and subtracting rows, eliminate the nonzero elements (except for the m_{ij}'s) in the last r rows. The result is the Smith Normal Matrix.

To show that this procedure will indeed transform B to S, we must show the following:

(1) A type (i) or (ii) column created from a type (iii) column cannot be identical to another type (i) or (ii) column or to a unit vector from A_3.

(2) No type (iii) column can remain after step 9 is completed. Statement (1) is true because otherwise we would have linear dependence, an impossibility since B is a basis. (The linear dependence would arise, in the case of identical type (i) or (ii) columns, by adding some of the last r columns to the type (i) or (ii) columns.) Statement (2) also follows from the linear independence of the basis vectors. Suppose, for instance, that we have k type (iii) columns such that no type (i) or (ii) columns exist that can be used to convert the type (iii) columns to types (i) and (ii) as described in step 9. This implies a linear dependence among the first m+n+mn-r rows containing 1*'s* (in type (iii) columns). For example, add those rows from the first n rows of B having a nonzero entry in any of these k type (iii) columns. Since these nonzero entries (1*'s*) are located independently, the result will be a row with exactly k 1*'s*; these 1*'s* will be in the columns corresponding to the k type (iii) columns. Similarly, add those rows from rows n+1 through n+m having a nonzero entry in any of the k type (iii) columns. This will result in a row having exactly k ones as described above, and hence we have linear dependence, a contridiction; therefore, no type (iii) columns can remain after step 9.

We see that divisibility is guaranteed such that $q_i|q_{i+1}$ (where q_i represents the *ith* diagonal element of S) since $m_{ij}=\mu$, $\forall(i,j)$, and the diagonal elements in the columns corresponding to A_1 and A_3 are 1*'s*. That is, the Smith Normal Matrix will have the form illustrated in Figure 8 where

$$S = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & \bigcirc & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \mu & & \\ & \bigcirc & & & & \ddots & \\ & & & & & & \mu \end{bmatrix} .$$

Of course, we know from Theorem 4.2 exactly what S's structure will be without having to actually generate S from B. However, our true motivation for applying a Smith Normal Algorithm is to generate RN and Rb and thereby establish the group problem constraint

$$\{RN \bmod S\cdot 1\}x_N = \{Rb \bmod S\cdot 1\} \bmod S\cdot 1.$$

The Row Transformation Matrix

As diagonalization of B proceeds according to the method outlined in the preceding sections, the only row operations performed (besides row interchanges) occur in steps 7 and 11; these operations are addition and subtraction. In step 7, row $m+in+j$ is added to rows j and $n+i$. No further row operations are performed on row $m+in+j$ (except possibly for row interchange). In step 9, as a type (i) or (ii) column is created from a type (iii) column, the resultant column has either a -1 in row $m+kn+j$ and a 1 in row $m+\ell n+j$ $(k\neq\ell)$ or a 0 in row $m+kn+j$ (in the case where $k=\ell$). After converting a type (iii) column, no further column operations are performed on that column. Therefore, in step 11 only addition and subtraction of rows will be necessary to transform the nonzero elements to 0's in the last r rows.

After diagonalizing the first m+n rows, row 1 is added to or subtracted from those rows containing a 1 or -1 in column 1. In general, to transform nonzero elements in the last r rows of column k, row k is added to or subtracted from those rows containing a 1 or -1 in column k. The effect of this in transforming I to R is to place -1 or 1 in the k*th* column in the rows that contained nonzero elements (in the matrix being transformed from B to S). Row interchanges change only the locations of elements in R, not the values of the elements themselves; hence, we have proved the following lemma.

Lemma 4.1. In the FCTP where $m_{ij}=\mu$ $\forall(i,j)$, elements of R are exclusively members of $\{0,1,-1\}$.

The RN Matrix

We partition N similar to the way we partitioned the coefficient matrix A earlier. Let $N = [N_1,N_2,N_3]$ where N_1 is composed of columns from A_1; N_2, of columns from A_2; N_3, of columns from A_3.

Theorem 4.4. Elements of RN are exclusively members of $\{0,1,-1,\mu,-\mu\}$.

Proof. A column of N_1 is $\alpha_{x_{ij}}$ as described earlier; i.e.

$$\alpha_{x_{ij}} = \begin{array}{c} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \\ \leftarrow \text{row } n+i \\ \\ \\ \\ \leftarrow \text{row } m+in+j \\ \\ \\ \\ \end{array}$$

where $j=1,\ldots,n$ and $i=1,\ldots,m$. Pre-multiplying N_1 by R corresponds to performing elementary row operations on N_1; these row operations are exactly those operations applied to B to create S. Row operations are, except for row interchanges, as follows:

1. Those operations employed in step 7 (adding row $m+in+j$ to rows j and $n+i$);

2. Those operations performed in step 11 (adding or subtracting some of the first $m+n$ rows from some of the last r rows).

Type 1 row operations will occur only if $x_{ij} \in x_B$; type 2 row operations will be performed on row $m+in+j$ only if $x_{ij} \in x_B$. Hence, by the nature of the row operations, row j will never be subtracted from row $n+i$ nor will row j or row $n+i$ be added to or subtracted from row $m+in+j$ if $x_{ij} \in x_N$. It follows that elements of RN_1 are of $\{0,1,-1\}$.

A column of N_2 is composed of all 0's except for an m_{ij} ($=\mu$)

element. Since the elements of R are of $\{0,1,-1\}$, elements of RN_2 are of $\{0,\mu,-\mu\}$.

Columns of N_3 are independent (positive and negative) unit vectors; R is nonsingular and unimodular with elements exclusively of $\{0,1,-1\}$. Therefore, elements of RN_3 are exclusively of $\{0,1,-1\}$, and Theorem 4.4 is true.

RN Mod S·1

We are ultimately interested in the form and structure of the group problem (P5). Among the several consequences of the properties we have noted for R and RN, we have the following theorem.

Theorem 4.5. In the FCTP where $m_{ij}=\mu$ $\forall(i,j)$, elements of RN mod S·1 are exclusively members of $\{0,1,-1\}$.

Proof. Elements of RN are exclusively members of $\{0,1,-1,\mu,-\mu\}$ by Theorem 4.4. The elements of RN mod S·1 are, therefore, exclusively members of $\{0,1,-1\}$ since $\{0,1,-1,\mu,-\mu\}\text{mod } 1 = \{0\}$ and $\{0,1,-1,\mu, -\mu\}\text{mod } \mu = \{0,1,-1\}$. Thus, Theorem 4.5 is true.

Consider RN_2, the portion of RN corresponding to $y_{ij} \in y_N$. Recall that the elements of RN_2 are of $\{0,\mu,-\mu\}$, which means that the elements of RN_2 mod μ are all zeros. Thus, the following theorem is true.

Theorem 4.6. The group elements associated with the $y_{ij} \in y_N$ are always null.

Consequently, we might proceed to set $y_N = 0$ and deal with the group problem in terms of x_N and s_N. Recall that s(i,j) denotes the slack variable corresponding to the "route capacity constraint" for route (i,j), and let $g(y_{ij})$ denote the group element associated with $y_{ij} \in y_N$. Suppose $y_{ij} \in y_N$ were to imply that $y_{ij}=0$ permanently; i.e., $g(y_{ij}) = 0 \Rightarrow y_{ij} = 0$. We see that $y_{ij} = 0 \Rightarrow x_{ij} = 0 \Rightarrow s(i,j) = 0$; however, we can show by counter-example that $y_{ij} \in y_N$ does not mean $y_{ij}=0$ in the optimal solution.

COUNTER-EXAMPLE. In experimental problem T-6, $y_{12}, y_{13} \in y_N$; however, the optimal solution to the problem has $y_{13}=1$ (and $x_{13}=25$).

Thus, a solution procedure for the FCTP group problem must recognize that, while the group element corresponding to a $y_{ij} \in y_N$ is null, the optimal solution may include $y_{ij}=1$. As we will see, this is handled quite easily in the procedure developed later in this chapter. Essentially, if x_{ij} becomes positive, we set $y_{ij}=1$.

We shall now turn our attention to a solution procedure that takes advantage of the theorems of this present chapter.

A Group Theoretic Solution Procedure

A procedure for solving (P3') is presented in this section. We distinguish three phases of this procedure as follows:

Phase I: Solve the associated LP of (P3').

Phase II: Establish the group problem (P5).

Phase III: Solve (P5) such that a feasible (and hence optimal) solution to (P3') is achieved.

Phase I: The Associated Linear Program

Phase I includes the initialization steps as well as the solution of the associated LP of (P3'). The steps of this phase are as follows:

1. Set $m_{ij} = \mu = \max\{\min(D_j, S_i)\}$, $\forall(i,j)$.

2. Solve the associated LP of (P3'); thus, determine B, N, x_B, x_N, y_B, y_N, s_B, s_N, $B^{-1}N$, $B^{-1}b$, and c_N^*.

3. Call the optimal LP objective value z_{LP}^*, and calculate the objective value where those $y_{ij} \in y_B$ which are positive are set equal to 1. Call this recalculated objective value $\lambda(x^*,y^*)$ as we did in Chapter II. As in the case of Balinski's approximation method, $\lambda(x^*,y^*)$ is an upper bound on the objective function of (P3'). Furthermore, $Z_u = \lambda(x^*,y^*) - z_{LP}^*$ is an upper bound on the objective of (P5).

Phase II: The Group Representation

Phase II includes calculation of the row transformation matrix, development of $R[N_1,N_3] \bmod \mu$, elimination of dominated variables, and establishment of a "flag set," F, to coordinate variable interrelationships in Phase III. The steps of this phase are as follows:

1. Calculate RN and Rb.

2. Determine $R[N_1,N_3] \bmod \mu$ and Rb mod μ.

3. Eliminate column vectors of $R[N_1,N_3] \bmod \mu$ corresponding to $s(D_j) \in s_N$ if $c_j \geq 0 \quad \forall j$. ($s(D_j) > 0$ implies that demand is oversatisfied; since there is no motivation to oversatisfy demand in the FCTP as stated, we may set $s(D_j) = 0$ permanently).

4. Establish what we will call the "flag set" $F = \{t_j | t_j$ corresponds to $x_{ij} \in x_N$ when $y_{ij} \in y_N$ or t_j corresponds to $s(i,j) \in s_N$ when $y_{ij} \in y_N$ but $x_{ij} \in x_B\}$. (In Phase III, $t_j > 0 \Rightarrow y_{ij} = 1$ where $t_j \in F$ and t_j corresponds to either x_{ij} or $s(i,j)$.)

5. Calculate an upper bound u_j on each group variable t_j such that

$$u_j = \begin{cases} \min\left\{\left[\dfrac{\bar{Z}_u}{c_j}\right], \mu - 1\right\} & \text{if } t_j \notin F, \\ \\ \min\left\{\left[\dfrac{\bar{Z}_u - c^*(y_{ij})}{c_j}\right], \mu - 1\right\} & \text{if } t_j \in F, \end{cases}$$

where [a] means "the greatest integer contained in a."

6. Order the group variables t(g) corresponding to x_N and the remaining s_N such that $u_1 \leq \ldots \leq u_n$. Call these reordered variables $w_1, \ldots, w_n$.

Phase III: Optimization Method

Given an upper bound, Z_u, on the objective and the variables ordered such that $u_1 \leq \ldots \leq u_n$, the method outlined in this section produces the optimal solution to (P3').

The step-wise quasi-enumerative procedure begins by defining what we call "necessity collections," denoted NC_i, $\forall i$, as the set of variables for which the group elements have nonzero *ith* components. Investigation progresses from the smallest NC_i (the one with the least number of variables) through the succeedingly larger NC_i*'s* in an attempt to reduce the size of the problem. When a better feasible solution is identified for (P5), feasibility is immediately checked in the original problem (P3'); i.e, we determine whether $x_B = B^{-1}b - B^{-1}Nx_N \geq 0$. If feasibility exists in (P3') the better feasible solution replaces the "incumbent" solution S; if feasibility does not exist in the original problem, the procedure backtracks and searches for another feasible solution better than the incumbent solution. Interrelationships among the variables are acknowledged and exploited.

The procedure continues until all combinations of variable values have been implicitly enumerated. Since there exist only a finite number of combinations of variable values and the method never reiterates any particular combination, the method produces the optimal solution in a finite number of steps. We outline the optimization procedure as follows:

1. (Establish Necessity Collections.) Let $NC_i = \{w_j | a_{ij} \neq 0\}$, $\forall i$. Reorder the components of the group elements such that NC_1 (the set of variables for which the group elements have nonzero first components) contains the least number of variables; NC_2, the next least, and so forth. Define order (NC_i) as the number of variables in NC_i. If Order (NC_k) = Order (NC_ℓ), let the NC containing the smallest indexed variable be assigned the smaller index. Set $b' = b$, $k = 1$, $\bar{S} = S = \Phi$, $i = 2$, $LCC^- = LCC^+ = \Phi$, where k is the index of the necessity collection being investigated, and LCC denotes the "least cost combination" of variable values that will satisfy a particular right-hand-side element; the corresponding "least cost" is denoted by LC.

2. (Determine the Least Total Cost, LC*, that can possibly be incurred with respect to succeeding necessity collections.)

(a) Set $\ell = \min j \ni w_j \in NC_i$ and $a_{ij} = -1$ (where a_{ij} denotes the *ith* component of the *jth* group element). If no such j exists, set $C^- = \infty$, determine $\ell = \min j \ni w_j \in NC_i$ and $a_{ij} = 1$, set $C^+ = 0$, and go to (e); otherwise, set $C^- = 0$, and continue.

(b) If $b_i' = 0$, set $w_\ell = 0$; otherwise, set $w_\ell = \min\{\mu - b_i', u_\ell\}$, calculate $b_i' = b_i' - \mu + w_\ell$, and augment LCC^- with w_ℓ.

(c) Calculate $C^- = C^- + c_\ell w_\ell$ $\left(\text{or } C^- = C^- + c_\ell w_\ell + c^*(y_{ij})\right.$ if $w_\ell \in F \Rightarrow y_{ij} = 1\left.\right)$. If $b_i' \neq 0$, determine next larger $j \ni w_j \in NC_i$, $w_j \notin \bar{S}$, and $a_{ij} = -1$; set $\ell = j$ and return to (b); however, if none exists, set $C^- = \infty$ and go to (d). If $b_i' = 0$, continue.

(d) Reset $b_i' = b_i$ and determine $\ell = \min j \ni w_j \in NC_i$, $w_j \notin \bar{S}$, and $a_{ij} = 1$, set $C^+ = 0$ and go to (e); if none exists, set $C^+ = \infty$ and go to (g).

(e) Set $w_\ell = \min\{b_i', u_\ell\}$, calculate $b_i' = b_i' - w_\ell$, and augment LCC^+ with w_ℓ.

(f) Calculate $C^+ = C^+ + c_\ell w_\ell$ or $C^+ = C^+ + c_\ell w_\ell + c^*(y_{ij})$ if $w_\ell \in F \Rightarrow y_{ij} = 1$. If $b_i' \neq 0$ and $C^+ < C^-$, then determine next larger $j \ni w_j \in NC_i$, $w_j \notin \bar{S}$, and $a_{ij} = 1$; set $\ell = j$ and return to (e). If none exists, set $C^+ = \infty$ and go to (g). If $b_i' \neq 0$ and $C^+ > C^-$, go to (g). If $b_i' = 0$, continue.

(g) Set $LC_i = \min\{C^-, C^+\}$. If $LC_i = C^-$, set $LCC_i = LCC^-$; if $LC_i = C^+$, set $LCC_i = LCC^+$. Reset $LCC^+ = \Phi$ and $LCC^+ = \Phi$. If $i=n$, go to (h); otherwise, set $i = i+1$ and return to (a).

(h) Calculate $LC^* = \sum_{i \in I} LC_i$ where $I = \{i | i>k$, LCC_i does not contain any $w_j \in NC_k\}$ such that no variable contributes to LC^* more than once; i.e., if $w_j \in LCC_\ell$ and $w_j \in LCC_h$, let $\max\{LC_\ell, LC_h\}$ be the contribution to LC^* by w_j and its attendant "least cost combination."

3. (Determine an upper bound, Z_k, on the cost of decisions with respect to Necessity Collection k.) Set the temporary upper bound on w_j,

$$u_j' = \begin{cases} \min\left\{u_j, \left[\dfrac{Z_k}{c_j}\right]\right\} & \text{if } w_j \notin F \\ \\ \min\left\{u_j, \left[\dfrac{Z_k - c^*(y_{ij})}{c_j}\right]\right\} & \text{if } w_j \in F \end{cases}$$

for each $j \ni w_j \in NC_k$. If k=1 and this is the first iteration or if $k>1$ and $\bar{S}=\Phi$, set $u_j = u_j'$, $\forall j \ni w_j \in NC_k$. If $u_j=0$, remove w_j from the problem and set $w_j=0$ permanently by augmenting S with w_j underlined (to denote a permanent assignment); delete w_j from all NC_i. If Order $(NC_k) = 1$ and $w_j \in NC_k$, set $w_j = a_{ij}b_i'$ permanently, set $b = b - g_jw_j$ (where g_j is the j*th* group element), and remove w_j from further consideration by augmenting S by w_j underlined and deleting w_j from all NC_i; revise Z_u and recalculate u_j. If $NC_k = \Phi$, set k = k+1 and return to 2; otherwise, set $j = \min j \ni w_j \notin S$, $u_j' = w_j = u_j$, augment $\bar{S}$ with w_j, set C=0, and continue.

4. (Enumerate combinations of values of variables.)

(a) Calculate $C = C + c_jw_j$ (plus corresponding $c^*(y_{ij})$ if $w_j \in F$), set $b' = b' - g_jw_j$.

(b) If $b' = 0$, go to (f); otherwise, remove w_j from all Necessity Collections. If more than one element remains in each NC_i, go to (c). If only one element, say w_p, remains in some NC_i, calculate a new temporary upper bound

$$u'_p = \begin{cases} \min\left\{u_p, \left[\frac{Z_u - C}{c_p}\right]\right\} & \text{if } w_p \notin F, \\ \min\left\{u_p, \left[\frac{Z_u - C - c^*(y_{ip})}{c_p}\right]\right\} & \text{if } w_p \in F, \end{cases}$$

and set $w_p = a_{ip} b'_i$. If $w_p > u'_p$, go to (e). If $w_p \leq u'_p$, augment $\bar{S}$ by w_p, set $b' = b' - g_p w_p$, $C = C + c_p w_p$ (plus $*(y_{ip})$ if $w_p \in F$), and determine whether there exists some NC_i such that Order $(NC_i) = 1$. Continue this process until there exists no NC_i such that Order $(NC_i) = 1$; then continue.

(c) Set $j = \min j \ni w_j \in \bar{S}$ and calculate a new temporary upper bound

$$u'_j = \begin{cases} \min\left\{u_j, \left[\frac{Z_u - C}{c_j}\right]\right\} & \text{if } w_j \notin F, \\ \min\left\{u_j, \left[\frac{Z_u - C - c^*(y_{ij})}{c_j}\right]\right\} & \text{if } w_j \in F. \end{cases}$$

(d) If $u'_j \neq 0$, set $w_j = u'_j$, augment $\bar{S}$ by w_j, and return to (a). If $u'_j = 0$, set $j = j-1$ and continue.

(e) Set $w_j = w_j - 1$, $C = C - c_j w_j$, $b' = b' + g_j w_j$, and return to (b).

(f) Calculate $[x_B, y_B, s_B]^T = B^{-1}b - B^{-1}N[x_N, y_N, s_N]^T$. If $[x_B, y_B, s_B]^T \ngeq 0$, return to (e). If $[x_B, y_B, s_B]^T \geq 0$, a better feasible solution has been found; therefore, set $S = \{\underline{w}_j\} + \bar{S}$ and $Z_u = C$, recalculate u_j $\forall j$, and return to (e).

In this procedure we see that backtracking occurs in step 4 in two cases:

1. In part (d) if $u_j'=0$, the procedure backtracks since there is no better solution to be found by continuing down the present path.

2. In part (f) we have identified a better solution to the group problem; this is the only irreducible solution that exists for the current set of fixed variables. This solution may or may not be feasible in the original problem, but in any event one cannot proceed down the tree any further.

The optimal solution will always be identified since we know that at least one feasible solution *can* be produced; i.e., the "rounded up" solution with cost $\lambda(x^*, y^*)$ found in Phase I is feasible and, if no better solution exists, will eventually be identified as the optimal solution.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The fixed-charge transportation problem (FCTP) has been investigated using group theory to produce a solution procedure that differs substantially from all other published methods (both exact and approximative) for this problem. Whereas the order of the group associated with the FCTP is quite large, a method is given for inducing reasonable subgroup orders. The resultant group problem has several special properties, not the least of which is that the components of the group elements are 0, 1, and -1. These special properties are exploited by the solution procedure developed for this specific purpose as part of the present investigation.

The main findings of this research are summarized as follows:

1. The order of the group associated with the fixed-charge transportation problem solved as a linear program is always equal to the product of the capacities of the routes included in the optimal LP basis (Theorem 4.1).

2. When the route capacities in the FCTP are equal, the orders of the resultant subgroups are equal; the subgroup orders are equal to the capacities (Theorem 4.2).

3. When route capacities are equal, the dimensionality of the group problem constraint equals the number of y_{ij} (0-1 variables)

included in the optimal LP basis (Theorem 4.3).

4. When route capacities are equal, components of the group elements are 0, 1, and -1 (Theorem 4.5).

5. The group elements corresponding to the $y_{ij} \epsilon y_N$ are null when route capacities are equal (Theorem 4.6).

6. Application of group theory to Benders' partitioning procedure in the case of the FCTP does not offer an effective approach to the problem (Appendix C).

These findings serve as a demonstration of the potential efficacy of the group theoretic approach focused on a particular type of problem. The underlying algebraic structures of some problems may not be apparent without the aid of group theory and related bodies of knowledge. On the other hand, the group theoretic approach is not a panacea; many problems are not susceptible to this approach.

Other Observed Group Properties

Six additional properties have been observed in respect to interrelationships among the group elements. These properties are demonstrated in Appendix G; they are not used directly in the solution procedure presented in the preceding chapter.

We introduce the following additional notation to facilitate discussion of the interrelationships among the variables in the FCTP group problem:

Let g(i,j) denote the group element associated with the slack variable s(i,j).

Let $g(S_i)$ denote the group element associated with the slack variable $s(S_i)$.

Let $g(D_i)$ denote the group element associated with the slack variable.

Property 1. If $x_{ij} \epsilon x_B$, $s(i,j) \epsilon s_N$, $s(S_i) \epsilon s_B$, and $s(D_j) \epsilon s_N$, then (a) $g(D_j) = g(i,j)$, and (b) $c^*\big(g(D_j)\big) - c^*\big(g(i,j)\big) = c_{ij}$.

Property 2. If $x_{ij} \epsilon x_N$, $s(i,j) \epsilon s_B$, $s(S_i) \epsilon s_B$, $s(D_j) \epsilon s_N$, then (a) $g(x_{ij}) = -g(D_j)$ and (b) $c^*\big(g(x_{ij})\big) + c^*\big(g(D_j)\big) = c_{ij}$.

Property 3. If $x_{ij} \epsilon x_N$, $s(i,j) \epsilon s_B$, $s(S_i) \epsilon s_N$, $s(D_j) \epsilon s_N$, then (a) $g(x_{ij}) + g(D_j) = g(S_i)$, and (b) $c^*\big(g(x_{ij})\big) + c^*\big(g(D_j)\big) - c^*\big(g(S_i)\big) = c_{ij}$.

Property 4. If $x_{ij} \epsilon x_N$, $s(i,j) \epsilon s_N$, $s(S_i) \epsilon s_N$, $s(D_j) \epsilon s_N$, then (a) $g(x_{ij}) + g(D_j) = g(i,j) + g(S_i)$, and (b) $c^*\big(g(x_{ij})\big) + c\big(g(D_j)\big) - c^*\big(g(i,j)\big) - c^*\big(g(S_i)\big) = c_{ij}$.

Property 5. If $x_{ij} \epsilon x_N$, $s(i,j) \epsilon s_N$, $s(S_i) \epsilon s_B$, $s(D_j) \epsilon s_N$, then (a) $g(x_{ij}) + g(D_j) = g(i,j)$, and (b) $c^*\big(g(x_{ij})\big) + c^*\big(g(D_j)\big) - c^*\big(g(i,j)\big) = c_{ij}$.

Properties 1 through 5 are summarized in Table 3. Additionally, we state a sixth property where $x_{ij} \epsilon x_B$, $\forall (i,j)$.

Table 3. Group Variable Interrelationships

I.P.	Conditions				Vector Relationship	Cost Relationship
	$x_{ij} \epsilon x_N$	$s(i,j) \epsilon s_N$	$s(S_i) \epsilon s_N$	$s(D_j) \epsilon s_N$		
1	No	Yes	No	Yes	$g(i,j)=g(D_j)$	$c^*\{g(D_j)\}-c^*\{g(i,j)\}=c_{ij}$
2	Yes	No	No	Yes	$g(x_{ij})=-g(D_j)$	$c^*\{g(x_{ij})\}+c^*\{g(D_j)\}=c_{ij}$
3	Yes	No	Yes	Yes	$g(x_{ij})+g(D_j)=g(S_i)$	$c^*\{g(x_{ij})\}+c^*\{g(D_j)\}-c^*\{g(S_i)\}=c_{ij}$
4	Yes	Yes	Yes	Yes	$g(x_{ij})+g(D_j)=g(S_i)+g(i,j)$	$c^*\{g(x_{ij})\}+c^*\{(D_j)\} - c^*\{g(i,j)\}-c^*\{g(S_i)\}=c_{ij}$
5	Yes	Yes	No	Yes	$g(x_{ij})+g(D_j)=g(i,j)$	$c^*\{g(x_{ij})\}+c^*\{g(D_j)\}-c^*\{g(i,j)\}=c_{ij}$.

Property 6. If $x_{ij} \epsilon x_B$, $\forall(i,j)$, $s(i,j) \epsilon s_N$, $s(\ell,j) \epsilon s_N$, $s(S_\ell) \epsilon s_N$, then (a) $g(i,j) = g(\ell,j) + g(S_\ell)$, $\ell \neq i$, and (b) $c^*[g(\ell,j)] + c^*[g(S_\ell)] - c^*[g(i,j)] = c_{ij} - c_{\ell j}$.

Formal proofs of these six properties are not presented in this thesis; however, we tentatively conclude that interrelationships that exist among the variables in (P3') are preserved in (P5).

Recommendations

Several interesting directions for future research and development are suggested by the results of this present study; additionally, several areas of curiosity have been aroused. We list some of these recommendations as follows:

1. Further investigation of the interrelationships among the elements of the FCTP group problem may yield additional useful information.

2. Investigation through the group theoretic approach of other fixed-charge problems including location-allocation problems is a natural direction for subsequent research.

3. The possible incorporation of duality theory in integer programs with the group theoretic approach may prove worthwhile; for example, the idea of sensitivity analysis in the group problem may be fruitful.

4. Further development and implementation of the solution method presented in this dissertation is desirable and strongly recommended.

APPENDICES

APPENDIX A

FUNDAMENTALS OF GROUP THEORY

This appendix provides a review of those parts of group theory that have been applied to integer programming. The reader desiring more than this rather basic foundation and framework is directed to such excellent works as those of MacLane and Birkoff [53], and Mostow, Sampson, and Meyer [54]. Before discussing group theory, we briefly review certain basic notations, definitions, and concepts of set theory, mapping, and integer number theory.

Set Theory Notation

We will use the following logical symbols from set theory.

Symbol	Meaning
$\Rightarrow$	"implies"
$\Leftrightarrow$	"if and only if"
$\forall$	"for every"
$\exists$	"there exists"
$\ni$	"such that"
$a \in A$	"the element a belongs to the set A"
$a \notin A$	"the element a does *not* belong to the set A"
$A \subset B$	"the set A is included in the set B"
$A \cap B$	"the intersection of the sets A and B"
$A \cup B$	"the union of sets A and B"

Symbol	Meaning
$\{x \mid P(x)\}$	"the set of x such that P(x)," i.e., the set of elements x satisfying the property P
$\{a,b,c\}$	"the set formed by the elements a,b,c"

Basic Concepts of Mapping

If S and T are nonempty sets, we define a *mapping* from S to T as a subset θ of $S\times T$ such that for every $s\epsilon S$ there exists a unique $t\epsilon T$ such that the ordered pair (s,t) is in θ.[1] This concept is often denoted by writing θ: $S \to T$. In other words, the mapping θ is a rule which associates any $s\epsilon S$ with some $t\epsilon T$. We call S the *domain* of θ and T the *range* of θ. The element t is called the *image* of s; the element s is called a *preimage* of t. Four basic types of mappings are designated as *into, onto, one-to-one,* and *matching* mappings.

A subset θ of $S\times T$ is called a mapping of S *into* T if (s,t_1) and $(s,t_2)\epsilon\theta$ occurs only if $t_1=t_2$, and for each $s\epsilon S$ there exists an element $(s,t)\epsilon\theta$.

θ is a mapping from S *onto* T if every element in T has at least one preimage in S; i.e., if for every $t\epsilon T$ there is at least one element $s\epsilon S$ for which $\theta s = t$. (We will often write $\theta s = t$ to mean θ: $s\to t$.)

θ is a *one-to-one* mapping if $\theta s = \theta'$ implies $s = s'$; i.e., distinct elements of S have distinct images in T (under θ).

θ is a *matching* of S and T (or θ matches S with T) if θ is both onto and one-to-one. Two sets are termed *equipotent* if there exists a matching one to the other.

---

[1]We recall that the cartesian product $S\times T = \{p \mid p=(s,t), s\epsilon S, t\epsilon T\}$.

Let $\theta : S \to T$ and $\Phi : T \to U$. We define $\theta \circ \Phi$, the *composition* of θ with Φ, as the mapping of S into U defined by $(\theta\circ\Phi)s = \Phi(\theta s)$ for all $s\epsilon S$. A *binary operation* "$\circ$" on a non-empty set S is a mapping of $S\times S$ into S; where $(s,t)\theta$ is the image of (s,t) under θ, we may write $s\circ t$, $s\cdot t$, st, $s+t$, or $s\times t$. A binary operation $\circ$ on a set S is called *commutative* whenever $x\circ y = y\circ x$ for all x, $y\epsilon S$. A binary operation $\circ$ on a set S is called *associative* whenever $(x\circ y)\circ z = x\circ(y\circ z)$ for all x, y, $z\epsilon S$.

A set S is said to have a unique *identity* element with respect to a binary operation $\circ$ on S if there exists an element $e\epsilon S$ with the property $e\circ x = x\circ e = x$ for every $x\epsilon S$. Consider the following example.

EXAMPLE: An identity element of Q with respect to addition is 0 since $0 + x = x + 0 = x$ for every $x\epsilon Q$; an identity element of Q with respect to multiplication is 1 since $1\cdot x = x\cdot 1 = x$ for every $x\epsilon Q$.

An element $y\epsilon S$ is called the (unique) *inverse* of $x\epsilon S$ provided $x\circ y = y\circ x = e$ where e is the identity element with respect to the binary operation $\circ$ on the set S.

Integer Number Theory

The Euclidean algorithm (sometimes called the division algorithm) states that for any two non-zero integers a and b, there exist unique integers q and r, called, respectively, quotient and remainder, such that $a = bq + r$ where $0 \leq r < |b|$. We denote by $b|a$ that b divides a such that $a = bq$; we call b a *divisor* of a. The positive integer c is said to be the *greatest common divisor* of a and b if (i) $c|a$ and $c|b$ and

(ii) any divisor of a and b is a divisor of c. The greatest common divisor of a and b is denoted by (a,b).

The integer $p > 1$ is a *prime number* if its only divisors are ± 1, $\pm p$. The integers a and b are *relatively prime* if $(a,b) = 1$. If a and b are relatively prime, we can find integers m and n such that $ma + nb = 1$. If a is relatively prime to b but $a|bc$, then $a|c$. Thus, if a prime number divides the product of certain integers, it must divide at least one of these integers.

The prime numbers serve as the building blocks for the set of integers; every integer $a \neq 0, \pm 1$ is either a prime or a composite. The *unique factorization theorem* states this precisely as follows: Any positive integer $a>1$ can be factored in a unique way as $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where $p_1 > p_2 > \ldots > p_t$ are distinct prime numbers and where each $\alpha_i \geq 1$.

The relation "*congruence modulo n*" is defined on all integers as $a \equiv b \bmod n$ if and only if $n|(a-b)$ where $n>0$. In other words, $a \equiv b \bmod n$ if and only if a and b have the same remainder when divided by n. We read $a \equiv b \bmod n$ as "a is congruent to b modulo n." The relation, congruence modulo n, defines an equivalence relationship on the set of integers and partitions the integers into n equivalence classes [0], [1], [2], ..., [n-1] called *residue classes modulo n* where

$$[r] = \{a : a \equiv r \bmod n\}.$$

If a and b are elements of the same residue class [s], then $a \equiv b \bmod n$. If [s] and [t] are distinct residue classes with $a \in [s]$ and $b \in [t]$, then

$a \not\equiv b \bmod n$. Let J_n be the set of all residue classes modulo n; that is, $J_n = \{[0],[1],\ldots,[n-1]\}$. We define "$\oplus$" (addition) and "$\otimes$" (multiplication) on the elements of J_n as follows:

$$[a] \oplus [b] = [a+b]$$

$$[a] \otimes [b] = [a \cdot b]$$

for every $[a],[b] \in J_n$.

The Concept of a Group

A non-empty set G on which a binary operation $\circ$ is defined is called a *group* if

(i) there exists an identity element (usually denoted by) $e \in G$ such that $a \circ e = e \circ a = a$ for $a \in G$.

(ii) for every choice of the elements a, b, $c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$;

(iii) every element $a \in G$ has an inverse $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$;

(iv) the combination (e.g., product or sum) of any two elements, a, $b \in G$, is a unique element $c = a \circ b$ which also belongs to the group, i.e., $c \in G$.

In other words, an algebraic system $(G,\circ,e)$ is a group if and only if $\circ$ is associative, e is an identity, each member of G possesses an inverse with respect to e, and the system is closed under the binary operation. We will sometimes write $(G,\circ)$ or, simply, G, to denote a group.

A group G is said to be *abelian* (or commutative) if for every $a, b \in G$, $a \circ b = b \circ a$. Most of our attention in this paper is directed to abelian groups where the binary operation is that of addition; i.e., $\circ = +$. In this additive case, we can define an abelian group[1] precisely as follows: An abelian group is a non-empty set G together with a binary operation + such that

(i) $(a+b) + c = a + (b+c)$ for all $a, b, c \in G$;

(ii) $a + b = b + a$ for all $a, b \in G$;

(iii) there exists an identity element denoted by 0, such that $a + 0 = a$ for all $a \in G$.

(iv) corresponding to each $a \in G$ there exists an element b such that $a + b = 0$. This b is unique and is denoted by $-a$. The element $-a$ is often termed the *negative of a.*

The *order* of a group G is the number of elements in G and is denoted by $|G|$. When the order of G is finite, we say that G is a *finite group.*

EXAMPLE: Let us illustrate some of these fundamental definitions by considering the abelian group (under the operation of addition) given as $G = \{0,1/6,2/6,3/6,4/6,5/6\} \pmod 1$. For any subset of G we see that the first group property holds; e.g., $(1/6 + 3/6) + 2/6 = 1/6 + (3/6 + 2/6)$; obviously, the second property also holds. In respect to the third property, we see that the identity element 0 exists such that $a + 0 = a$ for all $a \in G$. The fourth property is the most interesting. We see, for example, that $1/6 + 5/6 = 0$ and that $5/6 = -1/6$; likewise, $2/6 + 4/6 = 0$,

---

[1] Henceforth, unless otherwise noted, we assume that all groups discussed in this dissertation are abelian.

$3/6 + 3/6 = 0$, $4/6 + 2/6 = 0$, and $5/6 + 1/6 = 0$. The order of G is six since there are six elements in G; i.e., $|G| = 6$.

A subset H of a group G is said to be a *subgroup* of G if, under the binary operation in G, H itself forms a group. If H is a subgroup of G and K is a subgroup of H, then K is a subgroup of G. If (in abelian groups) the group relation is additive, a subgroup is characterized by the fact that it contains $a + b$ when it contains a and b, and $-a$ when it contains a. In other words, if a subgroup contains a and b, it must contain $a - b$. If Ω is any set of subgroups of a group G, the intersection of these subgroups is also a subgroup of G.

If $X \subseteq G$ let $gp(X) = \{g \in G | g = c_1 x_1 + \ldots + c_n x_n$ where for each i, $x_i \in X$ or $-x_i \in X\}$. Then gp(X) is a subgroup of G; in fact, it is the smallest subgroup of G containing X. If $G = gp(X)$, where X is a finite set, then G is called *finitely generated.*

Groups which can be generated by a single element are called *cyclic groups*. A group H is cyclic if we can find an element $x \in H$ such that $H = gp(x)$; the element x is then called a *generator* of H. We usually write gp(x) instead of gp({x}). More specifically we say that

$$gp(x) = \{t | t = rx,\ r = 0, \pm 1, \pm 2, \ldots\}.$$

An element rx of a finite cyclic group G of order n is a generator of G if and only if $(n,r) = 1$. Every subgroup of a cyclic group is itself a cyclic group. Every cyclic group is abelian; cyclic groups play an important role in the theory of abelian groups.

EXAMPLE: Let H = {0,2/6,4/6}. We see that H is a *subgroup* of G = {0,1/6,2/6,3/6,4/6,5/6} since the elements of H form a subset of the elements of G and since 2/6 + 4/6 = 0, 2/6 - 4/6 = -2/6 = 4/6, 4/6 - 2/6 = 2/6 = -4/6.

Likewise, L = {0,3/6} is a subgroup of G. On the other hand, {0,1/6,2/6} is not a subgroup of G even though the elements 0, 1/6, and 2/6 form a subset of the elements of G.

We see that G = gp(1/6); that is, G is a *cyclic group* since it can be generated by the single element 1/6. Likewise, H = gp(2/6) is a *cyclic subgroup* of order 3; L = gp(3/6) is cyclic subgroup of order 2. It is interesting to note that G = gp(5/6) = gp(1/6); that is, there are two *generators* of G. Likewise, H = gp(4/6) = gp(2/6).

Homomorphism, Isomorphism, Automorphism

Where S and T are both groups, a mapping $\theta : S \to T$ is a *homomorphism* (of S into T) if it preserves addition; i.e., $\theta(s_1+s_2) = \theta s_1 + \theta s_2$ for all s_1, $s_2 \epsilon S$. Every homomorphism $\theta : S \to T$ gives rise to two subgroups:

(1) The kernel of θ, Ker θ, is the set of all $s \epsilon S$ with $\theta s = 0$;

(2) The image of θ, Im θ, consists of all $t \epsilon T$ such that some $s \epsilon S$ satisfies $\theta s = t$.

The homomorphism θ is called an *isomorphism* if θ is one-to-one *and* onto; in this case, we write $S \cong T$.

An isomorphism of a group G with itself is called an *automorphism* of G. Thus, an automorphism θ of G is a one-to-one transformation of G onto itself such that $\theta(x+y) = (\theta x) + (\theta y)$ for all $x, y \epsilon G$. The automorphisms of any group G themselves form a group A.

Cosets and Factor Groups

In group theory, a *complex* is defined as an arbitrary set of elements of a group G. For an abelian group under the law of addition, by the sum of two complexes α and τ we understand the set of all sums $s + t$ where s is taken from α, and t from τ. If the complex α is a subgroup of G, and a an element of G, then the complex $a + \alpha$ is called a *left coset*, and the complex $\alpha + a$ a *right coset* (or a residue class) of α in G. However, there is no real distinction between right and left cosets in the present case, since G is abelian; i.e., $a + \alpha = \alpha + a$. Any two cosets are either disjoint or exactly the same. The union of all the cosets of α in G is G; thus, the cosets of α in G form a partition of G, and we can speak of *decomposing* G into cosets with respect to α.

The *factor group* of G by H is denoted by G/H, the collection of cosets of the subgroup H in G. Thus $G/H = \{g_1+H, g_2+H, \ldots, g_n+H\}$ where $G = \{g_1, g_2, \ldots, g_n\}$. If H is of order m and G is of order n, then the factor group G/H is of order n/m.

EXAMPLE: Where $G = \{0, 1/6, 2/6, 3/6, 4/6, 5/6\}$ and $H = \{0, 2/6, 4/6\}$, we have $G/H = \{(0,2/6,4/6),(1/6,3/6,5/6),(2/6,4/6,0),(3/6,5/6,1/6),(4/6,0,2/6),(5/6,1/6,3/6)\} = \{(0,2/6,4/6),(1/6,3/6,5/6)\}$. G/H is of order $6/3 = 2$.

We define *direct sum* $G \oplus H$ of any two groups G and H as the ordered pairs (g,h) with $G \epsilon G$, $h \epsilon H$; addition in $G \oplus H$ is defined by the formula

$$(g,h) \oplus (g',h') = (g+g',h+h').$$

$G \oplus H$ is a group. If G and H are finite groups, then $|G\oplus H| = |G||H|$. For example, if the order of G were 4 and the order of H were 12, then $|G\oplus H| = 48$.

Other Group Relationships

Let H be a subgroup of $G = gp(x)$. Then H is cyclic and either $H = gp(\ell x)$ where ℓx is the least positive multiple of x which lies in H or else $H = \{1\}$. If the order of G is $m<\infty$, then $\ell|m$ and the order of H is m/ℓ. If the order of G is infinite, H is infinite or $H = \{1\}$. Conversely, if ℓ is any positive integer dividing m, then $S = gp(\ell x)$ is of order m/ℓ. Consequently there is a subgroup of order q for any q that divides m. The number of distinct subgroups of G is the same as the number of distinct divisors of $m = |G|<\infty$. There is at most one subgroup of G of any given order for G finite.

Let us state the famous theorem due to Lagrange: If G is a finite group and α is a subgroup of G, then $|\alpha| \,\big|\, |G|$; i.e., the order of α divides the order of G. We define the *index* of α in G as the number of distinct cosets of α in G where α is a subgroup of G; we denote the index of α in G as $[G:\alpha]$. We define the order of an element a G as the least positive integer m such that $ma = e$. If G is a finite group and $a\epsilon G$, then $|a| \,\big|\, |G|$ and $|G|a = e$. Further, every group G of prime order p is cyclic. If G is a finite group, $|G| = |\alpha|\cdot[G:\alpha]$.

Any factor group of a cyclic group is cyclic; the homomorphic image of any cyclic group is cyclic. Any two cyclic groups of the same order are isomorphic. There exist cyclic groups of all orders, finite and infinite; however, there is one and only one subgroup of any given finite index $n>0$ in the infinite cyclic group. Let θ be a homomorphism of a cyclic group G. If $|G|<\infty$, $|\theta G| \,\big|\, |G|$. Furthermore, if H is any cyclic group such that $|H| \,\big|\, |G|$, there is a homomorphism of G onto H. If G is infinite cyclic, there is a homomorphism of G onto any cyclic group.

If $\theta : S \to T$ is a homomorphism of a group S into a group T, then Ker θ is a subgroup of S and $\Phi : \theta s \to (\text{Ker } \theta)s$ defines an isomorphism of θs onto S/Ker θ.

Every abelian group of finite order is isomorphic to a direct sum of cyclic groups of prime-power orders. We use this fact in integer programming when we write the direct sum expression

$$G \cong Z_{q_1} \oplus Z_{q_2} \oplus \dots \oplus Z_{q_r}$$

where Z_{q_i} = residue class of integers modulo q_i, and $q_1|q_2|\dots|q_r$ (q_i divides q_{i+1}).

APPENDIX B

EXPERIMENTAL RESULTS FOR THE FCTP WITH $m_{ij} = \mu \ \forall(i,j)$

A collection of 17 experimental 2×3 FCTP were solved wherein the optimal LP solutions were non-integer and the resultant group problems were formed. We present only the group problem constraint for each of the 17 problems in this Appendix. (These 17 problems are part of a set of 48 experimental problems; the problems are designated T-1 through T-50 with numbers T-13 and T-14 omitted.)

Problem T-6. $G \tilde{=} Z_{32} \oplus Z_{32} \oplus Z_{32} \oplus Z_{32}$

$$\begin{bmatrix}31\\0\\0\\0\end{bmatrix}t_1+\begin{bmatrix}31\\1\\31\\0\end{bmatrix}t_2+\begin{bmatrix}31\\1\\0\\31\end{bmatrix}t_3+\begin{bmatrix}31\\1\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\31\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\31\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\31\end{bmatrix}t_7=\begin{bmatrix}16\\16\\27\\25\end{bmatrix}\text{mod}\begin{bmatrix}32\\32\\32\\32\end{bmatrix}$$

Problem T-11. $G \simeq Z_{50} \oplus Z_{50} \oplus Z_{50} \oplus Z_{50}$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}t_1+\begin{bmatrix}1\\1\\0\\0\end{bmatrix}t_2+\begin{bmatrix}49\\49\\49\\0\end{bmatrix}t_3+\begin{bmatrix}49\\49\\0\\49\end{bmatrix}t_4+\begin{bmatrix}0\\49\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\49\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\49\end{bmatrix}t_7=\begin{bmatrix}25\\25\\33\\28\end{bmatrix}\text{mod}\begin{bmatrix}50\\50\\50\\50\end{bmatrix}$$

Problem T-15. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}121\\1\\1\\0\end{bmatrix}t_1+\begin{bmatrix}1\\121\\0\\1\end{bmatrix}t_2+\begin{bmatrix}1\\121\\121\\0\end{bmatrix}t_3+\begin{bmatrix}1\\121\\0\\0\end{bmatrix}t_4+\begin{bmatrix}121\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\121\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\121\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\121\end{bmatrix}t_8 = \begin{bmatrix}94\\4\\0\\35\end{bmatrix}\text{mod}\begin{bmatrix}122\\122\\122\\122\end{bmatrix}$$

Problem T-18. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}1\\0\\121\\121\\0\\1\end{bmatrix}t_1+\begin{bmatrix}121\\1\\0\\1\\121\\0\end{bmatrix}t_2+\begin{bmatrix}1\\121\\0\\121\\0\\0\end{bmatrix}t_3+\begin{bmatrix}1\\0\\0\\121\\0\\0\end{bmatrix}t_4+\begin{bmatrix}121\\0\\0\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\121\\0\\0\\0\\0\end{bmatrix}5_6+\begin{bmatrix}0\\0\\121\\0\\0\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\121\\0\\0\end{bmatrix}t_8+\begin{bmatrix}0\\0\\0\\0\\121\\0\end{bmatrix}t_9+\begin{bmatrix}0\\0\\0\\0\\0\\121\end{bmatrix}t_{10}=\begin{bmatrix}94\\0\\0\\9\\0\\35\end{bmatrix}\text{mod}\begin{bmatrix}122\\122\\122\\122\\122\\122\end{bmatrix}$$

Problem T-19. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}0\\1\\121\\0\\121\\1\end{bmatrix}t_1+\begin{bmatrix}121\\1\\0\\1\\121\\0\end{bmatrix}t_2+\begin{bmatrix}0\\121\\0\\121\\1\\0\end{bmatrix}t_3+\begin{bmatrix}0\\121\\0\\0\\1\\121\end{bmatrix}t_4+\begin{bmatrix}121\\0\\0\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\121\\0\\0\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\121\\0\\0\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\121\\0\\0\end{bmatrix}t_8+\begin{bmatrix}0\\0\\0\\0\\121\\0\end{bmatrix}t_9+\begin{bmatrix}0\\0\\0\\0\\0\\121\end{bmatrix}t_{10}+\begin{bmatrix}0\\121\\0\\0\\1\\0\end{bmatrix}t_{11}=\begin{bmatrix}0\\94\\0\\103\\28\\35\end{bmatrix}\text{mod}\begin{bmatrix}122\\122\\122\\122\\122\\122\end{bmatrix}$$

Problem T-20. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}0\\1\\121\\0\\121\\1\end{bmatrix} \quad \begin{bmatrix}1\\121\\0\\121\\1\\0\end{bmatrix} \quad \begin{bmatrix}121\\1\\0\\0\\121\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\\121\\0\end{bmatrix} \quad \begin{bmatrix}121\\0\\0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\121\\0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\121\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\121\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\\121\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\\0\\121\end{bmatrix} \qquad \begin{bmatrix}103\\113\\0\\0\\9\\35\end{bmatrix} \quad \begin{bmatrix}122\\122\\122\\122\\122\\122\end{bmatrix}$$

Problem T-22. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}1\\121\\1\\0\end{bmatrix} t_1 + \begin{bmatrix}1\\121\\1\\121\end{bmatrix} t_2 + \begin{bmatrix}0\\121\\1\\0\end{bmatrix} t_3 + \begin{bmatrix}121\\0\\0\\0\end{bmatrix} t_4 + \begin{bmatrix}0\\121\\0\\0\end{bmatrix} t_5 + \begin{bmatrix}0\\0\\121\\0\end{bmatrix} t_6 + \begin{bmatrix}0\\0\\0\\121\end{bmatrix} t_7 = \begin{bmatrix}15\\2\\120\\71\end{bmatrix} \text{mod} \begin{bmatrix}122\\122\\122\\122\end{bmatrix}$$

Problem T-23. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}1\\0\\1\\121\end{bmatrix} t_1 + \begin{bmatrix}121\\1\\0\\1\end{bmatrix} t_2 + \begin{bmatrix}121\\0\\121\\1\end{bmatrix} t_3 + \begin{bmatrix}121\\0\\0\\1\end{bmatrix} t_4 + \begin{bmatrix}121\\0\\0\\0\end{bmatrix} t_5 + \begin{bmatrix}0\\121\\0\\0\end{bmatrix} t_6 + \begin{bmatrix}0\\0\\121\\0\end{bmatrix} t_7 + \begin{bmatrix}0\\0\\0\\121\end{bmatrix} t_8 = \begin{bmatrix}2\\15\\0\\69\end{bmatrix} \text{mod} \begin{bmatrix}122\\122\\122\\122\end{bmatrix}$$

Problem T-25. $G \simeq Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122} \oplus Z_{122}$

$$\begin{bmatrix}121\\1\\0\\1\\121\\0\end{bmatrix} t_1 + \begin{bmatrix}0\\1\\121\\0\\121\\0\end{bmatrix} t_2 + \begin{bmatrix}0\\121\\0\\121\\1\\0\end{bmatrix} t_3 + \begin{bmatrix}0\\121\\0\\0\\1\\121\end{bmatrix} t_4 + \begin{bmatrix}0\\121\\0\\0\\1\\0\end{bmatrix} t_5 + \begin{bmatrix}121\\0\\0\\0\\0\\0\end{bmatrix} t_6 + \begin{bmatrix}0\\121\\0\\0\\0\\0\end{bmatrix} t_7 + \begin{bmatrix}0\\0\\121\\0\\0\\0\end{bmatrix} t_8 + \begin{bmatrix}0\\0\\0\\121\\0\\0\end{bmatrix} t_9 + \begin{bmatrix}0\\0\\0\\0\\121\\0\end{bmatrix} t_{10} + \begin{bmatrix}0\\0\\0\\0\\0\\121\end{bmatrix} t_{11} = \begin{bmatrix}0\\17\\0\\15\\105\\71\end{bmatrix} \text{mod} \begin{bmatrix}122\\122\\122\\122\\122\\122\end{bmatrix}$$

Problem T-27. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}154\\1\\1\\0\end{bmatrix}t_1+\begin{bmatrix}1\\154\\0\\1\end{bmatrix}t_2+\begin{bmatrix}1\\154\\154\\0\end{bmatrix}t_3+\begin{bmatrix}1\\154\\0\\0\end{bmatrix}t_4+\begin{bmatrix}154\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\154\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\154\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\154\end{bmatrix}t_8=\begin{bmatrix}143\\12\\122\\71\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\end{bmatrix}$$

Problem T-28. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}154\\1\\154\\1\\0\end{bmatrix}t_1+\begin{bmatrix}154\\0\\0\\1\\154\end{bmatrix}t_2+\begin{bmatrix}154\\0\\0\\1\\0\end{bmatrix}t_3+\begin{bmatrix}154\\0\\0\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\154\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\154\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\154\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\0\\154\end{bmatrix}t_8=\begin{bmatrix}58\\0\\0\\64\\71\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\\155\end{bmatrix}$$

Problem T-29. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}1\\0\\154\\154\\0\\1\end{bmatrix}t_1+\begin{bmatrix}0\\154\\1\\0\\1\\154\end{bmatrix}t_2+\begin{bmatrix}0\\0\\154\\0\\154\\1\end{bmatrix}t_3+\begin{bmatrix}0\\0\\154\\0\\0\\1\end{bmatrix}t_4+\begin{bmatrix}154\\0\\0\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\154\\0\\0\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\154\\0\\0\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\154\\0\\0\end{bmatrix}t_8+\begin{bmatrix}0\\0\\0\\0\\154\\0\end{bmatrix}t_9+\begin{bmatrix}0\\0\\0\\0\\0\\154\end{bmatrix}t_{10}=\begin{bmatrix}0\\0\\58\\0\\122\\13\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\\155\\155\end{bmatrix}$$

Problem T-30. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}154\\154\\154\\0\end{bmatrix}t_1+\begin{bmatrix}154\\154\\0\\154\end{bmatrix}t_2+\begin{bmatrix}154\\154\\0\\0\end{bmatrix}t_3+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\154\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\154\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\154\end{bmatrix}t_7=\begin{bmatrix}83\\83\\122\\71\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\end{bmatrix}$$

Problem T-31. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}154\\154\\154\\0\end{bmatrix}t_1+\begin{bmatrix}154\\154\\0\\154\end{bmatrix}t_2+\begin{bmatrix}1\\1\\0\\0\end{bmatrix}t_3+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\154\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\154\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\154\end{bmatrix}t_7=\begin{bmatrix}135\\135\\122\\71\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\end{bmatrix}$$

Problem T-32. $G \simeq Z_{155} \oplus Z_{155} \oplus Z_{155} \oplus Z_{155}$

$$\begin{bmatrix}154\\1\\154\\0\end{bmatrix}t_1+\begin{bmatrix}0\\154\\1\\154\end{bmatrix}t_2+\begin{bmatrix}0\\1\\154\\0\end{bmatrix}t_3+\begin{bmatrix}154\\0\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\154\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\154\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\154\end{bmatrix}t_7=\begin{bmatrix}0\\110\\12\\71\end{bmatrix}\bmod\begin{bmatrix}155\\155\\155\\155\end{bmatrix}$$

Problem T-37. $G \simeq Z_{71} \oplus Z_{71} \oplus Z_{71} \oplus Z_{71}$

$$\begin{bmatrix}70\\1\\70\\0\end{bmatrix}t_1+\begin{bmatrix}70\\1\\0\\70\end{bmatrix}t_2+\begin{bmatrix}70\\1\\0\\0\end{bmatrix}t_3+\begin{bmatrix}70\\0\\0\\0\end{bmatrix}t_4+\begin{bmatrix}0\\70\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\0\\70\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\0\\70\end{bmatrix}t_7=\begin{bmatrix}60\\66\\22\\0\end{bmatrix}\bmod\begin{bmatrix}71\\71\\71\\71\end{bmatrix}$$

Problem T-41. $G \simeq Z_{150} \oplus Z_{150} \oplus Z_{150} \oplus Z_{150}$

$$\begin{bmatrix}1\\0\\1\\149\end{bmatrix}t_1+\begin{bmatrix}149\\1\\0\\1\end{bmatrix}t_2+\begin{bmatrix}149\\0\\149\\1\end{bmatrix}t_3+\begin{bmatrix}149\\0\\0\\1\end{bmatrix}t_4+\begin{bmatrix}149\\0\\0\\0\end{bmatrix}t_5+\begin{bmatrix}0\\149\\0\\0\end{bmatrix}t_6+\begin{bmatrix}0\\0\\149\\0\end{bmatrix}t_7+\begin{bmatrix}0\\0\\0\\149\end{bmatrix}t_8=\begin{bmatrix}100\\50\\100\\50\end{bmatrix}\bmod\begin{bmatrix}150\\150\\150\\150\end{bmatrix}$$

APPENDIX C

BENDERS' PARTITIONING PROCEDURE AND GROUP THEORY IN THE FCTP

Benders' partitioning procedure [10] has been used by Spielberg [64] and others to solve the FCTP. Balinski [6] discusses the Benders approach in respect to the mixed integer problem. Unger [68] has applied a modified Benders' procedure to the capital budgeting problem. We summarize the background for Benders' method as follows:

Consider problem (P1) stated earlier; rewrite (P1) as

$$\min_{y}\{c_2y+\min_{x}[c_1x \mid A_1x \geq b-A_2y,\ x\geq 0]\}. \tag{1}$$

Given the vector y, the minimization over x is a linear program. This linear program can be replaced by its dual.

(P8)
$$\max_{u}\{u(b-A_2y) \mid uA_1 \leq c_1,\ u\geq 0\}$$

such that we may write

$$\min_{y}\{c_2y+\max_{u}[u(b-A_2y) \mid uA_1\leq c_1,\ u\geq 0]\} \tag{2}$$

We note two interesting features of (P8). First, the feasible region of u defined by $uA_1 \leq c_1$ is independent of y. Second, the maximum of $u(b-A_2y)$ always occurs on a vertex of the convex polytope defined by $uA_1 \le c_1$, provided that the convex polytope is bounded from above. Let u^k represent an extreme point of the convex polytope. Then we may write (P8) as

$$\text{(P8')} \qquad \text{Maximize} \quad u^k(b-A_2y)$$
$$\text{subject to} \quad u^k \ge 0 \ (k=1,\dots,K).$$

We are interested in the case where (P8) has a finite optimum solution. In the case where u goes to infinity for certain values of $(b-A_2y)$ we can add the constraint $\sum u_i \le M$ (where M is a very large positive constant) to the constraint set $uA_1 \le c_1$ [46].

We may substitute (P8') into (P1) to produce

$$\text{Minimize } z$$
$$\text{(P1')} \qquad \text{subject to } z \ge c_2y + \max_{u^k} u^k(b-A_2y)$$
$$u^k \ge 0 \quad (k=1,\dots,K).$$

Each u^k gives one constraint and (P1') is a pure integer program for a fixed number of u^k.

Benders' partitioning preserves the structure of the matrix A_1. This preservation proves helpful in the present investigation because of the unimodularity of A_1. A resultant algorithm, which terminates in a finite number of steps, may be stated as follows:

Step 0. Initialize with a $\bar{u} \geq 0$ such that $uA_1 \leq c_1$. This $\bar{u}$ does not have to be a vertex. (If none exists, then the original problem (P1) has no feasible solution.)

Step 1. Solve (P1') which we may write as

$$\text{Minimize} \quad z$$

$$\text{subject to } z \geq c_2 y + \bar{u}(b - A_2 y)$$

$$y \geq 0, \text{ integer.}$$

If z is unbounded from below, take a z to be any small value $\bar{z}$.

Step 2. Using $\bar{y}$ obtained in Step 1, solve the linear program

$$\underset{u}{\text{Maximize}} \quad u(b - A_2\bar{y})$$

$$\text{subject to} \quad uA_1 \leq c_1$$

$$u \geq 0.$$

If u is unbounded, add the constraint $\sum u_i \leq M$, where M is a large positive constant, and resolve this problem. Let the solution of this program be $\bar{\bar{u}}$. Determine whether $\bar{\bar{u}}(b - A_2 y) \geq \bar{z} - c_2\bar{y}$. If the equality holds, go to Step 3. If the equality does not hold, go to Step 1 and add the constraint

$$z \geq c_2 y + \bar{\bar{u}}(b - A_2 y)$$

to the existing set of constraints in (P1'). An inequality in (P1') can be dropped if the corresponding slack variable becomes positive.

Step 3. Use $\bar{y}$ obtained in Step 1. Solve the linear program

$$\text{Minimize} \quad c_1 x$$

$$\text{subject to} \quad A_1 x \geq b - A_2 \bar{y}$$

$$x \geq 0.$$

Let the solution be $\bar{x}$. We claim that $\bar{x}$ and $\bar{y}$ are then the optimum solution and $z^* = c_1 \bar{x} + c_2 \bar{y}$.

We solve the first 24 experimental problems by Benders method, calculating the order of the group corresponding to each integer subproblem created in the process. Some of the results are presented in Table 4.

It is noteworthy that the associated LP basis was unimodular for the first integer subproblem in each experimental problem.

It has been shown that route capacities and their interrelationships have a direct effect on the order of the group in the unpartitioned FCTP. We have been able to take advantage of this capacity relationship in the unpartitioned problem. Unfortunately, modification of route capacities does not appear to be helpful in the Benders problem since the group structure is dependent on the interrelationship of the fixed-charges, the route capacities, and the dual variables.

Table 4. Integer Subproblem Results

Problem	Number of Constraints $z \geq c_2y+\bar{\bar{u}}(b-A_2y)$	Order of Group for Corresponding Iteration
T-1	1	1
	2	1
T-2	1	1
	2	28
T-3	1	1
	2	12475
	3	12475
T-4	1	1
	2	27810
	3	349,015,500
T-5	3	1
T-6	1	1
	2	32000
T-7	2	33,920
	4	34,300,800
T-8	1	1
	2	13,930
T-9	1	1
	2	27,917
	3	381,848,726
T-10	1	1
	2	1
T-11	1	1
	2	50,000
	3	70,000,000
	4	13,916
T-12	1	1
	2	53,000
	3	53,000
T-15	1	1
	3	6,222
	4	36,078,816
	5	36,078,816
T-16	1	1
	2	61,244
T-17	1	1
	2	124,318
	3	6,351,157,984
T-18	1	1
	2	41,114
	3	2,472,842,644
T-19	1	1
	3	3,141,500,000
	4	$> 10^{10}$
	5	$> 10^{10}$
T-20	1	1
	3	6,083,200,600
T-21	1	1
T-22	1	1
T-23	1	1
T-24	1	1
	3	1,426,406,188
T-25	1	1
	3	457,500,000
T-26	1	1

APPENDIX D

BALAS' DUALITY THEORY FOR INTEGER PROGRAMS

Balas [2,3] has developed some important theory on duality in integer programs. In [2] Balas focuses on the applications of his theory to the fixed-charge problem; Gray [37] discusses these same applications in some detail; Ellwein [25] applies some fundamental results in his work on the fixed-charge location-allocation problem.

The basic result of Balas' theory exploited by Ellwein translates into the case of the fixed-charge transportation problem as follows:

A penalty may be incurred for not opening some route. This penalty occurs when $s_{ij} \leq 0$ where $s_{ij} = f_{ij} - u_{ij}m_{ij}$ and u_{ij} is the dual variable in the transportation subproblem (with all routes open) corresponding to the primal constraint $x_{ij} \leq m_{ij}y_{ij}$. We use this penalty concept such that $u_{ij}m_{ij} > f_{ij} \Rightarrow y_{ij} = 1$, i.e., we set the corresponding y_{ij} equal to 1 permanently.

Further, for those routes where $u_{ij}m_{ij} < f_{ij}$, we may solve the transportation problem where route (i,j) is not available and let $VMIN_{ij}$ be the total variable cost thus found. Calculate $\Delta V_{ij} = VMIN_{ij} - c_o$; $\Delta V_{ij} > f_{ij} \Rightarrow y_{ij} = 1$. When attempting to solve the transportation problem with route (i,j) deleted, if the solution becomes infeasible, then route (i,j) *must* be selected and $y_{ij} = 1$. We have applied Balas' penalty concept in the 48 experimental problems; seven of the 48 problems contain routes which must be open according to Balas' test. In

Table 5 we show the effects of these permanent openings on FMAX as defined in Chapter II (where FMAX ≥ fy, and FMAX' = FMAX - $\sum_{(i,j)\epsilon J} f_{ij}$, where $J = \{(i,j)|u_{ij}m_{ij}>f_{ij}\}$).

Table 5. Results of Balas' Penalty Test

Problem	$J=\{(i,j)\|u_{ij}m_{ij}>f_{ij}\}$	FMAX	FMAX'
T-2	{(1,1),(2,3)}	49	16
T-15	{(2,3)}	79	61
T-16	{(1,1)}	834	581
T-28	{(1,1)}	1053	800
T-33	{(1,1)}	49	31
T-45	{(1,1)}	49	31
T-46	{(1,1)}	1053	800

APPENDIX E

AN EXPERIMENT WITH SUBGROUP ORDER AND CAPACITY RELATIONSHIPS IN THE FCTP

Associated with Theorem 4.2 we have the following corollaries:

Corollary 1. q_m is maximized when $\gcd\{m_{ij}, \forall(i,j)\}$ is minimized.

Corollary 2. $\gcd\{m_{kl}\} < \gcd\{m_{ij}\} \Rightarrow q_m^{(k,l)} \geq q_m^{(i,j)}$; i.e., the maximum subgroup order will decrease as the corresponding $\gcd\{m_{ij}\}$ increases.

We demonstrate these corollaries by manipulating the route capacities of the 48 experimental problems with results shown in Table 6 below.

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's

Problem	Route Capacities $\binom{11}{21}$	$\binom{12}{22}$	$\binom{13}{23}$	$q_k, q_{k+1}, \ldots, q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-1 Original	32	27	25	21600	21600
A	32	28	32	4,32,224	28672
B	32	28	28	4,28,224	25088
C	32	32	26	2,32,416	26624
D	35	30	25	5,5,1050	26250
E	36	27	27	9,27,108	26244

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{Bmatrix}11\\21\end{Bmatrix}$	$\begin{Bmatrix}12\\22\end{Bmatrix}$	$\begin{Bmatrix}13\\23\end{Bmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-2 Original	50	33	28	66,23100	1524600
A	50	35	30	5,5,70,1050	1837500
B	54	36	30	6,18,36,540	2099520
C	56	35	28	7,7,140,280	1920800
T-3 Original	32	27	25	27,21600	583200
A	32	28	32	4,4,224,224	802816
B	32	28	28	4,28,28,224	702464
C	32	32	26	2,32,32,416	851968
D	35	30	25	5,5,30,1050	787500
E	36	27	27	9,27,27,108	708588
T-4 Original	32	27	25	27,21600	583200
A	32	28	32	4,4,224,224	802816
B	32	28	28	4,28,28,224	702464
C	32	32	26	2,32,32,416	851968
D	35	30	25	5,5,30,1050	787500
E	36	27	27	9,27,27,108	708588
T-5 Original	32	27	25	32,21600	691200
A	32	28	32	4,32,32,224	917504
B	32	28	28	4,4,224,224	802816
C	32	32	26	2,32,32,416	851968
D	35	30	25	5,5,35,1050	918750
E	36	27	27	9,9,108,108	944784
T-6 Original	32	27	25	21600,21600	466560000
A	32	28	32	4,4,32,32,224,224	822083584
B	32	28	28	4,4,28,28,224,224	629407744
C	32	32	26	2,2,32,32,416,416	708837376
D	35	30	25	5,5,5,5,1050,1050	689062500
E	36	27	27	9,9,27,27,108,108	688747536

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-7 Original	32	27	25	864,21600	18662400
A	32	28	32	4,4,32,224,224	25690112
B	32	28	28	4,4,28,224,224	22478848
C	32	32	26	2,32,32,32,416	27262976
D	35	30	25	5,5,5,210,1050	27562500
E	36	27	27	9,9,27,108,108	25509168
T-8 Original	50	33	30	2,23100	46200
A	50	35	30	5,10,1050	52500
B	54	36	30	6,18,540	58320
C	56	35	28	7,28,280	54880
T-9 Original	50	33	28	2,23100	46200
A	50	35	30	5,10,1050	52500
B	54	36	30	6,18,540	58320
C	56	35	28	7,28,280	54880
T-10 Original	50	33	28	2,1650,23100	76230000
A	50	35	30	5,5,10,350,1050	91875000
B	54	36	30	6,18,18,108,540	113374080
C	56	35	28	7,7,28,280,280	107564800
T-11 Original	50	33	28	2,2,23100,23100	2134440000
A	50	35	30	5,5,10,10,1050,1050	2756250000
B	54	36	30	6,6,18,18,540,540	3401222400
C	56	35	28	7,7,28,28,280,280	3011814400
T-12 Original	50	33	28	2,2,23100,23100	2134440000
A	50	35	30	5,5,10,10,1050,1050	2756250000
B	54	36	30	6,6,18,18,540,540	3401222400
C	56	35	28	7,7,28,28,280,280	3011814400

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-15 Original	103	122	35	103,439810	45300430
A	105	125	35	5,35,105,2625	48234375
B	105	140	35	35,35,105,420	54022500
C	104	124	36	4,4,104,29016	48282624
D	110	121	44	11,22,110,2420	64420400
E	108	132	36	12,36,108,1188	55427328
T-16 Original	103	122	35	12566,439810	5526652460
A	105	125	35	5,5,35,2625,2625	6029296875
B	105	140	35	35,35,35,420,420	7563150000
C	104	124	36	4,4,4,3224,29016	5987045376
D	110	121	44	11,11,22,1210,2420	7794868400
E	108	132	36	12,12,36,1188,1188	7316407296
T-17 Original	103	122	35	12566,439810	5526652460
A	105	125	35	5,5,35,2625,2625	6029296875
B	105	140	35	35,35,35,420,420	7563150000
C	104	124	36	4,4,4,3224,29016	5987045376
D	110	121	44	11,11,22,1210,2420	7794868400
E	108	132	36	12,12,36,1188,1188	7316407296
T-18 Original	103	122	35	12566,439810	5526652460
A	105	125	35	5,5,35,2625,2625	6029296875
B	105	140	35	35,35,35,420,420	7653150000
C	104	124	36	4,4,4,3224,29016	5987045376
D	110	121	44	11,11,22,1210,2420	7794868400
E	108	132	36	12,12,36,1188,1188	7316407296
T-19 Original	103	122	35	439810,439810	193432836100
A	105	125	35	5,5,35,35,2625,2625	211025390000
B	105	140	35	35,35,35,35,420,420	264710250000
C	104	124	36	4,4,4,4,29016,29016	215533644800
D	110	121	44	11,11,22,22,2420,2420	297723010000
E	108	132	36	12,12,36,36,1188,1188	263391000000

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-20 Original	103	122	35	122,439810	53656820
A	105	125	35	5,5,875,2625	57421875
B	105	140	35	35,35,140,420	72030000
C	104	124	36	4,4,124,29016	57567744
D	110	121	44	11,11,242,2420	70862440
E	108	132	36	12,12,396,1188	67744512
T-21 Original	15	122	71	129930	129930
A	15	125	75	5,75,375	140625
B	15	150	75	15,75,150	168750
C	16	136	72	8,8,2448	156672
T-22 Original	15	122	71	122,129930	15851460
A	15	125	75	5,25,375,375	17578125
B	15	150	75	15,75,150,150	25312500
C	16	136	72	8,8,136,2448	21307392
T-23 Original	15	122	71	71,129930	9225030
A	15	125	75	5,75,75,375	10546875
B	15	150	75	15,75,75,150	12656250
C	16	136	72	8,8,72,2448	11280384
T-24 Original	15	122	71	129930,129930	16881804900
A	15	125	75	5,5,75,75,375,375	19775390625
B	15	150	75	15,15,75,75,150,150	17797851560
C	16	136	72	8,8,8,8,2448,2448	24546115600
T-25 Original	15	122	71	8662,129930	1125453660
A	15	125	75	5,25,75,375,375	1318359375
B	15	150	75	15,75,75,150,150	1898437500
C	16	136	72	8,8,8,1224,2448	1534132224

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k, q_{k+1}, \ldots, q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-26 Original	15	122	71	122,129930	15851460
A	15	125	75	5,25,375,375	17578125
B	15	150	75	15,75,150,150	25312500
C	16	136	72	8,8,136,2448	21307392
T-27 Original	155	122	71	155,1342610	208104550
A	156	156	78	78,156,156,156	296120448
B	160	120	80	40,80,160,480	245760000
C	156	132	72	12,12,156,10296	231289344
T-28 Original	155	122	71	18910,1342610	25388760000
A	156	156	78	78,156,156,156,156	46194790000
B	160	120	80	40,40,80,480,480	29491200000
C	156	132	72	12,12,12,1716,10296	30530193410
T-29 Original	155	122	71	1342610,1342610	18026020000
A	156	156	78	78,78,156,156,156,156	36031936100
B	160	120	80	40,40,80,80,480,480	23592960000
C	156	132	72	12,12,12,12,10296,10296	21982000000
T-39 Original	155	122	71	18910,1342610	25388760000
A	156	156	78	78,156,156,156,156	46194790000
B	160	120	80	40,40,80,480,480	29491200000
C	156	132	72	12,12,12,1716,10296	30530193410
T-31 Original	155	122	71	1342610,1342610	18026020000
A	156	156	78	78,78,156,156,156,156	36031936100
B	160	120	80	40,40,80,80,480,480	23592960000
C	156	132	72	12,12,12,12,10296,10296	21982000000

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-32 Original	155	122	71	122,1342610	163798420
A	156	156	78	78,156,156,156	296120448
B	160	120	80	40,40,240,480	184320000
C	156	132	72	12,12,132,10296	195706368
T-33 Original	55	22	71	11,7810	85910
A	55	22	77	11,11,770	93170
B	55	25	75	5,25,825	103125
C	60	30	75	15,30,300	135000
T-34 Original	55	22	71	11,22,7810	1890020
A	55	22	77	11,11,22,770	2049740
B	55	25	75	5,25,25,825	2578125
C	60	30	75	15,30,30,300	4050000
T-35 Original	55	22	71	11,22,7810	1890020
A	55	22	77	11,11,22,770	2049740
B	55	25	75	5,25,25,825	2578125
C	60	30	75	15,30,30,300	4050000
T-36 Original	55	22	71	11,7810	85910
A	55	22	77	11,11,770	93170
B	55	25	75	5,25,825	103125
C	60	30	75	15,30,300	135000
T-37 Original	55	22	71	11,55,7810	4725050
A	55	22	77	11,11,55,770	5124350
B	55	25	75	5,5,275,825	5671875
C	60	30	75	15,30,60,300	8100000

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\binom{11}{21}$	$\binom{12}{22}$	$\binom{13}{23}$	$q_k, q_{k+1}, \ldots, q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-38 Original	55	22	71	11,22,7810	1890020
A	55	22	77	11,11,22,770	2049740
B	55	25	75	5,25,25,825	2578125
C	60	30	75	15,30,30,300	4050000
T-39 Original	50	100	150	50,50,50,300	37500000
A	51	100	149	51,759900	38754900
T-40 Original	50	100	150	50,50,50,150,300	5625000000
A	51	100	149	7599,759900	5774480100
T-41 Original	50	100	150	50,50,50,50,300,300	562500000000
A	51	100	149	759900,759900	577448010000
T-42 Original	50	100	150	50,50,50,100,300	3750000000
A	51	100	149	5100,759900	3875490000
T-43 Original	50	100	150	50,50,50,100,300	3750000000
A	51	100	149	5100,759900	3875490000
T-44 Original	50	100	150	50,50,50,100,300	3750000000
A	51	100	149	5100,759900	3875490000
T-45 Original	100	100	100	100,100,100,100	100000000
A	99	101	103	101,1029897	104019597
T-46 Original	100	100	100	100,100,100,100	100000000
A	99	101	103	101,1029897	104019597

Table 6. Route Capacities and Resultant Subgroup Orders for 48 (2×3) FCTP's (Continued)

Problem	Route Capacities $\begin{pmatrix}11\\21\end{pmatrix}$	$\begin{pmatrix}12\\22\end{pmatrix}$	$\begin{pmatrix}13\\23\end{pmatrix}$	$q_k,q_{k+1},\ldots,q_m$ Diagonal Elements In Smith Normal Matrix $(1,\ldots,1,q_k,q_{k+1},\ldots,q_m)$	$D=\lvert\det B\rvert=\prod_i q_i$
T-47 Original	100	100	100	100,100,100,100,100	10000000000
A	99	101	103	10403,1029897	10714020000
T-48 Original	100	100	100	100,100,100,100,100	10000000000
A	99	101	103	9999,1029897	10297940100
T-49 Original	100	100	100	100,100,100,100,100	10000000000
A	99	101	103	9999,1029897	10297940100
T-50 Original	100	100	100	100,100,100,100,100	10000000000
A	99	101	103	9999,1029897	10297940100

APPENDIX F

EXAMPLES OF R, B, N, AND RN MATRICES

This appendix presents examples of R, B, N, and RN for three 2×3 fixed-charge transportation problems.

<u>Problem T-6</u>

Row Transformation Matrix R

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

Basis Matrix B

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 32 & 0
\end{bmatrix}
$$

Problem T-6 (Continued)

Non-Basis Matrix N

$$
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
32 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

RN

$$
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
32 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
32 & 32 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
-32 & -32 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \\
32 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 32 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1
\end{bmatrix}
$$

Problem T-18

Row Transformation Matrix R

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Basis Matrix B

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 122 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 122 & 0
\end{bmatrix}
$$

Problem T-18 (Continued)

Non-Basis Matrix N

$$\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

RN

$$\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

Problem T-19

Row Transformation Matrix R

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Basis Matrix B

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 122 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 122 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 122 & 0
\end{bmatrix}
$$

Problem T-19 (Continued)

Non-Basis Matrix N

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

RN

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

APPENDIX G

DEMONSTRATION OF INTERRELATIONSHIP PROPERTIES AMONG VARIABLES IN THE FCTP GROUP PROBLEM

We show $[R_2][N_1,N_3] \bmod \mu$, list the applicable properties, and summarize the actual results for each of the 17 2×3 fixed-charge transportation problems referred to in the preceding appendices.

Problem T-6: $x_{ij} \epsilon x_B$, $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

c*	s(D₁)	s(D₂)	s(D₃)	s(S₂)	s(1,1)	s(1,2)	s(1,3)	s(2,1)	s(2,2)	s(2,3)
c*=39.91	39.22	38.53	2.06	7.91	8.22	8.53	8.84	9.16	9.47	
	-1	-1	-1	-1	-1	-1	-1	0	0	0
	0	1	1	1	0	1	1	-1	0	0
	0	-1	0	0	0	-1	0	0	-1	0
	0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\big(g(D_1)\big) - c^*\big(g(1,1)\big) = 39.90625 - 7.90625 = 32 = c_{11}$;
$g(1,2) = g(D_2)$, $c^*\big(g(D_2)\big) - c^*\big(g(1,2)\big) = 39.21875 - 8.21875 = 31 = c_{12}$;
$g(1,3) = g(D_3)$, $c^*\big(g(D_3)\big) - c^*\big(g(1,3)\big) = 38.53125 - 8.53125 = 30 = c_{13}$.

By Property 6

$g(1,1) = g(2,1) + g(S_2)$, $c_{11} - c_{21} = 3$,
$g(1,2) = g(2,2) + g(S_2)$, $c_{12} - c_{22} = 3$,
$g(1,3) = g(2,3) + g(S_2)$, $c_{13} - c_{23} = 3$.

<u>Problem T-11</u>: $x_{ij} \epsilon x_B$, $s(i,j) \epsilon s_N$, $\forall (i,j)$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

c^*=									
37.06	36.26	35.46	2.40	5.06	5.26	5.46	5.66	5.86	6.06
$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
1	0	0	1	1	0	0	0	-1	-1
0	0	0	1	0	0	0	-1	-1	-1
0	-1	0	0	0	-1	0	0	-1	0
0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(1,1)\} = 37.06 - 5.06 = 32 = c_{11}$;
$g(1,2) = g(D_2)$, $c^*\{g(D_2)\} - c^*\{g(1,2)\} = 36.26 - 5.26 = 31 = c_{12}$;
$g(1,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(1,3)\} = 35.46 - 5.46 = 30 = c_{13}$.

By Property 6

$g(1,1) = g(2,1) + g(S_2)$, $c_{11} - c_{21} = 3$,
$g(1,2) = g(2,2) + g(S_2)$, $c_{12} - c_{22} = 3$,
$g(1,3) = g(2,3) + g(S_2)$, $c_{13} - c_{23} = 3$.

<u>Problem T-15</u>: x_{13}, $x_{22} \epsilon x_N$; $s(1,3)$, $s(2,2) \epsilon s_B$; $s(S_1) \epsilon s_N$, $s(D_j) \epsilon s_N$, $\forall j$.

c^*=									
1.81	1.91	3.16	3.09	2.15	0.96	0.20	0.13	0.16	0.15
x_{13}	x_{22}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_1)$	$s(1,1)$	$s(1,2)$	$s(2,1)$	$s(2,3)$
1	-1	0	1	0	1	-1	0	0	0
-1	1	-1	-1	0	-1	0	0	-1	0
0	1	0	-1	0	0	0	-1	0	0
1	0	0	0	-1	0	0	0	0	-1

By Property 1

$g(2,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(2,1)\} = 3.163934 - 0.163934 = 3 = c_{21}$;
$g(2,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(2,3)\} = 2.147541 - 0.147541 = 2 = c_{23}$.

By Property 2

$g(x_{22}) = -g(D_2)$, $c_{22} = 5.00$

By Property 3

$g(x_{13}) + g(D_3) = g(S_1)$, $c_{13} = 3.00$.

<u>Problem T-18:</u> $x_{13}, x_{22} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_1) \epsilon s_N$, $s(D_j) \epsilon s_N$, $\forall j$.

c*=5.54	2.69	89.63	65.53	31.28	3.36	10.27	9.17	12.46	8.63	8.22	12.28
x_{13}	x_{22}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_1)$	s(1,1)	s(1,2)	s(1,3)	s(2,1)	s(2,2)	s(2,3)
1	-1	0	1	0	1	-1	0	0	0	0	0
0	1	0	-1	0	0	0	-1	0	0	0	0
-1	0	0	0	0	0	0	0	-1	0	0	0
-1	1	-1	-1	0	-1	0	0	0	-1	0	0
0	-1	0	0	0	0	0	0	0	0	-1	0
1	0	0	0	-1	0	0	0	0	0	0	-1

By Property 1

$g(2,1) = g(D_1)$, $c^*\big(g(D_1)\big) - c^*\big(g(2,1)\big) = 89.631441 - 8.631441 = 81 = c_{21}$;

$g(2,3) = g(D_3)$, $c^*\big(g(D_3)\big) - c^*\big(g(2,3)\big) = 31.28 - 12.28 = 19 = c_{22}$.

By Property 4

$g(x_{13}) + g(D_3) = g(S_1) + g(1,3)$, $c_{13} = 21.00$

By Property 5

$g(x_{22}) + g(D_2) = g(2,2)$, $c_{22} = 60.00$.

<u>Problem T-19:</u> $x_{11}, x_{13} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_1) \epsilon s_N$, $s(D_j) \epsilon s_N$, $\forall j$.

c* = 0	0	34.07	33.16	32.24	2.76	2.07	2.16	2.24	2.32	2.40	2.48
x_{11}	x_{13}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	s(1,1)	s(1,2)	s(1,3)	s(2,1)	s(2,2)	s(2,3)
-1	0	0	0	0	0	-1	0	0	0	0	0
1	1	-1	-1	-1	-1	0	-1	0	0	0	0
0	-1	0	0	0	0	0	0	-1	0	0	0
1	0	-1	0	0	0	0	0	0	-1	0	0
-1	-1	1	0	1	1	0	0	0	0	-1	0
0	1	0	0	-1	0	0	0	0	0	0	-1

By Property 1

$g(1,2) = g(D_2)$, $c^*\big(g(D_2)\big) - c^*\big(g(1,2)\big) = 33.16 - 2.16 = 31 = c_{12}$.

By Property 5

$g(x_{11}) + g(D_1) = g(1,1)$, $c_{11} = 32$

$g(x_{13}) + g(D_3) = g(1,3)$, $c_{13} = 30$.

Problem T-20: $x_{13}, x_{21} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall (i,j)$; $s(S_1) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

c*=19.43	19.87	122.55	131.51	120.63	6.53	40.03	38.98	37.53	36.43	35.51	34.63
x_{13}	x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	s(1,1)	s(1,1)	s(1,2)	s(1,3)	s(2,1)	s(2,2)	s(2,3)
0	1	-1	0	0	0	-1	0	0	0	0	0
1	-1	1	0	0	1	0	-1	0	0	0	0
-1	0	0	0	0	0	0	0	-1	0	0	0
0	-1	0	0	0	0	0	0	0	-1	0	0
-1	1	-1	-1	0	-1	0	0	0	0	-1	0
1	0	0	0	-1	0	0	0	0	0	0	-1

By Property 1

$g(2,2) = g(D_2)$, $c^*\left(g(D_2)\right) - c^*\left(g(2,2)\right) = 131.51 - 35.51 = 96 = c_{22}$;

$g(2,3) = g(D_3)$, $c^*\left(g(D_3)\right) - c^*\left(g(2,3)\right) = 120.63 - 34.63 = 86 = c_{23}$.

By Property 4

$g(x_{13}) + g(D_3) = g(s_1) + g(1,3)$, $c_{13} = 96$,

By Property 5

$g(x_{21}) + g(D_1) = g(2,1)$, $c_{21} = 106$.

Problem T-22: $x_{21} \epsilon x_N$; $s(2,1) \epsilon s_B$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	s(1,1)	s(1,2)	s(1,3)	s(2,2)	s(2,3)
1	-1	0	0	0	-1	0	0	0	0
-1	0	-1	-1	-1	0	-1	-1	0	0
1	0	0	1	1	0	0	1	-1	0
0	0	0	-1	0	0	0	-1	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\left(g(D_1)\right) - c^*\left(g(1,1)\right) = c_{11}$;

$g(1,2) = g(D_2)$, $c^*\left(g(D_2)\right) - c^*\left(g(1,2)\right) = c_{12}$;

$g(1,3) = g(D_3)$, $c^*\left(g(D_3)\right) - c^*\left(g(1,3)\right) = c_{13}$.

By Property 3

$g(x_{21}) + g(D_1) = g(S_2)$.

Problem T-23: $x_{12}, x_{21} \epsilon x_N$; $s(1,2)$, $s(2,1) \epsilon s_B$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{12}	x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,3)$	$s(2,2)$	$s(2,3)$
1	-1	0	-1	-1	-1	0	-1	0	0
0	1	-1	0	0	0	-1	0	0	0
1	0	0	-1	0	0	0	0	-1	0
-1	1	0	1	0	1	0	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(1,1)\} = c_{11}$;
$g(1,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(1,2)\} = c_{13}$.

By Property 2

$g(x_{12}) = -g(D_2)$.

By Property 3

$g(x_{21}) + g(D_1) = g(S_2)$.

Problem T-25: $x_{11}, x_{13} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall(i,j)$, $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{11}	x_{13}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
-1	0	0	0	0	0	-1	0	0	0	0	0
1	1	-1	-1	-1	-1	0	-1	0	0	0	0
0	-1	0	0	0	0	0	0	-1	0	0	0
1	0	-1	0	0	0	0	0	0	-1	0	0
-1	-1	1	0	1	1	0	0	0	0	-1	0
0	1	0	0	-1	0	0	0	0	0	0	-1

By Property 1

$g(1,2) = g(D_2)$, $c^*\{g(D_2)\} - c^*\{g(1,2)\} = c_{12}$.

By Property 5

$g(x_{11}) + g(D_1) = g(1,1)$, $g(x_{13}) + g(D_3) = g(1,3)$.

<u>Problem T-27:</u> $x_{13}, x_{22} \epsilon x_N$; $s(1,3)$, $s(2,2) \epsilon s_B$; $s(S_1) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{13}	x_{22}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_1)$	$s(1,1)$	$s(1,2)$	$s(2,1)$	$s(2,3)$
1	-1	0	1	0	1	-1	0	0	0
-1	1	-1	-1	0	-1	0	0	-1	0
0	1	0	-1	0	0	0	-1	0	0
1	0	0	0	-1	0	0	0	0	-1

By Property 1

$g(2,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(2,1)\} = c_{21}$;

$g(2,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(2,3)\} = c_{23}$.

By Property 2

$g(x_{22}) = -g(D_2)$.

By Property 3

$g(x_{13}) + g(D_3) = g(S_1)$.

<u>Problem T-28:</u> $x_{21} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall (i,j)$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
-1	0	-1	-1	-1	0	-1	-1	0	0	0
1	-1	0	0	0	-1	0	0	0	0	0
-1	0	0	0	0	0	0	0	-1	0	0
1	0	0	1	1	0	0	1	0	-1	0
0	0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(1,1)\} = c_{11}$;

$g(1,2) = g(D_2)$, $c^*\{g(D_2)\} - c^*\{g(1,2)\} = c_{12}$;

$g(1,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(1,3)\} - c_{13}$.

By Property 3

$g(x_{21}) + g(D_1) = g(S_2) + g(2,1)$.

<u>Problem T-29</u>: $x_{12}, x_{21} \epsilon x_N$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{12}	x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
0	1	-1	0	0	0	-1	0	0	0	0	0
-1	0	0	0	0	0	0	-1	0	0	0	0
1	-1	0	-1	-1	-1	0	0	-1	0	0	0
0	-1	0	0	0	0	0	0	0	-1	0	0
1	0	0	-1	0	0	0	0	0	0	-1	0
-1	1	0	1	0	1	0	0	0	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*[g(D_1)] - c^*[g(1,1)] = c_{11}$,

$g(1,3) = g(D_3)$, $c^*[g(D_3)] - c^*[g(1,3)] = c_{13}$.

By Property 4

$g(x_{21}) + g(D_1) = g(S_2) + g(2,1)$.

By Property 5

$g(x_{12}) + g(D_2) = g(1,2)$.

<u>Problem T-30</u>: $x_{ij} \epsilon x_B$, $\forall(i,j)$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_1) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_1)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
0	-1	-1	-1	1	0	0	0	-1	-1
-1	-1	-1	-1	0	0	0	-1	-1	-1
0	-1	0	0	0	-1	0	0	-1	0
0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(2,1) = g(D_1)$, $c^*[g(D_1)] - c^*[g(2,1)] = c_{21}$,

$g(2,2) = g(D_2)$, $c^*[g(D_2)] - c^*[g(2,2)] = c_{22}$,

$g(2,3) = g(D_3)$, $c^*[g(D_3)] - c^*[g(2,3)] = c_{23}$.

By Property 6

$g(2,1) = g(1,1) + g(S_1)$,

$g(2,2) = g(1,2) + g(S_1)$,

$g(2,3) = g(1,3) + g(S_1)$.

<u>Problem T-31</u>: $x_{ij} \epsilon x_B$, $\forall(i,j)$; $s(i,j) \epsilon s_N$, $\forall(i,j)$, $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
1	0	0	1	1	0	0	0	-1	-1
0	0	0	1	0	0	0	-1	-1	-1
0	-1	0	0	0	-1	0	0	-1	0
0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\big(g(D_1)\big) - c^*\big(g(1,1)\big) = c_{11}$,

$g(1,2) = g(D_2)$, $c^*\big(g(D_2)\big) - c^*\big(g(1,2)\big) = c_{12}$,

$g(1,3) = g(D_3)$, $c^*\big(g(D_3)\big) - c^*\big(g(1,3)\big) = c_{13}$.

By Property 6

$g(1,1) = g(2,1) + g(S_2)$,

$g(1,2) = g(2,2) + g(S_2)$,

$g(1,3) = g(2,3) + g(S_2)$.

<u>Problem T-32</u>: $x_{ij} \epsilon x_B$, $\forall(i,j)$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_1) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_1)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
-1	0	0	0	-1	0	0	-1	0	0
1	0	0	1	0	-1	-1	1	0	0
-1	-1	0	-1	0	0	1	-1	-1	0
0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(2,1) = g(D_1)$, $c^*\big(g(D_1)\big) - c^*\big(g(2,1)\big) = c_{21}$,

$g(2,2) = g(D_2)$, $c^*\big(g(D_2)\big) - c^*\big(g(2,2)\big) = c_{22}$,

$g(2,3) = g(D_3)$, $c^*\big(g(D_3)\big) - c^*\big(g(2,3)\big) = c_{23}$.

By Property 6

$g(2,1) = g(1,1) + g(S_1)$,

$g(2,2) = g(1,2) + g(S_1)$,

$g(2,3) = g(1,3) + g(S_1)$.

Problem T-37: $x_{ij} \epsilon x_B$, $\forall(i,j)$; $s(i,j) \epsilon s_N$, $\forall(i,j)$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,2)$	$s(1,3)$	$s(2,1)$	$s(2,2)$	$s(2,3)$
-1	-1	-1	-1	-1	-1	-1	0	0	0
0	1	1	1	0	1	1	-1	0	0
0	-1	0	0	0	-1	0	0	-1	0
0	0	-1	0	0	0	-1	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(1,1)\} = c_{11}$,

$g(1,2) = g(D_2)$, $c^*\{g(D_2)\} - c^*\{g(1,2)\} = c_{12}$,

$g(1,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(1,3)\} = c_{13}$.

By Property 6

$g(1,1) = g(2,1) + g(S_2)$,

$g(1,2) = g(2,2) + g(S_2)$,

$g(1,3) = g(2,3) + g(S_2)$.

Problem T-41: $x_{12}, x_{21} \epsilon x_N$; $s(1,2)$, $s(2,1) \epsilon s_B$; $s(S_2) \epsilon s_N$; $s(D_j) \epsilon s_N$, $\forall j$.

x_{12}	x_{21}	$s(D_1)$	$s(D_2)$	$s(D_3)$	$s(S_2)$	$s(1,1)$	$s(1,3)$	$s(2,2)$	$s(2,3)$
1	-1	0	-1	-1	-1	0	-1	0	0
0	1	-1	0	0	0	-1	0	0	0
1	0	0	-1	0	0	0	0	-1	0
-1	1	0	1	0	1	0	0	0	-1

By Property 1

$g(1,1) = g(D_1)$, $c^*\{g(D_1)\} - c^*\{g(1,1)\} = c_{11}$,

$g(1,3) = g(D_3)$, $c^*\{g(D_3)\} - c^*\{g(1,3)\} = c_{13}$.

By Property 2

$g(x_{12}) = -g(D_2)$.

By Property 3

$g(x_{21}) + g(D_1) = g(S_2)$.

APPENDIX H

AN EXAMPLE OF THE SOLUTION PROCEDURE

In this appendix we present an abbreviated example of the group theoretic procedure outlined in Chapter IV.

Consider a fixed-charge transportation problem with two sources and three destinations where the following information pertains:

$$D_1 = 103,\ D_2 = 122,\ D_3 = 35;\ S_1 = 150,\ S_2 = 200.$$

$$c = [2,2,3,3,5,2].$$

$$f = [25,16,17,20,16,18].\quad \mu = 122.$$

From Phase I:

$$Z_u = 39.80 \text{ where } z^*_{LP} = 634.20 \text{ and } \lambda(x^*,y^*) = 674.$$

In Phase II:

Minimize $0.13w_1 + 0.15w_2 + 0.16w_3 + 0.20w_4 + 0.96w_5 + 1.81w_6 + 1.91w_7$

subject to $\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}w_1+\begin{bmatrix}0\\-1\\0\\0\end{bmatrix}w_2+\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}w_3+\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}w_4+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}w_5+\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}w_6+\begin{bmatrix}1\\0\\1\\-1\end{bmatrix}w_7=\begin{bmatrix}0\\35\\9\\94\end{bmatrix}$

$w_j \geq 0$, integer, $\forall j$.

$u_1 = 121 = u_2 = u_3 = u_4,\ u_5 = 41,\ u_6 = 12 = u_7.$

$F = \{w_6, w_7\}$ where $w_6 \rightarrow x_{13},\ w_7 \rightarrow x_{22}$ and $y_{13}, y_{22} \in y_N$.

In Phase III:

1. $NC_1 = \{w_1,w_7\}$, $NC_2 = \{w_2,w_6\}$,

   $NC_3 = \{w_3,w_5,w_6,w_7\}$, $NC_4 = \{w_4,w_5,w_6,w_7\}$.

2. NC_2: $LC_2 = 13.05$, $LCC_2 = \{w_2\}$

   NC_3: $LC_3 = 18.08$, $LCC_3 = \{w_3\}$

   NC_4: $LC_4 = \underline{\ 5.06}$, $LCC_4 = \{w_4\}$

   $LC^* = 36.73 < Z_u' = 39.80$

3. $Z_1 = 39.80 - 36.73 = 3.07$.

   $u_1' = 23$, $u_7' = 0 \Rightarrow S = \{w_7{=}0\}$.

   $NC_1 = \{w_1,w_7\} - \{w_7\} = \{w_1\}$.

   Since Order $(NC_1) = 1$, set

   $w_1 = a_1, b_1' = (-1)(0) = 0$

   $S = \{w_1{=}0, w_2{=}0\}$, $Z = 0$.

2. NC_3: $LC_3 = 18.08$, $LCC_3 = \{w_3\}$

   $LC_4 = \underline{\ 5.60}$, $LCC_4 = \{w_4\}$

   $LC^* = 23.68 < Z_u' - z = 39.80$

3. $Z_2 = 39.80 - 23.68 = 16.12$

   $u_2' = 107$, $u_6' = 0 \Rightarrow S = S + \{w_6{=}0\}$

   $NC_2 = \{w_2,w_6\} - \{w_6\} = \{w_2\}$

   Since Order $(NC_2) = 1$, set

   $w_2 = a_{22}b_2' = (-1)35 = -35 \equiv 87 \pmod{122}$

   $S = \{w_1{=}0, w_2{=}87, w_6{=}0, w_7{=}0\}$, $z = 13.05$.

Effectively, the "remaining problem" appears as follows:

$$\text{Minimize} \quad 0.16w_3 + 0.20w_4 + 0.96w_5$$

$$\text{subject to} \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} w_4 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} w_5 = \begin{bmatrix} 9 \\ 94 \end{bmatrix}$$

$$w_3, w_4, w_5 \geq 0, \text{ integer}$$

where $NC_3 = \{w_3, w_5\}$, $NC_4 = \{w_4, w_5\}$,

$Z_u' = 39.80 - 13.05 = 26.75$, $u_3 = 121$, $u_4 = 121$, $u_5 = 27$.

2. NC_4: $LC_4 = \underline{5.60}$, $LCC_4 = \{w_4\}$,

$LC^* = 5.60 < Z_u'$

3. $Z_3 = 26.75 - 5.60 = 21.15$

$u_3' = 121$, $u_5' = 21$

4. $w_3 = 121$, $b' = \begin{bmatrix} 8 \\ 94 \end{bmatrix}$, $u_5' = 7 < w_5 = 114$

$w_3 = 120$, $b' = \begin{bmatrix} 7 \\ 94 \end{bmatrix}$, $u_5' = 7 < w_5 = 115$

$\vdots$

$w_3 = 113$, $b' = \begin{bmatrix} 0 \\ 94 \end{bmatrix}$, $u_5' = 9 > w_5 = 0$

$NC_4 = \{w_4\}$, $u_4' = 43 > w_4 = 28$

Since $b' = 0$, check feasibility in the original problem. The current solution produces a feasible solution in the original problem; i.e., $w_2 = 87$, $w_3 = 113$, $w_4 = 28$, $w_1 = w_5 = w_6 = w_7 = 0$ is the current solution to the group problem; the solution in the original problem is

$$x_{11} = 28,\ x_{12} = 122,\ x_{21} = 75,\ x_{23} = 35,$$
$$y_{11} = 1,\ y_{12} = 1,\ y_{21} = 1,\ y_{23} = 1,$$
$$x_{13} = 0,\ x_{22} = 0,\ y_{13} = 0,\ y_{22} = 0, \text{ total cost } = \$674.$$

Bounds are recalculated and the procedure continues; however, no other solutions are identified, and the incumbent is optimal.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Balas, Egon, "A Note on the Asymptotic Theory of Integer Programming and the 0-1 Case," Management Sciences Research Report No. 249, Carnegie-Mellon University (May 1971).

2. Balas, Egon, "Duality in Discrete Programming," Technical Report No. 67-5 (Revised Version), Carnegie-Mellon University (1967).

3. Balas, Egon, "Minimax and Duality for Linear and Nonlinear Mixed-Integer Programming," Management Sciences Research Report No. 176, Carnegie-Mellon University (1969).

4. Balinski, M. L., "Fixed Cost Transportation Problems," *NRLQ*, Vol. 8, No. 1 (1961), pp. 41-54.

5. Balinski, M. L. and P. Wolfe, "On Bender's Decomposition and a Plant Location Problem," Mathematical Working Paper, ARO-27 (December 1963).

6. Balinski, M. L., "Integer Programming: Methods, Uses, Computation," *Management Science*, Vol. 12 (1965), pp. 253-313.

7. Balinski, M. L., "On Recent Developments in Integer Programming," *Proceedings of the International Symposium on Mathematical Programming*, Princeton University (August 1967).

8. Balinski, M. L. and K. Spielberg, "Methods for Integer Programming: Algebraic, Combinatorial, and Enumerative," *Progress in Operations Research*, Volume 3, edited by J. Aronofsky, John Wiley and Sons (1968), pp. 195-292.

9. Baxter, W. W., "Group Theoretic Branch and Bound Algorithm for the Traveling Salesman Problem," unpublished Master's thesis, M.I.T. (1968).

10. Benders, J. F., "Partitioning Procedures for Solving Mixed-Variables Programming Problems," *Numerische Mathematik*, Vol. 4 (1962), pp. 238-252.

11. Bradley, Gordon H., "Equivalent Integer Programs and Canonical Problems," *Management Science*, Vol. 17, No. 5 (1971), pp. 354-366.

12. Bradley, Gordon H., "Equivalent Integer Programs," Presented to the IFORS Meeting, September 15, 1968.

13. Bradley, Gordon H. and P. M. Wahi, "An Algorithm for Integer Linear Programming: A Combined Algebraic and Enumeration Approach," Report No. 29, Department of Administrative Sciences, Yale University (December 1969).

14. Bradley, Gordon H., "Algorithms for Hermite and Smith Normal Matrices and Linear Diophantine Equations," Report No. 34, Department of Administrative Sciences, Yale University (June 1970).

15. Bradley, Gordon H., "Algorithm and Bound for the Greatest Common Divisor of n Integers," *Communications of the ACM*, Vol. 13, No. 7 (July 1970), pp. 433-436.

16. Bradley, G. H., "Transformation of Integer Programs to Knapsack Problems," Report No. 37, Department of Administrative Sciences, Yale University (November 1970).

17. Brooks, R. and A. M. Geoffrion, "Finding Everett's Lagrange Multipliers by Linear Programming," *Operations Research*, Vol. 14, pp. 1149-1153 (1966).

18. Campbell, H. G., *An Introduction to Matrices, Vectors, and Linear Programming*, Appleton-Century-Crofts, 1965.

19. Cooper, L. and C. Drebes, "An Approximate Solution Method for the Fixed Charge Problem," *NRLQ*, Vol. 14, No. 1 (1967), pp. 101-113.

20. Cooper, L. and A. M. Olson, "Random Perturbations and the MI-MII Heuristics for the Fixed-Charge Problem," Report No. COO-1493-7, Department of Applied Mathematics and Computer Science, Washington University (1968).

21. Cullen, C. G., *Matrices and Linear Transformations*, Addison-Wesley, 1966.

22. Denzler, D. R., "An Approximative Algorithm for the Fixed Charge Problem," *NRLQ*, Vol. 16, No. 3 (1969), pp. 411-416.

23. Devine, M. and F. Glover, "Generating the Nested Faces of the Gomory Polyhedron," Research Report R-69-1, School of Business, University of Texas, Austin (December 1969).

24. Dwyer, P. S., "Use of Completely Reduced Matrices in Solving Transportation Problems with Fixed Charges," *NRLQ*, Vol. 13 (1966), pp. 239-313.

25. Ellwein, Leon B., *Fixed Charge Location--Allocation Problems with Capacity and Configuration Constraints*, Technical Report No. 70-2, Department of Industrial Engineering, Stanford University (August 1970).

26. Geoffrion, A. M., "Integer Programming by Implicit Enumeration and Balas Method," *SIAM Review,* Vol. 9 (1967), pp. 178-190.

27. Gilmore, P. C. and R. E. Gomory, "The Theory and Computation of Knapsack Functions," *Operations Research,* Vol. 14 (November-December 1966), pp. 1045-1074.

28. Glover, Fred, "Faces of an Integer Polyhedron for Cyclic Groups--Part I," Department of Statistics and Operations Research, University of Texas, Austin (May 1968).

29. Glover, Fred, "Integer Programming over a Finite Additive Group," *SIAM Journal of Control,* Vol. 7 (1969), pp. 213-231.

30. Glover, Fred, and Litzler, Lee, "Extension of an Asymptotic Integer Programming Algorithm to the General Integer Programming Problem," Working Paper, School of Business, University of Texas, Austin (October 1969).

31. Gomory, R. E., "An Algorithm for Integer Solutions to Linear Programs," pages 269-302 in *Recent Advances in Mathematical Programming*, edited by Graves and Wolfe, McGraw-Hill (1963).

32. Gomory, R. E., "All-Integer Integer Programming Algorithm," pp. 193-206 in *Industrial Scheduling,* edited by Muth and Thompson, Prentice-Hall (1963).

33. Gomory, R. E., "On the Relation Between Integer and Non-Integer Solutions to Linear Programs," *Proceedings of the National Academy of Sciences,* Vol. 53, No. 2 (February 1965), pp. 260-265.

34. Gomory, R. E., "Faces of an Integer Polyhedron," *Proceedings of the National Academy of Sciences,* Vol. 57, No. 1 (January 1967), pp. 16-18.

35. Gomory, R. E., "Some Polyhedra Related to Combinatorial Problems," *Linear Algebra and Its Applications,* Vol. 2 (October 1969), pp. 451-558.

36. Gorry, G. A. and J. F. Shapiro, "An Adaptive Group Theoretic Algorithm for Integer Programming Problems," *Management Science,* Vol. 17, No. 5 (1971), pp. 285-306.

37. Gray, Paul, "Mixed Integer Programming Algorithms for Site Selection and Other Fixed-Charge Problems Having Capacity Constraints," Technical Report No. 101, Department of Operations Research and Statistics, Stanford University (1967).

38. Gray, Paul, "Exact Solution of the Fixed-Charge Transportation Problem" (forthcoming in *Operations Research*).

39. Hadley, G., *Linear Algebra,* Addison-Wesley, 1961.

40. Hadley, G., *Nonlinear and Dynamic Programming,* Addison-Wesley (1964).

41. Herstein, I. N., *Topics in Algebra,* Blaisdell (1964).

42. Hirsch, W. M. and G. B. Dantzig, "The Fixed Charge Problem," *NRLQ,* Vol. 15, No. 3 (1968), pp. 413-424.

43. Hirsch, W. M. and A. J. Hoffman, "Extreme Varieties, Concave Functions and the Fixed Charge Problem," *Communications on Pure and Applied Mathematics,* Vol. 19, No. 3 (August 1961), pp. 355-369.

44. Hu, T. C., "Decomposition Algorithm for Shortest Paths in a Network," *Operations Research,* Vol. 16 (January-February 1968), pp. 91-102.

45. Hu, T. C., "On the Asymptotic Integer Algorithm," *MRC Report 946,* University of Wisconsin, Madison (1968).

46. Hu, T. C., *Integer Programming and Network Flows,* Addison-Wesley (1969), 452 pages.

47. Jacobson, Nathan, *Lectures in Abstract Algebra, Volume II--Linear Algebra,* D. Van Nostrand, 1953.

48. Johnson, E. L., and R. E. Gomory, "Some Continuous Functions Related to Corner Polyhedra," Rough Draft of Forthcoming IBM Report, Received June 18, 1971.

49. Jones, A. P. and R. M. Soland, "A Branch-and-Bound Algorithm for Multi-Level Fixed-Charge Problems," *Management Science,* Vol. 16, No. 1 (1969), pp. 67-76.

50. Kortanek, K. O. and R. G. Jeroslow, "An Exposition on the Constructive Decomposition of the Group of Gomory Cuts and Gomory's Round-Off Algorithm," Technical Report No. 39, Department of Operations Research, Cornell University (January 1968).

51. Kuhn, H. W. and W. J. Baumol, "An Approximate Algorithm for the Fixed-Charge Transportation Problem," *NRLQ,* Vol. 9, No. 1 (1962), pp. 1-15.

52. Little, J. D. C., Murty, K. G., Sweeney, D. W., and Karel, C., "An Algorithm for the Traveling Salesman Problem, *Operations Research,* Vol. 11 (1963), pp. 972-989.

53. MacLane, S. and G. Birkhoff, *Algebra,* MacMillan, 1967.

54. Mostow, G. D., Sampson, J. H. and Meyer, J. P., *Fundamental Structures of Algebra* (Especially Chapter 10), McGraw-Hill, 1963.

55. Murty, K. G., "Solving the Fixed Charge Problem by Ranking Extreme Points," *ORSA,* Vol. 16, No. 2 (1968), pp. 268-279.

56. Robers, P. and L. Cooper, "A Study of the Fixed Charge Transportation Problem," Report No. COO-1493-9, Department of Applied Mathematics and Computer Science, Washington University (1969).

57. Saaty, T. L., *Optimization in Integers and Related Extremal Problems,* McGraw-Hill, 1970.

58. Shapiro, J. F. and H. M. Wagner, "A Finite Renewal Algorithm for the Knapsack and Turnpike Models," *Operations Research,* Vol. 15 (March-April, 1967), pp. 319-341.

59. Shapiro, J. F., "Dyamic Programming Algorithms for the Integer Programming Problem--I: The Integer Programming Problem Viewed as a Knapsack Type Problem," *Operations Research,* Vol. 16 (1968), pp. 103-121.

60. Shapiro, J. F., "Group Theoretic Algorithms for the Integer Programming Problem--II: Extension to a General Algorithm," *Operations Research,* Vol. 16 (1968), pp. 928-947.

61. Shapiro, J. F., "Generalized Lagrange Multipliers in Integer Programming," Working Paper, Sloan School of Management, M.I.T. (March 1969; Revised January 1970).

62. Shapiro, J. F., "Turnpike Theorems for Integer ProgrammingProblems," *Operations Research,* Vol. 18 (May-June 1970), pp. 432-440.

63. Smith, H. J. S., "On Systems of Linear Indeterminate Equations and Congruences," *Philos. Trans. Roy. Soc. London Ser. A,* CLI (1861), 293-326.

64. Spielberg, Kurt, "On the Fixed-Charge Transportation Problem," *Proceedings of the 19th National Conference,* ACM (1964), pp. A1. 1-1 to A1. 1-13.

65. Steinberg, David I., "The Fixed Charge Problem," Report No. COO-1493-15, Department of Applied Mathematics and Computer Science, Washington University (1969).

66. Steinberg, David I., "The Fixed Charge Problem," *NRLQ,* Vol. 17, No. 2 (1970), pp. 217-236.

67. Thiriez, H., "Airline Crew Scheduling: A Group Theoretic Approach," Report R-69, Flight Transportation Laboratory, M.I.T. (October 1969).

68. Unger, V. E., "Capital Budgeting and Mixed Zero-One Integer Programming," *AIIE Transactions*, Vol. 2, No. 1 (1970), pp. 28-36.

69. Walker, Warren, "Adjacent Extreme Point Algorithms for the Fixed-Charge Problem," Technical Report No. 40, Center for Environmental Quality Management, Cornell University (1968).

70. White, W. W., "On a Group Theoretic Approach to Linear Integer Programming," Operations Research Center Report 66-27, University of California, Berkeley (1966).

71. Wolsey, L. A., "Mixed Integer Programming: Discretization and the Group Theoretic Approach," Technical Report No. 42, Operations Research Center, Massachusetts Institute of Technology (1969).

72. Wolsey, L. A., "Group Representational Theory in Integer Programming," Technical Report No. 41, Operations Research Center, M.I.T. (February 1969).

73. Wolsey , L. A., "Classification and Simplification of General Mixed Integer Programming Problems," presented to the 17th International Conference of TIMS, London, England (July 1970).

74. Wolsey, L. A., "Extensions of the Group Theoretic Approach in Integer Programming," Working Paper, Manchester Business School (October 1970).

VITA

Curtis Johnston Tompkins, son of Mr. and Mrs. Joseph B. Tompkins, was born in Roanoke, Virginia, on July 14, 1942. He was graduated from William Byrd High School in Vinton, Virginia, in 1960.

He received his B.S. degree in industrial engineering from Virginia Polytechnic Institute in June, 1965. During his undergraduate career, the author was employed on the Cooperative Program by Appalachian Power Company and was a member of the Corps of Cadets where he served as Commander of the Regimental Band. He was listed in "Who's Who in American Universities and Colleges, 1963-64" and was a Distinguished Military Graduate.

The author joined E. I. DuPont de Nemours in June, 1965, and remained with that company until July, 1967, as an industrial engineer at the Spruance Fibers Plant in Richmond, Virginia. He entered the graduate school at Virginia Polytechnic Institute in January, 1966, and returned to the DuPont Company on alternating quarters, completing requirements for the degree of Master of Science in Industrial Engineering in September, 1967. His thesis was entitled "Integer Cutting Stock Procedures."

Mr. Tompkins entered the doctoral program in the Department of Industrial Engineering at Virginia Tech in the Fall of 1967. After completing two quarters of doctoral course work at Virginia Tech, he became a graduate teaching assistant in the Program in Hospital and Medical

Systems in the School of Industrial and Systems Engineering at the Georgia Institute of Technology, concurrently entering the Ph.D. program of the same school. In June, 1969, he became an Instructor with the Health Systems Research Center and in the School of Industrial and Systems Engineering at Georgia Tech.

He became a Director of Resource Management Corporation in November, 1969, and co-founded Health Resources Corporation in January, 1970. He currently is a Director of these two Atlanta-based firms.

From July, 1970, until July, 1971, Mr. Tompkins was an Instructor in the School of Industrial and Systems Engineering. He completed the degree requirements in the doctoral program at Georgia Tech in the Summer of 1971. He will join the faculty of the Graduate School of Business Administration at the University of Virginia in September, 1971.

The author is a member of Phi Kappa Phi, Alpha Pi Mu, the Operations Research Society of America, the Institute of Management Sciences, and the Hospital Management Systems Society.

Mr. Tompkins married the former Mary Katherine Hasle in September, 1964. Their first son, Robert Kiel, was born in Atlanta in May, 1968. Their second son, Joseph Harold, was born in Atlanta in January, 1970.