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RESPONSE OF SECOND ORDER NONLINEAR  
SERVOMECHANISMS WITH FIXED CRITICAL POINTS

A THESIS

Presented to  
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by  
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RESPONSE OF SECOND ORDER NONLINEAR  
SERVOMECHANISMS WITH FIXED CRITICAL POINTS

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#### DEDICATION

This thesis is dedicated to my wife Peggy whose encouragement and patience have made this work possible.

#### ACKNOWLEDGMENTS

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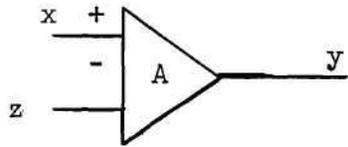
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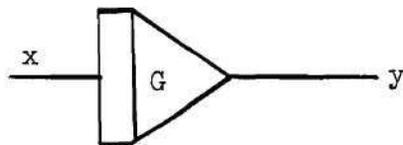
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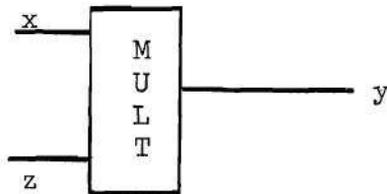
LIST OF SYMBOLS



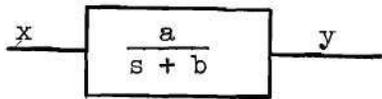
$$y = A(x - z)$$



$$y = G \int_0^t x \, dt$$



$$y = xz$$



$$ax = \frac{dy}{dt} + by$$

s

Laplace Transform variable

$\wp(u)$

Weierstrasse Pe-function

$J_\nu$

Bessel function of the first kind of order  $\nu$

$Y_\nu$

Bessel function of the second kind

$I_\nu$

Modified Bessel function of the first kind

$K_\nu$

Modified Bessel function of the second kind

## SUMMARY

It has been stated in other works that the transient response of a linear servomechanism could be improved by altering the characterizing equation in a nonlinear fashion. Several control systems whose characterizing equations are nonlinear have been proposed and examined. However, each such control system has been an isolated example, and no general statements have been made concerning their responses. The problem of improving the transient response of a linear system by intentionally altering its characterizing equation in a nonlinear fashion is here attacked by examining a subclass of the class of second order differential equations for their suitability as characterizing equations. The members of this subclass have the common feature that, when examined as a function of complex variable, their solutions have critical points (essential singularities) that are fixed (with respect to the constants of integration).

Examining this subclass allows use to be made of the extensive work done by mathematicians between 1810 and 1926 in the field of complex differential equations. The mathematical problem concerns the existence of solutions with fixed critical points of the class of differential equations

$$w'' = F(z, w, w'), \quad (1)$$

where  $F$  is rational in  $w'$ , algebraic in  $w$  and analytic in  $z$ . While it

has been shown that there are fifty canonical forms of equation (1) admitting to such solutions, the elements of the subclass examined here are those differential equations which can be expressed as

$$w'' = (Aw + B)w' + Cw^3 + Dw^2 + Ew + F . \quad (2)$$

A set of necessary conditions for equation (1) to have such solutions is that it be reducible to equation (2) by a suitable transformation where A and C have the pairs of constant values

- (a)  $A = 0; C = 0$
- (b)  $A = 0; C = 2$
- (c)  $A = -2; C = 0$
- (d)  $A = -1; C = 1$
- (e)  $A = -3; C = 1$

A control system with the characterizing equation of the form of equation (2) is referred to as a Type II system and is subdivided into Cases (a) to (e), which designate the numerical values of A and C. A servomechanism was then synthesized to have a characterizing equation of the Type II system. This procedure gave rise to five characterizing equations which featured nonlinear damping, nonlinear restoring force or a combination of both. In each case the solution of the characterizing equation admits to fixed critical points in the complex plane.

The question proposed in this thesis, "Can the transient response of a linear servomechanism be improved by altering its characterizing equation in a nonlinear fashion so that the solution has fixed critical points?" is answered with a qualified "yes." Three of the servomechanisms

synthesized, those with characterizing equations of the Type II system, Cases (a), (c) and (e), have transient responses which are faster than that of the linear servomechanism. The characterizing equation of the Type II Case (a) system does not have a solution with fixed critical points for every value of  $F$ . The characterizing equation of the Type II system, Cases (c) and (e), have such solutions for all values of  $F$ . One disadvantage of these systems is their asymmetrical control action, i.e., the response for a positive  $F$  is not the same as for a negative  $F$ . None of these three characterizing equations have a stable response for every value of  $F$ . An analytical advantage of these nonlinear characterizing equations over those cited in Chapter I is that they have solutions which are available in a closed form.

From the viewpoint of simplicity and speed of response, the characterizing equation of the Type II Case (c) system is the most promising developed here. The characterizing equation of the Type II Case (e) system is more complicated than Case (c), and, since a theoretical study of its speed of response showed no significant advantage over Case (c), it was not realized.

While the characterizing equation of the Type II Case (a) system did not admit to the proper solution for every value of  $F$ , its response was stable for positive values of  $F$  and hence was realized. Its response is sluggish for small inputs but relatively fast for large inputs. A servomechanism with this type of control action is referred to as a "variable-gain servo" and is useful when the control system has small-amplitude oscillations such as those caused by backlash in gears. This

type of control action has been proposed in other papers, but the characterizing equations of the Type II system, Cases (c) and (e), are contributions of this thesis.

Although the five characterizing equations proposed in this work were synthesized, no general method of synthesization has been developed. This is still a function of the ingenuity of the synthesizer.

## CHAPTER I

### INTRODUCTION

Physical systems in general can be described or characterized by either ordinary or partial differential equations involving an input quantity such as force, voltage or pressure, and an output quantity such as displacement, torque or flow. The output quantity does not react instantaneously to a change in the input quantity, but rather has a time delay. The fundamental task of a control engineer is to alter the basic characterizing equation of a physical system so as to obtain a desired response to a given change in the input quantity.

The theory of linear servomechanisms is concerned with improving the response of systems which can be characterized by means of ordinary linear differential equations with constant coefficients. As an example, a first order linear system with an input variable  $\theta_i$  and an output variable  $\theta_o$  has the characterizing equation\*

$$K\theta_i(t) = B\theta_o'(t) + \theta_o(t) .$$

With the system in its quiescent state, the characterizing equation becomes

$$K\bar{\theta}_i(s) = (Bs + 1)\bar{\theta}_o(s) ,$$

---

\* The primes denote differentiation with respect to time.

where  $s$  is the Laplace transform variable. It is customary to represent this equations by means of the block diagram as shown in Fig. 1. This diagram is often referred to as the open loop or uncompensated system. It is this basic system that the control engineer wishes to alter. The response of the uncompensated system to an input change of the form

$$\theta_i(t) = |\theta_i| ; t > 0$$

$$\theta_i(t) = 0 ; t < 0 ,$$

which is referred to as a step function, is

$$\theta_o(t) = K|\theta_i|(1 - e^{-t/B}) .$$

The response of this system can be improved by the addition of a linear controller and feedback, as shown in block diagram form in Fig. 2.

The characterizing equation of this compensated system can be normalized and written as

$$\frac{\bar{\theta}_o}{\bar{\theta}_i}(s) = \frac{KG}{B[s^2 + 2\zeta\omega_n s + \omega_n^2]}$$

where

$\zeta$  = damping ratio,

$\omega_n$  = undamped angular frequency, and

$G$  = gain of the controller.

For  $\theta_i(t)$  a step function there are three types of responses, depending upon the value of  $\zeta$ . These responses are as follows:

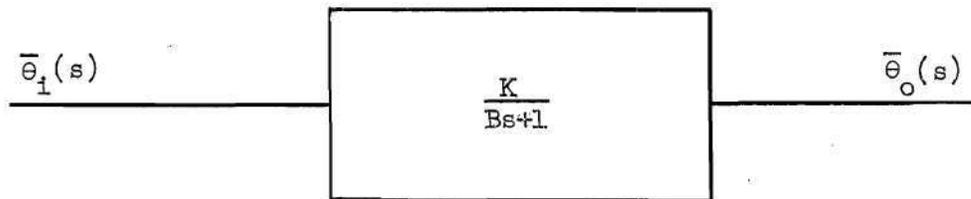


Figure 1. Block Diagram of Uncompensated System

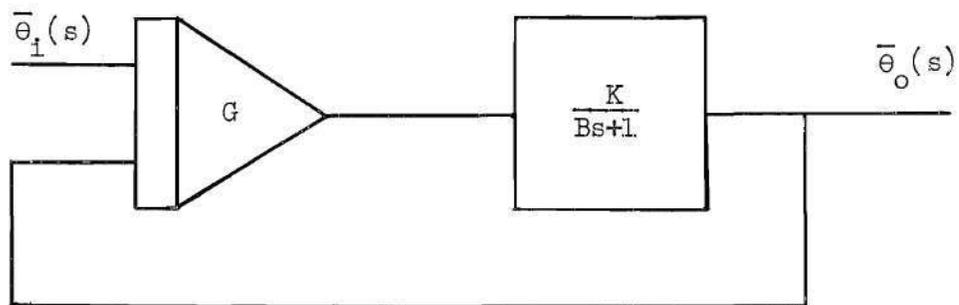


Figure 2. Block Diagram of Compensated System

(a)  $\zeta < 1$ , underdamped case

$$\theta_o(t) = \int_0^t \frac{\frac{KG}{B} |\theta_i| \epsilon^{-\zeta \omega_n t}}{\omega_n \sqrt{1-\zeta^2}} \sin \left[ \omega_n \sqrt{1-\zeta^2} t \right] dt.$$

(b)  $\zeta = 1$ , critically damped case

$$\theta_o(t) = \int_0^t \frac{KG}{B} |\theta_i| \epsilon^{-\omega_n t} dt.$$

(c)  $\zeta > 1$ , overdamped case

$$\theta_o(t) = \int_0^t \frac{\frac{KG}{B} |\theta_i| \epsilon^{-\zeta \omega_n t}}{\omega_n \sqrt{\zeta^2-1}} \sinh \left[ \omega_n \sqrt{\zeta^2-1} t \right] dt.$$

In general, the engineer adjusts the controller gain,  $G$ , to obtain the underdamped response. The smaller the damping ratio, the faster the system response, but the higher the overshoot. Hence the engineer must compromise between speed of response and overshoot. It has been found that, in many cases, values of  $\zeta$  between .4 and .6 give the best practical response. Thus the speed of response of a linear system is limited by the amount of overshoot permitted.

As the need for faster responding systems arose, engineers turned their attention to the possibility of improving system response by altering the basic characterizing equation in a nonlinear fashion. In 1950, D. McDonald<sup>1</sup> published a paper in which nonlinear control systems with the following characterizing equations were discussed:

$$y'' + \frac{2\zeta\omega_n}{1+a|y|} y' + \omega_n^2 y = 0 ,$$

$$y'' + \frac{2\zeta\omega_n}{1+|y+b\frac{dy}{dt}|} y' + \omega_n^2 y = 0 ,$$

and

$$y'' + 2\zeta\omega_n y' + \frac{\omega_n^2}{1+a\frac{dy}{dt}} y = 0 .$$

Special attention was given to the use of the phase-plane plots as an aid in designing these systems. In this paper the author introduced the concept of a "dual mode" control system in which the response was linear for small errors and nonlinear for large errors.

In the same year J. B. Lewis<sup>2</sup> investigated the possibility of improving the transient response of a second order linear system by introducing feedback which caused the damping ratio to vary during the period of the transient. This variation was achieved by the multiplication of two variables in the feedback path. The characterizing equation was of the form

$$y'' + ay' + by y' + cy = d .$$

Although the two authors utilized the second order linear system as a basic system and altered either the undamped natural frequency or the damping ratio in a nonlinear fashion, they developed no general method of synthesizing a nonlinear control system as there is for a linear control system.

The ease with which linear control systems are designed stems directly from the properties of linear differential equations with constant coefficients. One such property is that they satisfy the superposition theorem, as a result of which, if  $C_1(t)$  is the response of the system to an excitation  $r_1(t)$  and  $C_2(t)$  is the response to  $r_2(t)$ , then the response to  $r_1(t) + r_2(t)$  is  $C_1(t) + C_2(t)$ . Hence certain test signals, such as a step function or sinusoidal functions of varying frequency, can be used to measure the response of the control system. The question of stability is clearly defined in linear systems. The driving functions and initial conditions have no effect upon the stability. Graphical techniques, such as the root locus or Nyquist plots, are used to design systems with prescribed figures of merit, such as  $M_p$  (maximum overshoot),  $\omega_R$  (undamped angular frequencies) and  $\zeta$  (damping ratio).

The statements in the preceding paragraph do not hold for a nonlinear control system. Although the solutions to all linear differential equations with constant coefficients are available, only a small number of nonlinear equations have been solved. The solution of a linear equation is a linear function of the constants of integration, but this is not true for nonlinear equations. For a second order nonlinear equation three conditions may arise:

- (a) The solution is an algebraic or, in particular, a rational function of the constants of integration.
- (b) The general solution is a semi-transcendental function of the constants of integration.
- (c) Neither (a) nor (b) is true. The general solution is then said to be an essentially transcendental function of the constants of integration.

Because of these constants of integration, the response of a second order nonlinear system will not be faster than the linear system for all input signal amplitudes; however, there may be a range of amplitudes for which the nonlinear system reacts faster than the linear system.

The stability of the nonlinear system depends upon both the form of the input signal and its amplitude. It is usual to consider the input quantity as one of the following types of functions:

- (1) Step functions
- (2) Ramp functions
- (3) Sinusoidal functions
- (4) Random functions.

In a linear system driven by a sinusoidal signal, there are no other frequencies present except that of the input quantity. However, the output quantity of a nonlinear system driven by a sine wave contains frequencies other than those of the input sine wave. In this thesis, the input quantity will be taken to be a step function.

There are two techniques available for the design of nonlinear systems. These are the describing-function technique, first developed in this country by Kochenburger,<sup>3</sup> and the phase-plane portrait, suggested by L. A. MacColl.<sup>4</sup> These procedures are not general and are only applicable to certain types of nonlinearities and forcing functions.

Despite these difficulties, the demand for faster acting systems has forced attention to methods of improving system responses by the intentional addition of nonlinearities to a basic open loop linear system.

The basic purpose of this thesis is to investigate the possibility of improving the response of a second order linear system by the addition of certain nonlinearities. The nonlinearities are chosen so that the characterizing equation of the system, when expressed as a function of a complex variable, satisfies a set of necessary conditions for the solution of the differential equation to have fixed critical points (essential singularities fixed with respect to the initial conditions).

## CHAPTER II

### PRELIMINARY CONSIDERATIONS

General.--The general second-order linear control-system excited by a step-function input has the characterizing equation

$$y'' + By' + Ey = F ,$$

where  $y(t)$  is the output quantity and  $F$  is the amplitude of the step function input. The term  $Ey$  is often referred to as the restoring force of the control system and  $By'$  as the damping force.

The general second-order nonlinear control-system excited as above has the characterizing equation

$$y'' + Q(y',y)y' + P(y',y)y = F . \quad (1)$$

The term  $Q(y',y)y'$  is the nonlinear damping force and the term  $P(y',y)y$  is the nonlinear restoring force.

Although an analytic solution of equation (1) is not available for all forms of  $Q$  and  $P$ , it is logical to ask if anything can be said about the nature of the solution. The attempt to answer this question forms the basis of this thesis. Before continuing this discussion, the mathematical background will be developed.

Mathematical background.--The nonlinear complex differential equation<sup>\*</sup>

---

\*The primes denote differentiation with respect to the complex variable  $z$ .

$$[(W')^2 - W W'']^2 + 4W(W')^3 = 0 \quad (2)$$

has the solution

$$W = C_1 \epsilon^{\frac{1}{z-C_2}},$$

where  $C_1$  and  $C_2$  are constants of integration. This solution has an essential singularity which depends upon the constant of integration  $C_2$ . But the linear differential equation

$$W'' - W' + \frac{W}{4} = 0, \quad (3)$$

which has the solution

$$W = (C_1 + C_2 z) \epsilon^{z/2},$$

has an essential singularity at infinity which is not a function of the constants of integration.

Equations (2) and (3) are special cases of the general second order complex differential equation

$$W'' = F(z, W, W'), \quad (4)$$

where  $F$  is rational in  $W'$ , algebraic in  $W$  and analytic in  $z$ . Between 1887 and 1926 mathematicians<sup>5,6,7,8</sup> extensively studied the existence of solutions to equation (4) where critical points (essential singularities) were fixed with respect to the initial conditions. In 1894 G. Mittag-Leffler<sup>9</sup> studied the differential equation

$$W'' = (AW + B) W' + CW^3 + DW^2 + EW + F. \quad (5)$$

In 1926 E. L. Ince<sup>10</sup> showed that, for equation (4) to have a solution with fixed critical points, it is necessary that equation (4) be of the form

$$W'' = [A(z)W + B(z)] W' + C(z)W^3 + D(z)W^2 + E(z)W + F(z), \quad (6)$$

and that this equation be reducible by suitable transformations to equation (5) where  $A(z)$  and  $C(z)$  have one of the following pairs of constant values:

- (a)  $A = 0; C = 0$
- (b)  $A = 0; C = 2$
- (c)  $A = -2; C = 0$
- (d)  $A = -1; C = 1$
- (e)  $A = -3; C = -1$

The sufficient conditions were then determined by integration or otherwise.

Summary.--The question asked at the beginning of this chapter is now partially answered. Given an ordinary differential equation

$$y'' = f(t, y, y'), \quad (7)$$

one can replace the real variable  $t$  by the complex variable  $z$ , and the real function  $y(t)$  by some new function  $W$  of the complex variable  $z$ .

Thus, equation (7) is transformed into equation (4). The solution of the complex differential equation can be classified as to the absence or presence of fixed critical points. This classification was chosen to designate a subclass of the class of second-order differential

equations to be examined for their suitability as characterizing equations of a control system. This subclass has the advantage that the solutions of its elements are available.

An examination of the nonlinear systems discussed in Chapter I shows that their characterizing equations expressed in the complex plane have solutions with movable critical points. The basic purpose of this thesis is to investigate the possibility of improving the transient response of a control system with a step function input signal by constraining the characterizing equation expressed in the complex plane to have fixed critical points.

Procedure.--The basic unaltered control system to be utilized in this thesis is discussed in the Appendix and is characterized by a first order linear differential equation. Feedback and a nonlinear controller will be added to this system so that the characterizing equation of the altered system, when expressed as a complex differential equation, will have fixed critical points.

The general control system whose characterizing equation is given by

$$y'' + [Ay + B]y' + Cy^3 + Dy^2 + Ey + F$$

will be referred to as a Type II system and will be subdivided into Cases (a) to (e) which will indicate the pair of constant values of A and C tabulated on page 11. The procedure is as follows.

- (1) The response of the system will be examined in the complex plane and the location of the poles and essential singularities will be determined.

- (2) The response of the system will then be examined as a function of time. Special attention will be given to the determination of the constants of integration.
- (3) The stability of the system will be examined. This will be correlated with the location of the poles of the solution in the complex plane.
- (4) A physical system utilizing the basic control system will then be synthesized to have the same characterizing equation as in step 2.
- (5) The response of this system will be compared with the response of the linear system.

### CHAPTER III

#### RESPONSE OF FIRST ORDER, FIRST DEGREE SYSTEMS WITH FIXED CRITICAL POINTS

General.--While this thesis is not primarily concerned with first order characterizing equations, it seems proper to include a brief discussion of them in this chapter. It has been shown<sup>11</sup> that for a differential equation of the form

$$W' = f(W, z) ,$$

where  $f(W, z)$  is rational in  $W$ , to have a solution with fixed critical points, it must be the generalized Riccati equation

$$W' = f_1(z) + f_2(z)W + f_3(z)W^2 .$$

Response in the complex domain.--When  $f_1$ ,  $f_2$  and  $f_3$  are constants, the Riccati equation becomes

$$W' = DW^2 + EW + F .$$

By means of the substitutions

$$W = -\frac{1}{D} \frac{V'}{V} ,$$

its solution is obtained and may be written as

$$W = -\frac{1}{D} \frac{C_1 \alpha \epsilon \left( \sqrt{\frac{E^2}{4} - DF} \right) z + \beta \epsilon \left( \sqrt{\frac{E^2}{4} - DF} \right) z}{C_1 \epsilon \left( \sqrt{\frac{E^2}{4} - DF} \right) z + \epsilon \left( \sqrt{\frac{E^2}{4} - DF} \right) z}, \quad (8)$$

where

$$\alpha = \frac{E}{2} + \sqrt{\frac{E^2}{4} - DF},$$

and

$$\beta = \frac{E}{2} - \sqrt{\frac{E^2}{4} - DF}.$$

The poles of equation (8) are movable and occur at values of  $z$

where

$$z = \frac{1}{2 \sqrt{\frac{E^2}{4} - DF}} \ln(-1/C_1),$$

and  $C_1$  is the constant of integration.

Response in the real domain.--As a function of a real variable  $y(t)$ ,

Riccati's equation can be written

$$y' = Dy^2 + Ey + F,$$

and has the solution

$$y = \frac{1}{D} \frac{C_1 \alpha \epsilon \left( \sqrt{\frac{E^2}{4} + FD} \right) t + \beta \epsilon - \left( \sqrt{\frac{E^2}{4} + FD} \right) t}{C_1 \epsilon \left( \sqrt{\frac{E^2}{4} + FD} \right) t + \epsilon - \left( \sqrt{\frac{E^2}{4} + FD} \right) t} . \quad (9)$$

It should be noted that E and D have been chosen negative to insure that the solution cannot become oscillatory for positive values of F.  $C_1$  can be obtained from the relationship

$$C_1 = - \frac{\beta/D - y(0)}{-y(0) + \alpha/D} .$$

If the denominator of equation (9) becomes zero,  $y(t)$  is not bounded.

The unbounded response will occur for values of  $C_1$  where

$$-1 < C_1 < 0 . \quad (10)$$

If inequality (10) occurs, equation (8) has a pole on the positive real axis.

Physical system with a Riccati characterizing equation.---The physical system of Fig. 3 has the equation of motion

$$u = e_s - K_f \omega$$

and

$$u' + K_f a G^2 u^2 + bu = be_s , \quad (11)$$

where the parameters are defined in Appendix A.

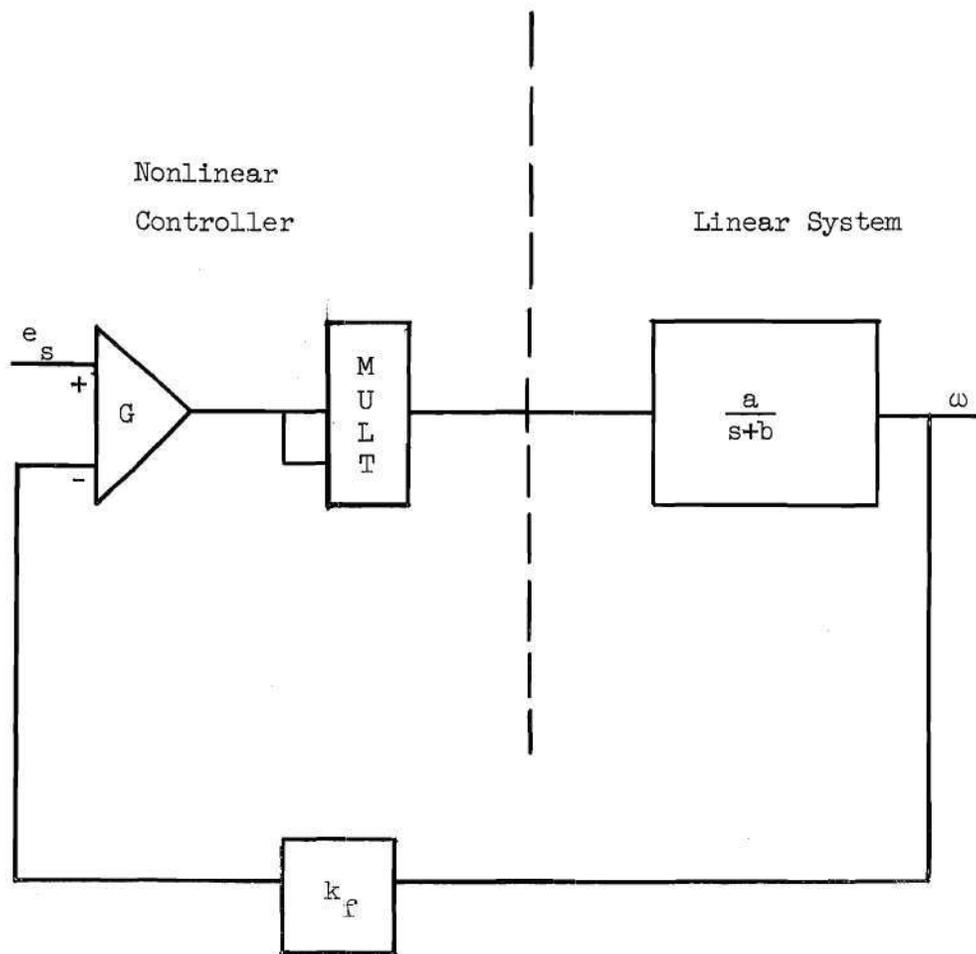


Figure 3. Physical System with Riccati Type  
Characterizing Equation

This is the Riccati characterizing equation with

$$F = be_s$$

$$D = K_f a G^2$$

and

$$E = b.$$

The solution of equation (11) is of the form of equation (9).

Comparison of linear and nonlinear systems.---The linear system has the characterizing equation

$$u' + [K_f a G + b]u = be_s,$$

which has the solution

$$u = \frac{be_s}{b + K_f a G} \left[ 1 - e^{-(b + K_f a G)t} \right].$$

The steady state error and the time constant of the two systems form the basis of comparison.

(a) Steady state error.---This is defined as

$$\lim_{t \rightarrow \infty} u = u_{s.s.},$$

and is

$$u_{s.s.} = \frac{be_s}{K_f a G + b} \cdot \cdot \cdot \cdot \cdot \text{linear system,}$$

or

$$u_{s.s.} = \frac{be_s}{2K_f a G^2} + \sqrt{\frac{b^2}{4K_f^2 a^2 G^4} + \frac{be_s}{K_f a G^2}} \dots \dots \dots \text{nonlinear system.}$$

(b) Time constant.--This is defined as the reciprocal of the coefficient of  $t$  in the exponent of the exponential and is

$$\tau = \frac{1}{b + K_f a G} \dots \dots \dots \text{linear system,}$$

or

$$\tau = \frac{1}{\sqrt{\frac{b^2}{4} + K_f a b G^2 e_s}} \dots \dots \dots \text{nonlinear system.}$$

For a high gain system, i.e.,  $G \gg 1$  and  $e_s > \frac{aK_f}{b}$ , the nonlinear system has a smaller steady state error than the linear system. For values of input signal where

$$e_s > \frac{(b + K_f a G)^2}{K_f a b G^2} - \frac{b^2}{4K_f a b G^2},$$

the nonlinear system has a smaller time constant than the linear system.

However, while the linear system always has a finite value of output for all values of input signal amplitude  $F$ , the nonlinear system will have an unbounded output if  $F$  approaches the value

$$u = -E/D \quad .$$

## CHAPTER IV

### TYPE II CASE (a) SYSTEMS

$$A = 0, C = 0$$

Response in the complex domain.--The Type II Case (a) differential equation in the complex domain is

$$w'' = Bw' + Dw^2 + Ew + F. \quad (12)$$

Following the example of Ince<sup>12</sup>, the following substitution is made in equation (12):

$$w = W(Z)\lambda(z) + \mu(z),$$

and

$$Z = \phi(z)$$

where

$$\frac{2\lambda'}{\lambda} = \frac{\phi''}{\phi'} = B$$

$$D\lambda = 6\phi'^2$$

and

$$2D\mu = \frac{\lambda''}{\lambda} - \frac{B\lambda'}{\lambda} - E.$$

With these substitutions, equation (12) becomes

$$W'' = 6W^2 + S(Z), \quad (13)$$

where

$$S(Z) = \frac{-\frac{1}{2500D} \left[ 625E^2 - 36B^2 \right] + F}{\lambda^2 \phi^2}. \quad (14)$$

$$\phi = \frac{5}{B} e^{(B/5)z} \quad (15)$$

$$\lambda = \frac{6}{D} e^{2(B/5)z}$$

and

$$\mu = -\frac{1}{50D} \left[ 6B^2 + 25E \right]. \quad (17)$$

But Ince<sup>13</sup> has shown that for the solution of equation (13) to have fixed critical points,  $S(Z)$  must be zero. This can happen in the following two cases.

Case i.  $B = E = F = 0$ . Equation (12) has the solution

$$W = \frac{6}{D} \wp(z - k, 0, h), \quad (18)$$

which is the Weierstrasse<sup>14</sup> Pe-function and is doubly periodic. The parameters  $k$  and  $h$  are the constants of integration;  $0$  and  $h$  are the invariants of the function.

Equation (18) has the series expansion<sup>15</sup>

$$W = \frac{6}{D} \left[ \frac{1}{(z - k)^2} + C_2(z - k)^2 + \dots + C_\lambda(z - k)^{2\lambda-2} + \dots \right], \quad (19)$$

where

$$c_2 = \frac{1}{20} g^2 = 0$$

$$c_3 = \frac{1}{28} h$$

and

$$c_\lambda = \frac{3}{(2\lambda + 1)(\lambda - 3)} \sum_{v=2}^{v=\lambda-2} c_v c_{\lambda-v} ; (\lambda > 3) . \quad (20)$$

Hence it is seen that equation (19) has a double order pole at

$$z = k$$

and thus is movable with respect to the initial conditions. The essential singularity at infinity, however, is fixed.

Case ii. For this case none of the parameters are zero and

$$F = \frac{1}{2500 D} \left[ 625E^2 - 36B^2 \right] .$$

Substituting equations (14) to (17) into equation (18) leads to

$$W = \frac{6}{D} e^{\frac{2B}{5} z} \mathcal{G} \left[ \left( \frac{5}{D} e^{\frac{B}{5} z} - k \right) ; o, h \right] + \frac{1}{50 D} \left[ 6B^2 + 25E \right] . \quad (21)$$

Thus, from equation (19), equation (21) has the series expansion

$$W = \frac{6}{D} u^2 \left[ \frac{1}{\left( \frac{5}{B} u - k \right)^2} + \dots c_\lambda \left( \frac{5}{B} u - k \right)^{2\lambda-2} + \dots \right] + \frac{1}{50 D} \left[ 6B^2 + 25E \right] \quad (22)$$

where  $C_\lambda$  is as defined in equation (20) and

$$u = e^{\frac{B}{5}z} .$$

The solution of equation (21) has a fixed essential singularity at infinity but has an infinite number of second order movable poles at

$$z = \frac{5}{B} \ln\left(\frac{B}{5}k\right) .$$

Response in the real domain.---Since the basic purpose of this thesis is to utilize the differential equations studies here as characterizing equations, their solutions must satisfy the following set of necessary conditions:

- (a) The solution  $f(t)$  must be bounded, i.e.,

$$f(t) \leq M; t \geq 0$$

- (b) The solution  $f(t)$  must have a finite limit, i.e.,

$$\lim_{t \rightarrow \infty} f(t) = B .$$

Systems described by characterizing equations whose solutions satisfy the conditions above will be called stable systems. If the complex function  $w(z)$  in equation (12) is replaced by a real function  $y(t)$ , case i and case ii can be examined in the real domain as follows.

Case 1. Equation (18) becomes

$$y = \frac{6}{D} \mathcal{L} \left[ (t - k) ; 0, h \right] . \quad (23)$$

The constant of integration  $h$  can be found from the first derivative of equation (23) as

$$(y')^2 = 4y^3 - h .$$

Since equation (23) does not satisfy condition (a) or (b), it is not suitable for control purposes.

Case ii. By the same type of substitutions as utilized in solving equation (12), the differential equation

$$y'' = By' + Dy^2 + Ey + F \quad (24)$$

can be expressed as

$$y_1'' = 6y_1^2 ;$$

hence,

$$y_1'^2 = 4y_1^3 - h .$$

The constant of integration  $h$  can be found from the initial condition by the relationship

$$h = 4 \left[ \frac{y(o) - u}{-6/D} \right]^3 - \left[ \frac{y'(o) + \frac{2B}{5} (y(o) - u)}{6/D} \right]^2 .$$

The parameters  $B$ ,  $D$  and  $E$  have been chosen negative and

$$F = - \frac{1}{2500 D} \left[ 625E^2 - 36B^2 \right] .$$

Hence the solution to equation (24) is

$$y = -\frac{6}{D} e^{-\frac{2B}{5}t} \wp \left[ \left( \frac{5}{B} e^{-\frac{B}{5}t} - k \right); 0, 1 \right] - \frac{1}{50D} \left[ 25E - 6B^2 \right]. \quad (25)$$

If a system characterized by equation (25) is to be stable, the argument of the Weierstrasse Pe-function must never be zero. Thus if

$$0 \leq k \leq 5/B ,$$

equation (25) is unstable. (It should be noted that this condition corresponds to a pole on the real axis for equation (22).)

It should be emphasized that equation (25) is the solution to equation (24) for only one value of F. Hence for any other value of F, equation (24) has a solution with movable critical points.

Stability of Type II Case (a) systems.---Systems of the type designated by case i are not stable for all initial conditions. The unbounded response occurs at  $t = k$ . In the complex plane the solution has a second order pole at  $z = k$ .

Systems of the form denoted by case ii are conditionally stable, depending upon the initial conditions. The unstable response occurs for

$$0 \leq k \leq 5/B .$$

In the complex plane the solution has an infinite number of second order poles at

$$z = 5/B \ln\left(\frac{B}{5} k\right) .$$

In both cases, the response is a semi-transcendental function of the constants of integration.

Since the response is not the same for a positive step function and a negative step function, this system has an asymmetrical characterizing equation.

But for the system with the characterizing equation

$$u'' + Bu' - Du^2 + Eu = -F, \quad (26)$$

let

$$u = -y.$$

Then equation (26) becomes

$$y'' + By' + Dy^2 + Ey = F. \quad (27)$$

Thus the response of equation (26) is the negative of equation (27).

Type II Case (a) systems with movable critical points.--A differential equation of the form of equation (24) has movable critical points for

$$F \neq + \frac{1}{2500D} \left[ 625E^2 - 36B^2 \right].$$

For such an equation a solution is not available. As a function of time, the equation can be studied by a phase plane plot. It can be shown<sup>16</sup> that this equation has two singular points, one at  $\frac{dy}{dt} = y = 0$  which is a center, and one at  $y = -\frac{E}{D}$  which is a col. Thus the system is stable if the initial conditions are such that subsequently  $y > -F/D$ . If  $y < -E/D$ , instability occurs because of the nonlinear restoring force  $Ey + Dy^2$ .

Physical system with a Type II Case (a) characterizing equation.--The characterizing equation of a separately excited d.c. motor has been

developed<sup>1</sup>. Consider now the control system as shown in block diagram in Fig. 4. To obtain the characterizing equation of this system, it is convenient to introduce the variable

$$u = \int_0^t (e_s - K_f \omega) dt ,$$

which is the integral of the error of the system. The equations of motion are

$$a \left[ G^2 u + GK_u \right] = \frac{d\omega}{dt} + b\omega ,$$

$$\omega = \frac{e_s - u'}{K_f} ,$$

and

$$\omega' = - \frac{u''}{K_f} .$$

Hence,

$$u'' + bu' + K_f a G^2 u^2 + K_f K_G a u = b e_s . \quad (28)$$

Now let

$$B = - b$$

$$D = - K_f a G^2$$

---

<sup>1</sup>See Appendix A.

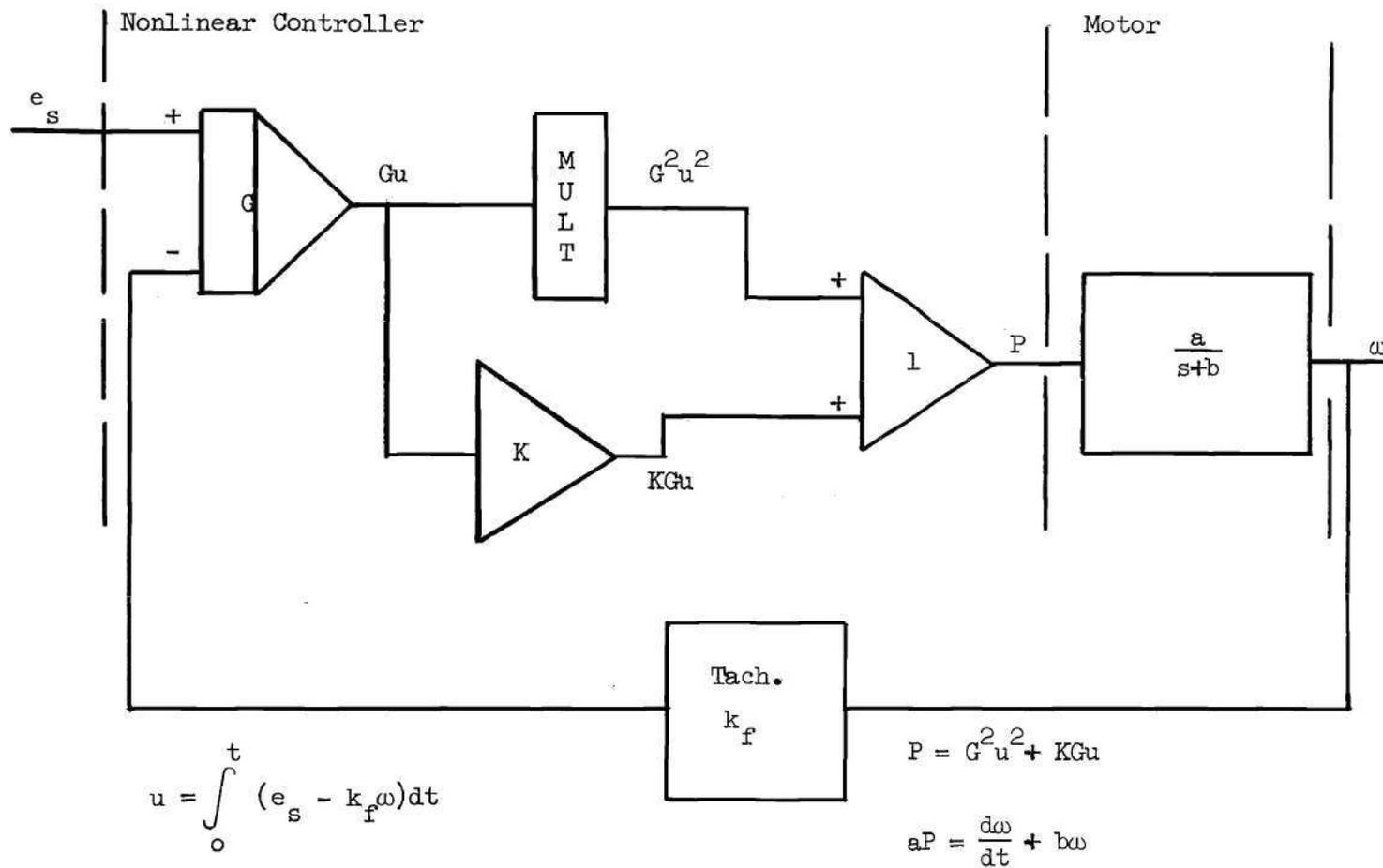


Figure 4. Physical System with Type II Case (a) Characterizing Equation

$$E = -K_f K G a$$

and

$$F = b e_s$$

and equation (28) becomes

$$u'' = B u' + D u^2 + E u + F, \quad (29)$$

which is a Type II Case (a) characterizing equation. If

$$e_s = - \frac{1}{2500abK_f G^2} \left[ 625(K_f K a G)^2 - 366^2 \right],$$

then equation (29) has a solution whose critical points in the complex plane are fixed.

Theoretical comparison of linear and nonlinear systems.--For purposes of comparison, the same values of parameters will be utilized in both the linear and nonlinear systems. These values are

$$G = 10$$

$$K = 10$$

$$a = 43.6$$

$$b = 4.07$$

and

$$K_f = .01$$

Hence for fixed critical points

$$e_s = - 2.7$$

With the system starting from rest, the initial conditions are

$$u(0) = 0 \quad (30)$$

and

$$u'(0) = -2.7 \quad (31)$$

The linear system has the characterizing equation

$$u'' + bu' + K_f a G u = b e_s ,$$

with the initial conditions as in equations (30) and (31).

In Fig. 5 is shown the analogue computer setup for determining the response of the linear system. For the nonlinear system, the analogue computer setup of Fig. 7 was used. In Fig. 6 is shown the theoretical response of the linear system for various values of  $e_s$ .

In Fig. 8 is shown the theoretical response of the nonlinear system with fixed critical points. It is seen that this is not a stable system. This response occurs because of the movable poles crossing the real axis.

In Fig. 9 is shown the theoretical response of the nonlinear system for positive values of  $e_s$ .

Actual physical system.--The control system as shown in Fig. 4 was realized by utilizing an analogue computer as the nonlinear controller. The setup for the controller is shown in Fig. 10. To obtain operation in the linear region of the motor, it was necessary to change the values of the system parameters from those utilized in the theoretical case. The response of the system as shown in Fig. 9 is for values of

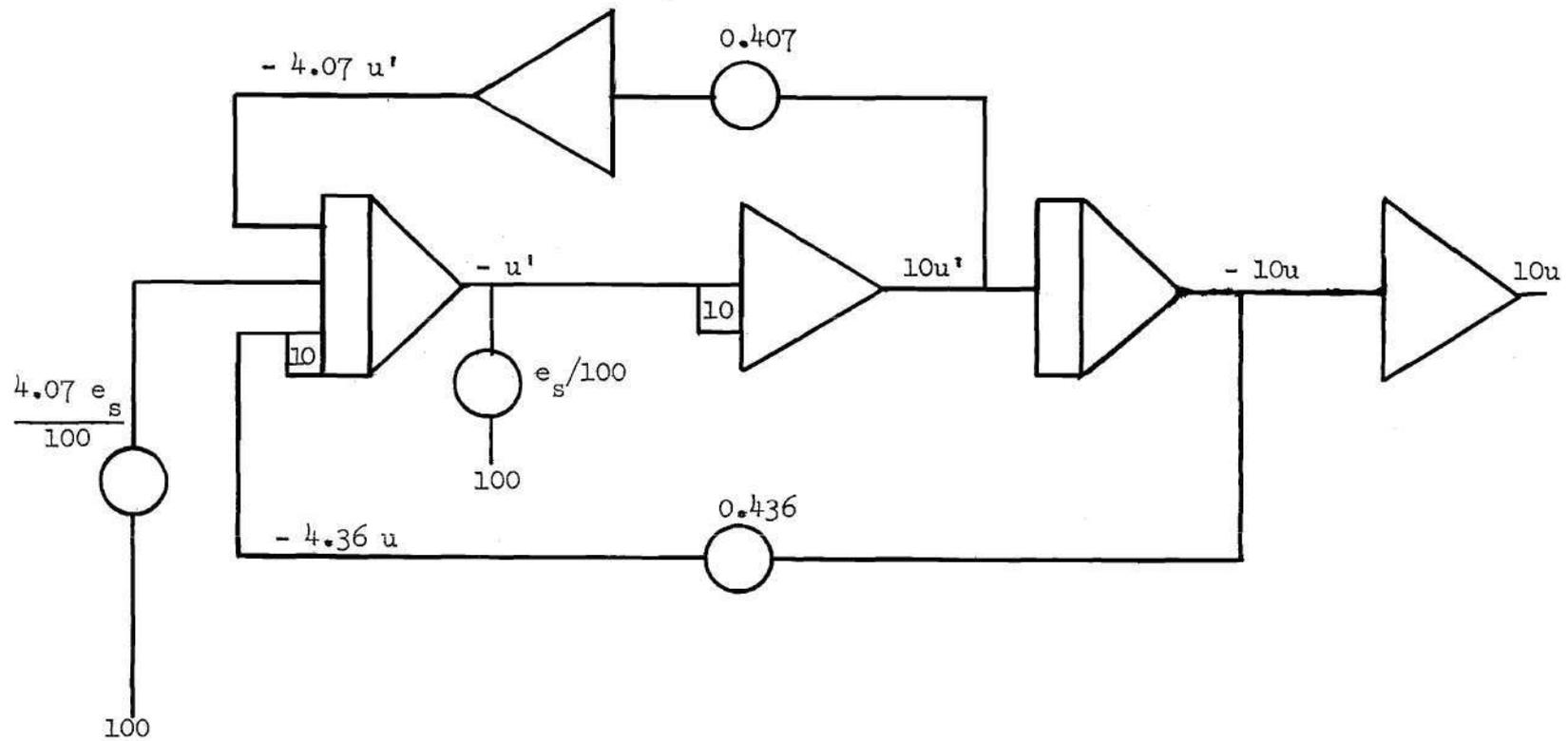


Figure 5. Computer Setup for Linear System

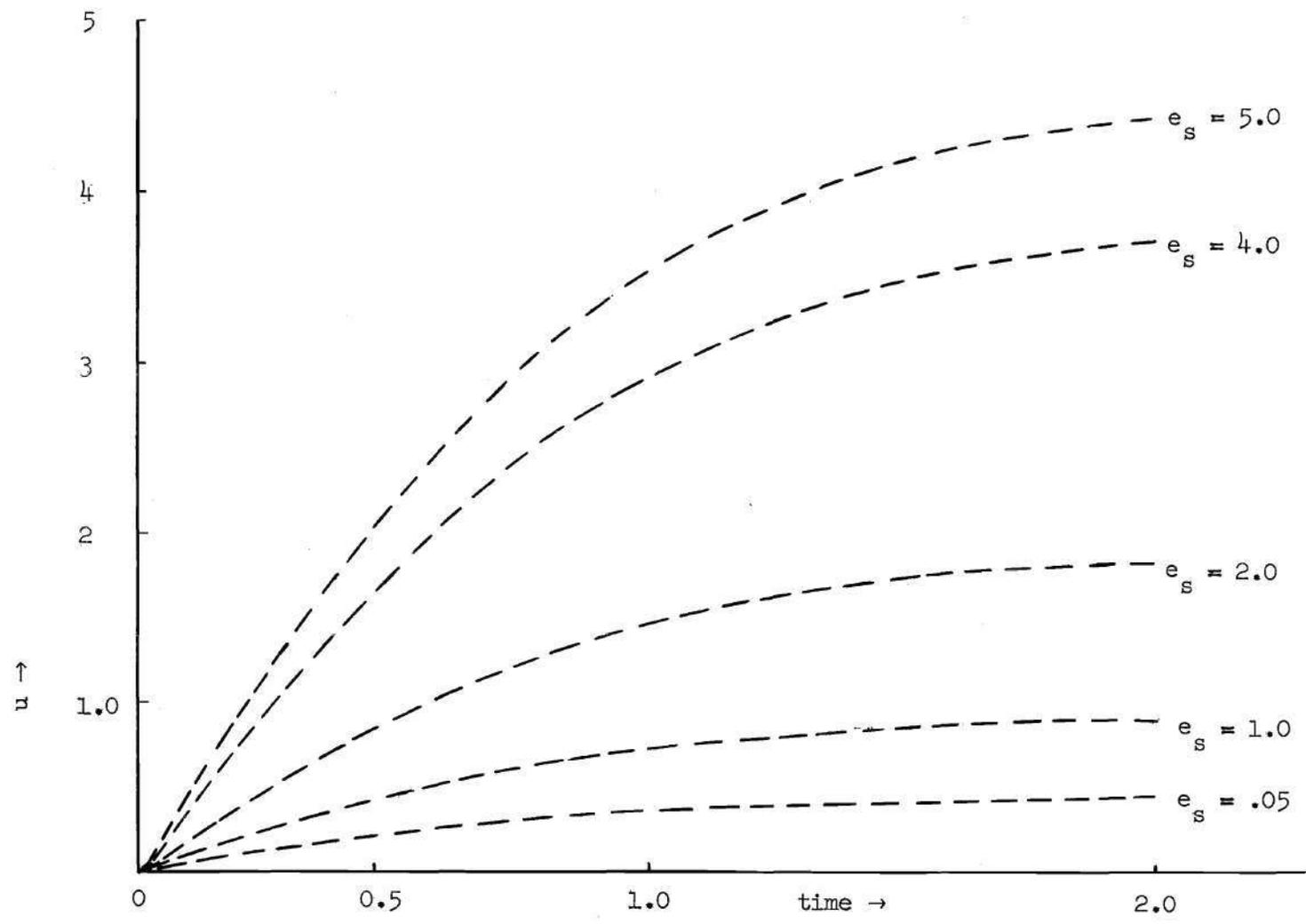


Figure 6. Response of Theoretical System

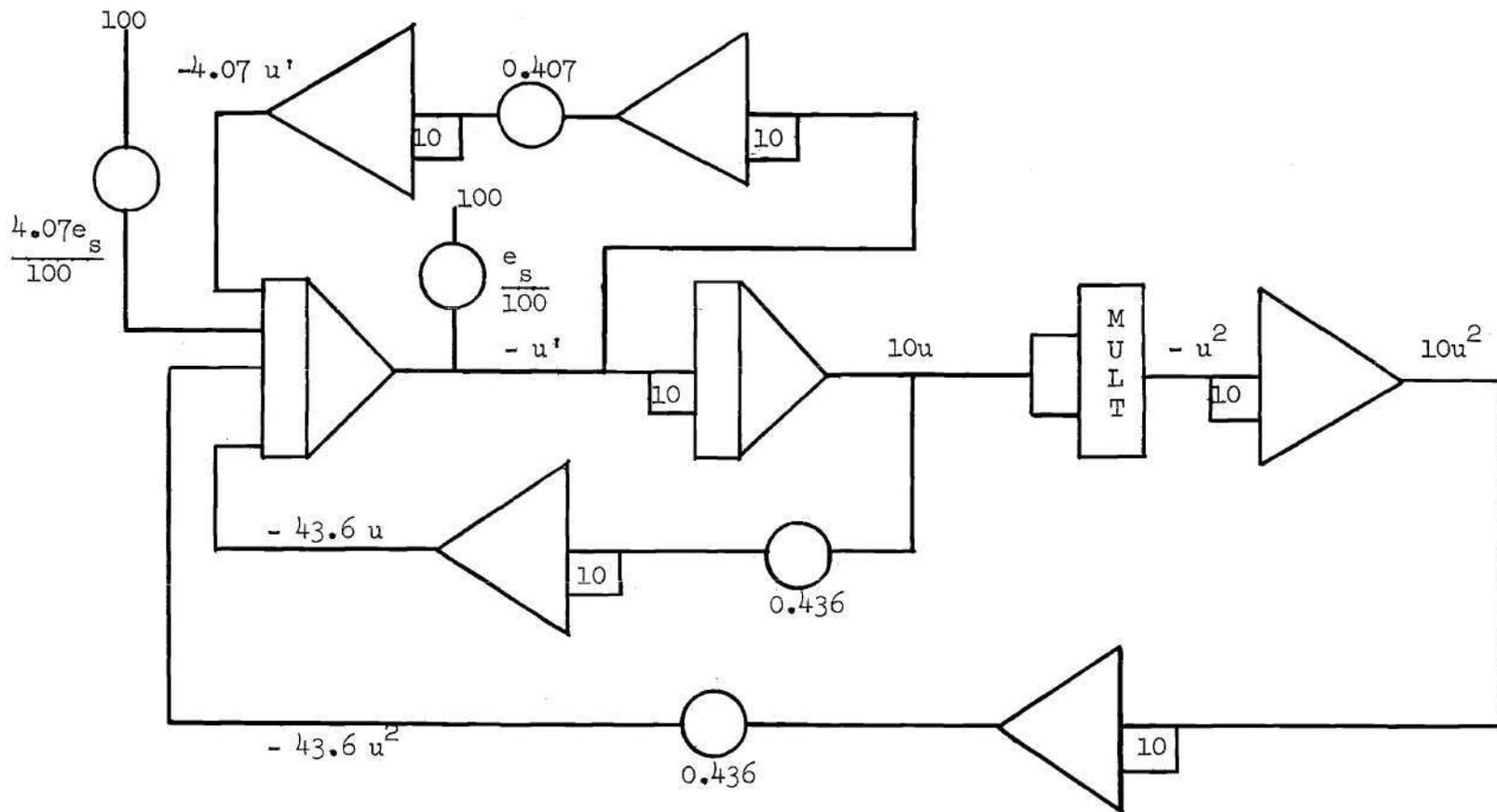


Figure 7. Computer Setup for Nonlinear System

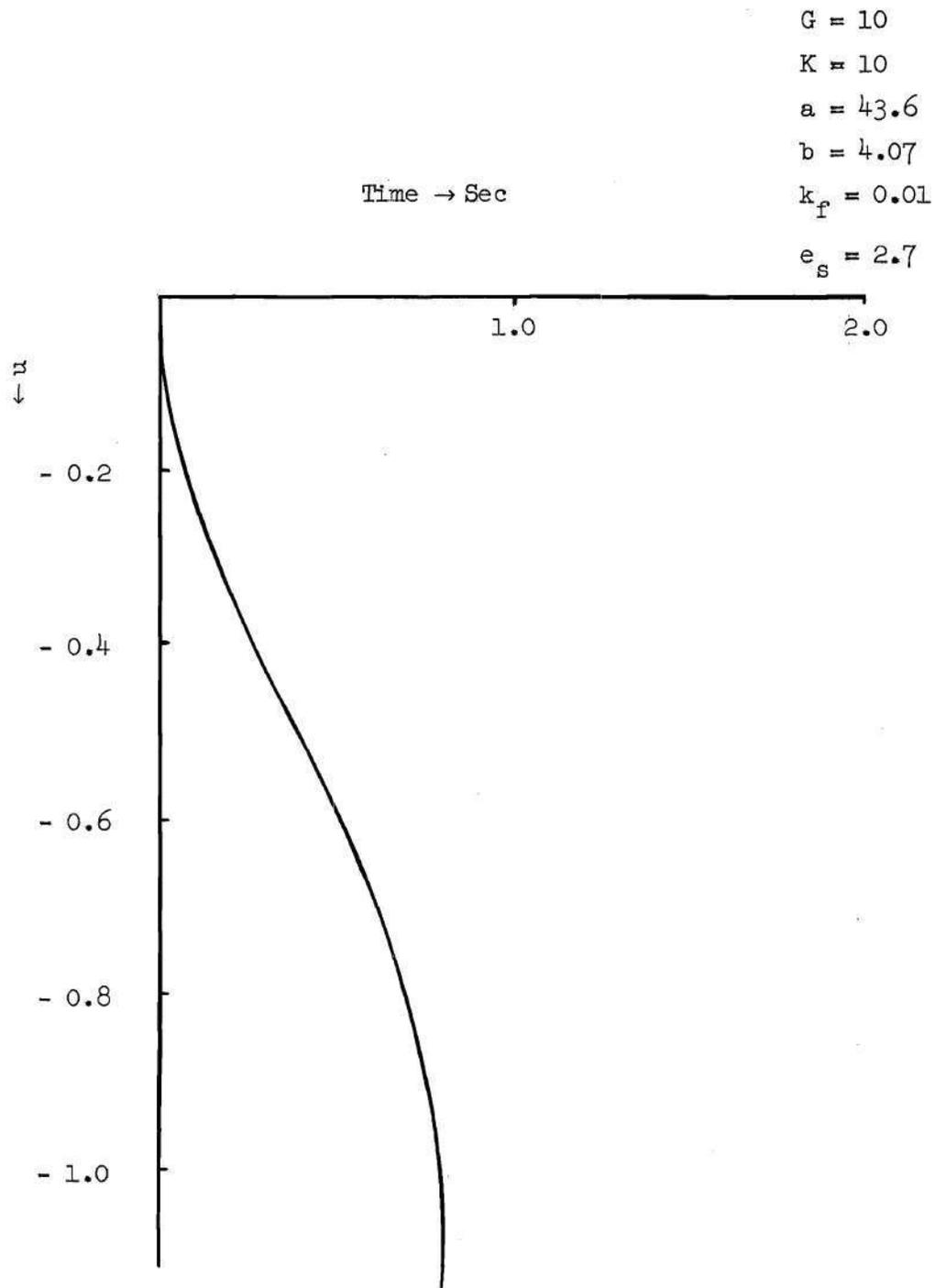


Figure 8. Theoretical Response of Nonlinear System with Fixed  
Critical Points

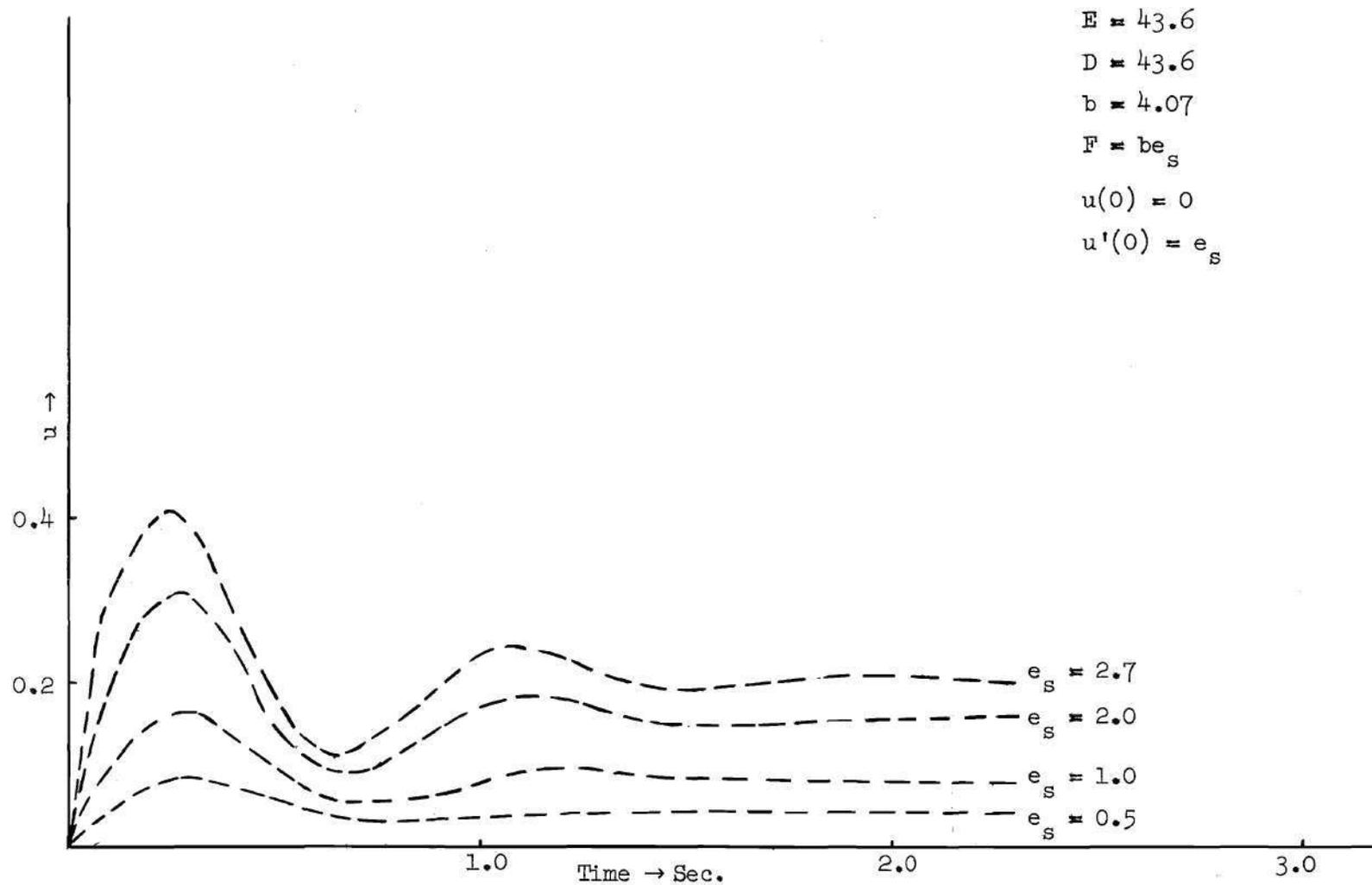


Figure 9. Theoretical Response of Nonlinear System with Movable Critical Points

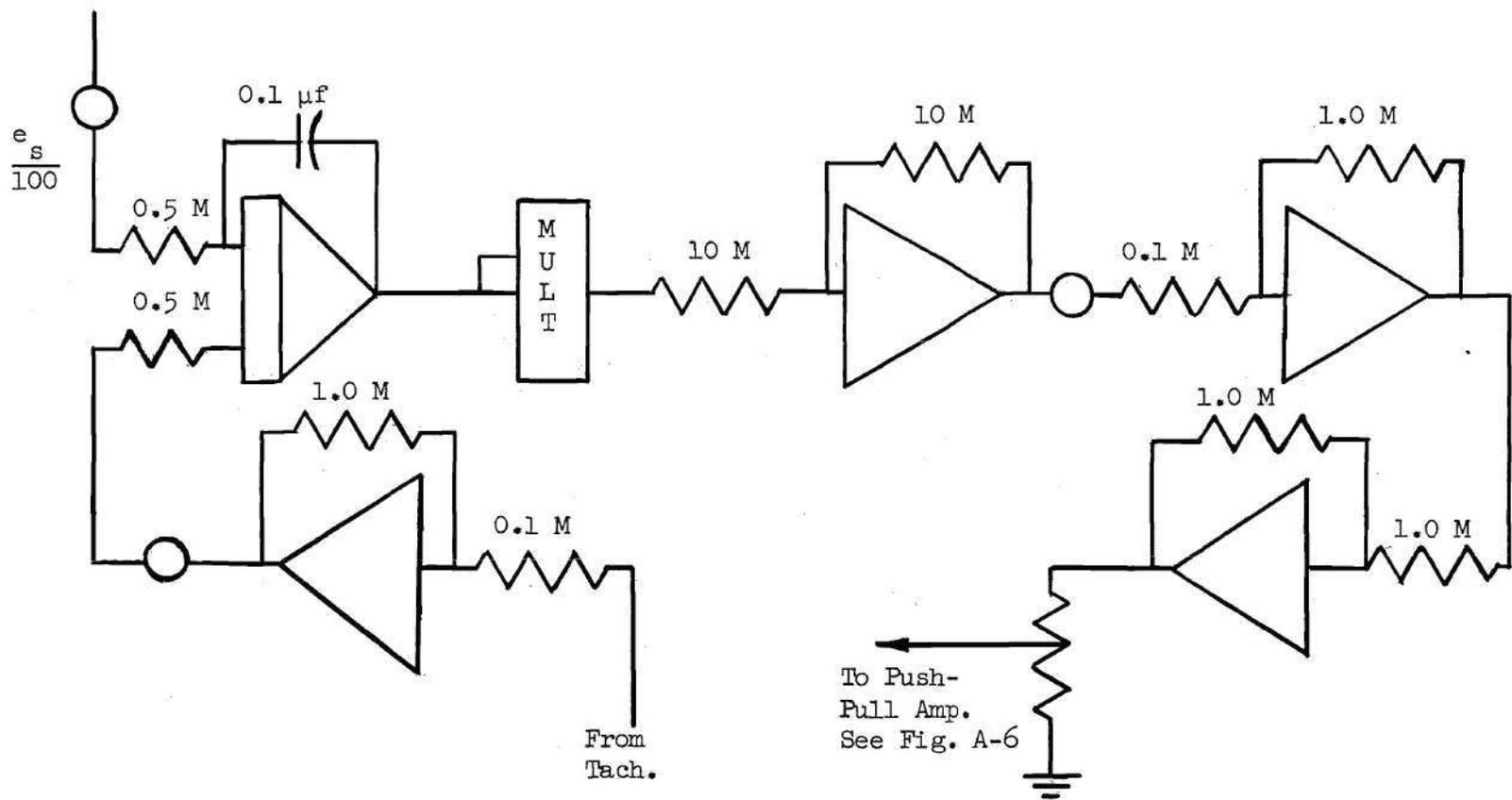


Figure 10. Nonlinear Controller for Type II Case (a) System

$$K = 0$$

$$K_f = .0084$$

$$b = 4.07$$

$$a = 31.4$$

and

$$G = 8.2 \text{ .}$$

Thus for signals above 0.5 volts the system is very oscillatory, having a maximum overshoot of 200% at 0.5 volts input.

Finally, it is of interest to note that the response of the system of Fig. 10 can be made symmetrical by altering the characterizing equation to

$$u'' + Bu' + D|u|u = F \text{ .}$$

This response has movable critical points, however, and is not considered further in this thesis.

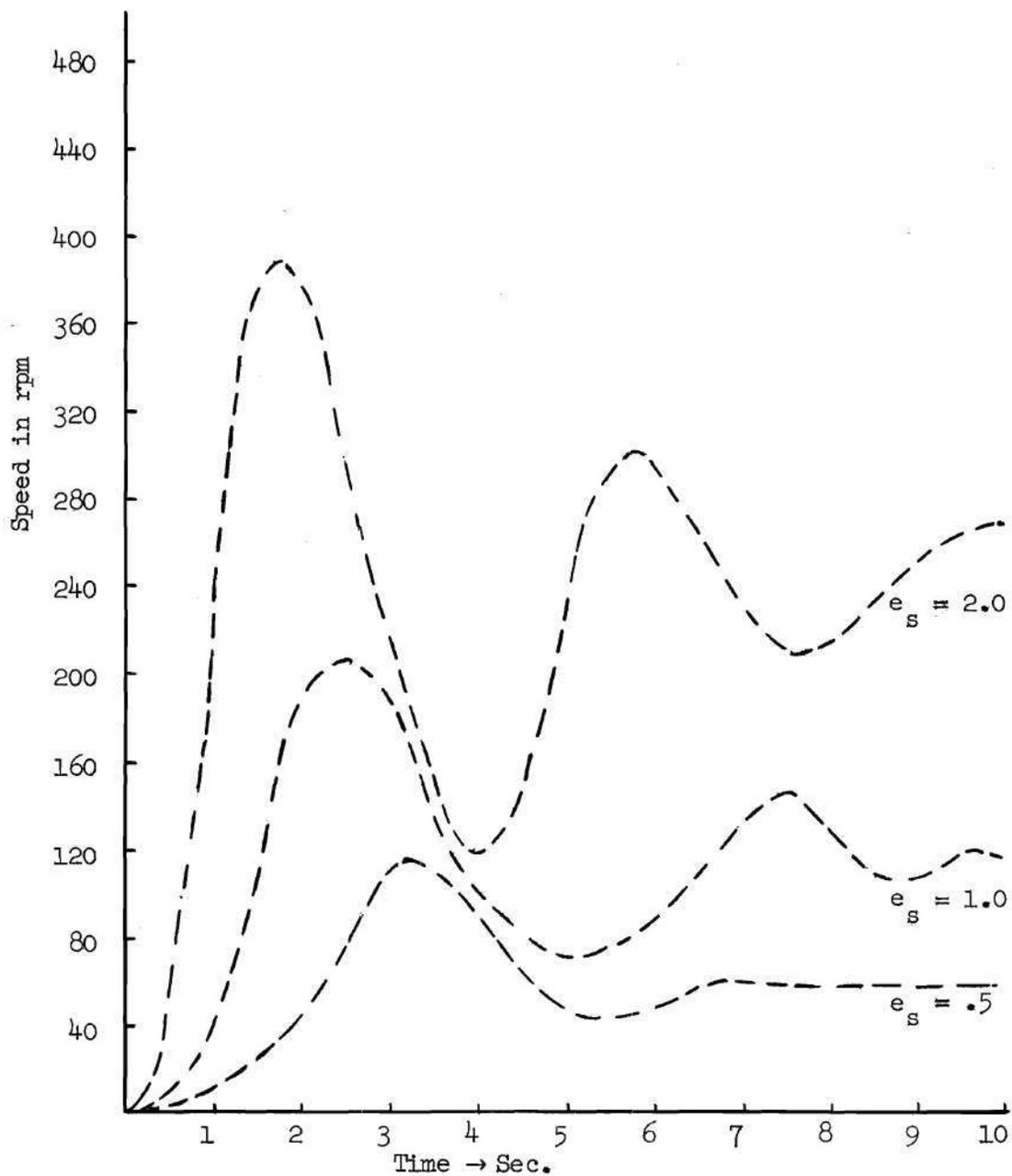


Figure 11. Actual Response of a Type II Class (a) System

## CHAPTER V

### TYPE II CASE (b) SYSTEMS

$$A = 0, C = 2$$

Response in the complex domain.--The form of the differential equation as a function of a complex variable  $w(z)$  is

$$w'' = Bw' + 2w^3 + Dw^2 + Ew + F. \quad (32)$$

There are two cases in which the solution of equation (32) has fixed critical points<sup>17,18</sup>. They are examined individually below.

Case 1.  $B = 0$ . With the substitutions

$$\lambda = 1$$

$$6\mu = -D$$

and

$$w = \lambda W + \mu,$$

equation (32) becomes

$$W'' = 2W^3 + RW + S \quad (33)$$

where

$$R = E - D^2/6$$

and

$$S = F + FD^3/216 - \frac{ED}{6}.$$

The first integral of equation (33) is

$$(W')^2 = W^4 + RW^2 + 2SW + h, \quad (34)$$

where  $h$  is a constant of integration. Equation (34) is in the standard form of a Jacobian Elliptic function<sup>19</sup>. These functions have the following properties<sup>20</sup>:

- (1) The zeros of  $\text{Sn}(u)$  are  $\dots 2MK + 2NiK^1$ ,
- (2) The zeros of  $\text{Cn}(u)$  are  $\dots (2M+1)K + 2NiK^1$ ,
- (3) The zeros of  $\text{dn}(u)$  are  $\dots (2M+1)K + (2N+1)iK^1$ ,
- (4) The poles of all three functions are  $\dots 2MK + (2N+1)iK^1$ .

$M$  and  $N$  are integers including zero, and  $K$  and  $K^1$  are the real and imaginary quarter period. The Jacobian Elliptic functions are also doubly periodic. The solution of equation (34) will be a semi-transcendental function of the constants of integration.

Case ii.  $B \neq 0$ . To insure fixed critical points in the solution, equation (32) must be in the form

$$W'' + 3aW' - 2W^3 + 2a^2W = 0. \quad (35)$$

To solve this equation, let

$$W(Z) = w e^{aZ}$$

and

$$Z = -\frac{1}{a} e^{-aZ}.$$

Equation (35) now becomes

$$W'' = 2W^3 . \quad (36)$$

The first integral of (36) is

$$(W')^2 = W^4 - h . \quad (37)$$

With the substitutions

$$s^2 = \frac{W^2}{\sqrt{h}} ,$$

$$k^2 = -1 ,$$

and

$$u = \pm \sqrt{-h^{1/2}} ,$$

equation (37) becomes

$$(s')^2 = [1 - s^2][1 - k^2 s^2] , \quad (38)$$

which has the solution

$$s = \text{Sn}(u + K_2 | m = -1) .$$

Hence the solution to (35) is

$$W = -jaK_1 e^{-az} \text{Sn}(K_1 e^{-az} + K_2 | m = -1) ,$$

where

$$K_1 = - \frac{\sqrt{-h^{1/2}}}{a} .$$

By means of the identities<sup>21</sup>

$$\operatorname{Sn}(u| - 1) = \frac{1}{\sqrt{2}} \operatorname{sd}(\sqrt{2} u | \frac{1}{2})$$

and

$$\operatorname{dn}^2 = \frac{1}{1 + \operatorname{msd}^2 u},$$

equation (38) may be rewritten as

$$W = -jaK_1 e^{-az} \left[ \frac{1 - \operatorname{dn}[\sqrt{2}(K_1 e^{-az} + K_2) | 1/2]}{(\operatorname{dn}[\sqrt{2}(K_1 e^{-az} + K_2) | 1/2])^2} \right]^{1/2}. \quad (39)$$

Equation (39) has fixed critical points, but the poles are movable with respect to the constants of integration. These poles occur at values of the argument where

$$K_1 e^{-az} + K_2 = (2M + 1)K + (2N + 1)iK^1,$$

where

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - 1/2 \sin^2 \theta}}$$

and

$$K^1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - 1/2 \sin^2 \theta}}$$

From equation (38) it is deduced that the solution of equation (35) is a semi-transcendental function of the constants of integration.

Response in the real domain.--As a function of a real variable  $y(t)$ , equation (35) (case ii) becomes

$$y'' + 3ay' - 2y^3 + 2a^2y = 0 . \quad (40)$$

To solve this equation the substitutions of case ii may be used. Equation (40) then becomes

$$(y_1')^2 = y_1^4 - h , \quad (41)$$

where

$$y_1 = ye^{+at}$$

and

$$T = -\frac{1}{a}e^{-at} .$$

There are three forms that the solution to equation (40) may take, depending upon the value of the constant of integration  $h$ . The relationship between the initial conditions and  $h$  is given by

$$h = y^4(0) - [ay(0) + y'(0)]^2 .$$

These three forms will now be examined individually.

(a)  $h = 0$ . The solution to equation (40) can be expressed as

$$y = \frac{ae^{-at}}{K_2 + e^{-at}} .$$

This solution is unstable if the constant of integration is such that

$$-1 \leq K_2 \leq 0 .$$

(b)  $h > 0$ . This response can be obtained from equation (39) and the identities<sup>22</sup>

$$\operatorname{dn}(ju|m) = \operatorname{dc}(u|m_1)$$

and

$$\operatorname{Cn}^2(u|m) = \frac{m_1}{\operatorname{dc}^2 u - m} .$$

Hence the solution is

$$y = \sqrt{h^{1/2}} \epsilon^{-at} \left[ \frac{\operatorname{Cn}^2\left(-\sqrt{\frac{h^{1/2}}{a}} \epsilon^{-at} + K_2|1/2\right) - 1}{\operatorname{Cn}^2\left(-\sqrt{\frac{h^{1/2}}{a}} \epsilon^{-at} + K_2|1/2\right) + 1} \right]^{1/2}, \quad (42)$$

and this solution is always stable.

(c)  $h < 0$ . Equation (40) now has the solution

$$y = -j \sqrt{jh^{1/2}} \epsilon^{-at} \operatorname{Sn} \left\{ -\sqrt{\frac{jh^{1/2}}{a}} \epsilon^{-at + K_2|1/2} \right\} ,$$

and, since the response is imaginary, it is unstable.

Stability of Type II Case (b) systems.---Systems characterized by case i have no damping and hence their response is periodic. Their motion can be described by means of the Jacobian elliptic function.

Systems characterized by case ii have positive damping but a nonlinear restoring force. From the solution to the characterizing

equation, it is seen that the stability of this system is severely restricted by the initial conditions. The stable response occurs only when

$$y^4(0) \geq \left[ ay(0) + y'(0) \right]^2 .$$

The restriction on the stable response severely limits the use of a Type II Class (b) characterizing equation.

Physical system with a Type II Case (b) characterizing equation.---In Fig. 12 is shown a control system which has a Type II Case (b) characterizing equation. By utilizing the variable

$$u = \int_0^t (e_s - K_f \omega) dt ,$$

the equations of motion are

$$a \left[ -x_1 u^3 + x_2 u \right] = W' + b\omega . \quad (43)$$

But,

$$\omega = \frac{e_s - u'}{K_f}$$

and

$$\omega' = - \frac{u''}{K_f} .$$

Hence equation (43) may be expressed as

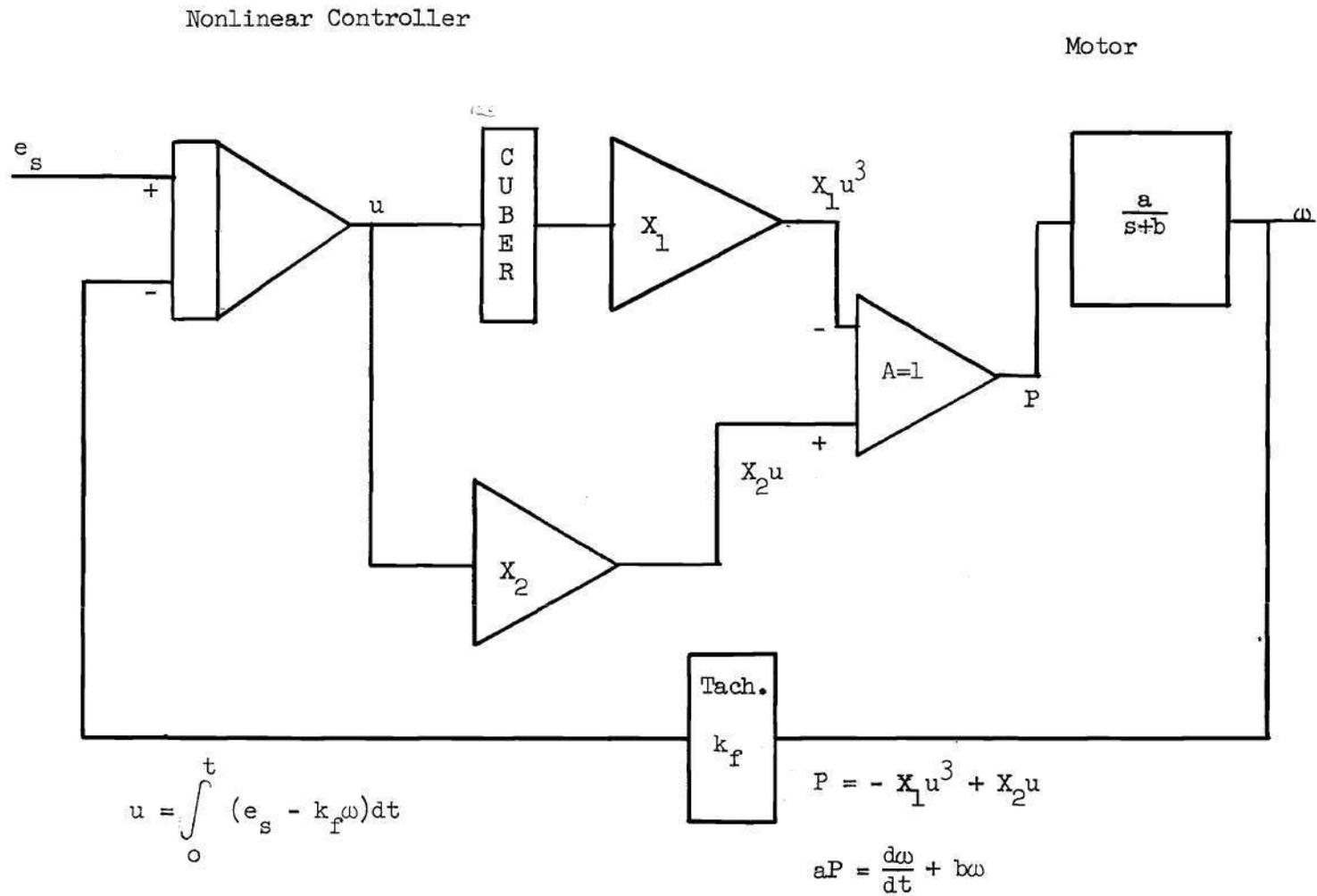


Figure 12. Physical System with Type II Case (b) Characterizing Equation

$$u'' + bu' + K_f a x_2 u - a K_f x_1 u^3 = b e_s . \quad (44)$$

With

$$x_1 = \frac{2}{a K_f} ,$$

$$x_2 = \frac{2b^2}{9a K_f} ,$$

and

$$e_s = 0 ,$$

equation (44) becomes the Type II Case (b) characterizing equation

$$u'' + bu' + \frac{2b^2}{9} u - 2u^3 = 0 .$$

The stable response occurs if

$$u^4(o) \leq \left[ au(o) + u'(o) \right]^2 .$$

It should be emphasized that a solution is available to the characterizing equation of Fig. 12 only if the input signal is zero. For a value of input voltage, a solution is not available, and furthermore the solution has movable critical points in the complex domain.

Summary of Type II Case (b) characterizing equations.--Like the Case (a) system, the Type II Case (b) characterizing equation has fixed critical points for only one value of input amplitude, i.e., zero. Hence it is

not suitable for purposes of this thesis. A further disadvantage is its limited stability, which is due to the nature of the nonlinear restoring force  $-2y^2 + 2a^2y$ . Unlike the Type II Case (a) system, it has an unstable response because the solution of the characterizing equation is imaginary.

## CHAPTER VI

### TYPE II CASE (c) SYSTEM

$$A = -2; C = 0$$

Response in the complex domain.--The form of the Type II Case (c) system in the complex domain is

$$W'' = (-2W + B)W' + DW^2 + EW + F. \quad (45)$$

For equation (45) to have a solution with fixed critical points, it has been shown<sup>23</sup> that it must be in the form

$$W'' + (2W + P)W' + PW^2 = F. \quad (46)$$

The first integral of equation (46) is

$$W' + W^2 = u, \quad (47)$$

where

$$u' = -Pu + F. \quad (48)$$

For P and F constant, the solution to equation (48) is

$$u = \frac{F}{P} + K_1 e^{-Pz}, \quad (49)$$

where  $K_1$  is the constant of integration.

Equation (47) is then

$$W' + W^2 = \frac{F}{P} + K_1 e^{-Pz}. \quad (50)$$

Equation (50) is Ricatti's differential equation and can be solved by the substitution

$$W = \frac{V'}{V} .$$

With this substitution, equation (50) becomes

$$V'' = \left[ \frac{F}{P} + K_1 \epsilon^{-Pz} \right] V . \quad (51)$$

With the substitution

$$Z = + \frac{2 \sqrt{-K_1}}{P} \epsilon^{-(P/2)z} ,$$

equation (51) becomes

$$Z^2 V'' + ZV' + [Z^2 - (\sqrt{\frac{4F}{P^3}})^2] V = 0 ,$$

which is Bessel's equation<sup>24</sup> of order  $\nu = \sqrt{\frac{4F}{P^3}}$ , and has the solution

$$V = A_1 J_\nu(Z) + B_1 Y_\nu(Z) .$$

Hence the solution to equation (46) is

$$W = + \sqrt{-K_1} \epsilon^{-\frac{P}{2}z} \left[ \frac{A_1 J'_\nu\left(\frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z}\right) + Y'_\nu\left(\frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z}\right)}{A_1 J_\nu\left(\frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z}\right) + Y_\nu\left(\frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z}\right)} \right] , \quad (52)$$

where

$$J_{\nu}(u) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}u\right)^{\nu+2r}}{r! \Gamma(\nu+r+1)}$$

$$Y_{\nu}(u) = \frac{J_{\nu}(u) \cos \nu\pi - J_{-\nu}(u)}{\sin \nu\pi}$$

$$J'_{\nu}(u) = - \frac{\nu J_{\nu}(u) + u J_{\nu-1}(u)}{u}$$

and

$$Y'_{\nu}(u) = - \frac{\nu Y_{\nu}(u) + u Y_{\nu-1}(u)}{u} .$$

Equation (52) has fixed critical points, but the poles which are determined by  $\nu$ ,  $\sqrt{-K_1}$  and  $A_1$  are movable. These poles occur for values of argument and order where

$$A_1 J_{\nu} \left( \frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z} \right) = - Y_{\nu} \left( \frac{2}{P} \sqrt{-K_1} \epsilon^{-\frac{P}{2}z} \right) , \quad (53)$$

provided the numerator of equation (52) does not have common factors with equation (53).

Response in the real domain.--Equation (46) can be expressed as a function of a real variable  $y(t)$  in the form

$$y'' + (2y + P)y' + Py^2 = F . \quad (54)$$

From equation (52) the solution to equation (54) is

$$y = -\sqrt{-K_1} \epsilon^{-\frac{P}{2}t} \left[ \frac{A_1 J_{\nu}'\left(\frac{2}{P}\sqrt{-K_1} \epsilon^{-\frac{P}{2}t}\right) + Y_{\nu}'\left(-\frac{2}{P}\sqrt{-K_1} \epsilon^{-\frac{P}{2}t}\right)}{A_1 J_{\nu}\left(\frac{2}{P}\sqrt{-K_1} \epsilon^{-\frac{P}{2}t}\right) + Y_{\nu}\left(\frac{2}{P}\sqrt{-K_1} \epsilon^{-\frac{P}{2}t}\right)} \right] \cdot (55)$$

The constant of integration  $K_1$  can be obtained from the initial conditions by the relationship

$$y'(0) + y^2(0) - F/P = K_1 .$$

There are three cases of equation (55) to be considered, depending upon the value of  $K_1$ . These cases will now be examined individually.

(a)  $K_1 = 0$ . From equation (50)

$$y' + y^2 = F/P \quad (56)$$

and hence,

$$y = \sqrt{F/P} \left[ \frac{\epsilon^{\sqrt{F/P} t} - C^2 \epsilon^{-\sqrt{F/P} t}}{\epsilon^{\sqrt{F/P} t} + C^2 \epsilon^{-\sqrt{F/P} t}} \right] \cdot (57)$$

$C$  is the constant of integration and is obtained from the relationship

$$C^2 = \frac{-y(0) + \sqrt{F/P}}{y(0) + \sqrt{F/P}} .$$

For negative values of  $F$ , equation (56) is unstable. For  $F = 0$ , the solution to equation (56) is

$$y = \frac{1}{(t + k)},$$

and is stable if

$$y(0) > 0.$$

For

$$y(0) < \sqrt{F/P},$$

$C^2$  is positive and the denominator of equation (57) is never zero and hence the system is stable. But, for

$$y(0) > \sqrt{F/P},$$

the denominator of equation (57) can be zero and hence the system can become unstable. However, by analogy with the linear system, it is seen that the nonlinear system is stable if the input signal allows the response to reach a steady state value without becoming negative.

Case (b).  $K_1$  negative. From equation (55), the response for Case (b) is

$$y = -\sqrt{K_1} \epsilon^{-\frac{P}{2}t} \left[ \frac{A_J Y'_v(u) + Y'_v(u)}{A_J Y_v(u) + Y_v(u)} \right], \quad (58)$$

where

$$u = +\frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t}$$

and

$$\nu = \frac{2}{P} \sqrt{F/P} .$$

If the order of equation (58) is a non integer when  $F > 0$ , it may be re-written

$$y = -\frac{P}{2}u \left[ -\frac{A_\nu J_\nu(u)}{u} + \frac{AuU_{\nu-1}(u)}{u} - \frac{\nu(J_\nu(u) \cos \nu\pi - J_\nu(u))}{u \sin \nu\pi} \right. \\ \left. + \frac{J_{\nu-1}(u) \cos(\nu-1)\pi - J_{\nu+1}(u)}{u \sin(\nu-1)\pi} \right] + AJ_\nu(u) + \frac{J_\nu(u) \cos \nu\pi - J_{-\nu}(u)}{\sin \nu\pi} , (59)$$

where

$$u = +\frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} .$$

The steady state value of equation (59) can be found from the relationships

$$\lim_{u \rightarrow 0} y(u) = \lim_{t \rightarrow \infty} y(t) ,$$

$$\lim_{u \rightarrow 0} \left[ J_\nu(u) \right] = \frac{(1/2 u)^\nu}{\Gamma(\nu+1)} ,$$

$$\lim_{u \rightarrow 0} \left[ J_{\nu-1}(u) \right] = \frac{(1/2 u)^{\nu-1}}{\Gamma(\nu)} ,$$

$$\lim_{u \rightarrow 0} \left[ J_{-\nu}(u) \right] = \frac{(1/2 u)^{-\nu}}{\Gamma(-\nu+1)} ,$$

and

$$\lim_{u \rightarrow 0} \left[ J_{-\nu+1}(u) \right] = \frac{(1/2 u)^{-\nu+1}}{\Gamma(-\nu+2)} .$$

Hence,

$$y_{s.s.} = \lim_{t \rightarrow \infty} y(t) = \sqrt{F/P} ,$$

and equation (59) has a finite, real, steady state value for all non integral values of  $\nu$ . A similar procedure for integral values of  $\nu$  shows the same result as equation (60). For  $F < 0$ , equation (60) is imaginary, and hence equation (54) is the characterizing equation of an unstable system.

However, equation (58) is not necessarily bounded for all values of  $t$ , the unstable response occurring for values of  $t$  where

$$A J_{\nu} \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right) = - Y_{\nu} \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right) . \quad (61)$$

This condition is analogous to a pole of equation (52) occurring on the real axis. By analogy with the linear system, it is seen that, if the input signal  $F$  in equation (58) is such as to cause  $y$  to become more positive, the system will always be stable since the damping coefficient  $(2y + P)$  and the restoring force  $Py^2$  are always positive. Conversely, if the signal  $F$  is decreased, the system will become unstable if  $y$  becomes negative. This is due to the inability of the restoring force to reverse. The time of the instability can be determined from equation (61).

Case (c).  $K_1$  positive. It is desirable to present the solution to equation (54) in real instead of imaginary form, and hence the modified Bessel functions must be used. Thus, its solution is

$$y = -\sqrt{K_1} \epsilon^{-\frac{P}{2}t} \left[ \frac{A I'_\nu\left(\frac{2}{P}\sqrt{K_1} \epsilon^{-\frac{P}{2}t}\right) + K'_\nu\left(\frac{2}{P}\sqrt{K_1} \epsilon^{-\frac{P}{2}t}\right)}{A I_\nu\left(\frac{2}{P}\sqrt{K_1} \epsilon^{-\frac{P}{2}t}\right) + K_\nu\left(\frac{2}{P}\sqrt{K_1} \epsilon^{-\frac{P}{2}t}\right)} \right], \quad (62)$$

where

$$I_\nu(u) = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}u\right)^{\nu+2r}}{r! \Gamma(\nu+r+1)},$$

and

$$K_\nu(u) = \frac{1}{2} \pi \frac{I_{-\nu}(u) - I_\nu(u)}{\sin \nu\pi}.$$

The recurrence formulae of Bessel's function of the first kind also hold true for the modified Bessel functions.

It a system whose characterizing equation is of the Type II Case (c) form has reached equilibrium at the value

$$y_{s.s.} = \sqrt{F/P},$$

and if the input amplitude  $F$  is suddenly changed to zero, its response will be given by equation (62) with

$$\nu = 0.$$

Hence,

$$y = \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \left[ \frac{-A I_1 \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right) + K_1 \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right)}{A I_0 \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right) + K_0 \left( \frac{2}{P} \sqrt{K_1} \epsilon^{-\frac{P}{2}t} \right)} \right], \quad (63)$$

where

$$A = \frac{-K_1 \left( \frac{2}{P} \sqrt{K_1} \right) + K_0 \left( \frac{2}{P} \sqrt{K_1} \right) \sqrt{\frac{F}{K_1 P}}}{\sqrt{\frac{F}{K_1 P} I_0 \left( \frac{2}{P} \sqrt{K_1} \right) + I_1 \left( \frac{2}{P} \sqrt{K_1} \right)}}.$$

For

$$\left| -A I_0 \left( \frac{2}{P} \sqrt{K_1} \right) \right| > K_0 \left( \frac{2}{P} \sqrt{K_1} \right),$$

equation (63) is unbounded and hence Case (c) is unstable.

Stability of Type II Case (c) systems.---The Type II Case (c) system is the first system that has been examined in this thesis whose solution has fixed critical points for all values of amplitude  $F$  of the input signal. It is also stable for a rather large range of operating conditions, restricted only by the necessity of  $y$  never becoming zero. Hence it shows promise for use as a characterizing equation of a physical system and will be examined further.

Synthesis of a Type II Case (c) characterizing equation.---In Fig. 3 is shown the block diagram of a speed control system utilizing a nonlinear controller. In terms of the variable

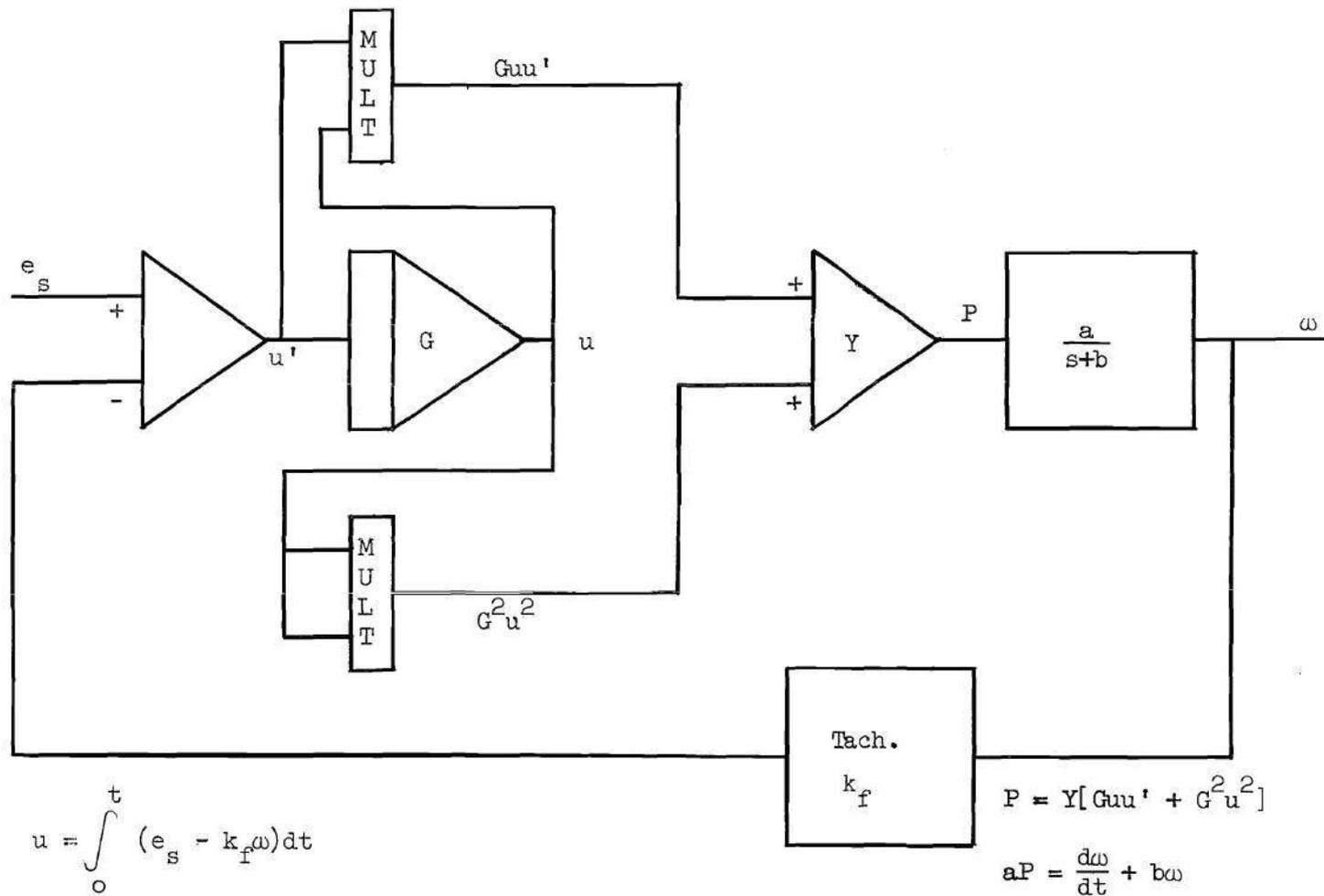


Figure 13. Physical System with a Type II Case (c) Characterizing Equation

$$u = \int_0^t (e_s - k_f \omega) dt,$$

the characterizing equation of the system is

$$u'' + [ak_f YG u + b]u' + ak_f YG^2 y^2 = be_s . \quad (64)$$

With the substitutions

$$u = xy ,$$

$$x = 2/ak_f YG ,$$

$$G = b/2 ,$$

$$P = b ,$$

and

$$F = \frac{abk_f YGe_s}{2} ,$$

equation (64) becomes the Type II Case (c) characterizing equation

$$y'' + [2y + P]y' + Py^2 = F . \quad (65)$$

For the system starting from rest, the initial conditions are

$$y(0) = 0$$

and

$$y'(0) = \frac{abk_f Ye_s}{4} .$$

The constant of integration  $K_1$  is

$$K_1 = \frac{abk_f Y e_s}{4} - \frac{abk_f Y e_s}{4} = 0 .$$

Thus the response of the system is given by

$$y = \sqrt{\frac{abk_f Y e_s}{4}} \tanh \left( \sqrt{\frac{abk_f Y e_s}{4}} s t \right) .$$

If this system settles out at the value given above and another signal  $e_{sa}$  is applied, the initial conditions are

$$y(0) = \sqrt{\frac{abk_f Y e_s}{4}}$$

and

$$y'(0) = \frac{abk_f Y}{4} (e_{sa} - e_s) .$$

Thus the constant of integration  $K_1$  is

$$K_1 = \frac{abk_f Y}{4} (e_{sa} - e_s) + \frac{abk_f Y e_s}{4} - \frac{abk_f Y e_{sa}}{4} = 0 .$$

From Case (a) of equation (54) the other constant of integration  $C^2$  is

$$C^2 = \frac{\sqrt{e_s} - \sqrt{e_{sa}}}{\sqrt{e_s} + \sqrt{e_{sa}}} ,$$

and the response is

$$y = \sqrt{F/P} \frac{\epsilon^{\sqrt{F/P} t} + C^2 \epsilon^{-\sqrt{F/P} t}}{\epsilon^{\sqrt{F/P} t} + C^2 \epsilon^{-\sqrt{F/P} t}},$$

where

$$\sqrt{F/P} = \sqrt{\frac{abk_f Y e_{sa}}{4}}.$$

The control system that has been synthesized in this chapter is very misleading. While the characterizing equation seems to be second order, the initial conditions are such that the equation is truly first order. Specifically, it is the Ricatti characterizing equation discussed in Chapter III. The real difficulty lies in the relationship between the initial conditions and the constant of integration  $K_1$ . Hence, to utilize this characterizing equation, it is necessary to alter the initial conditions.

With this in mind, the control system shown in Fig. 14 is developed. The equations of motion are

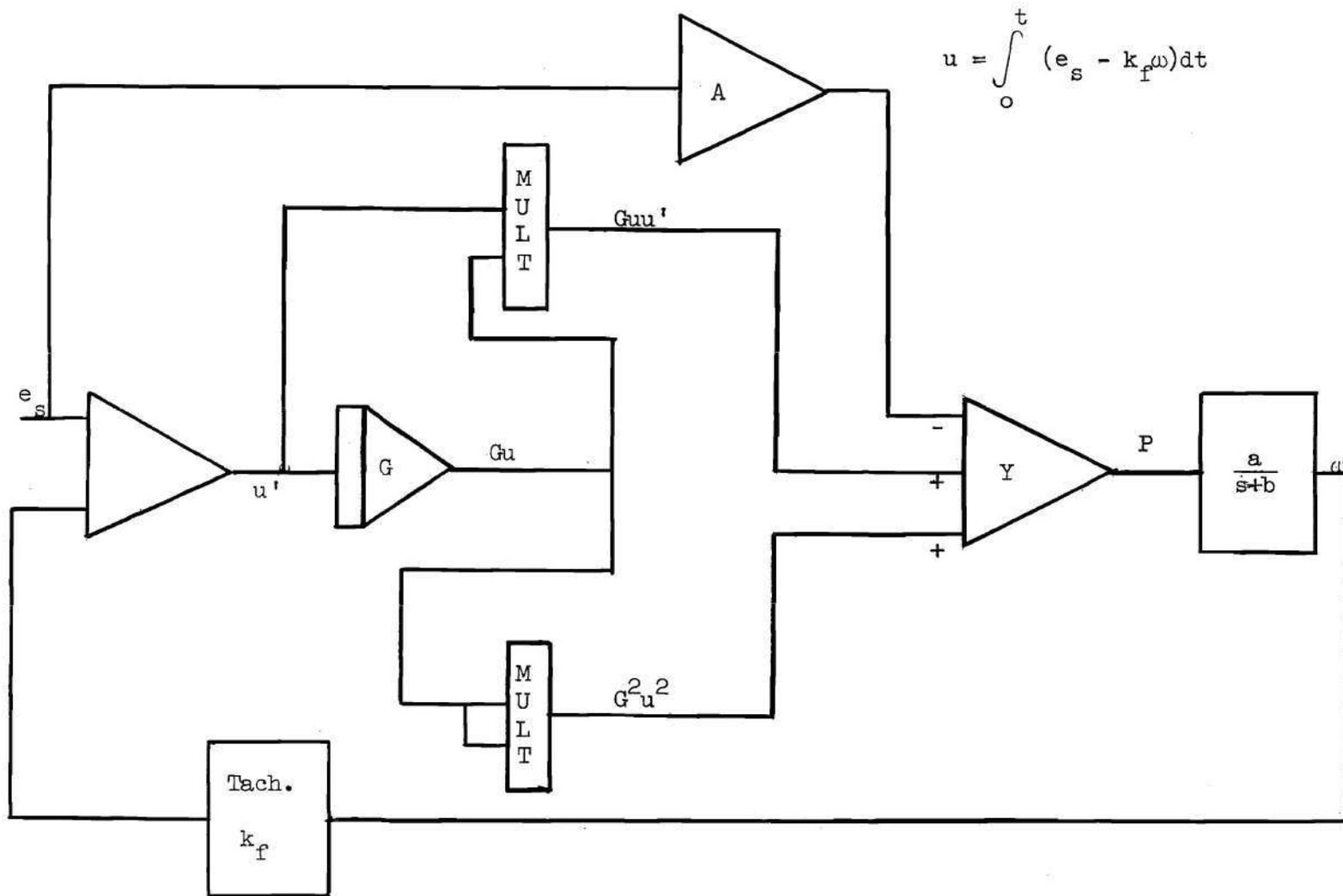
$$u = \int_0^t (e_s - k_f \omega) dt$$

and

$$u'' + [ak_f Y G u + b]u' + aYk_f G^2 u^2 = be_s + k_f a Y A e_s. \quad (67)$$

With

$$u = \frac{2}{ak_f Y G} y,$$



$$u = \int_0^t (e_s - k_f \omega) dt$$

Figure 14. Physical System with a Type II Case (c) Characterizing Equation  
 $K_1 \neq 0$

$$G = b/2 ,$$

$$P = b ,$$

$$A = b/aY ,$$

and

$$F = \frac{ab^2 k_f Y_e}{2 s} ,$$

equation (67) becomes the Type II Case (c) characterizing equation

$$y'' + [2y + P]y' + Py^2 = F .$$

For the system starting from rest, the initial conditions are

$$y(0) = 0$$

and

$$y'(0) = \frac{abk_f Y_e}{4 s} .$$

The constant of integration  $K_1$  is

$$K_1 = \frac{abk_f Y_e}{4 s} - \frac{abk_f Y_e}{2 s} = - \frac{abk_f Y_e}{4 s} .$$

The response for the system of Fig. 14 is then

$$u = - \sqrt{\frac{4}{abk_f Y_e}} e^{-bt} \left[ \frac{A J_v'(T) + Y_v'(T)}{A J_v(T) + Y_v(T)} \right] , \quad (68)$$

where

$$T = + \frac{b}{2} \sqrt{\frac{abk_f Y e_s}{4}} e^{-bt} ,$$

and

$$v = \frac{2}{b} \sqrt{\frac{abk_f Y e_s}{2}} .$$

The system of equation (67), after the transient has passed and a signal  $e_{sa}$  has been applied, has the initial conditions

$$y(0) = \sqrt{abk_f Y G/2}$$

and

$$y'(0) = \frac{abk_f Y}{4} (e_{sa} - e_s) .$$

Hence,

$$K_1 = - \frac{abk_f Y}{4} (e_{sa} - e_s) .$$

For

$$e_{sa} > e_s ,$$

the response is of the same form as equation (68). However, for

$$e_{sa} < e_s$$

the response is of the form of equation (62), i.e.,

$$y = \sqrt{K_1} \epsilon^{-\frac{b}{2}t} \left[ \frac{A I_V \left( \frac{2}{b} \sqrt{K_1} \epsilon^{-\frac{b}{2}t} \right) + K_V \left( \frac{2}{b} \sqrt{K_1} \epsilon^{-\frac{b}{2}t} \right)}{A I_V \left( \frac{2}{b} \sqrt{K_1} \epsilon^{-\frac{b}{2}t} \right) + K_V \left( \frac{2}{b} \sqrt{K_1} \epsilon^{-\frac{b}{2}t} \right)} \right], \quad (69)$$

where

$$v = \sqrt{\frac{2ak_f Y_e}{b} s_a}.$$

Because of the nonlinear restoring force, the Type II Case (c) characterizing equation synthesized here can lead to an unstable system if the input signal is decreased. This condition is, of course, undesirable and for this reason the control system of Fig. 15 is synthesized. With the variable

$$u = \int_0^t (e_s - k_f \omega) dt,$$

the characterizing equation is

$$u'' + [k_f a Y G u + b] u' + A a Y k_f G u + a k_f Y G^2 u^2 = b e_s. \quad (70)$$

For

$$u = 2/k_f a Y G y,$$

equation (70) becomes

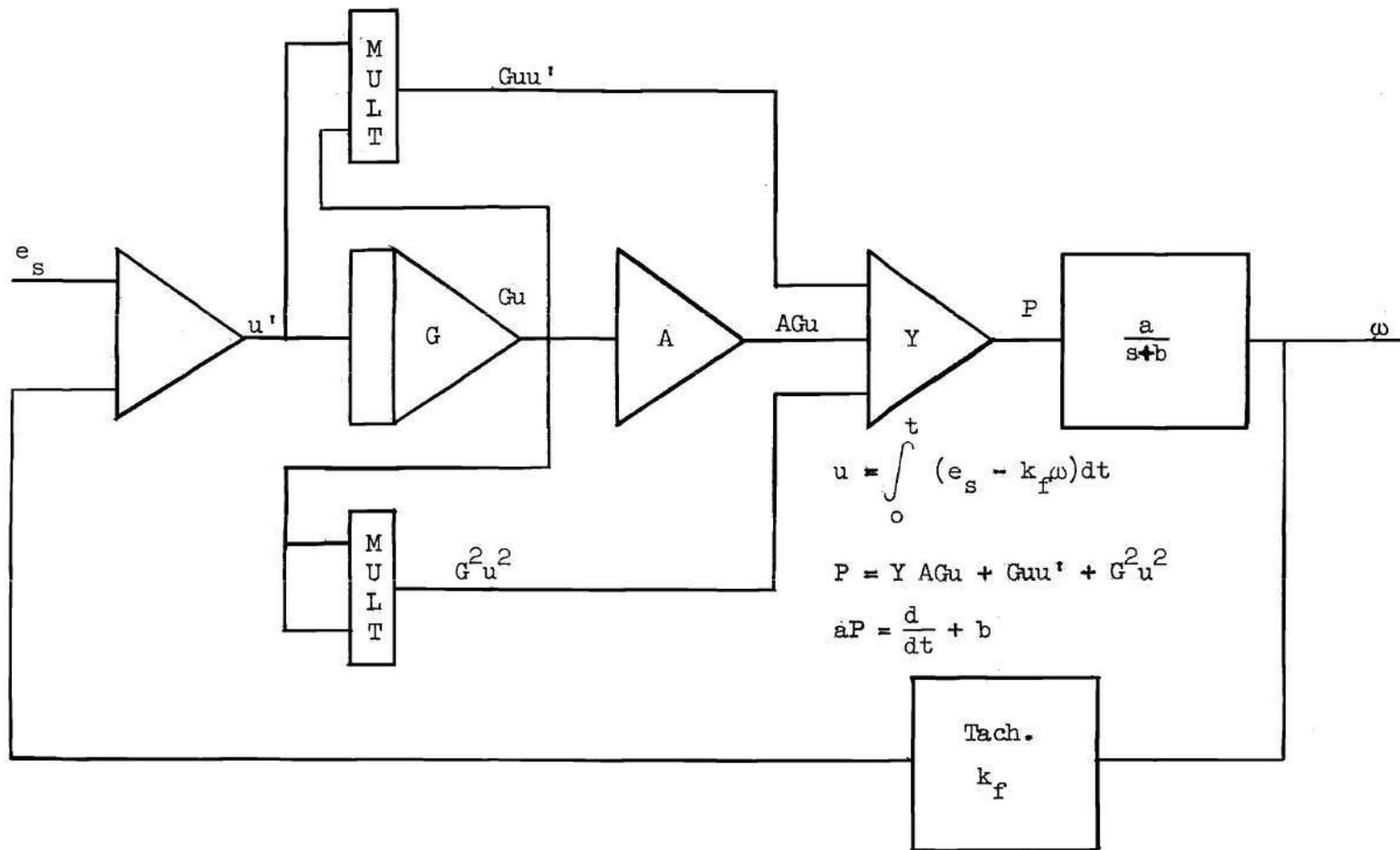


Figure 15. Physical System with a Type II Case (c) Characterizing Equation  
Having a Stable Isolated Singularity

$$y'' + [2y + b]y' + AaYk_f Gy + 2Gy^2 = \frac{k_f aYGbe_s}{2} . \quad (71)$$

When the square of the restoring force  $AaYk_f Gy + 2Gy^2$  is completed, equation (70) becomes

$$y'' + [2y + b]y' + 2G \left( y^2 + \frac{AaYk_f}{2}y + \frac{(AaYk_f)^2}{16} \right) = \frac{k_f aYGbe_s}{2} + \frac{(AaYk_f)^2 G}{8} . \quad (72)$$

For

$$x = y + \frac{AaYk_f}{4} ,$$

equation (72) becomes

$$x'' + \left[ 2x - \frac{AaYk_f}{2} + b \right]x' + 2Gx^2 = \frac{k_f aYGbe_s}{2} + \frac{(AaYk_f)^2 G}{8} . \quad (73)$$

For equation (73) to be a Type II Case (c) system it is necessary that

$$b - \frac{AaYk_f}{2} = 2G . \quad (74)$$

Hence, when

$$G = b/4 , \quad (75)$$

and with

$$A = \frac{b}{aYk_f} , \quad (76)$$

equation (73) becomes the Type II Case (c) characterizing equation

$$x'' + [2x + \frac{b}{2}]x' + \frac{b}{2}x^2 = \frac{k_f ab^2 Y e_s}{8} + \frac{b^3}{32}. \quad (77)$$

For the system starting from rest, the initial conditions are

$$x(0) = b/4,$$

and

$$x'(0) = \frac{k_f ab Y e_s}{8}.$$

Thus

$$-K_1 = + \frac{k_f ab Y e_s}{8}$$

and the response of equation (70) is

$$u = - \frac{8}{k_f ab Y} \left[ \sqrt{-K_1} \epsilon^{-\frac{b}{4}t} \left\{ \frac{A J_{\nu}'(\frac{4}{b} \sqrt{-K_1} \epsilon^{-\frac{b}{4}t}) + Y_{\nu}'(\frac{4}{b} \sqrt{-K_1} \epsilon^{-\frac{b}{4}t})}{A J_{\nu}(\frac{4}{b} \sqrt{-K_1} \epsilon^{-\frac{b}{4}t}) + Y_{\nu}(\frac{4}{b} \sqrt{-K_1} \epsilon^{-\frac{b}{4}t})} \right\} + \frac{b}{4} \right]. \quad (78)$$

If the system has settled to the steady state value of equation (78) and a signal  $e_{sa}$  is then applied, the initial conditions are

$$x^2(0) = \frac{k_f ab Y e_s}{4} + \frac{b^2}{16}$$

and

$$x'(0) = \frac{k_f abY}{8} [e_s - e_{sa}] .$$

Thus

$$K_1 = \frac{3}{8} f_f abY (e_s - e_{sa}) ,$$

and the response of equation (78) is

$$u = - \frac{8}{k_f abY} \left[ \sqrt{K_1} \epsilon^{-\frac{b}{4}t} \left\{ \frac{AI_V \left( \frac{4}{b} \sqrt{K_1} \epsilon^{-\frac{b}{4}t} \right) + K_V \left( \frac{4}{b} \sqrt{K_1} \epsilon^{-\frac{b}{4}t} \right)}{AI_V \left( \frac{4}{b} \sqrt{K_1} \epsilon^{-\frac{b}{4}t} \right) + K_V \left( \frac{4}{b} \sqrt{K_1} \epsilon^{-\frac{b}{4}t} \right)} \right\} + \frac{b}{4} \right] . \quad (79)$$

The question of the stability of the system for  $F = 0$  can be answered by examining equation (72) with the substitutions of equation (74) to (76),

$$y'' + [2y + b]y' + \frac{b}{2}y^2 + \frac{b^2}{8}y = \frac{k_f ab^2 Y e_s}{8} .$$

With the system unexcited but with the initial conditions

$$y(0) = y_{s.s.},$$

and

$$y'(0) = k_f aYG/2 e_s ,$$

it can be expressed as

$$\frac{dp}{dy} = \frac{\frac{b}{8}y^2 - bp + Q(p,y)}{p} , \quad (80)$$

where

$$p = \frac{dy}{dt}$$

and

$$Q(p,y) = -b/2 y^2 - 2yp .$$

Equation (80) has a point of equilibrium at

$$y = 0$$

and

$$p = 0 .$$

The nature of the equilibrium point is determined by first finding the characteristic equation. This is

$$S + (b)S + \frac{b^2}{8}S = 0 .$$

The roots are then

$$S_1, S_2 = -b/2 \pm \sqrt{\frac{b^2}{4} - \frac{b^2}{8}}$$

and are negative real. Thus the equilibrium point<sup>25</sup> of (80) is a stable node, and equation (79) is stable when

$$e_{sa} = 0 .$$

The characterizing equations of Fig. 13 and 14 do not have isolated singularities, and hence the nature of their singularities are not known.

Theoretical comparison of linear and nonlinear system.--Three nonlinear systems whose characterizing equations are of the Type II Case (c) form have been synthesized in this chapter. They are

$$u'' + \left( \frac{abk_f Y}{2} u + b \right) u' + \frac{ab^2 k_f}{4} Y u^2 = be_s$$

$$u'' + \left( \frac{ab}{2} k_f Y u + b \right) u' + \frac{ab^2}{4} k_f Y u^2 = 2be_s$$

and

$$u'' + \left( \frac{k_f ab}{4} Y u + b \right) u' + \frac{b^2}{4} u + \frac{ab^2}{16} k_f Y u^2 = be_s .$$

With the parameter values of

$$Y = 10$$

$$k_f = .01$$

$$a = 43.6$$

and

$$b = 4.07 ,$$

these equations became

$$u'' + (8.85u + 4.07)u' + 18u^2 = 4.07e_s \quad (81)$$

$$u'' + (8.85u + 4.07)u' + 18u^2 = 8.14e_s \quad (82)$$

and

$$u'' + (4.425u + 4.07)u' + 4.15u + 4.54u^2 = 4.07e_s \quad (83)$$

and the linear system has the characterizing equation

$$u'' + 4.07u' + 4.36u = 4.07e_s .$$

The solutions to these four equations are available, but it is more convenient to obtain their response by means of an analogue computer. In Fig. 16 is shown the computer arrangement for obtaining these responses, which are shown in Figs. 17 to 20. (The response to equation (81) was not obtained since it is not truly a second order differential equation.) In Fig. 21 is shown a comparison between the two nonlinear systems and the linear system. This shows that, for the value of input signal indicated, the Type II Case (c) characterizing equation with a stable isolated singularity has a faster response than the linear system by a factor of 2 to 1. The Type II Case (c) characterizing equation with a bias signal has a response that is not significantly better than the linear system for the indicated input signal.

In an actual control system it would not be desirable to utilize the Type II Case (c) characterizing equation with the bias signal because of the inability of the restoring force to change sign. If the controller should have a d.c. drift in the wrong direction, the system would be unable to correct for this and would become unstable. The Type II Case (c)

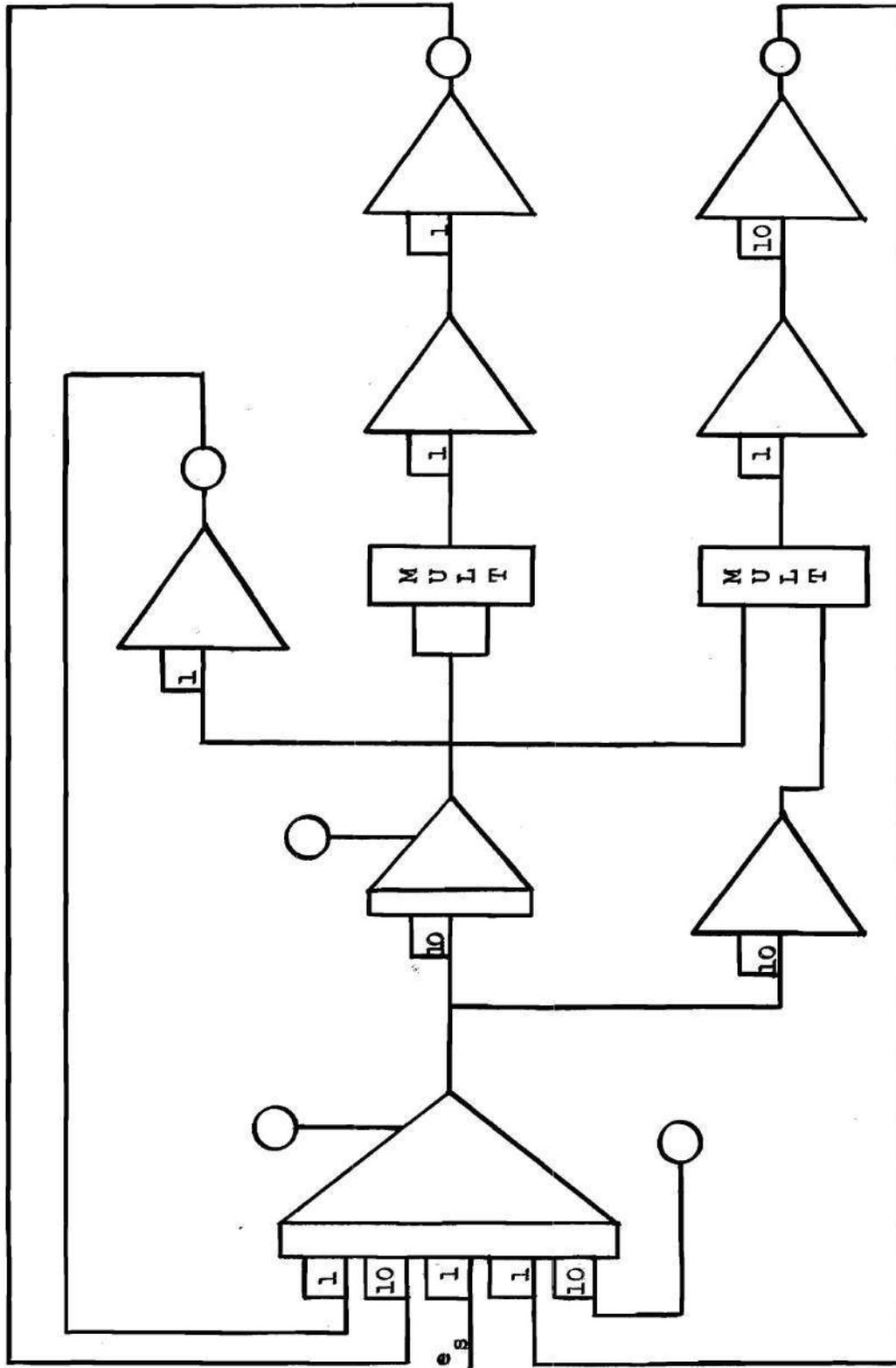


Figure 16. Computer Arrangement for Obtaining the Response of the Type II Case (c) System

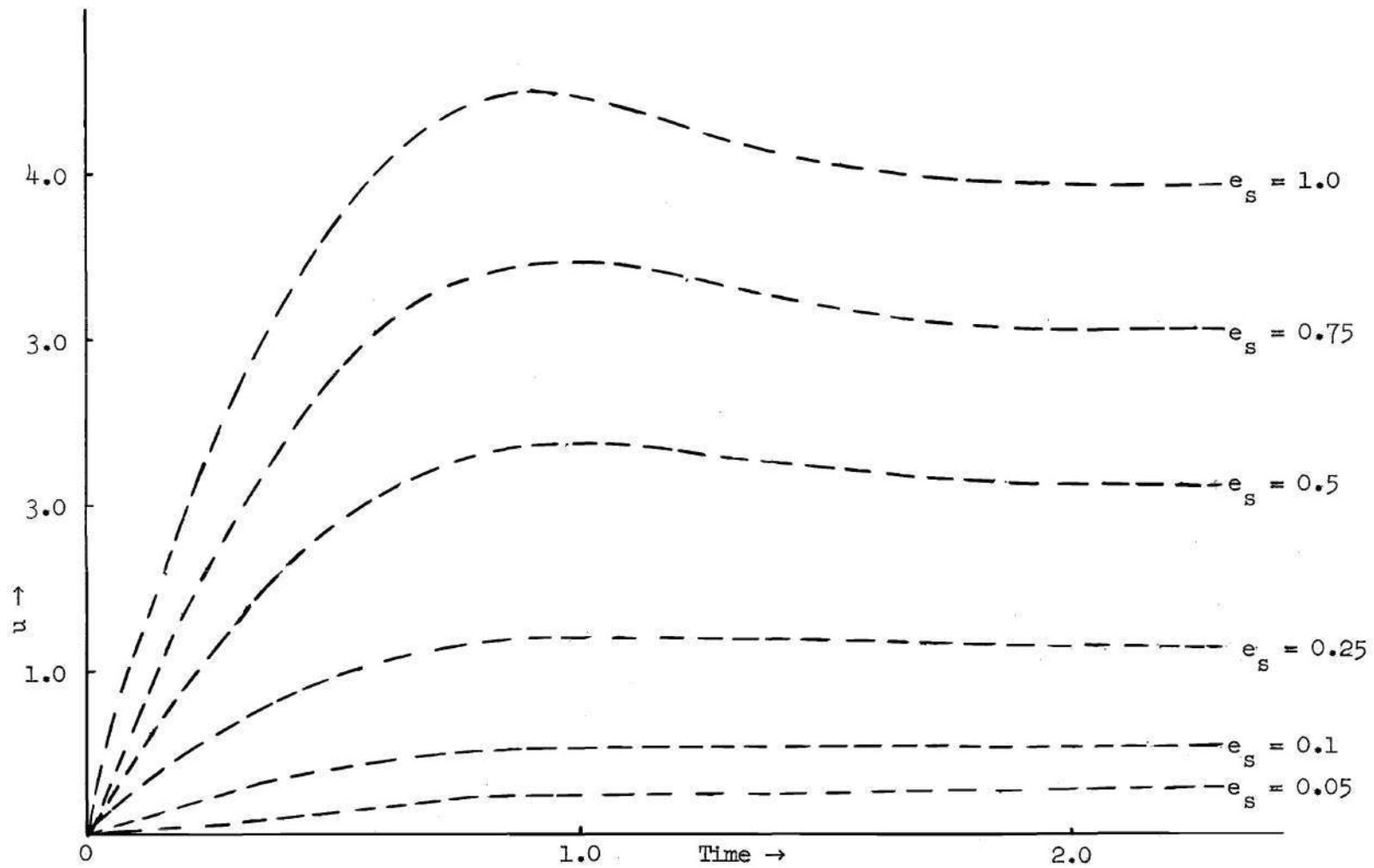


Figure 17. Type II Case (c) System with Stable Singularity

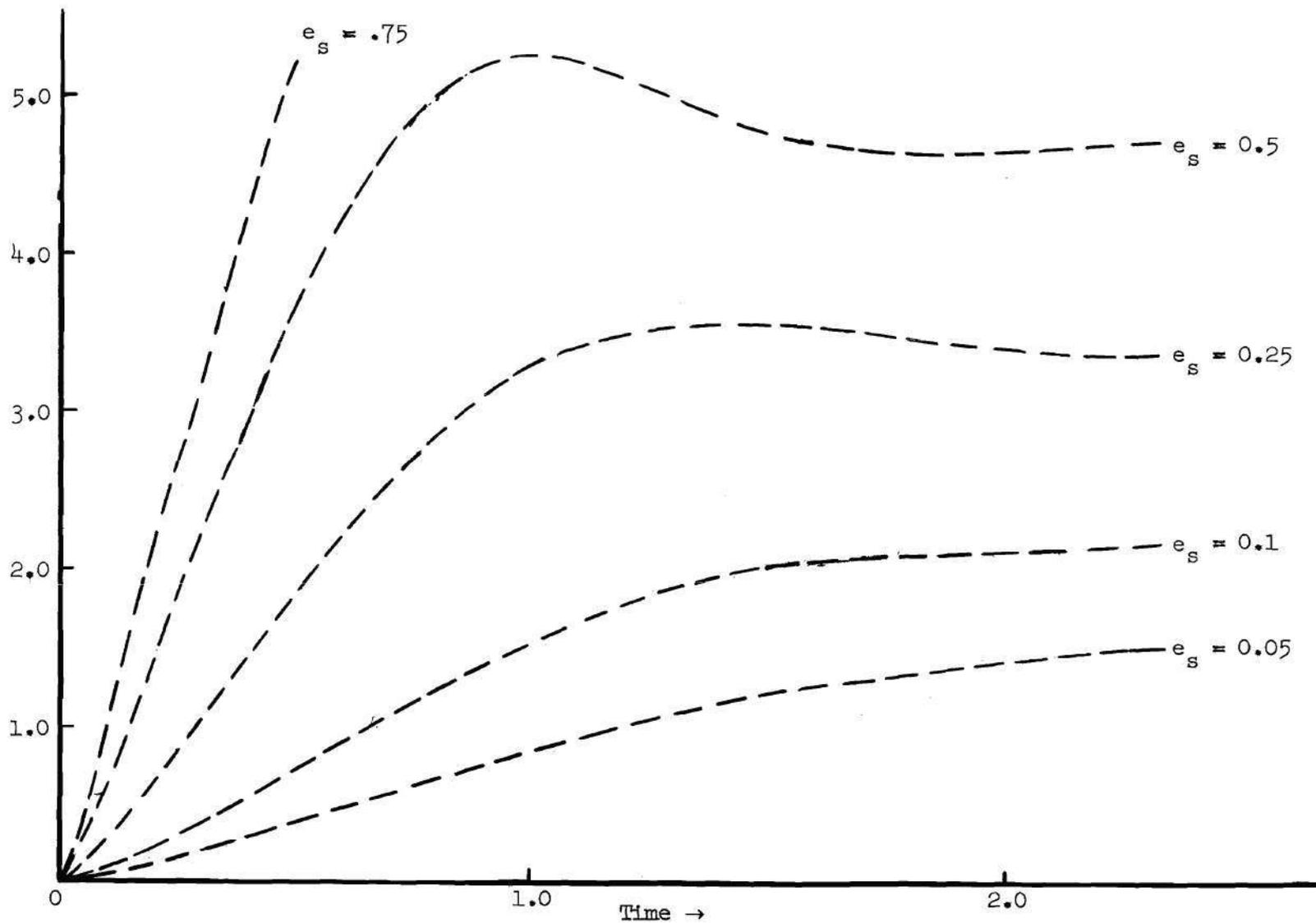


Figure 18. Response of Type II Case (c) System with Bias Signal

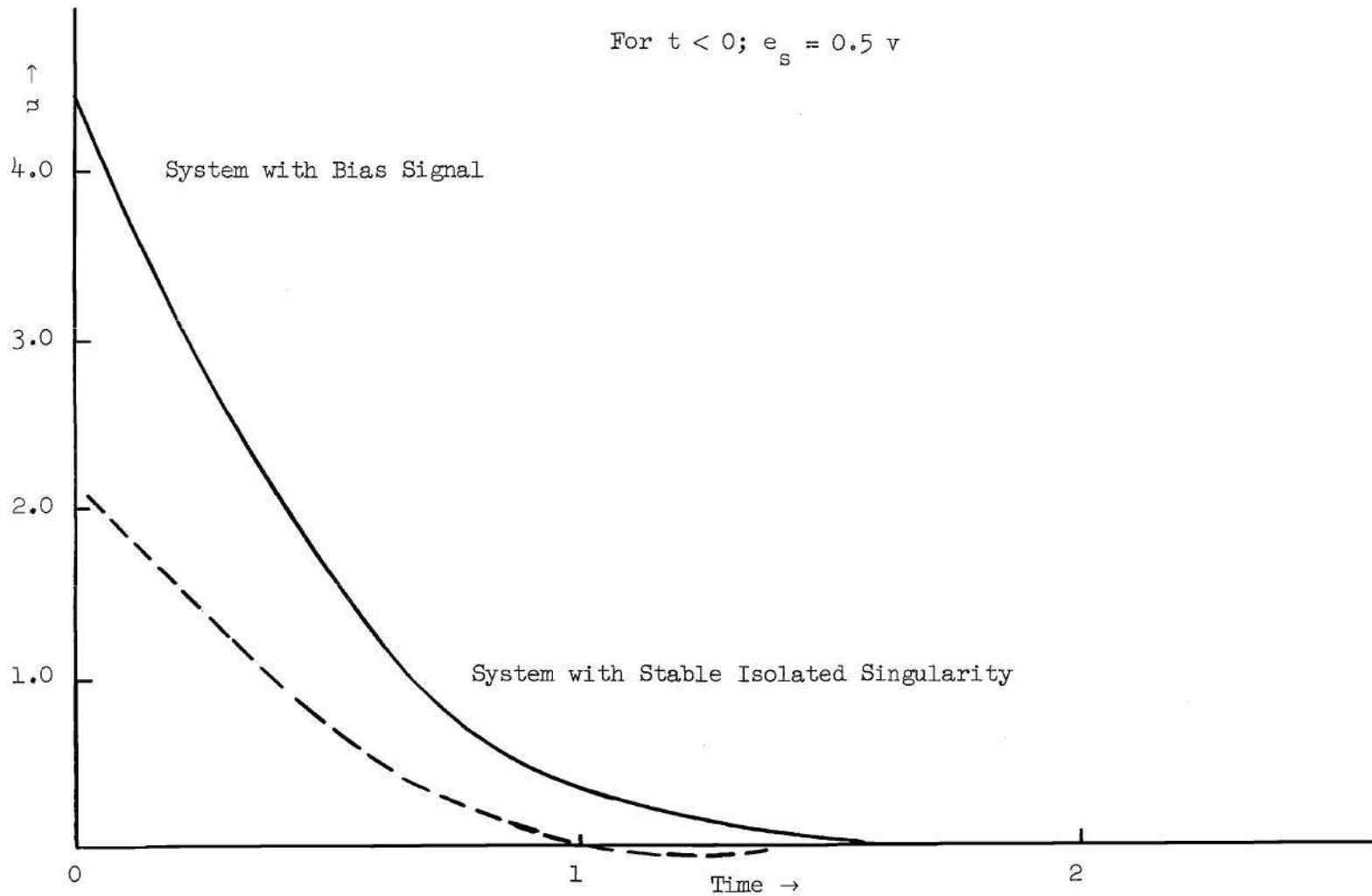


Figure 19. Comparison of System with Bias Signal and System with Stable Isolated Singularity for  $e_s = 0$ .

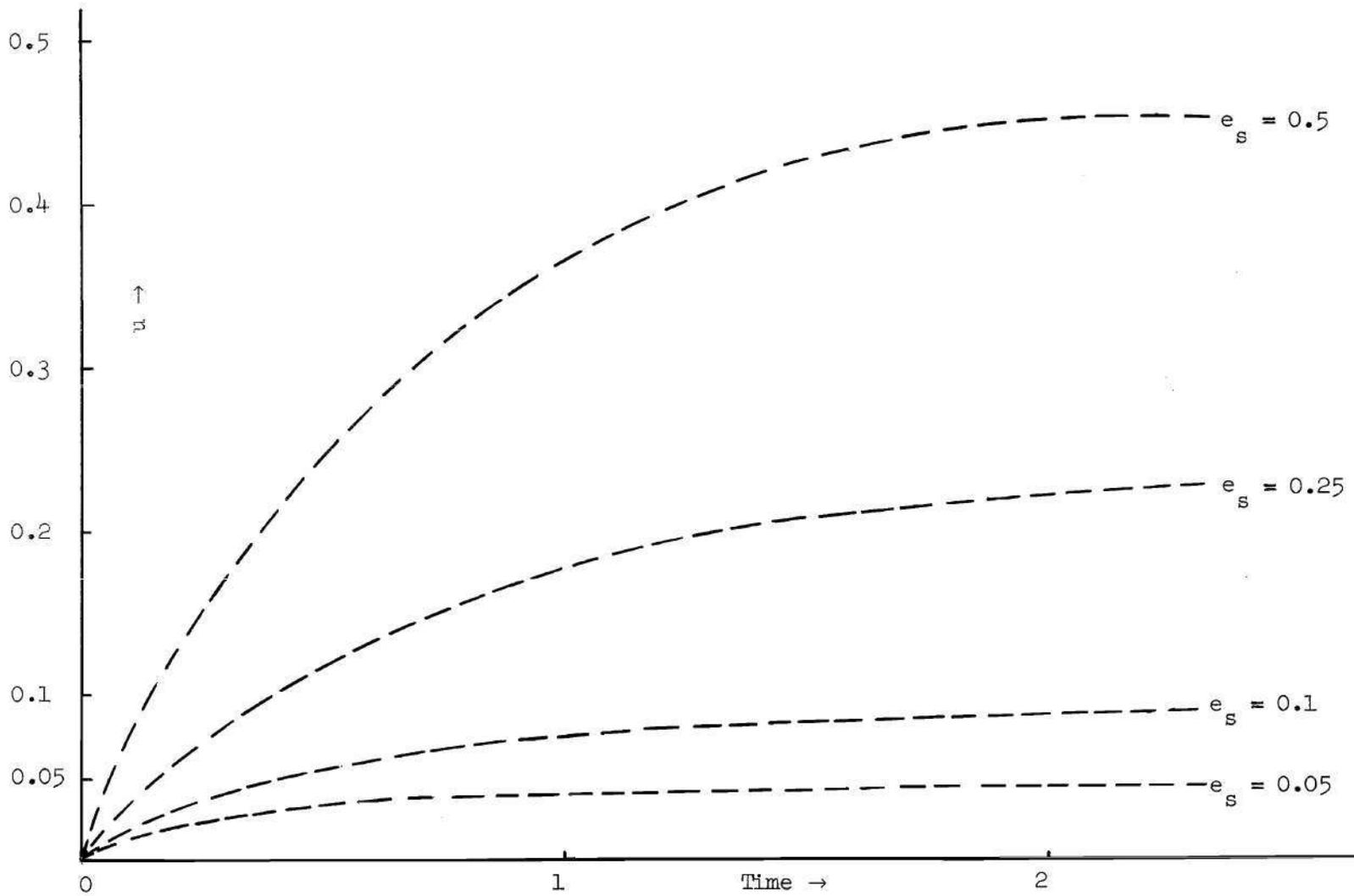


Figure 20. Response of Linear System

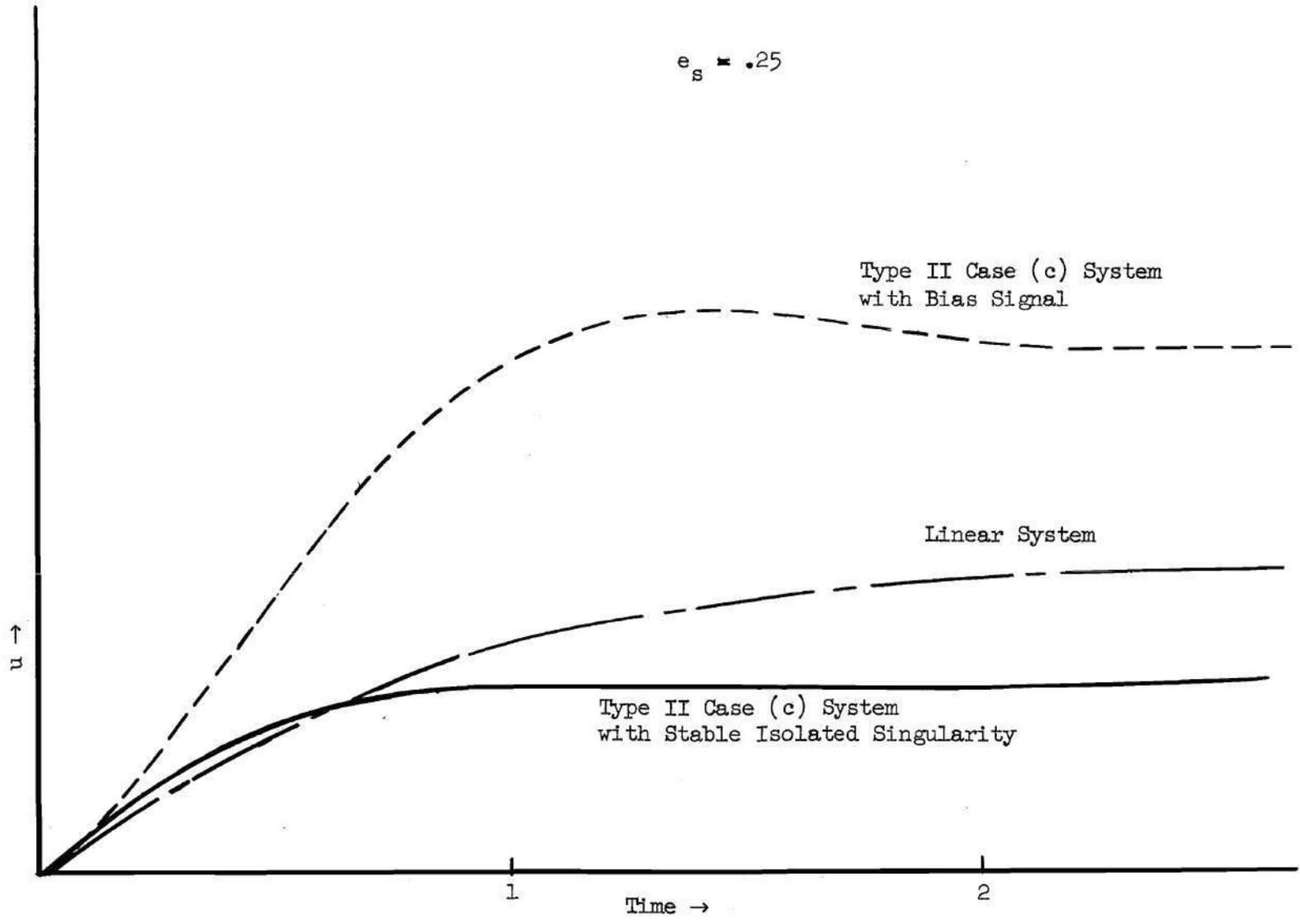


Figure 21. Comparison of Linear and Nonlinear Systems

system with a stable isolated singularity can only correct for a limited amount of drift, but it would still have more control than the system with a bias signal. It should also be emphasized that the nonlinear system does not react faster than the linear system for all values of input signal, since its response is a semi-transcendental function of the constants of integration. Also, the nonlinear system is asymmetrical and can only control for values of input signal where

$$F \geq 0 .$$

Realization of a Type II Case (c) characterizing equation with a stable isolated singularity.--To realize a Type II Case (c) system with a stable isolated singularity, the linear first order system of Fig. A-6 was utilized with an analogue computer serving as the nonlinear controller. The nonlinear controller arrangement is shown in Fig. 22. When the actual system was tested, it was found that signals smaller than one volt were not sufficient to run the motor because of the quiescent frictional force of the brushes. In Fig. 23 is shown the theoretical response of the system for signals larger than one volt. Fig. 24 shows the actual response of the system. In Figs. 24 and 25 are shown the speed responses of the linear and nonlinear system.

The most serious practical difficulty was due to drift in the electronic multiplier that produced the squared term of equation (83). To the system this drift was the same as an initial condition. The other difficulty was the limited range of input signal over which the linear system was truly linear. This region was restricted to a fairly narrow range of one to five volts. One distinct advantage of the nonlinear

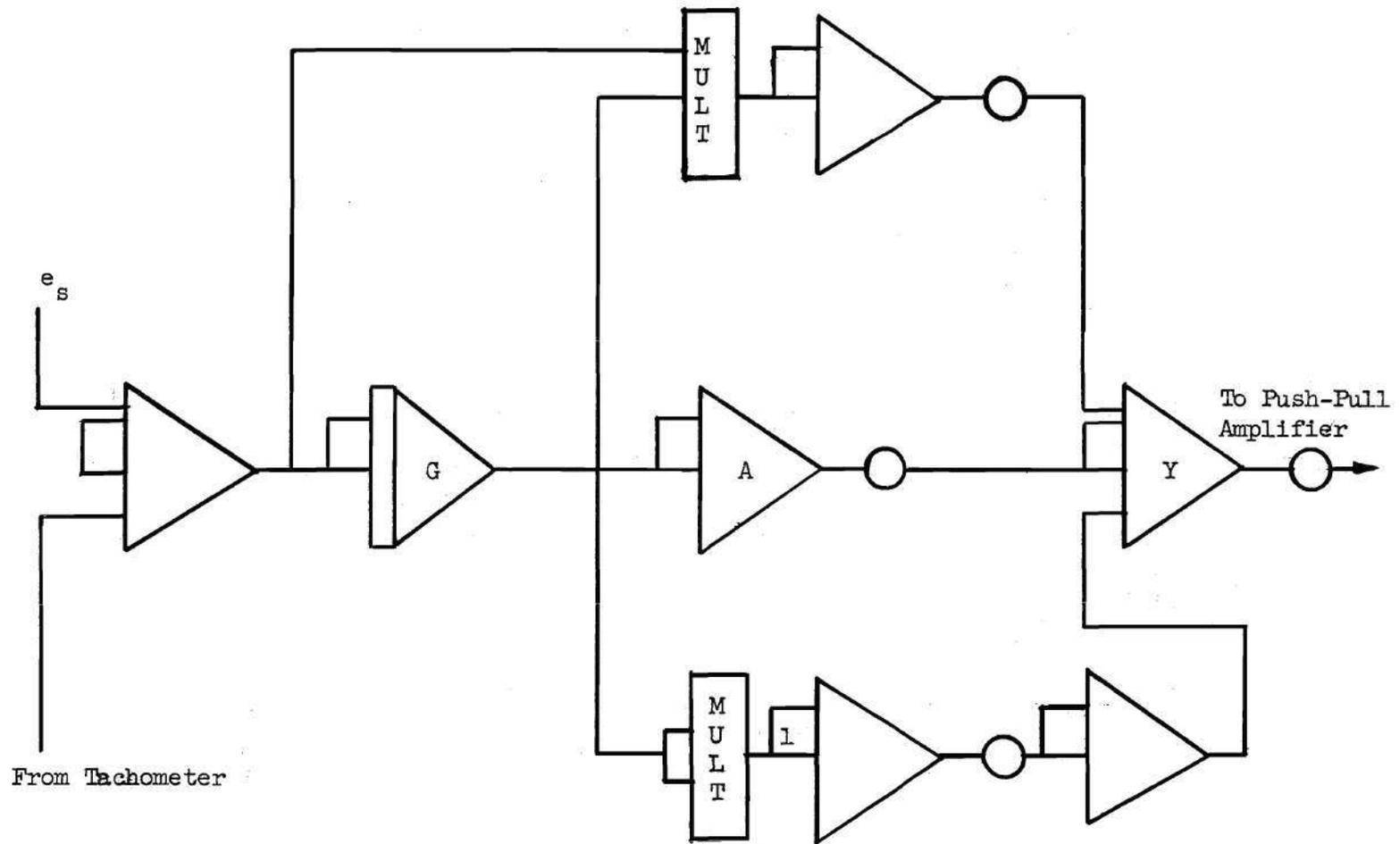


Figure 22. Nonlinear Controller for Type II Case (c) System with Stable Isolated Singularity

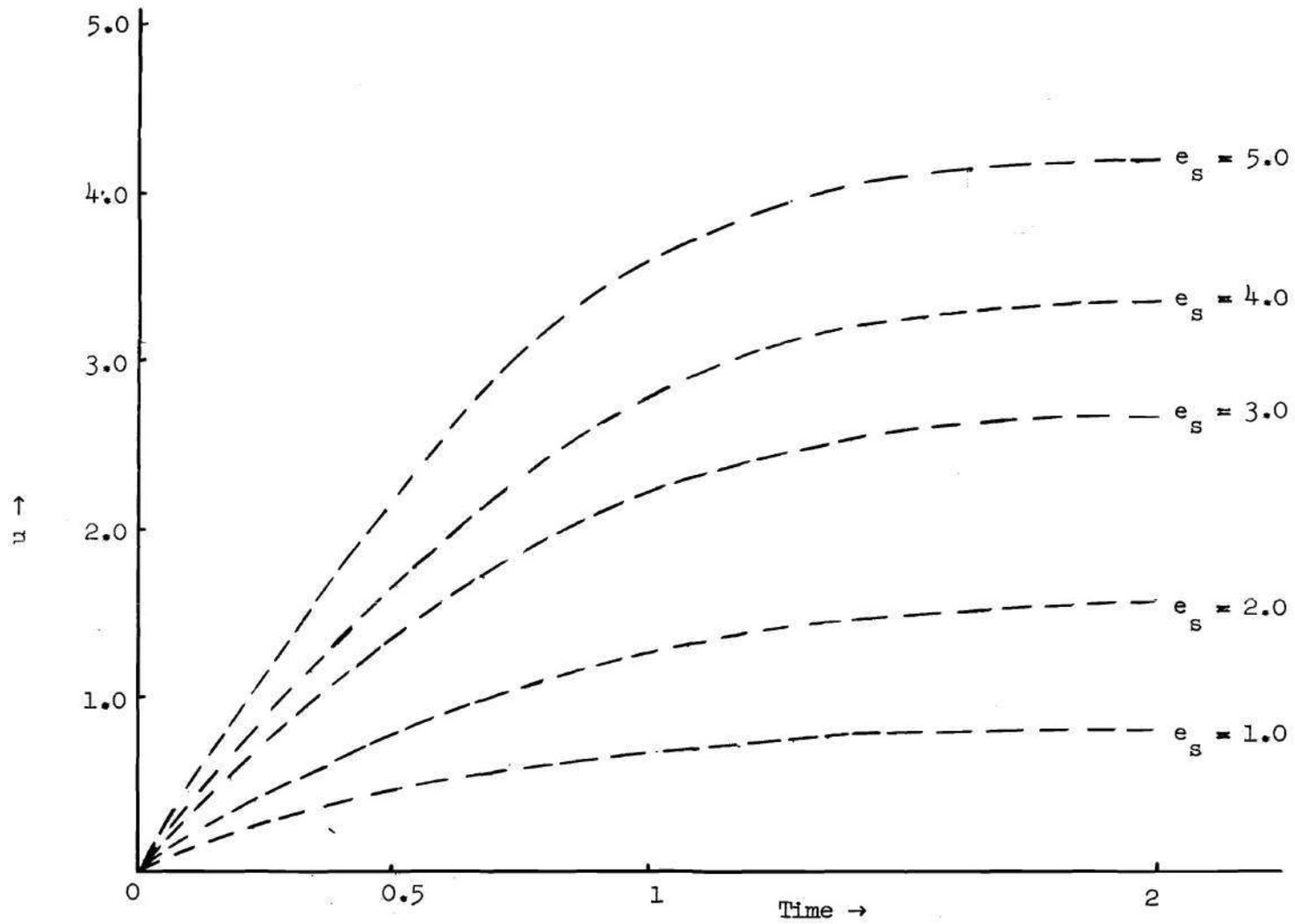


Figure 23. Response of Physical Linear System

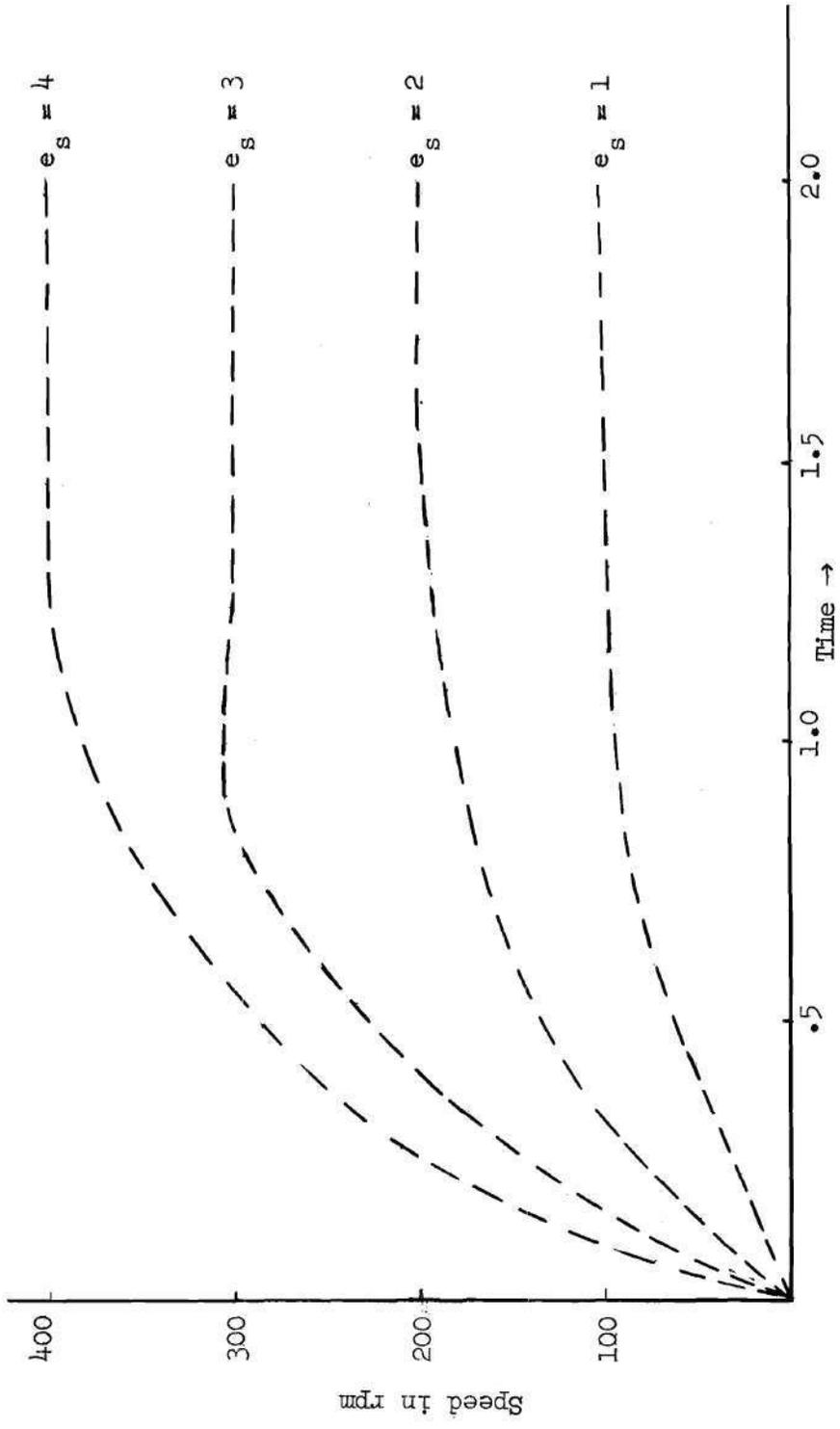


Figure 24. Theoretical Response of Linear System

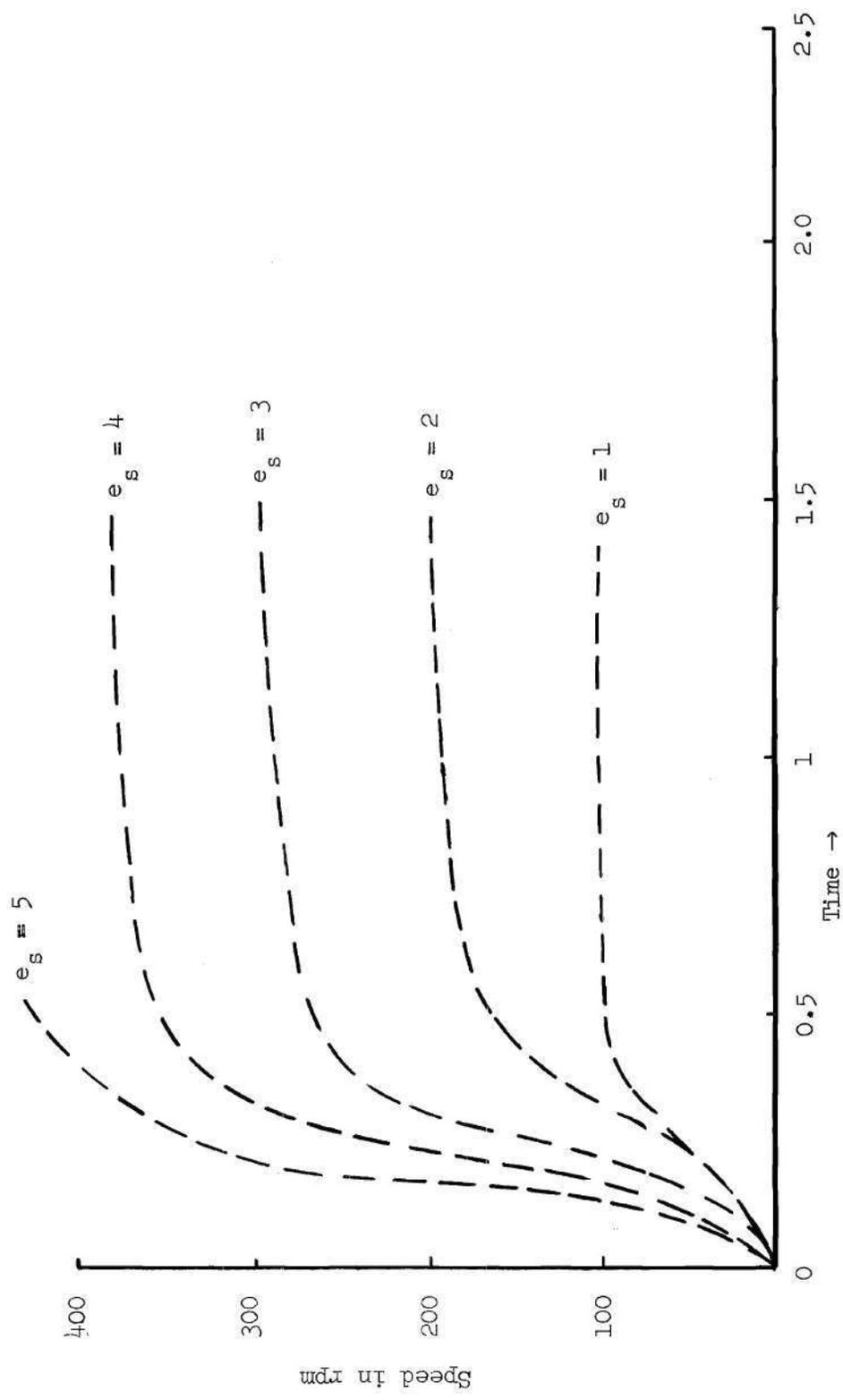


Figure 25. Response of Nonlinear System

system is the high initial force which breaks the brush friction much faster than does the linear system. This is due to the damping term  $G_{uu}'$ .

The linear term cannot be increased indefinitely, for from equation (74) it is seen that

$$A = (b - 2G) \frac{2}{aYk_f} .$$

This equation also places the restriction on  $G$  of

$$G < b/2 .$$

In the system tested it was found that the optimum system response occurred for parameters other than those necessary for a Type II Case (c) system.

For control purposes, the chief disadvantage of this system is that it is asymmetrical and is good for only one polarity of an input signal. If the signal goes negative and the polarity of the squared term is reversed, the characterizing equation is then

$$y'' + [-2G + P]y' - Py^2 = F$$

and will be stable. However, the damping will decrease with an increase in  $y$ . A better procedure would be to alter the equation to

$$y'' + [2|y| + P]y' + P|y|y = F . \quad (84)$$

Equation (84) is no longer a Type II Case (c) system now, however, but it is symmetrical.

## CHAPTER VII

### TYPE II CASE (d) SYSTEMS

$$A = -1; C = 1$$

Response in the complex domain.--The form of the Type II Case (d) differential equation is<sup>26,27</sup>

$$w'' = [-w + B]w' + w^3 + Dw^2 + Ew + F. \quad (85)$$

By a linear substitution of the form

$$w(z) = W(z) + \mu(z),$$

equation (85) can be expressed as

$$W'' = -W W' + W^3 + a[3W' + W^2] + RW + S, \quad (86)$$

where

$$3\mu + D = a$$

$$-\mu + B = 3a$$

$$R = 3\mu^2 + 2D\mu + E$$

and

$$S = \mu^3 + D\mu^2 + E\mu + F.$$

It has been shown that there are three cases of equation (86) which admit to a solution with fixed critical points. These three cases will now be examined individually.

case 1.  $a = R = S = 0$ . Equation (86) now becomes

$$W'' + W W' - W^3 = 0, \quad (87)$$

and has the solution

$$W = \frac{\wp'(z + C_1; 0, C_2)}{\wp(z + C_1; 0, C_2)}, \quad (88)$$

where  $C_1$  and  $C_2$  are constants of integration. This is a doubly periodic function with fixed critical points but with movable poles. The poles occur at values of the argument where

$$z = -C_1.$$

case 2.  $R \neq 0$ ,  $a = S = 0$ . Equation (86) now becomes

$$W'' + W W' - RW - W^3 = 0, \quad (89)$$

which has the solution

$$W = \frac{1}{2} \sqrt{-R/3} \frac{\wp'(u; 12, C_1)}{\wp(u; 12, C_1) - 1}, \quad (90)$$

where

$$u = \frac{3}{2} \sqrt{-R/3} + C_2,$$

and  $C_1$  and  $C_2$  are constants of integration. This is a doubly periodic function with fixed critical points but with movable poles. The poles occur at values of the argument where

$$z = -2 C_1 \sqrt{-3 R}$$

or

$$\wp(u; 12, C_1) = 1, \quad (91)$$

provided equation (91) has no common factors with the numerator of equation (90).

Case 3.,  $S = 0$ ,  $R = -2a^2$ . Equation (86) now becomes

$$W'' + (W + 3a) - W^3 + 2a^2 W = 0 \quad (92)$$

and has the solution

$$W = C_1 \epsilon^{-az} \frac{\wp'(u; 0, 1)}{\wp(u; 0, 1)}, \quad (93)$$

where

$$u = C_1 a \epsilon^{-az} + C_2,$$

and  $C_1$  and  $C_2$  are constants of integration. This is again a doubly periodic function with fixed critical points but with movable poles. There are an infinite number of simple poles at values of the argument where

$$z = -\frac{1}{a} \ln \left( -\frac{C_1}{C_2} a \right).$$

Response in the real domain.--As a function of a real variable  $y(t)$ , equation (85) becomes

$$y'' = (-y + B)y' + y^3 + Dy^2 + Fy + F . \quad (94)$$

Unfortunately, from the standpoint of the purpose of this thesis, the three cases of equation (94) which have solutions with fixed critical points in the complex plane do not have a constant term. Thus this system is not suitable for the characterizing equation of a physical system. Furthermore, cases 1 and 2 give rise to solutions which are doubly periodic and hence unstable.

For case 3, equation (94) can be expressed as

$$y'' + (y + a)y' - y^3 + ay^2 + 2a^2y = 0 , \quad (95)$$

which has the solution

$$y = c_1 \frac{\epsilon^{-at} \wp'(u; 0, 1)}{\wp(u; 0, 1)} , \quad (96)$$

where

$$u = \frac{c_1}{a} \epsilon^{-at} + c_2 .$$

The constants of integration  $c_1$  and  $c_2$  can be determined from the relationships

$$y(0) = c_1 \frac{\wp'(\frac{c_1}{a} + c_2; 0, 1)}{\wp(\frac{c_1}{a} + c_2; 0, 1)}$$

and

$$y'(0) = -ay(0) - c_1^2 \left[ \frac{\varphi''\left(\frac{c_1}{a} + c_2; 0, 1\right)}{\varphi^2\left(\frac{c_1}{a} + c_2; 0, 1\right)} - \frac{y^2(0)}{c_1^2} \right] .$$

This system becomes unstable if

$$\frac{c_1}{a} e^{-at} = -c_2 .$$

As an alternate approach to the question of stability, consider equation (95) in the phase plane with the characterizing equation

$$\frac{dp}{dy} = - \frac{2a^2y - ap + Q(p,y)}{p} ,$$

where

$$p = \frac{dy}{dt}$$

and

$$Q(p,y) = y^3 - ay^2 - yp .$$

The characteristic equation is

$$s^2 + as + 2a^2 = 0 ,$$

and hence the roots which are

$$s_1, s_2 = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 2a^2}$$

are negative imaginary, and the equilibrium point at the origin of the phase plane is a stable spiral point. The response of equation (95) is thus stable, provided the initial conditions are such that

$$y(t) > -a$$

for all values of  $t$ .

The characterizing equation

$$y'' + (y + a)y' - y^3 + ay^2 + 2a^2y = F \quad (97)$$

satisfies only the necessary conditions for the solution to have fixed critical points in the complex plane, and no solution is available. The nonlinear restoring force  $-y^3 + ay^2 + 2a^2y$  causes the system to become unstable if the amplitude  $F$  of the input signal is such that  $y(t) \geq a$ . This system is then stable for small displacements and is also an asymmetrical control system. When  $F = 0$ , the solution has fixed critical points in the complex plane, and the system is stable if the initial conditions are such that

$$y(t) > -a.$$

Physical system with a Type II Case (d) characterizing equation.--In

Fig. 26 is shown a physical system with a Type II Case (d) characterizing equation. The equations of motion may be expressed as

$$u = \int_0^t (e_s - k_f \omega) dt$$

and

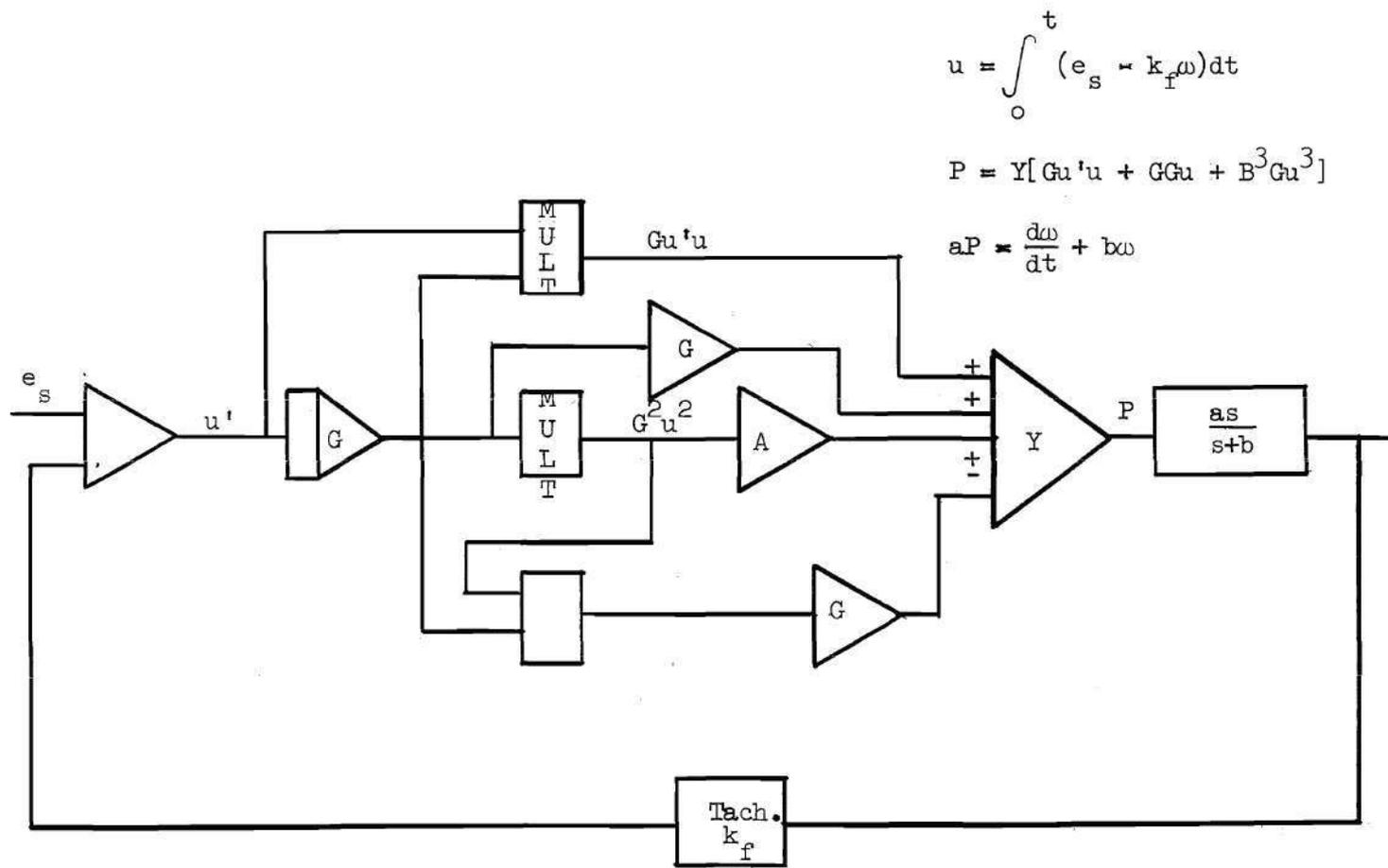


Figure 26. Physical System with Type II Case (d) Characterizing Equation

$$k_f a y [G u' u + A G^2 u^2 - B G^3 u^3 + C G u] = -u'' + b[e_s - u'] . \quad (98)$$

With the substitutions

$$u = \lambda y ,$$

$$\lambda = 1 Y / K_f a G ,$$

$$B = k_f a Y / G ,$$

$$A = b / G ,$$

$$C = 2b^2 / K_f a Y G ,$$

and

$$e_s = 0 ,$$

equation (98) becomes the Type II Case (d) characterizing equation

$$y'' + y'[y + b] - y^3 + by^2 + 2b^2 y = 0 . \quad (99)$$

The initial conditions are

$$y'(0) = k_f a G Y e_{sb}$$

and

$$y(0) = T$$

where T is the smallest positive root of

$$-T^2 + bT^2 + 2b^2T = \lambda b e_{sb} ,$$

and  $e_{sb}$  is the value of input signal before the transient response starts.

The solution to equation (99) is given by equation (96).

Since this system is unsuitable for control purposes, it will not be realized or studied further in this thesis.

## CHAPTER VIII

### TYPE II CASE (e) SYSTEMS

$$A = -3; C = -1$$

Response in the complex domain.--The general form of the Type II Case (e) system is<sup>28</sup>

$$w'' + (-3w + B)w' - w^3 + Dw^2 + Ew + F . \quad (100)$$

It has been shown that for equation (100) to have a solution with fixed critical points, it must be in the form

$$w'' + 3w w' + w^3 = B[w' + w^2] + Ew + F . \quad (101)$$

Equation (101) has the solution

$$w = \frac{u'}{u} ,$$

where  $u$  is the solution of the linear differential equation

$$u'' + Bu' + Eu + Fu = 0 . \quad (102)$$

If  $p_1, p_2, p_3$  are the roots of the characteristic equation of equation (102)

$$p^3 - Bp^2 - Ep - F = 0 , \quad (103)$$

then the solution to equation (101) is

$$w = \frac{C_1 p_1 \epsilon^{p_1 z} + C_2 p_2 \epsilon^{p_2 z} + p_3 \epsilon^{p_3 z}}{C_1 \epsilon^{p_1 z} + C_2 \epsilon^{p_2 z} + \epsilon^{p_3 z}}. \quad (104)$$

The response is then a semi-transcendental function of the constants of integration  $C_1$  and  $C_2$ . (It is assumed that there are no multiple roots of equation (104)).

For multiple roots, equation (104) becomes

$$w = \frac{\epsilon^{p_1 z} (C_1 p_1 + C_2 + p_1 C_2 z) + p_2 \epsilon^{p_2 z}}{(C_1 + C_2 z) \epsilon^{p_1 z} + \epsilon^{p_2 z}} \quad (105)$$

or

$$w = \frac{C_2 + p_1 C_1 + z(C_2 p_1 + 2) + p_2 z^2}{C_1 + C_2 z + z^2}. \quad (106)$$

Equations (104) and (105) are semi-transcendental functions of the constants of integration, whereas equation (106) is an algebraic function of its constants. Each of the three equations has critical points which are fixed but poles which are movable. If the numerator has no common factors with the denominator, these poles occur at values of the argument such that

$$C_1 \epsilon^{(p_1 - p_3)z} + C_2 \epsilon^{(p_2 - p_3)z} + 1 = 0 \quad (107)$$

$$(C_1 + C_2 z) \epsilon^{(p_1 - p_2)z} + 1 = 0 \quad (108)$$

$$C_1 + C_2 z + z^2 = 0 . \quad (109)$$

In equation (109), these poles occur at

$$z = -\frac{C_2}{2} \pm \sqrt{\frac{C_2^2}{4} - C_1} ,$$

provided

$$-\frac{pC_1 + C_2}{C_2 + 2} \neq -\frac{C_2}{2} \pm \sqrt{\frac{C_2^2}{4} - C_1} .$$

It is seen that a pole can occur at a positive real value of  $z$ , depending upon the value of  $C_1$  and  $C_2$ .

In equations (107) and (108), a sufficient condition for the absence of positive real poles is

$$C_1 > 0$$

and

$$C_2 > 0 .$$

Response in the real domain.--As a function of a real variable  $y(t)$ , equation (101) becomes

$$y'' + [3y + B]y' + y^3 + By^2 + Ey = F . \quad (110)$$

The presence of fixed critical points does not depend upon the sign of the parameters  $B$  and  $E$ ; but, since this equation is ultimately to be used as the characterizing equation of a physical system,  $B$  and  $E$  have been

chosen negative.  $F$  will be positive or negative, depending upon the direction of the input signal. The solutions to equation (110) are given by equations (104) to (106) with  $\omega$  replaced by  $y$  and  $z$  replaced by  $t$ .

For a stable physical system, the roots of the characteristic equation (104) must not be pure imaginary. The characteristic equation (104) becomes

$$p^3 + Bp^2 + Ep - F = 0 \quad (111)$$

and cannot have a pure imaginary factor. The discriminant of (111) is

$$\Delta = -18BEF + 4B^3F + B^2E^2 - 4E^3 - 27F^2,$$

and thus the locations of the roots vary with  $F$ . For  $F$  large there will be two conjugate imaginary-roots and a single real-root. For  $F$  small the location of the roots will be essentially determined by  $E^2[B - 4E]$ . For one value of  $F$ , there will be three real-roots. Hence, if  $B > 4E$ , the response will be analogous to that of the linear system, in that there are three types of responses depending upon the value of  $F$ , i.e.,

- |                     |                   |
|---------------------|-------------------|
| (1) Overdamped      | ( $F$ small).     |
| (2) Critical damped | ( $\Delta = 0$ ). |
| (3) Underdamped     | ( $F$ large).     |

For unequal roots and the system starting from rest, the initial conditions are related to the constants of integration by

$$0 = C_1p_1 + C_2p_2 + p_3$$

and

$$y'(0) = \frac{C_1 p_1^2 + C_2 p_2^2 + p_3^2}{C_1 + C_2 + 1} .$$

By analogy with the linear system, the Type II Case (e) system has a nonlinear damping coefficient  $(3y + B)$  and a nonlinear restoring force  $y^3 + By^2 + Ey$  and hence will be stable for positive values of displacement  $y$ . However, if  $y$  goes negative the system can become unstable. The origin of the phase plane is a stable point of equilibrium, but, if the displacement is such that

$$y(t) \rightarrow T ,$$

where  $T$  is the solution of

$$-T^3 + BT^2 - ET = 0 ,$$

the system will become unstable. Thus the Type II Case (e) system is asymmetrical.

The Type II Case (e) system is the second system examined in this thesis whose solution has fixed critical points in the complex plane for all values of input amplitude  $F$ . Hence it shows promise for use as the characterizing equation of a physical system. However, it has the disadvantage that it is not stable for all negative values of  $F$  and is an asymmetrical control system. It will now be studied further by synthesizing a physical system to have such a response.

Physical system with a Type II Case (e) characterizing equation.---The equation of motion of the system shown in block diagram form in Fig. 27 may be expressed as

$$u = \int_0^t (e_s - k_f \omega) dt$$

and

$$+ k_f a Y [AGu + Gu'u + \frac{bG}{3}u^2 + \frac{ak_f}{9}G^2u^3] = u'' + b[e_s - u'] . \quad (112)$$

With the substitutions

$$u = \lambda y ,$$

$$YAGak_f = E ,$$

$$\lambda = 3/aGYk_f ,$$

$$B = b ,$$

and

$$F = YabGk_f e_s / 3 ,$$

equation (112) becomes the Type II Case (e) characterizing equation

$$y'' + [3y + b] + y^3 + by^2 + Ey = F .$$

(It is not necessary to have a linear term for this to be a Type II Case (e) system, but drift in the multiplier can cause the system to become unstable without this term.)

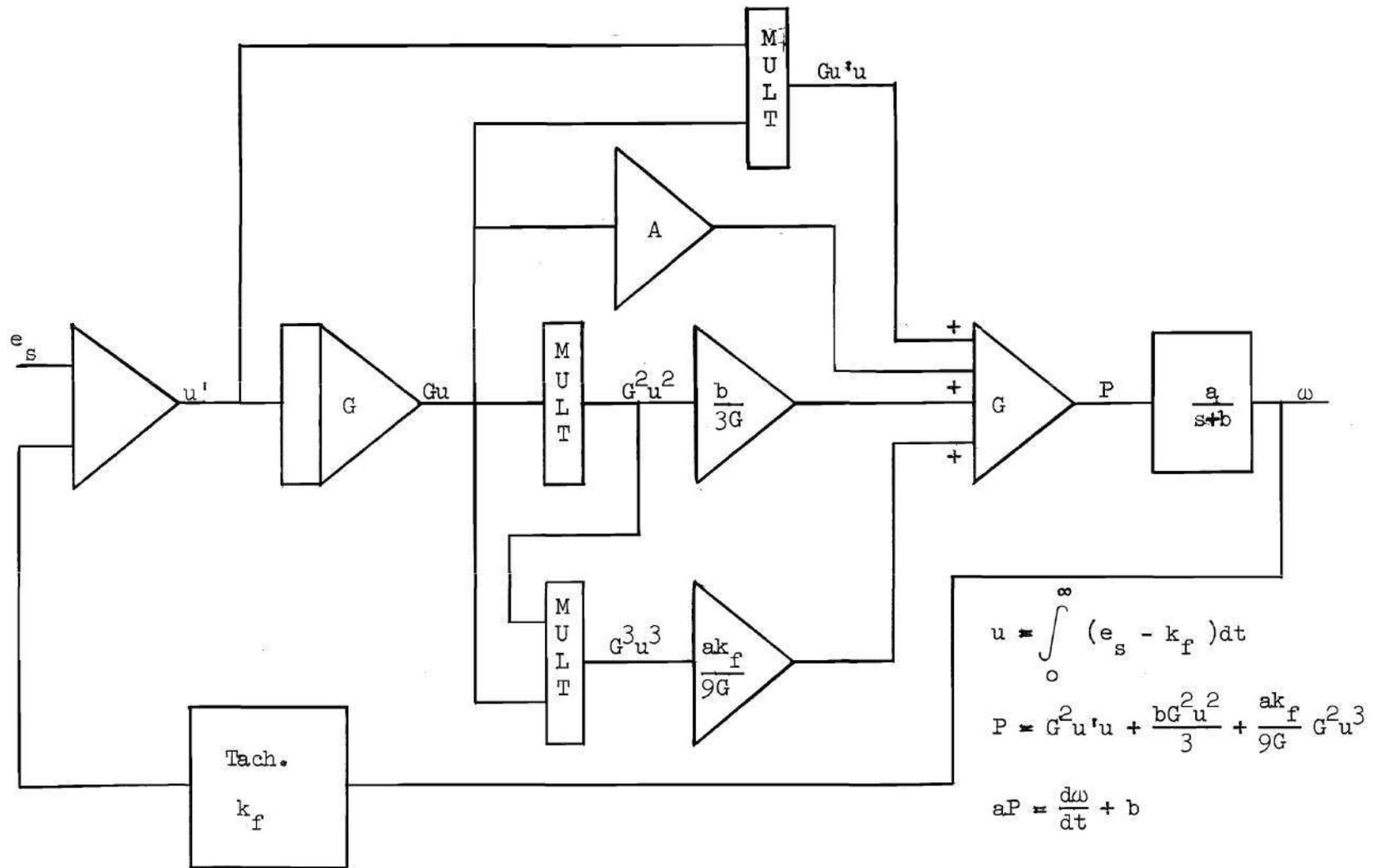


Figure 27. Physical System with Type II Case (e) Characterizing Equation

For the system starting from rest, the initial conditions are

$$y(0) = 0$$

and

$$y'(0) = \frac{aG^2 k_f}{3} e_s .$$

The solution is

$$u = \frac{3}{aG^2 k_f} \left[ \frac{C_1 p_1 \epsilon^{p_1 t} + C_2 p_2 \epsilon^{p_2 t} + p_3 \epsilon^{p_3 t}}{C_1 \epsilon^{p_1 t} + C_2 \epsilon^{p_2 t} + \epsilon^{p_3 t}} \right] ,$$

where  $p_1$ ,  $p_2$  and  $p_3$  are roots of

$$p^3 + Bp^2 + Ep - F = 0 ,$$

and it is assumed there are no multiple roots.

If the system is at a steady state value  $y_{s.s.}$  and the input signal is suddenly taken to zero, the initial conditions are

$$y(0) = y_{s.s.}$$

and

$$y'(0) = -e_s \frac{aG^3}{3} k_f ,$$

where  $e_s$  is the signal before the upset. The roots of the characteristic equation are now obtained from

$$p^3 + Bp^2 + Ep = 0$$

and are

$$p_1 = 0$$

and

$$p_2, p_3 = -B/2 \pm \sqrt{\frac{B^2}{4} - E}.$$

the response of the system will be of the form

$$u = \frac{3}{aG^2 k_f} \epsilon^{-B/2 t} \left[ \frac{C_2 p_2 \epsilon^{(\sqrt{B^2/4 - E})t} + p_3 \epsilon^{-(\sqrt{B^2/4 - E})t}}{C_1 + C_2 \epsilon^{(\sqrt{B^2/4 - E})t} + \epsilon^{-(\sqrt{B^2/4 - E})t}} \right]. \quad (113)$$

For

$$B^2 < 4E,$$

equation (113) may be rewritten as

$$u = \frac{3\epsilon^{-B/2 t}}{aG^2 k_f} \left[ \frac{C_2 p_2 \epsilon^{j(\sqrt{E - B^2/4})t} + p_3 \epsilon^{-j(\sqrt{E - B^2/4})t}}{C_1 + C_2 \epsilon^{j(\sqrt{E - B^2/4})t} + \epsilon^{-j(\sqrt{E - B^2/4})t}} \right],$$

where

$$C_2 = \frac{p_3}{p_2} \left[ \frac{y'(0) + y^2(0) - p_3 y(0)}{y'(0) + y^2(0) - p_2 y(0)} \right],$$

and

$$c_1 = -\frac{p_3}{p_2} \left[ \frac{y'(0) + y^2(0) - p_3 y(0)}{y'(0) + y^2(0) - p_2 y(0)} \right] \left[ \frac{p_2 - y(0)}{y(0)} \right] + \frac{p_3 - y(0)}{y(0)} .$$

Comparison of linear and nonlinear systems. ---With the system parameters adjusted to

$$k_f = .01 ,$$

$$b = 4.07 ,$$

$$a = 43.6 ,$$

$$G = 10 ,$$

$$Y = 1 ,$$

and

$$A = 1 ,$$

the Type II Case (e) characterizing equation becomes

$$u'' + [4.36u + 4.07]u' + 2.12u^3 + 5.91u^2 + 4.36u = 4.07e_s . \quad (114)$$

With

$$u = .686 y ,$$

equation (114) becomes

$$y'' + [3y + 4.07]y' + y^3 + 4.07y^2 + 4.36y = 5.92e_s .$$

The linear system has the characterizing equation

$$u'' + 4.07u' + 4.36u = 4.07e_s .$$

The analogue arrangement necessary to obtain the nonlinear response is shown in Fig. 31. The theoretical response of this system is shown in Figs. 32 to 37.

The solution to the Type II Case (e) system is in a form that is easier to utilize than the Type II Case (c) system; however, the response of this system is not significantly faster than the Case (e) system. Furthermore, since there is a cubic term present in the differential equation, it will be more difficult to realize. For this reason, the Type II Case (e) system was not realized, and attention was focused on realizing the Type II Case (c) system.

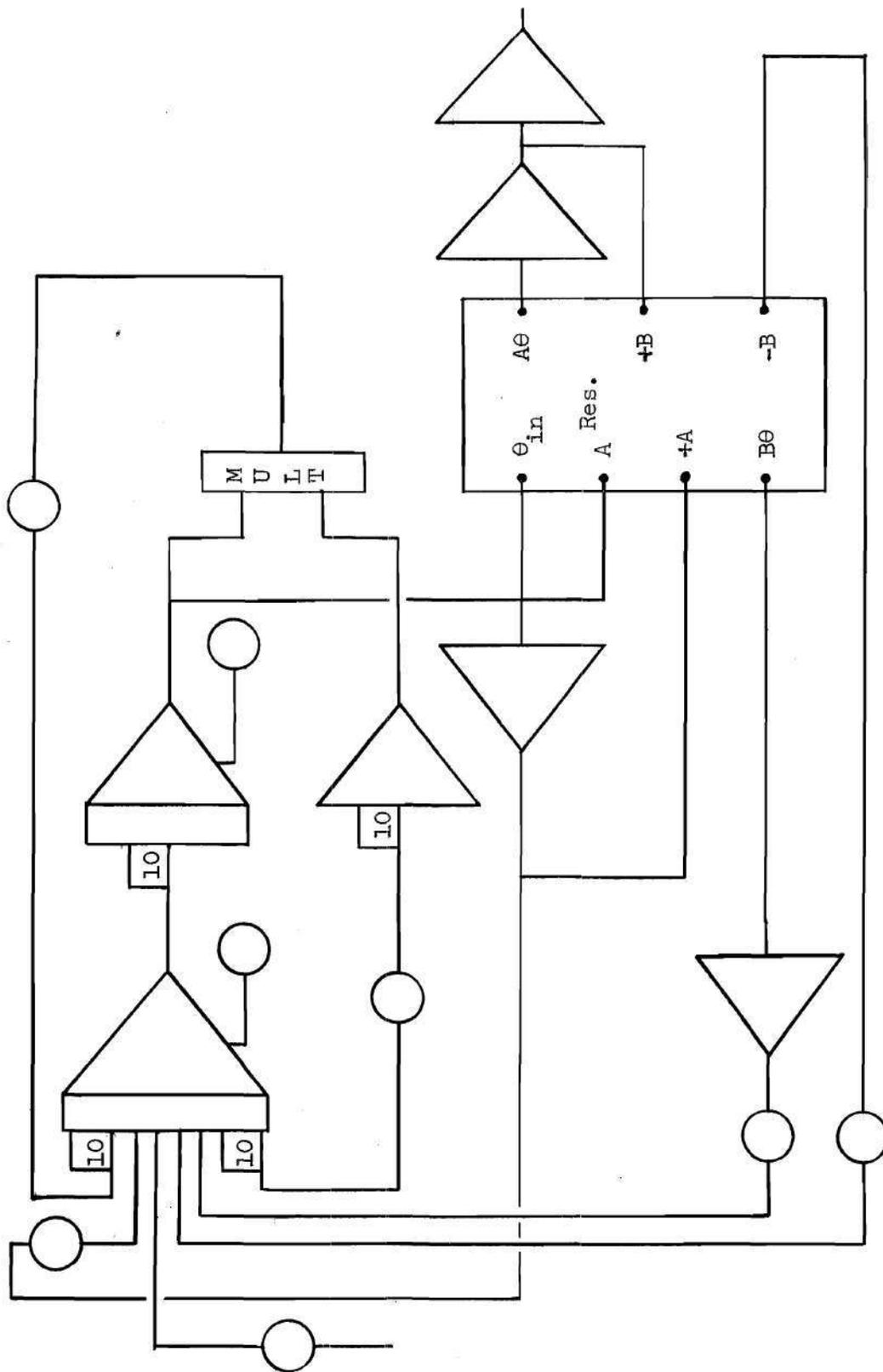


Figure 28. Computer Arrangement for Type II Case (e) System

Linear Term = 20

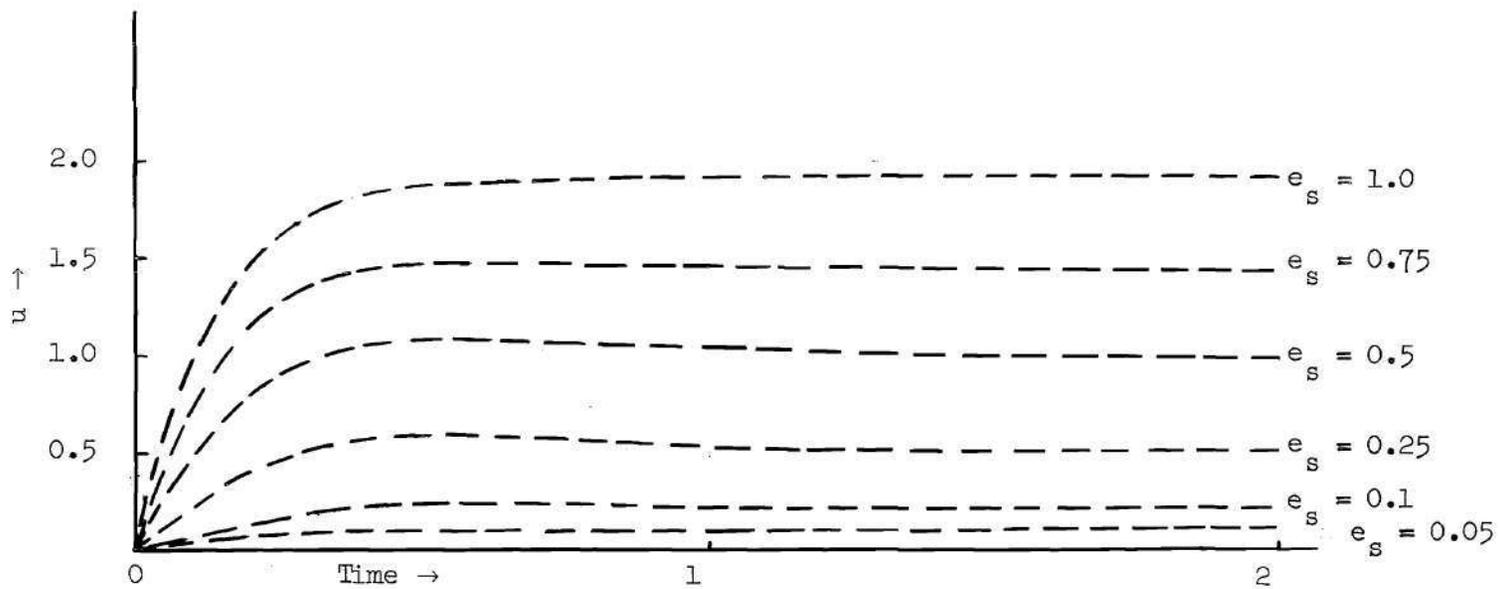


Figure 29. Response of Type II Case (e) System

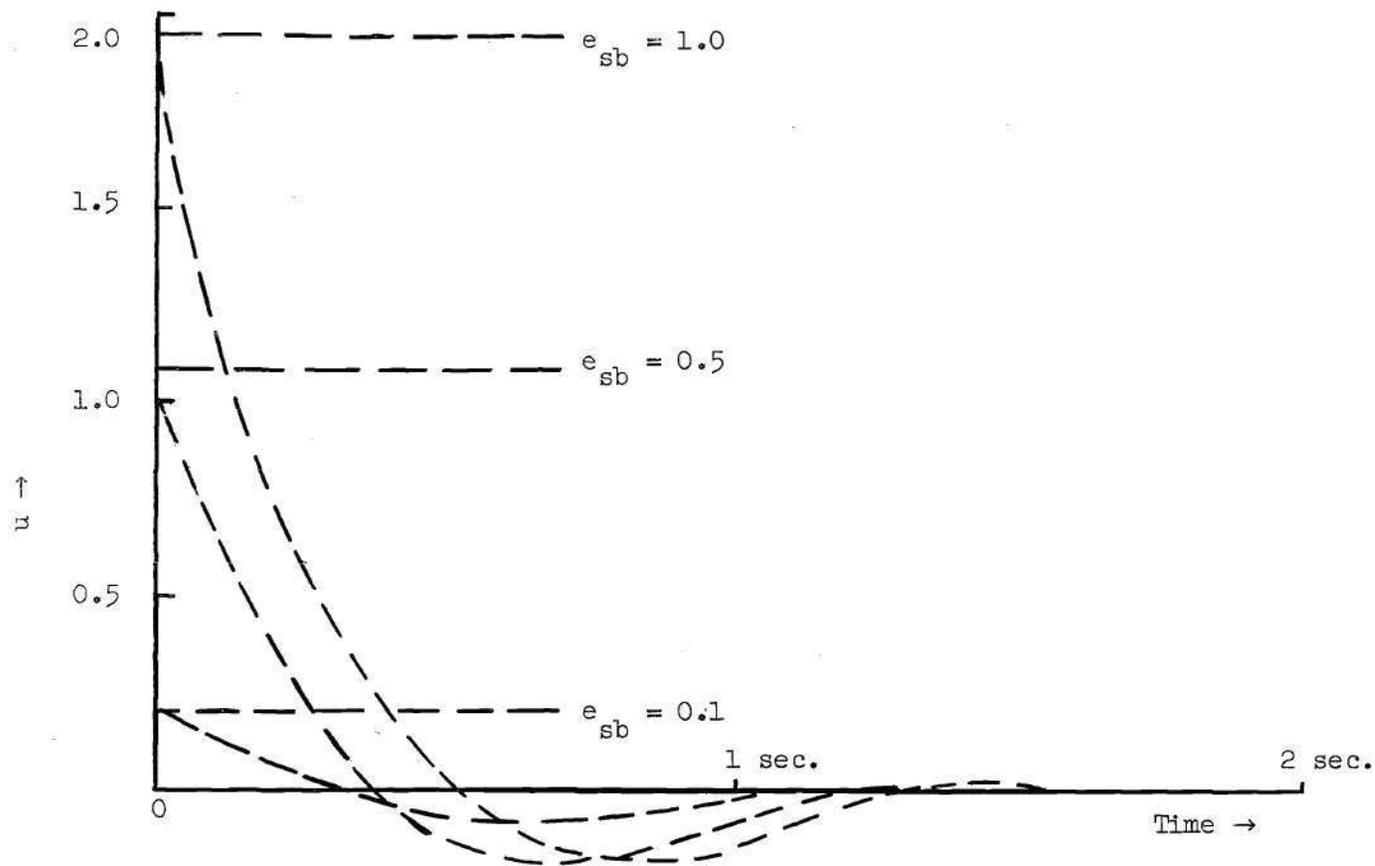


Figure 30. Decay of  $u$  for a Linear Term of 20

Linear Term = 10

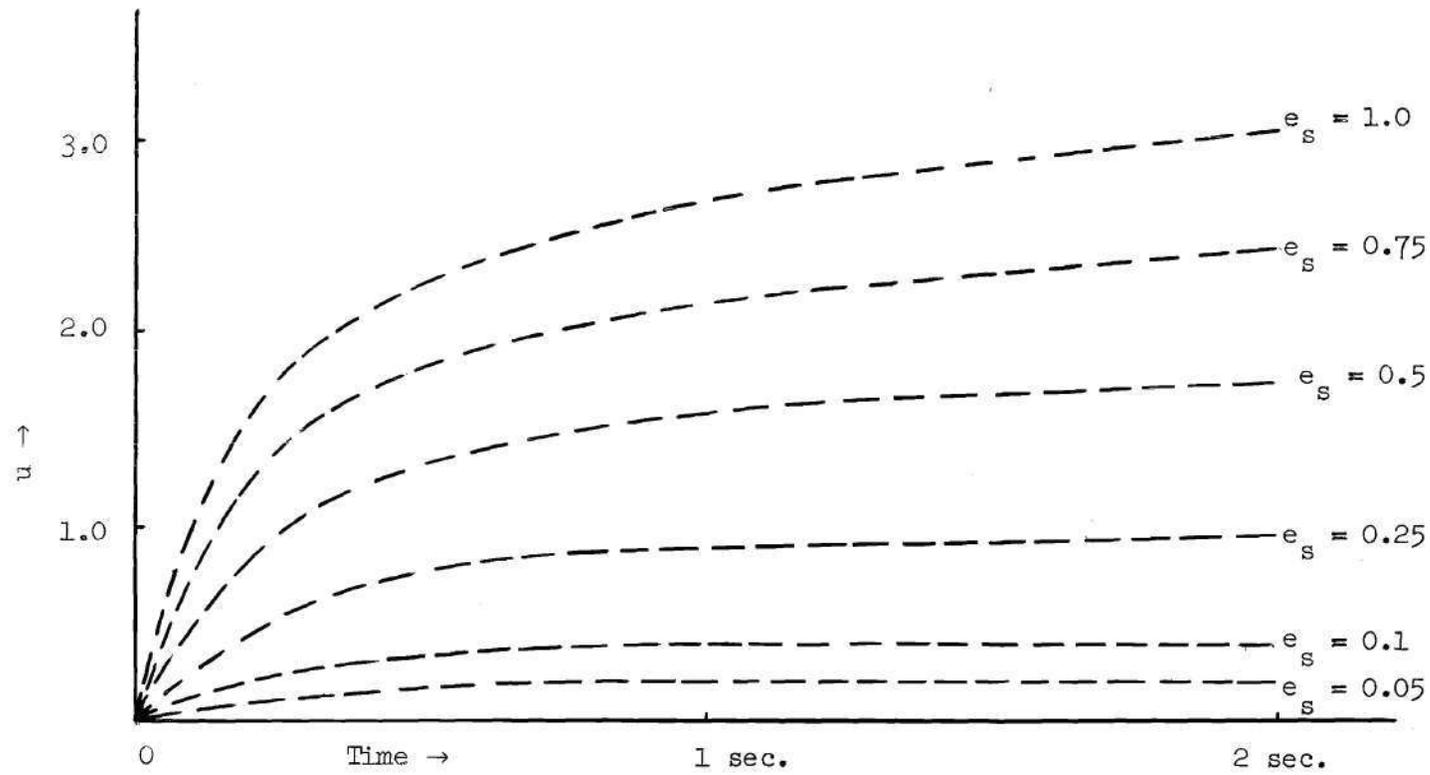


Figure 31. Response of a Type II Case (e) System

Linear Term  $\approx 10$

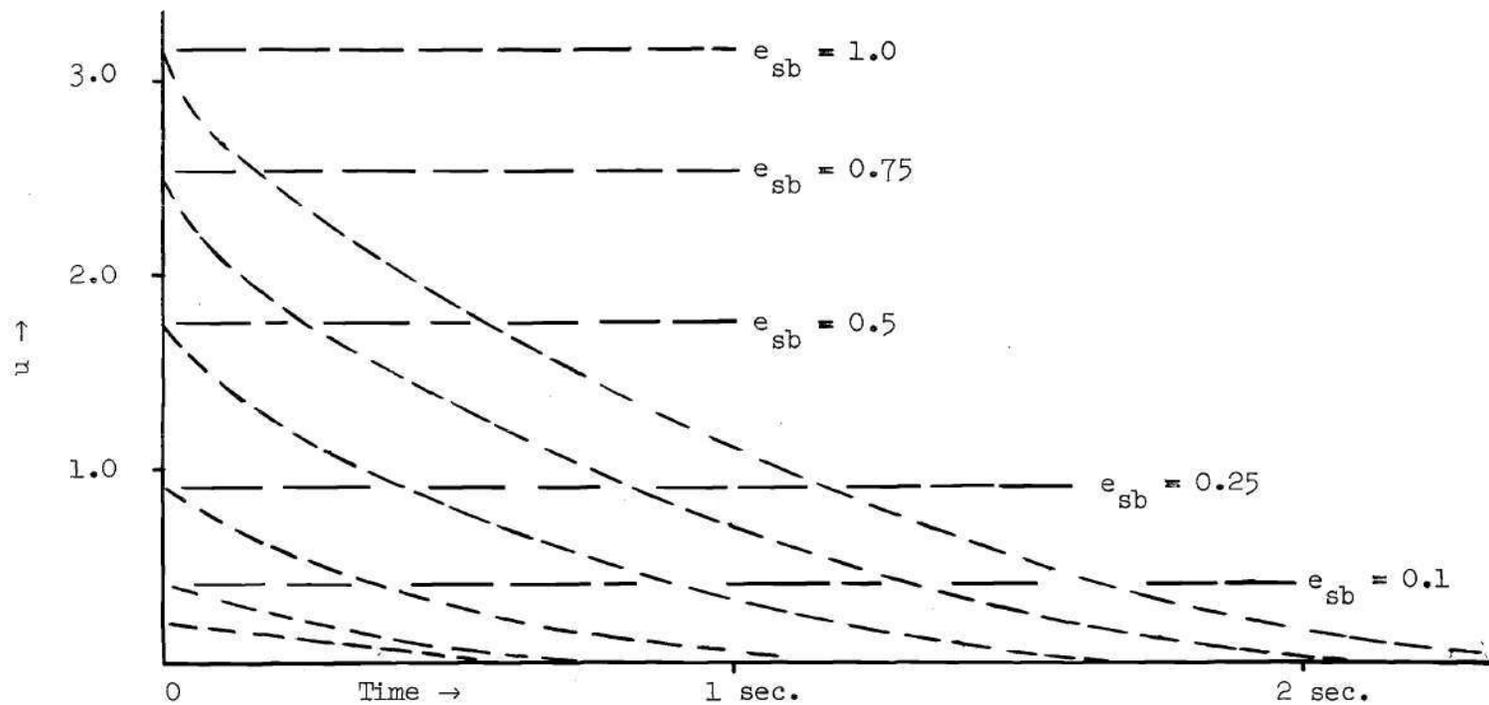


Figure 32. Decay of  $u$  for a Linear Term of 10

Linear Term = 0

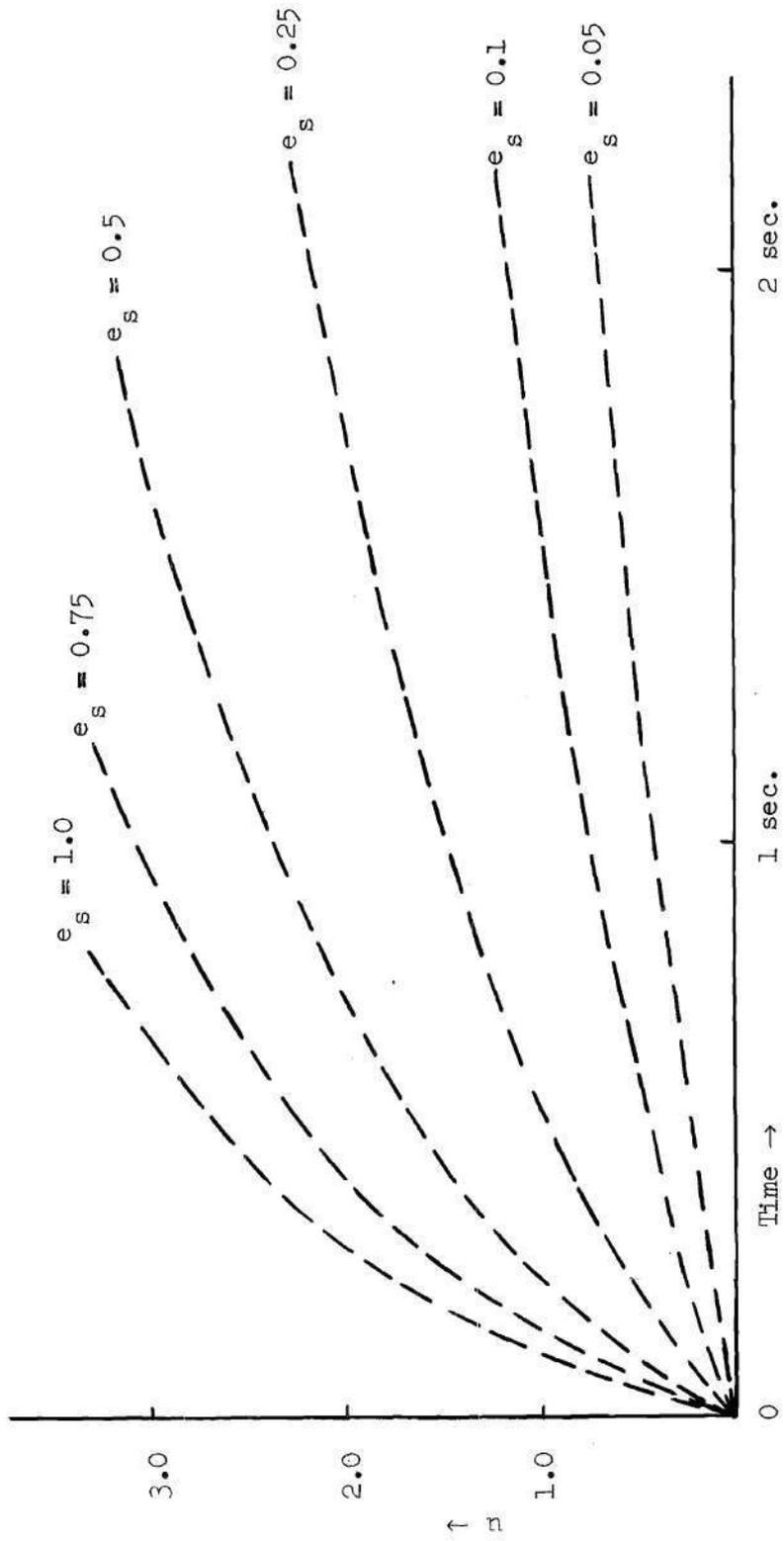


Figure 33. Response of a Type II Case (e) System

$$e_s = .25$$

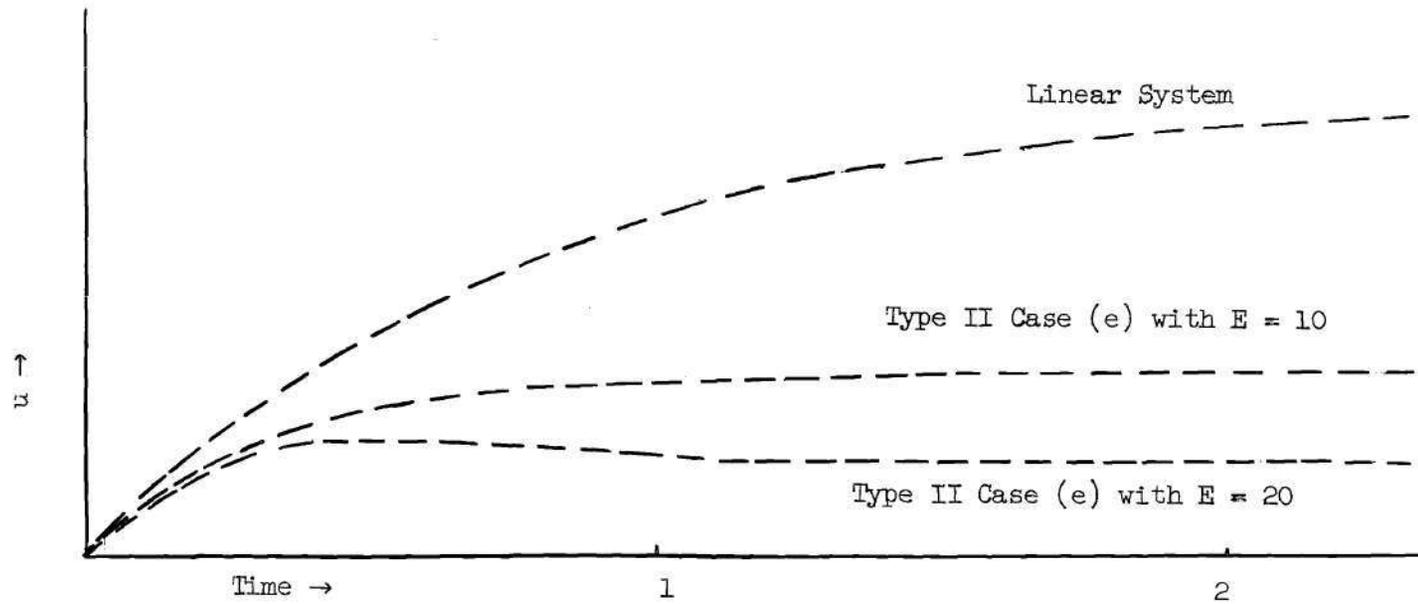


Figure 34. Comparison of Linear and Nonlinear System

## CHAPTER IX

### DISCUSSION AND RELATED PROBLEMS

Equally as important as the characterizing equations developed here is the insight gained into the fundamental reasons for the difference in response between linear and nonlinear servomechanisms. The subclass of differential equations examined contains the linear differential equation which is a special example of the Type II Case (a) system with

$$D = 0 .$$

It is the only member of this subclass whose response is a linear function of the constants of integration. Other members of the Type II system, as summarized in Tables 1 and 2, have responses which are semi-transcendental functions of the constants of integration and hence have poles whose locations are functions of these constants. Since all members of the Type II system have fixed critical points, it is the presence of these poles which are summarized in Table 3 that allows the response of systems with nonlinear characterizing equations to be faster (in some cases) than those systems characterized by linear equations. However, if these poles ever cross the positive real axis of the complex plane, the response of the characterizing equation becomes unstable. Unfortunately, the transcendental functions which enter into the solutions of the Type II system are such that asymmetrical control action is produced.

It is not to be inferred that movable critical points are detrimental to speed of response. The characterizing equation of Lewis' nonlinear servomechanism,

$$y'' + y'[Ay + b] + Ey = F, \quad (115)$$

satisfies the necessary conditions for a Type II Case (c) system, but not the sufficient conditions and hence has movable critical points. It can be shown that a critical point is due to a logarithmic term in the solution. Since the restoring force is linear, equation (115) will have symmetrical restoring action, but asymmetrical damping.

The type of instability encountered in the nonlinear servomechanisms synthesized in this paper is quite different from that experienced in linear servomechanisms. A second order linear servomechanism excited by a step function has a solution of the form

$$y = C_1 e^{p_1 t} + C_2 e^{p_2 t} + A.$$

If  $p_1$  or  $p_2$  has a positive real component, the response becomes unbounded and hence unstable. This type of instability is due to the presence of an essential singularity in the complex plane. Consider now the characterizing equation of the Type II Case (a) system which has a response of the form

$$y = Ae^{pt} \int [Be^{pt} + k_1, 0, k_2] - C.$$

If  $p$  has a positive real component, this response is unstable for the same reason as the linear response. However, if

$$B\epsilon^{pt} + k = 0 ,$$

the response will then become unstable because of the pole crossing the positive real axis of the complex plane. Some idea now emerges as to a stability criterion for nonlinear servomechanisms. Such a criterion should insure that the solution of the characterizing equation have the following features in the complex plane:

- (1) The critical points are such that the solution approaches zero along the positive real axis.
- (2) The poles do not cross the positive axis.
- (3) The solution has a real value along the positive real axis.

This thesis has shown that characterizing equations of the Type II system, excluding the linear equation, do not have these properties for every value of input signal amplitude. However, such a characterizing equation is physically realizable, for consider the equation

$$u'' + u' \left[ \frac{au'}{u-F} + b \right] + Cu = CF .$$

With the substitutions

$$y = u - F$$

and

$$y = T \frac{1}{1+at} ,$$

the solution can be obtained, and is

$$u = F + [Ae^{-\frac{bt}{2}} \sin(\omega_n t + \phi)]^{\frac{1}{1+a}} \quad (116)$$

where

$$\omega_n = \sqrt{(a+1)c - \frac{b^2}{4}},$$

and  $A$  and  $\phi$  are constants of integration. This is a symmetrical control system and if  $\frac{1}{1+a}$  is an odd integer, and if  $-1 < a < 0$ , it reacts faster than the linear system for small upsets, but is quite oscillatory for large upsets. In the complex plane, equation (116) has fixed critical points and no poles and satisfies the criteria stated above.

While the characterizing equations examined here are not stable for all signal amplitudes, some show promise for use as characterizing equations on nonlinear servomechanisms. Furthermore, the method of investigation has illustrated some fundamental differences between the response of linear and nonlinear servomechanisms. Other subclasses of the Type II system might be examined for their applicability as characterizing equations.

Related problems.--It has been shown that there are fifty canonical forms of equation (1) admitting to a solution with fixed critical points. One subclass has been examined here and the results have been promising. The members of the remaining subclasses should be examined for suitability as characterizing equations.

The acceptance of the response of a servomechanism to a unit step function as a valid criteria of its transient response has been questioned, and it has been proposed that the response to a ramp function contains

more useful information. The control systems synthesized here can be examined for a ramp input and a comparison should be made between the information gained by both inputs concerning the transient response of a nonlinear servomechanism.

The effect of a movable critical point upon the transient response of a nonlinear servomechanism has not been discussed here. However, it plays an important but obscure role. The obscurity stems from the complexity of the transcendental function which constitutes the solution of such a characterizing equation. Only a few characterizing equations with such solutions have been solved, but the characterizing equation of Lewis' nonlinear servomechanism seems to lend itself to a series solution which has been outlined by E. L. Ince. Examination of this solution might shed some light upon this effect.

Table 1

Summary of the Characterizing Equations of the Type II System  $y'' + [Ay + B]y' + Cy^3 + Dy^2 + Ey = F$ 

Case	Type of Control Damping Force	Restoring Force	Value of F	Role of Constants of Integration	Suitability for Control Systems
a	$By'$	$Ey + Dy^2$	$\frac{36B^2 - 625E^2}{2500D}$	Semi-transcendental	Some
b	$3ay'$	$-2y^3 + 2a^2y$	0	Semi-transcendental	None
c	$(2y + P)y'$	$Py^2$	Any	Semi-transcendental	Excellent
d	$(y + 3a)y'$	$-y^3 + 2a^2y$	0	Semi-transcendental	None
e	$(3y + B)y'$	$y^3 + By^2 + Ey$	Any	Semi-transcendental	Excellent

Table 2

Solutions of the Characterizing Equations of the Type II System  $y'' + [Ay + B]y' + Cy^3 + Dy^2 + Ey = F$ 

Case	Solutions
a	$y = -\frac{6}{D} \epsilon^{-\frac{2B}{5}t} \mathcal{F}\left[\left(\frac{5}{B} \epsilon^{-\frac{B}{5}t} - k\right); 0, 1\right] - \frac{1}{50D} \left[25E - 6B^2\right]$
b	$-jk_1 \epsilon^{-at} \operatorname{Sn}\left[k_1 \epsilon^{-at} + k_2   k^2 = -1\right]$
c	$-\sqrt{-k_1} \epsilon^{-\frac{p}{2}t} \left[ \frac{AJ'_v(u) + Y'_v(u)}{AJ_v(u) + Y_v(u)} \right]; u = \frac{2}{p} \sqrt{-k_1} \epsilon^{-\frac{p}{2}t}; = \frac{4F}{p^3}$
d	$k_1 u \frac{\mathcal{F}\left[\frac{k_1 u}{a} + k_2; 0, 1\right]}{\mathcal{F}\left[\frac{k_1 u}{a} + k_2; 0, 1\right]}; u = \epsilon^{-at}$
e	$y = \frac{u'}{u}; u'''' + Bu''' + Eu'' - Fu' = 0$

Table 3

Pole Location of the Solutions of the Characterizing Equations of the Type II System

$$w'' = [Aw + B]w' + Cw^3 + Dw^2 + Ew + F$$

Case	Conditions	Location
a	$B = E = F = 0$ $B, E, F, \neq 0$	$z = k$ $z = \frac{5}{B} \ln\left(\frac{B}{5} k\right)$
b	$B = 0$ $B \neq 0$	$2MK + i(2N + 1)K^{\frac{1}{2}}$ ; M, N integers $K_1 e^{-az} + K_2 = (2M + 1)K + i(2N + 1)K^{\frac{1}{2}}$
c		$A_1 J_{\nu}\left(\frac{2}{P} \sqrt{-K_1} e^{-\frac{P}{2}z}\right) = -Y_{\nu}\left(\frac{2}{P} \sqrt{-K_1} e^{-\frac{P}{2}z}\right)$
d	$a = R = S = 0$ $R \neq 0, a = S = 0$ $S = 0, R = -2a^2$	$z = -C_1$ $z = -2C_1 \sqrt{-3/R}, \quad (u; 12, C_1) = 1$ $z = -\frac{1}{a} \ln[-C_1/C_2(a)]$
e		$u''''(z) - Bu''(z) - Eu'(z) - Fu(z) = 0$

APPENDIX

## APPENDIX

Characterizing equation of a separately excited d.c. motor with viscous friction and inertia.--The schematic diagram of a separately excited d.c. motor with viscous friction and inertia is shown in Fig. A-1. With the assumptions listed in this figure, the equations of motion of the system can be expressed as

$$T_e = \text{electrical torque} = k_a i_a, \quad (\text{A-1})$$

$$T_m = \text{mechanical torque} = J\omega' + f\omega, \quad (\text{A-2})$$

and

$$e_t - \frac{1}{a} R_a i_a = k_m \omega, \quad (\text{generated voltage}). \quad (\text{A-3})$$

Substituting equations (A-2) and (A-1) into (A-3) leads to

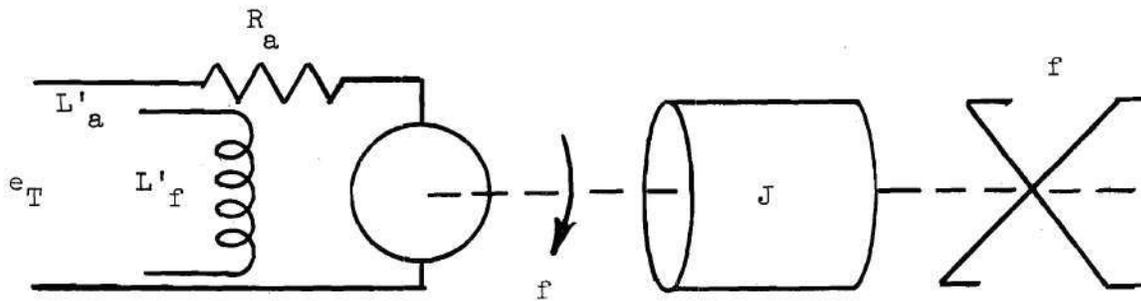
$$\frac{k_t e_t}{J R_a} = \omega' + \left( \frac{f}{J} + \frac{k_t k_m}{J R_a} \right) \omega. \quad (\text{A-4})$$

With the substitutions

$$a = \frac{k_t}{J R_a} \quad (\text{A-5})$$

and

$$b = \frac{f}{J} + \frac{k_t k_m}{J R_a} \quad (\text{A-6})$$



### Parameters

$J$  = Polar moment of inertia; slug-ft<sup>2</sup>

$f$  = Friction coefficient; ft-lb/rad./sec.

$k_T$  = Torque constant; ft-lb/amp.

$k_M$  = Generated voltage constant; volt/rad./sec.

$R_a$  = Armature resistance; ohms

$e_T$  = Terminal voltage

$I_a$  = Armature current

$\omega$  = Angular velocity; rad./sec.

### Assumptions

- (1) Negligible armature inductance
- (2) Linear armature resistance
- (3) No magnetic saturation

Figure A-1. Separately Excited d.c. Motor with Inertia  
and Viscous Friction

equation (A-4) becomes the characterizing equation

$$ae_t = \omega' + b\omega . \quad (\text{A-7})$$

It is customary to show this equation in block diagram form in Fig. A-2 where  $s$  is the symbol for the Laplace transform. This is the block diagram of the unaltered or open loop system. For  $e_t$  a step function, the response of equation (A-7) becomes

$$\omega = \frac{ae_t}{b} [1 - e^{-bt}] . \quad (\text{A-8})$$

In order to improve the response of this system, it is necessary to alter the characterizing equation (A-7) of the system. This can be accomplished as in Fig. A-3 by means of a linear controller which integrates the error of the system. If the error of the system is defined as

$$\epsilon = e_s - k_f\omega , \quad (\text{A-9})$$

then the output of the controller is

$$u = \int_0^t (e_s - k_f\omega) dt . \quad (\text{A-10})$$

In terms of the variable  $u$ , the characterizing equation of the compensated system becomes

$$u'' + bu' + k_f aG_1 = be_s , \quad (\text{A-11})$$

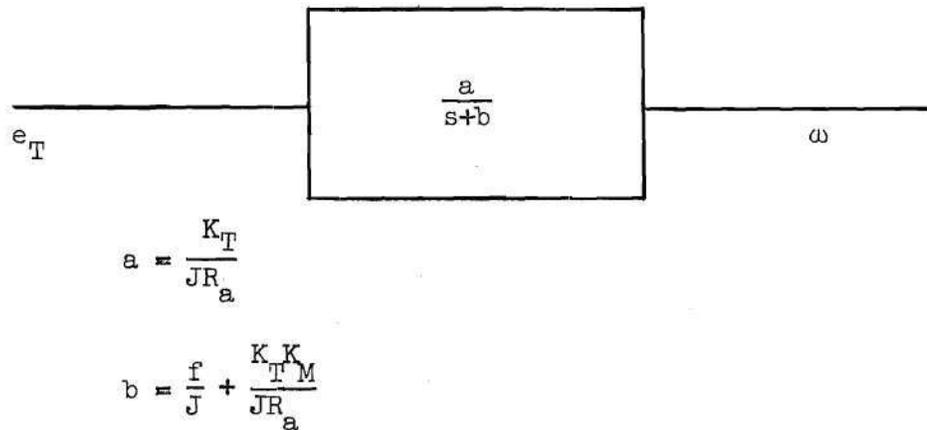


Figure A-2. Block Diagram of Open Loop System

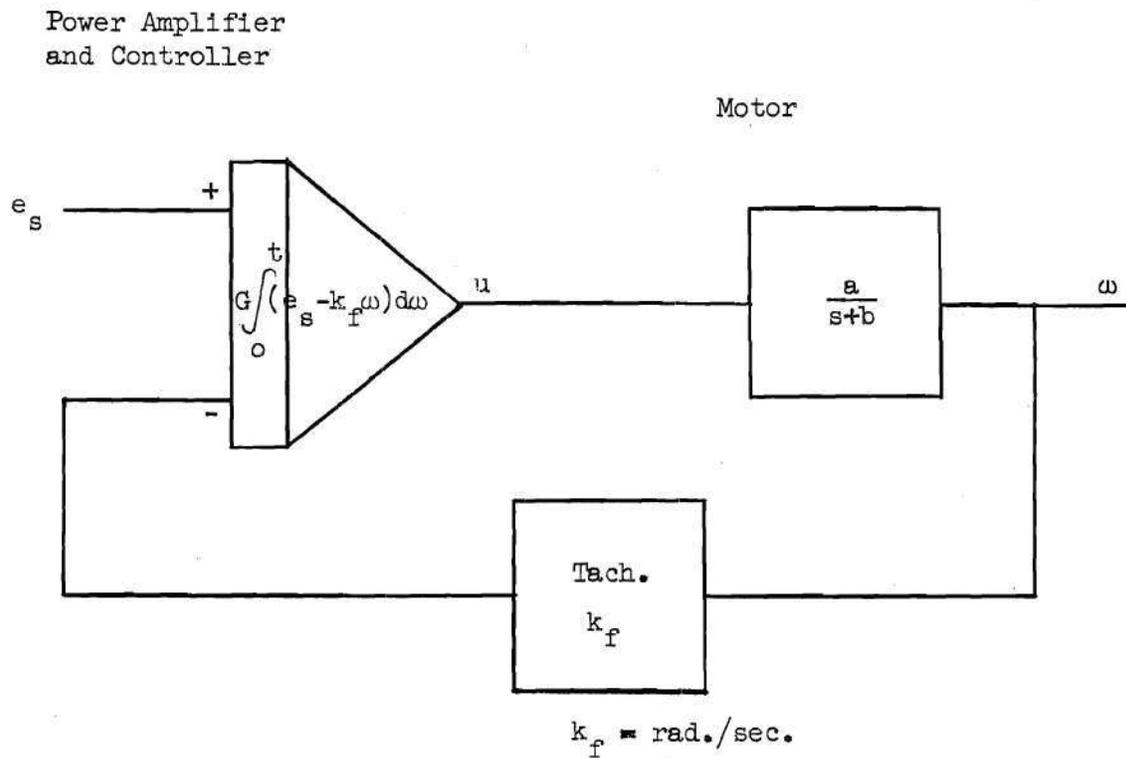


Figure A-3. Block Diagram of Compensated System

where

$$u' = e_s - k_f \omega . \quad (\text{A-12})$$

The addition of the linear controller has altered the characterizing equation to a second order linear differential equation. The solution to equation (A-11) is

$$u = [C_1 e^{\beta t} + C_2 e^{-\beta t}] e^{-bt} + \frac{be_s}{ak_f G} , \quad (\text{A-13})$$

where  $C_1$  and  $C_2$  are constants of integration and

$$\beta = \pm \sqrt{\frac{b^2}{4} - k_f a G} . \quad (\text{A-14})$$

Since the response of  $u$  is a linear function of the constants of integration, they will not change the form of the differential equation as they are changed by the initial conditions.

It is customary to choose the controller gain  $G$  such that

$$k_f a G > \frac{b^2}{4} . \quad (\text{A-15})$$

Equation (A-13) can then be expressed as

$$u = A e^{-\frac{b}{2}t} \sin(\beta t + \phi) + \frac{be_s}{k_f a G} , \quad (\text{A-16})$$

where  $A$  and  $\phi$  are constants of integration which are related to the initial conditions by

$$u|_{t=0} = A \sin \phi + \frac{be}{k_f a G} s, \quad (\text{A-17})$$

and

$$u'|_{t=0} = A \beta \cos \phi - \frac{b}{2} A \sin \phi. \quad (\text{A-18})$$

Equation (A-16) is the underdamped or oscillatory response.

Physical system utilized.--In Fig. A-4 are shown the physical systems utilized in this thesis. The power amplifier consists of a push-pull d.c. amplifier and an amplidyne. The viscous friction and inertia load were simulated by means of a d.c. generator with a resistive load. By adjusting the load resistance of the generator, the effective viscous friction could be altered. It was assumed that the time constants of the amplidyne were negligible compared to the d.c. motor and its load. Figs. A-5 to A-8 show that the assumptions of linearity are within reason. The armature resistance of the motor can be considered a constant only over a limited range of speed. For this reason the change in speed was kept to 300 rpm or less. An analogue computer was used to obtain the integration and the error of the system. The responses of the systems utilized in this thesis were portrayed on the x-y recorder of the analogue computer.

Evaluation of motor time constants.--The characterizing equation of the physical system has the steady state solution

$$\omega_{s.s.} = \frac{e}{b} e_t. \quad (\text{A-19})$$

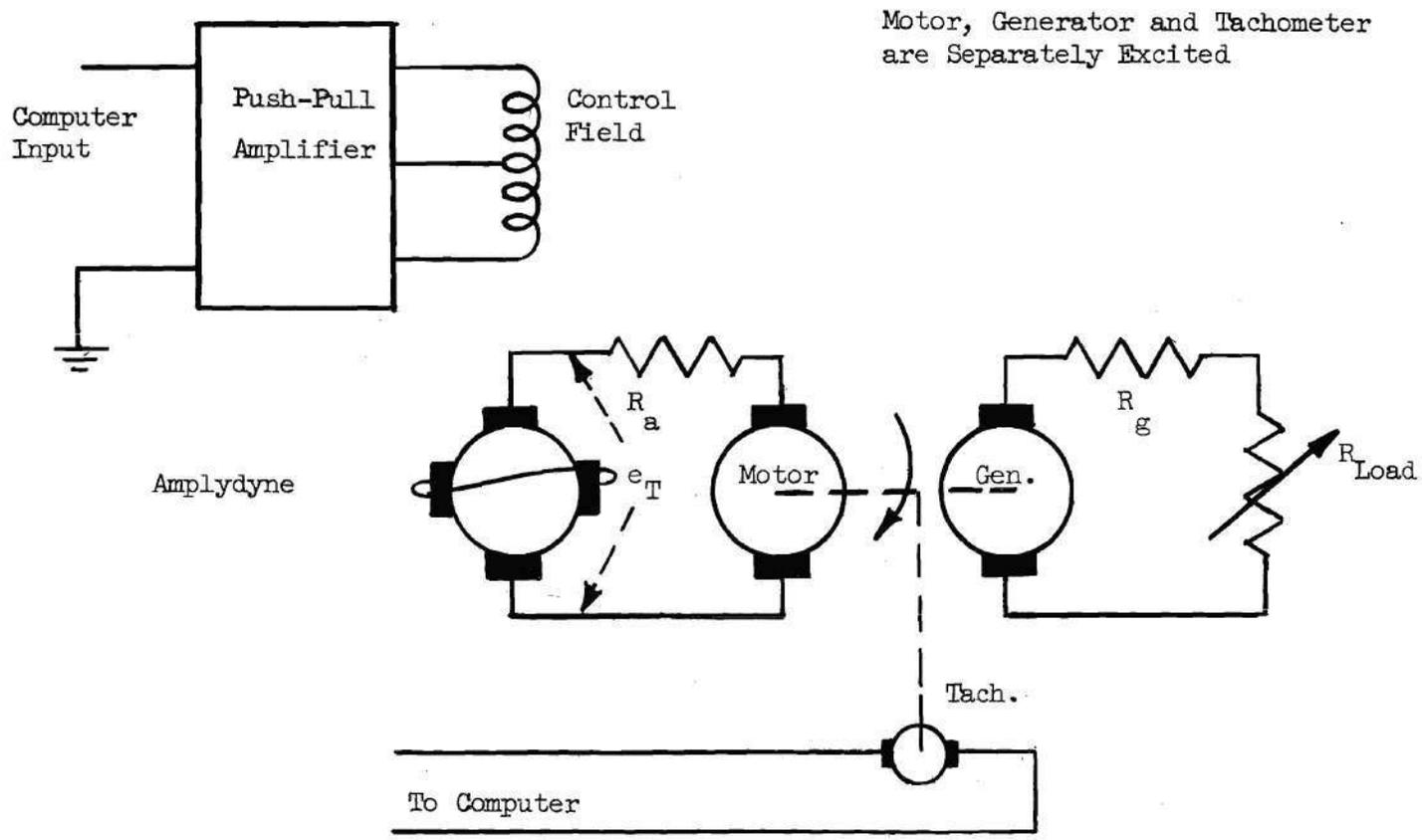


Figure A-4. Schematic Diagram of the Physical System

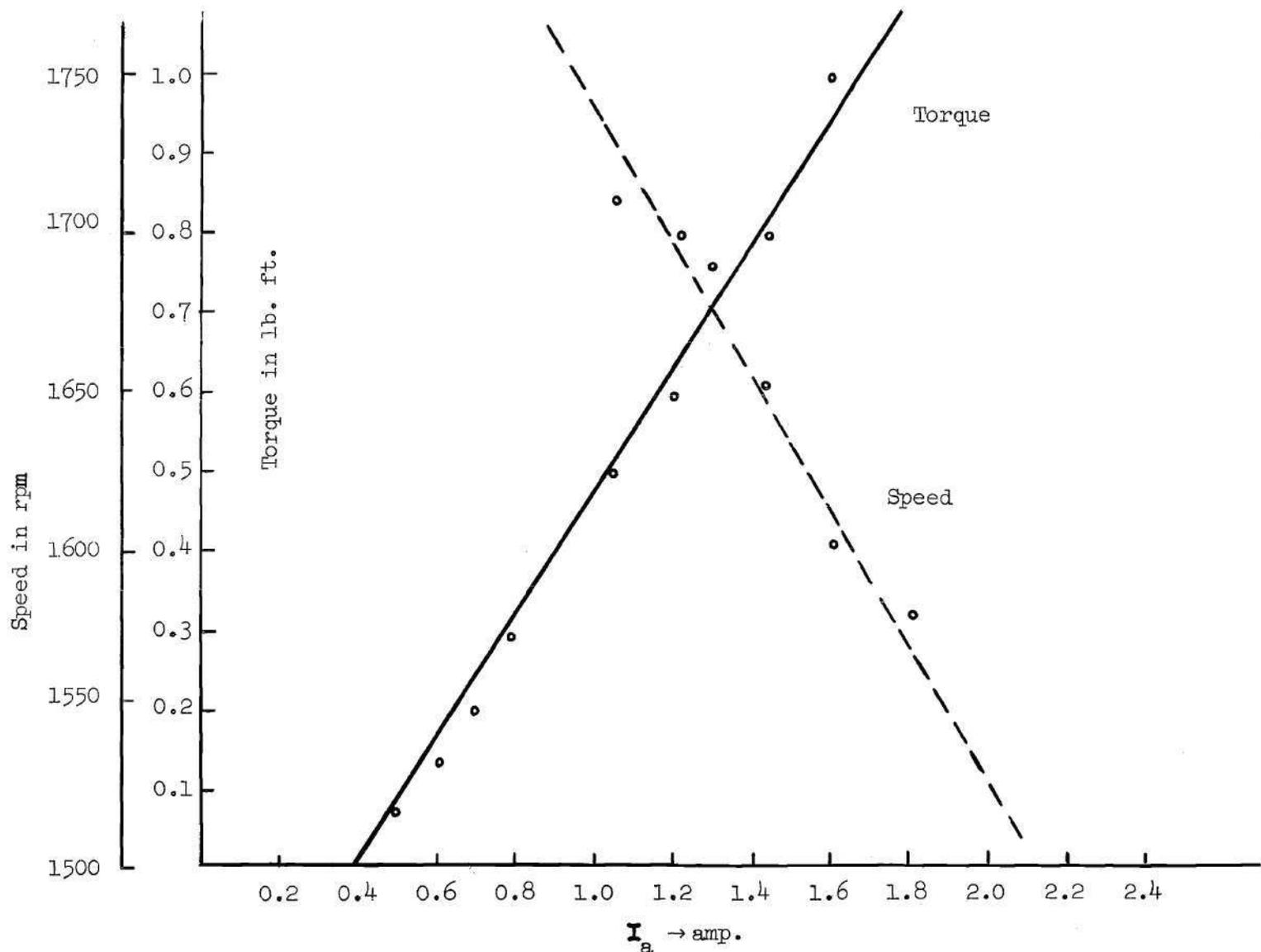


Figure A-5. Speed-Torque Relationship of the Motor

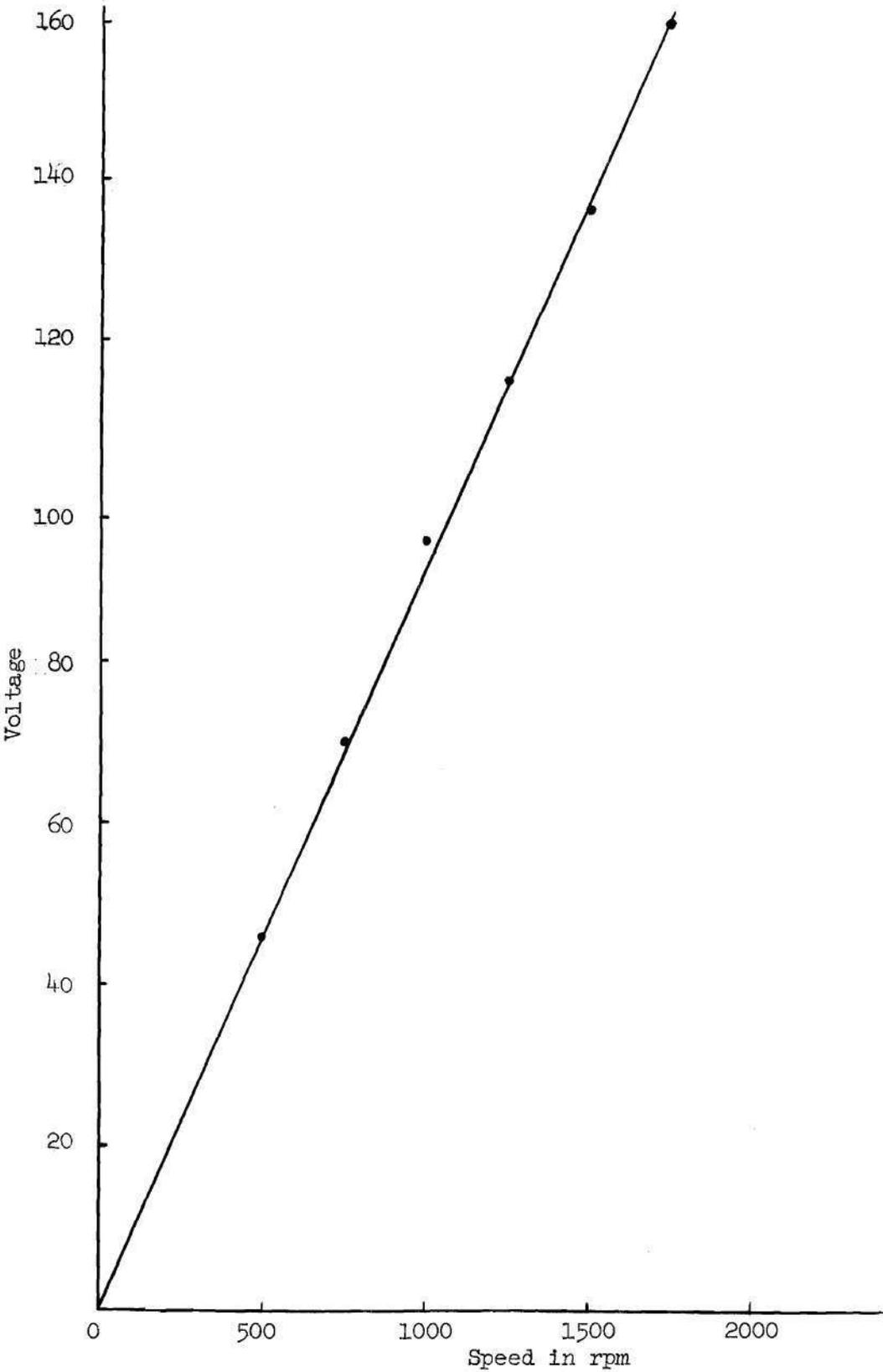


Figure A-6. No-Load Generated Voltage vs. Speed

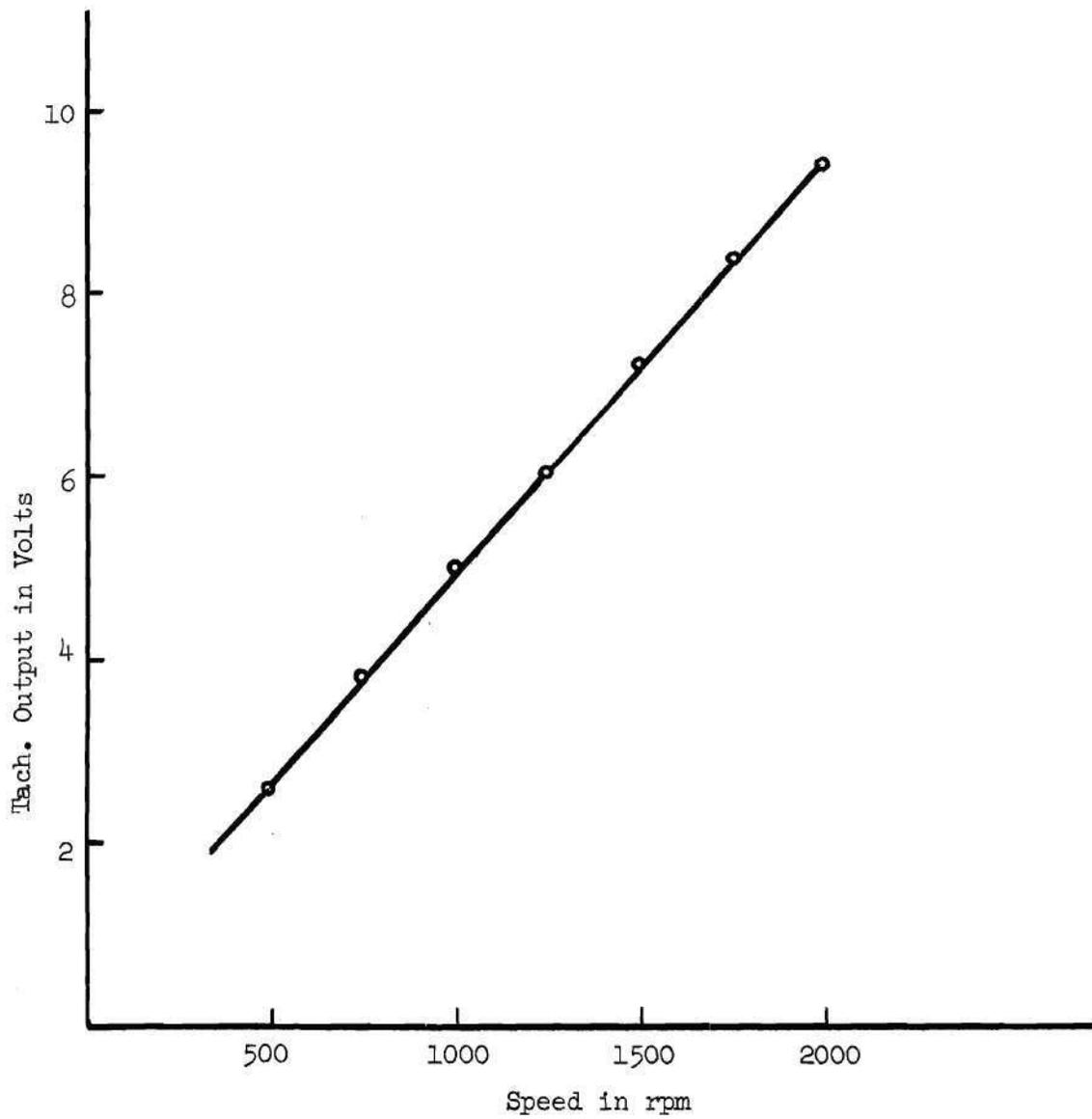


Figure A-7. Speed-Voltage Response of the Tachometer

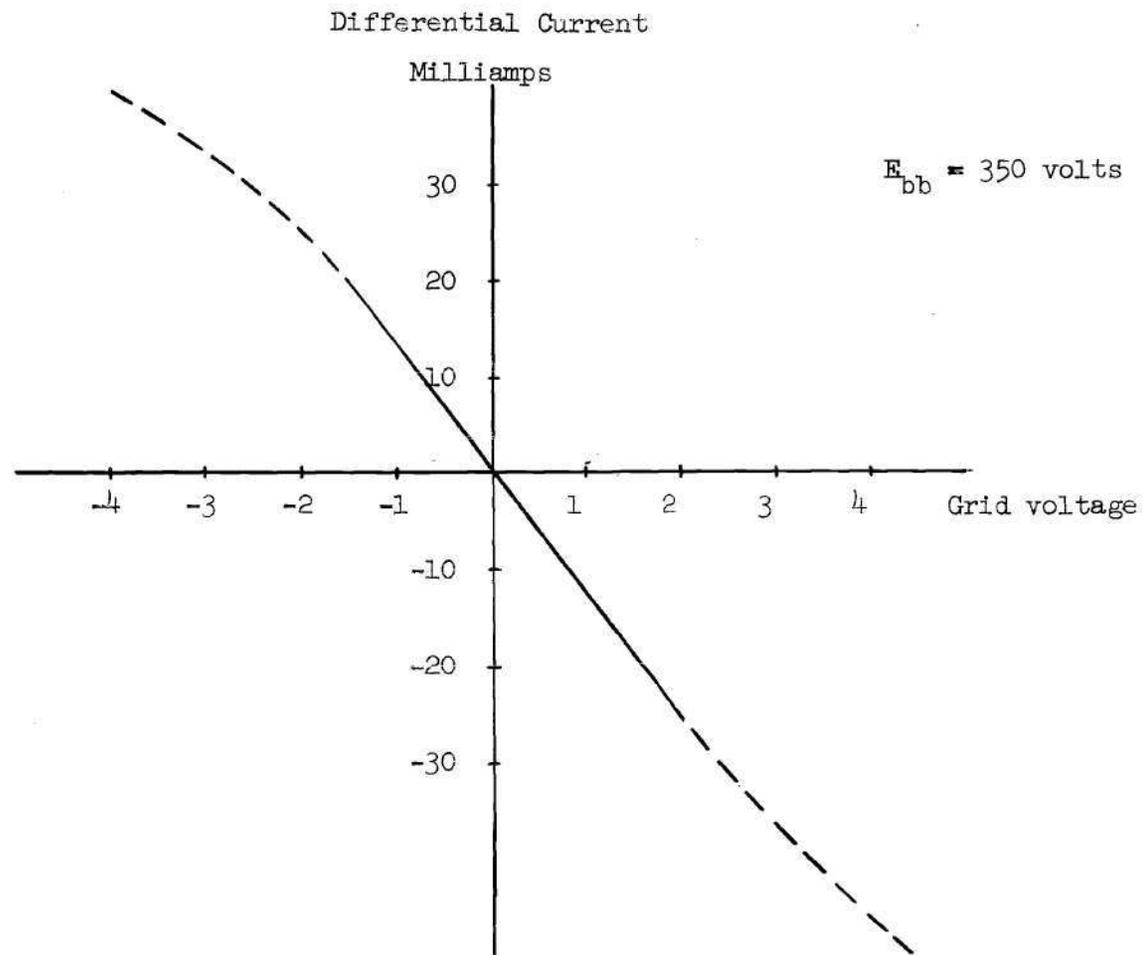


Figure A-8. Response of the Push-Pull Amplifier.

If the system is operating at some steady state speed  $\omega_{s.s.}$  and the excitation is removed, the resulting decay of speed is given by

$$\omega = \omega_{s.s.} e^{-bt} \quad (A-20)$$

Taking the logarithm to the base 10 of both sides of equation (A-20) leads to

$$\log_{10}\left(\frac{\omega}{\omega_{s.s.}}\right) = -bt \log_{10} e, \quad (A-21)$$

which plots as a straight line. The slope of this line is

$$-b \log_{10} e$$

from which  $b$  may be calculated. Once  $b$  is known, equation (A-19) can be utilized to obtain  $a$ . In Fig. A-9 is shown a typical decay curve of this system and in Fig. A-10 is a plot of equation (A-21). Since the slope of this curve is 1.77,

$$b = 43.6$$

and

$$a = 43.6 .$$

Response of physical system.--Typical values utilized in this thesis are

$$G = 10$$

$$K_f = .01$$

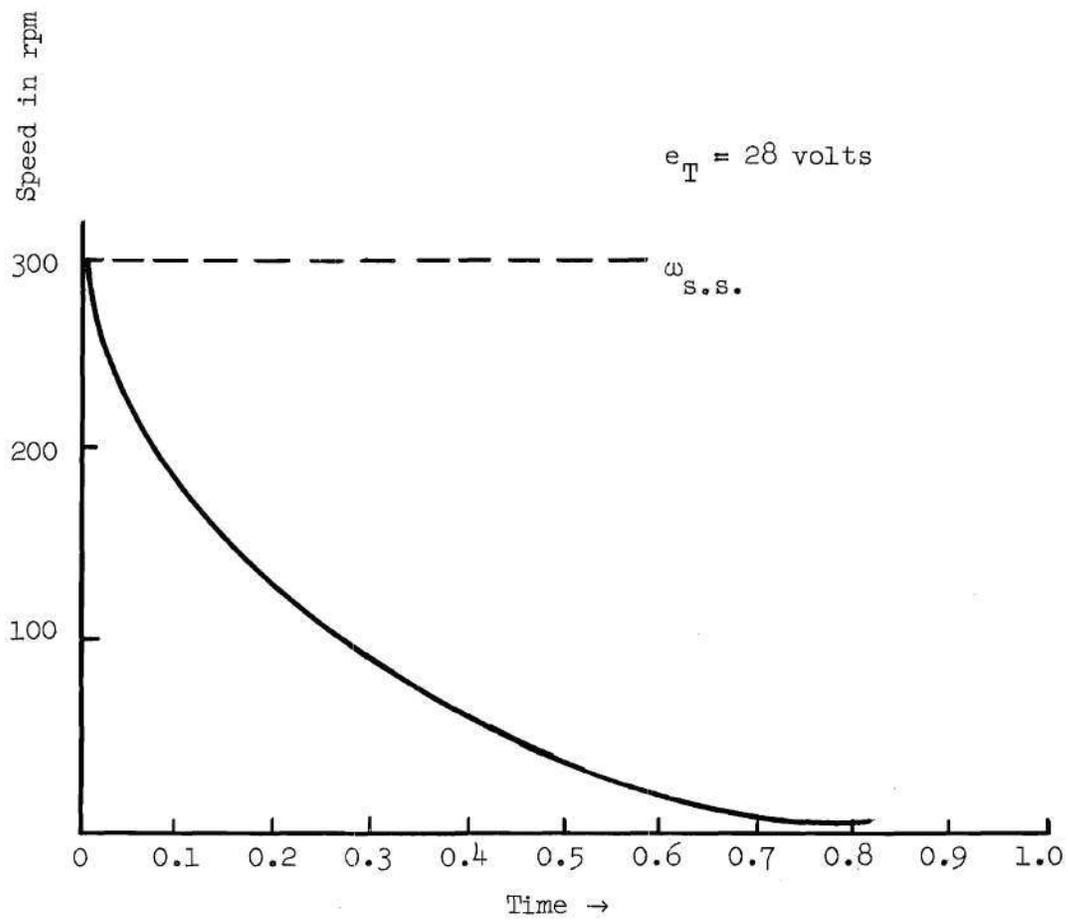


Figure A-9. Decay of Speed

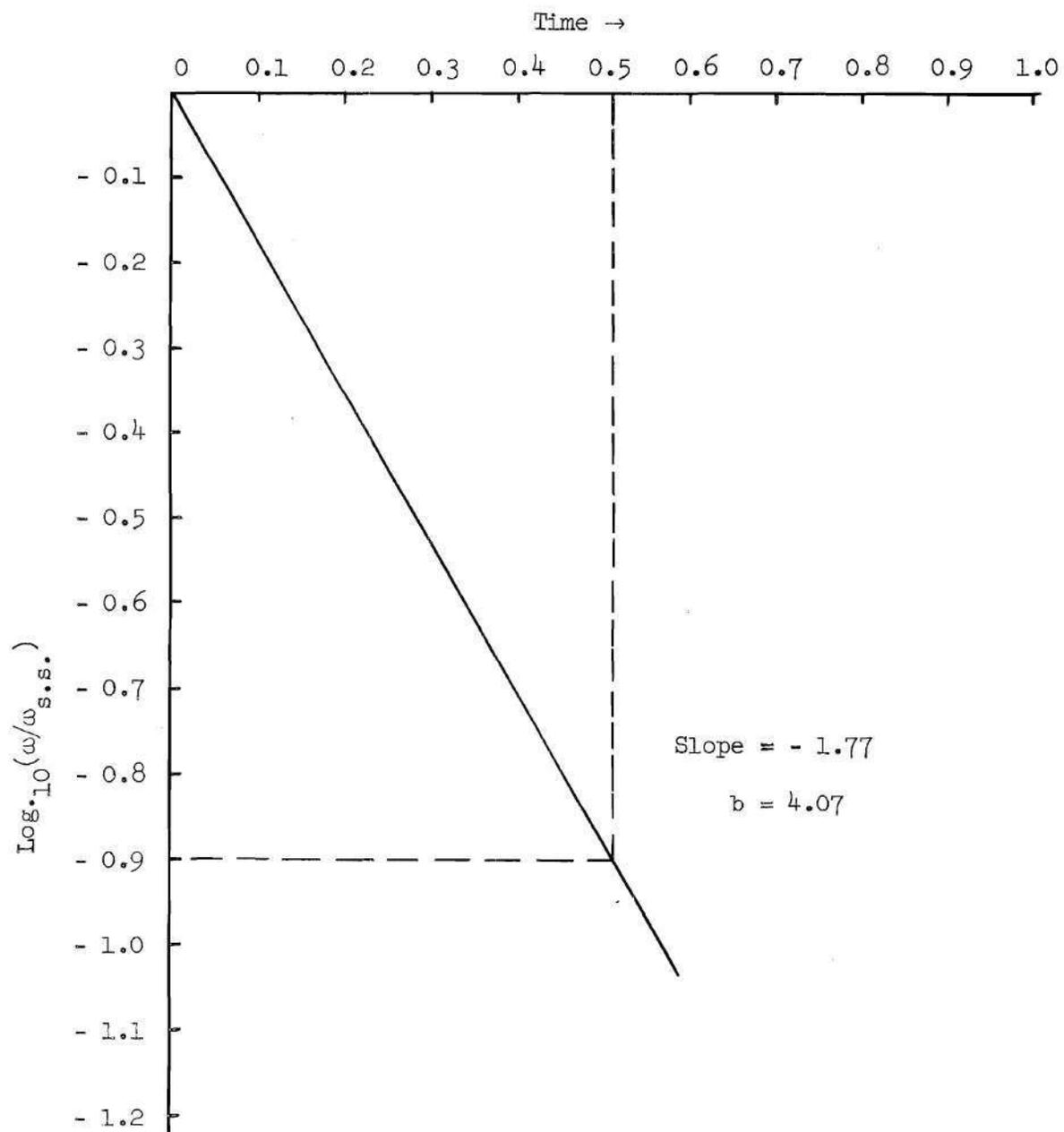


Figure A-10. Calculation of b

and

$$e_s = 1 \text{ volt.}$$

Hence

$$\beta = .6$$

and

$$b/2 = 2.035 .$$

For the system starting from rest, the initial conditions are

$$u(0) = 0$$

$$u'(0) = 1$$

and hence

$$A = 1.61$$

and

$$\phi = -35.4^\circ .$$

Thus the response of this system to a 1 volt step function is

$$u = 1.61e^{-2.035t} \sin(.6t - 35.4^\circ) + .931 .$$

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## VITA

John Milton Bailey, Jr. was born in Memphis, Tennessee, on June 3, 1925. He is the son of John M. and Ruth G. Bailey. He was married in 1949 to Miss Peggy Johnston of Memphis, Tennessee, and has two children.

He attended public schools in Memphis, Tennessee, where he graduated from Memphis Technical High School in 1943. He attended Davidson College at Davidson, North Carolina, and received the Bachelor of Science degree in Physics in 1949. He attended the University of Tennessee at Knoxville, Tennessee, where he received the Master of Science degree in Electrical Engineering in 1952. His master's thesis was in the field of linear servomechanisms. He has attended the Georgia Institute of Technology since 1954.

His experience includes three years as an instrument engineer with E. I. DuPont at Orange, Texas, where he was engaged in developing process control instrumentation. From 1954 to 1956 he was an instructor in the School of Electrical Engineering at the Georgia Institute of Technology, Atlanta, Georgia, and from 1956 to 1959 he was an Assistant Professor. He also served as a consultant for the Georgia Power Company in Atlanta, Georgia.